CALIFORNIA STATE UNIVERSITY

FULLERTON

Department of Mathematics

Exploration: Computer Graphics

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Math 489A/B
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§1. Transforming Points in the Plane

We explore representing points $(x, y)^\top$ in the plane by 3D-vectors of the form $(x, y, 1)^\top$, and transform them by means of 3×3 matrices. In particular, we perform various linear transformations on the letter "b" defined by the vector

$$\mathbf{b} = \begin{pmatrix} 2 & 2 & 2 & 2.2 & 2.2 \\ 0.4 & 0.2 & 0 & 0 & 0.2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

1.1. The Advantages of Representing 2D Points as 3D Points

We know that there are many plane transformations we can perform by representing points as $(x, y)^\top$ and using only 2×2 matrices. In fact, rotation of points about the origin can be done by applying the the following matrix.

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

We can also scale by a factor k by using kI_2, where I_2 is the 2×2 identity matrix (reflections can be handled using a similar idea).

So, why would we want to represent points as $(x, y, 1)^\top$ and use 3×3 matrices to perform plane transformations? It turns out that finding a 2×2 matrix that represents a shift within the plane is very difficult. For example, we might try to find a 2×2 matrix A with the property that

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + s \\ y \end{pmatrix}.$$

Then, $a_{11}x + a_{12}y = x + s$ and $a_{21}x + a_{22}y = y$. It follows that $a_{21} = 0$ and $a_{22} = 1$. But how do we select constants a_{11} and a_{12}? In this case, no constant values will satisfy the given property. Thus, no 2×2 matrix with constant entries can be constructed that will shift the point horizontally by s units.

However, we will discover that a 3×3 matrix with constant entries *can* be constructed that will shift points as we desire.

1.2 The Matrix Transformations

Throughout this paper, we will be referring to the following matrices:

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad L = R^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad D = U^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_x = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

§2. Applying the Matrix Transformations[1]

2.1 The Effect of A

Calculating the matrices, we find the following expressions:

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad A^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad A^3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad A^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By examining Figure 1, we see that each application of A rotates points 90° counterclockwise around the origin.

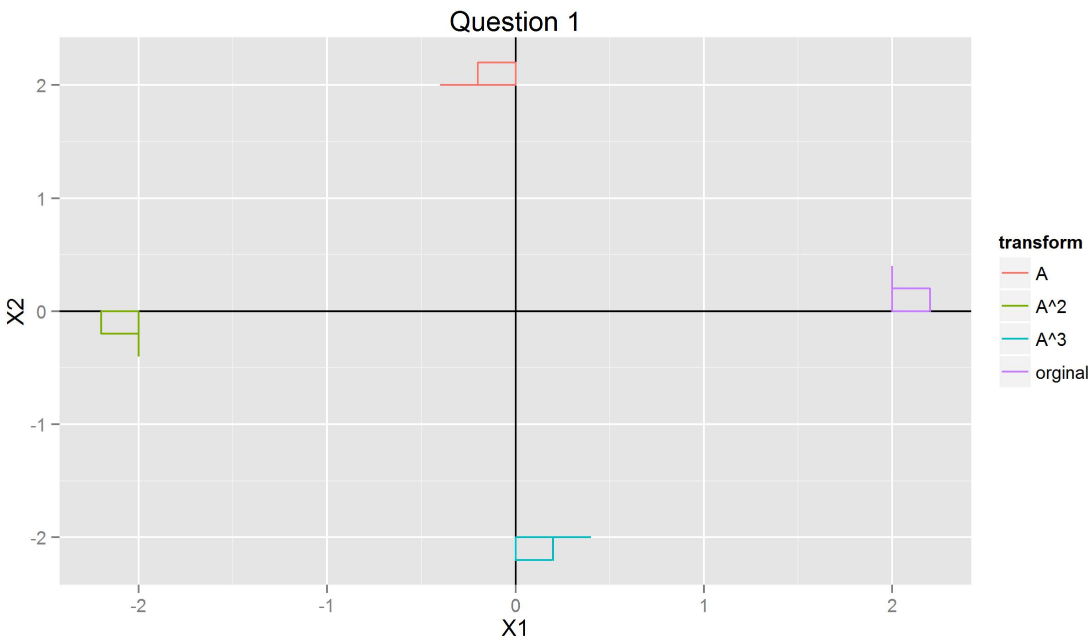

Figure 1: A rotates the original "b" by 90° counterclockwise around the origin.

2.2 The Effect of B

First, observe that B is identical to A^3. In particular,

$$B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = A^3.$$

Thus, B rotates points counterclockwise around the origin by 270°. More simply, this is a rotation *clockwise* around the origin by 90°. This is seen in Figure 2.

[1]See `https://github.com/nbsullivan/math489/blob/master/project3.R`to view a link to the R code utilized for the creation of plots.

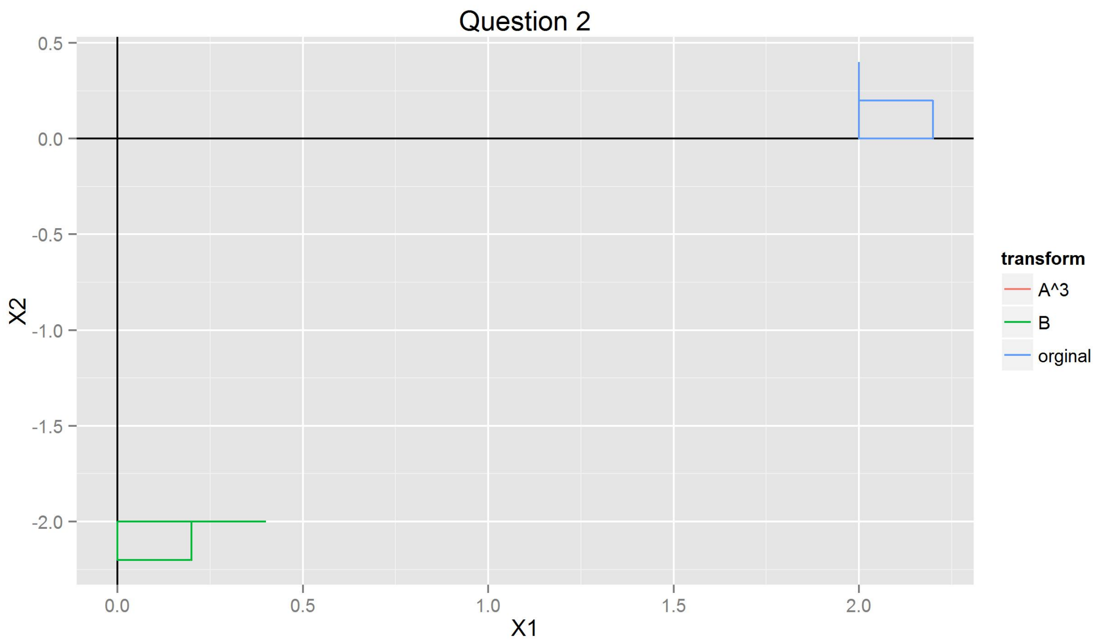

Figure 2: B rotates "b" by 90° clockwise around the origin. B is plotted over A^3.

2.3 The Effect of U, D, L, and R

Consider the effects of applying these four matrices to the original "b." As seen in Figure 3, these matrices seem to have received their names based on the direction of the resulting shifts (i.e., U for *up*, D for *down*, L for *left*, and R for *right*).

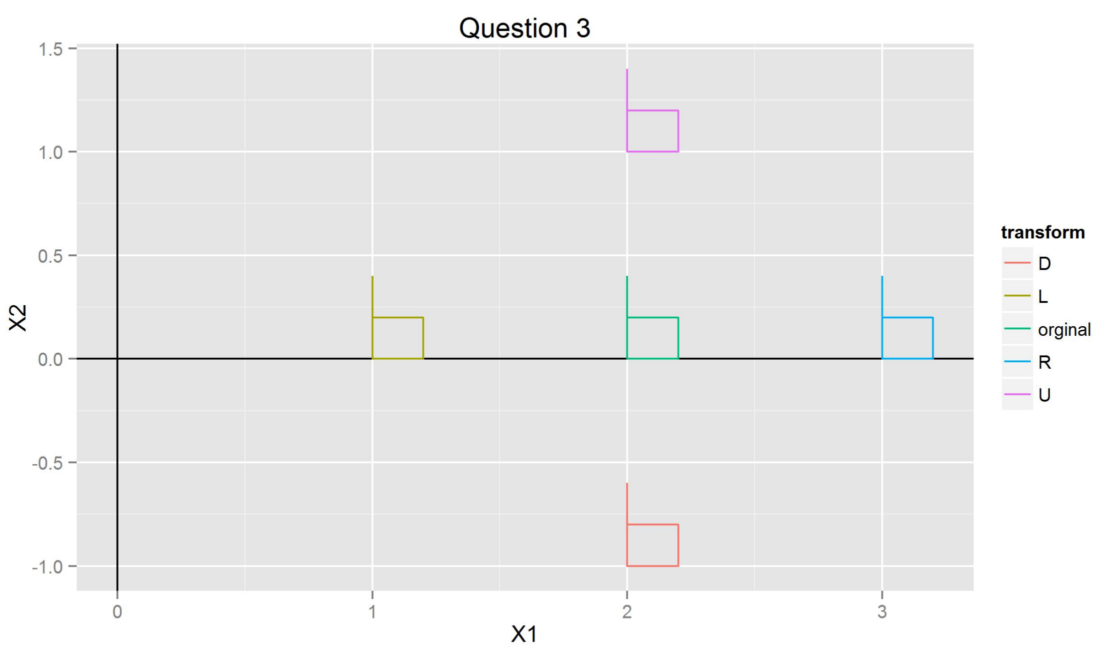

Figure 3: Each matrix corresponds to an orthogonal shift of 1 unit.

2.4 The Effect of P_x and P_y

By observing the effect of the matrices on the original "b" in Figure 4, we see that P_x is reflection about the y-axis, while P_y is reflection about the x-axis. It seems that the "x" and "y" in P_x and P_y, respectively, refer to the motion of the points, rather than the axis of reflection.

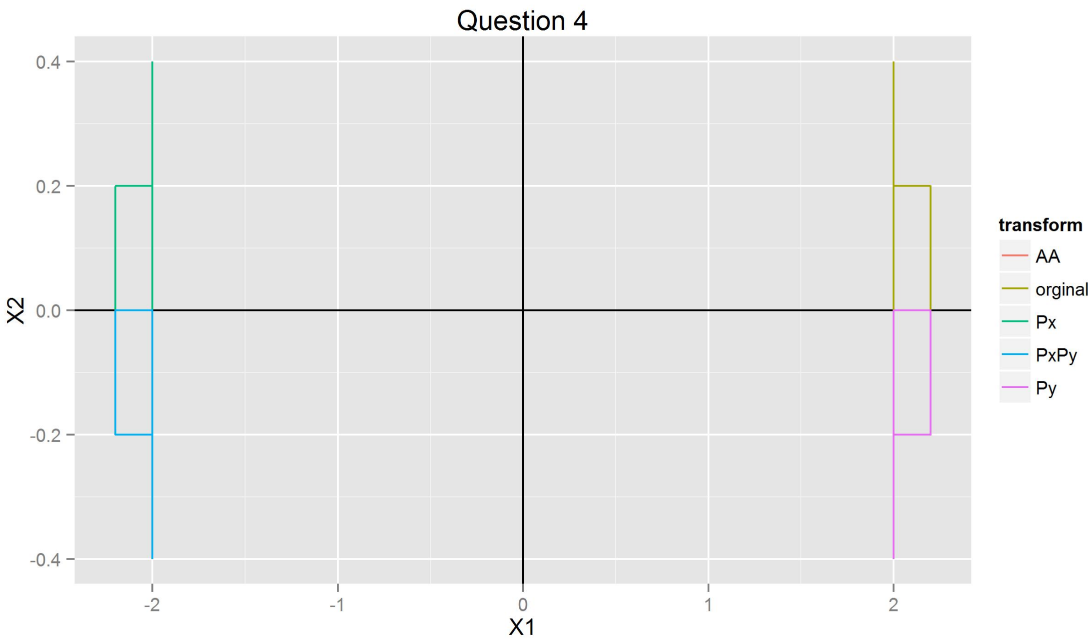

Figure 4: P_x reflects about the y-axis; P_y reflects about the x-axis. P_xP_y is plotted over A^2.

We also note that A^2 is the same as P_xP_y. In particular,

$$A^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = P_xP_y.$$

Summary of Matrices

In the table below, we summarize the action of each matrix we have discussed.

Matrix	Description
A	Rotation counterclockwise by 90° about the origin.
B	Rotation clockwise by 90° about the origin.
U	Shift upwards by 1 unit.
D	Shift downwards by 1 unit.
L	Shift left by 1 unit.
R	Shift right by 1 unit.
P_x	Reflection over y-axis.
P_y	Reflection over x-axis.

2.5 Turning a "b" into "d"

The matrix $R^4 P_x$ reflects points about the y-axis, and then shifts them to the right by 4 units. As seen in Figure 5, this takes the original "b" at $(2,0)^\intercal$ and produces a "d" at the point $(2,0)^\intercal$.

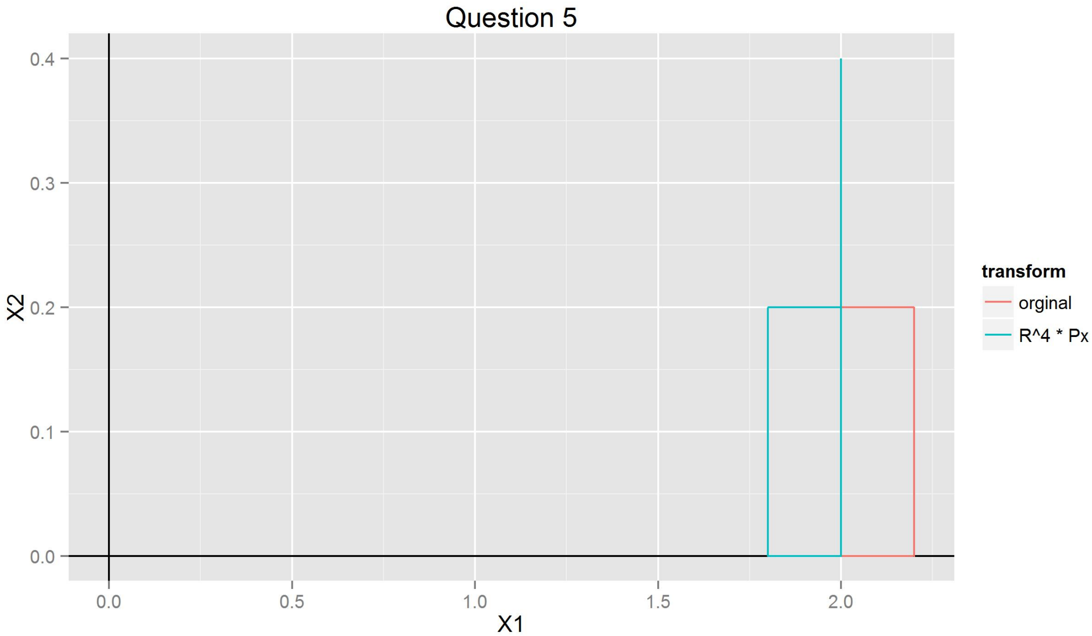

Figure 5: Reflecting "b" across the line $x = 2$.

2.6 More ways to turn "b" into "d"

The reading of this particular exercise suggests that matrices $R^2 P_x L^2$ and $P_x L^4$ satisfy exercise 5 just like $R^4 P_x$. Calculating $R^4 P_x$ explicitly, we find

$$R^4 P_x = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In similar fashion, we identify matrix $R^2 P_x L^2$.

$$R^2 P_x L^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This matrix shifts points to the left by 2 units, reflects them about the y-axis, and finally shifts them to the right by 2 units.

Finally, we will identify matrix $P_x L^4$.

$$P_x L^4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This matrix shifts points to the left by 4 units, and then reflects them about the y-axis.

Therefore, we see that $R^4 P_x = R^2 P_x L^2 = P_x L^4$, and thus could have been used as methods toward solving exercise 5. Figure 6 shows the result graphically.

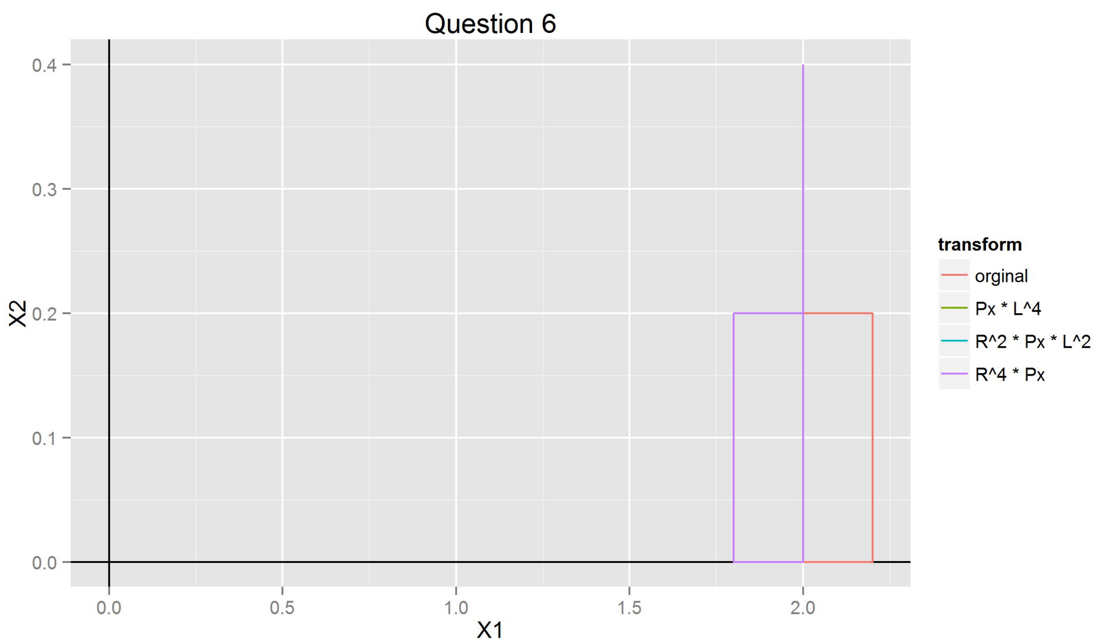

Figure 6: The transformations $R^4 P_x, R^2 P_x L^2$, and $P_x L^4$ are equivalent. Note that $R^4 P_x$ is plotted over $R^2 P_x L^2$ and $P_x L^4$.

2.7 Turning "b" into "p"

The chosen matrix is

$$LU^3 P_y = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

This matrix takes points and reflects them over the x-axis, shifts them up by 3 units, and then shifts them left by 1 unit. As can be seen in Figure 7, this takes the original "b" at $(2,0)^\intercal$ and transforms it into a "p" at $(1,3)^\intercal$.

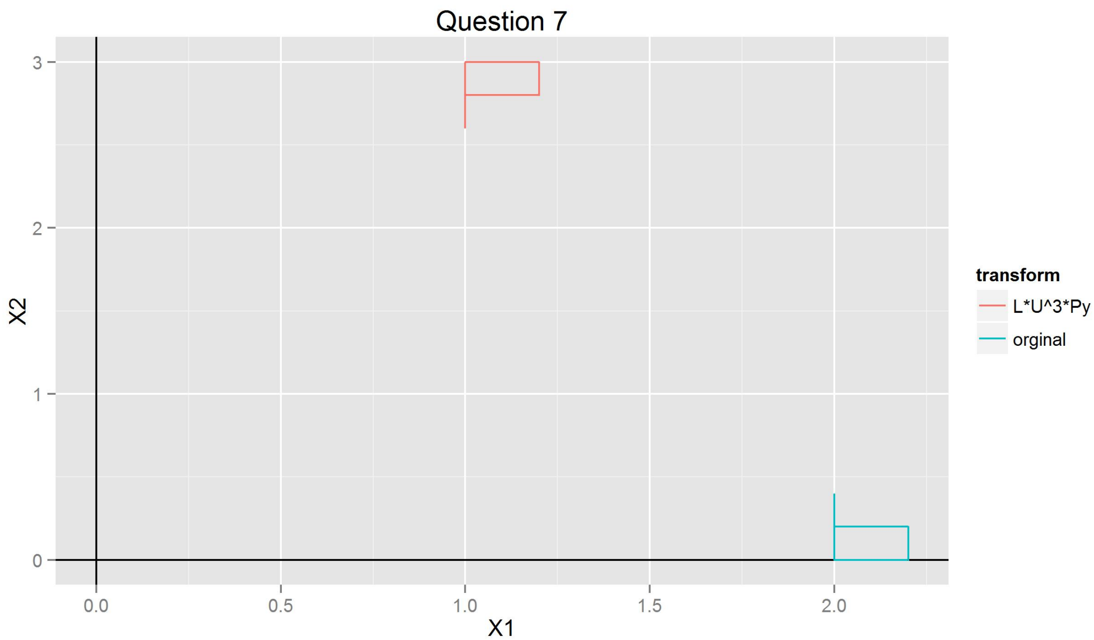

Figure 7: Transforming "b" into a "p".

2.8 More ways to turn "b" into "p"

This time, we do not use P_y. Instead, we use the matrix

$$U^3 L P_x A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^3 \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

We can see, computationally, that this matrix is exactly the same as LU^3P_y used previously. Thus, we can take the original "b" at $(2,0)^\intercal$ and transform it into a "p" at $(1,3)^\intercal$ by first rotating it by 180° about the origin (at this stage, the "b" looks like a "q"), followed by reflecting it over the y-axis (now the "q" becomes a "p"), then shifting it left 1 unit, and upwards 3 units. Comparing Figures 7 and 8, we can see that the effect of these two matrices is, indeed, identical.

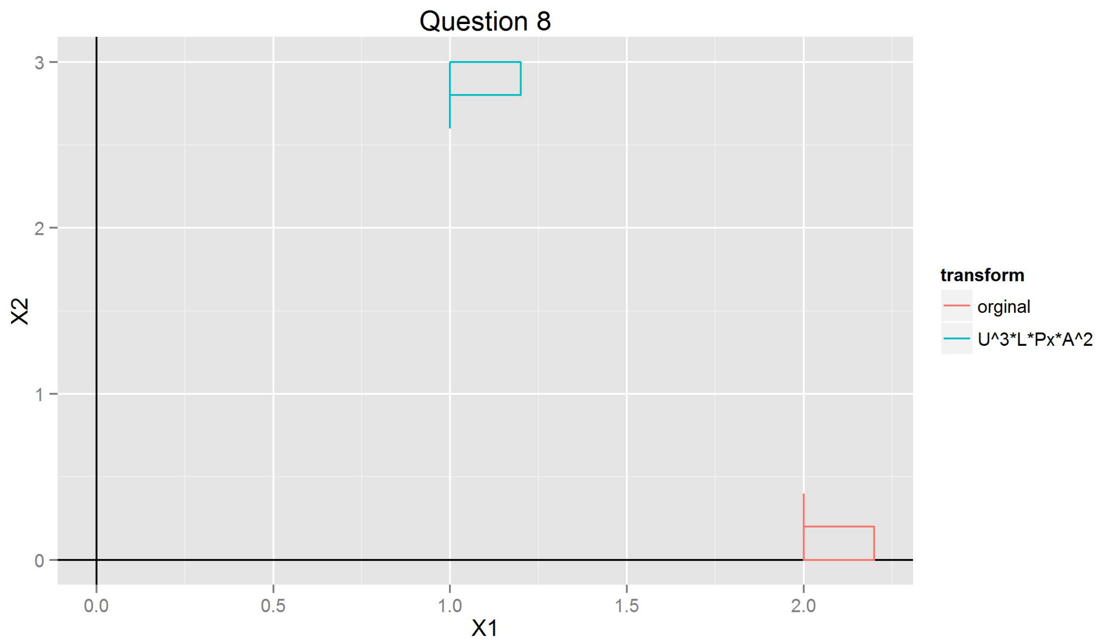

Figure 8: $U^3 L P_x A^2$ is equivalent to $L U^3 P_y$

2.9 Turning "b" into "q"

To turn the "b" into a "q" with upper right corner at $(-1,-2)^\intercal$ without using A or B (that is, without rotation), we use the following matrix:

$$DP_xP_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

This reflects points about the x-axis, followed by reflection over the y-axis, and finally shifts down by 1 unit.

We can also perform this transformation without using any reflections with

$$DA^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since the resulting matrices are both the same, this shows that we can also produce the same effect by rotating 180°, followed by a downward shift of 1 unit (see Figure 9). In fact, we showed previously in §2.3 that $A^2 = P_xP_y$.

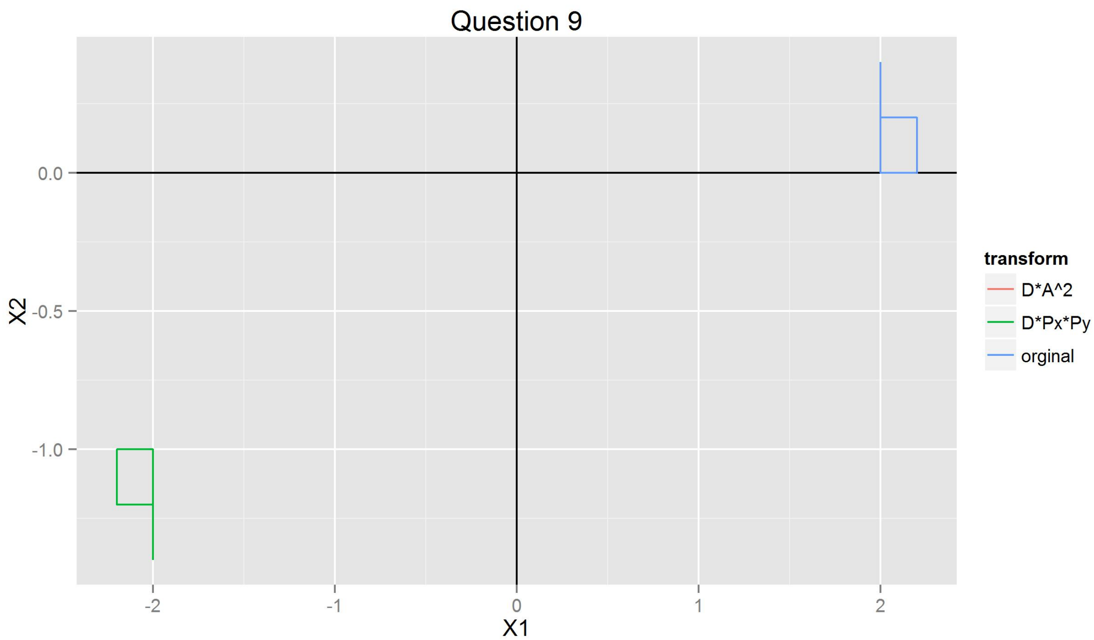

Figure 9: DP_xP_y is equivalent to DA^2. Note that DP_xP_y is plotted over DA^2.

2.10 Turning "b" into a BIG "d"

Define $E_{(2,2)} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which scales by a factor of 2 in both the horizontal and vertical direction. To transform 'b' into a 'd' that is double the original size and located at the origin, we take

$$E_{(2,2)}R^2P_x = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

That is, we first reflect the "b" over the y-axis, then shift it to the right by 2 units; finally, we scale it by a factor of 2. The result is shown in Figure 10.

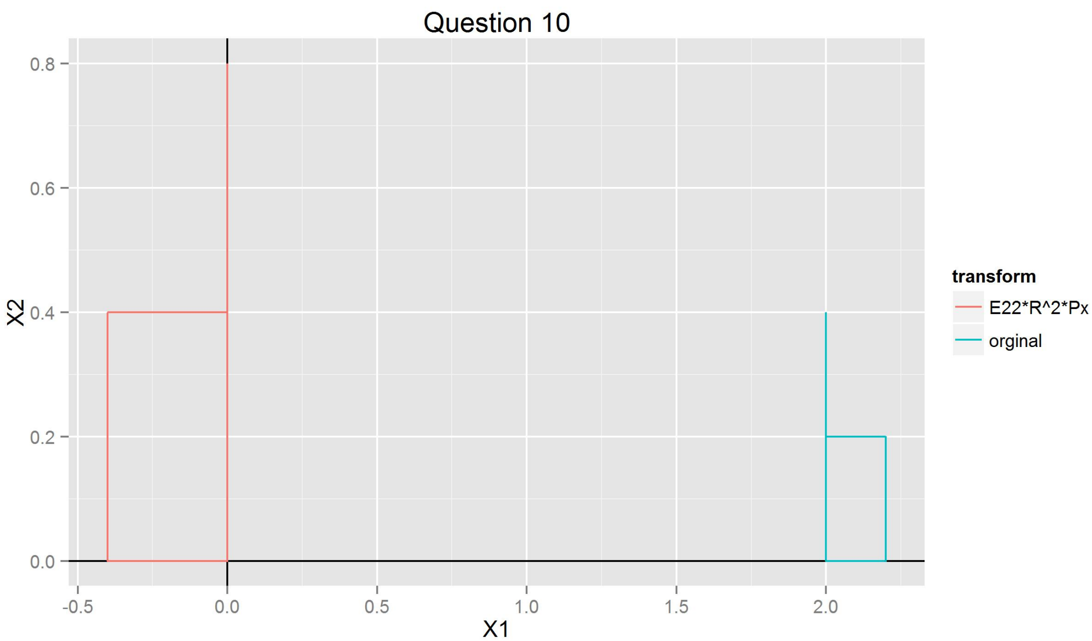

Figure 10: Transforming "b" into a "d" that is twice as large.

§3. A Noninteger Matrix

We are given $Z = \begin{pmatrix} 0.8 & -0.6 & 0 \\ 0.6 & 0.8 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Observe the effect of applying Z three times in Figure 11.

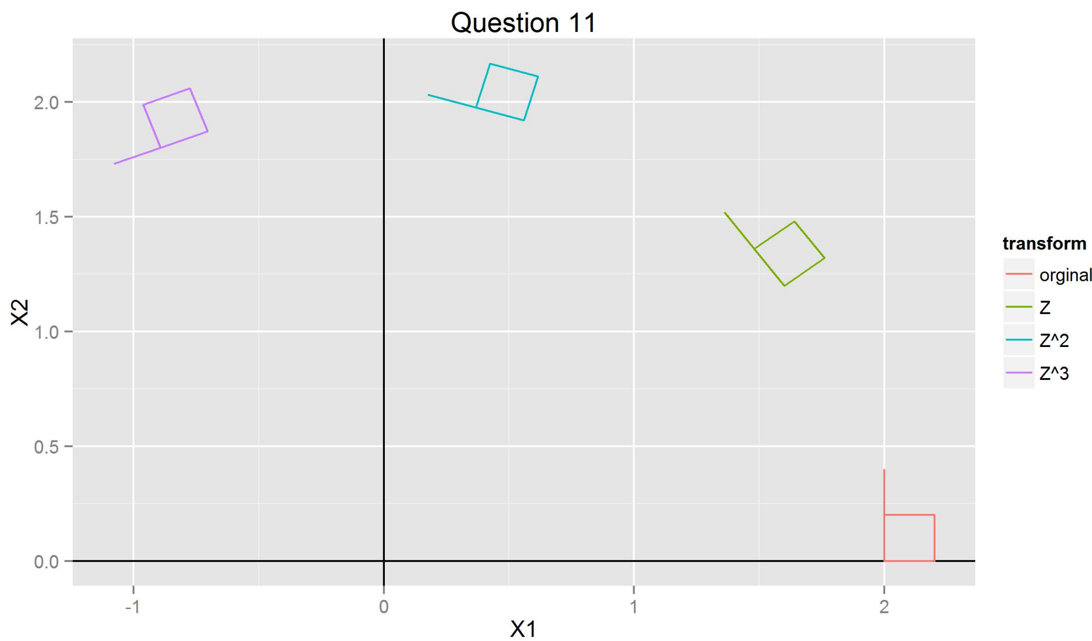

Figure 11: Rotation resulting from applications of Z

Note that Z is of the rotation matrix form $R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$. In particular, notice that $0.8^2 + 0.6^2 = 1$. We take $\cos(\theta) = 0.8$, then $\sin(\theta) = 0.6$ by the Pythagorean identity: $\cos^2(\theta) + \sin^2(\theta) = 1$. Thus, Z is the matrix that rotates points by an angle $\theta = \arccos(0.8) \approx 36.87°$ counterclockwise around the origin. Figure 11 shows the results of applying Z three times.

By applying Z repeatedly, can we ever return "b" to its original position? The answer appears to be, no. This is equivalent to saying that $Z^n \neq I$ for any integer n (or that Z is of infinite order/period). While we have not proven the result, we can say that there are no solutions up to $n = 1 \times 10^9$ using the following MATLAB script:

```
%Define Z
Z = [0.8 -0.6 0; 0.6 0.8 0; 0 0 1];

for n = 1:1*10^9
    A = Z^n;
   if A == eye(3) %check if A is the identity
       display('Found it!')
       n
   end
end
```

§4. Conclusion

In this project, we analyzed to what extent we could modify a geometric representation of the letter "b" in three-dimensional space. However, in order to comply with the transformations presented, we began by representing two-dimensional points of the form $(x, y)^T$ as three-dimensional points of the form $(x, y, 1)^T$. This allowed us to perform shifts within the plane using simple matrices. Given a set of five coordinates, we illustrated the lowercase letter "b" and applied matrix transformations $A, B, U, D, L, R, P_x, P_y$ to rotate, shift, and reflect our letter "b" as needed. Throughout the course of this project, we were able to identify alternate approaches to achieve an equivalent transformation. In the last portion of this project, we were given a slightly different situation; whereas opposed to merely reflecting and shifting the letter "b" about an axis, we performed a counterclockwise rotation about the origin by a given angle $\theta \approx 36.87°$.