ECE3311 – Principles of Communication Systems

Project 05

Due: 5:00 PM, Wednesday 13 December 2017

1 Project Objective & Learning Outcomes

In this project, we will investigate the physical layer of a generic digital communication system using Orthogonal Frequency Division Multiplexing (OFDM) over channels that introduce noise. From this project, it is expected that the following learning outcomes are achieved:

- Understand how multicarrier modulation operates by sending multiple signals across different carrier frequencies at the same time.
- Learn about the orthogonal frequency division multiplexing concept by leveraging the Inverse Discrete Fourier Transform (IDFT) and DFT.
- Master Quadrature Amplitude Modulation (QAM) and how it operates as a 2-dimensional modulation scheme.

2 Modulation and Demodulation of QAM symbols

Quadrature Amplitude Modulation (QAM) is the technique of transmitting data on two quadrature carriers (*i.e.*, they are 90° out of phase with each other, making them *orthogonal*). In what follows, we will consider a digital implementation of the modulation, so that the carrier signals are $\cos(\omega_k n)$ and $\sin(\omega_k n)$, with carrier frequency ω_k. The amplitude of each of these two carriers is specified by the sequence of input bits, and is changed every $2N$ samples, which is called the *QAM symbol period*. The amplitudes can take on only a finite number of possible values. Figure 1 represents a discrete-time model of the QAM modulator, where the sequence of input bits $d[m]$ is used to determine the amplitudes of the two carriers. For each symbol period, D bits of input are taken from the input bit stream $d[m]$ and used to select one of the 2^D combinations of amplitudes for the two carriers. Calling these amplitudes $a[\ell]$ and $b[\ell]$ respectively, where ℓ represents the symbol time index, these amplitudes are kept constant for the duration of a symbol period by the upsampling and rectangular window filtering shown in Figure 1, yielding the piecewise constant signals $a'[n]$ and $b'[n]$. One can then write the expression of the modulated signal as

$$s[n] = a'[n]\cos(\omega_k n) + b'[n]\sin(\omega_k n) \tag{1}$$

where $\omega_k = 2\pi k/2N$ is the carrier frequency, and $2N$ is the period of the symbol. Note that we have to limit the possible carrier frequencies to integer multiples of $2\pi/2N$ in order to keep orthogonality between the sinusoidal and cosinusoidal carriers in a digital implementation (see derivations below).

Question 1 (5 points): Prove that the carriers $\cos(\omega_k n)$ and $\sin(\omega_k n)$ are orthogonal over the symbol period[1] . Are there any values of k that are exceptions? Why?

[1] Two real sequences $h[n]$ and $g[n]$ are said to be orthogonal over a period 0 to P if $\sum_{n=0}^{P-1} h[n]g[n] = 0$.

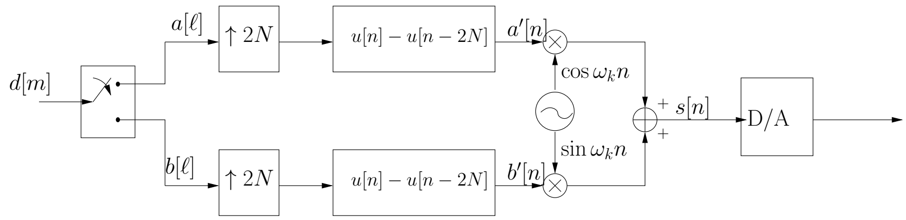


Figure 1: Rectangular QAM modulator (multirate model of a digital implementation).

Matlab Task 1 (10 points): Implement the QAM modulator as shown in Figure 1, given an arbitrary ω_k.

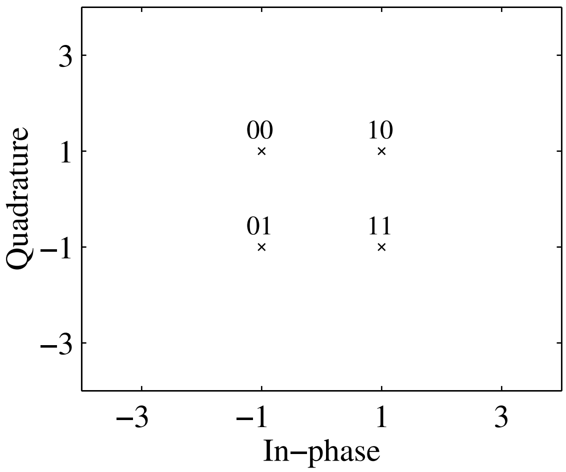
In this project, we will be dealing with rectangular QAM signal constellations, such as those shown in Figures 2(a), 2(b), and 2(c) for 4-QAM, 16-QAM, and 64-QAM, respectively. This bascially amounts to saying that $a[\ell], b[\ell] \in \{\pm(2k-1)E, k = 1, \ldots, 2^{D/2-1}\}$, where E is some positive constant that scales the energy of the sent signals, and $D/2$ is the number of bits used to represent the amplitude level of one of the carriers during a symbol.

One of the advantages of QAM signalling is the fact that demodulation is relatively simple to perform. From Figure 3, the received signal, $r[n]$, is split into two streams and each multiplied by carriers, $\cos(\omega_k n)$ and $\sin(\omega_k n)$, followed by a summation block (implemented here through filtering by a rectangular window followed by downsampling). This process produces estimates of the in-phase and quadrature amplitudes, $\hat{a}[l]$ and $\hat{b}[l]$, namely

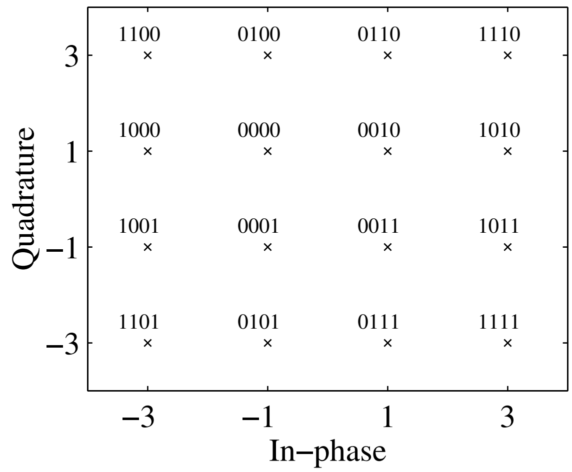
$$
\begin{aligned}
\hat{a}[\ell] &= \sum_{n=2\ell N}^{2\ell N+2N-1} r[n]\cos(\omega_k n) \\
&= \sum_{n=2\ell N}^{2\ell N+2N-1} a'[n]\cos(\omega_k n)\cos(\omega_k n) + \sum_{n=2\ell N}^{2\ell N+2N-1} b'[n]\sin(\omega_k n)\cos(\omega_k n) \\
&= \sum_{n=2\ell N}^{2\ell N+2N-1} a'[n]\cos\left(\frac{2\pi kn}{2N}\right)\cos\left(\frac{2\pi kn}{2N}\right) + \sum_{n=2\ell N}^{2\ell N+2N-1} b'[n]\sin\left(\frac{2\pi kn}{2N}\right)\cos\left(\frac{2\pi kn}{2N}\right) \\
&= \sum_{n=2\ell N}^{2\ell N+2N-1} \frac{a'[n]}{2}
\end{aligned}
\tag{2}
$$

and

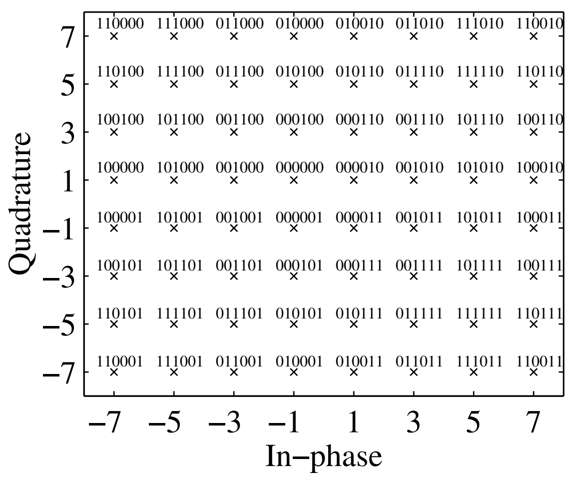
$$
\begin{aligned}
\hat{b}[\ell] &= \sum_{n=2\ell N}^{2\ell N+2N-1} r[n]\sin(\omega_k n) \\
&= \sum_{n=2\ell N}^{2\ell N+2N-1} a'[n]\cos(\omega_k n)\sin(\omega_k n) + \sum_{n=2\ell N}^{2\ell N+2N-1} b'[n]\sin(\omega_k n)\sin(\omega_k n) \\
&= \sum_{n=2\ell N}^{2\ell N+2N-1} a'[n]\cos\left(\frac{2\pi kn}{2N}\right)\sin\left(\frac{2\pi kn}{2N}\right) + \sum_{n=2\ell N}^{2\ell N+2N-1} b'[n]\sin\left(\frac{2\pi kn}{2N}\right)\sin\left(\frac{2\pi kn}{2N}\right) \\
&= \sum_{n=2\ell N}^{2\ell N+2N-1} \frac{b'[n]}{2}
\end{aligned}
\tag{3}
$$



(a) 4-QAM signal constellation



(b) Rectangular 16-QAM signal constellation



(c) Rectangular 64-QAM signal constellation

Figure 2: Three types of QAM signal constellations. The in-phase values indicate the amplitude of the carrier $\cos(\omega_k n)$ while the quadrature values indicate the amplitude of the carrier $\sin(\omega_k n)$.

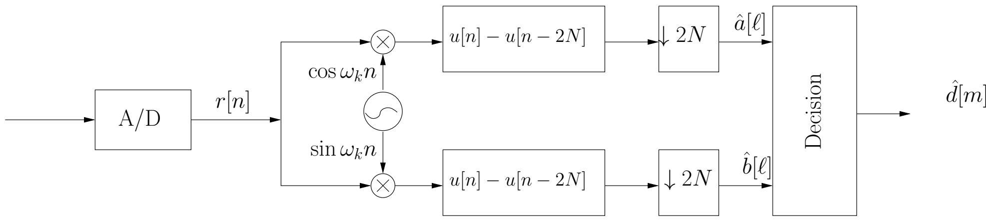


Figure 3: Rectangular QAM demodulator.

where, due to the orthogonality of the two carriers, the cross terms vanish, leaving the desired amplitude (after some trigonometric manipulation). The bits etsimated from $\hat{a}[\ell]$ and $\hat{b}[\ell]$ bits are then multiplexed together, forming the reconstructed version of $d[m]$, $\hat{d}[m]$.

Question 2 (5 points): Prove that Eqs. (2) and (3) are true. What are the exceptions for the carrier frequencies for perfect recovery? Why is k constrained to be below N?

Matlab Task 2 (15 points): Implement the rectangular QAM demodulator as shown in Figure 3, given an arbitrary ω_k. Test the cascade of the QAM modulator and demodulator to see if the input is completely recovered at the output. Implement the constellations of 4-QAM, 16-QAM and 64-QAM. also implement the special, degenerate case of BPSK, where only the cosine carrier is modulated with one bit.

We have so far dealt with modulation and demodulation in an ideal setting. Thus, one should expect that $\hat{d}[m] = d[m]$ with probability of 1. However, in the next subsection we will examine a physical phenomenon that distorts the transmitted signal, resulting in transmission errors.

3 Noise

By definition, *noise* is an undesirable disturbance accompanying the received signal that may distort the information carried by the signal. Noise can originate from human-made and natural sources, such as thermal noise due to the thermal agitation of electrons in transmission lines, antennas, or other conductors.

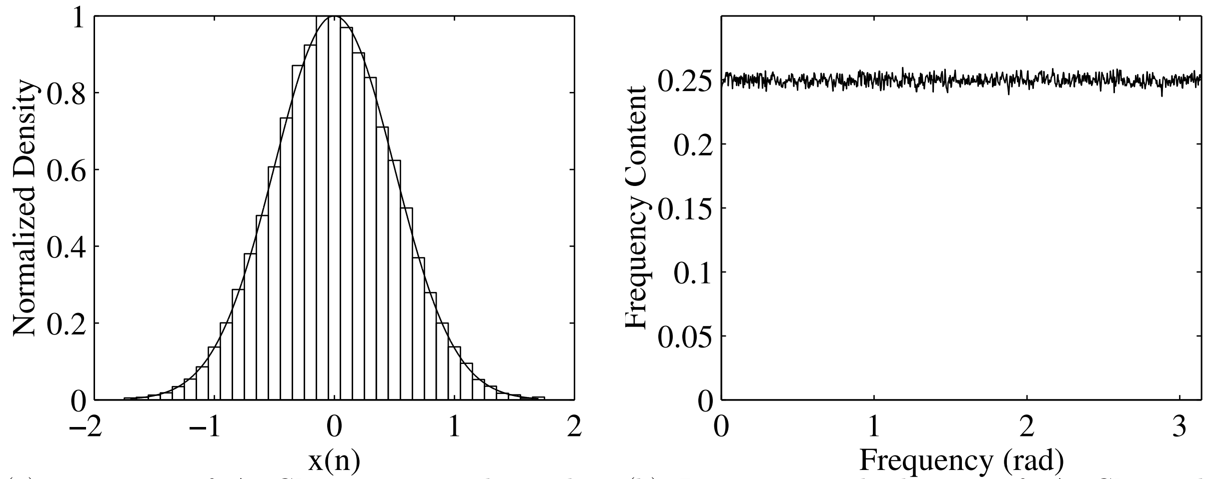
The combination of such sources of noise is known to have a Gaussian distribution, as shown in Figure 4(a). A histogram of zero-mean Gaussian noise with a variance of $\sigma_n^2 = 0.25$ is shown, with the corresponding continuous probability density function (pdf) superimposed on it. The continuous Gaussian pdf is defined as

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{(x-\mu_n)^2}{2\sigma_n^2}} \tag{4}$$

where μ_n and σ_n^2 are the mean and variance.

In this work, we will also make the assumption that the noise introduced by the channel is *white*, which means that the noise has frequency content that is approximately flat, as shown in Figure 4(b) for the case of zero-mean white Gaussian Noise with a variance of $\sigma_n^2 = 0.25$. Although in reality this assumption may not hold in certain cases, it does help simplify the process of demodulation.

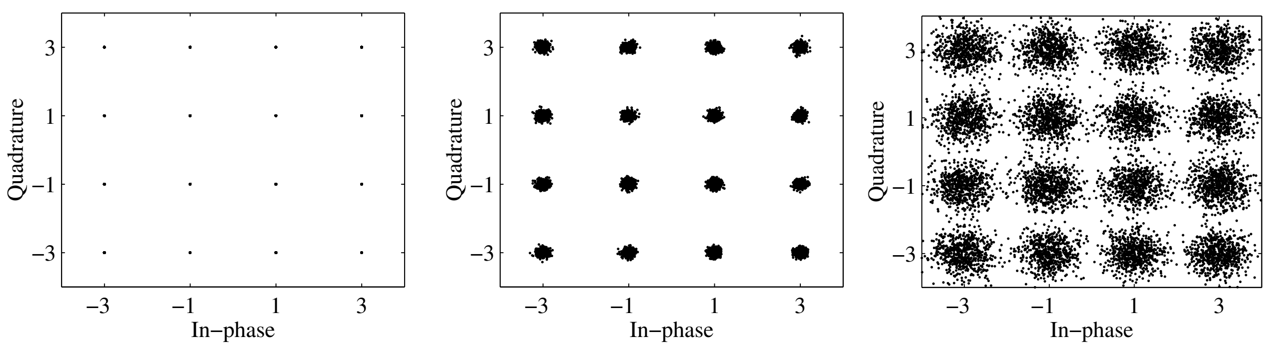
Thus, when noise is added to the transmitted signal, the receiver has to make a decision on what has been transmitted based on the received signal. Usually this is accomplished via a "nearest neighbour" rule with a known set of symbols. However, if a large amount of noise is added to the



(a) Histogram of AWGN superimposed on the probability density function for a Gaussian random variable of N(0,0.25)

(b) Power spectral density of AWGN with N(0,0.25)

Figure 4: Time and frequency domain properties of AWGN



(a) AWGN $N(0, 0.00001)$

(b) AWGN $N(0, 0.01)$

(c) AWGN $N(0, 0.25)$

Figure 5: 16-QAM signal constellations with varying amounts of noise

signal, there is a possibility that the received symbol might be shifted closer to a symbol other than the correct one, resulting in an error. As seen in Figures 5(a), 5(b), and 5(c), where additive white Gaussian noise is included with the signal, it is readily observable that as the noise power increases, the constellation points become fuzzy until they begin overlapping with each other. It is at this point that the system will begin to experience errors.

Matlab Task 3 (10 points): Implement an AWGN generator that accepts as inputs: the variance σ^2, the mean μ, and the number of random points R. Verify that it works by creating a histogram of its output and compare against the Gaussian pdf as described by Eq. (4) for $\mu = 3$, $\sigma^2 = 2$.

4 Probability of Bit Error

The introduction of noise, which is a stochastic signal, to the transmitted signal means that the overall received signal is also stochastic. Thus, transmitting the same information over and over again may

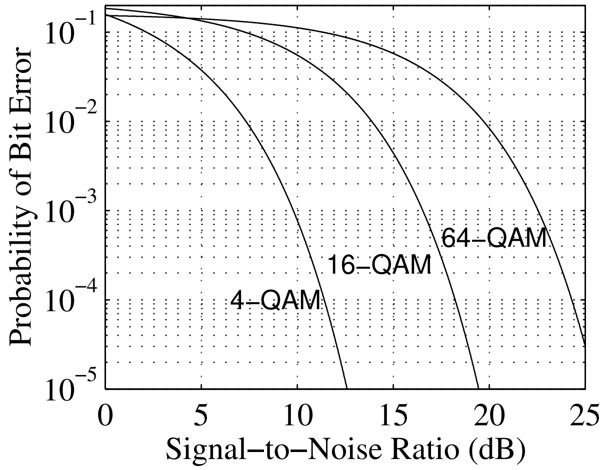


Figure 6: Probability of Bit Error curves for 4-QAM, rectangular 16-QAM, and rectangular 64-QAM

yield different results each time, assuming that the same modulation scheme and receiver design are employed.

In order to quantify the performance of a particular system set-up (*e.g.*, modulation scheme, receiver design, etc...), many use the ratio between the number of bit errors that occur and the total number of bits transmitted. This ratio is an estimate of the *Probability of Bit Error* or *Bit Error Rate* (BER) and is dependent on the noise variance, the modulation scheme employed, and the receiver design.

In Figure 6, the BER curves for rectangular 4-QAM, 16-QAM, and 64-QAM are presented when additive white Gaussian noise (AWGN) is introduced. Notice how as the signal-to-noise ratio (SNR) increases (noise power decreases), the probability of bit error decreases, as expected. Note that the SNR is the ratio of symbol energy to the noise energy, namely

$$\gamma = \frac{E_s}{\sigma_n^2}, \tag{5}$$

where $E_s = \frac{1}{N}\sum_{n=0}^{2N-1} s^2[n]$.

Matlab Task 4 (10 points): Verify that rectangular QAM modulator/demodulator works by introducing the appropriate amount of noise and comparing the resulting BER with the BER in Figure 6 at 10^{-3}. What values of SNR did you employ?

5 The OFDM Principle

As we have seen thus far, signals can be transmitted on a single carrier frequency. For instance, Eq. (1) modulates to the carrier frequency ω_k. However, the transmitted signal may only use up a small portion of the total available bandwidth. To increase bandwidth efficiency and throughput, it is possible to send additional QAM signals in other portions of the unused bandwidth simultaneously. An example of this is shown in Figure 7, where we have taken the QAM modulator of Figure 1 and put several of them in parallel, each with a different carrier frequency. The data for each modulator came from portions of a high speed bit stream. This is principle behind multicarrier modulation.

Question 3 (5 points): Draw a schematic for an multiplexed QAM demodulator as in Figure 7. Show that the subcarriers of the orthogonally multiplexed QAM system are orthogonal.

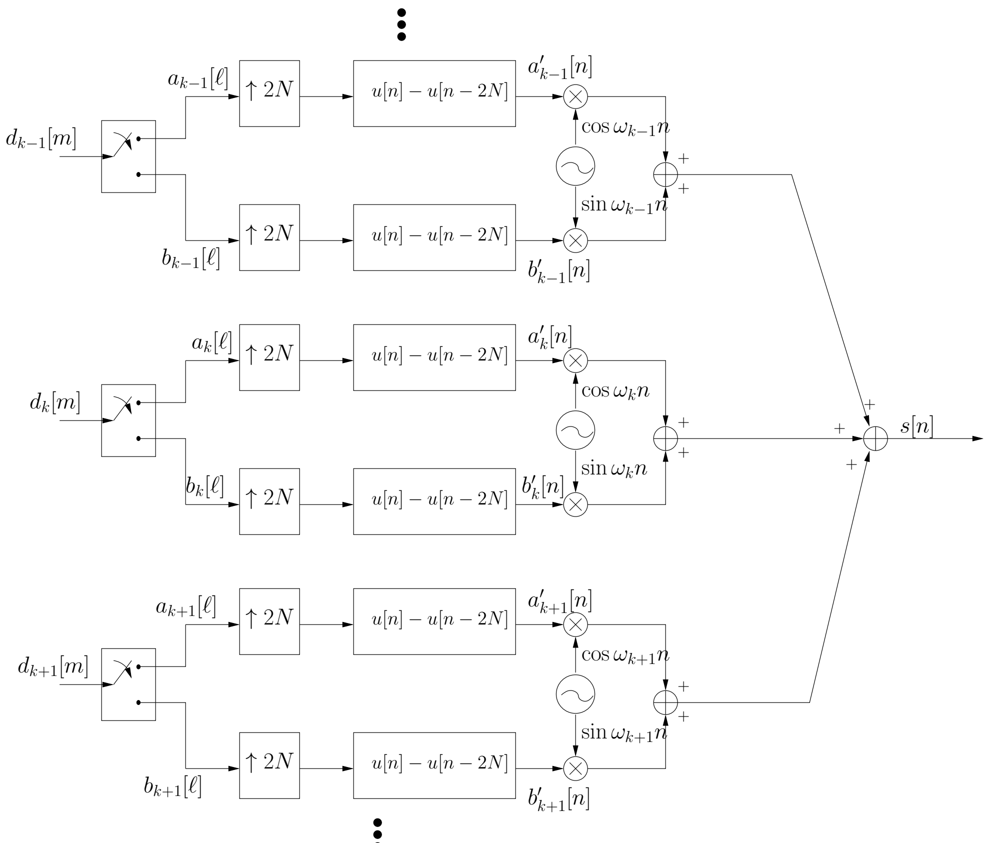


Figure 7: Transmitter of an orthogonally multiplexed QAM system

Matlab Task 5 (15 points): Implement Figure 7 and its corresponding receiver. Verify that it works under ideal conditions. What carrier frequencies did you not use in the implementation?

Orthogonal Frequency Division Multiplexing (OFDM) is an efficient type of multicarrier modulation, which employs the discrete Fourier transform (DFT) and inverse DFT (IDFT) to modulate and demodulate the data streams. Since the carriers used in Figure 7 are sinusoidal function of $2\pi kn/2N$, it should come as no surprise that a $2N$-point DFT or IDFT can carry out the same modulation, since it contains also summations of terms of the form $e^{\pm 2\pi kn/2N}$. The set-up of an OFDM system is presented in Figure 8. A high-speed digital input, $d[m]$, is demultiplexed into N subcarriers using a commutator. The data on each subcarrier is then modulated into an M-QAM symbol, which maps a group of $\log_2(M)$ bits at a time. Unlike the representation of Eq. (1), for subcarrier k we will rearrange $a_k[\ell]$ and $b_k[\ell]$ into real and imaginary components such that the output of the "modulator" block is $p_k[\ell] = a_k[\ell] + jb_k[\ell]$. In order for the output of the IDFT block to be real, given N subcarriers we must use a $2N$-point IDFT, where terminals $k = 0$ and $k = N$ are "don't care" inputs. For the subcarriers $1 \leq k \leq N-1$, the inputs are $p_k[\ell] = a_k[\ell] + jb_k[\ell]$, while for the subcarriers $N+1 \leq k \leq 2N-1$, the inputs are $p_k[\ell] = a_{2N-k}[\ell] + jb_{2N-k}[\ell]$.

Question 4 [BONUS] (3 points): Why are $k = 0$ and $k = N$ "don't care" inputs?

The IDFT is then performed, yielding

$$s[2\ell N + n] = \frac{1}{2N} \sum_{k=0}^{2N-1} p_k[\ell] e^{j(2\pi nk/2N)}, \tag{6}$$

where this time $2N$ consecutive samples of $s[n]$ constitute an OFDM symbol, which is a sum of N different QAM symbols.

This results in the data being modulated on several subchannels. This is achieved by multiplying each data stream by a $\sin(Nx)/\sin(x)$, several of which are shown in Figure 9.

The subcarriers are then multiplexed together using a commutator, forming the signal $s[n]$, and transmitted to the receiver. Once at the receiver, the signal is demultiplexed into $2N$ subcarriers of data, $\hat{s}[n]$, using a commutator and a $2N$-point DFT, defined as

$$\bar{p}_k[\ell] = \sum_{n=0}^{2N-1} \hat{s}[2\ell N + n] e^{-j(2\pi nk/2N)}, \tag{7}$$

is applied to the inputs, yielding the estimates of $p_k[\ell]$, $\bar{p}_k[\ell]$. The output of the equalizer, $\hat{p}_k[\ell]$, then passed through a demodulator and the result multiplexed together using a commutator, yielding the reconstructed high-speed bit stream, $\hat{d}[m]$.

Question 5 [BONUS] (5 points): Show that the OFDM configuration and the orthogonally multiplexed QAM system are identical. Which of the two implementations is faster if the OFDM system employs FFTs? Prove it by comparing the number of operations.

Matlab Task 6 [BONUS] (15 points): Implement Figure 8 using IFFT and FFT blocks using `ifft` and `fft`. The size of the input/output should be a variable.

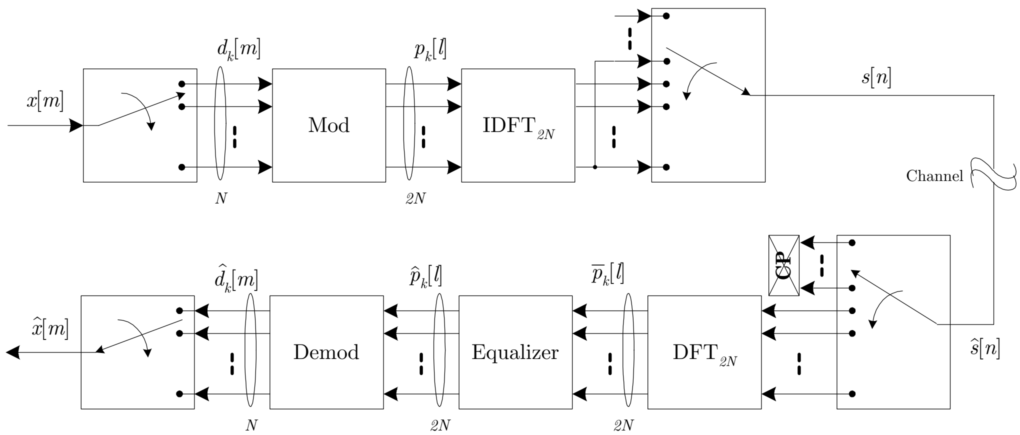


Figure 8: Overall schematic of an Orthogonal Frequency Division Multiplexing System

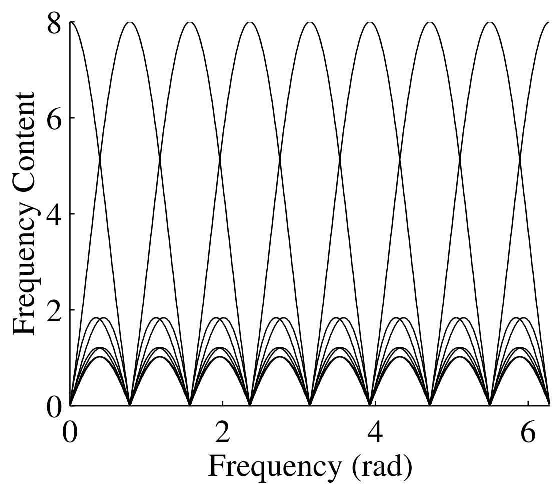


Figure 9: Characteristics of Orthogonal Frequency Division Multiplexing: Frequency response of OFDM subcarriers.

6 Final Report Format & Content

Each experiment report should possess the following format:

- A cover page (2 points) that includes the course number, project number, names and WPI ID numbers, submission date.
- A narrative (10 points) of the process taken during this experiment and the experiences encountered by the student. Figures, plots, schematics, diagrams, snippets of source code, and other visuals are highly encouraged.
- Responses to all questions indicated in the project handout. Please make sure that the responses are of sufficient detail.
- A summary (5 points) that contains all lessons learned from this project.
- All source code (5 points) generated (as an appendix).

This single document must be electronically submitted in **PDF format** (no other formats will be accepted) via the ECE3311 CANVAS website by the due date. Failure to submit this report by the specified due date and time will result in a grade of "0%".