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Department of Mathematics

Exploration: Circles and Ellipses

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Math 489A/B
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§1. Plotting Transformations by A

We consider the equations $\mathbf{x}(n+1) = A\mathbf{x}(n)$,where $A = \begin{pmatrix} 0.995 & -0.1 \\ 0.1 & 0.995 \end{pmatrix}$. Let $\mathbf{x}(0) = (1,0)^\top$. We solve the equation $\mathbf{x}(n+1) = A\mathbf{x}(n)$ graphically and display the plots below. The points appear to be creating a circle of radius 1. However, as we will see later in §7, this is actually a slowly growing spiral.

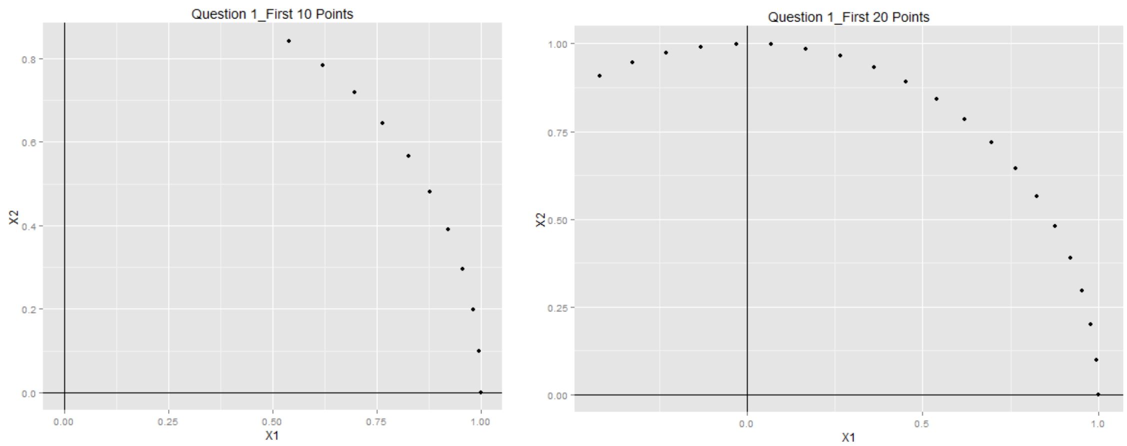


Figure 1: (Left) First 10 points, (Right) First 20 points.

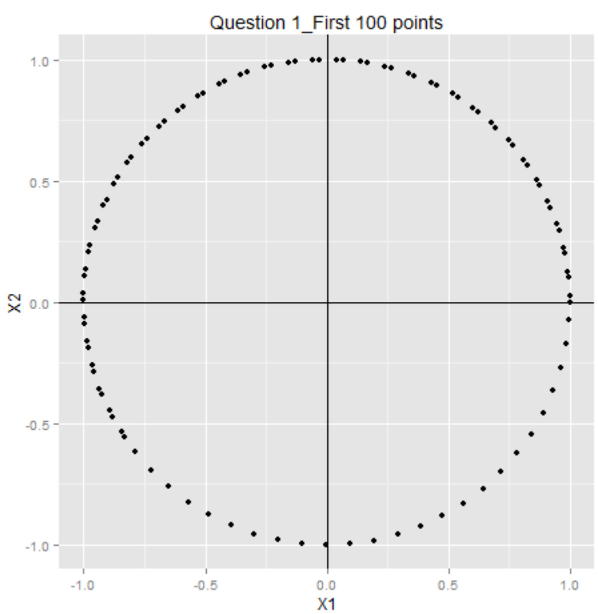


Figure 2: First 100 points.

§2. Plotting Transformations by B

Now, we consider the equations $\mathbf{x}(n+1) = B\mathbf{x}(n)$, where $B = \begin{pmatrix} 1 & -0.1 \\ 0.1 & 0.99 \end{pmatrix}$. Let $\mathbf{x}(0) = (1, 0)^\top$ as before. We would predict that B produces an elliptical plot. This should be expected since B is not exactly of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ required for a simple rotation followed by stretching. Nevertheless, it closely approximates this form. The result is shown below.

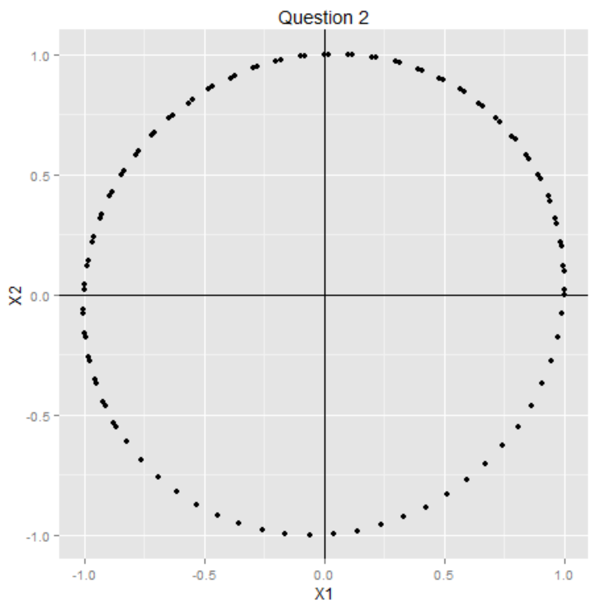


Figure 3: First 100 points of B. Note that the figure has an elliptic form (slightly stretched along the direction of $(1, 1)^\top$).

§3. Computing P

We use one of the eigenvectors of B to construct $P = (\mathbf{v}_R, \mathbf{v}_I)$ and apply P^{-1} to the points in step 2. Let $B = \begin{pmatrix} 1 & -0.1 \\ 0.1 & 0.99 \end{pmatrix}$. Then, for an equation of the form $B\vec{x} = \lambda\vec{x}$, $(B - \lambda I)\vec{x} = \vec{0} \implies$

$$\begin{pmatrix} 1-\lambda & -0.1 \\ 0.1 & 0.99-\lambda \end{pmatrix} \vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

In order to have nontrivial solutions, we require

$$\det(B - \lambda I) = 0,$$

which yields the following characteristic equation.

$$p_B(\lambda) = \lambda^2 - \frac{199}{100}\lambda + 1 = 0.$$

Upon solving for λ, we get that $\lambda_1 = 0.995 + 0.0999i$ and $\lambda_2 = 0.995 - 0.0999i$. Since we are dealing with two equations of the form $B\vec{x}_1 = \lambda_1\vec{x}_1$ and $B\vec{x}_2 = \lambda_2\vec{x}_2$, upon solving for eigenvectors $\vec{x}_1$ and $\vec{x}_2$, our calculations yield $\vec{x}_1 = (0.7071, 0.0354 - 0.7062i)^\top$ and $\vec{x}_2 = (0.7071, 0.0354 + 0.7062i)^\top$.

We will specifically consider λ_1 and $\vec{x}_1$ for the next portion of this exercise. Because we are asked to structure our eigenvector $\vec{x}_1$ such that $\mathbf{v}_R$ and $\mathbf{v}_I$ represent the real and imaginary portions of the eigenvector, respectively, we will define the following.

$$\vec{x}_1 = (0.7071, 0.0354 - 0.7062i)^\top \implies \mathbf{v}_R = \begin{pmatrix} 0.7071 \\ 0.0354 \end{pmatrix}, \mathbf{v}_I = \begin{pmatrix} 0 \\ -0.7062 \end{pmatrix}$$

Then, we will create a matrix P such that $P = (\mathbf{v}_R, \mathbf{v}_I)$. Therefore,

$$P = \begin{pmatrix} 0.7071 & 0 \\ 0.0354 & -0.7062 \end{pmatrix}.$$

It follows that

$$P^{-1} = \begin{pmatrix} 1.4142 & 0 \\ 0.0708 & -1.4160 \end{pmatrix}.$$

From the reading, we know that P^{-1} is a change of basis matrix from the standard basis to the basis $\mathcal{B} = \{\mathbf{v}_R, \mathbf{v}_I\}$. Under this basis, B is a transformation that rotates and stretches vectors. Hence, we expect that applying P^{-1} to the points generated by B in step 2, will transform the elliptic plot into a circular plot. This is shown below.

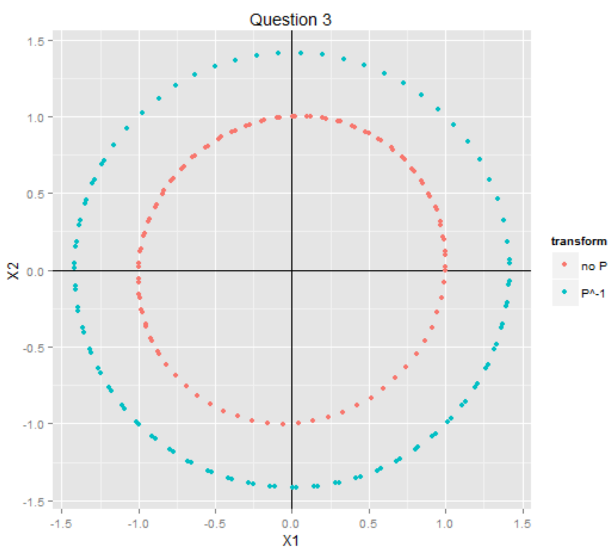


Figure 4: Effect of P^{-1} on the points produced in step 2.

§4. Diagonalizing A and B

We find all the eigenvalues and eigenvectors of A and B below.

4.1 Diagonalizing A

The eigenvalues are $\lambda_1 = 0.995 + 0.1i$ and $\lambda_2 = 0.995 - 0.1i$ with corresponding eigenvectors $\mathbf{a_1} = (1, -i)^\top$ and $\mathbf{a_2} = (1, i)^\top$. Thus, we diagonalize A as:

$$A = VDV^{-1} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} 0.995 + 0.1i & 0 \\ 0 & 0.995 - 0.i \end{pmatrix} \begin{pmatrix} 1/2 & 1/2i \\ 1/2 & -1/2i \end{pmatrix}.$$

Consider the eigenvector $\mathbf{a_2} = (1, i)^\top$. We have $\mathbf{v}_R = (1, 0)^\top$ and $\mathbf{v}_I = (0, 1)^\top$. The real and imaginary parts are orthogonal and have the same length.

4.2 Diagonalizing B

The eigenvalues are $\lambda_1 = 0.995 + 0.0999i$ and $\lambda_2 = 0.995 - 0.0999i$ with corresponding eigenvectors $\mathbf{b_1} = (0.7071, 0.0354 - 0.7062i)^\top$ and $\mathbf{b_2} = (0.7071, 0.0354 + 0.7062i)^\top$. Thus, we diagonalize B as:

$$\begin{aligned} B &= VDV^{-1} \\ &= \begin{pmatrix} 0.7071 & 0.7071 \\ 0.0354 - 0.7062i & 0.0354 + 0.7062i \end{pmatrix} \begin{pmatrix} 0.995 + 0.0999i & 0 \\ 0 & 0.995 - 0.0999 \end{pmatrix} \begin{pmatrix} 0.7071 - 0.0354i & 0.7080i \\ 0.7071 + 0.0354i & -0.7080i \end{pmatrix}. \end{aligned}$$

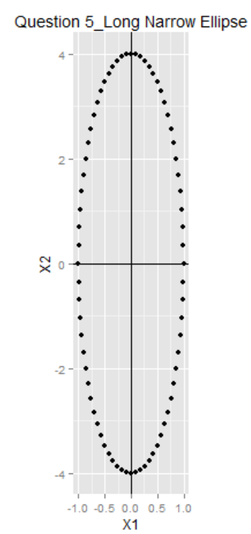
Consider the eigenvector $\mathbf{b_2} = (0.7071, 0.0354 + 0.7062i)^\top$. We have $\mathbf{v}_R = (0.7071, 0.0354)^\top$ and $\mathbf{v}_I = (0, 0.7062)^\top$. The real and imaginary parts are not orthogonal and do not have the same length.

§5. A Narrow Ellipse

We now find a matrix that produces a long, narrow ellipse. Consider the rotation matrix

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The resultant vectors $R^i\vec{x}$ lie on a circle of radius 1 for all nonnegative powers of R. Suppose we take $\theta = 5°$. Then, plotting many points solving $\mathbf{x}(n+1) = R\mathbf{x}(n)$ will produce a circular graph of radius 1. We can turn this graph into a long narrow ellipse by using a change of basis. For instance, take $V = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$. This stretches the y-axis by a factor of 4, but leaves the x-axis unchanged. Applying this to our circle will result in a strong skewing effect and produce a long, narrow ellipse.



§6. Computing the Angle of Rotation of A and B

Now, we will find the angle of rotation for A and B. Moreover, we will explore the values of n such that $\mathbf{x}(n)$ is close to the x_1 axis. Since $0.995 \pm 0.1i$ compose a pair of eigenvalues with eigenvectors $\mathbf{a}_1$ and $\mathbf{a}_2$, then matrix A rotates by an angle $\tan^{-1}(\frac{0.1}{0.995})^\circ \approx 5.7391^\circ$. Since $0.995 \pm 0.0999i$ compose a pair of eigenvalues with eigenvectors $\mathbf{b}_1$ and $\mathbf{b}_2$, then matrix B rotates by an angle $\tan^{-1}(\frac{0.0999}{0.995})^\circ \approx 5.7334^\circ$.

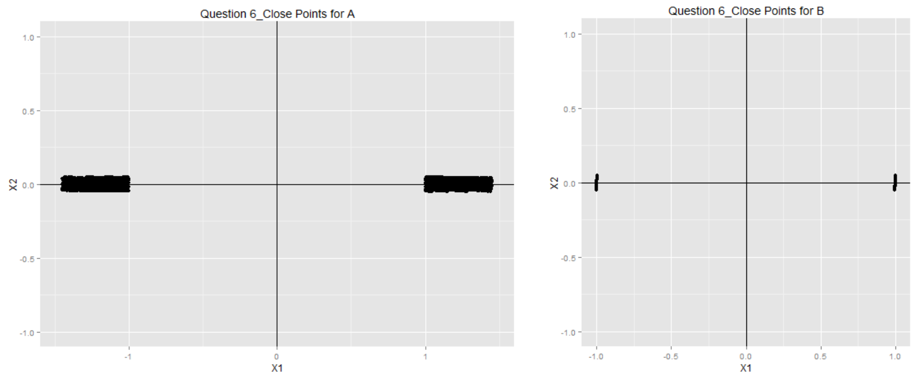


Figure 5: (Left) 'Close' points of A, (Right) 'Close' points of B. Tolerance of $\epsilon = 0.05$

We define a 'close point to the x_1 axis' to be one that satisfies $|x_2| < 0.05$ where $\mathbf{x} = (x_1, x_2)^\intercal$. Below we display the output from R for the first 100 values of n such that $\mathbf{x}(n)$ is close to the x_1 axis for the A and B matrices. We plot these values with an approximating smoothing function in the following figure.

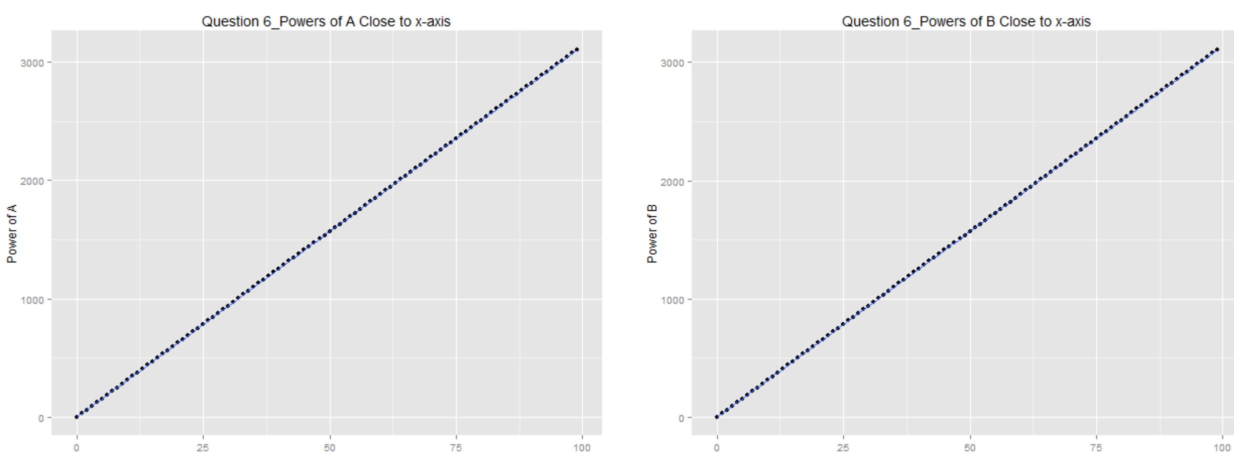


Figure 6: (Left) Fitting to 'close' powers of A, (Right) Fitting to 'close' powers of B.

Some sample powers of A yielding close points are:

$$0, 31, 63, 94, 125, 157, 188, 220, 251, 282, 314, 345, 376, 408, 439, 470, \ldots$$

while some sample powers of B yielding close points are:

$$0, 31, 63, 94, 126, 157, 188, 220, 251, 283, 314, 345, 377, 408, 440, 471, \ldots$$

§7. Examining A more Closely

Whenever a matrix has eigenvalues of the form $a \pm bi$, we can compute the stretching factor by $\sqrt{a^2 + b^2}$. Denote the stretching factor of a matrix as $\text{scale}(M)$ for some matrix M. Then, $\text{scale}(A) \approx 1.0000125$. Even though this factor is only slightly larger than 1, we will see an enlarging spiral after many applications of A.

In each of the four following figures, we plot up to the 100,000th point. In the first figure, we plot every one of these points and observe a densely packed spiral. The second figure displays the plot of every 314th point, which helps us see the shape of the spiral more clearly. The third figure shows the plot of every 3,137th point. The fourth figure displays every 1,000th point. Observe the interesting 'galaxy' pattern that emerges. In fact, an 'arm' of this pattern is associated with every 17,000th point.

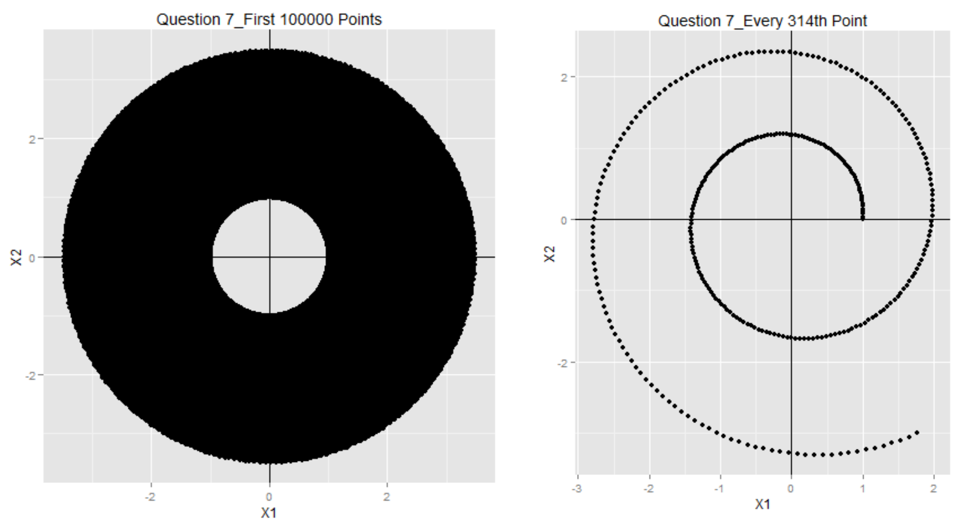


Figure 7: (Left) Plotting the first 100,000 points, (Right) Every 314th point.

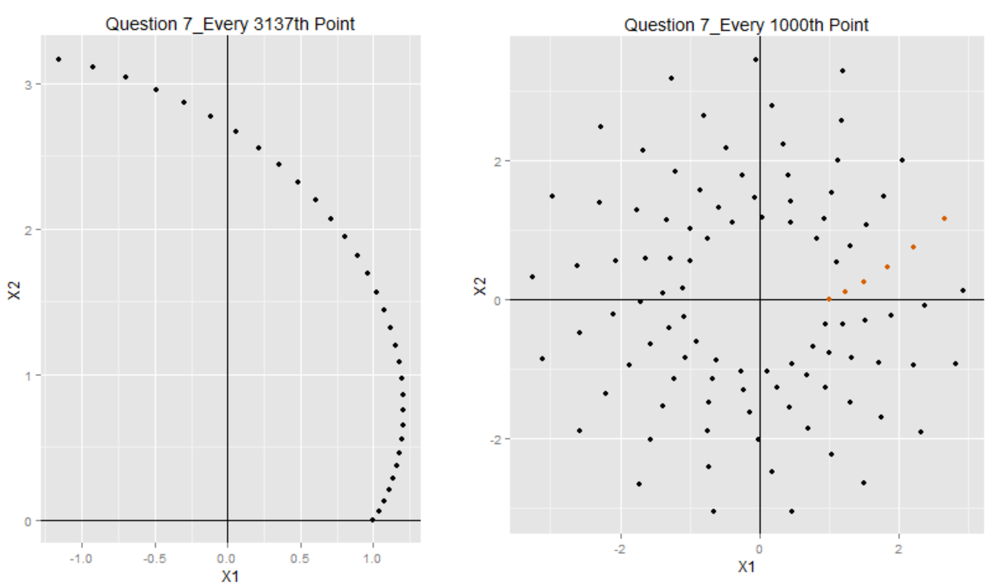


Figure 8: (Left) Every 3,137th point, (Right) Every 1,000th point with every 17,000th point in red.

§8. Examining Multiples of A and B

Consider the following matrices: $A_+ = 1.01A$, $A_- = 0.99A$, $B_+ = 1.01B$, and $B_- = 0.99B$. We have:

$$A_+ = \begin{pmatrix} 1.00495 & -0.101 \\ 0.101 & 1.00495 \end{pmatrix}, \quad A_- = \begin{pmatrix} 0.98505 & -0.099 \\ 0.099 & 0.98505 \end{pmatrix}, \quad B_+ = \begin{pmatrix} 1.01 & -0.101 \\ 0.101 & 0.9999 \end{pmatrix}, \quad B_- = \begin{pmatrix} 0.99 & -0.099 \\ 0.099 & 0.9801 \end{pmatrix}$$

It follows that

$$\text{scale}(A_+) \approx 1.01, \quad \text{scale}(A_-) \approx 0.99, \quad \text{scale}(B_+) = 1.01, \quad \text{scale}(B_-) = 0.99.$$

Therefore, A_+ and B_+ will expand, spiraling outward, while A_- and B_- will contract, spiraling inward.

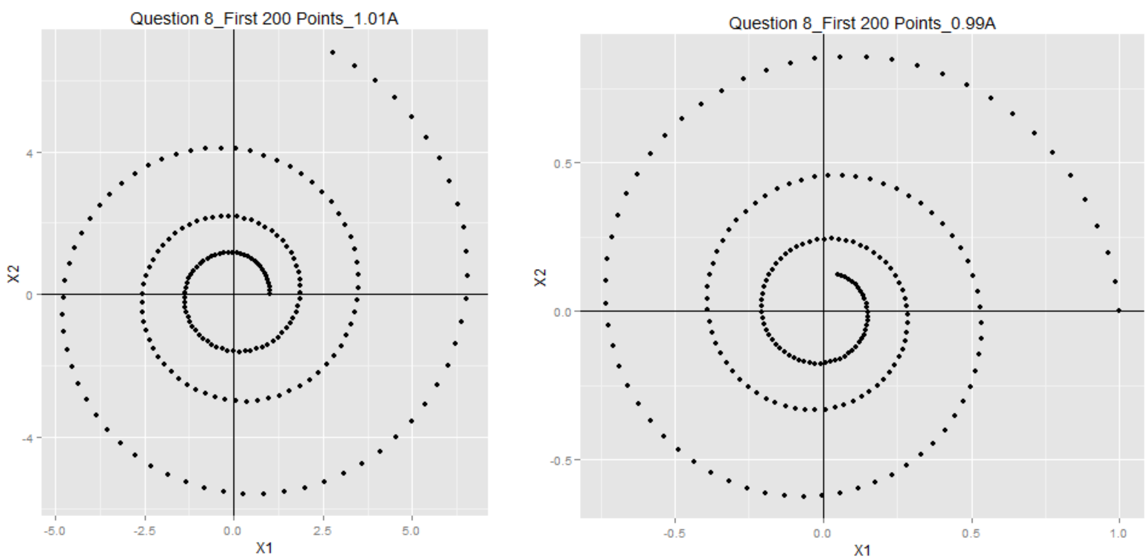


Figure 9: (Left) A_+ spirals outward counterclockwise. (Right) A_- spirals inward counterclockwise.

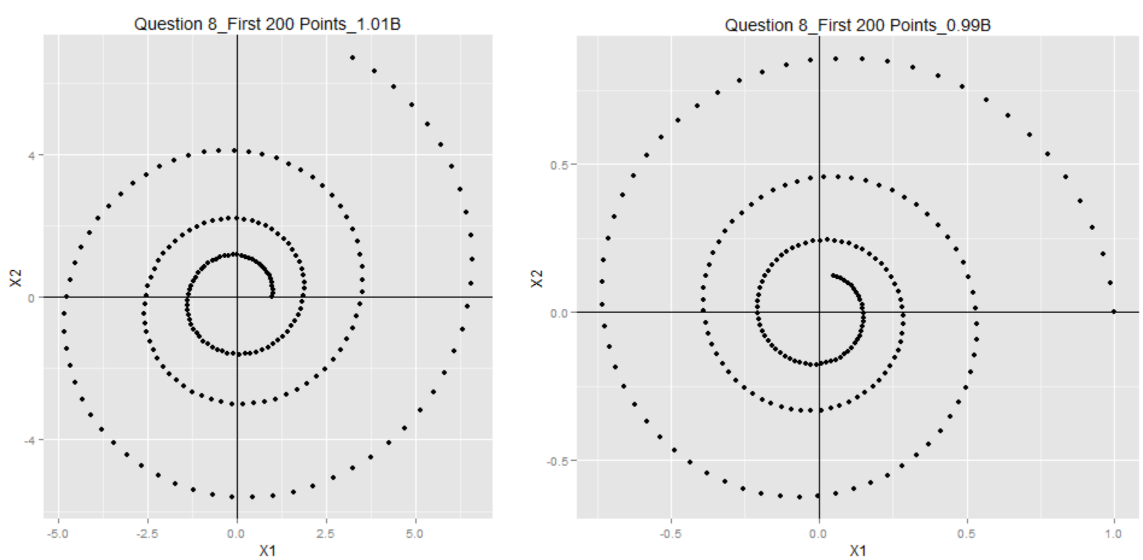


Figure 10: (Left) B_+ spirals outward counterclockwise. (Right) B_- spirals inward counterclockwise.

§9. Conclusion

In this project we investigated the effect of matrices of (or similar to) the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ (c.f. §1,7,8). It was found that such matrices transform points by rotating and stretching them. The amount of rotation and stretching can be computed by finding the eigenvalues and eigenvectors of each matrix (c.f. §4). Given a matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with complex eigenvalues $a \pm bi$, the amount of rotation is calculated by $\tan^{-1}(b/a)$, while the stretching factor is given by $\sqrt{a^2 + b^2}$. The resulting transformations are circles or circular spirals.

In the case of B, we do not have a perfectly circular transformation (c.f. §2). In particular, we discover that B produces a slightly elliptical graph. By taking an eigenvector of B, say $\vec{x}_1$, we can split it into its real and imaginary parts: $\vec{x}_1 = \mathbf{v}_R + i\mathbf{v}_I$ (c.f. §3). Constructing the matrix $P = \begin{pmatrix} \mathbf{v}_R & \mathbf{v}_I \end{pmatrix}$ provides a change of basis matrix, which converts B into a transformation of the desired form. That is, B within the basis $\{\mathbf{v}_R, \mathbf{v}_I\}$, B has the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

Furthermore, we described a method to construct a long, narrow ellipse (c.f. §5). All that is required is to first produce a circle, and then transform these points by a scaling matrix. In general, the matrix $\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$ stretches points horizontally by a factor c and vertically by a factor d.

Lastly, we found points $\mathbf{x}(n)$ which lie close to the x_1-axis when transformed by powers of A and B (c.f. §6). We graphed an approximating smoothing function fitting the powers of A and B, respectively, and listed some sample powers satisfying the condition.