Fourier–Mukai numbers of K3 categories of very general special cubic fourfolds

Yu-Wei Fan   Kuan-Wen Lai

Abstract

We give counting formulas for the numbers of Fourier–Mukai partners of K3 categories of very general special cubic fourfolds.

1 Introduction

Let $X \subseteq \mathbb{P}^5$ be a smooth cubic hypersurface over $\mathbb{C}$. Then its bounded derived category of coherent sheaves has a semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle$$

where the subcategory $\mathcal{A}_X$ is called the *K3 category of* X. In contrast to its common usage in the literature, we say two cubic fourfolds are *Fourier–Mukai partners*, or *FM-partners* for short, if their K3 categories are equivalent. The number of FM-partners of a cubic fourfold up to isomorphism is finite [Huy17, Theorem 1.1]. Moreover, this number is equal to 1 if the cubic is very general, or more precisely, if the lattice

$$H^{2,2}(X, \mathbb{Z}) \coloneqq H^4(X, \mathbb{Z}) \cap H^2(X, \Omega_X^2)$$

is spanned by the square of the hyperplane class $h \coloneqq c_1(\mathcal{O}_X(1))$ [Huy17, Theorem 1.5 (i)].

A cubic fourfold X is *special* if $H^{2,2}(X, \mathbb{Z})$ has rank at least 2. In this situation, there exists a rank 2 saturated sublattice $h^2 \in K \subseteq H^{2,2}(X, \mathbb{Z})$. Special cubic fourfolds having such a sublattice K with $\mathrm{disc}(K) = d$ form an irreducible divisor $\mathcal{C}_d$ in the moduli space, which is nonempty if and only if $d \geq 8$ and $d \equiv 0, 2 \pmod 6$ [Has00, Theorem 1.0.1]. In this paper, we call $X \in \mathcal{C}_d$ *very general* if $H^{2,2}(X, \mathbb{Z})$ has rank exactly 2.

To state our main theorem, let us first consider the ring $\mathbb{Z}_{2d}$ of integers modulo $2d$ and denote its subset of square roots of unity as

$$\left(\mathbb{Z}_{2d}^{\times}\right)_2 \coloneqq \left\{ n \in \mathbb{Z}_{2d}^{\times} \;\middle|\; n^2 \equiv 1 \pmod{2d} \right\}.$$

Using the Chinese remainder theorem, one can verify that

$$\left|\left(\mathbb{Z}_{2d}^{\times}\right)_2\right| = \begin{cases} 4 & \text{if} \quad d = 2^{a+1} \\ 2^{k+1} & \text{if} \quad d = 2p_1^{e_1} \cdots p_k^{e_k} \\ 2^{k+2} & \text{if} \quad d = 2^{a+1} p_1^{e_1} \cdots p_k^{e_k} \end{cases}$$

where $a, k \geq 1$ and every p_i is an odd prime.

2020 Mathematics Subject Classification 14F08, 14J35, 14J70
Keywords: Fourier–Mukai partners, K3 categories, special cubic fourfolds

Theorem 1.1. *Let $X \in \mathcal{C}_d$ be a very general member and $\mathrm{FM}(X)$ be the set of isomorphism classes of FM-partners of X.*

- *If $d \not\equiv 0 \pmod 3$, then $|\mathrm{FM}(X)| = \frac{1}{4}\left|\left(\mathbb{Z}_{2d}^{\times}\right)_2\right|$.*
- *If $d \equiv 0 \pmod 3$ and $d \not\equiv 0 \pmod 9$, then $|\mathrm{FM}(X)| = \frac{1}{8}\left|\left(\mathbb{Z}_{2d}^{\times}\right)_2\right|$.*
- *If $d \equiv 0 \pmod 9$ and $\frac{d}{18} \equiv 1 \pmod 3$, then $|\mathrm{FM}(X)| = \frac{1}{4}\left|\left(\mathbb{Z}_{2d}^{\times}\right)_2\right|$.*
- *If $d \equiv 0 \pmod 9$ and $\frac{d}{18} \equiv 2 \pmod 3$, then $|\mathrm{FM}(X)| = \frac{1}{2}\left|\left(\mathbb{Z}_{2d}^{\times}\right)_2\right|$.*
- *If $d \equiv 0 \pmod{27}$, then $|\mathrm{FM}(X)| = \frac{3}{4}\left|\left(\mathbb{Z}_{2d}^{\times}\right)_2\right|$.*

The first two cases, namely, the cases when $d \not\equiv 0 \pmod 9$, were proved in our earlier work [FL23, Proposition 2.6]. Therefore, we will treat only the cases when $d \equiv 0 \pmod 9$ in this paper. This condition implies $d \equiv 0 \pmod{18}$ and we will mostly work with

$$d' := \frac{d}{18}$$

throughout the paper. According to the theorem, if $d \not\equiv 0 \pmod{27}$, then $|\mathrm{FM}(X)|$ is a power of 2. If $d \equiv 0 \pmod{27}$, then $|\mathrm{FM}(X)| = 3 \cdot 2^n$ for some $n \geq 0$. It would be interesting to understand the origin of the factor of 3 from a geometric perspective.

Our proof of the theorem is mainly based on works by Addington–Thomas [AT14] and Huybrechts [Huy17] as these works turn the original problem into a problem about counting certain overlattices. In Section 2, we briefly review the background and set up necessary notations along the way. In Section 3, we introduce the overlattices produced naturally by FM-partners and translate the counting problem for one to the other. In Section 4, we count the number of overlattices and finish the proof of the main theorem.

Acknowledgments The second author is supported by the ERC Synergy Grant HyperK (ID: 854361).

2 Mukai lattices of the K3 categories

The topological K-theory $K_{\mathrm{top}}(X)$ a cubic fourfold is a free abelian group equipped with a natural integral bilinear pairing $\chi(\,\cdot\,,\,\cdot\,)$. It contains the subgroup

$$K_{\mathrm{top}}(\mathcal{A}_X) := \{\kappa \in K_{\mathrm{top}}(X) \mid \chi([\mathcal{O}_X(i)], \kappa) = 0 \ \text{ for } \ i = 0, 1, 2\},$$

called the *Mukai lattice of* $\mathcal{A}_X$, and it has a polarized Hodge structure of K3 type. Indeed, the pairing $\chi(\,\cdot\,,\,\cdot\,)$ is symmetric on $K_{\mathrm{top}}(\mathcal{A}_X)$ and so turns it into a lattice. On the other hand, the Mukai vector defines an embedding

$$K_{\mathrm{top}}(X) \longhookrightarrow H^*(X, \mathbb{Q}) : E \longmapsto \mathrm{ch}(E) \cdot \sqrt{\mathrm{td}(X)}$$

which endows $K_{\mathrm{top}}(\mathcal{A}_X)$ a Hodge structure by taking the restriction of the Hodge structure on $H^4(X, \mathbb{Z})$. By [Huy17, Theorem 1.5 (iii)], two very general special cubic fourfolds are FM-partners if and only if there exists a Hodge isometry between their Mukai lattices.

Following our previous work [FL23], we define $\widetilde{H}(\mathcal{A}_X,\mathbb{Z}) \coloneqq K_{\mathrm{top}}(\mathcal{A}_X)(-1)$. As an abstract lattice, it is unimodular of signature $(4,20)$, so we have

$$\widetilde{H}(\mathcal{A}_X,\mathbb{Z}) \cong E_8(-1)^{\oplus 2} \oplus U^{\oplus 4} \qquad \text{where} \qquad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.1}$$

Let us further define $N(\mathcal{A}_X) \coloneqq \widetilde{H}^{1,1}(\mathcal{A}_X,\mathbb{Z})$ and $T(\mathcal{A}_X) \coloneqq N(\mathcal{A}_X)^{\perp \widetilde{H}(\mathcal{A}_X,\mathbb{Z})}$. The projections of the classes $[\mathcal{O}_{\mathrm{line}}(1)]$ and $[\mathcal{O}_{\mathrm{line}}(2)]$ to $K_{\mathrm{top}}(\mathcal{A}_X)$ induces two elements $\lambda_1,\lambda_2 \in N(\mathcal{A}_X)$ which span the sublattice

$$A_2(X) \coloneqq \langle \lambda_1,\lambda_2\rangle \cong \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \subseteq N(\mathcal{A}_X).$$

By [AT14, Proposition 2.3], there is a commutative diagram of Hodge structures

$$\begin{tikzcd}
A_2(X)^{\perp \widetilde{H}(\mathcal{A}_X,\mathbb{Z})} \arrow[r, "\sim"] & \langle h^2\rangle^{\perp H^4(X,\mathbb{Z})(-1)} = H^4(X,\mathbb{Z})_{\mathrm{prim}}(-1) \\
T(\mathcal{A}_X) \arrow[u, hook] \arrow[r, "\sim"] & H^{2,2}(X,\mathbb{Z})^{\perp H^4(X,\mathbb{Z})(-1)} \arrow[u, hook]
\end{tikzcd}$$

where the horizontal maps are isomorphisms.

From now on, we fix a very general $X \in \mathcal{C}_d$ with $d \equiv 0 \pmod 9$. Let us denote for short that $T \coloneqq T(\mathcal{A}_X)$ and $S \coloneqq N(\mathcal{A}_X) \cap A_2(X)^{\perp N(\mathcal{A}_X)}$. Then a direct computation using the fact that $d \equiv 0 \pmod 6$ gives

$$H^{2,2}(X,\mathbb{Z}) \cong \begin{pmatrix} 3 & 0 \\ 0 & 6d' \end{pmatrix} \qquad \text{whence} \qquad N(\mathcal{A}_X) \cong \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & -6d' \end{pmatrix}.$$

Denote by $\{\lambda_1,\lambda_2,\ell\}$ the standard basis for $N(\mathcal{A}_X)$ so that $\ell^2 = -6d'$. Then

$$S = \langle \ell \rangle \qquad \text{whence} \qquad S^*/S = \left\langle \frac{\ell}{6d'} \right\rangle \cong \mathbb{Z}_{6d'}. \tag{2.2}$$

Let us also give an explicit formula for the discriminant group of T.

Lemma 2.1. *There exists $t_1,t_2 \in T$ with $t_1^2 = -6$, $t_2^2 = 6d'$, and $t_1t_2 = 0$ such that*

$$T^*/T = \left\langle \frac{t_1}{3} \right\rangle \oplus \left\langle \frac{t_2}{6d'} \right\rangle \cong \mathbb{Z}_3 \oplus \mathbb{Z}_{6d'}.$$

Proof. As a result of [Nik79, Theorem 1.14.4], we can fix an isomorphism (2.1) such that the basis elements for $N(\mathcal{A}_X)$ are identified as

$$\lambda_1 = e_1 + f_1, \qquad \lambda_2 = e_2 + f_2 - e_1, \qquad \ell = e_3 - (3d')f_3$$

where $\{e_i,f_i\}$, $i=1,2,3$, are the standard bases for the last three copies of U. This isomorphism identifies T with the sublattice $E_8(-1)^{\oplus 2} \oplus U \oplus A_2(-1) \oplus \mathbb{Z}(6d')$ where

$$A_2(-1) = \langle e_1 - f_1 - e_2,\, e_2 - f_2\rangle \qquad \text{and} \qquad \mathbb{Z}(6d') = \langle e_3 + (3d')f_3\rangle.$$

Take $t_1 \coloneqq e_1 - f_1 - 2e_2 + f_2$ and $t_2 \coloneqq e_3 + (3d')f_3$ respectively from these factors. Then a direct computation shows that they satisfy the requirements. $\square$

3 FM-partners and two types of overlattices

Let us retain the setting from the previous section and define $\mathcal{M}_{S,T}$ to be the set of even overlattices $L \supseteq S \oplus T$ with $\mathrm{disc}(L) = 3$ such that $S, T \subseteq L$ are both saturated. Our goal is to turn the original counting problem to the counting on the set $\mathcal{M}_{S,T}$. Let us start by showing that the elements of $\mathcal{M}_{S,T}$ can be divided into two types.

Using $[S^* : S] = 6d'$ and $[T^* : T] = 18d'$, one can deduce from the chain of inclusions

$$S \oplus T \subseteq L \subseteq L^* \subseteq S^* \oplus T^*$$

that $[L\colon S \oplus T] = [S^* \oplus T^* : L^*] = 6d'$. Now, as $S, T \subseteq L$ are saturated, the projections

$$L^*/(S \oplus T) \longrightarrow S^*/S \cong \mathbb{Z}_{6d'} \tag{3.1}$$

$$L^*/(S \oplus T) \longrightarrow T^*/T \cong \mathbb{Z}_3 \oplus \mathbb{Z}_{6d'} \tag{3.2}$$

are both surjective. In particular, the second map is an isomorphism as the orders of the two groups are both $18d'$. Hence, there exist integers b_1 and b_2 such that

$$L^*/(S \oplus T) = \left\langle \frac{b_1\ell + t_1}{3} \right\rangle \oplus \left\langle \frac{b_2\ell + t_2}{6d'} \right\rangle \tag{3.3}$$

where ℓ, t_1, t_2 are as in (2.2) and Lemma 2.1.

The subgroup $L/(S \oplus T) \subseteq L^*/(S \oplus T)$ has index 3, so its image under (3.2) followed by the projection to the factor $\mathbb{Z}_{6d'}$ is either $\mathbb{Z}_{2d'}$ or $\mathbb{Z}_{6d'}$. We can use (3.3) to deduce an explicit expression for $L/(S \oplus T)$ in each of the two cases as follows:

(I) If the image is $\mathbb{Z}_{2d'}$, then

$$L/(S \oplus T) = \left\langle \frac{b_1\ell + t_1}{3} \right\rangle \oplus \left\langle \frac{b_2\ell + t_2}{2d'} \right\rangle.$$

In view of the surjectivity of (3.1), we may assume that

$$0 \le b_1 < 3 \qquad \text{and} \qquad 0 \le b_2 < 2d' \quad \text{with} \quad \gcd(b_1, 3) = \gcd(b_2, 2d') = 1. \tag{3.4}$$

(II) If the image is $\mathbb{Z}_{6d'}$, then

$$L/(S \oplus T) = \left\langle \frac{kt_1}{3} + \frac{b_3\ell + t_2}{6d'} \right\rangle \quad \text{for some} \quad 0 \le k < 3.$$

Here $b_3 = 2d'kb_1 + b_2$. In view of the surjectivity of (3.1), we may assume that

$$0 \le b_3 < 6d' \quad \text{with} \quad \gcd(b_3, 6d') = 1. \tag{3.5}$$

In order to relate the set $\mathcal{M}_{S,T}$ to the set of FM-partners, let us consider the set $\widetilde{\mathrm{FM}}(X)$ of triples (Y, ϕ, ψ) where

- $Y \in \mathrm{FM}(X)$,

- $\phi\colon S \longrightarrow S_Y \coloneqq N(\mathcal{A}_Y) \cap A_2(Y)^{\perp N(\mathcal{A}_Y)}$ is an isometry of rank 1 lattices,
- $\psi\colon T \longrightarrow T_Y \coloneqq T(\mathcal{A}_Y)$ is a Hodge isometry.

Lemma 3.1. *The forgetful map*

$$\widetilde{\mathrm{FM}}(X) \longrightarrow \mathrm{FM}(X) : (Y, \phi, \psi) \longmapsto Y$$

is 4*-to-*1*. Therefore, we have* $|\mathrm{FM}(X)| = \frac{1}{4}|\widetilde{\mathrm{FM}}(X)|$.

Proof. For each $Y \in \mathrm{FM}(X)$, there are exactly two isometries $\phi\colon S \longrightarrow S_Y$ which are different by a sign. On the other hand, there exists a Hodge isometry $\psi\colon T \longrightarrow T_Y$ due to [Huy17, Theorem 1.5 (iii)]. Because the only Hodge isometries on T are ± 1 [Huy16, Corollary 3.3.5], the only Hodge isometries from T to T_Y are $\pm\psi$. This shows that the preimage over $Y \in \mathrm{FM}(X)$ consists of $(Y, \pm\phi, \pm\psi)$, so the statement follows. □

For each $(Y, \phi, \psi) \in \widetilde{\mathrm{FM}}(X)$, one can verify that the pullback

$$L_{(Y,\phi,\psi)} \coloneqq (\phi \oplus \psi)^* \left(A_2(Y)^{\perp \widetilde{H}(\mathcal{A}_Y, \mathbb{Z})} \right) \subseteq S^* \oplus T^*$$

is an element of $\mathcal{M}_{S,T}$. This defines a map

$$\widetilde{\mathrm{FM}}(X) \longrightarrow \mathcal{M}_{S,T} : (Y, \phi, \psi) \longmapsto L_{(Y,\phi,\psi)}. \tag{3.6}$$

Due to the Torelli theorem [Voi86], if (Y, ϕ, ψ) and (Y', ϕ', ψ') have the same image under this map, then $Y \cong Y'$. Hence, the forgetful map in Lemma 3.1 factors as

$$\begin{tikzcd} \widetilde{\mathrm{FM}}(X) \arrow[r, "L_\bullet"] \arrow[dr, "4:1"'] & \mathcal{M}_{S,T} \arrow[d] \\ & \mathrm{FM}(X). \end{tikzcd}$$

This shows that every fiber of (3.6) is contained in a fiber of the forgetful map. Moreover, the map is surjective due to the surjectivity of the period map [Laz10, Theorem 1.1]; see the end of [FL23, Proof of Lemma 2.7] for the details.

Lemma 3.2. *Map* (3.6) *is* 2*-to-*1*. As a result, we have* $|\mathrm{FM}(X)| = \frac{1}{2}|\mathcal{M}_{S,T}|$.

Proof. It is easy to see that $L_{(Y,\phi,\psi)} = L_{(Y,-\phi,-\psi)}$ for every $(Y, \phi, \psi) \in \widetilde{\mathrm{FM}}(X)$. As a consequence, the fiber $\{(Y, \pm\phi, \pm\psi)\} \subseteq \widetilde{\mathrm{FM}}(X)$ over each $Y \in \mathrm{FM}(X)$ is mapped by (3.6) as the set $\{L_{(Y,\phi,\psi)}, L_{(Y,-\phi,\psi)}\}$. To prove the statement, it suffices to verify $L_{(Y,\phi,\psi)} \neq L_{(Y,-\phi,\psi)}$, that is, one gets a different lattice after replacing ℓ with $-\ell$.

First suppose that $L \coloneqq L_{(Y,\phi,\psi)}$ satisfies (I). Assume, to the contrary, that L remains the same after replacing ℓ with $-\ell$. This assumption implies that

$$\frac{b_1\ell + t_1}{3} = \pm\left(\frac{-b_1\ell + t_1}{3}\right)$$

within the $\mathbb{Z}_3$ factor of $L/(S\oplus T)\cong \mathbb{Z}_3\oplus\mathbb{Z}_{2d'}$, which gives $\frac{2b_1}{3}\ell=0$ or $\frac{2}{3}t_1=0$. Note that the latter case does not occur. In the former case, we get $b_1\equiv 0 \pmod 3$, but this contradicts the fact that $\gcd(b_1,3)=1$.

Now suppose that L satisfies (II). Assume again to the contrary that L remains the same after replacing ℓ with $-\ell$. In this case, we get

$$\frac{kt_1}{3}+\frac{b_3\ell+t_2}{6d'}=\pm\left(\frac{kt_1}{3}+\frac{-b_3\ell+t_2}{6d'}\right)\ \in\ L/(S\oplus T)\cong\mathbb{Z}_{6d'},$$

which implies $\frac{b_3}{3d'}\ell=0$ or $\frac{2k}{3}t_1+\frac{2}{3d'}t_2=0$. The latter case does not occur. In the former case, we get $b_3\equiv 0 \pmod{3d'}$, but this contradicts the fact that $\gcd(b_3,6d')=1$.

We have proved the first statement, which gives $|\widetilde{\mathrm{FM}}(X)|=2|\mathcal{M}_{S,T}|$. It then follows from Lemma 3.1 that $|\mathrm{FM}(X)|=\frac{1}{4}|\widetilde{\mathrm{FM}}(X)|=\frac{1}{2}|\mathcal{M}_{S,T}|$. □

4 Counting the number of overlattices

It remains to count the number of elements in $\mathcal{M}_{S,T}$, or more precisely, to count the number of lattices of the forms (I) and (II) which are even. Let us start by proving a basic property about square roots of unity.

Lemma 4.1. *Let n be a positive integer. Then the number of integers $0\le b<2n$ which satisfy $b^2\equiv 1 \pmod{4n}$ is equal to $\frac{1}{2}\left|\left(\mathbb{Z}_{4n}^\times\right)_2\right|$.*

Proof. For each b satisfying the hypothesis, the integers b and $b+2n$ represent distinct elements in $\left(\mathbb{Z}_{4n}^\times\right)_2$. This proves the statement. □

Let us first count even overlattices of Type (I).

Lemma 4.2. *Elements in $\mathcal{M}_{S,T}$ of Type (I) exist only if $d'\equiv 2 \pmod 3$. If this condition holds, then there are $\left|\left(\mathbb{Z}_{4d'}^\times\right)_2\right|$ many of them.*

Proof. An overlattice of Type (I) has the form

$$L=S+T+\left\langle\frac{b_1\ell+t_1}{3}\right\rangle+\left\langle\frac{b_2\ell+t_2}{2d'}\right\rangle$$

where (b_1,b_2) are as in (3.4). Such a lattice is even, that is, belongs to $\mathcal{M}_{S,T}$, if and only if

$$\left(\frac{b_1\ell+t_1}{3}\right)^2=-\frac{2}{3}\left(b_1^2d'+1\right)\ \in\ 2\mathbb{Z}\qquad\Longleftrightarrow\qquad d'\equiv 2\pmod 3$$

and

$$\left(\frac{b_2\ell+t_2}{2d'}\right)^2=-\frac{3}{2d'}\left(b_2^2-1\right)\ \in\ 2\mathbb{Z}\qquad\Longleftrightarrow\qquad b_2^2\equiv 1\pmod{4d'}.$$

From here, we see that even overlattices of Type (I) exist only if $d'\equiv 2 \pmod 3$. In this situation, the number of such lattices equals the number of pairs (b_1,b_2) which satisfy

$$b_1\in\{1,2\}\qquad\text{and}\qquad 0\le b_2<2d'\quad\text{with}\quad b_2^2\equiv 1\pmod{4d'}.$$

By Lemma 4.1, the number of choices for b_2 is equal to $\frac{1}{2}\left|\left(\mathbb{Z}_{4d'}^\times\right)_2\right|$. Since there are two choices for b_1, the number of desired (b_1,b_2) is equal to $\left|\left(\mathbb{Z}_{4d'}^\times\right)_2\right|$. □

Recall that an overlattice of Type (II) has the form

$$L = S + T + \left\langle \frac{b_3\ell + 2d'kt_1 + t_2}{6d'} \right\rangle$$

where $0 \leq k < 3$ and b_3 is as in (3.5). This lattice is even if and only if

$$\left(\frac{b_3\ell + 2d'kt_1 + t_2}{6d'}\right)^2 = -\frac{1}{6d'}\left(b_3^2 + 4d'k^2 - 1\right) \in 2\mathbb{Z}. \tag{4.1}$$

Let us count the number of such overlattices case-by-case.

Lemma 4.3. *There are $\frac{1}{2}\left|\left(\mathbb{Z}_{12d'}^{\times}\right)_2\right|$ many elements in $\mathcal{M}_{S,T}$ of Type (II) with $k = 0$.*

Proof. When $k = 0$, condition (4.1) reduces to

$$-\frac{1}{6d'}\left(b_3^2 - 1\right) \in 2\mathbb{Z} \qquad \Longleftrightarrow \qquad b_3^2 \equiv 1 \pmod{12d'}.$$

By Lemma 4.1, the number of choices for b_3 is equal to $\frac{1}{2}\left|\left(\mathbb{Z}_{12d'}^{\times}\right)_2\right|$. □

Now we consider the cases when $k = 1, 2$.

Lemma 4.4. *Elements in $\mathcal{M}_{S,T}$ of Type (II) with $k = 1$ exist only if $d' \equiv 0 \pmod 3$. If this holds, then there are $\frac{1}{2}\left|\left(\mathbb{Z}_{4d'}^{\times}\right)_2\right|$ many of them. The same statement holds for $k = 2$.*

Proof. When $k = 1$ (resp. $k = 2$), condition (4.1) is equivalent to

$$b_3^2 + 4d' - 1 \equiv 0 \pmod{12d'}. \tag{4.2}$$

Since $b_3^2 \equiv 1 \pmod 3$, the above relation modulo 3 gives $d' \equiv 0 \pmod 3$. Now write

$$b_3 = b_4 + 2d'm \quad \text{with} \quad 0 \leq b_4 < 2d'.$$

Then (3.5) holds if and only if $m \in \{0, 1, 2\}$ and $\gcd(b_4, 2d') = 1$. From (4.2), we get

$$b_3^2 + 4d' - 1 \equiv b_4^2 + 4d'(mb_4 + d'm^2 + 1) - 1 \equiv 0 \pmod{12d'}. \tag{4.3}$$

This relation modulo $4d'$ gives

$$b_4^2 \equiv 1 \pmod{4d'}. \tag{4.4}$$

We claim that for each such b_4, whether $m = 0, 1, 2$ is uniquely determined. Indeed, if we write $b_4^2 = 1 + 4d'r$ with r an integer, then

$$b_4^2 + 4d'(mb_4 + d'm^2 + 1) - 1 = 4d'(r + mb_4 + d'm^2 + 1).$$

Inserting this into (4.3) with $d' \equiv 0 \pmod 3$ in mind reduces the relation to

$$r + mb_4 + 1 \equiv 0 \pmod 3.$$

Then the claim follows as $b_4 \not\equiv 0 \pmod 3$. By Lemma 4.1, the number of $0 \leq b_4 < 2d'$ satisfying (4.4) is equal to $\frac{1}{2}\left|\left(\mathbb{Z}_{4d'}^{\times}\right)_2\right|$. This completes the proof. □

Proof of Theorem 1.1. By Lemmas 4.2, 4.3, 4.4, the numbers of elements in $\mathcal{M}_{S,T}$ of different types and values of d' can be organized into a table:

	$d' \equiv 0 \pmod 3$	$d' \equiv 1 \pmod 3$	$d' \equiv 2 \pmod 3$
(I)	0	0	$\left\|\left(\mathbb{Z}_{4d'}^{\times}\right)_2\right\|$
(II) with $k=0$	$\frac{1}{2}\left\|\left(\mathbb{Z}_{12d'}^{\times}\right)_2\right\|$	$\frac{1}{2}\left\|\left(\mathbb{Z}_{12d'}^{\times}\right)_2\right\|$	$\frac{1}{2}\left\|\left(\mathbb{Z}_{12d'}^{\times}\right)_2\right\|$
(II) with $k=1$	$\frac{1}{2}\left\|\left(\mathbb{Z}_{4d'}^{\times}\right)_2\right\|$	0	0
(II) with $k=2$	$\frac{1}{2}\left\|\left(\mathbb{Z}_{4d'}^{\times}\right)_2\right\|$	0	0
$\left\|\mathcal{M}_{S,T}\right\|$	$\frac{3}{2}\left\|\left(\mathbb{Z}_{4d'}^{\times}\right)_2\right\|$	$\left\|\left(\mathbb{Z}_{4d'}^{\times}\right)_2\right\|$	$2\left\|\left(\mathbb{Z}_{4d'}^{\times}\right)_2\right\|$

The formulas can then be deduced from Lemma 3.2 and a direct computation. □

References

[AT14] Nicolas Addington and Richard Thomas, *Hodge theory and derived categories of cubic fourfolds*, Duke Math. J. **163** (2014), no. 10, 1885–1927.

[FL23] Yu-Wei Fan and Kuan-Wen Lai, *New rational cubic fourfolds arising from Cremona transformations*, Algebr. Geom. **10** (2023), no. 4, 432–460.

[Has00] Brendan Hassett, *Special cubic fourfolds*, Compositio Math. **120** (2000), no. 1, 1–23.

[Huy16] D. Huybrechts, *Lectures on K3 surfaces*, Cambridge Studies in Advanced Mathematics, vol. 158, Cambridge University Press, Cambridge, 2016.

[Huy17] Daniel Huybrechts, *The K3 category of a cubic fourfold*, Compos. Math. **153** (2017), no. 3, 586–620.

[Laz10] Radu Laza, *The moduli space of cubic fourfolds via the period map*, Ann. of Math. (2) **172** (2010), no. 1, 673–711.

[Nik79] V. V. Nikulin, *Integer symmetric bilinear forms and some of their geometric applications*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 1, 111–177, 238.

[Voi86] Claire Voisin, *Théorème de Torelli pour les cubiques de* $\mathbf{P}^5$, Invent. Math. **86** (1986), no. 3, 577–601.

Y.-W. Fan, Yau Mathematical Sciences Center
Tsinghua University
Beijing 100084, P. R. China
Email: `ywfan@mail.tsinghua.edu.cn`

K.-W. Lai, Mathematisches Institut der Universität Bonn
Endenicher Allee 60, 53121 Bonn, Deutschland
Email: `kwlai@math.uni-bonn.de`