CORRIGENDUM TO "TRANSVERSE LINES TO SURFACES OVER FINITE FIELDS"

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In [ADL21, Theorem 4.5], we claim that, given a reduced and geometrically irreducible hypersurface $X = \{F = 0\} \subset \mathbb{P}^n$ over $\mathbb{F}_q$, if it is Frobenius nonclassical, then it is nonreflexive. If we denote by $\gamma\colon X \dashrightarrow X^*$ the Gauss map of X, then X is nonreflexive if and only if, at a general point $P \in X$, the pullback of differentials

$$d\gamma_P^*\colon (\gamma^*\Omega_{X^*}) \otimes \mathbb{F}_q(P) \longrightarrow \Omega_X \otimes \mathbb{F}_q(P) \tag{0.1}$$

is not injective. In our original argument, we intended to prove this property via [ADL21, Lemma 4.6], which shows that the determinant of the Hessian matrix

$$H_F := (F_{ij}) \quad \text{where} \quad F_{ij} = \frac{\partial^2 F}{\partial X_j \partial X_i}$$

is zero modulo F. While [ADL21, Lemma 4.6] is correct, this is not sufficient to prove [ADL21, Theorem 4.5]. The subtle error is the following *incorrect* assertion: a hypersurface X is nonreflexive if and only if the determinant of the Hessian matrix of X vanishes identically on X. Let us explain why this claim fails when $\deg(X) \equiv 1 \mod \operatorname{char}(\mathbb{F}_q)$.

Indeed, if we denote $d = \deg(F)$ and $F_i = \partial F / \partial X_i$, then a computation with Euler's formula shows

$$\begin{pmatrix} d(d-1)F & (d-1)F_1 & \cdots & (d-1)F_n \\ (d-1)F_1 & F_{11} & \cdots & F_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ (d-1)F_n & F_{1n} & \cdots & F_{nn} \end{pmatrix} = \begin{pmatrix} X_0 & X_1 & \cdots & X_n \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \cdot H_F \cdot \begin{pmatrix} X_0 & 0 & \cdots & 0 \\ X_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ X_n & 0 & \cdots & 1 \end{pmatrix} \tag{0.2}$$

from which we see that $\det(H_F)$ is constantly zero whenever $d \equiv 1 \mod \operatorname{char}(\mathbb{F}_q)$. Moreover, we actually assume that the Gauss map is finite in our original argument, but we mistakenly ignored this assumption in the statement of [ADL21, Theorem 4.5]. In this erratum, we fix the statement and prove it without using [ADL21, Lemma 4.6].

Theorem 0.1. *Let $X \subset \mathbb{P}^n$ be a reduced and geometrically irreducible hypersurface over $\mathbb{F}_q$ such that $\dim(X) = \dim(X^*)$. If X is Frobenius nonclassical, then it is nonreflexive. In particular, every smooth Frobenius nonclassical hypersurface is nonreflexive.*

Our strategy of proof goes as follows: Let I and I' be the ideal sheaves for X and X^*, respectively. Then there is a commutative diagram for sheaves of differentials:

$$\begin{array}{ccccccc} \gamma^*\left(I'/I'^2\right) & \longrightarrow & \gamma^*\left(\Omega_{(\mathbb{P}^n)^*}|_{X^*}\right) & \longrightarrow & \gamma^*\Omega_{X^*} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle d\gamma^*} & & \\ I/I^2 & \longrightarrow & \Omega_{\mathbb{P}^n}|_X & \longrightarrow & \Omega_X & \longrightarrow & 0. \end{array} \tag{0.3}$$

Note that $\gamma = \Gamma|_X$ where Γ is the polar map

$$\Gamma\colon \mathbb{P}^n \dashrightarrow (\mathbb{P}^n)^* : [X_0 : \cdots : X_n] \longmapsto \left[\frac{\partial F}{\partial X_0} : \cdots : \frac{\partial F}{\partial X_n}\right].$$

We have $\gamma^*\left(\Omega_{(\mathbb{P}^n)^*}|_{X^*}\right) = \left(\Gamma^*\Omega_{(\mathbb{P}^n)^*}\right)|_X$ and the vertical arrow in the middle of (0.3) is induced by the pullback of differentials $d\Gamma^*\colon \Gamma^*\Omega_{(\mathbb{P}^n)^*} \longrightarrow \Omega_{\mathbb{P}^n}$. Let $U \subset \mathbb{P}^n$ be an open neighborhood of P where $I(U) = (f)$. In order to prove that (0.1) is not injective, we will prove that the image of the linear map

$$d\Gamma_P^*\colon (\Gamma^*\Omega_{(\mathbb{P}^n)^*}) \otimes \mathbb{F}_q(P) \longrightarrow \Omega_{\mathbb{P}^n} \otimes \mathbb{F}_q(P) \tag{0.4}$$

has dimension at most $n-2 = \dim(X) - 1$ modulo df.

1. Proof of the theorem

Let $[Y_0 : \cdots : Y_n]$ be homogeneous coordinates for $(\mathbb{P}^n)^*$ so that the polar map Γ can be written as $Y_i = F_i$, and let $y_i = Y_i/Y_0$ be the affine coordinates for the chart $\{Y_0 \neq 0\}$. Assume without loss of generality that the point $P \in X$ belongs to the open subset

$$U := \{X_0 \neq 0\} \cap \{F_0 \neq 0\} \subset \mathbb{P}^n.$$

If we write $x_i := X_i/X_0$ and let $f_i = f_i(x_1, \ldots, x_n)$ be the dehomogenization of F_i with respect to X_0, then $\Gamma|_U$ can be expressed as $y_i = f_i/f_0 = F_i/F_0$. In this setting, the map of differentials $d\Gamma^*|_U$ sends each dy_i to

$$dy_i = \sum_{j=1}^n \frac{\partial(f_i/f_0)}{\partial x_j} dx_j = \sum_{j=1}^n \left(\frac{(\partial f_i/\partial x_j) f_0 - f_i(\partial f_0/\partial x_j)}{f_0^2}\right) dx_j = \sum_{j=1}^n \left(\frac{F_{ij}F_0 - F_iF_{0j}}{F_0^2}\right) dx_j$$

This linear map corresponds to the square matrix M_F/F_0 where M_F is given by

$$M_F := \begin{pmatrix} F_{11} - \frac{F_1}{F_0}F_{01} & F_{21} - \frac{F_2}{F_0}F_{01} & \cdots & F_{n1} - \frac{F_n}{F_0}F_{01} \\ F_{12} - \frac{F_1}{F_0}F_{02} & F_{22} - \frac{F_2}{F_0}F_{02} & \cdots & F_{n2} - \frac{F_n}{F_0}F_{02} \\ \vdots & \vdots & \ddots & \vdots \\ F_{1n} - \frac{F_1}{F_0}F_{0n} & F_{2n} - \frac{F_2}{F_0}F_{0n} & \cdots & F_{nn} - \frac{F_n}{F_0}F_{0n} \end{pmatrix}.$$

Now we extend the above matrix to the following one

$$H_F'' := \begin{pmatrix} 0 & F_1 & F_2 & \cdots & F_n \\ F_1 & F_{11} - \frac{F_1}{F_0}F_{01} & F_{21} - \frac{F_2}{F_0}F_{01} & \cdots & F_{n1} - \frac{F_n}{F_0}F_{01} \\ F_2 & F_{12} - \frac{F_1}{F_0}F_{02} & F_{22} - \frac{F_2}{F_0}F_{02} & \cdots & F_{n2} - \frac{F_n}{F_0}F_{02} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_n & F_{1n} - \frac{F_1}{F_0}F_{0n} & F_{2n} - \frac{F_2}{F_0}F_{0n} & \cdots & F_{nn} - \frac{F_n}{F_0}F_{0n} \end{pmatrix}.$$

Lemma 1.1. *Let $(0, a_1, \ldots, a_n)$ be a nonzero vector where $a_i \in \overline{\mathbb{F}_q}$. If*

$$(0, a_1, \ldots, a_n) \cdot H_F''(P) = 0 \qquad \textit{where} \qquad P \in U,$$

then the column space of $M_F(P)$ has dimension $\leq n-2$ modulo $(F_1(P), \ldots, F_n(P))^t$.

Proof. The hypothesis implies that

$$(a_1, \dots, a_n) \cdot M_F(P) = 0 \qquad \text{and} \qquad (a_1, \dots, a_n) \cdot (F_1(P), \dots, F_n(P))^t = 0.$$

The first equation implies that of $\mathrm{rk}(M_F(P)) \le n-1$. If $\mathrm{rk}(M_F(P)) \le n-2$, the proof is done. If $\mathrm{rk}(M_F(P)) = n-1$, the second equation above implies that $(F_1(P), \dots, F_n(P))^t$ belongs to the column space of $M_F(P)$, which proves the claim. □

The matrix H_F'' is related to the Hessian matrix H_F in the following way: If we restrict the matrix on the left hand side of (0.2) to $X = \{F = 0\}$ and then divide its first row and first column by $(d-1)$, we will get

$$H_F' := \begin{pmatrix} 0 & F_1 & \cdots & F_n \\ F_1 & F_{11} & \cdots & F_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ F_n & F_{1n} & \cdots & F_{nn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \frac{F_{01}}{F_0} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{F_{0n}}{F_0} & 0 & \cdots & 1 \end{pmatrix} \cdot H_F''. \tag{1.1}$$

However, we cannot divide by $d-1$ in the field when $d \equiv 1 \mod p$. We see that the matrix H_F'' is the correct replacement for the usual Hessian in positive characteristic, especially when $d \equiv 1 \mod p$.

On the other hand, X is Frobenius nonclassical means that there exists a polynomial R that satisfies

$$FR = X_0^q F_0 + X_1^q F_1 + \cdots + X_n^q F_n. \tag{1.2}$$

Lemma 1.2. *Let $P = [1 : X_1 : \cdots : X_n] \in X \cap U$. Then*

$$(1 - d + R, X_1 - X_1^q, X_2 - X_2^q, \cdots, X_n - X_n^q) \cdot H_F'(P) = 0.$$

Proof. Subtracting Euler's formula $dF = X_0F_0 + \cdots + X_nF_n$ by (1.2) gives

$$(d - R)F = (X_0 - X_0^q)F_0 + (X_1 - X_1^q)F_1 + \cdots + (X_n - X_n^q)F_n. \tag{1.3}$$

Taking partial derivatives of both sides with respect to X_i followed by a rearrangement gives

$$-R_iF = (1 - d + R)F_i + (X_0 - X_0^q)F_{0i} + (X_1 - X_1^q)F_{1i} + \cdots + (X_n - X_n^q)F_{ni} \tag{1.4}$$

Then the statement follows by a straightforward computation with (1.3), (1.4), the hypothesis that $X_0 = 1$, and the fact that $F(P) = 0$. □

Proof of Theorem 0.1. Pick a general $P = [1 : X_1 : \cdots : X_n] \in X \cap U$. First we compute

$$\begin{aligned} &(1 - d + R, X_1 - X_1^q, X_2 - X_2^q, \cdots, X_n - X_n^q) \cdot \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \frac{F_{01}}{F_0} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{F_{0n}}{F_0} & 0 & \cdots & 1 \end{pmatrix} \\ &= \left((1 - d + R) + \sum_{j=1}^{n} \left(\frac{F_{0j}}{F_0}(X_j - X_j^q) \right), X_1 - X_1^q, \dots, X_n - X_n^q \right) \\ &\overset{(1.4)}{=} (0, X_1 - X_1^q, \dots, X_n - X_n^q) \end{aligned}$$

Lemma 1.2 and relation (1.1) implies that $(0, X_1 - X_1^q, \dots, X_n - X_n^q) \cdot H_F''(P) = 0$, which implies that $(X_1 - X_1^q, \dots, X_n - X_n^q) \cdot M_F(P) = 0$. By applying Lemma 1.1, we conclude that the image of $d\gamma_P^*$ has dimension $\leq n-2$ modulo $\sum_{i=1}^n F_i(P)dx_i = df$, thus it cannot be injective. This shows that the Gauss map γ is inseparable, whence X is nonreflexive. $\square$

References

[ADL21] Shamil Asgarli, Lian Duan, and Kuan-Wen Lai, *Transverse lines to surfaces over finite fields*, Manuscripta Math. **165** (2021), no. 1-2, 135–157.