MATH 3800 F

Assignment 2

Krystian Wojcicki, 101001444

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1. (0, 0), (2, 5), (4, 7)

$$y_1 = S_1(x_1) : 0 = a_1$$
$$y_2 = S_1(x_2) : 5 = a_1 + 2b_1 + 4c_1 + 8d_1$$
$$y_2 = S_2(x_2) : 5 = a_2 + 2b_2 + 4c_2 + 8d_2$$
$$y_3 = S_2(x_3) : 7 = a_2 + 4b_2 + 16c_3 + 64d_2$$
$$S_1\prime(x) = b_1 + 2xc_1 + 3x^2d_1$$
$$S_2\prime(x) = b_2 + 2xc_2 + 3x^2d_2$$
$$S_1\prime\prime(x) = 2c_1 + 6xd_1$$
$$S_2\prime\prime(x) = 2c_2 + 6xd_2$$
$$S_1\prime(x_2) = S_2\prime(x_2) : b_1 + 4c_1 + 12d_1 = b_2 + 4c_2 + 12d_2$$
$$S_1\prime\prime(x_2) = S_2\prime\prime(x_2) : 2c_1 + 12d_1 = 2c_2 + 12d_2$$
$$S_1\prime\prime(x_1) = 0 : 2c_1 + 0 = 0$$
$$S_2\prime\prime(x_3) = 0 : 2c_2 + 24d_2 = 0$$

$a_1 = 0$ $c_1 = 0$ $c_2 = 12d_1$ $d_1 = -d_2$

Verify that $f(x) = x^3 + 8x - 7$ has a root in $[0, 1]$. Use fixed-point iteration to find this root to 5 decimal places. Start with $x_0 = 0.75$, but verify that your chosen $g(x)$ will work before you begin.:

First verify that $f(a) * f(b) < 0$.
$f(0) = 0^3 + 8 * 0 - 7 = -7, f(1) = 1^3 + 8 * 1 - 7 = 2 \rightarrow f(a) * f(b) \rightarrow -7 * 2 < 0$.
Therefore since there is a sign change there must also be a root in the interval $[0, 1]$.

$f(x) = x^3 + 8x - 7 \rightarrow \frac{7-x^3}{8}$. Take $g(x) = \frac{7-x^3}{8}$

Since $|g\prime(x)| = |\frac{-3x^2}{8}| = |-\frac{3}{8}x^2| \leq \frac{3}{8} < 1$ on $[0, 1]$ then $x_{n+1} = g(x_n)$ will generate a convergent sequence.

$$x_0 = 0.75, x_1 = g(0.75) = \frac{7 - 0.75^3}{8} = 0.82227$$
$$x_2 = g(0.82227) = 0.80551$$
$$x_3 = g(0.80551) = 0.80967$$
$$x_4 = g(0.80967) = 0.80865$$
$$x_5 = g(0.80865) = 0.80890$$
$$x_6 = g(0.80890) = 0.80884$$
$$x_7 = g(0.80884) = 0.80885$$
$$x_8 = g(0.80885) = 0.80885$$

Check $f(0.80885) \simeq -0.00002$ therefore very close to 0 and the root is 0.80885.

2. **Use Newton's Method to find the point of intersection of the curves $y = x^3$ and $y = cos(x)$ to 6 decimal places. Start with $x_0 = 1$:**

If $x^3 = cos(x)$ then $f(x) = x^3 - cos(x) = 0$. So $f\prime(x) = 3x^2 + sin(x)$.

So $x_{n+1} = x_n - \frac{f(x_n)}{f\prime(x_n)} = x_n - \frac{x^3 - cos(x)}{3x^2 + sin(x)} = \frac{2x^3 + x * sin(x) + cos(x)}{3x^2 + sin(x)}$

$$x_0 = 1, x_1 = \frac{2 * 1^3 + 1 * sin(1) + cos(1)}{3 * 1^2 + sin(1)} = 0.880333$$
$$x_2 = g(0.880333) = 0.865684$$
$$x_3 = g(0.865684) = 0.865474$$
$$x_4 = g(0.865474) = 0.865474$$

Check $f(0.865474) \simeq -9.96 * 10^{-8}$ which is very close to 0 therefore okay, in addition $x^3 = 0.865474^3 \simeq 0.65 \simeq cos(0.865464) = cos(x)$, and the root solution to the intersection of the curves is $x = 0.865474$.

3. **Solve the following dynamical systems. Find and classify any equilibria.:**

Firstly we know that when $a_{n+1} = r * a_n + b$ then $a_n = r^n(a_0 - a) + a, a = \frac{b}{1-r}$ if $r \neq 1$.

(a) $a_{n+1} = (2/5)a_n, a_0 = 10$**:** $\rightarrow b = 0, r = 2/5 \rightarrow a = \frac{0}{1-r} \rightarrow a = 0$. Therefore the solution is $a_n = (2/5)^n(10)$ and the equilibrium is when $a = 0$ and since $|r| < 1$ the solution is stable.

(b) $a_{n+1} = (3/5)a_n + 100, a_0 = 20$**:** $\rightarrow b = 100, r = 3/5 \rightarrow a = \frac{100}{1-3/5} = 250$. Therefore the solution is $a_n = (3/5)^n(20 - 250) + 250$ and the equilibrium is when $a = \frac{100}{1-3/5} = 250$ and since $|r| < 1$ the solution is stable.

(c) $a_{n+1} = (-2/3)a_n + 500, a_0 = 25$**:** $\rightarrow b = 500, r = -2/3 \rightarrow a = \frac{500}{1--2/3} \rightarrow a = 300$. Therefore the solution is $a_n = (-2/3)^n(25 - 300) + 300$ and the equilbirium is when $a = \frac{500}{1--2/3} = 300$ and since $|r| < 1$ the solution is stable.

(d) $a_{n+1} = 3a_n - 30, a_0 = 0$**:** $\rightarrow b = -30, r = 3 \rightarrow a = \frac{-30}{1-3} \rightarrow a = 15$. Therefore the solution is $a_n = 3^n(0 - 15) + 15$ and the equilibrium is when $a = \frac{-30}{1-3} = 15$ and since $|r| > 1$ the solution is unstable.

4. **Suppose that Owls have Mice for their primary food source in a wildlife sanctuary. If M_n is the Mouse population after n years and O_n is the Owl, the following model has been suggested:**

$$M_{n+1} = 1.3M_n - 0.002O_nM_n$$
$$O_{n+1} = 0.6O_n + 0.0004O_nM_n$$

These equations can be rewritten as the following

$$M_{n+1} = M_n + \Delta M_n, \Delta M_n = M_n(0.3 - 0.002O_n)$$
$$O_{n+1} = O_n + \Delta O_n, \Delta O_n = O_n(-0.4 + 0.0004M_n)$$

(a) **What type of interaction is this ? How do the coefficients tell us ?:**

$$M_{n+1} = \underbrace{1.3}_{\text{Mouse population grows if no owls}} M_n - \underbrace{0.002}_{\text{Mouse population decreases with presence of owls}} O_nM_n$$
$$O_{n+1} = \underbrace{0.6}_{\text{Owl population decreases if no mice}} O_n + \underbrace{0.0004}_{\text{Owl population increased by presence of mice}} O_nM_n$$

The above mentioned interactions can also be seen from the Δ Equations.

$$\Delta M_n = M_n(\underbrace{0.3}_{\text{Unbounded growth in mouse population in absence of predator}} - \underbrace{0.002}_{\text{Predator population decreases mouses growth rate}} O_n)$$
$$\Delta O_n = O_n(\underbrace{-0.4}_{\text{Absence of prey causes predators to die off}} + \underbrace{0.0004}_{\text{The more prey the more the growth rate of the predator increases}} M_n)$$

Based on the coefficients this is a predator prey interaction as the Mice populations growth rate is decreased by an increase in the population of Owls.

(b) **Find the equilibrium values ?**
As taught in class when we have a system of equtions as follows:

$$\Delta N_n = 0 = N(a - bP) \rightarrow N = 0 \text{ or } P = \frac{a}{b}$$
$$\Delta P_n = 0 = P(cN - d) \rightarrow P = 0 \text{ or } N = \frac{d}{c}$$

Then the system has two fixed/equilbriums $(P, N) \rightarrow$ (0,0) and $(\frac{a}{b}, \frac{d}{c})$. Alternatively this can be rewritten using the original equations as follows:

$$x_{n+1} = ax_n - bx_n y_n$$
$$y_{n+1} = cy_n + dx_n y_n$$

Then the system has two fixed points (0,0) and $(\frac{1-c}{d}, \frac{a-1}{b})$.
In our case this gives equilibrium values of $(M, O) \rightarrow (\frac{1-c}{d}, \frac{a-1}{b}) \rightarrow (\frac{1-0.6}{0.0004}, \frac{1.3-1}{0.002}) \rightarrow (1000, 150)$ or $(\frac{a}{b}, \frac{d}{c}) \rightarrow (\frac{0.4}{0.0004}, \frac{0.3}{0.002}) \rightarrow (1000, 150)$ and the trivial solution of $(0, 0)$.

Putting in $M_0 = 1000$ and $O_0 = 150$ back into the equations to confirm they are indeed the equilbiriums.
$M_1 = 1.3 * M_0 - 0.002 * O_0 * M_0 = 1.3 * 1000 - 0.002 * 150 * 1500 = 1000$
$O_1 = 0.6 * O_0 + 0.0004 * O_0 * M_0 = 0.6 * 150 + 0.0004 * 150 * 1000 = 150.$

Therefore $(1000, 150)$ is an equilibrium value and its trivial to see $(0, 0)$ is obviously an equilibrium value.

(c) **Predict the long-term outcome if $M_0 = 1200$ and $O_0 = 100$**

```
double mouse = 1200;
double owl = 100;

for(int i = 0; i < 100; i++) {
    System.out.println("m_" + i + ": " + mouse + " o_" + i + ": " + owl);
    double newMouse = 1.3 * mouse - 0.002 * owl * mouse;
    double newOwl = 0.6 * owl + 0.0004 * owl * mouse;

    mouse = Math.max(0, newMouse);
    owl = Math.max(0, newOwl);
}
```

```
m_0: 1200.0 o_0: 100.0
m_1: 1320.0 o_1: 108.0
m_2: 1430.88 o_2: 121.824
m_3: 1511.51294976 o_3: 142.820610048
m_4: 1533.2164315276495 o_4: 172.04244666087007
m_5: 1465.624748704614 o_5: 208.7367904527881
m_6: 1293.4525612104476 o_6: 247.61399669278296
m_7: 1040.934413145911 o_7: 276.6791813012039
m_8: 777.2049746547647 o_8: 281.2094612677063
m_9: 573.2516826166988 o_9: 256.14863364752284
m_10: 451.5519169248868 o_10: 212.424234283878
m_11: 395.1763516179801 o_11: 165.82276864720134
m_12: 382.6707836449873 o_12: 125.70535587999817
m_13: 401.2644846525417 o_13: 94.66472034518726
m_14: 445.67264960012704 o_14: 71.99306829674777
m_15: 515.2037614788562 o_15: 56.02997757831045
m_16: 612.03117951467 o_16: 45.164728628554876
m_17: 740.3560890990823 o_17: 38.1557260311307
m_18: 905.9652676265191 o_18: 34.192965259136
m_19: 1115.7995700705999 o_19: 32.90683472425657
m_20: 1377.1045770163605 o_20: 34.43107364963785
m_21: 1695.4055718922612 o_21: 39.62471983558419
m_22: 2069.6673018721012 o_22: 50.64682021891821
m_23: 2480.923356931952 o_23: 72.31691923170686
m_24: 2866.374895964932 o_24: 115.15524514834534
m_25: 3066.1311570906023 o_25: 201.12438862176907
m_26: 2752.6229954097735 o_26: 367.3441349346634
m_27: 1556.0900679325755 o_27: 624.870446180824
m_28: 78.20769821919339 o_28: 763.8641457271253
m_29: 0.0 o_29: 482.21451007207077
m_30: 0.0 o_30: 289.32870604324245
m_31: 0.0 o_31: 173.59722362594547
m_32: 0.0 o_32: 104.15833417556728
m_33: 0.0 o_33: 62.495000505340364
m_34: 0.0 o_34: 37.49700030320422
m_35: 0.0 o_35: 22.49820018192253
m_36: 0.0 o_36: 13.498920109153518
m_37: 0.0 o_37: 8.099352065492111
m_38: 0.0 o_38: 4.859611239295266
m_39: 0.0 o_39: 2.91576674357716
m_40: 0.0 o_40: 1.749460046146296
m_41: 0.0 o_41: 1.0496760276877775
m_42: 0.0 o_42: 0.6298056166126664
m_43: 0.0 o_43: 0.37788336996759986
m_44: 0.0 o_44: 0.22673002198055991
```

```
m_45: 0.0 o_45: 0.13603801318833594
m_46: 0.0 o_46: 0.08162280791300157
m_47: 0.0 o_47: 0.04897368474780094
m_48: 0.0 o_48: 0.029384210848680564
m_49: 0.0 o_49: 0.01763052650920834
m_50: 0.0 o_50: 0.010578315905525004
m_51: 0.0 o_51: 0.006346989543315002
m_52: 0.0 o_52: 0.003808193725989001
m_53: 0.0 o_53: 0.002284916235593401
m_54: 0.0 o_54: 0.0013709497413560404
m_55: 0.0 o_55: 8.225698448136242E-4
m_56: 0.0 o_56: 4.935419068881745E-4
m_57: 0.0 o_57: 2.9612514413290467E-4
m_58: 0.0 o_58: 1.776750864797428E-4
m_59: 0.0 o_59: 1.0660505188784568E-4
m_60: 0.0 o_60: 6.39630311327074E-5
m_61: 0.0 o_61: 3.837781867962444E-5
m_62: 0.0 o_62: 2.3026691207774664E-5
m_63: 0.0 o_63: 1.3816014724664797E-5
m_64: 0.0 o_64: 8.289608834798877E-6
m_65: 0.0 o_65: 4.9737653008793265E-6
m_66: 0.0 o_66: 2.9842591805275958E-6
m_67: 0.0 o_67: 1.7905555083165574E-6
m_68: 0.0 o_68: 1.0743333049899343E-6
m_69: 0.0 o_69: 6.445999829939605E-7
m_70: 0.0 o_70: 3.867599897963763E-7
m_71: 0.0 o_71: 2.3205599387782577E-7
m_72: 0.0 o_72: 1.3923359632669545E-7
m_73: 0.0 o_73: 8.354015779601727E-8
m_74: 0.0 o_74: 5.012409467761036E-8
m_75: 0.0 o_75: 3.0074456806566214E-8
m_76: 0.0 o_76: 1.8044674083939727E-8
m_77: 0.0 o_77: 1.0826804450363835E-8
m_78: 0.0 o_78: 6.496082670218301E-9
m_79: 0.0 o_79: 3.8976496021309806E-9
m_80: 0.0 o_80: 2.3385897612785884E-9
m_81: 0.0 o_81: 1.403153856767153E-9
m_82: 0.0 o_82: 8.418923140602918E-10
m_83: 0.0 o_83: 5.05135388436175E-10
m_84: 0.0 o_84: 3.03081233061705E-10
m_85: 0.0 o_85: 1.81848739837023E-10
m_86: 0.0 o_86: 1.091092439022138E-10
m_87: 0.0 o_87: 6.546554634132828E-11
m_88: 0.0 o_88: 3.927932780479697E-11
m_89: 0.0 o_89: 2.356759668287818E-11
m_90: 0.0 o_90: 1.4140558009726907E-11
m_91: 0.0 o_91: 8.484334805836145E-12
m_92: 0.0 o_92: 5.090600883501686E-12
m_93: 0.0 o_93: 3.054360530101012E-12
m_94: 0.0 o_94: 1.832616318060607E-12
m_95: 0.0 o_95: 1.0995697908363643E-12
m_96: 0.0 o_96: 6.597418745018186E-13
m_97: 0.0 o_97: 3.9584512470109117E-13
m_98: 0.0 o_98: 2.3750707482065467E-13
m_99: 0.0 o_99: 1.425042448923928E-13
```

As can be seen from the above screenshots. There is first some cycling in the populations and at $n = 29$ the mice are completely wiped out, afterwards the owls die off as well due to the fact there are no mice left.