ST509\_Midterm\_HJ

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# 1. Introduction

$$
y\_i |x\_i \sim{} Poisson(\mu\_{\beta}(x\_i)) \\
log(\mu\_\beta{(x\_i)}) = \beta\_0 + \beta^TX \\
-\frac{1}{N} \sum^{N}\_{i=1}\{{y\_{i}(\beta\_0 + \beta^Tx\_i) - e^{\beta\_0+\beta^Tx\_i}}\} + \lambda ||\beta||\_1
$$

When the response variable Y is nonnegative and represents a count, its mean will be positive and the Poisson likelihood is often used for inference The l\_{1}-penalized negative log-likelihood is given by Typically, We do not penalize the intercept beta0. It is easy to see that this enforces the constraint that the average fitted value is equal to the mean response :

# 2. Competing Methods

## Poisson Regression

$$
log(\mu\_\beta{(x\_i)}) = \beta\_{0} + \beta^{TX} \\
$$

$$
\mu\_\beta({x\_i})= \lambda\_{i} = exp(\beta\_{0} + \beta^{TX}) \\
$$

## 1. Unpenalized Poisson Regression

##&= \frac{\exp^{-\exp(\beta\_0 + \beta^T X)} \times \exp(\beta\_0 + \beta^T X)^{y\_i}}{y\_i!} \\  
##X\_i \beta &= \beta\_0 + \beta^T X\_i \\  
  
##L(\beta) &= \prod\_{i=1}^n \frac{e^{-\exp(X\_i \beta)} \exp(X\_i \beta)^{y\_i}}{y\_i!} \\  
##l(\beta) &= \sum\_{i=1}^n \left(-\exp(X\_i \beta) + y\_i X\_i \beta - \log(y\_i!)\right) \\  
  
#\beta\_{OLS} &= \arg \min\_{\beta} l'(\beta)   
#\end{align}

library(MASS)  
generate\_data <- function(n, p, nu) {  
 mu <- rep(0, p)  
 sigma <- outer(1:p, 1:p, FUN = function(i, j) nu^(abs(i-j)))  
 x <- mvrnorm(n=n, mu=mu, Sigma=sigma)  
 beta0 = 1  
 beta = c(rep(1,3), rep(0, p-3))  
 mu\_x <- exp(beta0 + x %\*% beta)  
 y <- rpois(n, mu\_x)  
 return(list(x = x, y = y))  
}  
set.seed(2024020409)  
train\_data <- generate\_data(500, 50, 0.7)  
test\_data <- generate\_data(500, 50, 0.7)

my\_poi <- function(X, y, init = NULL, max\_iter = 1000, eps = 1e-5) {  
 # Scaling  
 y\_sd <- sd(y); X\_sd <- apply(X, 2, sd)  
 y <- y /y\_sd; X <- t(t(X)/ X\_sd)  
 # Add intercept term  
 X <- cbind(1, X) ; n <- nrow(X); p <- ncol(X)   
 # INIT  
 if (is.null(init)) init <- rep(0, p);  
 beta <- init   
 # Iteration  
 for (iter in 1:max\_iter) {  
 eta <- X %\*% beta  
 w <- exp(eta) # Mean = Variance  
   
 # Limit w   
 w <- pmin(w, 1e6)  
 w <- pmax(w, 1e-6)  
   
 z <- eta + (y-w)/w  
 # IRLS  
 X\_tilde <- diag(c(sqrt(w))) %\*% X  
 z\_tilde <- diag(c(sqrt(w))) %\*% z  
 qr\_obj <- qr(X\_tilde)  
 new\_beta <- backsolve(qr\_obj$qr, qr.qty(qr\_obj, z\_tilde)) # Check Convergence  
 if (max(abs(new\_beta - beta))/ max(abs(beta)) < eps) break  
 beta <- new\_beta  
 }  
 # Warning  
 if (iter == max\_iter) warning("Algorithm may not have converged!")  
 # Restore beta coef  
 beta <- c(beta) \* c(1, 1/X\_sd) + c(log(y\_sd), rep(0, p-1))  
 # Result  
 list(X\_tilde = dim(X\_tilde), est = t(beta), iterations = iter)  
}

train\_data <- generate\_data(500, 50, 0.7)  
y <- train\_data$y ; x <-train\_data$x  
my\_poi(x, y, init = rep(3, 51))

## $X\_tilde  
## [1] 500 51  
##   
## $est  
## [,1] [,2] [,3] [,4] [,5] [,6] [,7]  
## [1,] 0.9825567 1.003834 0.9952995 1.014792 -0.01483829 0.02260662 -0.03286948  
## [,8] [,9] [,10] [,11] [,12] [,13]  
## [1,] 0.01534072 -0.009470657 0.01184592 4.996025e-05 -0.03623518 0.03685734  
## [,14] [,15] [,16] [,17] [,18] [,19]  
## [1,] -0.008090208 0.008525065 -0.0003296124 0.006840637 -0.03829035 0.04146782  
## [,20] [,21] [,22] [,23] [,24] [,25]  
## [1,] -0.01138962 -0.005552458 0.001087698 0.0008345985 0.004879776 0.004859562  
## [,26] [,27] [,28] [,29] [,30] [,31]  
## [1,] -0.02066295 0.03018091 -0.01809376 0.002426706 -0.006006737 -0.0090391  
## [,32] [,33] [,34] [,35] [,36] [,37]  
## [1,] 0.00201061 -0.01714482 0.02848699 -0.01523338 -0.01929848 0.02232584  
## [,38] [,39] [,40] [,41] [,42] [,43]  
## [1,] 0.008763946 0.002053838 0.01180394 0.01938962 -0.01431178 -0.01536064  
## [,44] [,45] [,46] [,47] [,48] [,49]  
## [1,] 0.02951308 -0.03035566 0.0002294486 -0.008802028 -0.007037254 -0.008278157  
## [,50] [,51]  
## [1,] 0.01793886 -0.007693518  
##   
## $iterations  
## [1] 161

for (i in seq(0.1, 3, 0.5)){  
   
 # Define Initial Beta  
 init = rep(i, 51)  
   
 # my\_poi  
 result = my\_poi(x, y, init = init)  
   
 # print  
 print(i)  
 print(result)  
   
}

## [1] 0.1  
## $X\_tilde  
## [1] 500 51  
##   
## $est  
## [,1] [,2] [,3] [,4] [,5] [,6] [,7]  
## [1,] 0.9825952 1.003829 0.9952839 1.014785 -0.01483794 0.02261185 -0.03286902  
## [,8] [,9] [,10] [,11] [,12] [,13]  
## [1,] 0.01533666 -0.009465885 0.01185055 3.685509e-05 -0.03623076 0.03686773  
## [,14] [,15] [,16] [,17] [,18] [,19]  
## [1,] -0.008099618 0.008529082 -0.0003259425 0.0068286 -0.03828348 0.04147563  
## [,20] [,21] [,22] [,23] [,24] [,25]  
## [1,] -0.01139421 -0.005548972 0.001087802 0.0008312723 0.004887542 0.004858939  
## [,26] [,27] [,28] [,29] [,30] [,31]  
## [1,] -0.020666 0.03018648 -0.01809607 0.002429568 -0.006008412 -0.009036358  
## [,32] [,33] [,34] [,35] [,36] [,37]  
## [1,] 0.002008238 -0.0171496 0.02849514 -0.01523996 -0.01929958 0.02233258  
## [,38] [,39] [,40] [,41] [,42] [,43]  
## [1,] 0.008767953 0.00205486 0.0118 0.01938935 -0.01430794 -0.01536184  
## [,44] [,45] [,46] [,47] [,48] [,49]  
## [1,] 0.02951776 -0.03036411 0.0002384951 -0.008802477 -0.007036759 -0.00828489  
## [,50] [,51]  
## [1,] 0.01793475 -0.007687592  
##   
## $iterations  
## [1] 10  
##   
## [1] 0.6  
## $X\_tilde  
## [1] 500 51  
##   
## $est  
## [,1] [,2] [,3] [,4] [,5] [,6] [,7]  
## [1,] 0.9825873 1.003838 0.995282 1.014787 -0.01483538 0.02261606 -0.03287671  
## [,8] [,9] [,10] [,11] [,12] [,13]  
## [1,] 0.01534183 -0.009470989 0.01185652 3.898594e-05 -0.03622622 0.03685359  
## [,14] [,15] [,16] [,17] [,18] [,19]  
## [1,] -0.008088954 0.008525032 -0.0003236105 0.006831602 -0.03828754 0.04147646  
## [,20] [,21] [,22] [,23] [,24] [,25]  
## [1,] -0.01138955 -0.005561338 0.001092402 0.000825092 0.004890592 0.004865252  
## [,26] [,27] [,28] [,29] [,30] [,31]  
## [1,] -0.0206659 0.0301787 -0.01809186 0.00241935 -0.00600741 -0.009039012  
## [,32] [,33] [,34] [,35] [,36] [,37]  
## [1,] 0.00201139 -0.01715077 0.02849282 -0.01523349 -0.01930145 0.0223268  
## [,38] [,39] [,40] [,41] [,42] [,43]  
## [1,] 0.008769815 0.002057028 0.01181125 0.01938418 -0.01431658 -0.01535879  
## [,44] [,45] [,46] [,47] [,48] [,49]  
## [1,] 0.02952028 -0.03037087 0.0002398591 -0.008800539 -0.007044089 -0.008280081  
## [,50] [,51]  
## [1,] 0.017944 -0.007692163  
##   
## $iterations  
## [1] 35  
##   
## [1] 1.1  
## $X\_tilde  
## [1] 500 51  
##   
## $est  
## [,1] [,2] [,3] [,4] [,5] [,6] [,7]  
## [1,] 0.9825566 1.003834 0.9952996 1.014792 -0.0148383 0.0226066 -0.03286946  
## [,8] [,9] [,10] [,11] [,12] [,13]  
## [1,] 0.01534072 -0.009470656 0.0118459 4.99893e-05 -0.0362352 0.03685734  
## [,14] [,15] [,16] [,17] [,18] [,19]  
## [1,] -0.008090208 0.008525064 -0.0003296284 0.006840662 -0.03829036 0.0414678  
## [,20] [,21] [,22] [,23] [,24] [,25]  
## [1,] -0.01138962 -0.005552437 0.001087687 0.0008346216 0.00487975 0.004859545  
## [,26] [,27] [,28] [,29] [,30] [,31]  
## [1,] -0.02066295 0.03018092 -0.01809376 0.002426723 -0.006006734 -0.0090391  
## [,32] [,33] [,34] [,35] [,36] [,37]  
## [1,] 0.002010607 -0.0171448 0.02848698 -0.01523338 -0.01929847 0.02232584  
## [,38] [,39] [,40] [,41] [,42] [,43]  
## [1,] 0.00876393 0.00205383 0.01180392 0.01938963 -0.01431177 -0.01536064  
## [,44] [,45] [,46] [,47] [,48] [,49]  
## [1,] 0.02951306 -0.03035562 0.0002294216 -0.008802032 -0.007037237 -0.00827815  
## [,50] [,51]  
## [1,] 0.01793885 -0.007693522  
##   
## $iterations  
## [1] 62  
##   
## [1] 1.6  
## $X\_tilde  
## [1] 500 51  
##   
## $est  
## [,1] [,2] [,3] [,4] [,5] [,6] [,7]  
## [1,] 0.9825566 1.003834 0.9952996 1.014792 -0.0148383 0.0226066 -0.03286946  
## [,8] [,9] [,10] [,11] [,12] [,13]  
## [1,] 0.01534072 -0.009470656 0.0118459 4.998393e-05 -0.03623519 0.03685734  
## [,14] [,15] [,16] [,17] [,18] [,19]  
## [1,] -0.008090208 0.008525064 -0.0003296255 0.006840657 -0.03829036 0.0414678  
## [,20] [,21] [,22] [,23] [,24] [,25]  
## [1,] -0.01138962 -0.005552441 0.001087689 0.0008346174 0.004879754 0.004859548  
## [,26] [,27] [,28] [,29] [,30] [,31]  
## [1,] -0.02066295 0.03018092 -0.01809376 0.00242672 -0.006006735 -0.0090391  
## [,32] [,33] [,34] [,35] [,36] [,37]  
## [1,] 0.002010608 -0.01714481 0.02848698 -0.01523338 -0.01929847 0.02232584  
## [,38] [,39] [,40] [,41] [,42] [,43]  
## [1,] 0.008763933 0.002053832 0.01180392 0.01938963 -0.01431177 -0.01536064  
## [,44] [,45] [,46] [,47] [,48] [,49]  
## [1,] 0.02951306 -0.03035562 0.0002294266 -0.008802031 -0.00703724 -0.008278152  
## [,50] [,51]  
## [1,] 0.01793885 -0.007693521  
##   
## $iterations  
## [1] 88  
##   
## [1] 2.1  
## $X\_tilde  
## [1] 500 51  
##   
## $est  
## [,1] [,2] [,3] [,4] [,5] [,6] [,7]  
## [1,] 0.9825567 1.003834 0.9952995 1.014792 -0.01483829 0.02260662 -0.03286948  
## [,8] [,9] [,10] [,11] [,12] [,13]  
## [1,] 0.01534072 -0.009470657 0.01184592 4.99633e-05 -0.03623518 0.03685734  
## [,14] [,15] [,16] [,17] [,18] [,19]  
## [1,] -0.008090208 0.008525064 -0.0003296141 0.00684064 -0.03829036 0.04146782  
## [,20] [,21] [,22] [,23] [,24] [,25]  
## [1,] -0.01138962 -0.005552456 0.001087697 0.0008346009 0.004879773 0.00485956  
## [,26] [,27] [,28] [,29] [,30] [,31]  
## [1,] -0.02066295 0.03018091 -0.01809376 0.002426708 -0.006006737 -0.0090391  
## [,32] [,33] [,34] [,35] [,36] [,37]  
## [1,] 0.00201061 -0.01714482 0.02848699 -0.01523338 -0.01929848 0.02232584  
## [,38] [,39] [,40] [,41] [,42] [,43]  
## [1,] 0.008763944 0.002053837 0.01180394 0.01938962 -0.01431178 -0.01536064  
## [,44] [,45] [,46] [,47] [,48] [,49]  
## [1,] 0.02951308 -0.03035565 0.0002294458 -0.008802028 -0.007037252 -0.008278156  
## [,50] [,51]  
## [1,] 0.01793886 -0.007693519  
##   
## $iterations  
## [1] 114  
##   
## [1] 2.6  
## $X\_tilde  
## [1] 500 51  
##   
## $est  
## [,1] [,2] [,3] [,4] [,5] [,6] [,7]  
## [1,] 0.9825569 1.003834 0.9952994 1.014792 -0.01483828 0.02260667 -0.03286952  
## [,8] [,9] [,10] [,11] [,12] [,13]  
## [1,] 0.01534073 -0.009470659 0.01184599 4.989218e-05 -0.03623513 0.03685732  
## [,14] [,15] [,16] [,17] [,18] [,19]  
## [1,] -0.008090206 0.008525066 -0.0003295748 0.00684058 -0.03829033 0.04146787  
## [,20] [,21] [,22] [,23] [,24] [,25]  
## [1,] -0.01138962 -0.005552508 0.001087724 0.0008345432 0.004879838 0.004859601  
## [,26] [,27] [,28] [,29] [,30] [,31]  
## [1,] -0.02066298 0.0301809 -0.01809374 0.002426665 -0.006006744 -0.0090391  
## [,32] [,33] [,34] [,35] [,36] [,37]  
## [1,] 0.002010618 -0.01714486 0.02848703 -0.01523339 -0.0192985 0.02232585  
## [,38] [,39] [,40] [,41] [,42] [,43]  
## [1,] 0.008763983 0.002053856 0.01180398 0.01938959 -0.01431181 -0.01536063  
## [,44] [,45] [,46] [,47] [,48] [,49]  
## [1,] 0.02951312 -0.03035575 0.0002295123 -0.00880202 -0.007037293 -0.008278172  
## [,50] [,51]  
## [1,] 0.01793889 -0.007693509  
##   
## $iterations  
## [1] 140