

How to do a factor analysis with the psych package

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Contents

0.1	Jump starting the <i>psych</i> package to do factor analysis –a guide for the impatient	2
1	Overview of this and related documents	3
2	Input the data	3
2.1	Data input from the clipboard	3
2.2	Basic descriptive statistics	4
3	Item and scale analysis	5
3.1	Dimension reduction through factor analysis and cluster analysis	5
3.1.1	Minimum Residual Factor Analysis	7
3.1.2	Principal Axis Factor Analysis	8
3.1.3	Weighted Least Squares Factor Analysis	8
3.1.4	Principal Components analysis (PCA)	14
3.1.5	Hierarchical and bi-factor solutions	14
3.1.6	Item Cluster Analysis: iclust	18
3.2	Confidence intervals using bootstrapping techniques	21
3.3	Comparing factor/component/cluster solutions	21
3.4	Determining the number of dimensions to extract.	27
3.4.1	Very Simple Structure	28
3.4.2	Parallel Analysis	30
3.5	Factor extension	30
4	Psychometric Theory	33
5	SessionInfo	33

0.1 Jump starting the *psych* package to do factor analysis –a guide for the impatient

You have installed *psych* (section ??) and you want to use it for a factor analysis without reading much more. What should you do?

If you have already read the [getting started](#) documentation and have R (?) and the *psych* package ([Revelle, 2012](#)) installed then just

1. Activate the *psych* package:

```
library(psych)
```

2. Input your data (section 2.1). Go to your friendly text editor or data manipulation program (e.g., Excel) and copy the data to the clipboard. Include a first line that has the variable labels. Paste it into *psych* using the `read.clipboard.tab` command:

```
myData <- read.clipboard.tab()
```

3. Make sure that what you just read is right. Describe it (section 2.2) and perhaps look at the first and last few lines:

```
describe(myData)  
headTail(myData)
```

4. Test for the number of factors in your data using parallel analysis (`fa.parallel`, section 3.4.2) or Very Simple Structure (`vss`, 3.4.1) .

```
fa.parallel(myData)  
vss(myData)
```

5. Factor analyze (see section 3.1) the data with a specified number of factors (the default is 1), the default method is minimum residual, the default rotation for more than one factor is oblimin. There are many more possibilities (see sections 3.1.1-3.1.3). Compare the solution to a hierarchical cluster analysis using the ICLUST algorithm ([Revelle, 1979](#)) (see section 3.1.6). Also consider a hierarchical factor solution to find coefficient ω (see 3.1.5).

```
fa(myData)  
iclust(myData)  
omega(myData)
```

You might find reading this entire [over view vignette](#) helpful to get a broader understanding of what can be done in R using the *psych*. Remember that the help command (?) is available for every function. Try running the examples for each help page.

1 Overview of this and related documents

This is one of a number of “How To use the *psych* package to do X” documents. The entire set may be found at personality-project.org/r/psych. Some of these are extracted from the [overview of psych](#) vignette.

2 Input the data

A number of *psych* functions facilitate the entry of data and finding basic descriptive statistics.

Remember, to run any of the *psych* functions, it is necessary to make the package active by using the `library` command:

```
> library(psych)
```

The other packages, once installed, will be called automatically by *psych*.

It is possible to automatically load *psych* and other functions by creating and then saving a “.First” function: e.g.,

```
.First <- function(x) {library(psych)}
```

2.1 Data input from the clipboard

There are of course many ways to enter data into R. Reading from a local file using `read.table` is perhaps the most preferred. However, many users will enter their data in a text editor or spreadsheet program and then want to copy and paste into R. This may be done by using `read.table` and specifying the input file as “clipboard” (PCs) or “pipe(pbpaste)” (Macs). Alternatively, the `read.clipboard` set of functions are perhaps more user friendly:

`read.clipboard` is the base function for reading data from the clipboard.

`read.clipboard.csv` for reading text that is comma delimited.

`read.clipboard.tab` for reading text that is tab delimited (e.g., copied directly from an Excel file).

`read.clipboard.lower` for reading input of a lower triangular matrix with or without a diagonal. The resulting object is a square matrix.

`read.clipboard.upper` for reading input of an upper triangular matrix.

`read.clipboard.fwf` for reading in fixed width fields (some very old data sets)

For example, given a data set copied to the clipboard from a spreadsheet, just enter the command

```
> my.data <- read.clipboard()
```

This will work if every data field has a value and even missing data are given some values (e.g., NA or -999). If the data were entered in a spreadsheet and the missing values were just empty cells, then the data should be read in as a tab delimited or by using the `read.clipboard.tab` function.

```
> my.data <- read.clipboard(sep="\t")    #define the tab option, or
> my.tab.data <- read.clipboard.tab()    #just use the alternative function
```

For the case of data in fixed width fields (some old data sets tend to have this format), copy to the clipboard and then specify the width of each field (in the example below, the first variable is 5 columns, the second is 2 columns, the next 5 are 1 column the last 4 are 3 columns).

```
> my.data <- read.clipboard.fwf(widths=c(5,2,rep(1,5),rep(3,4)))
```

2.2 Basic descriptive statistics

Before doing any analysis, it is important to make sure that your data are what you think they are. You should examine the basic descriptive statistics using the `describe` function.

Consider the data set `sat.act` which includes data from 700 web based participants on 3 demographic variables and 3 ability measures.

`describe` reports means, standard deviations, medians, min, max, range, skew, kurtosis and standard errors for integer or real data. Non-numeric data, although the statistics are meaningless, will be treated as if numeric (based upon the categorical coding of the data), and will be flagged with an *.

`describeBy` reports descriptive statistics broken down by some categorizing variable (e.g., gender, age, etc.)

```
> library(psych)
> data(sat.act)
> describe(sat.act) #basic descriptive statistics
```

	var	n	mean	sd	median	trimmed	mad	min	max	range	skew	kurtosis	se
gender	1	700	1.65	0.48	2	1.68	0.00	1	2	1	-0.61	-1.62	0.02
education	2	700	3.16	1.43	3	3.31	1.48	0	5	5	-0.68	-0.07	0.05
age	3	700	25.59	9.50	22	23.86	5.93	13	65	52	1.64	2.42	0.36
ACT	4	700	28.55	4.82	29	28.84	4.45	3	36	33	-0.66	0.53	0.18

SATV	5	700	612.23	112.90	620	619.45	118.61	200	800	600	-0.64	0.33	4.27
SATQ	6	687	610.22	115.64	620	617.25	118.61	200	800	600	-0.59	-0.02	4.41

3 Item and scale analysis

The main functions in the *psych* package are for analyzing the structure of items and of scales and for finding various estimates of scale reliability. These may be considered as problems of dimension reduction (e.g., factor analysis, cluster analysis, principal components analysis) and of forming and estimating the reliability of the resulting composite scales.

3.1 Dimension reduction through factor analysis and cluster analysis

Parsimony of description has been a goal of science since at least the famous dictum commonly attributed to William of Ockham to not multiply entities beyond necessity¹. The goal for parsimony is seen in psychometrics as an attempt either to describe (components) or to explain (factors) the relationships between many observed variables in terms of a more limited set of components or latent factors.

The typical data matrix represents multiple items or scales usually thought to reflect fewer underlying constructs². At the most simple, a set of items can be thought to represent a random sample from one underlying domain or perhaps a small set of domains. The question for the psychometrician is how many domains are represented and how well does each item represent the domains. Solutions to this problem are examples of *factor analysis* (FA), *principal components analysis* (PCA), and *cluster analysis* (CA). All of these procedures aim to reduce the complexity of the observed data. In the case of FA, the goal is to identify fewer underlying constructs to explain the observed data. In the case of PCA, the goal can be mere data reduction, but the interpretation of components is frequently done in terms similar to those used when describing the latent variables estimated by FA. Cluster analytic techniques, although usually used to partition the subject space rather than the variable space, can also be used to group variables to reduce the complexity of the data by forming fewer and more homogeneous sets of tests or items.

At the data level the data reduction problem may be solved as a *Singular Value Decomposition* of the original matrix, although the more typical solution is to find either the

¹Although probably neither original with Ockham nor directly stated by him (Thorburn, 1918), Ockham's razor remains a fundamental principal of science.

²Cattell (1978) as well as MacCallum et al. (2007) argue that the data are the result of many more factors than observed variables, but are willing to estimate the major underlying factors.

principal components or *factors* of the covariance or correlation matrices. Given the pattern of regression weights from the variables to the components or from the factors to the variables, it is then possible to find (for components) individual *component* or *cluster scores* or estimate (for factors) *factor scores*.

Several of the functions in *psych* address the problem of data reduction.

fa incorporates five alternative algorithms: *minres factor analysis*, *principal axis factor analysis*, *weighted least squares factor analysis*, *generalized least squares factor analysis* and *maximum likelihood factor analysis*. That is, it includes the functionality of three other functions that will be eventually phased out.

factor.minres Minimum residual factor analysis is a least squares, iterative solution to the factor problem. minres attempts to minimize the residual (off-diagonal) correlation matrix. It produces solutions similar to maximum likelihood solutions, but will work even if the matrix is singular.

factor.pa Principal Axis factor analysis is a least squares, iterative solution to the factor problem. PA will work for cases where maximum likelihood techniques (**factanal**) will not work. The original communality estimates are either the squared multiple correlations (**smc**) for each item or 1.

factor.wls Weighted least squares factor analysis is a least squares, iterative solution to the factor problem. It minimizes the (weighted) squared residual matrix. The weights are based upon the independent contribution of each variable.

principal Principal Components Analysis reports the largest *n* eigen vectors rescaled by the square root of their eigen values.

factor.congruence The congruence between two factors is the cosine of the angle between them. This is just the cross products of the loadings divided by the sum of the squared loadings. This differs from the correlation coefficient in that the mean loading is not subtracted before taking the products. **factor.congruence** will find the cosines between two (or more) sets of factor loadings.

vss Very Simple Structure [Revelle and Rocklin \(1979\)](#) applies a goodness of fit test to determine the optimal number of factors to extract. It can be thought of as a quasi-confirmatory model, in that it fits the very simple structure (all except the biggest *c* loadings per item are set to zero where *c* is the level of complexity of the item) of a factor pattern matrix to the original correlation matrix. For items where the model is usually of complexity one, this is equivalent to making all except the largest loading for each item 0. This is typically the solution that the user wants to interpret. The analysis includes the MAP criterion of [Velicer \(1976\)](#) and a χ^2 estimate.

fa.parallel The parallel factors technique compares the observed eigen values of a cor-

relation matrix with those from random data.

`fa.plot` will plot the loadings from a factor, principal components, or cluster analysis (just a call to plot will suffice). If there are more than two factors, then a SPLOM of the loadings is generated.

`fa.diagram` replaces `fa.graph` and will draw a path diagram representing the factor structure. It does not require `Rgraphviz` and thus is probably preferred.

`fa.graph` requires `Rgraphviz` and will draw a graphic representation of the factor structure. If factors are correlated, this will be represented as well.

`iclust` is meant to do item cluster analysis using a hierarchical clustering algorithm specifically asking questions about the reliability of the clusters (Revelle, 1979). Clusters are formed until either coefficient α Cronbach (1951) or β Revelle (1979) fail to increase.

3.1.1 Minimum Residual Factor Analysis

The factor model is an approximation of a correlation matrix by a matrix of lower rank. That is, can the correlation matrix, ${}_n\vec{R}_n$ be approximated by the product of a factor matrix, ${}_n\vec{F}_k$ and its transpose plus a diagonal matrix of uniqueness.

$$R = FF' + U^2 \quad (1)$$

The maximum likelihood solution to this equation is found by `factanal` in the `stats` package. Five alternatives are provided in `psych`, all of them are included in the `fa` function and are called by specifying the factor method (e.g., `fm="minres"`, `fm="pa"`, `fm="wls"`, `fm="gls"` and `fm="ml"`). In the discussion of the other algorithms, the calls shown are to the `fa` function specifying the appropriate method.

`factor.minres` attempts to minimize the off diagonal residual correlation matrix by adjusting the eigen values of the original correlation matrix. This is similar to what is done in `factanal`, but uses an ordinary least squares instead of a maximum likelihood fit function. The solutions tend to be more similar to the MLE solutions than are the `factor.pa` solutions. `min.res` is the default for the `fa` function.

A classic data set, collected by Thurstone and Thurstone (1941) and then reanalyzed by Bechtoldt (1961) and discussed by McDonald (1999), is a set of 9 cognitive variables with a clear bi-factor structure Holzinger and Swineford (1937). The minimum residual solution was transformed into an oblique solution using the default option on rotate which uses an oblimin transformation (Table 1). Alternative rotations and transformations include "none", "varimax", "quartimax", "bentlerT", and "geominT" (all of which are orthogonal

rotations). as well as “promax”, “oblimin”, “simplimax”, “bentlerQ”, and “geominQ” and “cluster” which are possible oblique transformations of the solution. The default is to do a oblimin transformation, although prior versions defaulted to varimax. The measures of factor adequacy reflect the multiple correlations of the factors with the best fitting linear regression estimates of the factor scores (Grice, 2001).

3.1.2 Principal Axis Factor Analysis

An alternative, least squares algorithm, `factor.pa`, does a Principal Axis factor analysis by iteratively doing an eigen value decomposition of the correlation matrix with the diagonal replaced by the values estimated by the factors of the previous iteration. This OLS solution is not as sensitive to improper matrices as is the maximum likelihood method, and will sometimes produce more interpretable results. It seems as if the SAS example for PA uses only one iteration. Setting the `max.iter` parameter to 1 produces the SAS solution.

The solutions from the `fa`, the `factor.minres` and `factor.pa` as well as the `principal` functions can be rotated or transformed with a number of options. Some of these call the *GPArotation* package. Orthogonal rotations are `varimax` and `quartimax`. Oblique transformations include `oblimin`, `quartimin` and then two targeted rotation functions `Promax` and `target.rot`. The latter of these will transform a loadings matrix towards an arbitrary target matrix. The default is to transform towards an independent cluster solution.

Using the Thurstone data set, three factors were requested and then transformed into an independent clusters solution using `target.rot` (Table 2).

3.1.3 Weighted Least Squares Factor Analysis

Similar to the minres approach of minimizing the squared residuals, factor method “wls” weights the squared residuals by their uniquenesses. This tends to produce slightly smaller overall residuals. In the example of weighted least squares, the output is shown by using the `print` function with the `cut` option set to 0. That is, all loadings are shown (Table 3).

The unweighted least squares solution may be shown graphically using the `fa.plot` function which is called by the generic `plot` function (Figure 1. Factors were transformed obliquely using a oblimin. These solutions may be shown as item by factor plots (Figure 1 or by a structure diagram (Figure 2.

A comparison of these three approaches suggests that the minres solution is more similar to a maximum likelihood solution and fits slightly better than the pa or wls solutions. Comparisons with SPSS suggest that the pa solution matches the SPSS OLS solution, but

Table 1: Three correlated factors from the Thurstone 9 variable problem. By default, the solution is transformed obliquely using oblimin. The extraction method is (by default) minimum residual.

```
> f3t <- fa(Thurstone,3,n.obs=213)
> f3t
Factor Analysis using method = minres
Call: fa(r = Thurstone, nfactors = 3, n.obs = 213)
Standardized loadings (pattern matrix) based upon correlation matrix
```

	MR1	MR2	MR3	h2	u2
Sentences	0.91	-0.04	0.04	0.82	0.18
Vocabulary	0.89	0.06	-0.03	0.84	0.16
Sent.Completion	0.83	0.04	0.00	0.73	0.27
First.Letters	0.00	0.86	0.00	0.73	0.27
4.Letter.Words	-0.01	0.74	0.10	0.63	0.37
Suffixes	0.18	0.63	-0.08	0.50	0.50
Letter.Series	0.03	-0.01	0.84	0.72	0.28
Pedigrees	0.37	-0.05	0.47	0.50	0.50
Letter.Group	-0.06	0.21	0.64	0.53	0.47

```

MR1 MR2 MR3
SS loadings      2.64 1.86 1.50
Proportion Var   0.29 0.21 0.17
Cumulative Var   0.29 0.50 0.67
Proportion Explained 0.44 0.31 0.25
Cumulative Proportion 0.44 0.75 1.00

With factor correlations of
MR1 MR2 MR3
MR1 1.00 0.59 0.54
MR2 0.59 1.00 0.52
MR3 0.54 0.52 1.00

Test of the hypothesis that 3 factors are sufficient.

The degrees of freedom for the null model are 36 and the objective function was 5.2 with Chi Square of 1081.97
The degrees of freedom for the model are 12 and the objective function was 0.01

The root mean square of the residuals (RMSR) is 0.01
The df corrected root mean square of the residuals is 0.02

The harmonic number of observations is 213 with the empirical chi square 0.58 with prob < 1
The total number of observations was 213 with MLE Chi Square = 2.82 with prob < 1

Tucker Lewis Index of factoring reliability = 1.027
RMSEA index = 0 and the 90 % confidence intervals are NA NA
BIC = -61.51
Fit based upon off diagonal values = 1
Measures of factor score adequacy
```

	MR1	MR2	MR3
Correlation of scores with factors	0.96	0.92	0.90
Multiple R square of scores with factors	0.93	0.85	0.81
Minimum correlation of possible factor scores	0.86	0.71	0.63

Table 2: The 9 variable problem from Thurstone is a classic example of factoring where there is a higher order factor, g, that accounts for the correlation between the factors. The extraction method was principal axis. The transformation was a targeted transformation to a simple cluster solution.

```
> f3 <- fa(Thurstone,3,n.obs = 213,fm="pa")
> f3o <- target.rot(f3)
> f3o
Call: NULL
Standardized loadings (pattern matrix) based upon correlation matrix
```

	PA1	PA2	PA3	h2	u2
Sentences	0.89	-0.03	0.07	0.81	0.19
Vocabulary	0.89	0.07	0.00	0.80	0.20
Sent.Completion	0.83	0.04	0.03	0.70	0.30
First.Letters	-0.02	0.85	-0.01	0.73	0.27
4.Letter.Words	-0.05	0.74	0.09	0.57	0.43
Suffixes	0.17	0.63	-0.09	0.43	0.57
Letter.Series	-0.06	-0.08	0.84	0.69	0.31
Pedigrees	0.33	-0.09	0.48	0.37	0.63
Letter.Group	-0.14	0.16	0.64	0.45	0.55

	PA1	PA2	PA3
SS loadings	2.45	1.72	1.37
Proportion Var	0.27	0.19	0.15
Cumulative Var	0.27	0.46	0.62
Proportion Explained	0.44	0.31	0.25
Cumulative Proportion	0.44	0.75	1.00

	PA1	PA2	PA3
PA1	1.00	0.02	0.08
PA2	0.02	1.00	0.09
PA3	0.08	0.09	1.00

Table 3: The 9 variable problem from Thurstone is a classic example of factoring where there is a higher order factor, g, that accounts for the correlation between the factors. The factors were extracted using a weighted least squares algorithm. All loadings are shown by using the cut=0 option in the `print.psych` function.

```
> f3w <- fa(Thurstone,3,n.obs = 213,fm="wls")
> print(f3w,cut=0,digits=3)
Factor Analysis using method = wls
Call: fa(r = Thurstone, nfactors = 3, n.obs = 213, fm = "wls")
Standardized loadings (pattern matrix) based upon correlation matrix
```

	WLS1	WLS2	WLS3	h2	u2
Sentences	0.905	-0.034	0.040	0.822	0.178
Vocabulary	0.890	0.066	-0.031	0.835	0.165
Sent.Completion	0.833	0.034	0.007	0.735	0.265
First.Letters	-0.002	0.855	0.003	0.731	0.269
4.Letter.Words	-0.016	0.743	0.106	0.629	0.371
Suffixes	0.180	0.626	-0.082	0.496	0.504
Letter.Series	0.033	-0.015	0.838	0.719	0.281
Pedigrees	0.381	-0.051	0.464	0.505	0.495
Letter.Group	-0.062	0.209	0.632	0.527	0.473

```

WLS1 WLS2 WLS3
SS loadings      2.647 1.864 1.488
Proportion Var   0.294 0.207 0.165
Cumulative Var    0.294 0.501 0.667
Proportion Explained 0.441 0.311 0.248
Cumulative Proportion 0.441 0.752 1.000

With factor correlations of
WLS1 WLS2 WLS3
WLS1 1.000 0.591 0.535
WLS2 0.591 1.000 0.516
WLS3 0.535 0.516 1.000

Test of the hypothesis that 3 factors are sufficient.

The degrees of freedom for the null model are 36 and the objective function was 5.198 with Chi Square of 1081.968
The degrees of freedom for the model are 12 and the objective function was 0.014

The root mean square of the residuals (RMSR) is 0.006
The df corrected root mean square of the residuals is 0.014

The harmonic number of observations is 213 with the empirical chi square 0.531 with prob < 1
The total number of observations was 213 with MLE Chi Square = 2.886 with prob < 0.996

Tucker Lewis Index of factoring reliability = 1.0264
RMSEA index = 0 and the 90 % confidence intervals are NA NA
BIC = -61.45
Fit based upon off diagonal values = 1
Measures of factor score adequacy
```

	WLS1	WLS2	WLS3
Correlation of scores with factors	0.964	0.923	0.902
Multiple R square of scores with factors	0.929	0.853	0.814
Minimum correlation of possible factor scores	0.858	0.706	0.627

```
> plot(f3t)
```

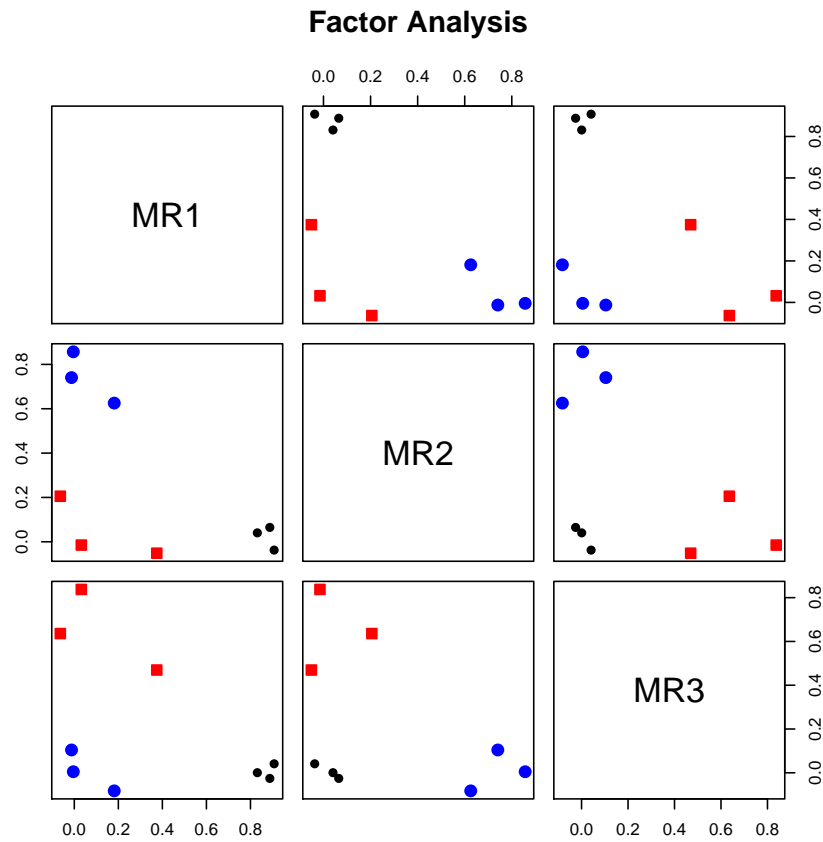


Figure 1: A graphic representation of the 3 oblique factors from the Thurstone data using `plot`. Factors were transformed to an oblique solution using the `oblimin` function from the `GPArotation` package.

```
> fa.diagram(f3t)
```

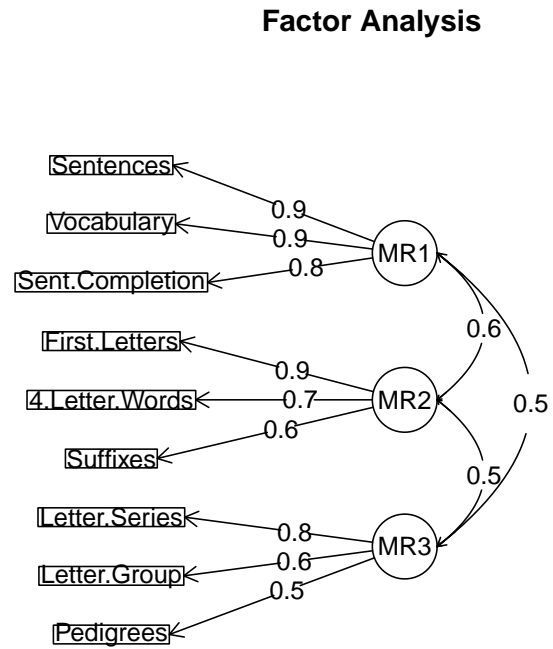


Figure 2: A graphic representation of the 3 oblique factors from the Thurstone data using `fa.diagram`. Factors were transformed to an oblique solution using `oblimin`.

that the minres solution is slightly better. At least in one test data set, the weighted least squares solutions, although fitting equally well, had slightly different structure loadings. Note that the rotations used by SPSS will sometimes use the “Kaiser Normalization”. By default, the rotations used in psych do not normalize, but this can be specified as an option in `fa`.

3.1.4 Principal Components analysis (PCA)

An alternative to factor analysis, which is unfortunately frequently confused with *factor analysis*, is *principal components analysis*. Although the goals of *PCA* and *FA* are similar, *PCA* is a descriptive model of the data, while *FA* is a structural model. Psychologists typically use *PCA* in a manner similar to factor analysis and thus the `principal` function produces output that is perhaps more understandable than that produced by `princomp` in the *stats* package. Table 4 shows a *PCA* of the Thurstone 9 variable problem rotated using the `Promax` function. Note how the loadings from the factor model are similar but smaller than the principal component loadings. This is because the *PCA* model attempts to account for the entire variance of the correlation matrix, while *FA* accounts for just the *common variance*. This distinction becomes most important for small correlation matrices. Also note how the goodness of fit statistics, based upon the residual off diagonal elements, is much worse than the `fa` solution.

3.1.5 Hierarchical and bi-factor solutions

For a long time structural analysis of the ability domain have considered the problem of factors that are themselves correlated. These correlations may themselves be factored to produce a higher order, general factor. An alternative (Holzinger and Swineford, 1937; Jensen and Weng, 1994) is to consider the general factor affecting each item, and then to have group factors account for the residual variance. Exploratory factor solutions to produce a hierarchical or a bifactor solution are found using the `omega` function. This technique has more recently been applied to the personality domain to consider such things as the structure of neuroticism (treated as a general factor, with lower order factors of anxiety, depression, and aggression).

Consider the 9 Thurstone variables analyzed in the prior factor analyses. The correlations between the factors (as shown in Figure 2 can themselves be factored. This results in a higher order factor model (Figure 3). An alternative solution is to take this higher order model and then solve for the general factor loadings as well as the loadings on the residualized lower order factors using the *Schmid-Leiman* procedure. (Figure 4). Yet another solution is to use structural equation modeling to directly solve for the general and group factors.

Table 4: The Thurstone problem can also be analyzed using Principal Components Analysis. Compare this to Table 2. The loadings are higher for the PCA because the model accounts for the unique as well as the common variance. The fit of the off diagonal elements, however, is much worse than the **fa** results.

```
> p3p <-principal(Thurstone,3,n.obs = 213,rotate="Promax")
> p3p
Principal Components Analysis
Call: principal(r = Thurstone, nfactors = 3, rotate = "Promax", n.obs = 213)
Standardized loadings (pattern matrix) based upon correlation matrix
```

	PC1	PC2	PC3	h2	u2
Sentences	0.92	0.01	0.01	0.86	0.14
Vocabulary	0.90	0.10	-0.05	0.86	0.14
Sent.Completion	0.91	0.04	-0.04	0.83	0.17
First.Letters	0.01	0.84	0.07	0.78	0.22
4.Letter.Words	-0.05	0.81	0.17	0.75	0.25
Suffixes	0.18	0.79	-0.15	0.70	0.30
Letter.Series	0.03	-0.03	0.88	0.78	0.22
Pedigrees	0.45	-0.16	0.57	0.67	0.33
Letter.Group	-0.19	0.19	0.86	0.75	0.25

	PC1	PC2	PC3
SS loadings	2.83	2.19	1.96
Proportion Var	0.31	0.24	0.22
Cumulative Var	0.31	0.56	0.78
Proportion Explained	0.41	0.31	0.28
Cumulative Proportion	0.41	0.72	1.00


```
With component correlations of
  PC1  PC2  PC3
PC1 1.00 0.51 0.53
PC2 0.51 1.00 0.44
PC3 0.53 0.44 1.00

Test of the hypothesis that 3 components are sufficient.

The degrees of freedom for the null model are 36 and the objective function was 5.2
The degrees of freedom for the model are 12 and the objective function was 0.62
The total number of observations was 213 with MLE Chi Square = 127.9 with prob < 1.6e-21

Fit based upon off diagonal values = 0.98
```

```

> om.h <- omega(Thurstone,n.obs=213,sl=FALSE)
> op <- par(mfrow=c(1,1))

```

Omega

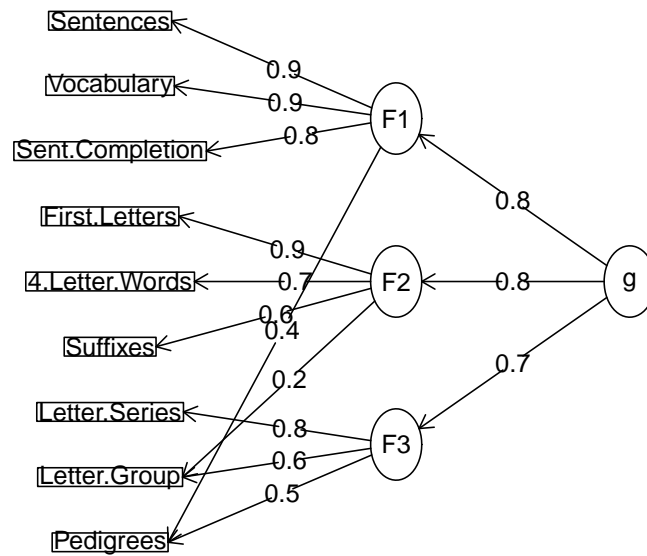


Figure 3: A higher order factor solution to the Thurstone 9 variable problem


```
> om <- omega(Thurstone,n.obs=213)
```

Omega

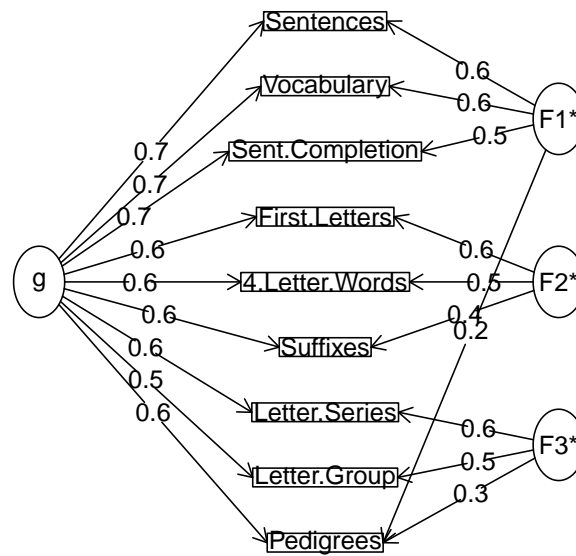


Figure 4: A bifactor factor solution to the Thurstone 9 variable problem

Yet another approach to the bifactor structure is to use the `bifactor` rotation function in either *psych* or in *GPArotation*. This does the rotation discussed in [Jennrich and Bentler \(2011\)](#).

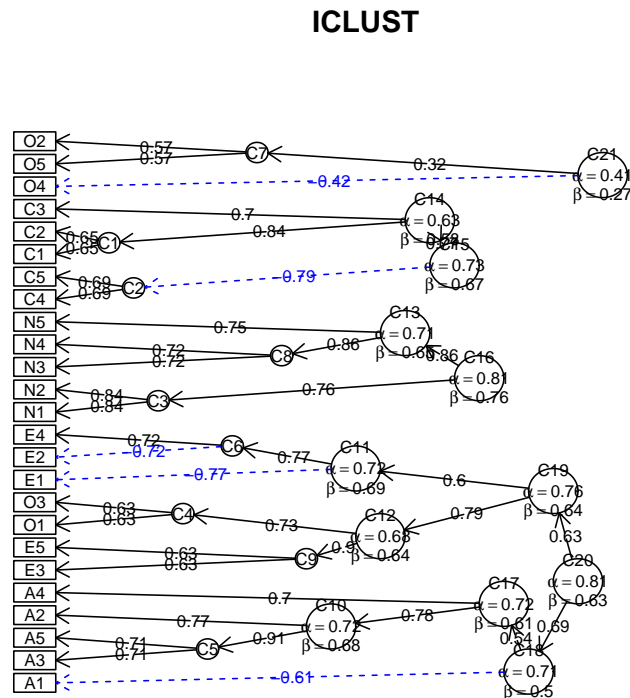
3.1.6 Item Cluster Analysis: `iclust`

An alternative to factor or components analysis is *cluster analysis*. The goal of cluster analysis is the same as factor or components analysis (reduce the complexity of the data and attempt to identify homogeneous subgroupings). Mainly used for clustering people or objects (e.g., projectile points if an anthropologist, DNA if a biologist, galaxies if an astronomer), clustering may be used for clustering items or tests as well. Introduced to psychologists by [Tryon \(1939\)](#) in the 1930's, the cluster analytic literature exploded in the 1970s and 1980s ([Blashfield, 1980](#); [Blashfield and Aldenderfer, 1988](#); [Everitt, 1974](#); [Hartigan, 1975](#)). Much of the research is in taxonomic applications in biology ([Sneath and Sokal, 1973](#); [Sokal and Sneath, 1963](#)) and marketing ([Cooksey and Soutar, 2006](#)) where clustering remains very popular. It is also used for taxonomic work in forming clusters of people in family ([Henry et al., 2005](#)) and clinical psychology ([Martinet and Ferrand, 2007](#); [Mun et al., 2008](#)). Interestingly enough it has had limited applications to psychometrics. This is unfortunate, for as has been pointed out by e.g. ([Tryon, 1935](#); [Loevinger et al., 1953](#)), the theory of factors, while mathematically compelling, offers little that the geneticist or behaviorist or perhaps even non-specialist finds compelling. [Cooksey and Soutar \(2006\)](#) reviews why the `iclust` algorithm is particularly appropriate for scale construction in marketing.

Hierarchical cluster analysis forms clusters that are nested within clusters. The resulting *tree diagram* (also known somewhat pretentiously as a *rooted dendritic structure*) shows the nesting structure. Although there are many hierarchical clustering algorithms in R (e.g., `agnes`, `hclust`, and `iclust`), the one most applicable to the problems of scale construction is `iclust` ([Revelle, 1979](#)).

1. Find the proximity (e.g. correlation) matrix,
2. Identify the most similar pair of items
3. Combine this most similar pair of items to form a new variable (cluster),
4. Find the similarity of this cluster to all other items and clusters,
5. Repeat steps 2 and 3 until some criterion is reached (e.g., typically, if only one cluster remains or in `iclust` if there is a failure to increase reliability coefficients α or β).
6. Purify the solution by reassigning items to the most similar cluster center.

```
> data(bfi)
> ic <- iclust(bfi[1:25])
```



19

Table 5: The summary statistics from an iclust analysis shows three large clusters and smaller cluster.

```
> summary(ic) #show the results
ICLUST (Item Cluster Analysis)Call: iclust(r.mat = bfi[1:25])
ICLUST
```

```
Purified Alpha:
  C20  C16  C15  C21
0.80 0.81 0.73 0.61
```

```
Guttman Lambda6*
  C20  C16  C15  C21
0.82 0.81 0.72 0.61
```

```
Original Beta:
  C20  C16  C15  C21
0.63 0.76 0.67 0.27
```

```
Cluster size:
C20 C16 C15 C21
 10   5   5   5
```

```
Purified scale intercorrelations
reliabilities on diagonal
correlations corrected for attenuation above diagonal:
      C20   C16   C15   C21
C20  0.80 -0.291  0.40 -0.33
C16 -0.24  0.815 -0.29  0.11
C15  0.30 -0.221  0.73 -0.30
C21 -0.23  0.074 -0.20  0.61
```

The previous analysis (Figure 5) was done using the Pearson correlation. A somewhat cleaner structure is obtained when using the `polychoric` function to find polychoric correlations (Figure 6). Note that the first time finding the polychoric correlations some time, but the next three analyses were done using that correlation matrix (`r.poly$rho`). When using the console for input, `polychoric` will report on its progress while working using `progressBar`.

Table 6: The `polychoric` and the `tetrachoric` functions can take a long time to finish and report their progress by a series of dots as they work. The dots are suppressed when creating a Sweave document.

```
> data(bfi)
> r.poly <- polychoric(bfi[1:25]) #the ... indicate the progress of the function
```

A comparison of these four cluster solutions suggests both a problem and an advantage of clustering techniques. The problem is that the solutions differ. The advantage is that the structure of the items may be seen more clearly when examining the clusters rather than a simple factor solution.

3.2 Confidence intervals using bootstrapping techniques

Exploratory factoring techniques are sometimes criticized because of the lack of statistical information on the solutions. Overall estimates of goodness of fit including χ^2 and RMSEA are found in the `fa` and `omega` functions. Confidence intervals for the factor loadings may be found by doing multiple bootstrapped iterations of the original analysis. This is done by setting the `n.iter` parameter to the desired number of iterations. This can be done for factoring of Pearson correlation matrices as well as polychoric/tetrachoric matrices (See Table 8). Although the example value for the number of iterations is set to 20, more conventional analyses might use 1000 bootstraps. This will take much longer.

3.3 Comparing factor/component/cluster solutions

Cluster analysis, factor analysis, and principal components analysis all produce structure matrices (matrices of correlations between the dimensions and the variables) that can in turn be compared in terms of the *congruence coefficient* which is just the cosine of the angle between the dimensions

$$c_{f_i f_j} = \frac{\sum_{k=1}^n f_{ik} f_{jk}}{\sum f_{ik}^2 \sum f_{jk}^2}.$$

Consider the case of a four factor solution and four cluster solution to the Big Five problem.

```
> ic.poly <- iclust(r.poly$rho,title="ICLUST using polychoric correlations")
> iclust.diagram(ic.poly)
```

ICLUST using polychoric correlations

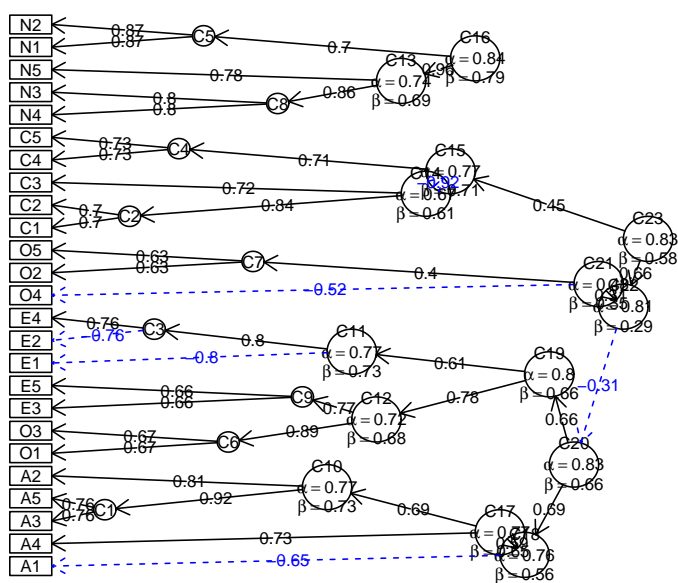


Figure 6: ICLUST of the BFI data set using polychoric correlations. Compare this solution to the previous one (Figure 5) which was done using Pearson correlations.

```
> ic.poly <- iclust(r.poly$rho,5,title="ICLUST using polychoric correlations for nclusters=5")
> iclust.diagram(ic.poly)
```

ICLUST using polychoric correlations for nclusters=5

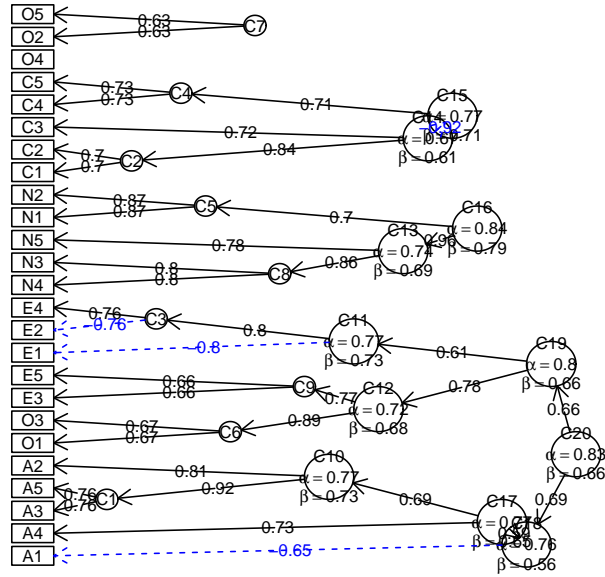


Figure 7: ICLUST of the BFI data set using polychoric correlations with the solution set to 5 clusters. Compare this solution to the previous one (Figure 6) which was done without specifying the number of clusters and to the next one (Figure 8) which was done by changing the beta criterion.

```
> ic.poly <- iclust(r.poly$rho,beta.size=3,title="ICLUST beta.size=3")
```

ICLUST beta.size=3

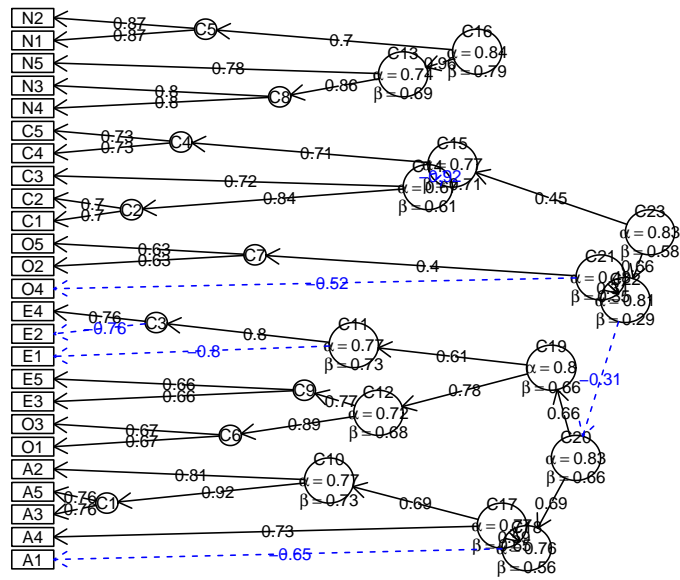


Figure 8: ICLUST of the BFI data set using polychoric correlations with the beta criterion set to 3. Compare this solution to the previous three (Figure 5, 6, 7).

Table 7: The output from `iclust` includes the loadings of each item on each cluster. These are equivalent to factor structure loadings. By specifying the value of `cut`, small loadings are suppressed. The default is for `cut=0.3`

```
> print(ic,cut=.3)
ICLUST (Item Cluster Analysis)
Call: iclust(r.mat = bfi[1:25])

Purified Alpha:
  C20  C16  C15  C21
0.80 0.81 0.73 0.61

G6* reliability:
  C20  C16  C15  C21
0.77 0.81 0.59 0.45

Original Beta:
  C20  C16  C15  C21
0.63 0.76 0.67 0.27

Cluster size:
C20 C16 C15 C21
 10   5   5   5

Item by Cluster Structure matrix:
      0   P   C20   C16   C15   C21
A1 C20 C20
A2 C20 C20 0.59
A3 C20 C20 0.65
A4 C20 C20 0.43
A5 C20 C20 0.65
C1 C15 C15          0.54
C2 C15 C15          0.62
C3 C15 C15          0.54
C4 C15 C15          0.31 -0.66
C5 C15 C15 -0.30 0.36 -0.59
E1 C20 C20 -0.50
E2 C20 C20 -0.61 0.34
E3 C20 C20 0.59          -0.39
E4 C20 C20 0.66
E5 C20 C20 0.50          0.40 -0.32
N1 C16 C16          0.76
N2 C16 C16          0.75
N3 C16 C16          0.74
N4 C16 C16 -0.34 0.62
N5 C16 C16          0.55
O1 C20 C21          -0.53
O2 C21 C21          0.44
O3 C20 C21 0.39          -0.62
O4 C21 C21          -0.33
O5 C21 C21          0.53

With eigenvalues of:
C20 C16 C15 C21
3.2 2.6 1.9 1.5

Purified scale intercorrelations
reliabilities on diagonal
correlations corrected for attenuation above diagonal:
      C20  C16  C15  C21
C20 0.80 -0.29 0.40 -0.33
C16 -0.24 0.81 -0.29 0.11
C15 0.30 -0.22 0.73 -0.30
C21 -0.23 0.07 -0.20 0.61

Cluster fit = 0.68   Pattern fit = 0.96   RMSR = 0.05
NULL
```

Table 8: An example of bootstrapped confidence intervals on 10 items from the Big 5 inventory. The number of bootstrapped samples was set to 20. More conventional bootstrapping would use 100 or 1000 replications.

```
> fa(bfi[1:10], 2, n.iter=20)
Factor Analysis with confidence intervals using method = minres
Call: fa(r = bfi[1:10], nfactors = 2, n.iter = 20)
Factor Analysis using method = minres
Call: fac(r = r, nfactors = nfactors, n.obs = n.obs, rotate = rotate,
  scores = scores, residuals = residuals, SMC = SMC, covar = covar,
  missing = FALSE, impute = impute, min.err = min.err, max.iter = max.iter,
  symmetric = symmetric, warnings = warnings, fm = fm, alpha = alpha,
  oblique.scores = oblique.scores, np.obs = np.obs)
Standardized loadings (pattern matrix) based upon correlation matrix
  MR2  MR1  h2  u2
A1  0.07 -0.40 0.15 0.85
A2  0.02  0.65 0.44 0.56
A3 -0.03  0.77 0.57 0.43
A4  0.15  0.44 0.26 0.74
A5  0.02  0.62 0.39 0.61
C1  0.57 -0.05 0.30 0.70
C2  0.62 -0.01 0.39 0.61
C3  0.54  0.03 0.30 0.70
C4 -0.66  0.01 0.43 0.57
C5 -0.57 -0.05 0.35 0.65

SS loadings          MR2  MR1
Proportion Var       1.80 1.77
Cumulative Var       0.18 0.18
Proportion Explained 0.50 0.50
Cumulative Proportion 0.50 1.00

With factor correlations of
  MR2  MR1
MR2 1.00 0.32
MR1 0.32 1.00

Test of the hypothesis that 2 factors are sufficient.

The degrees of freedom for the null model are 45 and the objective function was 2.03 with Chi Square of 5664.89
The degrees of freedom for the model are 26 and the objective function was 0.17

The root mean square of the residuals (RMSR) is 0.04
The df corrected root mean square of the residuals is 0.07

The harmonic number of observations is 2762 with the empirical chi square 403.38 with prob < 2.6e-69
The total number of observations was 2800 with MLE Chi Square = 464.04 with prob < 9.2e-82

Tucker Lewis Index of factoring reliability = 0.865
RMSEA index = 0.078 and the 90 % confidence intervals are 0.071 0.084
BIC = 257.67
Fit based upon off diagonal values = 0.98
Measures of factor score adequacy
  MR2  MR1
Correlation of scores with factors 0.86 0.88
Multiple R square of scores with factors 0.74 0.77
Minimum correlation of possible factor scores 0.49 0.54

Coefficients and bootstrapped confidence intervals
  low  MR2 upper  low  MR1 upper
A1  0.03  0.07  0.10 -0.44 -0.40 -0.36
A2 -0.03  0.02  0.05  0.61  0.65  0.70
A3 -0.05 -0.03  0.00  0.71  0.77  0.81
A4  0.10  0.15  0.19  0.40  0.44  0.49
A5 -0.02  0.02  0.06  0.57  0.62  0.66
C1  0.52  0.57  0.61 -0.09 -0.05 -0.02
C2  0.58  0.62  0.65 -0.05 -0.01  0.04
C3  0.48  0.54  0.59 -0.01  0.03  0.07
C4 -0.71 -0.66 -0.61 -0.03  0.01  0.03
C5 -0.63 -0.57 -0.53 -0.09 -0.05 -0.01

Interfactor correlations and bootstrapped confidence intervals
  lower estimate upper
1  0.25  0.32  0.36
>
```

```
> f4 <- fa(bfi[1:25], 4, fm="pa")
> factor.congruence(f4, ic)
```

	C20	C16	C15	C21
PA1	0.92	-0.32	0.44	-0.40
PA2	-0.26	0.95	-0.33	0.12
PA3	0.35	-0.24	0.88	-0.37
PA4	0.29	-0.12	0.27	-0.90

A more complete comparison of oblique factor solutions (both minres and principal axis), bifactor and component solutions to the Thurstone data set is done using the `factor.congruence` function. (See table 9).

Table 9: Congruence coefficients for oblique factor, bifactor and component solutions for the Thurstone problem.

```
> factor.congruence(list(f3t, f3o, om, p3p))
```

	MR1	MR2	MR3	PA1	PA2	PA3	g	F1*	F2*	F3*	h2	PC1	PC2	PC3
MR1	1.00	0.06	0.09	1.00	0.06	0.13	0.72	1.00	0.06	0.09	0.74	1.00	0.08	0.04
MR2	0.06	1.00	0.08	0.03	1.00	0.06	0.60	0.06	1.00	0.08	0.57	0.04	0.99	0.12
MR3	0.09	0.08	1.00	0.01	0.01	1.00	0.52	0.09	0.08	1.00	0.51	0.06	0.02	0.99
PA1	1.00	0.03	0.01	1.00	0.04	0.05	0.67	1.00	0.03	0.01	0.69	1.00	0.06	-0.04
PA2	0.06	1.00	0.01	0.04	1.00	0.00	0.57	0.06	1.00	0.01	0.54	0.04	0.99	0.05
PA3	0.13	0.06	1.00	0.05	0.00	1.00	0.54	0.13	0.06	1.00	0.53	0.10	0.01	0.99
g	0.72	0.60	0.52	0.67	0.57	0.54	1.00	0.72	0.60	0.52	0.99	0.69	0.58	0.50
F1*	1.00	0.06	0.09	1.00	0.06	0.13	0.72	1.00	0.06	0.09	0.74	1.00	0.08	0.04
F2*	0.06	1.00	0.08	0.03	1.00	0.06	0.60	0.06	1.00	0.08	0.57	0.04	0.99	0.12
F3*	0.09	0.08	1.00	0.01	0.01	1.00	0.52	0.09	0.08	1.00	0.51	0.06	0.02	0.99
h2	0.74	0.57	0.51	0.69	0.54	0.53	0.99	0.74	0.57	0.51	1.00	0.71	0.56	0.49
PC1	1.00	0.04	0.06	1.00	0.04	0.10	0.69	1.00	0.04	0.06	0.71	1.00	0.06	0.00
PC2	0.08	0.99	0.02	0.06	0.99	0.01	0.58	0.08	0.99	0.02	0.56	0.06	1.00	0.05
PC3	0.04	0.12	0.99	-0.04	0.05	0.99	0.50	0.04	0.12	0.99	0.49	0.00	0.05	1.00

3.4 Determining the number of dimensions to extract.

How many dimensions to use to represent a correlation matrix is an unsolved problem in psychometrics. There are many solutions to this problem, none of which is uniformly the best. Henry Kaiser once said that “a solution to the number-of factors problem in factor analysis is easy, that he used to make up one every morning before breakfast. But the problem, of course is to find *the* solution, or at least a solution that others will regard quite highly not as the best” [Horn and Engstrom \(1979\)](#).

Techniques most commonly used include

- 1) Extracting factors until the chi square of the residual matrix is not significant.
- 2) Extracting factors until the change in chi square from factor n to factor n+1 is not significant.
- 3) Extracting factors until the eigen values of the real data are less than the corresponding

eigen values of a random data set of the same size (parallel analysis) `fa.parallel` (Horn, 1965).

4) Plotting the magnitude of the successive eigen values and applying the scree test (a sudden drop in eigen values analogous to the change in slope seen when scrambling up the talus slope of a mountain and approaching the rock face (Cattell, 1966).

5) Extracting factors as long as they are interpretable.

6) Using the Very Structure Criterion (`vss`) (Revelle and Rocklin, 1979).

7) Using Wayne Velicer’s Minimum Average Partial (`MAP`) criterion (Velicer, 1976).

8) Extracting principal components until the eigen value < 1 .

Each of the procedures has its advantages and disadvantages. Using either the chi square test or the change in square test is, of course, sensitive to the number of subjects and leads to the nonsensical condition that if one wants to find many factors, one simply runs more subjects. Parallel analysis is partially sensitive to sample size in that for large samples the eigen values of random factors will be very small. The scree test is quite appealing but can lead to differences of interpretation as to when the scree “breaks”. Extracting interpretable factors means that the number of factors reflects the investigators creativity more than the data. `vss`, while very simple to understand, will not work very well if the data are very factorially complex. (Simulations suggests it will work fine if the complexities of some of the items are no more than 2). The eigen value of 1 rule, although the default for many programs, seems to be a rough way of dividing the number of variables by 3 and is probably the worst of all criteria.

An additional problem in determining the number of factors is what is considered a factor. Many treatments of factor analysis assume that the residual correlation matrix after the factors of interest are extracted is composed of just random error. An alternative concept is that the matrix is formed from major factors of interest but that there are also numerous minor factors of no substantive interest but that account for some of the shared covariance between variables. The presence of such minor factors can lead one to extract too many factors and to reject solutions on statistical grounds of misfit that are actually very good fits to the data. This problem is partially addressed later in the discussion of simulating complex structures using `sim.structure` and of small extraneous factors using the `sim.minor` function.

3.4.1 Very Simple Structure

The `vss` function compares the fit of a number of factor analyses with the loading matrix “simplified” by deleting all except the `c` greatest loadings per item, where `c` is a measure

of factor complexity [Revelle and Rocklin \(1979\)](#). Included in `vss` is the MAP criterion (Minimum Absolute Partial correlation) of [Velicer \(1976\)](#).

Using the Very Simple Structure criterion for the bfi data suggests that 4 factors are optimal (Figure 9). However, the MAP criterion suggests that 5 is optimal.

```
> vss <- vss(bfi[1:25],title="Very Simple Structure of a Big 5 inventory")
```

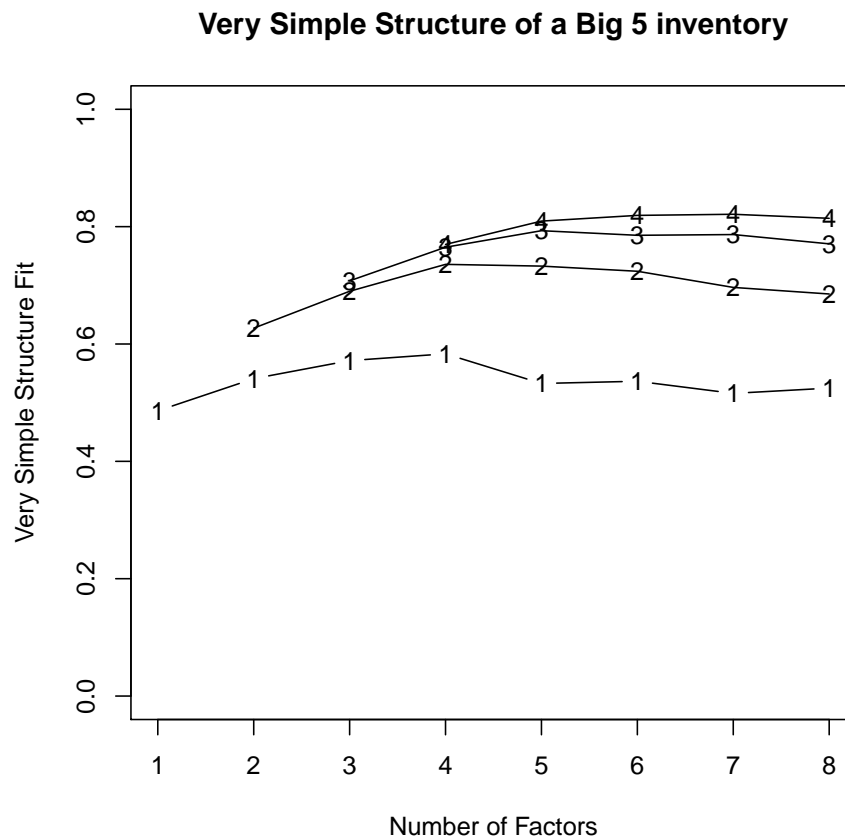


Figure 9: The Very Simple Structure criterion for the number of factors compares solutions for various levels of item complexity and various numbers of factors. For the Big 5 Inventory, the complexity 1 and 2 solutions both achieve their maxima at four factors. This is in contrast to parallel analysis which suggests 6 and the MAP criterion which suggests 5.

```
> vss

Very Simple Structure of  Very Simple Structure of a Big 5 inventory
Call: VSS(x = x, n = n, rotate = rotate, diagonal = diagonal, fm = fm,
  n.obs = n.obs, plot = plot, title = title)
VSS complexity 1 achieves a maximum of 0.58 with 4 factors
```

```

VSS complexity 2 achieves a maximum of 0.74 with 4 factors

The Velicer MAP criterion achieves a minimum of 0.01 with 5 factors

Velicer MAP
[1] 0.02 0.02 0.02 0.02 0.01 0.02 0.02 0.02

Very Simple Structure Complexity 1
[1] 0.49 0.54 0.57 0.58 0.53 0.54 0.52 0.52

Very Simple Structure Complexity 2
[1] 0.00 0.63 0.69 0.74 0.73 0.72 0.70 0.69

```

3.4.2 Parallel Analysis

An alternative way to determine the number of factors is to compare the solution to random data with the same properties as the real data set. If the input is a data matrix, the comparison includes random samples from the real data, as well as normally distributed random data with the same number of subjects and variables. For the BFI data, parallel analysis suggests that 6 factors might be most appropriate (Figure 10). It is interesting to compare `fa.parallel` with the `paran` from the *paran* package. This latter uses smcs to estimate communalities. Simulations of known structures with a particular number of major factors but with the presence of trivial, minor (but not zero) factors, show that using smcs will tend to lead to too many factors.

A more tedious problem in terms of computation is to do parallel analysis of *polychoric* correlation matrices. This is done by `fa.parallel.poly`. By default the number of replications is 20. This is appropriate when choosing the number of factors from dichotomous or polytomous data matrices.

3.5 Factor extension

Sometimes we are interested in the relationship of the factors in one space with the variables in a different space. One solution is to find factors in both spaces separately and then find the structural relationships between them. This is the technique of structural equation modeling in packages such as *sem* or *lavaan*. An alternative is to use the concept of *factor extension* developed by (Dwyer, 1937). Consider the case of 16 variables created to represent one two dimensional space. If factors are found from eight of these variables, they may then be extended to the additional eight variables (See Figure 11).

Another way to examine the overlap between two sets is the use of *set correlation* found by `set.cor` (discussed later).

```
> fa.parallel(bfi[1:25],main="Parallel Analysis of a Big 5 inventory")
Parallel analysis suggests that the number of factors = 6 and the number of components = 6
```

Parallel Analysis of a Big 5 inventory

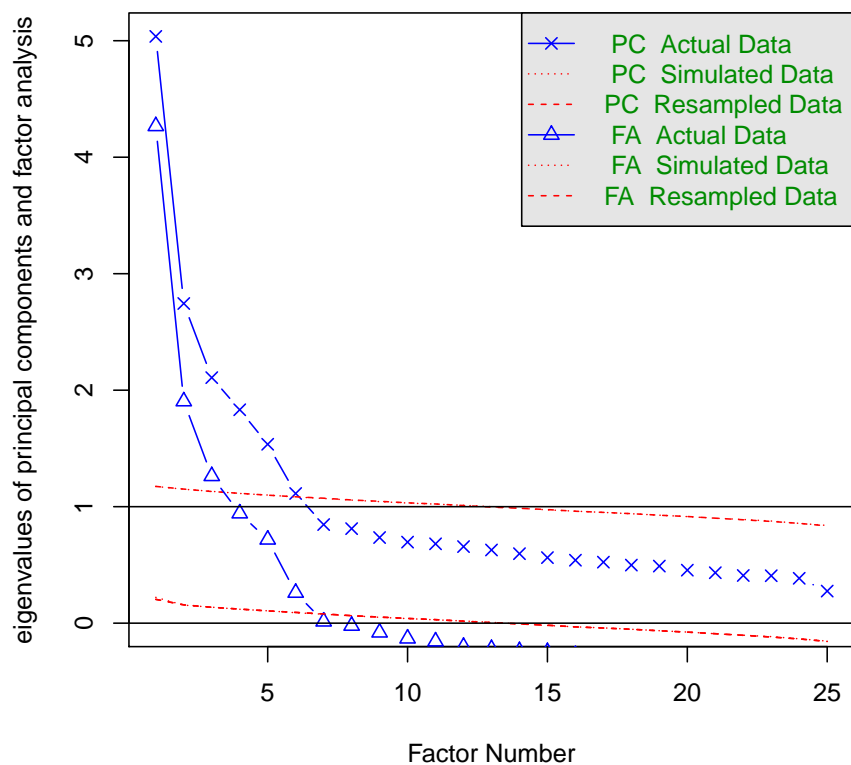


Figure 10: Parallel analysis compares factor and principal components solutions to the real data as well as resampled data. Although vss suggests 4 factors, MAP 5, parallel analysis suggests 6. One more demonstration of Kaiser's dictum.

```

> v16 <- sim.item(16)
> s <- c(1,3,5,7,9,11,13,15)
> f2 <- fa(v16[s],2)
> fe <- fa.extension(cor(v16)[s,-s],f2)
> fa.diagram(f2,fe=fe)

```

Factor analysis and extension

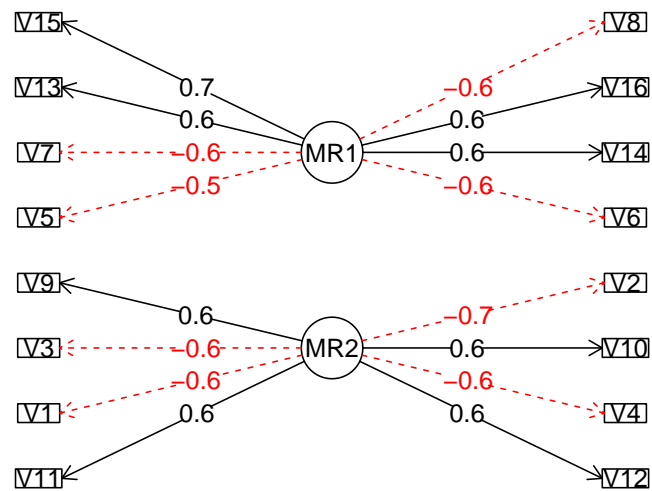


Figure 11: Factor extension applies factors from one set (those on the left) to another set of variables (those on the right). `fa.extension` is particularly useful when one wants to define the factors with one set of variables and then apply those factors to another set. `fa.diagram` is used to show the structure.

4 Psychometric Theory

The *psych* package has been developed to help psychologists do basic research. Many of the functions were developed to supplement a book (<http://personality-project.org/r/book> An introduction to Psychometric Theory with Applications in R (Revelle, prep) More information about the use of some of the functions may be found in the book .

For more extensive discussion of the use of *psych* in particular and R in general, consult <http://personality-project.org/r/r.guide.html> A short guide to R.

5 SessionInfo

This document was prepared using the following settings.

```
> sessionInfo()

R Under development (unstable) (2013-01-13 r61644)
Platform: i386-apple-darwin9.8.0/i386 (32-bit)

locale:
[1] en_US.UTF-8/en_US.UTF-8/en_US.UTF-8/C/en_US.UTF-8/en_US.UTF-8

attached base packages:
[1] stats      graphics  grDevices  utils      datasets  methods   base

other attached packages:
[1] sem_3.1-0      matrixcalc_1.0-3    mvtnorm_0.9-9994    GPArotation_2012.3-1
[5] MASS_7.3-23    psych_1.3.01

loaded via a namespace (and not attached):
[1] tools_3.0.0
```

References

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