Math 138 - January 4^{th} 2015

Review of Integration

Given a, b ϵ $\mathbb{R}$. a < b and n ϵ $\mathbb{N}$ we divide the interval [a, b] into n subintervals [x_{i-1}, x_i], i = 1 ... n of equal width $\Delta x = \frac{b-a}{n}$. amd choose sample points:

$$x_i * \epsilon [x_{i-1}, x_i], \text{ i = 1 ... n}$$

For a function f on the closed interval [a, b]. the definite integral of f from a to b is:
$\int_a^b f(x)dx = \lim n \to \infty \sum_{i=1}^n f(x_i*)\Delta x$

We say that f is integrable if the above limit exists and is independant of the sample points x_i*.
Recall that continuous functions are integrable.
Furthermore, $\lim n \to \infty \sum_{i=1}^n f(x_i*)\Delta x$ is the sum of the areas of the rectangles, and $\int_a^b f(x)dx$ is the area under a curve between two x values.

Fundamental Theorem of Calculus Suppose f is continous on [a,b].

I: If $g(x) = \int_a^x f(t)dt$, then g is differentiable and $g'(x) = f(x), x\epsilon$ [a, b]

II: $\int_a^b f(x)dx = F(b) - F(a)$ where F is **any** antiderivative of $f(x)$, i.e, $F'(x) = f(x), x$ ϵ [a, b]

Example 1: Find the derivative of g(x), where:
a) $g(x) = \int_1^x ln(1+t^2)dt$
b) $g(x) = \int_1^{3x+2} \frac{t}{1+t^3}dt$

Solution a): $g'(x) = ln(1+x^2)$ By FTC(I)
Solution b): Write $g(x) = f(u(x))$, where $f(u) = \int_1^u \frac{t}{1+t^3}dt$ and $u(x) = 3x+2$ By the chain rule and FTC(I),

$$\begin{aligned} g'(x) &= f'(u(x)) \cdot u'(x) \\ &= \frac{u(x)}{1+u(x^3)} \cdot 3 \\ &= \frac{3(3x+2)}{1+(3x+2)^3} \end{aligned} \tag{1}$$

Example 2: Evaluate the integrals: a) $\int_1^9 \sqrt{x}dx$ b) $\int_{\frac{\pi}{6}} \pi sin\theta d\theta$
Solutions: a)

$$\begin{aligned}\int_1^9 \sqrt{x}dx &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\Big|_1^9 \\ &= \frac{2}{3}(9^{\frac{3}{2}} - 1) \\ &= \frac{2}{3}(26) \\ &= \frac{52}{3}\end{aligned} \tag{2}$$

If f has an antriderivative F on [a, b] (i.e $F'(x) = f(x)$) the indefinite integral of f is $\int f(x)dx = F(x) + C$ By FTC(II) the definite integral,
$\int_a^b f(x)dx = \int f(x)dx\Big|_a^b$

Examples:
1) $\int sec^2 x dx = tanx + C$
2) $\int \frac{1}{x}dx = ln|x| + C$

Substitution Rule: If $a = g(x)$ is differentiable and f is continuous on the range of u, then $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$

Example:

$$\begin{aligned}\int_1^e \frac{ln(x)}{x}dx &= \int_1^e lnx\frac{dx}{x} \\ &= \int_{u(1)=0}^{u(e)=1} u \cdot du \\ &= \frac{u^2}{2}\Big|_0^1 \\ &= \frac{1}{2}\end{aligned} \tag{3}$$