MATH 3800 F

Assignment 4

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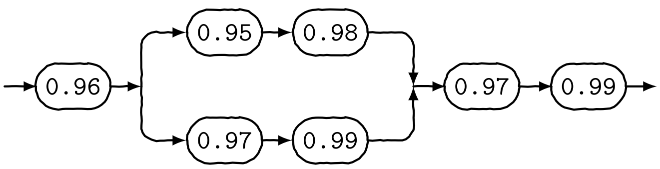
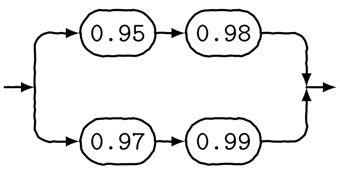
Winter 2020

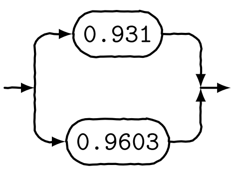
1. **Consider the following design proposal for a Mars lander module. What is the system reliability ?**



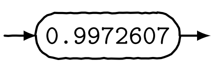
Parallel subsystem:



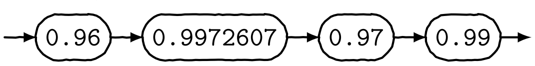
is equivalent to



Which is equivalent to



Meaning the entire system can be represented as



Meaning the entire systems reliability $R_s = (0.96)(0.9972607)(0.97)(0.99)$

2. **Find the Cholesky decomposition and use it to solve the system of equations.**

$$9x + 6y + 12z = 17.4 \tag{1}$$
$$6x + 13y + 11z = 23.6 \tag{2}$$
$$12x + 11y + 26z = 30.8 \tag{3}$$

$A = \begin{bmatrix} 9 & 6 & 12 \\ 6 & 13 & 11 \\ 12 & 11 & 26 \end{bmatrix}, b = \begin{bmatrix} 17.4 \\ 23.6 \\ 30.8 \end{bmatrix}$

$A = GG^T = \begin{bmatrix} g_{11} & 0 & 0 \\ g_{21} & g_{22} & 0 \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} g_{11} & g_{21} & g_{31} \\ 0 & g_{22} & g_{32} \\ 0 & 0 & g_{33} \end{bmatrix}$

$g_{11}^2 = 9 \Rightarrow g_{11} = 3$

$g_{11}g_{21} = 6 \Rightarrow g_{21} = 2$
$g_{11}g_{31} = 12 \Rightarrow g_{31} = 4$
$g_{21}^2 + g_{22}^2 = 13 \Rightarrow g_{22} = 3$
$g_{31}g_{21} + g_{22}g_{32} = 11 \Rightarrow g_{32} = 1$
$g_{31}^2 + g32^2 + g_{33}^2 = 26 \Rightarrow g_{33} = 3$

Therefore $G = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 1 & 3 \end{bmatrix}, G^T = \begin{bmatrix} 3 & 2 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

$A\bar{x} = b \Rightarrow GG^T\bar{x} = b$ and let $G^T\bar{x} = \bar{y}$ then we have that $G\bar{y} = b$ and $\begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 1 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 17.4 \\ 23.6 \\ 30.8 \end{bmatrix}$

Meaning $y_1 = 17.4/3 = 5.8, y_2 = (23.6 - 2 * 5.8)/3 = 4, y_3 = (30.8 - 4 * 5.8 - 1 * 4)/3 = 1.2$

Then $G^T\bar{x} = \bar{y} \Rightarrow \begin{bmatrix} 3 & 2 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5.8 \\ 4 \\ 1.2 \end{bmatrix}$

So $z = 1.2/3 = 0.4, y = (4 - 0.4)/3 = 1.2, x = (5.8 - 4 * 0.4 - 2 * 1.2)/3 = 0.6$, or $\bar{x} = \begin{bmatrix} 0.6 \\ 1.2 \\ 0.4 \end{bmatrix}$

Check just to make sure, substitute $\bar{x}$ back into the original set of equations $9(0.6) + 6(1.2) + (12) = 17.4, 6(0.6) + 13(1.2) + 11(0.4) = 23.6, 12(0.6) + 11(1.2) + 26(0.4) = 30.8$

3. **Use 5 iterations of Gauss-Seidel to solve the system, starting with $\mathbf{x = y = z = 1}$**

$$10x + y + z = 6 \tag{4}$$
$$x + 10y + z = 6 \tag{5}$$
$$x + y + 10z = 6 \tag{6}$$

Since the system is already diagonally dominant we can just directly solve for x,y,z.

```python
# Python 3
import numpy as np
np.set_printoptions(precision=4)

x = np.array([1, 1, 1, 1, 1, 1], dtype='float64')
y = np.array([1, 1, 1, 1, 1, 1], dtype='float64')
z = np.array([1, 1, 1, 1, 1, 1], dtype='float64')

for i in range(1, 6):
  x[i] = np.divide(np.subtract(6, np.add(y[i - 1], z[i - 1])), 10)
  y[i] = np.divide(np.subtract(6, np.add(x[i], z[i - 1])), 10)
  z[i] = np.divide(np.subtract(6, np.add(x[i], y[i])), 10)

print("x:", x)
print("y:", y)
print("z:", z)
```

```
x: [1.     0.4    0.5026 0.5002 0.5    0.5   ]
y: [1.     0.46   0.4983 0.5    0.5    0.5   ]
z: [1.     0.514  0.4999 0.5    0.5    0.5   ]
```

Check just to make sure, substitute $x = 0.5, y = 0.5, z = 0.5$ back into the system. $10(0.5) + (0.5) + (0.5) = 6, (0.5) + 10(0.5) + (0.5) = 6, (0.5) + (0.5) + 10(0.5) = 6$

4. **Consider the survival of whales. If the number of whales falls below a minum survival level m, the species will become extinct. The population is also limited by the carrying capacity M of the environment.**

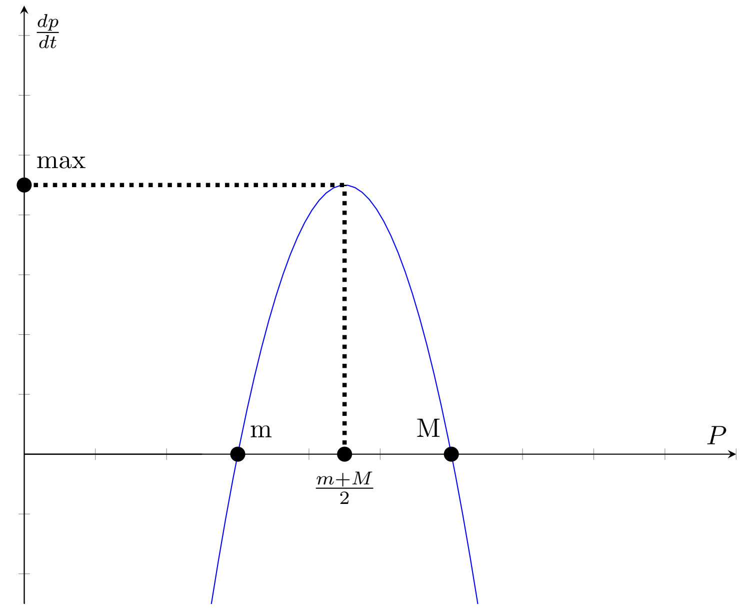
   (a) **(a) discuss the following model for the whale population $P(t)$ ($k > 0$ is a constant): $\frac{dp}{dt} = k(M - P)(P - m)$**

   Equilibrium when $\frac{dp}{dt} = 0$ which occurs when $P = m$ or $P = M$.

   It is easy to see that $\frac{dp}{dt} < 0$ if $P < m$ or if $P > M$
   $\frac{dp}{dt} > 0$ if $m < P < M$.

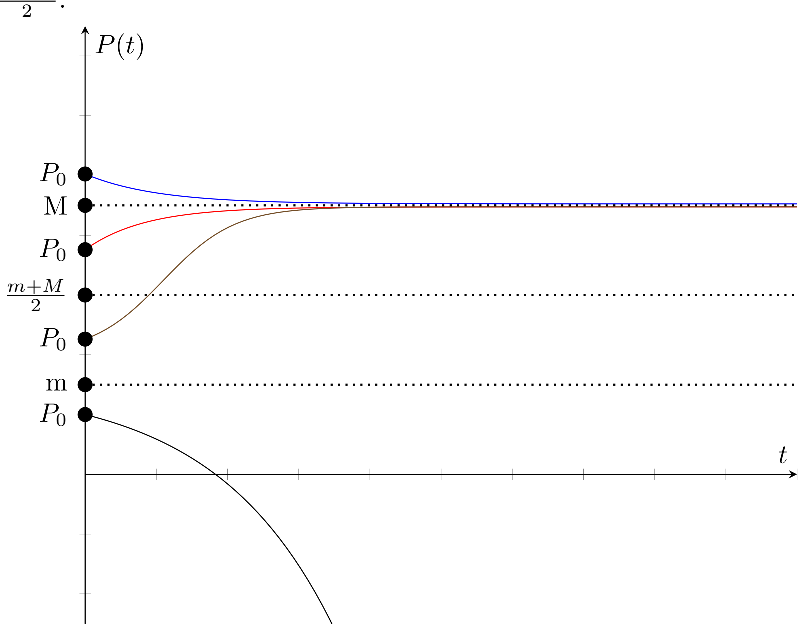
   Therefore if $P < m$ then the population will decline and go to extinction. But if $P > m$ then the population will approach the converging capacity which is M.

   (b) **(b) graph dP/dt vs P and P vs t, considering cases where $P_0 < m, m < P_0 < M$ and $P_0 > M$**



$\frac{dp}{dt}$ will reach a max value at $\frac{m+M}{2}$

$\frac{dp}{dt}$ is increasing for $P < \frac{m+M}{2}$ and decreasing when $P > \frac{m+M}{2}$. So $P(t)$ will have an inflection point at $\frac{m+M}{2}$.



The blue line shows P is decreasing and concave up. The horizontal line at M shows the carrying capacity and an equilibrium solution. The red line shows P is increasing and concave down, the entire portion between M and $\frac{m+M}{2}$ is concave down.. The horizontal line at $\frac{m+M}{2}$ shows the inflection point. The brown line shows P is increasing and concave up (between the third P_0 and $\frac{m+M}{2}$). The horizontal line at m shows minimum survival level and an equilibrium solution. Below that the black line shows P is decreasing and concave down.

(c) **(c) solve the model and show that $P(t) \Rightarrow M$ as $t \Rightarrow \infty$ (provided what ?)**

$\frac{dp}{dt} = k(M-P)(P-m) = \frac{dp}{(M-P)(P-m)} = kdt$

Partial Fractions gives us $\frac{1}{(M-P)(P-m)} = \frac{a}{M-P} + \frac{b}{P-m} \Rightarrow a = b = \frac{1}{M-m}$

Meaning we have $(\frac{1}{M-P} + \frac{1}{P-m})dp = k(M-m)dt$

Integrate on both sides $\int(\frac{1}{M-P} + \frac{1}{P-m})dp = \int k(M-m)dt$

$-ln|M-P| + ln|P-m| = k(M-m)t + C$

Take to e^x, $\frac{P-m}{M-P} = Ke^{k(M-m)t}$

Solving for P we get $P(t) = \frac{m+MKe^{k(M-m)t}}{1+Ke^{k(M-m)t}}$

$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{m+MKe^{k(M-m)t}}{1+Ke^{k(M-m)t}}$, using L'hopital's rule $\Rightarrow \lim_{t\to\infty} \frac{MKe^{k(M-m)t}}{Ke^{k(M-m)t}} = M$ Provided that $P_0 > m$.

(d) **(d) discuss how you would test the model and how you would determine m and M**

We could estimate M from a plot of data and estimate $\frac{m+M}{2}$ by looking for the change in concavity and the maximum growth location. We could then estimate for m utilizing M. Then if we plot $ln\frac{P-m}{M-P}$ versus t we should have a linear function which we can use to test the model. If the population falls below m, the population will go to zero rapidly. So care must be taken to ensure $P > m$ at all times.