Math 489AB Exam 2 Power Vectors

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Text

Applied Linear Algebra: The Decoupling Principle. 2nd ed. Lorenzo Sadun

Notation

A	Matrix
$p_A(\lambda)$	Characteristic Polynomial
λ	Eigenvalue
$\mathbf{v}_i$	Eigenvector
$\mathbf{w}_i$	Power vector

Concepts

- A power vector $\mathbf{w}$ of order p satisfies $(A - \lambda I)^p \mathbf{w} = \mathbf{0}$, but $(A - \lambda I)^{p-1}\mathbf{w} \neq \mathbf{0}$. [Definition]
- A power vector of order p can systematically generate p linearly independent power vectors of progressively smaller order. [Theorem - Exercise 8 and 9]

Algorithms/Interpretations

1. Focus on a Generating Power Vector

- Identify a power vector of order p, say $\mathbf{w}_p$. This can be tricky, we show how in another section.
- Generate p linearly independent power vectors of progressively smaller order by performing:

$$\begin{aligned} \mathbf{w}_p &\quad \text{Order } p \\ (A - \lambda I)\mathbf{w}_p = \mathbf{w}_{p-1} &\quad \text{Order } p-1 \\ (A - \lambda I)^2\mathbf{w}_p = \mathbf{w}_{p-2} &\quad \text{Order } p-2 \\ \vdots &\quad \vdots \\ (A - \lambda I)^{p-1}\mathbf{w}_p = \mathbf{w}_1 &\quad \text{Order } 1 \end{aligned}$$

- We now have a linearly independent set of p vectors: $\{\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_p\}$.

2. Focus on a Generating Eigenvector

- Identify an eigenvector $\mathbf{v}$. We already know how to do this.
- Assume a power vector of the next order exists, say $\mathbf{w}_2$. If it does, then it it possible that it is such that

$$(A - \lambda I)\mathbf{w}_2 = \mathbf{v}.$$

  Solve this equation by taking RREF of $\left[A - \lambda I \mid \mathbf{v}\right]$. This can be repeated:

$$(A - \lambda I)\mathbf{w}_{i+1} = \mathbf{w}_i.$$

Finding Power Vectors and Their Orders

- We start with a matrix A. Find the eigenvalues from $p_A(\lambda) = 0$.
- Start with an eigenvalue λ_i. $(A - \lambda I)$ yields the space of all power vectors of order 1 (namely the eigenvectors), E_{λ_i}.
- $(A - \lambda I)^2$ yields the space of all power vectors of order 2 and lower, $\tilde{E}_{\lambda_i}$. Note that:

$$\tilde{E}_{\lambda_i} = (\text{Set of Order 2 Power Vectors}) \cup (\text{Set of Order 1 Power Vectors})$$

- Similarly $(A - \lambda I)^3$ yields $\tilde{\tilde{E}}_{\lambda_i}$ where

$$\tilde{\tilde{E}}_{\lambda_i} = (\text{Set of Order 3 Power Vectors}) \cup (\text{Set of Order 2 Power Vectors}) \cup (\text{Set of Order 1 Power Vectors})$$

Jordan Canonical Form Cases

We list the cases that require Jordan Canonical form for a 3×3 matrix.

- Case 1: $p_A(\lambda) = (\lambda - \lambda_i)^3$, $m_g(\lambda_i) = 2$. Can be annoying!
- Case 2: $p_A(\lambda) = (\lambda - \lambda_i)^3$, $m_g(\lambda_i) = 1$.
- Case 3: $p_A(\lambda) = (\lambda - \lambda_i)^2(\lambda - \lambda_j)$, $m_g(\lambda_i) = m_g(\lambda_j) = 1$. Can be annoying!

Warnings

- Often when we form eigenvectors we will get RREF results like $(-1/2, 1, 3)$. Some of us like to choose a scaled version for simplicity like $(-1, 2, 6)$. DO NOT EVER SCALE WITH JORDAN FORM PROBLEMS! The Jordan Canonical form is VERY picky and will not like you. Always use the exact vectors that come out of the RREF process.
- Challenges of Method 1: Can be time-consuming to sift through all the power spaces.
- Challenges of Method 2: Solutions to the augmented matrix are not always obvious.
- Challenges to both Methods: If there are two eigenvectors, the problems tend to become tricky.

Parallel Example (Easy)

Let $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$. We find $p_A(\lambda) = (\lambda - 1)^3$.

Power Vector to Eigenvector

$$(A - 1I) = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \to E_1 = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$(A - 1I)^2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \tilde{E}_1 = E_1 \cup \operatorname{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$(A - 1I)^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \tilde{\tilde{E}}_1 = E_1 \cup \tilde{E}_1 \cup \operatorname{Span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

So, $(0, 0, 1)^\top$ is order 3, $(0, 1, 0)^\top$ is order 2, and $(1, 0, 0)^\top$ is order 1. These will not create a Jordan Canonical basis, however! We use the process:

$$\mathbf{w}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{w}_2 = (A - 1I)\mathbf{w}_3 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

$$\mathbf{w}_1 = (A - 1I)^2\mathbf{w}_3 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

Thus, a Jordan Canonical basis is

$$\mathcal{B} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$$

$$\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Eigenvector to Power vector

$$(A - 1I) = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \to E_1 = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

We will choose the eigenvector $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. We will assume that there is some power vector of order 2. It must be

$$\mathbf{v}_1 = (A - 1I)\mathbf{w}_1.$$

Solving the RREF of $\left[A - \lambda I \mid \mathbf{v}\right]$ gives

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

We now assume that there is a power vector of order 3. It must be

$$\mathbf{w}_1 = (A - 1I)\mathbf{w}_2.$$

Solving the RREF of $\left[A - \lambda I \mid \mathbf{w}_1\right]$ gives

$$\mathbf{w}_2 = \begin{pmatrix} 1 \\ -1/2 \\ 1/2 \end{pmatrix}.$$

Thus, a Jordan Canonical basis is

$$\mathcal{B} = \{\mathbf{v}_1, \mathbf{w}_1, \mathbf{w}_2\}$$

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1/2 \\ 1/2 \end{pmatrix} \right\}$$

Parallel Example (Medium)

Let $A = \begin{pmatrix} 5 & -3 & -1 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. We find $p_A(\lambda) = (\lambda - 5)^2(-\lambda)$.

Power Vector to Eigenvector

$$(A - 5I) = \begin{pmatrix} 0 & -3 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & -5 \end{pmatrix} \to E_5 = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$(A - 5I)^2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -5 \\ 0 & 0 & 25 \end{pmatrix} \to \tilde{E}_5 = E_5 \cup \text{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$(A - 0I) = \begin{pmatrix} 5 & -3 & -1 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to E_0 = \text{Span}\left\{ \begin{pmatrix} 2/25 \\ -1/5 \\ 1 \end{pmatrix} \right\}$$

Note that using higher powers of $(A - \lambda I)^i$ yields redundant information (same span). So, $(0,1,0)^\top$ is order 2, $(1,0,0)^\top$ is order 1, as is $(2/25, -1/5, 1)^\top$. We use the process:

$$\mathbf{w}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathbf{w}_2 = (A - 5I)\mathbf{w}_3 = \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{w}_1 = (A - 5I)^2\mathbf{w}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Uh oh... not enough vectors! Where is $\mathbf{w}_1$? We have a partial basis, $\{\mathbf{w}_2, \mathbf{w}_3\}$. Well, let's just find a power vector somewhere that is not a member of the span of this set. Then, we should have a full basis, and hopefully a Jordan Canonical one. The only choice that seems logical is the eigenvector associated with E_0. In fact, this works! Thus, a Jordan Canonical basis is

$$\mathcal{B} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$$

$$\mathcal{B} = \left\{ \begin{pmatrix} 2/25 \\ -1/5 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Eigenvector to Power vector

$$(A - 5I) = \begin{pmatrix} 0 & -3 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & -5 \end{pmatrix} \to E_5 = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$(A - 0I) = \begin{pmatrix} 5 & -3 & -1 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to E_0 = \text{Span}\left\{ \begin{pmatrix} 2/25 \\ -1/5 \\ 1 \end{pmatrix} \right\}$$

We have the eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{u}_1 = \begin{pmatrix} 2/25 \\ -1/5 \\ 1 \end{pmatrix}$. We will assume that there is some power vector of order 2. It must be

$$\mathbf{v}_1 = (A - 5I)\mathbf{w}_1 \text{ or } \mathbf{u}_1 = (A - 0I)\mathbf{w}_1.$$

There is no solution for $\mathbf{u}_1$. Solving the RREF of $\left[A - 5I \mid \mathbf{v}_1\right]$ gives

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ -1/3 \\ 0 \end{pmatrix}.$$

Well, now we have 3 linearly independent power vectors. Thus, a Jordan Canonical basis is

$$\mathcal{B} = \{\mathbf{v}_1, \mathbf{w}_1, \mathbf{u}_1\}$$

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1/3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2/25 \\ -1/5 \\ 1 \end{pmatrix} \right\}$$

Parallel Example (Hard)

Let $A = \begin{pmatrix} 1 & -3 & -1 \\ 1 & 5 & 1 \\ -2 & -6 & 0 \end{pmatrix}$. We find $p_A(\lambda) = -(\lambda-2)^3$.

Power Vector to Eigenvector

$$(A-2I) = \begin{pmatrix} -1 & -3 & -1 \\ 1 & 3 & 1 \\ -2 & -6 & -2 \end{pmatrix}$$
$$\to E_2 = \text{Span}\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$
$$(A-2I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \tilde{E}_2 = E_2 \cup \text{Span}\left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \right\}$$

Note that the strange vector $(1,3,1)^\top$ was chosen as the cross product of the spanning vectors in E_2. $(0,0,1)^\top$ is another suitable choice. So, $(1,3,1)^\top$ is order 2, $(-3,1,0)^\top$ is order 1, as is $(-1,0,1)^\top$. We use the process:

$$\mathbf{w}_3 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$
$$\mathbf{w}_2 = (A-2I)\mathbf{w}_3 = \begin{pmatrix} -11 \\ 11 \\ -22 \end{pmatrix}$$
$$\mathbf{w}_1 = (A-2I)^2\mathbf{w}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Uh oh... not enough vectors! Where is $\mathbf{w}_1$? We have a partial basis, $\{\mathbf{w}_2, \mathbf{w}_3\}$. Well, let's just find a power vector somewhere that is not a member of the span of this set. Then, we should have a full basis, and hopefully a Jordan Canonical one. Looks like either of the original eigenvectors will do. Let's just take $(-3,1,0)^\top$ In fact, this works! Thus, a Jordan Canonical basis is

$$\mathcal{B} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$$
$$\mathcal{B} = \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -11 \\ 11 \\ -22 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \right\}$$

Eigenvector to Power vector

$$(A-2I) = \begin{pmatrix} -1 & -3 & -1 \\ 1 & 3 & 1 \\ -2 & -6 & -2 \end{pmatrix}$$
$$\to E_2 = \text{Span}\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

We have the eigenvectors $\mathbf{v}_1 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. Heads up that since the eigenvectors come from the same space there will be trouble! We will assume that there is some power vector of order 2. It must be

$$a\mathbf{v}_1 + b\mathbf{v}_2 = (A-2I)\mathbf{w}_1.$$

There is no solution for many choices of a and b. In fact, observe that linear combinations of the columns of $(A-2I)$ must always be multiples of $(-1,1,-2)^\top$. Then, solving the RREF of $\left[A-5I \mid \mathbf{v}_1 - 2\mathbf{v}_2\right]$ gives

$$\mathbf{w}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Well, now we have 3 linearly independent power vectors. Be careful to use the eigenvector we built $\mathbf{v}_1 - 2\mathbf{v}_2$. Thus, a Jordan Canonical basis is

$$\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_1 - 2\mathbf{v}_2, \mathbf{w}_1\}$$
$$\mathcal{B} = \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$