Any small mistakes get (-1)

only answers get no point !.

Student ID:____________________ **Name:** Suggested Solutions

1. **[20 points]** Let the random variable X (folded normal distribution) have the pdf

$$f(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2}, \quad 0 < x < \infty$$

3 pts

(a) Find the mean of X.

$$E[X] = \int_0^\infty x \frac{2}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$$= -\frac{2}{\sqrt{2\pi}} e^{-x^2/2} \Big|_0^\infty$$

$$= -\frac{2}{\sqrt{2\pi}} [0 - 1] = \sqrt{\frac{2}{\pi}}$$

7 pts

(b) Find the variance of X.

$$E[X^2] = \int_0^\infty \frac{2}{\sqrt{2\pi}} x^2 e^{-x^2/2} dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \underbrace{x}_{u} \cdot \underbrace{x e^{-x^2/2} dx}_{dv} \qquad v = -e^{-x^2/2}$$

integration by parts

$$= \sqrt{\frac{2}{\pi}} \left\{ \left[-x e^{-x^2/2} \Big|_0^\infty \right] - \int_0^\infty -e^{-x^2/2} dx \right\}$$

$$= \sqrt{\frac{2}{\pi}} \left\{ (0 - 0) + \int_0^\infty e^{-x^2/2} dx \right\}$$

$$= \underbrace{\int_0^\infty \frac{2}{\sqrt{2\pi}} e^{-x^2/2} dx}_{\text{Folded Normal}} = 1, \quad \text{Then } Var(X) = 1 - \frac{2}{\pi}.$$

Wrong integration process (-3)

showing correct answers and wrong answers will get points off

- correct $E(X^2)$ and no answer for $Var(X)$ → +2
- correct integration process but wrong answers → +2
- wrong integration process and wrong answers → +1
- correct integration process, no answer for $E(X^2)$, $Var(X)$ → +2
- $\int x^2 e^{-\frac{x^2}{2}} dx$ no integration process but correct answers → -3

10 pts (c) Find the transformation of $g(X) = Y$ and values of α and β so that $Y \sim Gamma(\alpha, \beta)$.

$$* \quad f_Y(y) = \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-y/\beta}, \quad 0<y<\infty \quad \text{for } Y \sim Gamma(\alpha,\beta)$$

Let $Y = X^2$. because $x>0$, $X = \sqrt{Y}$.

Then $f_Y(y) = f_X(\sqrt{y}) \left|\frac{\partial}{\partial y}\sqrt{y}\right|$

$$= \frac{2}{\sqrt{2\pi}} e^{-y/2} \cdot \frac{1}{2\sqrt{y}}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{y}} e^{-y/2}$$

$$= \frac{1}{\Gamma(\frac{1}{2})\, 2^{\frac{1}{2}}} y^{\frac{1}{2}-1} e^{-y/2}, \quad y>0.$$

without range -1

Thus $\alpha = \frac{1}{2}$, $\beta = 2$, $g(x) = x^2$.

no working process -3

- only ' $f_X(g^{-1}(y)) \cdot \left|\frac{d}{dy} g^{-1}(y)\right|$ ' $\to +1$
- " and one of $\alpha, \beta, g(x)$ $\to +2$

2. [**15 points**] Let X have a zero-truncated Poisson distribution with parameter λ. That is,

$$f_X(x) = \frac{\lambda^x e^{-\lambda}}{x!(1-e^{-\lambda})}, \quad x = 1, 2, 3, \ldots$$

10 pts (a) Find the mgf of X.

$$Ee^{tx} = \sum_{x=0}^{\infty} e^{tx}\left(\frac{e^{-\lambda}}{1-e^{-\lambda}}\right)\frac{\lambda^x}{x!} = \sum_{x=1}^{\infty} \frac{e^{-\lambda}}{1-e^{-\lambda}} \frac{(\lambda e^t)^x}{x!}$$

$$= \sum_{x=1}^{\infty} \underbrace{\frac{(\lambda e^t)^x e^{-\lambda e^t}}{x!(1-e^{-\lambda e^t})}}_{\text{pmf of truncated Poisson}(\lambda e^t)} \times \frac{e^{\lambda e^t} e^{-\lambda}(1-e^{-\lambda e^t})}{(1-e^{-\lambda})}$$

$$= \frac{e^{\lambda(e^t-1)}(1-e^{-\lambda e^t})}{1-e^{-\lambda}} \quad \text{—①}$$

$$\left(\text{or, } \frac{e^{-\lambda}(e^{\lambda e^t}-1)}{1-e^{-\lambda}}, \text{ or } \frac{e^{\lambda e^t}-1}{e^{\lambda}-1}\right)$$

② ③

①, ②, ③ are the same.

5 pts (b) Calculate E(X).

From ③, $\frac{\partial}{\partial t} Ee^{tx} = \frac{\lambda e^t \cdot e^{\lambda e^t}}{e^{\lambda}-1}$.

when $t=0$, it becomes $\frac{\lambda e^{\lambda}}{e^{\lambda}-1}$ //

(b)
+1 → wrong calculation + answer
+3 → differentiation is correct, (mgf) but wrong answer
(notation mistake → −1)

+9 → (a) · correct answer but, notation mistake ex) range of x 3 in $\sum_x$, no explation about distribution when sum equals to zero
+6 → · include the process of using taylor expansion, or the distribution of poisson & zero-truncated poisson correctly but wrong answer
+3 → · wrong calculation (misusing taylor & poisson)
+1 → only the answer

3. **[20 points]** Let X follows Pareto distribution. Its pdf is

$$f_X(x) = \beta\alpha^\beta x^{-(\beta+1)}, \quad \alpha > 0, \ \beta > 0, \ \alpha \le x < \infty$$

7 pts

(a) Find EX^r.

$$EX^r = \int_\alpha^\infty \beta \alpha^\beta x^{-(\beta-r+1)} dx$$

$$= \beta\alpha^\beta \int_\alpha^\infty x^{-(\beta-r+1)} dx$$

$$= \beta\alpha^\beta \cdot \left(-\frac{1}{\beta-r} x^{-(\beta-r)} \Big|_\alpha^\infty\right)$$

$$= \beta\alpha^\beta \left(-\frac{1}{\beta-r}\left(0 - \alpha^{-(\beta-r)}\right)\right) \quad \text{when } \beta > r.$$

$$= \frac{\beta\alpha^r}{\beta-r}, \quad \text{if } \beta > r$$

−2 w/o this range

−2 wrong this range

* −1 wrong integral range / wrong calculation.

* −1 wrong answer

* −2 wrong process / no more calculation.

* $EX^r = \frac{d}{dx^r} M_X^r(t)\Big|_{t=0}$, with wrong answer gets 1 pts.

5pts (b) Find the mgf of $Y = log(X)$ for $|t| < \beta$.

$$M_Y(t) = Ee^{tY} = Ee^{t\log x} = EX^t$$

$$= \frac{\beta\alpha^t}{\beta - t} \quad \text{for } |t|<\beta.$$

* <transformation>, -1 wrong integral range.
* <transformation>, -1 Pareto pdf
* correct process with no answer -2pts
  ↳ wrong calculation
* wrong answer (only) -1 pts

8pts (c) Using the result in (b), find E(Y).

$$E[Y] = \frac{\partial}{\partial t} M_Y(t)\Big|_{t=0}.$$

$$\frac{\partial}{\partial t} M_Y(t) = \frac{\partial}{\partial t} \beta\alpha^t \cdot (\beta - t)^{-1}$$

$$= \frac{\partial}{\partial t} \beta \cdot e^{t\log\alpha} (\beta - t)^{-1}$$

$$= \beta \cdot \log\alpha \, e^{t\log\alpha} (\beta - t)^{-1} - \beta\alpha^t (\beta - t)^{-2}(-1).$$

$$\frac{\partial}{\partial t} M_Y(0) = \beta(\log\alpha) \cdot \frac{1}{\beta} + \beta\beta^{-2}$$

$$= \log\alpha + \frac{1}{\beta}.$$

* $\boxed{\frac{\partial}{\partial t}\alpha^t = \alpha^t \text{ gets } -3}$

⊕ -3 mistake differentiation.

✓ Correct process with no answer -4pts (→ (b) wrong answer)

✓ wrong calculation with correct answer -1 pts

✓ wrong calculation with wrong answer -2pts

4. **[15 points]** Let X have the negative binomial distribution with pmf

$$f(x) = \binom{r+x-1}{x} p^r (1-p)^x, \quad x = 0, 1, 2, ..., \quad 0 < p < 1, \quad r > 0$$

5pts (a) Find the mgf of X.

$$Ee^{tX} = \sum_{x=0}^{\infty} \binom{r+x-1}{x} p^r (e^t(1-p))^x$$

correct form of e^{tx} & pmf (+1)

$$= \sum_{x=0}^{\infty} \underbrace{\binom{r+x-1}{x} [1-e^t(1-p)]^r [e^t(1-p)]^x}_{\text{pmf of } NB(r,\, e^t(1-p))} \cdot \frac{p^r}{1-e^t(1-p)}$$

correct process (+2)

$$= \left[\frac{p}{1-(1-p)e^t}\right]^r \quad \text{where } 0 < e^t(1-p) < 1$$

→범위?

correct answer (+1)

answer (+2)

range (+1)

$$\Rightarrow t < -\log(1-p)$$

-1 w/o t

- no working process gets almost zero points
- correct Ee^{tx} with pmf form : +1
- correct calculation process : +2
- correct answer [form of mgf of x : +1 ; range of t : +1]

10 pts (b) Let $Y = 2pX$. Show that, as $p \downarrow 0$, the mgf of Y converges to that of a chi squared random variable with $2r$ degrees of freedom by showing that

$$\lim_{p\to 0} M_Y(t) = \left(\frac{1}{1-2t}\right)^r. \qquad \text{(Don't need to show that } t < 1/2.\text{)}$$

$$M_Y(t) = E e^{t(2pX)} = E e^{(2pt)X} \longrightarrow +2 \text{ points}$$

$M_Y(t)$ form

$$= \left[\frac{p}{1-(1-p)e^{2pt}}\right]^n \longrightarrow +2 \text{ points}$$

4 points

By L'Hopital's rule

Now, $\displaystyle \lim_{p\to 0} \frac{p}{1-(1-p)e^{2pt}} = \lim_{p\to 0} \frac{1}{e^{2pt} - 2t(1-p)e^{2pt}}$

process, l'hopital

$$= \frac{1}{1-2t}$$

5 points

$$\therefore \lim_{p\to 0} M_Y(t) = \left(\frac{1}{1-2t}\right)^n.$$

1 points

* wrong mgf (-4)

wrote relationship between mgf

, [$M_X(2pt) = M_Y(t)$; +2

correct $M_Y(t)$ form ; +2

- only answer gets 1 point

, wrote jacobian (continuous transformation) [with all process ; -9

just wrote, but solved with right process ; -1

, without l'hopital / solving process & only result ; -3

, used l'hopital but wrong calculation ; -2

5. **[15 points]** Let X have the pdf

$$f(x) = \frac{4}{\beta^3\sqrt{\pi}} x^2 e^{-x^2/\beta^2}, \quad 0 < x < \infty, \ \beta > 0$$

Verify that $f(x)$ is a pdf.

$$\int_0^\infty f(x)dx = \int_0^\infty \frac{4}{\beta^3\sqrt{\pi}} \underbrace{x}_{u} \cdot \underbrace{x e^{-x^2/\beta^2}}_{dv} dx$$

By integration by parts,

$$V = -\frac{\beta^2}{2} e^{-x^2/\beta^2}$$

$$\int_0^\infty f(x)dx = \frac{4}{\beta^3\sqrt{\pi}}\left[-\frac{\beta^2}{2} x e^{-x^2/\beta^2} \Big|_0^\infty - \int_0^\infty -\frac{\beta^2}{2} e^{-x^2/\beta^2} dx \right]$$

$$= \frac{4}{\beta^3\sqrt{\pi}}\left[-\frac{\beta^2}{2}(0-0) + \frac{\beta^2}{2}\int_0^\infty e^{-x^2/\beta^2} dx \right]$$

$$= \frac{2}{\beta\sqrt{\pi}} \int_0^\infty e^{-x^2/\beta^2} dx$$

$$= \frac{1}{\beta\sqrt{\pi}} \int_{-\infty}^\infty e^{-x^2/\beta^2} dx$$

-3.

$$\left(= \frac{1}{\beta\sqrt{\pi}} \sqrt{\pi\beta^2} \int_{-\infty}^\infty \underbrace{\frac{1}{\sqrt{2\pi(\beta^2/2)}} e^{-x^2/2(\beta^2/2)}}_{\text{pdf of } N(0, \frac{\beta^2}{2})} dx \right.$$

$$= \frac{\sqrt{\pi\beta^2}}{\beta\sqrt{\pi}} = 1.$$

notation ~~[illegible]~~ mistake -1

(integration by parts) setting mistake -10

without explanation about distribution (normal, gamma ..) -3

without explanation -3

6. [**15 points**] The mgf of $f(x) = \frac{2x}{c^2}, \quad 0 < x < c$ can be expressed as

$$\frac{2}{c^2t^2}a(t).$$

Find $a(t)$.

$$M_X(t) = \int_0^c e^{tx} f(x)\,dx = \int_0^c e^{tx}\frac{2x}{c^2}\,dx = \frac{2}{c^2}\int_0^c \underbrace{x}_{u}\underbrace{e^{tx}}_{dv}\,dx$$

$dv.\ v = \frac{1}{t}e^{tx}$

Then, by integration by parts

$$M_X(t) = \frac{2}{c^2}\left[\frac{x}{t}e^{tx}\Big|_0^c - \int_0^c \frac{1}{t}e^{tx}\,dx\right]$$

⇒ (-5) if this is wrong

$$= \frac{2}{c^2}\left[\frac{c}{t}e^{tc} - \frac{1}{t^2}e^{tx}\Big|_0^c\right]$$

$$= \frac{2}{c^2}\left[\frac{c}{t}e^{tc} - \frac{1}{t^2}(e^{tc}-1)\right]$$

$$= \frac{2}{ct}e^{tc} - \frac{2}{c^2t^2}(e^{tc}-1)$$

Or, $\frac{2}{c^2t^2}(cte^{tc} - e^{tc} + 1)$

$\therefore a(t) = cte^{ct} - e^{ct} + 1$

$(= e^{ct}(ct-1)+1)$

almost all wrong +1.

No working process : -10 ~ -15

∴ such as Integration by parts.

mgf of f(x) is not $\frac{2x}{c^2}$, f(x) is $\frac{2x}{c^2}$. ⇒ (-10)