FROBENIUS NONCLASSICAL HYPERSURFACES

SHAMIL ASGARLI, LIAN DUAN, AND KUAN-WEN LAI

Abstract. A smooth hypersurface over a finite field $\mathbb{F}_q$ is called Frobenius nonclassical if the image of every geometric point under the q-th Frobenius endomorphism remains in the unique hyperplane tangent to the point. In this paper, we establish sharp lower and upper bounds for the degrees of such hypersurfaces, give characterizations for those achieving the maximal degrees, and prove in the surface case that they are Hermitian when their degrees attain the minimum. We also prove that the set of $\mathbb{F}_q$-rational points on a Frobenius nonclassical hypersurface form a blocking set with respect to lines, which indicates the existence of many $\mathbb{F}_q$-points.

Contents

1. Introduction

In the celebrated paper [SV86], Stöhr and Voloch used Weierstrass order-sequences to obtain an upper bound on the number of rational points for curves embedded in an arbitrary projective space over finite fields. The concept of Frobenius nonclassical curves was introduced naturally in their work as those curves whose order sequence behaves differently from "most" curves. Afterwards, Hefez and Voloch [HV90] extensively studied the properties of

2020 Mathematics Subject Classification 14G05, 14G15, 14J70, 51E21.
Keywords: blocking sets, Frobenius nonclassical hypersurfaces, Hermitian varieties

Frobenius nonclassical curves. In addition to showing that such curves are always nonreflexive, they also computed the precise number of $\mathbb{F}_q$-points on Frobenius nonclassical *plane* curves. It turns out that Frobenius nonclassical plane curves have many $\mathbb{F}_q$-points. Inspired by this observation, points on Frobenius nonclassical plane curves have been used to construct new complete arcs in the field of combinatorial geometry [GPTU02, Bor09a, BMT14].

As a natural generalization of the plane curve case, our earlier work [ADL21] introduced the concept of Frobenius nonclassical hypersurfaces, with a particular emphasis on the surface case. We found that if a hypersurface is *not* Frobenius nonclassical, then it is easier to prove a Bertini-type theorem such as the existence of a smooth hyperplane section over the ground field [ADL21, Theorem 2.2][ADL22, Theorem 1.4]. The purpose of the present paper is to provide a systematic study of Frobenius nonclassical hypersurfaces in arbitrary dimensions. We extend some of the results known for plane curves such as the lower and upper bounds on the degree to higher dimensions, and give evidences for the abundance of $\mathbb{F}_q$-points on these hypersurfaces.

Definition 1.1. Let $X \subset \mathbb{P}^n$ be a hypersurface defined by a polynomial F over the finite field $\mathbb{F}_q$. We say that X is *Frobenius nonclassical* if the q-th Frobenius morphism

$$\Phi\colon \mathbb{P}^n \longrightarrow \mathbb{P}^n : [x_0 : \cdots : x_n] \longmapsto [x_0^q : \cdots : x_n^q]$$

maps every $\overline{\mathbb{F}_q}$-point $P \in X$ into the embedded tangent subspace

$$T_P X := \left\{ \sum_{i=0}^{n} x_i \left(\frac{\partial F}{\partial x_i}(P) \right) = 0 \right\} \subset \mathbb{P}^n.$$

Equivalently, X is Frobenius nonclassical if F divides the polynomial

$$F_{1,0} := \sum_{i=0}^{n} x_i^q \frac{\partial F}{\partial x_i}.$$

Remark 1.2. One can interpret $F_{1,0}$ as the directional derivative $D_{\mathbf{x}^q} F$ of the function F in the direction $\mathbf{x}^q = (x_0^q, \dots, x_n^q)$. The notation $F_{1,0}$ is originated from our earlier work [ADL21] where it is a special case of $F_{a,b} = \sum_i x_i^{q^a} (\partial F / \partial x_i)^{q^b}$.

By definition, $\mathbb{F}_q$-hyperplanes are Frobenius nonclassical over $\mathbb{F}_q$. Below we present some examples that are not hyperplanes:

Example 1.3. For $n = 2m+1$, the smooth hypersurface $X \subset \mathbb{P}^n$ defined by

$$F = \sum_{i=0}^{m} \left(x_{2i}^q x_{2i+1} - x_{2i} x_{2i+1}^q \right)$$

is Frobenius nonclassical over $\mathbb{F}_q$ since $F_{1,0} = 0$. The hypersurface X has several remarkable properties. For example, X is space-filling in the sense that $X(\mathbb{F}_q) = \mathbb{P}^n(\mathbb{F}_q)$, that is, X passes through every $\mathbb{F}_q$-point of the ambient projective space. Moreover, if $H \subset \mathbb{P}^n$ is any two-dimensional plane defined over $\mathbb{F}_q$, then either $H \subset X$ or $X \cap H$ consists of a union of $q+1$ distinct $\mathbb{F}_q$-lines passing through a common point. This last property was carefully examined in [ADL21, Example 3.4] in the special case when $n = 3$.

Example 1.4 (Hermitian varieties)**.** For a square q, the hypersurface $X \subset \mathbb{P}^n$ defined by

$$F = \sum_{i=0}^{r} x_i^{\sqrt{q}+1} \qquad \text{where} \qquad 0 \le r \le n \tag{1.1}$$

is Frobenius nonclassical over $\mathbb{F}_q$ as one can verify that $F_{1,0} = (F)^{\sqrt{q}}$. This is an example of a *Hermitian variety* defined as follows: Consider the involution $x \mapsto x^{\sqrt{q}} = \overline{x}$ on $\mathbb{F}_q$. For a scalar matrix $H = (h_{ij})$, let us write $\overline{H} = (\overline{h_{ij}})$. A Hermitian variety is a hypersurface in $\mathbb{P}^n$ defined by a polynomial of the form

$$F = \overline{\mathbf{x}} \cdot H \cdot \mathbf{x}^t$$

where $\mathbf{x} = (x_0, \dots, x_n)$ and H is a scalar matrix that satisfies $\overline{H}^t = H \neq 0$. In fact, every such F is projectively equivalent to (1.1) over $\mathbb{F}_q$ by [BC66, Theorem 4.1].

Our first main result concerns lower bounds on the degree of a Frobenius nonclassical hypersurface. The result below was proved in [HV90, Proposition 6] for smooth plane curves, and in [BH17, Corollary 3.2] when the curves are geometrically irreducible.

Theorem 1.5. *Let $X \subset \mathbb{P}^n$, where $n \geq 2$, be a Frobenius nonclassical hypersurface of degree $d \geq 2$ over $\mathbb{F}_q$ of characteristic p which is smooth at $\mathbb{F}_q$-points. Then*

- *$d \geq p + 1$, and*
- *$d \geq \sqrt{q} + 1$ provided that X is reduced.*

In the case that X is a reduced curve or a smooth surface, the condition $d = \sqrt{q} + 1$ is attained only if X is Hermitian.

The two lower bounds in Theorem 1.5 are sharp due to Examples 1.3 and 1.4, respectively. Note that the bound $d \geq p + 1$ is not redundant with respect to $d \geq \sqrt{q} + 1$ as the former is stronger than the latter when $q = p$. The reducedness for the second bound is necessary because, if the assumption is dropped, then one can easily construct counterexamples using $\mathbb{F}_q$-hyperplanes and p-th powers of pointless hypersurfaces; see Remark 2.5 for concrete examples of pointless hypersurfaces. We will prove the two bounds respectively in Theorems 2.9 and 3.10. For the last statement, the curve case is essentially established by Borges and Homma [BH17, Corollary 3.2] as mentioned above, but we still need to treat the case when the curve is potentially reducible; see Lemma 3.11. We treat the surface case in Proposition 3.13. It is worth mentioning that an alternative characterization of Hermitian surfaces via the number of rational points was proved by Homma and Kim [HK16].

Our next main result concerns upper bounds on the degree.

Theorem 1.6. *Let $X \subset \mathbb{P}^n$, where $n \geq 2$, be a smooth Frobenius nonclassical hypersurface of degree $d \geq 2$ over $\mathbb{F}_q$. Then $d \leq q + 2$ and, in the cases $d = q + 1$ and $d = q + 2$, the defining polynomial F has the following forms:*

(1) *$d = q + 1$ and n is odd if and only if there exists a nondegenerate skew-symmetric matrix (A_{ij}) with entries in $\mathbb{F}_q$ and zero diagonal such that*

$$F = \sum_{i,j=0}^{n} x_i^q A_{ij} x_j.$$

(2) *$d = q + 1$ and n is even if and only if $p = 2$ and, up to a $\mathrm{PGL}_{n+1}(\mathbb{F}_q)$-action,*

$$F_{1,0} = x_0^{q-1} F \qquad \text{and} \qquad F = x_0 G + \sum_{i,j=1}^{n} x_i^q B_{ij} x_j$$

where $\frac{\partial G}{\partial x_0} = 0$ and $(B_{ij})_{1 \leq i,j \leq n}$ is a nondegenerate skew-symmetric matrix with entries in $\mathbb{F}_q$ and zero diagonal.

(3) *$d = q+2$ if and only if $p = n = 2$ and, upon rescaling F by a nonzero constant,*

$$F = x_0x_1x_2(x_0^{q-1} + x_1^{q-1} + x_2^{q-1}) + G(x_0^2, x_1^2, x_2^2)$$

for some polynomial G.

Furthermore, condition (1) or (3) occurs if and only if $F_{1,0}$ is the zero polynomial.

Example 1.3 is a special case of Theorem 1.6 (1). For concrete examples that are classified by (2) and (3) in the theorem, one can find them in Examples 4.10 and 4.11, respectively. As an immediate consequence of Theorems 1.5 and 1.6:

Corollary 1.7. *Let $X \subset \mathbb{P}^n$, where $n \geq 2$, be a smooth Frobenius nonclassical hypersurface of degree d over $\mathbb{F}_q$ of characteristic p. Then*

$$\max\{p+1, \sqrt{q}+1\} \leq d \leq \begin{cases} q+1 & \text{if} \quad p \text{ is odd}, \\ q+2 & \text{if} \quad p = 2. \end{cases}$$

In particular, when p is odd, the hypersurface X is Frobenius nonclassical over the prime field $\mathbb{F}_p$ implies that $d = p+1$ and vice versa.

Examples characterized by Theorem 1.6 (1) are *multi-Frobenius nonclassical* in the following sense: Let $X = \{F = 0\} \subset \mathbb{P}^n$ be such an example. If we consider X as a variety over the quadratic extension $\mathbb{F}_{q^2}$, then

$$F_{1,0} := \sum_{i=0}^{n} x_i^{q^2} \frac{\partial F}{\partial x_i} = \sum_{i,j=0}^{n} x_i^q A_{ij} x_j^{q^2} = -F^q.$$

This shows that X is Frobenius nonclassical over not only $\mathbb{F}_q$ but also $\mathbb{F}_{q^2}$. In fact, if we pick an $\alpha \in \mathbb{F}_{q^2} \setminus \{0\}$ that satisfies $\alpha^q = -\alpha$, then $X = \{\alpha F = 0\}$ is a Hermitian variety over $\mathbb{F}_{q^2}$. The classification of multi-Frobenius nonclassical plane curves was carried out by Borges in [Bor09b].

Notice that not every Frobenius nonclassical hypersurface is Hermitian. Example 4.11, which belongs to Theorem 1.6 (3), provides smooth such examples in characteristic 2. For smooth examples in arbitrary characteristics, one can consider

$$F = \sum_{i=0}^{n} x_i^{q^r + \cdots + q + 1} \qquad \text{where} \qquad r \geq 2$$

and view it as a polynomial over $\mathbb{F}_{q^{r+1}}$. Then $F_{1,0} = \sum_{i=0}^{n} x_i^{q^{r+1} + \cdots + q} = F^q$. Therefore, the hypersurface $X = \{F = 0\} \subset \mathbb{P}^n$ is Frobenius nonclassical over $\mathbb{F}_{q^{r+1}}$ and it is clearly not Hermitian over any ground field.

Hermitian varieties, which include examples in Theorem 1.6 (1) as discussed above, are defined by polynomials of the form

$$F = \sum_{i=0}^{n} x_i^{q'} L_i$$

where $L_0, \ldots, L_n$ are linear polynomials and q' is a power of the characteristic of the ground field. Such polynomials are called *Frobenius forms* in [KKP+22]. In the last part of Section 3, we show that Frobenius nonclassical hypersurfaces over $\mathbb{F}_q$ of degree $\sqrt{q}+1$ are defined by Frobenius forms under certain assumptions, and then prove that they must be Hermitian in that situation. This result, together with Theorem 1.5, suggests the following conjecture:

Conjecture 1.8. A Frobenius nonclassical hypersurface $X \subset \mathbb{P}^n$, where $n \geq 2$, over $\mathbb{F}_q$ of degree $\sqrt{q}+1$ is Hermitian provided that it is reduced and smooth at $\mathbb{F}_q$-points.

Smooth Frobenius nonclassical hypersurfaces are nonreflexive [ADL21, Theorem 4.5], so their degrees are either congruent to 0 or 1 modulo the characteristic of the ground field [Kle86, page 191]. For curves, the possibility of being congruent to 0 has been excluded by Pardini [Par86, Corollary 2.2], so it is natural to ask if the same condition holds in higher dimensions. We prove that this is true with certain additional assumptions. To be more concrete, we say that a hypersurface $X \subset \mathbb{P}^n$ over $\mathbb{F}_q$ has *separated variables* if, up to a projective transformation over $\mathbb{F}_q$, its defining polynomial has the form

$$F(x_0, \cdots, x_n) = G(x_0, \ldots, x_m) + H(x_{m+1}, \ldots, x_n)$$

for some $m \in \{0, \ldots, n-1\}$. Frobenius nonclassical components of plane curves with separated variables were studied by Borges in [Bor16].

Our final result establishes the congruence condition on the degree of a Frobenius nonclassical hypersurface with separated variables.

Theorem 1.9. *Let $X = \{F = 0\} \subset \mathbb{P}^n$ be a smooth Frobenius nonclassical hypersurface of degree d over $\mathbb{F}_q$ with separated variables. Then $d \equiv 1 \pmod{p}$ where $p = \mathrm{char}(\mathbb{F}_q)$.*

Organization of the paper. The present paper is organized as follows. In Section 2, we discuss existence and multitude of $\mathbb{F}_q$-rational points, and prove the lower bound $d \geq p+1$. Section 3 is devoted to the proof of the lower bound $d \geq \sqrt{q}+1$. In the same section, we also show that smooth Frobenius nonclassical surfaces of degree $\sqrt{q}+1$ are Hermitian and provide evidences about this phenomenon in higher dimensions. The proof of the upper bound $d \leq q+2$ is given in Section 4 along with the classification result in the cases $d = q+2$ and $d = q+1$. Finally, Section 5 is devoted to the study of Frobenius nonclassical hypersurfaces with separated variables, and contains the proof of Theorem 1.9.

Throughout the paper, by saying that a hypersurface is not a p-th power, we mean its defining polynomial is not a p-th power. In many proofs within the paper, we frequently use the fact that an $\mathbb{F}_q$-irreducible component of a Frobenius nonclassical hypersurface is still Frobenius nonclassical, which is obvious from the geometric definition. In addition, Frobenius nonclassicality is preserved under taking $\mathbb{F}_q$-hyperplane sections, which is a useful property when proving by induction on the dimension.

Acknowledgements. We would like to thank Nathan Kaplan and Felipe Voloch for their informative comments. The first author was partially supported by an NSERC PDF award, and by a postdoctoral research fellowship from the University of British Columbia. The third author is supported by the ERC Synergy Grant HyperK.

2. Rational points and the universal lower bound on degree

In this section, we prove the universal lower bound $d \geq p+1$ in Theorem 1.5. One of the key ingredients is Lemma 2.2 which asserts the existence of an $\mathbb{F}_q$-rational point on a Frobenius nonclassical hypersurface of degree $d \leq q+1$. At the end of this section, we gather evidence for the abundance of rational points, and in particular, establish an explicit lower bound for the number of $\mathbb{F}_q$-points in Proposition 2.12. In the case of surfaces, Proposition 2.14 gives a refined lower bound for the number of rational points.

2.1. **Existence of rational points.** Here we prove that, under mild assumptions, every Frobenius nonclassical hypersurface contains $\mathbb{F}_q$-points. We first prove a Bertini-type result for hypersurfaces that are not a p-th power.

Lemma 2.1. *Let $X = \{F = 0\} \subset \mathbb{P}^n$, where $n \geq 2$, be a hypersurface of degree d over $\mathbb{F}_q$ of characteristic p such that $d \leq q+1$ and X is not a p-th power. Then, for every $1 \leq r \leq n-1$, there exists a linear subspace $H \subset \mathbb{P}^n$ over $\mathbb{F}_q$ of dimension r such that $H \not\subset X$ and the restriction $F|_H$ is not a p-th power.*

Proof. By induction, it suffices to prove the statement only for the case $r = n-1$, that is, for the case when H is an $\mathbb{F}_q$-hyperplane. Because F is not a p-th power, there exists a system of homogeneous coordinates $\{x_0, \dots, x_n\}$ such that

$$F = x_0^s G_s(x_1, \dots, x_n) + \sum_{m \neq s} x_0^m G_m(x_1, \dots, x_n)$$

where $s \not\equiv 0 \pmod p$ and $G_s \neq 0$. The hyperplanes in $\mathbb{P}^n$ of the form

$$H_{\underline{a}} := \{a_1 x_1 + \cdots + a_n x_n = 0\} \qquad \text{where} \qquad \underline{a} = (a_1, \dots, a_n) \in (\mathbb{F}_q)^n \setminus \{0\}$$

are parametrized by $\mathbb{P}^{n-1}_{\mathbb{F}_q}$. Thus there are $\frac{q^n-1}{q-1}$ many of them. If $G_s|_{H_{\underline{a}}} = 0$ for all $H_{\underline{a}}$, then the hypersurface $\{G_s = 0\}$ contains every $H_{\underline{a}}$ as a component, whence

$$\deg(G_s) \geq \frac{q^n-1}{q-1}.$$

But this is impossible since

$$\deg(G_s) = d - s \leq q < \frac{q^n-1}{q-1}$$

where the last inequality holds whenever $n \geq 2$. This shows that there exists $H_{\underline{a}}$ such that $G_s|_{H_{\underline{a}}} \neq 0$. Therefore, the coefficient of x_0^s in $F|_{H_{\underline{a}}}$ is nontrivial, which implies that $F|_{H_{\underline{a}}}$ is not a p-th power. $\square$

The next result explains how to find an $\mathbb{F}_q$-point on a Frobenius nonclassical hypersurface X by looking at its intersection with a suitable $\mathbb{F}_q$-line. If X contains all the $\mathbb{F}_q$-lines of $\mathbb{P}^n$, then clearly X contains all $\mathbb{F}_q$-points in $\mathbb{P}^n$. Thus, it is natural to assume that there exists at least one $\mathbb{F}_q$-line not contained in X.

Lemma 2.2. *Let $X = \{F = 0\} \subset \mathbb{P}^n$ be a Frobenius nonclassical hypersurface of degree d over $\mathbb{F}_q$ and let $p = \mathrm{char}(\mathbb{F}_q)$. Assume that there exists an $\mathbb{F}_q$-line $L \not\subset X$ such that the intersection $X \cap L$ is not a p-th power. Then $X \cap L$ contains at least one $\mathbb{F}_q$-point and the intersection multiplicity at every non-$\mathbb{F}_q$-point is divisible by p.*

Proof. Assume without loss of generality that the line L is given by

$$L = \{[x : y : 0 : 0 : \cdots : 0] \mid [x : y] \in \mathbb{P}^1\}.$$

Then $X \cap L$ is defined by the binary form $f(x, y) = F(x, y, 0, ..., 0)$. Note that $f(x, y) \not\equiv 0$ because $L \not\subset X$. Because X is Frobenius nonclassical,

$$F \mid x_0^q F_0 + ... + x_n^q F_n \qquad \text{where} \qquad F_i := \frac{\partial F}{\partial x_i},$$

which implies that $f(x,y)$ divides $x^q f_x(x,y) + y^q f_y(x,y)$. By Euler's formula, $f(x,y)$ also divides $xf_x(x,y) + yf_y(x,y)$. As a result,

$$f(x,y) \mid (x^q - x)f_x(x,y) + (y^q - y)f_y(x,y). \tag{2.1}$$

In both cases, $f(x,y)$ is not a p-th power, so $f_x(x,y)$ and $f_y(x,y)$ cannot be identically zero simultaneously. Assume without loss of generality that $f_x(x,y) \neq 0$. Note that this implies that $f_x(x,1) \not\equiv 0$. Substituting $y = 1$ into (2.1), we obtain

$$f(x,1) \mid (x^q - x)f_x(x,1). \tag{2.2}$$

Let $\alpha \in \overline{\mathbb{F}_q} \setminus \mathbb{F}_q$ be a non-$\mathbb{F}_q$-root of $f(x,1)$, so that $f(x,1) = (x-\alpha)^m g(x)$ for some $m \geq 1$ and $g(\alpha) \neq 0$. Then

$$f_x(x,1) = m(x-\alpha)^{m-1}g(x) + (x-\alpha)^m g'(x) = (x-\alpha)^{m-1}h(x)$$

where $h(x) = mg(x) + (x-\alpha)g'(x)$. Relation (2.2) now takes the form

$$(x-\alpha)^m g(x) \mid (x^q - x)(x-\alpha)^{m-1}h(x)$$

which implies that $x - \alpha$ divides $(x^q - x)h(x)$. If $m \not\equiv 0 \pmod{p}$, then $h(\alpha) = mg(\alpha) \neq 0$. The polynomial $(x - \alpha)$ does not divide $(x^q - x)$ since $\alpha \notin \mathbb{F}_q$. Hence

$$(x-\alpha) \mid h(x) = mg(x) + (x-\alpha)g'(x).$$

But this implies that $(x-\alpha)$ divides $g(x)$, a contradiction. We conclude that $m \equiv 0 \pmod{p}$. This shows that every non-$\mathbb{F}_q$-point in the intersection $X \cap L$ appears with multiplicity divisible by p.

Let us prove that there exists an $\mathbb{F}_q$-point in the intersection $X \cap L$. If this is not true, then every $P \in X \cap L$ is not defined over $\mathbb{F}_q$, whence appears with multiplicity divisible by p due to the above result. But this implies that $f = F|_L$ is a p-th power, which contradicts the assumption that $X \cap L$ is not a p-th power. This completes the proof. □

Corollary 2.3. *Let $X \subset \mathbb{P}^n$ be a Frobenius nonclassical hypersurface over $\mathbb{F}_q$ such that its degree $d \not\equiv 0 \pmod{p}$ where $p = \operatorname{char}(\mathbb{F}_q)$. Then X meets every $\mathbb{F}_q$-line $L \subset \mathbb{P}^n$ in at least one $\mathbb{F}_q$-point.*

Proof. If $L \subset X$, then the proof is done. Assume $L \not\subset X$. The condition $d \not\equiv 0 \pmod{p}$ implies that $X \cap L$ is not a p-th power. The desired result follows from Lemma 2.2. □

Corollary 2.4. *Let $X \subset \mathbb{P}^n$ be a Frobenius nonclassical hypersurface of degree d over $\mathbb{F}_q$ and let $p = \operatorname{char}(\mathbb{F}_q)$. If $d \leq q+1$ and F is not a p-th power, then X contains at least one $\mathbb{F}_q$-point.*

Proof. If X contains every $\mathbb{F}_q$-line in $\mathbb{P}^n$, then the conclusion is obvious. Otherwise, there exists an $\mathbb{F}_q$-line L such that $X \cap L$ is not a p-th power by Lemma 2.1. The result now follows from Lemma 2.2. □

Corollary 2.4 fails in general when $d \geq q+2$. Indeed, the curves in Theorem 1.6 (3) satisfy the hypothesis of the corollary except that $d = q+2$. Those curves contain no $\mathbb{F}_q$-point by Corollary 4.9.

Remark 2.5. In Corollary 2.4, the hypothesis that F is not a p-th power is necessary because, if this assumption is dropped, then the p-th power of any hypersurface over $\mathbb{F}_q$ that contains no $\mathbb{F}_q$-point would provide a counterexample. One way to construct a concrete example of a pointless hypersurface $Y = \{G = 0\} \subset \mathbb{P}^n$ is via the *norm polynomials* [LN96, Example 6.7] as follows. Let $\{\alpha_0, \alpha_1, ..., \alpha_n\}$ be a basis for $\mathbb{F}_{q^{n+1}}$ over $\mathbb{F}_q$. Consider the homogeneous polynomial over $\mathbb{F}_q$:

$$G = \prod_{i=0}^{n} (\alpha_0^{q^i} x_0 + \cdots + \alpha_n^{q^i} x_n).$$

For every $\mathbf{b} = (b_0, b_1, ..., b_n) \in \mathbb{F}_q^{n+1}$, it is easy to check that $G(\mathbf{b}) = N_{\mathbb{F}_{q^{n+1}}/\mathbb{F}_q}(\mathbf{b})$ is the usual norm map. Hence $G(\mathbf{b}) = 0$ implies $\mathbf{b} = \mathbf{0}$. Thus $Y = \{G = 0\} \subset \mathbb{P}^n$ contains no $\mathbb{F}_q$-points. Another way to construct a pointless hypersurface when $n + 1 \not\equiv 0 \pmod p$ is to take

$$G = \sum_{i=0}^{n} x_i^{q-1}$$

Because $a^{q-1} = 1$ for all $a \in \mathbb{F}_q^*$, it is clear that $Y = \{G = 0\} \subset \mathbb{P}^n$ has no $\mathbb{F}_q$-points.

2.2. **Proof of the lower bound $\mathbf{d \geq p + 1}$.** Let us start by establishing two fundamental lemmas. The first one provides a criterion for the geometric irreducibility of a Frobenius nonclassical hypersurface.

Lemma 2.6. *Let $X = \{F = 0\} \subset \mathbb{P}^n$ be a Frobenius nonclassical hypersurface over $\mathbb{F}_q$ which is irreducible over $\mathbb{F}_q$ and contains a smooth $\mathbb{F}_q$-point $P \in X$. Then X is geometrically irreducible.*

Proof. Since P is smooth, it is contained in a unique geometrically irreducible component $X' \subset X$. Under the q-th Frobenius endomorphism Φ, we have $P = \Phi(P) \in X' \cap \Phi(X')$. Both X' and $\Phi(X')$ are geometrically irreducible components of X that contain P, so we have $X' = \Phi(X')$, that is, X' is defined over $\mathbb{F}_q$. It follows that $X = X'$ as X is irreducible over $\mathbb{F}_q$. □

The next lemma gives a lower bound on the degree of a hypersurface which is almost space-filling and contains a smooth rational point.

Lemma 2.7. *Let $X \subset \mathbb{P}^n$, where $n \geq 2$, be a hypersurface over $\mathbb{F}_q$ that contains at least one smooth $\mathbb{F}_q$-point and satisfies $\#X(\mathbb{F}_q) \geq \#\mathbb{P}^n(\mathbb{F}_q) - 1$. Then $\deg(X) \geq q + 1$.*

Proof. We proceed by induction on n. If $n = 2$, then $X = C \subset \mathbb{P}^2$ is a plane curve containing a smooth $\mathbb{F}_q$-point $P \in C$. The tangent line $L = T_P C$ is defined over $\mathbb{F}_q$. We also have $L \neq C$ since $\#C(\mathbb{F}_q) \geq \#\mathbb{P}^2(\mathbb{F}_q) - 1 > \#\mathbb{P}^1(\mathbb{F}_q)$. Assume that $L \not\subset C$. Then L intersects C properly in at least $\#\mathbb{P}^1(\mathbb{F}_q) - 1 = q$ points and with multiplicity at least 2 at P. In particular, L meets C in at least $q + 1$ points counted with multiplicity. Thus $\deg(C) \geq q + 1$.

Let us treat the case when $L \subset C$. Since $\#C(\mathbb{F}_q) \geq \#\mathbb{P}^2(\mathbb{F}_q) - 1$, there is at most one $\mathbb{F}_q$-point not contained in C. Thus, we can find another $\mathbb{F}_q$-line $L' \neq L$ which also passes through P and satisfies $L'(\mathbb{F}_q) \subset C(\mathbb{F}_q)$. Note that $L' \not\subset C$ because C is smooth at $P \in L \cap L'$. It follows that $\deg(C) \geq \#L'(\mathbb{F}_q) = q + 1$.

Now let $X \subset \mathbb{P}^n$ be a hypersurface as in the statement with a smooth $P \in X(\mathbb{F}_q)$. There exists an $\mathbb{F}_q$-hyperplane $H \subset \mathbb{P}^n$ passing through P which intersects $T_P X$ properly. Then $Y := X \cap H$ is a hypersurface in $H \cong \mathbb{P}^{n-1}$ over $\mathbb{F}_q$ that contains P as a smooth $\mathbb{F}_q$-point and satisfies $\#Y(\mathbb{F}_q) \geq \#\mathbb{P}^{n-1}(\mathbb{F}_q) - 1$. This implies that $\deg(X) = \deg(Y) \geq q + 1$ by the induction hypothesis. □

We are now ready to establish the lower bound $d \geq p+1$ in Theorem 1.5. The proof will proceed by induction on the dimension of the hypersurface. The following result settles the initial case.

Lemma 2.8. *Let $C \subset \mathbb{P}^2$ be a Frobenius nonclassical curve of degree $d \geq 2$ over $\mathbb{F}_q$ of characteristic p which is smooth at $\mathbb{F}_q$-points. Then $d \geq p+1$.*

Proof. Because C is smooth at $\mathbb{F}_q$-points, it cannot contain an $\mathbb{F}_q$-line with multiplicity greater than or equal to 2. Let $C' \subset C$ be a curve irreducible over $\mathbb{F}_q$. Then C' is Frobenius nonclassical over $\mathbb{F}_q$. Notice that there is nothing to prove if $\deg(C') \geq q+1$, so we can assume $\deg(C') \leq q$. If C' is the p-th power of an $\mathbb{F}_q$-line, then it is singular at $\mathbb{F}_q$-points, a contradiction. Hence if C' is a p-th power, then $\deg(C') = mp$ for some $m \geq 2$, thus $\deg(C') \geq p+1$ as desired. Assume that C' is not a p-th power. Then Corollary 2.4 implies that C' contains an $\mathbb{F}_q$-point, which is smooth by hypothesis. It follows from Lemma 2.6 that C' is geometrically irreducible.

The geometric irreducibility of C' and the fact that it is smooth at $\mathbb{F}_q$-points force it to be reduced. Therefore, C is nonreflexive by [HV90, Proposition 1]. Pick a smooth non-$\mathbb{F}_q$-point $P \in C$. The non-reflexivity implies that $T_P C$ intersects C at P with multiplicity divisible by p. Since C' is Frobenius nonclassical, $T_P C$ also contains $\Phi(P) \neq P$ where Φ is the q-th Frobenius endomorphism. Therefore, $T_P C$ intersects C' in at least $p+1$ points counted with multiplicity, which yields $\deg(C) \geq \deg(C') \geq p+1$. □

Theorem 2.9. *Let $X \subset \mathbb{P}^n$, where $n \geq 2$, be a Frobenius nonclassical hypersurface of degree $d \geq 2$ over $\mathbb{F}_q$ of characteristic p which is smooth at $\mathbb{F}_q$-points. Then $d \geq p+1$.*

Proof. We proceed by induction on the dimension of X. The base case is proved in Lemma 2.8. For the inductive step, let $X \subset \mathbb{P}^n$ be a Frobenius nonclassical hypersurface that is smooth at $\mathbb{F}_q$-points. Then X can contain at most one $\mathbb{F}_q$-linear component, possibly some $H_0 \subset \mathbb{P}^n$. Let $H \subset \mathbb{P}^n$ be an $\mathbb{F}_q$-hyperplane distinct from H_0. Then $Y := X \cap H$ is Frobenius nonclassical with $\dim(Y) < \dim(X)$. If Y is smooth at $\mathbb{F}_q$-points, then the inductive hypothesis implies that $\deg(X) = \deg(Y) \geq p+1$.

Thus, we may assume that for every H as above, the section $Y = X \cap H$ is singular at some $\mathbb{F}_q$-point Q, which is equivalent to asserting that $H = T_Q Y$. This shows that each $\mathbb{F}_q$-hyperplane in $\mathbb{P}^n$, other than possibly H_0, is tangent to X at some $\mathbb{F}_q$-point. Notice that the Gauss map induced by X is well-defined at the set of $\mathbb{F}_q$-points as these points are smooth. It follows that the Gauss map is surjective at the level of $\mathbb{F}_q$-points away from H_0:

$$X(\mathbb{F}_q) \twoheadrightarrow (\mathbb{P}^n)^*(\mathbb{F}_q) \setminus \{H_0\} : P \longmapsto T_P X.$$

Hence $\#X(\mathbb{F}_q) \geq \#\mathbb{P}^n(\mathbb{F}_q) - 1$. By Lemma 2.7, we get $\deg(X) \geq q+1 \geq p+1$. □

The following result will be used later in the proof of Proposition 2.12. We include it here as it has the same flavor as Lemma 2.7.

Lemma 2.10. *Let $X \subset \mathbb{P}^n$ be a hypersurface over $\mathbb{F}_q$ which is space-filling, namely, it satisfies $X(\mathbb{F}_q) = \mathbb{P}^n(\mathbb{F}_q)$. Then $\deg(X) \geq q+1$.*

Proof. We proceed by induction on n. When $n = 1$, the conclusion follows since a space-filling subset $X \subset \mathbb{P}^1$ is defined by a binary form divisible by $x^q y - xy^q$. For the inductive step, let $X \subset \mathbb{P}^n$ be a hypersurface with $X(\mathbb{F}_q) = \mathbb{P}^n(\mathbb{F}_q)$ where $n \geq 2$. If X contains all the

$\mathbb{F}_q$-hyperplanes in $\mathbb{P}^n$, then

$$\deg(X) \geq \#(\mathbb{P}^n)^*(\mathbb{F}_q) = \sum_{i=0}^{n} q^i > q+1.$$

Otherwise, there exists an $\mathbb{F}_q$-hyperplane $H \subset \mathbb{P}^n$ such that $\dim(X \cap H) = \dim(H) - 1$. Now, $Y := X \cap H$ can be viewed as a hypersurface in $H \cong \mathbb{P}^{n-1}$ which is space-filling. Applying the induction hypothesis to Y, we obtain $\deg(X) = \deg(Y) \geq q+1$. □

2.3. **Blocking sets and abundance of rational points.** In this part, we provide evidence for the abundance of $\mathbb{F}_q$-points on Frobenius nonclassical hypersurfaces. This phenomenon was already observed for the case of smooth plane curves by Hefez–Voloch [HV90] and later extended to singular curves by Borges–Homma [BH17]. We will start by showing that the configuration of $\mathbb{F}_q$-points on a Frobenius nonclassical hypersurface possesses an interesting combinatorial structure.

To state our first result in this direction, we introduce a relevant definition from finite geometry.

Definition 2.11. Let $\mathcal{S}$ be a set of $\mathbb{F}_q$-points in $\mathbb{P}^n$. We say that $\mathcal{S}$ is a *blocking set with respect to lines* if $\mathcal{S} \cap L$ is non-empty for each $\mathbb{F}_q$-line $L \subset \mathbb{P}^n$. Such a blocking set $\mathcal{S}$ is called *trivial* if it contains all the $\mathbb{F}_q$-points on some $\mathbb{F}_q$-hyperplane. Otherwise, $\mathcal{S}$ is called *non-trivial*.

More generally, one can define the notion of a k-blocking set which is a set of $\mathbb{F}_q$-points in $\mathbb{P}^n$ which meets every $(n-k)$-dimensional space defined over $\mathbb{F}_q$. The definition above can then be viewed as the special case when $k = n-1$. See [KS20, Chapter 9] for a comprehensive account of finite geometry in higher dimensional spaces.

Proposition 2.12. *Let $X \subset \mathbb{P}^n$ be a Frobenius nonclassical hypersurface over $\mathbb{F}_q$, $q \neq 2$, of degree $d \leq q$ and let $p = \mathrm{char}(\mathbb{F}_q)$. Assume that $d \not\equiv 0 \pmod{p}$ and X contains no $\mathbb{F}_q$-linear component. Then $X(\mathbb{F}_q)$ is a non-trivial blocking set with respect to lines. Furthermore,*

$$\#X(\mathbb{F}_q) \geq \frac{q^n - 1}{q-1} + \sqrt{q} \cdot q^{n-2}$$

Proof. By Corollary 2.3, every $\mathbb{F}_q$-line L meets X in at least one $\mathbb{F}_q$-point, so $X(\mathbb{F}_q)$ is a blocking set with respect to lines. Next, let H be any $\mathbb{F}_q$-hyperplane. Then $Y := X \cap H$ is a hypersurface of degree $d \leq q$ inside $H \cong \mathbb{P}^{n-1}$. By Lemma 2.10, we know that $Y(\mathbb{F}_q) \neq H(\mathbb{F}_q)$, which means $X(\mathbb{F}_q)$ does not contain all of $H(\mathbb{F}_q)$. This shows that $X(\mathbb{F}_q)$ is a non-trivial blocking set. Finally, the lower bound on the number $\mathbb{F}_q$-points is a consequence of Heim's theorem on blocking sets ([Hei96], see also [HT15, Theorem 9.8]). □

Proposition 2.12 fails when $d \geq q+1$ in view of Theorem 1.6. Indeed, for the examples in (1) and (2) of the theorem, the former are space-filling, while the latter contain all the $\mathbb{F}_q$-points on the hyperplane $\{x_0 = 0\}$. Therefore, the sets of $\mathbb{F}_q$-points in these two cases form trivial blocking sets.

Next, we prove a lower bound on the number of $\mathbb{F}_q$-points on a smooth Frobenius nonclassical surface that depends on the degree of the surface. As a preparation, we prove a lower bound for curves.

Lemma 2.13. *Let $C \subset \mathbb{P}^2$ be a reduced Frobenius nonclassical curve over $\mathbb{F}_q$ of degree $d \le q+1$ which is smooth at all of its $\mathbb{F}_q$-points. Then*

$$\#C(\mathbb{F}_q) \ge d(q-d+2)$$

Proof. Let us write $C = C_1 \cup \cdots \cup C_m$ where C_i, $i = 1, \ldots, m$ are $\mathbb{F}_q$-irreducible. Since C is reduced, each C_i is not a p-th power. By Corollary 2.4 and Lemma 2.6, each C_i is geometrically irreducible. Denoting $d_i = \deg(C_i)$, we know that $\#C_i(\mathbb{F}_q) \ge d_i(q-d_i+2)$ by [BH17, Corollary 1.4]. Since C is smooth at all of its $\mathbb{F}_q$-points, $C(\mathbb{F}_q)$ is a disjoint union of $C_i(\mathbb{F}_q)$ for $i = 1, \ldots, m$. Therefore,

$$\begin{aligned} \#C(\mathbb{F}_q) = \sum_{i=1}^{m} \#C_i(\mathbb{F}_q) \ge \sum_{i=1}^{m} d_i(q-d_i+2) &= dq - \sum_{i=1}^{m} d_i^2 + 2d \\ &\ge dq - \left(\sum_{i=1}^{m} d_i\right)^2 + 2d = d(q-d+2) \end{aligned}$$

as claimed. □

We are now ready to establish a lower bound on the number of $\mathbb{F}_q$-points for surfaces. We will see that, when q is large, the bound is roughly

$$qd(q-d+2) \approx \mathrm{O}(dq^2)$$

If the surface is not linear, then this bound grows at least in the rate $\mathrm{O}(q^{\frac{5}{2}})$ in view of the lower bound $d \ge \sqrt{q}+1$ in Theorem 1.5. This is a direct analogue of the curve case, where the lower bound is $d(q-d+2) \approx \mathrm{O}(dq)$.

Proposition 2.14. *Suppose that $X \subset \mathbb{P}^3$ is a smooth Frobenius nonclassical surface over $\mathbb{F}_q$ of degree $d \le q+1$. Then*

$$\#X(\mathbb{F}_q) \ge \frac{(q^3+q^2+q+1)d(q-d+2)}{(q^2+q)+d(q-d+2)}$$

Proof. Let us call an $\mathbb{F}_q$-plane H "good" if $X \cap H$ is smooth at $\mathbb{F}_q$-points. Consider the set

$$\mathcal{I} = \{(H, P) \mid H \text{ is a good plane with } P \in (X \cap H)(\mathbb{F}_q)\}.$$

The number of good planes is at least

$$\#(\mathbb{P}^3)^*(\mathbb{F}_q) - \#X(\mathbb{F}_q) = (q^3+q^2+q+1) - \#X(\mathbb{F}_q).$$

For each good plane H, we observe that $\#(X \cap H)(\mathbb{F}_q) \ge d(q-d+2)$ by Lemma 2.13. Here, we are using the fact that a plane section of a smooth surface is a reduced curve (which is a consequence of Zak's theorem [Zak93, Corollary I.2.8]). Thus, we get a lower bound

$$\#\mathcal{I} \ge (q^3+q^2+q+1-\#X(\mathbb{F}_q)) \cdot d(q-d+2).$$

On the other hand, each $\mathbb{F}_q$-point of X is contained in at most q^2+q good planes. This gives us an upper bound,

$$\#\mathcal{I} \le \#X(\mathbb{F}_q) \cdot (q^2+q).$$

Combining the lower and the upper bounds, we obtain,

$$(q^3+q^2+q+1-\#X(\mathbb{F}_q)) \cdot d(q-d+2) \le \#X(\mathbb{F}_q) \cdot (q^2+q).$$

Rearranging this inequality, we get

$$\#X(\mathbb{F}_q) \geq \frac{(q^3+q^2+q+1)d(q-d+2)}{(q^2+q)+d(q-d+2)}.$$

as desired. □

Remark 2.15. In [HK13, Theorem 1.2], Homma and Kim proved an upper bound for the number of rational points on a hypersurface without an $\mathbb{F}_q$-linear component. According to [Tir17, Theorem 1 (1) and (2)], this upper bound is achieved by examples in Theorem 1.6 (1) and surface examples of degree $\sqrt{q}+1$ as in Theorem 1.5. This provides another evidence on the abundance of rational points on Frobenius nonclassical hypersurfaces.

3. Lower bounds on degree and Hermitian surfaces

In this section, we finish the proof of Theorem 1.5. For the lower bound $d \geq \sqrt{q}+1$, we will present a proof by contradiction, and therefore, will assume the existence of a hypersurface $X \subset \mathbb{P}^n$ as in the hypothesis except that $d \leq \sqrt{q}$. Our strategy consists of two steps:

I. We first find a 2-plane $H \subset \mathbb{P}^n$ such that $X \cap H$ contains a curve component C over $\mathbb{F}_q$ that is reduced of degree at least 2 and smooth at $\mathbb{F}_q$-points.
II. Then we prove that there exists a curve $C' \subset C$ over $\mathbb{F}_q$ of degree at least 2 that is geometrically irreducible. The curve C' is Frobenius nonclassical by construction. Hence $\deg(C') \geq \sqrt{q}+1$ by [BH17, Corollary 3.2], contradicting our assumption that $d \leq \sqrt{q}$.

For the last statement in the theorem, the curve case is done in [BH17, Corollary 3.2], so we will prove the assertion in the surface case. In the end of this section, we include a discussion on the Hermiticity in higher dimensions.

3.1. Linear sections that are smooth at $\mathbb{F}_q$-points.

Here we prove several Bertini-type results for reduced hypersurfaces over $\mathbb{F}_q$ that are smooth at $\mathbb{F}_q$-points, which will be used to establish Step I. in our strategy.

Lemma 3.1. *Let $X \subset \mathbb{P}^n$ where $n \geq 4$ be a reduced hypersurface over $\mathbb{F}_q$ of degree $d \leq \frac{q}{2}$ on which every $\mathbb{F}_q$-point is smooth. Then there exists an $\mathbb{F}_q$-hyperplane H such that $X \cap H$ is reduced of dimension $n-2$ and smooth at $\mathbb{F}_q$-points.*

Proof. Our strategy is to show that the number of $\mathbb{F}_q$-hyperplanes in $\mathbb{P}^n$ is greater than the number of $\mathbb{F}_q$-hyperplanes H that satisfy at least one of the following bad conditions:

- $X \cap H$ is not reduced.
- $X \cap H$ is not of dimension $n-2$.
- $X \cap H$ is singular at some $\mathbb{F}_q$-point.

By [ADL22, Proposition 4.6], the number of $\mathbb{F}_q$-hyperplanes H such that $X \cap H$ is not reduced or does not have dimension $n-2$ is at most

$$d(d-1)(q+1)^2+1.$$

On the other hand, $X \cap H$ is singular at an $\mathbb{F}_q$-point $P \in X$ implies that $H = T_P X$, so the number of hyperplanes H for which $X \cap H$ is singular at some $\mathbb{F}_q$-point is at most $\#X(\mathbb{F}_q)$, the total number of $\mathbb{F}_q$-points on X. It was proved independently by Serre [Ser91] and Sorensen [Sør94] that

$$\#X(\mathbb{F}_q) \leq dq^{n-1}+q^{n-2}+\cdots+q+1.$$

Thus, the number of bad hyperplanes is at most the sum of these contributions:

$$\big(dq^{n-1} + q^{n-2} + \cdots + q + 1\big) + d(d-1)(q+1)^2 + 1.$$

Because the total number of $\mathbb{F}_q$-hyperplanes in $\mathbb{P}^n$ is $\sum_{i=0}^{n} q^i$, the result follows if we can prove the following inequality:

$$\sum_{i=0}^{n} q^i > \big(dq^{n-1} + q^{n-2} + \cdots + q + 1\big) + d(d-1)(q+1)^2 + 1$$

which is equivalent to

$$q^n > (d-1)q^{n-1} + d(d-1)(q+1)^2 + 1. \tag{3.1}$$

Using the assumptions that $d \leq \frac{q}{2}$ and $n \geq 4$, we get

$$\begin{aligned}(d-1)q^{n-1} + d(d-1)(q+1)^2 + 1 &\leq \left(\frac{q}{2} - 1\right) q^{n-1} + \frac{q}{2}\left(\frac{q}{2} - 1\right)(q+1)^2 + 1 \\ &= \frac{q^n}{2} - q^{n-1} + \frac{q^4}{4} - \frac{3q^2}{4} - \frac{q}{2} + 1 < q^n.\end{aligned}$$

The last inequality can be proved by computing directly that the real function

$$f(x) = x^n - \left(\frac{x^n}{2} - x^{n-1} + \frac{x^4}{4} - \frac{3x^2}{4} - \frac{x}{2} + 1\right) \quad \text{where} \quad n \geq 4$$

satisfies $f(x) > 0$ for all $x \geq 2$. This proves inequality (3.1) and establishes the existence of an $\mathbb{F}_q$-hyperplane satisfying the desired conditions. $\square$

Lemma 3.2. *Let $X \subset \mathbb{P}^3$ be a reduced surface of degree d over $\mathbb{F}_q$ such that $2 \leq d \leq \sqrt{q}$ and every $\mathbb{F}_q$-point on X is smooth. Then there exists an $\mathbb{F}_q$-plane H such that $X \cap H$ contains a curve component C over $\mathbb{F}_q$ of degree ≥ 2 that is reduced and smooth at $\mathbb{F}_q$-points.*

Proof. The proof is similar to the proof of Lemma 3.1. The only difference is that we need to apply a refined bound on the number of $\mathbb{F}_q$-points on X by Homma and Kim [HK13]. The theorem [HK13, Theorem 1.2] in the case of surfaces states that

$$\#X(\mathbb{F}_q) \leq (d-1)q^2 + dq + 1 \tag{3.2}$$

provided that X has no $\mathbb{F}_q$-linear component. Because X is smooth at $\mathbb{F}_q$-points, it has at most one $\mathbb{F}_q$-plane as a component. Thus, we proceed according to two cases.

Assume that X has no $\mathbb{F}_q$-plane component. In this case, we can directly apply (3.2). As in the proof of Lemma 3.1, the number of bad planes defined over $\mathbb{F}_q$ is at most

$$\#X(\mathbb{F}_q) + d(d-1)(q+1)^2 + 1 \leq (d-1)q^2 + dq + 1 + d(d-1)(q+1)^2 + 1$$

The conclusion will follow if we can show that the total number of $\mathbb{F}_q$-planes exceeds the number of bad planes. Thus, it suffices to show that:

$$q^3 + q^2 + q + 1 > (d-1)q^2 + dq + 1 + d(d-1)(q+1)^2 + 1 \tag{3.3}$$

whenever $d \leq \sqrt{q}$. The right hand side of the inequality increases as d increases, so it suffices to establish the inequality (3.3) in the case $d = \sqrt{q}$. In other words, it suffices to prove the inequality

$$q^3 + q^2 + q + 1 > (\sqrt{q} - 1)q^2 + \sqrt{q} \cdot q + 1 + \sqrt{q}(\sqrt{q} - 1)(q+1)^2 + 1.$$

This inequality can be directly checked for each $q \geq 2$. This justifies (3.3) and furnishes an $\mathbb{F}_q$-plane H such that $C = H \cap X$ is a reduced curve of degree $d \geq 2$ that is smooth at $\mathbb{F}_q$-points.

Now assume that X has exactly one $\mathbb{F}_q$-plane component. Note that if $d = 2$, then X is a union of two $\mathbb{F}_q$-planes and thus contains singular $\mathbb{F}_q$-points, which contradicts our hypothesis. Hence $d \geq 3$ in this case. Write $X = Y \cup H_0$ where H_0 is an $\mathbb{F}_q$-plane. Since Y has no $\mathbb{F}_q$-plane component and Y is also Frobenius nonclassical, we can apply the same argument as in the previous case to Y. Note that inequality (3.3) still holds in this case as $\deg(Y) = \deg(X) - 1 < \sqrt{q}$. Thus, we obtain an $\mathbb{F}_q$-plane H such that $C = H \cap Y$ is a reduced curve of degree $d - 1 \geq 2$ that is smooth at $\mathbb{F}_q$-points. $\square$

Corollary 3.3. *Let $X \subset \mathbb{P}^n$, where $n \geq 3$, be a reduced hypersurface over $\mathbb{F}_q$ of degree d such that $2 \leq d \leq \sqrt{q}$ and every $\mathbb{F}_q$-point on X is smooth. Then there exists an $\mathbb{F}_q$-plane H such that $X \cap H$ contains a curve component C over $\mathbb{F}_q$ of degree ≥ 2 that is reduced and smooth at $\mathbb{F}_q$-points.*

Proof. The assumption $2 \leq \sqrt{q}$ is equivalent to $\sqrt{q} \leq \frac{q}{2}$, so we can apply Lemma 3.1 repeatedly until getting a linear subspace $H' \cong \mathbb{P}^3$ over $\mathbb{F}_q$ such that $X \cap H'$ is a reduced surface that is smooth at $\mathbb{F}_q$-points. By Lemma 3.2, there exists a plane $H \subset H'$ over $\mathbb{F}_q$ such that $X \cap H$ satisfies the desired property. $\square$

3.2. **Existence of transverse lines to plane curves.** As an intermediate step, we establish a few results that guarantee the existence of a transverse $\mathbb{F}_q$-line to a reduced plane curve with smooth $\mathbb{F}_q$-points under the assumption that $d \leq \sqrt{q}$.

Lemma 3.4. *Let $C \subset \mathbb{P}^2$ be a reduced and geometrically irreducible curve of degree d over $\overline{\mathbb{F}_q}$ that is smooth at $\mathbb{F}_q$-points. Then the number of $\mathbb{F}_q$-lines not transverse to C is at most*

$$\frac{1}{2}(d-1)(d-2) + d(d-1)q + 1.$$

Proof. A line L is not transverse to the curve C if and only if it passes through a singular point of C or is the tangent line at a smooth point of C. We will compute an upper bound for the number of $\mathbb{F}_q$-lines in each of the two categories.

By [Liu02, §7.5, Proposition 5.4], the number of singular points of a geometrically irreducible curve is at most $\frac{1}{2}(d-1)(d-2)$. By hypothesis, each singular point of C is *not* defined over $\mathbb{F}_q$. Thus, each singular point has at most one $\mathbb{F}_q$-line passing through it, and this gives a total contribution of

$$\frac{1}{2}(d-1)(d-2) \tag{3.4}$$

many non-transverse $\mathbb{F}_q$-lines passing through a singular point of C.

To estimate the number of tangent $\mathbb{F}_q$-lines to C, note that the dual curve C^* has degree at most $d(d-1)$. It follows that

$$\#\{\mathbb{F}_q\text{-lines tangent to } C\} \leq C^*(\mathbb{F}_q) \leq \deg(C^*)q + 1 \leq d(d-1)q + 1 \tag{3.5}$$

where the last inequality follows from the Serre–Sørensen bound that $\#E(\mathbb{F}_q) \leq \delta q + 1$ for any plane curve E of degree δ. Adding up (3.4) and (3.5) gives the desired bound. $\square$

Remark 3.5. In the proof above, we used the result $\#E(\mathbb{F}_q) \leq dq + 1$ for any plane curve $E \subset \mathbb{P}^2$. This result is a special case of the more general result for hypersurfaces independently proved by Serre [Ser91] and Sørensen [Sør94]. However, in those papers, the varieties are defined over $\mathbb{F}_q$, whereas in our lemma, E is a plane curve defined over $\overline{\mathbb{F}_q}$, and not necessarily over $\mathbb{F}_q$. The result $\#E(\mathbb{F}_q) \leq dq + 1$ nonetheless holds even in this more general case. Indeed, the same proof in Sørensen [Sør94, Theorem 2.1] goes through even if E is not defined over $\mathbb{F}_q$.

Next, we generalize the previous result without the hypothesis on geometric irreducibility.

Lemma 3.6. *Let $C \subset \mathbb{P}^2$ be a reduced curve of degree d over $\overline{\mathbb{F}_q}$ on which every $\mathbb{F}_q$-point is smooth. Then the number of $\mathbb{F}_q$-lines not transverse to C is at most*

$$\frac{1}{2}d^2 + qd^2 - qd + \frac{1}{2}d$$

Proof. Let us write $C = C_1 \cup C_2 \cup \cdots \cup C_m$ where each C_i is geometrically irreducible of degree d_i. Every $\mathbb{F}_q$-line not transverse to C is either not transverse to some C_i (Type I) or passes through an intersection of C_i and C_j for some $i \neq j$ (Type II). The number of non-transverse $\mathbb{F}_q$-lines of Type I can be bounded by applying Lemma 3.4 to each C_i and summing up the contributions:

$$\#\{\text{Type I non-transverse } \mathbb{F}_q\text{-lines}\} \leq \sum_{i=1}^{m} \left(\frac{1}{2}(d_i - 1)(d_i - 2) + d_i(d_i - 1)q + 1 \right).$$

To give an upper bound on the number of non-transverse $\mathbb{F}_q$-lines of Type II, we note that any point $P \in C_i \cap C_j$ is a singular point of C. Since C is smooth at $\mathbb{F}_q$-points, P is not defined over $\mathbb{F}_q$. Thus, there can be at most one $\mathbb{F}_q$-line that passes through P. Since $C_i \cap C_j$ has at most $d_i d_j$ distinct points by Bézout's theorem,

$$\#\{\text{Type II non-transverse } \mathbb{F}_q\text{-lines}\} \leq \sum_{i<j} d_i d_j.$$

Therefore, the number of $\mathbb{F}_q$-lines not transverse to C is at most

$$\begin{aligned}
&\sum_{i=1}^{m} \left(\frac{1}{2}(d_i - 1)(d_i - 2) + d_i(d_i - 1)q + 1 \right) + \sum_{i<j} d_i d_j \\
&= \sum_{i=1}^{m} \left(\frac{1}{2}(d_i^2 - 3d_i + 2) + (d_i^2 - d_i)q + 1 \right) + \sum_{i<j} d_i d_j \\
&= \frac{1}{2} \left(\sum_{i=1}^{m} d_i^2 + \sum_{i<j} 2d_i d_j \right) + \sum_{i=1}^{m} q d_i^2 - \sum_{i=1}^{m} \left(q + \frac{3}{2} \right) d_i + \sum_{i=1}^{m} 2 \\
&\leq \frac{1}{2}d^2 + qd^2 - \left(q + \frac{3}{2} \right) d + 2m \\
&= \frac{1}{2}d^2 + qd^2 - qd + \frac{1}{2}d + (2m - 2d) \\
&\leq \frac{1}{2}d^2 + qd^2 - qd + \frac{1}{2}d
\end{aligned}$$

where the last inequality follows from the fact that $m \leq d$. ☐

Corollary 3.7. *Let $C \subset \mathbb{P}^2$ be a reduced curve over $\overline{\mathbb{F}_q}$ of degree $d \leq \sqrt{q}$ that is smooth at $\mathbb{F}_q$-points. Then there exists an $\mathbb{F}_q$-line L transverse to C.*

Proof. It suffices to show that the total number of $\mathbb{F}_q$-lines in $\mathbb{P}^2$ exceeds the number of $\mathbb{F}_q$-lines not transverse to C. By Lemma 3.6 and the hypothesis that $d \leq \sqrt{q}$,

$$\begin{aligned}\#\{\mathbb{F}_q\text{-lines not transverse to } C\} &\leq \frac{1}{2}d^2 + qd^2 - \left(q - \frac{1}{2}\right) d < \frac{1}{2}d^2 + qd^2 \\ &\leq \frac{1}{2}\left(\sqrt{q}\right)^2 + q\left(\sqrt{q}\right)^2 = q^2 + \frac{q}{2} < q^2 + q + 1 = \#\{\mathbb{F}_q\text{-lines in } \mathbb{P}^2\}\end{aligned}$$

as desired. □

Remark 3.8. In [ADL22, Theorem 4.2], we proved that every reduced plane curve C of degree d in $\mathbb{P}^2$ admits a transverse $\mathbb{F}_q$-line once $q \geq \frac{3}{2}d(d-1)$. Corollary 3.7 relaxes the bound to $q \geq d^2$ at the cost of the additional hypothesis that C is smooth at its $\mathbb{F}_q$-points.

3.3. **Proof of the lower bound $\mathbf{d \geq \sqrt{q} + 1}$.** To establish Step II., we need one more result concerning the existence of a smooth $\mathbb{F}_q$-point and geometric irreducibility for Frobenius nonclassical curves.

Lemma 3.9. *Let $C \subset \mathbb{P}^2$ be a Frobenius nonclassical curve over $\mathbb{F}_q$ which is irreducible over $\mathbb{F}_q$ and admits a transverse $\mathbb{F}_q$-line L. Then L intersects C in $\mathbb{F}_q$-points only and C is geometrically irreducible.*

Proof. Assume, to the contrary, that there exists $P \in L \cap C$ *not* defined over $\mathbb{F}_q$. Note that P is a smooth point as L meets C transversely. Under the q-th Frobenius endomorphism Φ, we have $\Phi(P) \in T_P C \cap L$. As a result, both $T_P C$ and L contain the distinct points P and $\Phi(P)$, whence $L = T_P C$. This shows that L is tangent to C, a contradiction. Therefore, L intersects C in $\deg(C)$ many smooth $\mathbb{F}_q$-points, which implies that C is geometrically irreducible by Lemma 2.6. □

We are now ready to prove the lower bound $d \geq \sqrt{q} + 1$ in Theorem 1.5.

Theorem 3.10. *Let $X \subset \mathbb{P}^n$, where $n \geq 2$, be a reduced Frobenius nonclassical hypersurface of degree $d \geq 2$ over $\mathbb{F}_q$ of characteristic p which is smooth at $\mathbb{F}_q$-points. Then $d \geq \sqrt{q} + 1$.*

Proof. Assume, to the contrary, that there exists a reduced Frobenius nonclassical hypersurface $X \subset \mathbb{P}^n$ of degree d where $2 \leq d \leq \sqrt{q}$. By Corollary 3.3, there exists an $\mathbb{F}_q$-plane H such that $X \cap H$ contains a curve component C over $\mathbb{F}_q$ that is reduced of degree ≥ 2 and smooth at $\mathbb{F}_q$-points. The smoothness at $\mathbb{F}_q$-points forbids C from being a union of $\mathbb{F}_q$-lines, so there exists a curve $C' \subset C$ of degree ≥ 2 irreducible over $\mathbb{F}_q$.

Being an $\mathbb{F}_q$-component of a linear section over $\mathbb{F}_q$ implies that C' is Frobenius nonclassical as a plane curve in $H \cong \mathbb{P}^2$. On the other hand, C' is smooth at $\mathbb{F}_q$-points and we have $\deg(C') \leq \deg(C) \leq \deg(X) \leq \sqrt{q}$, so there exists an $\mathbb{F}_q$-line transverse to C' by Corollary 3.7. We conclude that C' is geometrically irreducible by Lemma 3.9. However, this implies $\deg(C') \geq \sqrt{q} + 1$ according to [BH17, Corollary 3.2], a contradiction. □

3.4. **Frobenius nonclassical surfaces of minimal degrees.** It turns out that smooth Frobenius nonclassical surfaces over $\mathbb{F}_q$ whose degrees attain the minimum $\sqrt{q} + 1$ are Hermitian, that is, they are projectively equivalent over $\mathbb{F}_q$ to the surface defined by

$$x_0^{\sqrt{q}+1} + x_1^{\sqrt{q}+1} + x_2^{\sqrt{q}+1} + x_3^{\sqrt{q}+1} = 0.$$

Our proof is built upon the curve case proved by Borges and Homma [BH17, Corollary 3.2], which in turn relies on the characterization of Hermitian curves [HKT08, Theorem 10.47].

The main idea of our proof in the surface case is to find sufficiently many $\mathbb{F}_q$-planes that cut out Hermitian curves on the surface.

Lemma 3.11. *Let $C \subset \mathbb{P}^2$ be a reduced Frobenius nonclassical curve over $\mathbb{F}_q$ of degree $d = \sqrt{q}+1$ which is smooth at $\mathbb{F}_q$-points. Then C is Hermitian.*

Proof. Write $C = C_1 \cup \cdots \cup C_r$ where each C_i is irreducible over $\mathbb{F}_q$. Note that each C_i is Frobenius nonclassical. Furthermore, each C_i has an $\mathbb{F}_q$-point P_i by Corollary 2.4, because $\deg(C_i) \leq \sqrt{q}+1$ and C_i is not a p-th power as C is a reduced curve. By hypothesis, C_i is smooth at P_i. Applying Lemma 2.6, we see that each C_i is geometrically irreducible. Observe that C cannot be a union of $\mathbb{F}_q$-lines due to our hypothesis. Hence, there exists $1 \leq i \leq r$ such that $\deg(C_i) \geq 2$. By [BH17, Corollary 3.2], we obtain that $\deg(C_i) \geq \sqrt{q}+1 = \deg(C)$ and thus $C = C_i$ is Hermitian. □

Lemma 3.12. *Let $X \subset \mathbb{P}^n$ be a hypersurface of degree d over $\mathbb{F}_q$ that is smooth at all its $\mathbb{F}_q$-points. Then the number of $\mathbb{F}_q$-hyperplanes $H \subset \mathbb{P}^n$ such that the intersection $X \cap H$ is proper and smooth at $\mathbb{F}_q$-points is at least $q^{n-1}(q-d+1)$.*

Proof. The number of $\mathbb{F}_q$-hyperplanes that are tangent to X at an $\mathbb{F}_q$-point or appear as a linear component of X is bounded by $\#X(\mathbb{F}_q)$. Using the Serre–Sørensen bound [Ser91, Sør94],

$$\#X(\mathbb{F}_q) \leq dq^{n-1} + \frac{q^{n-1}-1}{q-1}.$$

Therefore, the number of $\mathbb{F}_q$-hyperplanes as in the statement is at least

$$\frac{q^{n+1}-1}{q-1} - \left(dq^{n-1} + \frac{q^{n-1}-1}{q-1}\right) = \left(\frac{q^2-1}{q-1}\right) q^{n-1} - dq^{n-1} = q^{n-1}(q-d+1)$$

which gives the desired bound. □

Proposition 3.13. *Suppose that $X \subset \mathbb{P}^3$ is a smooth Frobenius nonclassical surface over $\mathbb{F}_q$ of degree $d = \sqrt{q}+1$. Then X is Hermitian.*

Proof. Note that X contains no $\mathbb{F}_q$-linear component due to the hypothesis. Let us call an $\mathbb{F}_q$-plane $H \subset \mathbb{P}^3$ "good" if $X \cap H$ is smooth at $\mathbb{F}_q$-points. By Lemma 3.12, X admits at least $q^3 - q^2\sqrt{q}$ many good planes. Let $H \subset \mathbb{P}^3$ be any good plane. Then $C := X \cap H$ is a Frobenius nonclassical curve over $\mathbb{F}_q$ of degree $\sqrt{q}+1$ that is smooth at $\mathbb{F}_q$-points. As X is smooth, C is a reduced curve by Zak's theorem [Zak93, Corollary I.2.8]. Thus C is Hermitian by Lemma 3.11.

Note that $q \geq 4$ since q is a square. We claim that there exist four good planes H_1, H_2, H_3, and H_4 that are linearly independent. Indeed, if this was not true, then the number of good planes would be at most $\#(\mathbb{P}^3)^*(\mathbb{F}_q) = q^2+q+1$, which is strictly less than $q^3 - q^2\sqrt{q}$ for $q \geq 4$, a contradiction. After a change of coordinates, we may assume $H_i = \{x_i = 0\}$ for $i = 0, 1, 2, 3$. Let F be a defining polynomial for X. Then the fact that $X \cap \{x_i = 0\}$ is Hermitian implies that

- For each i, $F|_{x_i=0}$ defines a Hermitian curve in the remaining three variables.
- F does not contain any monomials involving exactly three variables.

By collecting the monomials appropriately, we can express:

$$F = G + x_0x_1x_2x_3R \tag{3.6}$$

where G defines a Hermitian surface and $\deg(R) = \sqrt{q} - 3$.

Since X admits at least $q^3 - q^2\sqrt{q}$ good planes and we have used up 4 of those planes in the analysis above, there are still at least $q^3 - q^2\sqrt{q} - 4$ good planes remaining. If

$$W = \{a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 = 0\}$$

defines a good plane other than $H_1, \dots, H_4$, then, by imposing the relation $\sum_i a_ix_i = 0$ on equation (3.6), we deduce that

$$R|_W = 0.$$

Indeed, if we write, without loss of generality, that $x_0 = -a_0^{-1}(a_1x_1 + a_2x_2 + a_3x_3)$, then the term $x_0x_1x_2x_3R$ contributes a nonzero monomial of the form $x_1^i x_2^j x_3^k$ to $F|_W$. But such a term cannot appear in a polynomial defining a Hermitian curve.

Therefore, R is divisible by $\sum_i a_ix_i$. Since W is an arbitrary good plane different from $H_1, \dots, H_4$, we conclude that R is divisible by a product of distinct $q^3 - q^2\sqrt{q} - 4$ linear forms over $\mathbb{F}_q$. In particular,

$$\sqrt{q} - 3 = \deg(R) \geq q^3 - q^2\sqrt{q} - 4.$$

However, $\sqrt{q} - 3 < q^3 - q^2\sqrt{q} - 4$ for $q \geq 4$. This is a contradiction unless $R = 0$. We conclude that $F = G$ and thus X is a Hermitian surface. $\square$

3.5. **Evidence for the Hermiticity in higher dimensions.** For most examples of Frobenius nonclassical hypersurfaces that we know, their defining polynomials F satisfy the property that $F_{1,0}$ is proportional to a power of F. These include all examples in this paper except for the ones classified by Theorem 1.6 (2), and also the curve examples in [HV90, Theorem 2]. Therefore, it is reasonable to assume this condition while investigating the properties of Frobenius nonclassical hypersurfaces. In the following, we provide evidence for Conjecture 1.8 by showing that it holds in odd characteristics under this assumption. The proof relies on the main results from [KKP+22] about F-pure thresholds.

Proposition 3.14. *Let $F \in \mathbb{F}_q[x_0, \dots, x_n]$ be a homogeneous polynomial of degree $\sqrt{q} + 1$ which defines a reduced Frobenius nonclassical hypersurface that satisfies*

$$F_{1,0} := \sum_{i=0}^{n} x_i^q \frac{\partial F}{\partial x_i} = cF^{\sqrt{q}} \qquad \text{for some} \qquad c \in \mathbb{F}_q \setminus \{0\}.$$

Then F is a Frobenius form, namely, it is defined by the expression

$$F = \sum_{i=0}^{n} x_i^{\sqrt{q}} L_i$$

for some linear polynomials $L_0, \dots, L_n$.

Proof. Let $\mathrm{fpt}(F)$ and $\mathrm{fpt}(F_{1,0})$ denote the F-pure thresholds of F and $F_{1,0}$, respectively. First, we have $\mathrm{fpt}(F) \geq \frac{1}{\sqrt{q}}$ by [KKP+22, Theorem 1.1]. Next, the assumption $F_{1,0} = cF^{\sqrt{q}}$ implies that $\mathrm{fpt}(F_{1,0}) = \frac{1}{\sqrt{q}}\mathrm{fpt}(F)$ by [KKP+22, Proposition 2.2 (2)]. Moreover, we see that $\mathrm{fpt}(F_{1,0}) \leq \frac{1}{q}$ from the definition of $F_{1,0}$. Combining these relations, we obtain

$$\frac{1}{\sqrt{q}} \leq \mathrm{fpt}(F) = \sqrt{q} \cdot \mathrm{fpt}(F_{1,0}) \leq \frac{1}{\sqrt{q}}$$

which implies that $\mathrm{fpt}(F) = \frac{1}{\sqrt{q}}$. Thus F is a Frobenius form by [KKP+22, Theorem 4.3]. $\square$

Now we proceed to prove that the Frobenius form in Proposition 3.14 turns out to be Hermitian due to the Frobenius nonclassical property. First note that a Frobenius form over an arbitrary field k of positive characteristic p can be equivalently written as

$$F = \sum_{i,j=0}^{n} x_i^{q'} M_{ij} x_j = \mathbf{x}^{q'} \cdot M \cdot \mathbf{x}^t \tag{3.7}$$

where $\mathbf{x} = (x_0, \ldots, x_n)$, $\mathbf{x}^{q'} = (x_0^{q'}, \ldots, x_n^{q'})$, and $M = (M_{ij})$ is a matrix with entries in k.

Lemma 3.15. *Let F be a Frobenius form as in (3.7) over $k = \mathbb{F}_{q'^2}$. Suppose that $p \neq 2$ and that F divides the polynomial*

$$G := \sum_{i=0}^{n} x_i^{q'^2} \frac{\partial F}{\partial x_i} = \mathbf{x}^{q'} \cdot M \cdot (\mathbf{x}^{q'^2})^t.$$

Then M is either

- *a Hermitian matrix in the sense that $\overline{M} := (M_{ij}^{q'}) = M^t$, or*
- *a skew-Hermitian matrix in the sense that $\overline{M} := (M_{ij}^{q'}) = -M^t$.*

Proof. Because $p \neq 2$, we are allowed to write $M = M_1 + M_2$ where

$$M_1 = \frac{1}{2}(M + \overline{M}^t) \qquad \text{and} \qquad M_2 = \frac{1}{2}(M - \overline{M}^t).$$

Notice that M_1 is Hermitian and M_2 is skew-Hermitian. Now we have $F = F_1 + F_2$ where $F_1 = \mathbf{x}^{q'} \cdot M_1 \cdot \mathbf{x}^t$ and $F_2 = \mathbf{x}^{q'} \cdot M_2 \cdot \mathbf{x}^t$. One can verify directly that $G = F_1^{q'} - F_2^{q'}$. This relation, together with the fact that $F^{q'} = F_1^{q'} + F_2^{q'}$, implies

$$G + F^{q'} = 2F_1^{q'} \qquad \text{and} \qquad G - F^{q'} = -2F_2^{q'}.$$

Because F divides G, it divides the left hand sides of the above two equations, whence it divides both F_1 and F_2. Thus, there exist $c_1, c_2 \in \mathbb{F}_q$ such that $F_1 = c_1 F$ and $F_2 = c_2 F$, or equivalently, $M_1 = c_1 M$ and $M_2 = c_2 M$. If $c_1 = c_2 = 0$, then $M = 0$ and there is nothing left to prove. If $c_1 \neq 0$, then $M = c_1^{-1} M_1$, so M is Hermitian. If $c_2 \neq 0$, then $M = c_2^{-1} M_2$ and M is skew-Hermitian in this case. □

Corollary 3.16. *Let $X \subset \mathbb{P}^n$ be a reduced Frobenius nonclassical hypersurface over $\mathbb{F}_q$ of degree $\sqrt{q}+1$ and assume that $\operatorname{char}(\mathbb{F}_q) \neq 2$ and $F_{1,0} = cF^{\sqrt{q}}$ for some nonzero constant c. Then X is Hermitian.*

Proof. By Proposition 3.14, the hypersurface X is defined by a Frobenius form F. Applying Lemma 3.15 with $q' = \sqrt{q}$, we conclude that F is of the form (3.7) such that M is either Hermitian or skew-Hermitian. If M is Hermitian, then we have the desired result. Otherwise, we can pick $c \in \mathbb{F}_q$ that satisfies $c^{\sqrt{q}} = -c$. Then cM is Hermitian. Since $\{cF = 0\}$ defines the same hypersurface X, this completes the proof. □

4. Upper bounds on degree and characterizations

This section is devoted to the proof of Theorem 1.6. We will establish the degree bounds and give characterizations step by step, starting from the simplest case $F_{1,0} = 0$, then the case $d \not\equiv 0 \pmod{p}$ where $p = \operatorname{char}(\mathbb{F}_q)$, and eventually to the full generality. The main machinery involved in the proof is the Koszul complex.

4.1. **Hypersurfaces with vanishing $\mathbf{F_{1,0}}$.** Let $X = \{F = 0\} \subset \mathbb{P}^n$ be a Frobenius nonclassical hypersurface over $\mathbb{F}_q$ that satisfies the vanishing condition

$$F_{1,0} = \sum_{i=0}^{n} x_i^q \frac{\partial F}{\partial x_i} = 0.$$

Now consider the ring $R := \mathbb{F}_q[x_0, \dots, x_n]$, the sequence $\mathbf{x}^q := (x_0^q, \dots, x_n^q)$, and the associated Koszul complex

$$\begin{array}{l} K_\bullet(\mathbf{x}^q) : 0 \longrightarrow \bigwedge^{n+1} R^{n+1} \longrightarrow \cdots \longrightarrow \bigwedge^2 R^{n+1} \xrightarrow{\delta_2} R^{n+1} \xrightarrow{\delta_1} R \longrightarrow 0. \\ \hspace{22em} e_i \longmapsto x_i^q \end{array}$$

Here $\{e_0, \dots, e_n\}$ is the canonical basis for the free R-module $R^{n+1} = \bigoplus_{i=0}^n Re_i$. In this setting, the vanishing of $F_{1,0}$ is equivalent to the condition

$$\sum_{i=0}^{n} \frac{\partial F}{\partial x_i} e_i \in \ker \delta_1. \tag{4.1}$$

On the other hand, as the sequence $\mathbf{x}^q = (x_0^q, \dots, x_n^q)$ is regular [Mat89, Theorem 16.1], the complex $K_\bullet(\mathbf{x}^q)$ is exact at degree i for all $i \geq 1$ [Mat89, Theorem 16.5 (i)]. In particular,

$$H_1(K_\bullet(\mathbf{x}^q)) = \ker \delta_1 / \operatorname{im} \delta_2 = 0 \tag{4.2}$$

The following lemma is a consequence of these relations.

Lemma 4.1. *Retain the setting from above. Then*

$$\frac{\partial F}{\partial x_j} = \sum_{i=0}^{n} x_i^q G_{ij}, \qquad j = 0, \dots, n,$$

for some $G_{ij} \in \mathbb{F}_q[x_0, \dots, x_n]$ that satisfies $G_{ji} = -G_{ij}$ and $G_{ii} = 0$.

Proof. Relation (4.2) implies that the element (4.1) admits a preimage

$$\sum_{i<j} G_{ij} e_i \wedge e_j \in \bigwedge^2 R^{n+1}$$

under the differential δ_2. By setting $G_{ij} = -G_{ji}$ for $i > j$ and $G_{ii} = 0$, we get

$$\begin{aligned} \sum_{j=0}^{n} \left(\frac{\partial F}{\partial x_j} \right) e_j &= \delta_2 \left(\sum_{i<j} G_{ij} e_i \wedge e_j \right) = \sum_{i<j} G_{ij} \delta_2(e_i \wedge e_j) = \sum_{i<j} G_{ij}(x_i^q e_j - x_j^q e_i) \\ &= \sum_{i<j} x_i^q G_{ij} e_j - \sum_{j<i} x_i^q G_{ji} e_j = \sum_{i<j} x_i^q G_{ij} e_j + \sum_{j<i} x_i^q G_{ij} e_j = \sum_{j=0}^{n} \left(\sum_{i=0}^{n} x_i^q G_{ij} \right) e_j. \end{aligned}$$

Comparing both sides of the equation gives the desired equalities. □

Corollary 4.2. *Let $X = \{F = 0\} \subset \mathbb{P}^n$ be a Frobenius nonclassical hypersurface of degree d over $\mathbb{F}_q$ of characteristic p that satisfies $F_{1,0} = 0$.*

(1) *If F is not a p-th power, then $d \geq q + 1$.*

(2) *If $d \not\equiv 0 \pmod{p}$, then*

$$F = \sum_{i,j=0}^{n} x_i^q A_{ij} x_j$$

where A_{ij} are polynomials that satisfy $A_{ji} = -A_{ij}$ and $A_{ii} = 0$.

Proof. The polynomial F is not a p-th power if and only if

$$\frac{\partial F}{\partial x_j} \neq 0 \quad \text{for some} \quad j \in \{0, \ldots, n\}.$$

This implies that some G_{ij} in Lemma 4.1 is not the zero polynomial. Therefore, $\deg(G_{ij}) = d - 1 - q \geq 0$. Thus $d \geq q + 1$. This proves (1). If $d \not\equiv 0 \pmod{p}$, then Euler's formula and Lemma 4.1 imply that

$$F = d^{-1} \sum_{j=0}^{n} \frac{\partial F}{\partial x_j} x_j = d^{-1} \sum_{i,j=0}^{n} x_i^q G_{ij} x_j.$$

This proves (2) by setting $A_{ij} := d^{-1} G_{ij}$. $\square$

4.2. **Restrictions imposed by smoothness.** Let $X = \{F = 0\} \subset \mathbb{P}^n$ be a hypersurface over $\mathbb{F}_q$ and denote $R = \mathbb{F}_q[x_0, \ldots, x_n]$ and $R^{n+1} = \bigoplus_{i=0}^n Re_i$ as before. The Jacobian ideal

$$J_F = \left(\frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_n} \right) \subset R$$

defines a Koszul complex

$$\begin{array}{c} K_\bullet(J_F) : 0 \longrightarrow \bigwedge^{n+1} R^{n+1} \xrightarrow{\delta_{n+1}} \cdots \xrightarrow{\delta_3} \bigwedge^2 R^{n+1} \xrightarrow{\delta_2} R^{n+1} \xrightarrow{\delta_1} R \longrightarrow 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad e_i \longmapsto \frac{\partial F}{\partial x_i} \end{array}$$

In this setting, the length of a maximal regular sequence in J_F, that is, the *depth* of J_F, can be computed by [Mat89, Theorem 16.8]

$$\operatorname{depth}(J_F) = n + 1 - \max\{i \mid H_i(K_\bullet(J_F)) = \ker \delta_i / \operatorname{im} \delta_{i+1} \neq 0\}. \tag{4.3}$$

On the other hand, R is a polynomial ring over a field and thus is Cohen–Macaulay [Eis95, Proposition 18.9]. This implies that [Eis05, Theorem A2.38]

$$\operatorname{depth}(J_F) = \operatorname{codim}(J_F) \tag{4.4}$$

where $\operatorname{codim}(J_F)$ is the Krull codimension of the scheme $\{J_F = 0\} \subset \mathbb{P}^n$.

Lemma 4.3. *Let $X = \{F = 0\} \subset \mathbb{P}^n$ be a smooth Frobenius nonclassical hypersurface of degree d over $\mathbb{F}_q$ of characteristic p such that $d \not\equiv 0 \pmod{p}$. Then*

$$x_j^q - \left(\frac{F_{1,0}}{dF} \right) x_j = \sum_{i=0}^{n} \frac{\partial F}{\partial x_i} \beta_{ij} \quad \textit{for all} \quad j = 0, \ldots, n,$$

where $\beta_{ij} \in \mathbb{F}_q[x_0, \ldots, x_n]$ satisfy $\beta_{ji} = -\beta_{ij}$ and $\beta_{ii} = 0$. In particular, we have $d \leq q + 1$.

Proof. The assumption $d \not\equiv 0 \pmod p$ and Euler's formula imply that the singular locus of X coincides with $\{J_F = 0\} \subset \mathbb{P}^n$. Hence $\operatorname{codim}(J_F) = n+1$ since X is smooth. Using (4.3) and (4.4), we conclude that

$$\max\{i \mid H_i(K_\bullet(J_F)) = \ker \delta_i / \operatorname{im} \delta_{i+1} \neq 0\} = 0.$$

In particular,

$$H_1(K_\bullet(J_F)) = \ker \delta_1 / \operatorname{im} \delta_2 = 0. \tag{4.5}$$

Define $\alpha := d^{-1}(F_{1,0}/F) \in \mathbb{F}_q[x_0, \dots, x_n]$. This is well-defined as $d \not\equiv 0 \pmod p$ and X is Frobenius nonclassical. Rearranging the equation gives $F_{1,0} - \alpha d F = 0$, which can be expanded via Euler's formula as

$$\sum_{j=0}^{n} (x_j^q - \alpha x_j) \frac{\partial F}{\partial x_j} = 0.$$

This equation shows that

$$\sum_{j=0}^{n} (x_j^q - \alpha x_j) e_j \in \ker \delta_1.$$

By (4.5), this element admits a preimage $\sum_{i<j} \beta_{ij} e_i \wedge e_j \in \bigwedge^2 R^{n+1}$ under δ_2. By setting $\beta_{ij} = -\beta_{ji}$ for $i > j$ and $\beta_{ii} = 0$, we obtain

$$\begin{aligned}
\sum_{j=0}^{n} (x_j^q - \alpha x_j) e_j &= \delta_2\Big(\sum_{i<j} \beta_{ij} e_i \wedge e_j\Big) = \sum_{i<j} \beta_{ij} \delta_2(e_i \wedge e_j) = \sum_{i<j} \beta_{ij} \left(\frac{\partial F}{\partial x_i} e_j - \frac{\partial F}{\partial x_j} e_i \right) \\
&= \sum_{i<j} \beta_{ij} \frac{\partial F}{\partial x_i} e_j - \sum_{j<i} \beta_{ji} \frac{\partial F}{\partial x_i} e_j = \sum_{i<j} \beta_{ij} \frac{\partial F}{\partial x_i} e_j + \sum_{j<i} \beta_{ij} \frac{\partial F}{\partial x_i} e_j = \sum_{j=0}^{n} \left(\sum_{i=0}^{n} \beta_{ij} \frac{\partial F}{\partial x_i} \right) e_j.
\end{aligned}$$

Comparing both sides of the equation gives the desired relations.

To prove that $d \leq q+1$, first note that $x_j^q - x_j \alpha = 0$ implies that $\alpha = x_j^{q-1}$, which cannot hold for all $0 \leq j \leq n$. Hence there exists j such that $x_j^q - x_j \alpha \neq 0$. Therefore,

$$q = \deg(x_j^q - \alpha x_j) = \deg\left(\sum_{i=0}^{n} \frac{\partial F}{\partial x_i} \beta_{ij} \right) = d - 1 + \deg(\beta_{ij}).$$

This shows that $q \geq d-1$, or equivalently, $d \leq q+1$. □

Corollary 4.4. *Let $X = \{F = 0\} \subset \mathbb{P}^n$ be a smooth Frobenius nonclassical hypersurface of degree d over $\mathbb{F}_q$ of characteristic p such that $d \not\equiv 0 \pmod p$. Assume additionally that $F_{1,0} = 0$. Then there is a nondegenerate skew-symmetric matrix (A_{ij}) with entries in $\mathbb{F}_q$ and zeros along the diagonal such that*

$$F = \sum_{i,j=0}^{n} x_i^q A_{ij} x_j.$$

In particular, this situation occurs only when n is odd.

Proof. Corollary 4.2 (1) and Lemma 4.3 imply that $d = q+1$. Corollary 4.2 (2) then implies that $F = \sum_{i,j=0}^{n} x_i^q A_{ij} x_j$ where (A_{ij}) a skew-symmetric matrix with entries in $\mathbb{F}_q$ and zeros along the diagonal. The smoothness of X implies that (A_{ij}) is nondegenerate. This forces the size of (A_{ij}) to be even, so n must be odd. □

4.3. **Characterization in the case of degree $\mathbf{q+1}$.** Lemma 4.3 can be refined in the case that $d = q+1$ to obtain characterization for smooth Frobenius nonclassical hypersurfaces over $\mathbb{F}_q$ of degree $q+1$.

Proposition 4.5. *Let $X = \{F = 0\} \subset \mathbb{P}^n$ be a smooth Frobenius nonclassical hypersurface over $\mathbb{F}_q$ of degree $d = q+1$ and let $p = \mathrm{char}(\mathbb{F}_q)$.*

(1) *If n is odd, then there exists a nondegenerate skew-symmetric matrix (A_{ij}) with entries in $\mathbb{F}_q$ and zeros along the diagonal such that*

$$F = \sum_{i,j=0}^{n} x_i^q A_{ij} x_j$$

and vice versa. Notice that $F_{1,0} = 0$ in this case.

(2) *If n is even, then $p = 2$ and there exists a system of coordinates $\{y_0, \dots, y_n\}$ and a nondegenerate skew-symmetric matrix $(B_{ij})_{1 \le i,j \le n}$ with entries in $\mathbb{F}_q$ and zeros along the diagonal such that*

$$F_{1,0} = y_0^{q-1} F, \qquad F = y_0 \frac{\partial F}{\partial y_0} + \sum_{i,j=1}^{n} y_i^q B_{ij} y_j, \qquad \frac{\partial^2 F}{\partial y_0^2} = 0,$$

and vice versa.

Proof. Define $\alpha := F_{1,0}/F$. Lemma 4.3 together with the assumption $d = q+1$ implies that there exist $\beta_{ij} \in \mathbb{F}_q$ satisfying $\beta_{ij} = -\beta_{ji}$ and $\beta_{ii} = 0$ such that

$$x_j^q - \alpha x_j = \sum_{i=0}^{n} \frac{\partial F}{\partial x_i} \beta_{ij}, \quad j = 0, \dots, n. \tag{4.6}$$

The fact $d \not\equiv 0 \pmod{p}$ and the smoothness of X imply that $\frac{\partial F}{\partial x_i}$, $i = 0, \dots, n$, are linearly independent over $\mathbb{F}_q$. On the other hand, the $\mathbb{F}_q$-vector subspace

$$V = \mathrm{span}\{x_j^q - \alpha x_j \mid j = 0, \dots, n\} \subset \mathbb{F}_q[x_0, \dots, x_n]$$

has dimension either n or $n+1$ by Lemma 4.6. These properties with (4.6) imply that $\mathrm{rank}(\beta_{ij}) = \dim(V)$ which equals either n or $n+1$. Because the rank of a skew-symmetric matrix is always even, we have

$$\mathrm{rank}(\beta_{ij}) = \begin{cases} n+1 & \text{if } n \text{ is odd,} \\ n & \text{if } n \text{ is even.} \end{cases}$$

Assume that n is odd. Then (β_{ij}) is nondegenerate and thus admits an inverse (A_{ij}) that satisfies $A_{ij} = -A_{ji}$ and $A_{ii} = 0$ as well. Using (4.6), we obtain

$$\frac{\partial F}{\partial x_i} = \sum_{j=0}^{n} A_{ij}(x_j^q - \alpha x_j), \quad i = 0, \dots, n.$$

Then Euler's formula gives

$$F = \sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i} = \sum_{i,j=0}^{n} x_i A_{ij}(x_j^q - \alpha x_j) = \sum_{i,j=0}^{n} x_i A_{ij} x_j^q - \alpha \sum_{i,j=0}^{n} x_i A_{ij} x_j = \sum_{i,j=0}^{n} x_i A_{ij} x_j^q.$$

This proves one implication of (1). The converse and $F_{1,0} = 0$ can be verified directly.

Assume that n is even. The fact that $\dim(V) = n$ and Lemma 4.6 imply that there exists a system of coordinates $\{y_0, \ldots, y_n\}$ such that $\alpha = y_0^{q-1}$. In particular, the first equation in (2) holds. Lemma 4.3 in the new coordinates implies that there exist $\eta_{ij} \in \mathbb{F}_q$ satisfying $\eta_{ij} = -\eta_{ji}$ and $\eta_{ii} = 0$ such that

$$y_j^q - y_0^{q-1} y_j = \sum_{i=0}^{n} \frac{\partial F}{\partial y_i} \eta_{ij}, \quad j = 0, \ldots, n. \tag{4.7}$$

For $j = 0$, the above relation reduces to $0 = \sum_{i=0}^{n} \frac{\partial F}{\partial y_i} \eta_{i0}$. As $\frac{\partial F}{\partial y_i}$, $i = 0, \ldots, n$, are linearly independent over $\mathbb{F}_q$, we conclude that $\eta_{i0} = -\eta_{0i} = 0$ for $i = 0, \ldots, n$. It follows that the minor $(\eta_{ij})_{1 \le i,j \le n}$ has full rank n, and thus admits an inverse $(B_{ij})_{1 \le i,j \le n}$ that satisfies $B_{ij} = -B_{ji}$ and $B_{ii} = 0$. Collecting the relations in (4.7) for $j = 1, \ldots, n$, we obtain

$$\frac{\partial F}{\partial y_i} = \sum_{j=1}^{n} B_{ij}(y_j^q - y_0^{q-1} y_j), \quad i = 1, \ldots, n.$$

Applying Euler's formula, we get

$$\begin{aligned} F - y_0 \frac{\partial F}{\partial y_0} &= \sum_{i=1}^{n} y_i \frac{\partial F}{\partial y_i} = \sum_{i,j=1}^{n} y_i B_{ij}(y_j^q - y_0^{q-1} y_j) \\ &= \sum_{i,j=1}^{n} y_i B_{ij} y_j^q - y_0^{q-1} \sum_{i,j=1}^{n} y_i B_{ij} y_j = \sum_{i,j=1}^{n} y_i B_{ij} y_j^q. \end{aligned} \tag{4.8}$$

This proves the second equation in (2). Applying $\frac{\partial}{\partial y_0}$ to both sides of (4.8) gives

$$y_0 \frac{\partial^2 F}{\partial y_0^2} = 0 \qquad \text{whence} \qquad \frac{\partial^2 F}{\partial y_0^2} = 0. \tag{4.9}$$

This proves the third equation in (2).

Now we prove that $p = 2$. For the sake of simplicity, we denote

$$G := \frac{\partial F}{\partial y_0} \qquad \text{and} \qquad H := \sum_{i,j=1}^{n} y_i B_{ij} y_j^q.$$

Notice that $H_{1,0} = 0$. Rewrite (4.8) as $F = y_0 G + H$. Then a straightforward computation gives

$$F_{1,0} = y_0^q G + y_0 G_{1,0} + H_{1,0} = y_0^q G + y_0 G_{1,0}.$$

It follows that

$$y_0^q G + y_0 G_{1,0} = F_{1,0} = y_0^{q-1} F = y_0^{q-1}(y_0 G + H) = y_0^q G + y_0^{q-1} H.$$

Rearranging the terms and eliminating common factors to get

$$G_{1,0} = y_0^{q-2} H. \tag{4.10}$$

Recall from (4.9) that $\frac{\partial G}{\partial y_0} = 0$. Applying $\frac{\partial}{\partial y_0}$ to (4.10), the left hand side gives

$$\frac{\partial G_{1,0}}{\partial y_0} = \frac{\partial}{\partial y_0} \left(\sum_{i=0}^{n} y_i^q \frac{\partial G}{\partial y_i} \right) = \sum_{i=0}^{n} y_i^q \frac{\partial^2 G}{\partial y_0 \partial y_i} = \sum_{i=0}^{n} y_i^q \frac{\partial^2 G}{\partial y_i \partial y_0} = 0,$$

while the right hand side with the fact that $\frac{\partial H}{\partial y_0} = 0$ gives

$$\frac{\partial}{\partial y_0}(y_0^{q-2}H) = -2y_0^{q-3}H + y_0^{q-2}\frac{\partial H}{\partial y_0} = -2y_0^{q-3}H.$$

Hence $0 = -2y_0^{q-3}H$. We have $H \neq 0$ since $(B_{ij})_{1\leq i,j\leq n}$ is nondegenerate, so $p = 2$. This completes the proof of one implication in (2). The converse can be verified directly using the fact that X is smooth. □

Lemma 4.6. *Let $\alpha \in \mathbb{F}_q[x_0, \dots, x_n]$ be a homogeneous polynomial that is either constantly zero or nonzero of degree $q-1$. Consider the $\mathbb{F}_q$-vector space*

$$V = \operatorname{span}\{x_j^q - \alpha x_j \mid j = 0, \dots, n\} \subset \mathbb{F}_q[x_0, \dots, x_n]$$

which is of dimension at most $n+1$. Under the situation that $\dim(V) \leq n$, it can only happen that $\dim(V) = n$ and, in this case, there exists a system of coordinates $\{y_0, \dots, y_n\}$ such that $\alpha = y_0^{q-1}$ and that $\{y_j^q - y_0^{q-1}y_j \mid j = 1, \dots, n\}$ forms a basis for V.

Proof. The condition $\dim(V) \leq n$ means the polynomials $x_j^q - \alpha x_j$, $j = 0, \dots, n$, are linearly dependent, so there exists $(c_0, \dots, c_n) \in \mathbb{F}_q^{n+1} \setminus \{0\}$ such that

$$\sum_{j=0}^{n} c_j(x_j^q - \alpha x_j) = \sum_{j=0}^{n} c_j x_j^q - \alpha \sum_{j=0}^{n} c_j x_j = 0.$$

Since $c_j = c_j^q$ for all j, rearranging the above equation gives

$$\alpha \sum_{j=0}^{n} c_j x_j = \sum_{j=0}^{n} c_j x_j^q = \sum_{j=0}^{n} c_j^q x_j^q = \left(\sum_{j=0}^{n} c_j x_j\right)^q.$$

Eliminating common factors from both sides gives

$$\alpha = \left(\sum_{j=0}^{n} c_j x_j\right)^{q-1}.$$

As $(c_0, \dots, c_n)$ is a nonzero vector, we can set $y_0 = \sum_{j=0}^{n} c_j x_j$ and complete it to a coordinate system $\{y_0, \dots, y_n\}$. Notice that $\alpha = y_0^{q-1}$ in this setting.

The polynomials $y_i^q - y_0^{q-1}y_i$, $1 \leq i \leq n$, cut out the set of $\mathbb{F}_q$-points away from the hyperplane $\{y_0 = 0\}$, which implies that they are linearly independent over $\mathbb{F}_q$. To finish the proof, it is sufficient to show that they belong to V. Suppose that the transformation between the coordinate systems $\{x_0, \dots, x_n\}$ and $\{y_0, \dots, y_n\}$ is given by

$$y_i = \sum_{j=0}^{n} g_{ij} x_j \quad \text{where} \quad (g_{ij}) \in \mathrm{GL}_{n+1}(\mathbb{F}_q).$$

Then $y_i^q = \sum_{j=0}^{n} g_{ij} x_j^q$. Hence, for $i = 1, \dots, n$,

$$y_i^q - \alpha y_i = \sum_{j=0}^{n} g_{ij} x_j^q - \alpha \sum_{j=0}^{n} g_{ij} x_j = \sum_{j=0}^{n} g_{ij}(x_j^q - \alpha x_j) \in V.$$

This completes the proof. □

4.4. **The upper bound $\mathbf{d \le q + 2}$ and further characterization.** Let us proceed to study smooth Frobenius nonclassical hypersurfaces $X = \{F = 0\} \subset \mathbb{P}^n$ over $\mathbb{F}_q$ without the assumption that $\deg(X)$ is not divisible by $p = \mathrm{char}(\mathbb{F}_q)$. In order to deal with this general situation, we work with the quotient ring $\overline{R} := \mathbb{F}_q[x_0, \dots, x_n]/(F)$ and the ideal

$$\overline{J_F} := \left(\frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_n}\right) \subset \overline{R}$$

where the latter determines a Koszul complex

$$\begin{array}{c} K_\bullet(\overline{J_F}) : 0 \longrightarrow \bigwedge^{(n+1)} \overline{R}^{n+1} \xrightarrow{\delta_{n+1}} \cdots \xrightarrow{\delta_3} \bigwedge^2 \overline{R}^{n+1} \xrightarrow{\delta_2} \overline{R}^{n+1} \xrightarrow{\delta_1} \overline{R} \longrightarrow 0 \\ \hfill e_i \longmapsto \frac{\partial F}{\partial x_i}. \end{array}$$

The fact that $X \subset \mathbb{P}^n$ is a hypersurface implies that $\overline{R}$ is Cohen–Macaulay. (See, for example, [Eis95, Section 18.5].) By [Eis05, Theorem A2.38] and [Mat89, Theorem 16.8],

$$\mathrm{codim}(\overline{J_F}) = \mathrm{depth}(\overline{J_F}) = n + 1 - \max\left\{ i \,\middle|\, H_i(K_\bullet(\overline{J_F})) = \ker \delta_i / \operatorname{im} \delta_{i+1} \neq 0 \right\}.$$

The smoothness of X implies that $\mathrm{codim}(\overline{J_F}) = n$, so the above relation reduces to

$$\max\left\{ i \,\middle|\, H_i(K_\bullet(\overline{J_F})) = \ker \delta_i / \operatorname{im} \delta_{i+1} \neq 0 \right\} = 1.$$

In particular,

$$H_2(K_\bullet(\overline{J_F})) = \ker \delta_2 / \operatorname{Im} \delta_3 = 0. \tag{4.11}$$

Lemma 4.7. *Let $X = \{F = 0\} \subset \mathbb{P}^n$ be a smooth Frobenius nonclassical hypersurface of degree d over $\mathbb{F}_q$. Then $d \le q + 2$ and there exists*

$$\gamma = \sum_{i<j<k} \gamma_{ijk} e_i \wedge e_j \wedge e_k \in \bigwedge^3 \overline{R}^{n+1}$$

such that

$$x_i x_j^q - x_i^q x_j = \sum_{k=0}^{n} \gamma_{ijk} \frac{\partial F}{\partial x_k} \pmod{F}, \qquad 0 \le i < j \le n.$$

Proof. Let us consider the two elements in $\overline{R}^{n+1}$

$$\mathbf{x} := \sum_{i=0}^{n} x_i e_i \quad \text{and} \quad \mathbf{x}^q := \sum_{i=0}^{n} x_i^q e_i.$$

Euler's formula implies that $\mathbf{x} \in \ker \delta_1$. On the other hand, X is Frobenius nonclassical and so $\mathbf{x}^q \in \ker \delta_1$. These facts imply that

$$\delta_2(\mathbf{x} \wedge \mathbf{x}^q) = \left(\sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i} \right) \mathbf{x}^q - \left(\sum_{i=0}^{n} x_i^q \frac{\partial F}{\partial x_i} \right) \mathbf{x} = \delta_1(\mathbf{x})\mathbf{x}^q - \delta_1(\mathbf{x}^q)\mathbf{x} = 0.$$

That is, $\beta := \mathbf{x} \wedge \mathbf{x}^q \in \ker \delta_2$. By (4.11), there exists

$$\gamma = \sum_{i<j<k} \gamma_{ijk} e_i \wedge e_j \wedge e_k \in \bigwedge^3 \overline{R}^{n+1}$$

such that $\beta = \delta_3(\gamma)$.

For the sake of simplicity, we denote $F_i := \frac{\partial F}{\partial x_i}$ for $i = 0, \dots, n$. Expanding both sides of the relation $\beta = \delta_3(\gamma)$ gives

$$\begin{aligned}
\beta = \sum_{i<j} (x_i x_j^q - x_i^q x_j) e_i \wedge e_j &= \sum_{i<j<k} \gamma_{ijk} \delta_3(e_i \wedge e_j \wedge e_k) \\
&= \sum_{i<j<k} \gamma_{ijk} \left(F_i e_j \wedge e_k - F_j e_i \wedge e_k + F_k e_i \wedge e_j \right) \\
&= \sum_{i<j<k} \gamma_{ijk} F_i e_j \wedge e_k - \sum_{i<j<k} \gamma_{ijk} F_j e_i \wedge e_k + \sum_{i<j<k} \gamma_{ijk} F_k e_i \wedge e_j \\
&= \sum_{k<i<j} \gamma_{kij} F_k e_i \wedge e_j - \sum_{i<k<j} \gamma_{ikj} F_k e_i \wedge e_j + \sum_{i<j<k} \gamma_{ijk} F_k e_i \wedge e_j \\
&= \sum_{k<i<j} \gamma_{ijk} F_k e_i \wedge e_j + \sum_{i<k<j} \gamma_{ijk} F_k e_i \wedge e_j + \sum_{i<j<k} \gamma_{ijk} F_k e_i \wedge e_j \\
&= \sum_{i<j} \left(\sum_{k=0}^{n} \gamma_{ijk} F_k \right) e_i \wedge e_j.
\end{aligned}$$

Comparing both sides of the equality gives the desired relations

$$x_i x_j^q - x_i^q x_j = \sum_{k=0}^{n} \gamma_{ijk} F_k \pmod{F}, \qquad 0 \le i < j \le n. \tag{4.12}$$

Let us prove that $d \le q + 2$. Note that (4.12) is an equality between polynomials of degree $q + 1$ modulo F. If we assume, to the contrary, that $d > q + 2$, then in particular $\deg(F_k) = d - 1 > q + 1$, which implies that (4.12) holds without modulo F. Comparing the degrees on both sides gives $q + 1 \ge \deg(F_k) = d - 1$, or equivalently, $d \le q + 2$, which contradicts our assumption. Hence $d \le q + 2$. $\square$

Proposition 4.8. *Let $X = \{F = 0\} \subset \mathbb{P}^n$, where $n \ge 2$, be a smooth Frobenius nonclassical hypersurface of degree d over $\mathbb{F}_q$ of characteristic p such that $d = q + 2$. Then $p = n = 2$ and, upon rescaling F by a nonzero constant, we have*

$$\frac{\partial F}{\partial x_0} = x_1 x_2^q - x_1^q x_2, \quad \frac{\partial F}{\partial x_1} = x_2 x_0^q - x_2^q x_0, \quad \frac{\partial F}{\partial x_2} = x_0 x_1^q - x_0^q x_1.$$

In particular, we have $F_{1,0} = 0$ and

$$F = x_0 x_1 x_2 (x_0^{q-1} + x_1^{q-1} + x_2^{q-1}) + G(x_0^2, x_1^2, x_2^2)$$

for some polynomial G.

Proof. First of all, if $p \neq 2$, then $d = q + 2 \equiv 2 \not\equiv 0 \pmod{p}$. But this forces $d \le q + 1$ by Lemma 4.3, a contradiction. Hence $p = 2$.

Let us show that $n = 2$. By hypothesis, $\deg(F) = d = q + 2 > q + 1$. This implies that the relations from Lemma 4.7:

$$x_i x_j^q - x_i^q x_j = \sum_{k=0}^{n} \gamma_{ijk} \frac{\partial F}{\partial x_k} \tag{4.13}$$

hold without modulo F and also that $\gamma_{ijk} \in \mathbb{F}_q$. Note that the $n(n+1)/2$ polynomials $x_i x_j^q - x_i^q x_j$ for $0 \le i < j \le n$ are linearly independent over $\mathbb{F}_q$. On the other hand, the polynomials $\frac{\partial F}{\partial x_k}$, $0 \leq k \leq n$, span an $\mathbb{F}_q$-vector subspace in $\mathbb{F}_q[x_0, \dots, x_n]$ of dimension at most $n+1$. These facts with (4.13) imply that

$$\frac{n(n+1)}{2} \leq n+1, \qquad \text{or equivalently,} \qquad n \leq 2.$$

We have $n \geq 2$ by hypothesis, so $n = 2$.

Equation (4.13) under the condition $n = 2$ gives

$$\gamma_{120} \cdot \frac{\partial F}{\partial x_0} = x_1 x_2^q - x_1^q x_2, \quad \gamma_{201} \cdot \frac{\partial F}{\partial x_1} = x_2 x_0^q - x_2^q x_0, \quad \gamma_{012} \cdot \frac{\partial F}{\partial x_2} = x_0 x_1^q - x_0^q x_1.$$

By convention, we have $\gamma_{120} = \gamma_{201} = \gamma_{012}$. Denote this element by c. Then $c \neq 0$ and replacing F by $c^{-1}F$ gives the desired expressions for the partial derivatives. The vanishing of $F_{1,0}$ and the formula for F are straightforward computations using these expressions. $\square$

Corollary 4.9. *Let $X = \{F = 0\} \subset \mathbb{P}^n$, where $n \geq 2$, be a smooth Frobenius nonclassical hypersurface over $\mathbb{F}_q$ of degree $d = q+2$. Then X contains no $\mathbb{F}_q$-point.*

Proof. By Proposition 4.8, the partial derivatives of F vanish at every $\mathbb{F}_q$-point. If X contains any $\mathbb{F}_q$-point, then the point has to be singular, which cannot happen as we assume X to be smooth. $\square$

4.5. **Sharpness of the upper bounds.** Example 1.3 shows that the upper bound in Lemma 4.3 is sharp. In the following, we exhibit more examples of smooth Frobenius nonclassical hypersurfaces whose degrees attain the upper bounds in Lemmas 4.3 and 4.7. Then we finish the proof of Theorem 1.6 at the end of this section.

Example 4.10. Over $\mathbb{F}_4 = \mathbb{F}_2(a)$, where $a^2 + a + 1 = 0$, the plane curve $X \subset \mathbb{P}^2$ defined by

$$F = x\left((a+1)x^4 + x^2y^2 + x^2yz + x^2z^2 + y^4 + y^2z^2 + z^4\right) + y^4z + yz^4$$

provides an example for Proposition 4.5 (2) as well as Theorem 1.6 (2). The Hessian matrix of this example equals

$$\left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right) = \begin{pmatrix} 0 & x^2z & x^2y \\ x^2z & 0 & x^3 \\ x^2y & x^3 & 0 \end{pmatrix}$$

which has a zero determinant. In general, the determinant of the Hessian matrix of a smooth Frobenius nonclassical hypersurface X vanishes along X by [ADL21, Lemma 4.6]. But the Hessian matrix itself may not vanish along X due to this example.

Example 4.11. Over $\mathbb{F}_2$, the plane curve $X \subset \mathbb{P}^2$ defined by

$$(x+y+z)xyz + x^2y^2 + x^2z^2 + y^2z^2 + x^4 + y^4 + z^4 = 0$$

provides an example for Proposition 4.8 as well as Theorem 1.6 (3). This is the Dickson–Guralnick–Zieve curve for $q = 2$ [GKT19]. As another example, we have the plane curve over $\mathbb{F}_4$ defined by

$$(x^3 + y^3 + z^3)xyz + x^2y^2(x^2 + y^2) + y^2z^2(y^2 + z^2) + (a+1)x^6 + y^6 + az^6 = 0$$

where a is a multiplicative generator of $\mathbb{F}_4^*$ that satisfies $a^2 + a + 1 = 0$. Finally, let a be a multiplicative generator of $\mathbb{F}_8^*$ satisfying $a^3 + a + 1 = 0$. Then the plane curve defined by

$$\begin{aligned}(x^7 + y^7 + z^7)xyz + a^2x^{10} + x^6y^4 + x^4y^6 + (a+1)y^{10} + (a^2+1)x^8z^2 + x^4y^4z^2 \\ + ay^8z^2 + x^6z^4 + x^2y^4z^4 + y^6z^4 + x^4z^6 + y^4z^6 + (a^2+1)x^2z^8 + ay^2z^8 + az^{10} = 0\end{aligned}$$

provides an example over $\mathbb{F}_8$.

Proof of Theorem 1.6. The bound $d \leq q+2$ is a consequence of Lemma 4.7. The "only if" parts of (1) and (2) follow respectively from Proposition 4.5 (1) and (2). The "only if" part of (3) follows from Proposition 4.8. The converses of (1), (2), and (3) are consequences of the smoothness of X.

In (1) and (3), straightforward computations give $F_{1,0} = 0$. Conversely, let us assume $F_{1,0} = 0$. Then Corollary 4.2 (1) and the fact that $d \leq q+2$ imply $d = q+1$ or $d = q+2$. If $d = q+1$, then we are in case (1) by Corollary 4.4. If $d = q+2$, then case (3) occurs by Proposition 4.8. ☐

5. Frobenius nonclassical hypersurfaces with separated variables

In this section, we prove that a smooth Frobenius nonclassical hypersurface over $\mathbb{F}_q$ with separated variables has degree $1 \pmod p$ where $p = \mathrm{char}(\mathbb{F}_q)$. Let us first show that examples in Theorem 1.6 (3) do *not* have separated variables.

Lemma 5.1. *Smooth Frobenius nonclassical hypersurfaces over $\mathbb{F}_q$ of degree $d = q+2$ do not have separated variables.*

Proof. Let $X = \{F = 0\} \subset \mathbb{P}^2$ be such an example. Assume, to the contrary, that it has separated variables. Then there exists a system of coordinates $\{x_0, x_1, x_2\}$, a polynomial $H = H(x_0, x_1)$, and a constant $c \neq 0$ such that

$$F = H(x_0, x_1) + c x_2^{q+2}. \tag{5.1}$$

However, this implies $\frac{\partial F}{\partial x_2} = 0$, contradicting Proposition 4.8.

Here is another proof based on (5.1) without using Proposition 4.8: Write $H_0 := \frac{\partial H}{\partial x_0}$ and $H_1 := \frac{\partial H}{\partial x_1}$. Then the fact that $F_{1,0} = 0$ implies

$$H_{1,0} = x_0^q H_0 + x_1^q H_1 = 0, \qquad \text{or equivalently,} \qquad x_0^q H_0 = x_1^q H_1. \tag{5.2}$$

Hence x_0^q divides H_1 and x_1^q divides H_0. Since H_0 and H_1 have degree $q+1$, there exist linear forms L_0 and L_1 such that $H_0 = x_1^q L_0$ and $H_1 = x_0^q L_1$. Substituting these back into (5.2), we get $x_0^q x_1^q L_0 = x_1^q x_0^q L_1$, whence $L := L_1 = L_2$. If $L = 0$, then $H_0 = H_1 = 0$. But this implies that

$$\frac{\partial F}{\partial x_0} = H_0 = 0, \qquad \frac{\partial F}{\partial x_1} = H_1 = 0, \qquad \text{and} \qquad \frac{\partial F}{\partial x_2} = (q+2)c x_2^{q+1} = 0,$$

contradicting the assumption that X is smooth. Assume $L \neq 0$. By Euler's formula,

$$0 = (q+2)H = x_0 H_0 + x_1 H_1 = (x_0 x_1^q + x_1 x_0^q) L$$

which is a contradiction as both $x_0 x_1^q + x_1 x_0^q$ and L are nonzero ☐

As a preparation for the proof of Theorem 1.9, let us recall an elementary fact about polynomial arithmetic.

Lemma 5.2. *Let k be an arbitrary field, $g, h \in k$ be nonzero constants, $r(t) \in k[t]$ be a polynomial of degree m, and d be a positive integer. Suppose that*

$$(gt^d + h) \cdot r(t) = at^{d+m} + b$$

for some nonzero $a, b \in k$. Then d divides m.

Proof. Let us write $r(t) = \sum_{i=0}^{m} r_i t^i$. Then

$$(gt^d + h) \cdot r(t) = \sum_{i=0}^{m} gr_i t^{d+i} + \sum_{i=0}^{m} hr_i t^i = \sum_{i=d}^{d+m} gr_{i-d} t^i + \sum_{i=0}^{m} hr_i t^i$$
$$= \sum_{i=m+1}^{d+m} gr_{i-d} t^i + \sum_{i=d}^{m} (gr_{i-d} + hr_i) t^i + \sum_{i=0}^{d-1} hr_i t^i.$$

Due to the hypothesis, all but the constant and the leading coefficients in the last expression vanish. That is,

$$r_0 \neq 0 \qquad \text{and} \qquad r_i = 0 \quad \text{for} \quad 1 \leq i \leq d-1, \tag{5.3}$$

$$gr_{i-d} + hr_i = 0 \quad \text{for} \quad d \leq i \leq m, \tag{5.4}$$

$$r_{d+m} \neq 0 \qquad \text{and} \qquad r_{i-d} = 0 \quad \text{for} \quad m+1 \leq i \leq d+m-1. \tag{5.5}$$

Together with the assumption that $g, h \neq 0$, relations (5.3) and (5.4) imply that $r_i = 0$ if and only if $i \not\equiv 0 \pmod{d}$. Similarly, relations (5.4) and (5.5) imply that $r_i = 0$ if and only if $i \not\equiv d+m \equiv m \pmod{d}$. These two statements then imply that $i \equiv 0 \pmod{d}$ if and only if $i \equiv m \pmod{d}$ for $0 \leq i \leq m$, which happens only when d divides m. ☐

Proof of Theorem 1.9. If $F_{1,0} = 0$, then we are in case (1) or (3) of Theorem 1.6. If case (1) occurs, then $d = q+1 \equiv 1 \pmod{p}$. On the other hand, case (3) does not occur due to Lemma 5.1. This shows that the statement holds when $F_{1,0} = 0$.

Assume that $F_{1,0} \neq 0$. By hypothesis, there exists $m \in \{0, \dots, n-1\}$ such that

$$F(x_0, \cdots, x_n) = G(x_0, \dots, x_m) + H(x_{m+1}, \dots, x_n).$$

Note that X is smooth implies that G and H are not constantly zero. The fact that X is Frobenius nonclassical implies that there exists a polynomial R such that $FR = F_{1,0}$, or equivalently,

$$(G + H) \cdot R = G_{1,0} + H_{1,0}. \tag{5.6}$$

Let us extend $\mathbb{F}_q$ by formal variables $u_0, \dots, u_n$ to the function field $k := \mathbb{F}_q(u_0, \dots, u_n)$. Now consider the line $\ell \subset \mathbb{P}^n_k$ spanned by the points

$$[u_0 : \cdots : u_m : 0 : \cdots : 0] \qquad \text{and} \qquad [0 : \cdots : 0 : u_{m+1} : \cdots : u_n]$$

and express it in terms of the parametric equations with affine parameter t:

$$x_i = \begin{cases} u_i t & \text{for} \quad i = 0, \dots, m, \\ u_i & \text{for} \quad i = m+1, \dots, n. \end{cases}$$

The restriction of $(G + H)$ to ℓ has the form

$$(G + H)|_\ell = G(u_0 t, \dots, u_m t) + H(u_{m+1}, \dots, u_n)$$
$$= G(u_0, \dots, u_m) t^d + H(u_{m+1}, \dots, u_n) = gt^d + h \quad \text{where} \quad g, h \in k \setminus \{0\}.$$

Similarly, $(G_{1,0} + H_{1,0})|_\ell = at^{d+q-1} + b$ for some $a, b \in k$. As a result, restricting (5.6) to ℓ gives

$$(gt^d + h) \cdot r(t) = at^{d+q-1} + b \qquad \text{where} \qquad r(t) = R|_\ell \in k[t].$$

The assumption $F_{1,0} \neq 0$ implies that $r(t) \neq 0$. Together with the fact that $g, h \neq 0$, one can verify that $a, b \neq 0$. Lemma 5.2 then implies that d divides $q-1$.

According to [Kle86, page 191], the fact that X is nonreflexive [ADL21, Theorem 4.5] implies that $d \equiv 0$ or $1 \pmod{p}$. If $d \equiv 0 \pmod{p}$, then the fact that d divides $q-1$ implies that p divides 1, a contradiction. Therefore, we must have $d \equiv 1 \pmod{p}$. □

References

[ADL21] Shamil Asgarli, Lian Duan, and Kuan-Wen Lai, *Transverse lines to surfaces over finite fields*, Manuscripta Math. **165** (2021), no. 1-2, 135–157.

[ADL22] ———, *Transverse linear subspaces to hypersurfaces over finite fields*, arXiv e-prints (January 2022), available at https://arxiv.org/abs/2008.11306v2.

[BC66] R. C. Bose and I. M. Chakravarti, *Hermitian varieties in a finite projective space* $\mathrm{PG}(N, q^2)$, Canadian J. Math. **18** (1966), 1161–1182.

[BH17] Herivelto Borges and Masaaki Homma, *Points on singular Frobenius nonclassical curves*, Bull. Braz. Math. Soc. (N.S.) **48** (2017), no. 1, 93–101.

[BMT14] Herivelto Borges, Beatriz Motta, and Fernando Torres, *Complete arcs arising from a generalization of the Hermitian curve*, Acta Arith. **164** (2014), no. 2, 101–118.

[Bor09a] Herivelto Borges, *On complete (N, d)-arcs derived from plane curves*, Finite Fields Appl. **15** (2009), no. 1, 82–96.

[Bor09b] ———, *On multi-Frobenius non-classical plane curves*, Arch. Math. (Basel) **93** (2009), no. 6, 541–553.

[Bor16] ———, *Frobenius nonclassical components of curves with separated variables*, J. Number Theory **159** (2016), 402–425.

[Eis05] David Eisenbud, *The geometry of syzygies*, Graduate Texts in Mathematics, vol. 229, Springer-Verlag, New York, 2005. A second course in commutative algebra and algebraic geometry.

[Eis95] ———, *Commutative algebra. With a view toward algebraic geometry*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995.

[GKT19] Massimo Giulietti, Gábor Korchmáros, and Marco Timpanella, *On the Dickson-Guralnick-Zieve curve*, J. Number Theory **196** (2019), 114–138.

[GPTU02] Massimo Giulietti, Fernanda Pambianco, Fernando Torres, and Emanuela Ughi, *On complete arcs arising from plane curves*, Des. Codes Cryptogr. **25** (2002), no. 3, 237–246.

[Hei96] Udo Heim, *Blocking sets in finite projective spaces*, Mitt. Math. Semin. Gießen **226** (1996), 4–82 (German).

[HK13] Masaaki Homma and Seon Jeong Kim, *An elementary bound for the number of points of a hypersurface over a finite field*, Finite Fields Appl. **20** (2013), 76–83.

[HK16] ———, *The characterization of Hermitian surfaces by the number of points*, J. Geom. **107** (2016), no. 3, 509–521.

[HKT08] J. W. P. Hirschfeld, G. Korchmáros, and F. Torres, *Algebraic curves over a finite field*, Princeton Series in Applied Mathematics, Princeton University Press, Princeton, NJ, 2008.

[HT15] J. W. P. Hirschfeld and J. A. Thas, *Open problems in finite projective spaces*, Finite Fields Appl. **32** (2015), 44–81 (English).

[HV90] Abramo Hefez and José Felipe Voloch, *Frobenius nonclassical curves*, Arch. Math. (Basel) **54** (1990), no. 3, 263–273.

[KKP+22] Zhibek Kadyrsizova, Jennifer Kenkel, Janet Page, Jyoti Singh, Karen E. Smith, Adela Vraciu, and Emily E. Witt, *Lower bounds on the F-pure threshold and extremal singularities*, arXiv e-prints (2022), available at https://arxiv.org/abs/2009.13679v4.

[Kle86] Steven L. Kleiman, *Tangency and duality*, Proceedings of the 1984 Vancouver conference in algebraic geometry, 1986, pp. 163–225.

[KS20] György Kiss and Tamás Szőnyi, *Finite geometries*, CRC Press, Boca Raton, FL, 2020.

[Liu02] Qing Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné, Oxford Science Publications.

[LN96] Rudolf Lidl and Harald Niederreiter, *Finite fields*, 2nd ed., Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1996.

[Mat89] Hideyuki Matsumura, *Commutative ring theory*, Second, Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989. Translated from the Japanese by M. Reid.

[Par86] Rita Pardini, *Some remarks on plane curves over fields of finite characteristic*, Compositio Math. **60** (1986), no. 1, 3–17.

[Ser91] Jean-Pierre Serre, *Lettre à M. Tsfasman*, 1991, pp. 11, 351–353 (1992). Journées Arithmétiques, 1989 (Luminy, 1989).

[Sør94] Anders Bjært Sørensen, *On the number of rational points on codimension-*1 *algebraic sets in* $\mathbf{P}^n(\mathbf{F}_q)$, Discrete Math. **135** (1994), no. 1-3, 321–334.

[SV86] Karl-Otto Stöhr and José Felipe Voloch, *Weierstrass points and curves over finite fields*, Proc. London Math. Soc. (3) **52** (1986), no. 1, 1–19.

[Tir17] Andrea Luigi Tironi, *Hypersurfaces achieving the Homma-Kim bound*, Finite Fields Appl. **48** (2017), 103–116.

[Zak93] F. L. Zak, *Tangents and secants of algebraic varieties*, Translations of Mathematical Monographs, vol. 127, American Mathematical Society, Providence, RI, 1993. Translated from the Russian manuscript by the author.

S. Asgarli, Department of Mathematics
University of British Columbia
Vancouver, BC V6T1Z2, Canada
sham9292@gmail.com

L. Duan, Institute of Mathematical Sciences
ShanghaiTech University
No.393 Middle Huaxia Road, Pudong New District, Shanghai, China
l.duanzwz@gmail.com

K.-W. Lai, Mathematisches Institut der Universität Bonn
Endenicher Allee 60, 53121 Bonn, Deutschland
kwlai@math.uni-bonn.de