Grothendieck ring of birational permutations

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1 Introduction

Let X be a variety defined over an arbitrary field k and $\mathrm{Bir}_k(X)$ be the group of birational self-maps on X. An element of $\mathrm{Bir}_k(X)$ is called a *birational permutation* if it induces a permutation on the set $X(k)$ of k-rational points on X. Clearly, birational permutations form a subgroup $\mathrm{BBir}_k(X) \subseteq \mathrm{Bir}_k(X)$, and there is a canonical group homomorphism

$$\sigma\colon \mathrm{BBir}_k(X) \longrightarrow \mathrm{Sym}(X(k))$$

where $\mathrm{Sym}(X(k))$ is the symmetric group of the set $X(k)$.

The main goal of this note is to construct a Grothendieck ring $K_0(\mathcal{P}_k)$ over a perfect field k, where each element $[X, \alpha]$ in the ring is represented by a variety X equipped with a birational permutation α. We will prove the existence of a group homomorphism

$$\mathsf{p}\colon K_0(\mathcal{P}_k) \longrightarrow (\mathbb{Z}/2\mathbb{Z}, +, 0), \qquad \mathsf{p}([X, \alpha]) = \begin{cases} 0 & \text{if } \sigma(\alpha) \text{ is even} \\ 1 & \text{if } \sigma(\alpha) \text{ is odd} \end{cases}$$

which respects the ring structure via the following rule

$$\mathsf{p}([X, \alpha] \cdot [Y, \beta]) = |Y(k)| \cdot \mathsf{p}([X, \alpha]) + |X(k)| \cdot \mathsf{p}([Y, \beta]).$$

Note that this is analogous to Leibniz's rule.

We will carry out the construction of $K_0(\mathcal{P}_k)$ in Section 2 and formulate the homomorphism p in Section 3. In fact, we will proceed the construction for both birational self-maps and birational permutations parallelly as there is no much difference, and in the hope to inspire similar constructions for other meaningful types of birational self-maps. Inspired by the Weil conjecture, we will discuss shortly in Section 4 the rationality conjecture of certain zeta functions constructed from automorphisms of varieties.

1.1 Grothendieck ring of varieties: a brief review Here is a brief review on the definition of the ordinary Grothendieck ring, namely, the Grothendieck ring of varieties. Let $\mathcal{V}_k$ be the category of algebraic varieties over a perfect field k. Recall that the Grothendieck group $K_0(\mathcal{V}_k)$ is constructed by first taking the free abelian group generated by the isomorphism classes of objects in $\mathcal{V}_k$, and then taking quotient under the relations

$$[X] \sim [Z] + [X \setminus Z]$$

where Z is a closed subvariety of X. The group $K_0(\mathcal{V}_k)$ can be equipped with a ring structure by defining the ring multiplication to be

$$[X] \cdot [Y] = [X \times Y]. \tag{1.1}$$

The product on the right hand side is understood as a fiber product over $\mathrm{Spec}(k)$. In particular, the identity element of this multiplication is $[\mathrm{Spec}(k)]$.

Remark 1.1. In general, the direct product of two varieties over an arbitrary field k may be non-reduced, so (1.1) should be modified as

$$[X] \cdot [Y] = [(X \times Y)_{\mathrm{red}}].$$

For example, consider the function field $k = \mathbb{F}_p(t)$, where p is a prime, and the k-algebra

$$K := k[x]/(x^p - t) = k[\sqrt[p]{t}] = \mathbb{F}_p(\sqrt[p]{t}).$$

The algebra K is a field and thus reduced. However, the tensor product

$$K \otimes_k K = K \otimes_k k[x]/(x^p - t) = K[x]/(x^p - t)$$

is non-reduced as $x - \sqrt[p]{t}$ is a non-zero nilpotent. This can be avoided by assuming that k is perfect since in this case the tensor product of reduced k-algebras is always reduced [Bou03, Chapter V, §15.5, Theorem 3 (d)].

2 Construction of the Grothendieck ring

Let us denote by $\mathcal{B}_k$ (resp. $\mathcal{P}_k$) the category whose objects are pairs (X, α), where X is a variety over a perfect field k and $\alpha \in \mathrm{Bir}_k(X)$ (resp. $\alpha \in \mathrm{BBir}_k(X)$), and a morphism $\varphi\colon (X, \alpha) \to (Y, \beta)$ is given by a morphism $\varphi\colon X \to Y$ of varieties that satisfies $\varphi\alpha = \beta\varphi$, that is, the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, dashed, "\alpha"'] & Y \arrow[d, dashed, "\beta"] \\ X \arrow[r, "\varphi"] & Y. \end{tikzcd}$$

Two morphisms $\varphi\colon (X, \alpha) \to (Y, \beta)$ and $\psi\colon (Y, \beta) \to (Z, \gamma)$ compose as

$$\psi \circ \varphi\colon (X, \alpha) \longrightarrow (Z, \gamma),$$

which is well-defined as the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, dashed, "\alpha"'] & Y \arrow[r, "\psi"] \arrow[d, dashed, "\beta"] & Z \arrow[d, dashed, "\gamma"] \\ X \arrow[r, "\varphi"] & Y \arrow[r, "\psi"] & Z. \end{tikzcd}$$

In this setting, $\varphi\colon (X, \alpha) \to (Y, \beta)$ is an isomorphism if there exists $\psi\colon (Y, \beta) \to (X, \alpha)$ such that $\psi\varphi = \mathrm{id}_X$ and $\varphi\psi = \mathrm{id}_Y$, where the compositions are taking within $\mathcal{V}_k$.

Definition 2.1. Consider an embedding of a subvariety $\iota\colon Z \hookrightarrow X$. We say Z is *invariant* under $\alpha \in \mathrm{Bir}_k(X)$ (resp. $\mathrm{BBir}_k(X)$) if there exists $\alpha_Z \in \mathrm{Bir}_k(Z)$ (resp. $\mathrm{BBir}_k(X)$) such that the diagram commutes:

$$\begin{tikzcd} Z \arrow[r, hook, "\iota"] \arrow[d, dashed, "\alpha_Z"'] & X \arrow[d, dashed, "\alpha"] \\ Z \arrow[r, hook, "\iota"] & X. \end{tikzcd}$$

That is, there exists a morphism $\iota\colon (Z, \alpha_Z) \longrightarrow (X, \alpha)$ in the category $\mathcal{B}_k$ (resp. $\mathcal{P}_k$). We call the map α_Z a *restriction* of α to Z.

Lemma 2.2. *Let α_Z and α'_Z be restrictions of α to Z. Then $\alpha_Z = \alpha'_Z$.*

Proof. By definition, we have $\alpha\iota = \iota\alpha_Z$. Pre-composing both sides of this equation with α_Z^{-1}, we get $\alpha\iota\alpha_Z^{-1} = \iota$. Similarly, we have $\alpha\iota\alpha_Z'^{-1} = \iota$. On the other hand, if we pre-compose both sizes of $\alpha\iota = \iota\alpha_Z$ with $\alpha_Z'^{-1}$, we get $\alpha\iota\alpha_Z'^{-1} = \iota\alpha_Z\alpha_Z'^{-1}$, which implies $\iota\alpha_Z\alpha_Z'^{-1} = \iota$. Since ι is a monomorphism, we conclude that $\alpha_Z\alpha_Z'^{-1} = \mathrm{id}_Z$, hence $\alpha_Z = \alpha'_Z$. ◻

Remark 2.3. Let $\varphi\colon (X,\alpha) \to (Y,\beta)$ be an isomorphism in $\mathcal{B}_k$ (resp. $\mathcal{P}_k$), and let $\iota\colon Z \hookrightarrow X$ be a subvariety invariant under α. Note that the *image* of Z in Y should be understood as the subvariety given by the inclusion map $\varphi\iota\colon Z \hookrightarrow Y$. Now we have the commutative diagram

$$\begin{tikzcd} Z \arrow[r, hook, "\iota"] \arrow[d, dashed, "\alpha_Z"'] & X \arrow[r, "\varphi"] \arrow[d, dashed, "\alpha"] & Y \arrow[d, dashed, "\beta"] \\ Z \arrow[r, hook, "\iota"] & X \arrow[r, "\varphi"] & Y. \end{tikzcd}$$

This implies that the restriction of β to Z coincides with α_Z.

Remark 2.4. Let $U \subseteq X$ be an open subvariety. Then it is invariant under every $\alpha \in \mathrm{Bir}_k(X)$ as the restriction α_U always induces a birational self-map on U. Note that this is not true if we are working with $\mathcal{P}_k$ because α_U may not induce a bijection on $U(k)$. However, if a closed subvariety $Z \subseteq X$ is invariant under $\alpha \in \mathrm{BBir}_k(X)$, then the open complement $U := X \setminus Z$ is invariant under α as well.

Let $\{X,\alpha\}$ denote the isomorphism class of $(X,\alpha) \in \mathcal{B}_k$ (resp. $\mathcal{P}_k$), and define

$$G(\mathcal{B}_k) := \bigcup_{(X,\alpha)\in\mathcal{B}_k} \mathbb{Z}\{X,\alpha\} \qquad \left(\text{resp. } G(\mathcal{P}_k) := \bigcup_{(X,\alpha)\in\mathcal{P}_k} \mathbb{Z}\{X,\alpha\}\right)$$

to be the free abelian group generated by these isomorphism classes. This group can be equipped with a ring multiplication defined by

$$\{X,\alpha\}\cdot\{Y,\beta\} = \{X\times Y, \alpha\times\beta\} \tag{2.1}$$

where $\{\mathrm{Spec}(k), \mathrm{id}_{\mathrm{Spec}(k)}\}$ plays as the multiplicative identity.

Lemma 2.5. *The multiplication (2.1) is well-defined.*

Proof. Let (X',α') and (Y',β') be representatives of $\{X,\alpha\}$ and $\{Y,\beta\}$, respectively, so that there are isomorphisms

$$\varphi\colon (X,\alpha) \to (X',\alpha'), \qquad \psi\colon (Y,\beta) \to (Y',\beta').$$

Taking direct product in $\mathcal{V}_k$ gives an isomorphism $\varphi\times\psi\colon X\times Y \to X'\times Y'$ that satisfies

$$(\varphi\times\psi)(\alpha\times\beta) = (\varphi\alpha)\times(\psi\beta) = (\alpha'\varphi)\times(\beta'\psi) = (\alpha'\times\beta')(\varphi\times\psi).$$

Therefore, we have an isomorphism between pairs

$$\varphi\times\psi\colon (X\times Y, \alpha\times\beta) \to (X'\times Y', \alpha'\times\beta').$$

It follows that

$$\{X',\alpha'\}\cdot\{Y',\beta'\} = \{X'\times Y', \alpha'\times\beta'\} = \{X\times Y, \alpha\times\beta\} = \{X,\alpha\}\cdot\{Y,\beta\}$$

which completes the proof. ◻

Now consider the equivalence relation on $G(\mathcal{B}_k)$ (resp. $G(\mathcal{P}_k)$) generated by

$$\{X, \alpha\} \sim \{Z, \alpha_Z\} + \{U, \alpha_U\} \tag{2.2}$$

where $Z \subseteq X$ is an α-invariant closed subvariety, $U = X \setminus Z$, and α_Z and α_U are the restrictions of α to Z and U, respectively.

Lemma 2.6. *The relation* (2.2) *is well-defined.*

Proof. Let (Y, β) be any representative of the isomorphism class $[X, \alpha]$, so that there exists an isomorphism $\varphi\colon (X, \alpha) \to (Y, \beta)$, which gives commutative diagrams

$$\begin{tikzcd} Z \arrow[r, hook, "\iota"] \arrow[d, dashed, "\alpha_Z"'] & X \arrow[r, "\varphi"] \arrow[d, dashed, "\alpha"] & Y \arrow[d, dashed, "\beta"] \\ Z \arrow[r, hook, "\iota"] & X \arrow[r, "\varphi"] & Y \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, hook, "\kappa"] \arrow[d, dashed, "\alpha_U"'] & X \arrow[r, "\varphi"] \arrow[d, dashed, "\alpha"] & Y \arrow[d, dashed, "\beta"] \\ U \arrow[r, hook, "\kappa"] & X \arrow[r, "\varphi"] & Y. \end{tikzcd}$$

Consider Z and U as subvarieties of Y via the inclusions $\varphi\iota$ and $\varphi\kappa$, respectively. Then they are both invariant under β, and the restrictions of β to Z and U coincide with α_Z and α_U, respectively. By definition, we get $\{Y, \beta\} \sim \{Z, \alpha_Z\} + \{U, \alpha_U\}$ as desired. $\square$

Let $K_0(\mathcal{B}_k)$ (resp. $K_0(\mathcal{P}_k)$) be the quotient of $G(\mathcal{B}_k)$ (resp. $G(\mathcal{P}_k)$) by the relation (2.2), and denote by $[X, \alpha]$ the element in the quotient represented by $\{X, \alpha\}$.

Lemma 2.7. *The ring multiplication* (2.1) *descends to* $K_0(\mathcal{B}_k)$ *(resp.* $K_0(\mathcal{P}_k)$*).*

Proof. Pick $\{X, \alpha\}, \{Y, \beta\} \in G(\mathcal{B}_k)$ (resp. $G(\mathcal{P}_k)$), and let $Z \subseteq X$ be a closed subvariety invariant under α with open complement $U := X \setminus Z$. Note that $Z \times Y$ is a closed subvariety of $X \times Y$ invariant under $\alpha \times \beta$, and its complement in $X \times Y$ equals $U \times Y$. Moreover, the restrictions of $\alpha \times \beta$ to $Z \times Y$ and $U \times Y$ coincides with $\alpha_Z \times \beta$ and $\alpha_U \times \beta$, respectively. As a consequence,

$$\{X \times Y, \alpha \times \beta\} \sim \{Z \times Y, \alpha_Z \times \beta\} + \{U \times Y, \alpha_U \times \beta\}.$$

Therefore,

$$\begin{aligned} \{X, \alpha\} \cdot \{Y, \beta\} = \{X \times Y, \alpha \times \beta\} &\sim \{Z \times Y, \alpha_Z \times \beta\} + \{U \times Y, \alpha_U \times \beta\} \\ = \{Z, \alpha_Z\} \cdot \{Y, \beta\} + \{U, \alpha_U\} \cdot \{Y, \beta\} &= (\{Z, \alpha_Z\} + \{U, \alpha_U\}) \cdot \{Y, \beta\}. \end{aligned}$$

We conclude that

$$[X, \alpha] \cdot [Y, \beta] = ([Z, \alpha_Z] + [U, \alpha_U]) \cdot [Y, \beta],$$

which completes the proof. $\square$

Definition 2.8. We call $K_0(\mathcal{B}_k)$ (resp. $K_0(\mathcal{P}_k)$) the *Grothendieck ring of birational self-maps* (resp. *Grothendieck ring of birational permutations*).

3 The parity homomorphism

Given a finite set A, let us denote by $\mathbf{s}\colon \mathrm{Sym}(A) \longrightarrow (\mathbb{Z}/2\mathbb{Z}, +, 0)$ the group homomorphism that maps a permutation to its parity.

Lemma 3.1. *Let A, B be finite sets and $\alpha \in \mathrm{Sym}(A)$, $\beta \in \mathrm{Sym}(B)$. Then the parity of $\alpha \times \beta$ acting on $A \times B$ satisfies* $\mathbf{s}(\alpha \times \beta) = |B| \cdot \mathbf{s}(\alpha) + |A| \cdot \mathbf{s}(\beta)$.

Proof. The action of $\alpha \times \beta$ equals the composition $(\alpha \times \mathrm{id}_B)(\mathrm{id}_A \times \beta)$. Therefore,

$$\begin{aligned}\mathbf{s}(\alpha \times \beta) &= \mathbf{s}((\alpha \times \mathrm{id}_B)(\mathrm{id}_A \times \beta)) = \mathbf{s}((\alpha \times \mathrm{id}_B)) + \mathbf{s}((\mathrm{id}_A \times \beta)) \\ &= \mathbf{s}(\alpha^{|B|}) + \mathbf{s}(\beta^{|A|}) = |B| \cdot \mathbf{s}(\alpha) + |A| \cdot \mathbf{s}(\beta).\end{aligned}$$

□

Theorem 3.2. *Over a finite field k, there exists a group homomorphism*

$$\mathsf{p}\colon K_0(\mathcal{P}_k) \longrightarrow (\mathbb{Z}/2\mathbb{Z}, +, 0)$$

such that $\mathsf{p}([X,\alpha)]) = 0$ if α acts on $X(k)$ as an even permutation and $\mathsf{p}([X,\alpha]) = 1$ if the action is odd. Moreover,

1. *given $\alpha, \alpha' \in \mathrm{BBir}_k(X)$, we have $\mathsf{p}([X,\alpha\alpha']) = \mathsf{p}([X,\alpha]) + \mathsf{p}([X,\alpha'])$, and*
2. *it satisfies the Leibniz-type relation*

$$\mathsf{p}([X,\alpha] \cdot [Y,\beta]) = |Y(k)| \cdot \mathsf{p}([X,\alpha]) + |X(k)| \cdot \mathsf{p}([Y,\beta]).$$

Proof. Note that an isomorphism $\varphi\colon (X,\alpha) \xrightarrow{\sim} (Y,\beta)$ implies that $\alpha = \varphi^{-1}\beta\varphi$. Because ϕ is an isomorphism, it induces a bijection between $X(k)$ and $Y(k)$, thus the above equation implies that the permutations induced by α on $X(k)$ and β on $Y(k)$ have the same parity. Therefore, assigning to each $(X,\alpha) \in G(\mathcal{P}_k)$ the parity of α acting on $X(k)$ well defines a group homomorphism

$$\mathsf{p}_G\colon G(\mathcal{P}_k) \longrightarrow (\mathbb{Z}/2\mathbb{Z}, +, 0)\ , \quad \mathsf{p}_G(\{X,\alpha\}) = \begin{cases} 0 & \text{if the action is even,} \\ 1 & \text{if the action is odd.}\end{cases}$$

Let $Z \subseteq X$ be a closed subvariety invariant under $\alpha \in \mathrm{BBir}_k(X)$ and let $U := X \setminus Z$ be the complement. Since the action of α on $X(k)$ is a multiplication of its restrictions to $Z(k)$ and $U(k)$, which are disjoint permutations, we have

$$\mathsf{p}_G(\{X,\alpha\}) = \mathsf{p}_G(\{Z,\alpha_Z\}) + \mathsf{p}_G(\{U,\alpha_U\}).$$

Therefore, p_G factors through $K_0(\mathcal{B}_k)$ via the homomorphism

$$\mathsf{p}\colon K_0(\mathcal{B}_k) \longrightarrow (\mathbb{Z}/2\mathbb{Z}, +, 0)\ , \quad \mathsf{p}([X,\alpha]) = \mathsf{p}_G(\{X,\alpha\}).$$

Property (1) is trivial as it reflects how parities of permutations change under compositions. On the other hand, Lemma 3.1 implies that

$$\mathsf{p}([X,\alpha] \cdot [Y,\beta]) = \mathsf{p}([X \times Y, \alpha \times \beta]) = |Y(k)| \cdot \mathsf{p}([X,\alpha]) + |X(k)| \cdot \mathsf{p}([Y,\beta])$$

which proves property (2). □

Remark 3.3. This construction could possibly be extended so that an object is a pair $[X,\alpha]$ where X is a variety over k and $\alpha\colon X \dashrightarrow X$ is a "locally open" rational map. The *dynamical degree* of such a rational map α was proved to be invariant under birational conjugations [DS05]. This may be used to construct a homomorphism that maps $[X,\alpha]$ to the dynamical degree of α.

Question 3.4. Over a number field k, how does a birational permutation interact with the height of a k-rational point?

4 Parity zeta function of automorphisms

Let X be a smooth projective variety over a finite field $k = \mathbb{F}_q$. The Hasse–Weil zeta function of X is defined as

$$\zeta(X;s) = \exp\left(\sum_{m=1}^{\infty} \frac{|X(\mathbb{F}_{q^m})|}{m} q^{-ms}\right).$$

This function is usually considered as a formal power series in $t := q^{-s}$, so one may define

$$Z(X;t) = \exp\left(\sum_{m=1}^{\infty} \frac{|X(\mathbb{F}_{q^m})|}{m} t^{m}\right).$$

The (proved) Weil conjecture asserts that $Z(X,t)$ is a rational function in t.

Let α be an automorphism of X. Then α induces a permutation on $X(\mathbb{F}_{q^m})$ for all $m \geq 1$. Define $\mathsf{p}_m(\alpha)$ to be the parity of α when acting on $X(\mathbb{F}_{q^m})$. By modifying the coefficients in the zeta function with the factor $(-1)^{\mathsf{p}_m}$, we obtain the *parity zeta function*

$$\zeta(X,\alpha;s) := \exp\left(\sum_{m\geq 1} \frac{(-1)^{\mathsf{p}_m(\alpha)}|X(\mathbb{F}_{q^m})|}{m} q^{-ms}\right).$$

This function encodes how the parity induced by α alters via extensions of the ground field. One may also consider

$$Z(X,\alpha;t) := \exp\left(\sum_{m\geq 1} \frac{(-1)^{\mathsf{p}_m(\alpha)}|X(\mathbb{F}_{q^m})|}{m} t^{m}\right).$$

Question 4.1. Is $Z(X,\alpha,t)$ a rational function in t?

Let $X^n := X \times \cdots \times X$ be the direct product of $n \geq 0$ copies of X with $X^0 := \operatorname{Spec}(k)$. The symmetric product $S^n(X)$ is defined as the quotient of X^n by the group of permutations on the factors. Kapranov's motivic zeta function is defined as

$$\zeta_{\mathrm{mot}}(X;t) := \sum_{n\geq 0} [S^n(X)] t^n \in 1 + tK_0(\mathcal{V}_k)[t].$$

It is known that, if X is a smooth, geometrically connected, projective curve over a perfect field k carrying a k-rational point, then $\zeta_{\mathrm{mov}}(X;t)$ is rational (modulo certain relations, see Remark 4.3). Moreover, if X has genus g, the rational function has the form

$$\zeta_{\mathrm{mov}}(X;t) = \frac{f(t)}{(1-t)(1-[\mathbb{A}^1]t)}$$

where $f \in K_0(\mathcal{V}_k)[t]$ is a polynomial of degree $\leq 2g$.

Every $\alpha \in \operatorname{Aut}_k(X)$ induces an automorphism α^n on X^n, where $\alpha^0 := \mathrm{id}_{\operatorname{Spec}(k)}$. One can verify that α^n commutes with the permutations on the factors, so it descends to an automorphism $S^n(\alpha)$ on $S^n(X)$. We define in a similar way that

$$\zeta_{\mathrm{mot}}(X,\alpha;t) := \sum_{n\geq 0} [S^n(X), S^n(\alpha)] t^n \in 1 + tK_0(\mathcal{P}_k)[t].$$

Question 4.2. Assume that X is a smooth, geometrically connected, projective curve over a perfect field k carrying a k-rational point. Let α be an automorphism of X. Is $\zeta_{\mathrm{mot}}(X,\alpha;t)$ rational (modulo some suitable relation)?

Remark 4.3. The definitions in the last part about the motivic zeta functions are not in their original or most suitable forms. Let $R(\mathcal{V}_k) \subseteq K_0(\mathcal{V}_k)$ be the subgroup generated by elements of the form $[X]-[Y]$ where X and Y admit a morphism $h\colon X \to Y$ that is surjective and *radicial* (see [Mus11, A.3] for the definition). In the discussion about the rationality of $\zeta_{\mathrm{mov}}(X;t)$, the ring $K_0(\mathcal{V}_k)$ should be replaced by the quotient

$$\widetilde{K}_0(\mathcal{V}_k) := K_0(\mathcal{V}_k)/R(\mathcal{V}_k).$$

5 Connections to Ekedahl's work

Two varieties X and Y are *stably birational* if $X \times \mathbb{P}^n$ is birational to $Y \times \mathbb{P}^m$ for some $n, m \geq 0$. The materials in the previous sections are partially motivated by the following result: if two varieties X and Y over $\mathbb{F}_q$ are stably birational, then [Eke83]

$$|X(\mathbb{F}_q)| \equiv |Y(\mathbb{F}_q)| \mod q.$$

In general, one may anticipate a motivic version of this relation:

Conjecture 5.1. If two varieties X and Y are stably birational, then

$$[X] \equiv [Y] \mod [\mathbb{A}^1]$$

in the Grothendieck ring $K_0(\mathcal{V}_k)$.

In fact, this conjecture holds over an algebraically closed field of characteristic zero due to the weak factorization theorem [LL03].

Suppose that $k = \mathbb{F}_{2^r}$ where $r \geq 2$. It seems that Conjecture 5.1 could potentially provide an easy way to prove [ALNZ22, Theorem 1.1] and its generalization. The reason is as follows: consider the ring homomorphism

$$\mathsf{f}\colon K_0(\mathcal{P}_k) \longrightarrow K_0(\mathcal{V}_k) : [X,\alpha] \longmapsto [X].$$

If (X,α) and (Y,β) are birational to each other, then Conjecture 5.1 implies that

$$[X,\alpha] \equiv [Y,\beta] \mod \mathsf{f}^{-1}([\mathbb{A}^1]).$$

Now, we make the naive assumption:

$$[X,\alpha] \equiv [Y,\beta] \mod [\mathbb{A}^1,\delta] \quad \text{for some} \quad \delta \in \mathrm{Aut}_k(\mathbb{A}^1). \tag{5.1}$$

Every $\delta \in \mathrm{Aut}_k(\mathbb{A}^1)$ extends to an automorphism of $\mathbb{P}^1$ fixing a point. Thus δ acts as an even permutation on $\mathbb{A}^1(k)$ by [ALNZ22, Proposition 3.5]. This implies that for every $[Z,\gamma] \in K_0(\mathcal{V}_k)$, it holds that

$$\begin{aligned}\mathsf{p}([\mathbb{A}^1,\delta][Z,\gamma]) &= |Z(k)|\cdot \mathsf{p}([\mathbb{A}^1,\delta]) + |\mathbb{A}^1(k)|\cdot \mathsf{p}([Z,\gamma]) \\ &= |Z(k)|\cdot 0 + 0\cdot \mathsf{p}([Z,\gamma]) = 0.\end{aligned}$$

Together with (5.1), this implies that $\mathsf{p}([X,\alpha]) = \mathsf{p}([Y,\beta])$.

6 Measuring the difference of exceptional loci

Let $\mathcal{C}_k$ be the category of isomorphism classes of birational maps $[\alpha\colon X \dashrightarrow Y]$. In this category, a morphism between two objects $[\alpha\colon X \dashrightarrow Y]$ and $[\alpha'\colon X' \dashrightarrow Y']$ is a pair of morphisms $(f\colon X \longrightarrow X', g\colon Y \longrightarrow Y')$ such that the diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed, "\alpha"'] & X' \arrow[d, dashed, "\alpha'"] \\ Y \arrow[r, "g"] & Y'. \end{tikzcd}$$

One can define $K_0(\mathcal{C}_k)$, the Grothendieck ring associated with $\mathcal{C}_k$, in a similar way as defining $K_0(\mathcal{P}_k)$. More precisely, the ring $K_0(\mathcal{C}_k)$ is the free abelian group generated by isomorphism classes of objects in $\mathcal{C}_k$ subject to the relation

$$[\alpha\colon X \dashrightarrow Y] \sim [\alpha|_Z\colon Z \dashrightarrow W] + [\alpha|_{X\setminus Z}\colon X \setminus Z \dashrightarrow Y \setminus W]$$

for every closed subvariety $Z \subseteq X$ such that the restriction $\alpha|_Z$ is birational onto its image $W := \alpha(Z) \subseteq Y$. The multiplication on $K_0(\mathcal{C}_k)$ is given by

$$[\alpha\colon X \dashrightarrow Y] \cdot [\alpha'\colon X' \dashrightarrow Y'] = [\alpha \times \alpha'\colon X \times X' \dashrightarrow Y \times Y'].$$

Theorem 6.1 (Shinder–Lin)**.** *To each object $[\alpha\colon X \dashrightarrow Y] \in \mathcal{C}_k$, one can assign an invariant*

$$c(\alpha\colon X \dashrightarrow Y) := \sum_{F\in \mathrm{Ex}(\alpha^{-1})} [F] - \sum_{E\in \mathrm{Ex}(\alpha)} [E] \ \in \ K_0(\mathcal{V}_k)$$

such that for any birational maps $\alpha\colon X \dashrightarrow Y$ and $\beta\colon Y \dashrightarrow Z$, we have

$$c(\beta \circ \alpha) = c(\beta) + c(\alpha).$$

Question 6.2. Does the map $c\colon \mathcal{C}_k \longrightarrow K_0(\mathcal{V}_k)$ factor through $K_0(\mathcal{C}_k)$?

The map $\alpha \times \alpha'\colon X \times X' \dashrightarrow Y \times Y'$ factors in two ways as follows

$$\begin{tikzcd} & Y \times X' \arrow[dr, dashed, "\mathrm{id}_Y \times \alpha'"] & \\ X \times X' \arrow[ur, dashed, "\alpha \times \mathrm{id}_{X'}"] \arrow[dr, dashed, "\mathrm{id}_X \times \alpha'"'] & & Y \times Y' \\ & X \times Y' \arrow[ur, dashed, "\alpha \times \mathrm{id}_{Y'}"'] & \end{tikzcd}$$

Combining this with Theorem 6.1 gives

$$\begin{aligned} c(\alpha \times \alpha') &= c(\alpha \times \mathrm{id}_{X'}) + c(\mathrm{id}_Y \times \alpha') = c(\alpha) \cdot [X'] + [Y] \cdot c(\alpha') \\ &= c(\mathrm{id}_X \times \alpha') + c(\alpha \times \mathrm{id}_{Y'}) = [X] \cdot c(\alpha') + c(\alpha) \cdot [Y']. \end{aligned}$$

Now assume that $\alpha = \gamma \circ \beta^{-1}$ (resp. $\alpha' = \gamma' \circ \beta'^{-1}$) where β and γ (resp. β' and γ') are simple blowups, so that

$$c(\alpha) = [F] - [E] \qquad \text{and} \qquad c(\alpha') = [F'] - [E'].$$

It follows that

$$c(\alpha \times \alpha') = c(\alpha) \cdot [Z'] + [Z] \cdot c(\alpha') + [E] \cdot [E'] - [F] \cdot [F']$$

where Z and Z' are the graphs:

$$\begin{tikzcd} & Z \arrow[dl, "\beta"'] \arrow[dr, "\gamma"] & \\ X \arrow[rr, dashed, "\alpha", "\sim"'] & & Y \end{tikzcd} \qquad \begin{tikzcd} & Z' \arrow[dl, "\beta'"'] \arrow[dr, "\gamma'"] & \\ X' \arrow[rr, dashed, "\alpha'", "\sim"'] & & Y'. \end{tikzcd}$$

References

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