(a, b, 0) Class 25M

In order to be consistent with the exam table, the PMF will be denoted as p_n in this section.

$$\Pr(N=n)=p(n)=p_n$$

In this section, we will discuss each member of the $(a,\,b,\,0)$ class:

- Poisson
- Binomial
    - Bernoulli
- Negative binomial
    - Geometric

Poisson

Let N follow a ***Poisson*** distribution with mean λ, i.e.,

$$N\sim\text{Poisson}\;(\lambda)$$

Then, N has the following PMF:

$$p_n=\frac{e^{-\lambda}\lambda^n}{n!},\qquad n=0,\,1,\,2,\,\ldots$$

One key property of the Poisson distribution is that its mean equals its variance.

$$\mathrm{E}[N]=\mathrm{Var}[N]=\lambda$$

The sum of **independent** Poisson random variables is a new Poisson random variable. Assume we have k Poisson random variables that are independent of each other.

$$N_1 \sim \text{Poisson}\,(\lambda_1)$$

$$N_2 \sim \text{Poisson}\,(\lambda_2)$$

$$\vdots$$

$$N_k \sim \text{Poisson}\,(\lambda_k)$$

Let N be the sum of the Poisson random variables.

$$N = \sum_{i=1}^{k} N_i$$

Then, N follows a Poisson distribution with mean equal to the sum of the means of the k Poisson random variables.

$$N \sim \text{Poisson}\left(\sum_{i=1}^{k} \lambda_i\right)$$

Coach's Remarks

Sometimes, you might see the term ***Poisson process***. For the purpose of this exam, the term can be used interchangeably with "Poisson distribution."

In short, the Poisson process is a broader term that describes the counting of occurrences, while the Poisson distribution refers to the number of occurrences.

Binomial

A ***binomial*** distribution with parameters m and q means

- There are m independent trials.
- For each trial, an event of interest occurs with probability q.

Let N follow a binomial distribution with parameters m and q, i.e.

$$N \sim \text{Binomial}\,(m,\, q)$$

Then, N has the following PMF:

$$p_n = \binom{m}{n} q^n (1-q)^{m-n}, \qquad n = 0,\, 1,\, \ldots,\, m$$

The mean and variance are

$$\mathrm{E}[N] = mq$$

$$\mathrm{Var}[N] = mq(1-q)$$

Similar to the Poisson distribution, the sum of **independent** binomial random variables with the same parameter q is a binomial random variable.

$$N_i \sim \text{Binomial}\,(m_i,\, q)$$

$$N = \sum_{i=1}^{k} N_i$$

$$N \sim \text{Binomial}\left(\sum_{i=1}^{k} m_i,\, q\right)$$

Bernoulli

A ***Bernoulli*** distribution is a binomial distribution with $m = 1$. The PMF, mean and variance are simplified to

$$N \sim \text{Bernoulli}\,(q)$$

$$p_n = q^n(1-q)^{1-n}, \qquad n = 0,\, 1$$

$$\text{E}[N] = q$$

$$\text{Var}[N] = q(1-q)$$

The sum of m **i.i.d.** Bernoulli random variables follows a binomial distribution with parameters m and q.

$$N_1,\, N_2,\, \ldots,\, N_m \sim \text{Bernoulli}\,(q)$$

$$N = N_1 + N_2 + \ldots + N_m$$

$$N \sim \text{Binomial}\ (m,\ q)$$

Negative Binomial

Let N follow a ***negative binomial*** distribution with parameters r and β, i.e.,

$$N \sim \text{Negative Binomial}\ (r,\ \beta)$$

Then, N has the following PMF:

$$p_n = \begin{cases} \dfrac{1}{(1+\beta)^r}, & n = 0 \\ \dfrac{r(r+1)\ldots(r+n-1)}{n!} \cdot \dfrac{\beta^n}{(1+\beta)^{n+r}}, & n = 1,\ 2,\ \ldots \end{cases}$$

The PMF given in the exam table might not be intuitive. Some students prefer the following expression.

$$p_n = \binom{n+r-1}{n} \left(\frac{\beta}{1+\beta}\right)^n \left(\frac{1}{1+\beta}\right)^r, \qquad n = 0,\ 1,\ 2,\ \ldots$$

The equivalence of the two PMFs is explained in the appendix at the end of this section.

The mean and variance are

$$\mathrm{E}[N] = r\beta$$

$$\mathrm{Var}[N] = r\beta(1+\beta)$$

The sum of **independent** negative binomial random variables with the same parameter β is also a negative binomial random variable.

$$N_i \sim \text{Negative Binomial}\ (r_i,\ \beta)$$

$$N = \sum_{i=1}^{k} N_i$$

$$N \sim \text{Negative Binomial}\ \left(\sum_{i=1}^{k} r_i,\ \beta\right)$$

Geometric

A ***geometric*** distribution is a negative binomial distribution with $r = 1$. The PMF, mean and variance are simplified to

$$N \sim \text{Geometric}\ (\beta)$$

$$p_n = \frac{\beta^n}{(1+\beta)^{n+1}}, \qquad n = 0,\ 1,\ 2,\ \ldots$$

$$\mathrm{E}[N] = \beta$$

$$\mathrm{Var}[N] = \beta(1+\beta)$$

The sum of r **i.i.d.** geometric random variables follows a negative binomial distribution with parameters r and β.

$$N_1,\ N_2,\ \ldots,\ N_r \sim \text{Geometric }(\beta)$$

$$N = N_1 + N_2 + \ldots + N_r$$

$$N \sim \text{Negative Binomial }(r,\ \beta)$$

As stated earlier, the geometric distribution has the memoryless property. That means for $N \sim \text{Geometric }(\beta)$, $(N - d \mid N \ge d)$ follows the same distribution as N.

$$N - d \mid N \ge d \sim \text{Geometric }(\beta)$$

The memoryless property also implies the following:

- $\mathrm{E}[N - d \mid N \ge d] = \mathrm{E}[N] = \beta$
- $\mathrm{Var}[N - d \mid N \ge d] = \mathrm{Var}[N] = \beta(1 + \beta)$
- $\Pr(N - d = a \mid N \ge d) = \Pr(N = a)$

Example S2.4.1.1

An insured owns 3 cars in Year 1 and 4 cars in Year 2. Each car has a 10% chance of getting into an accident, and can only get into at most 1 accident per year.

The car accidents are independent of each other.

Calculate the probability that the insured gets into at most 1 accident in two years.

Solution

Let N_1 and N_2 be the numbers of accidents in first and second year, respectively.

$$N_1 \sim \text{Binomial}\ (3,\ 0.1)$$

$$N_2 \sim \text{Binomial}\ (4,\ 0.1)$$

Let N be the total number of accidents in two years, so $N = N_1 + N_2$. Because N_1 and N_2 share the same q, N follows a binomial distribution with $m = 3 + 4 = 7$.

$$N \sim \text{Binomial}\ (7,\ 0.1)$$

Therefore, the desired probability is

$$\begin{aligned}\Pr(N \le 1) &= p_0 + p_1 \\ &= \binom{7}{0}(0.1)^0(1-0.1)^7 + \binom{7}{1}(0.1)^1(1-0.1)^6 \\ &= 0.48 + 0.37 \\ &= \mathbf{0.85}\end{aligned}$$

■

Important Property

The discrete distributions introduced above are categorized as members of the $(a, b, 0)$ class for a reason. For each distribution, there exist the constants a and b

such that

$$\frac{p_n}{p_{n-1}} = a + \frac{b}{n}, \qquad n = 1,\, 2,\, \ldots$$

Besides calculating probabilities, the values of a and b determine which distribution the variable follows and provide a way to easily calculate the parameters.

The table below summarizes the signs of a and b for each distribution. The expressions for a and b can be found in the exam table.

Distribution	a	b
Poisson	0	+
Binomial	-	+
Negative binomial	+	+

An example would better illustrate this concept.

Example S2.4.1.2

For an $(a,\, b,\, 0)$ class distribution, you are given:

- $\dfrac{p_1}{p_0} = 2$
- $\dfrac{p_2}{p_1} = \dfrac{4}{3}$

Calculate the mean of the distribution.

Solution

Recall that

$$\frac{p_n}{p_{n-1}} = a + \frac{b}{n}$$

Use this relationship to form two equations.

$$\frac{p_1}{p_0} = a + \frac{b}{1} = 2 \qquad (1)$$

$$\frac{p_2}{p_1} = a + \frac{b}{2} = \frac{4}{3} \qquad (2)$$

Solve for a and b.

$(1) - (2)$:

$$\frac{b}{2} = \frac{2}{3} \quad \Rightarrow \quad b = \frac{4}{3}$$

Substitute into (1):

$$a + \frac{4}{3} = 2 \quad \Rightarrow \quad a = \frac{2}{3}$$

Both a and b are positive. That means the distribution is negative binomial. Use the a and b formulas provided in the exam table to determine the values of the parameters.

$$a = \frac{\beta}{1+\beta} = \frac{2}{3} \quad \Rightarrow \quad \beta = 2$$

$$b = \frac{(r-1)\beta}{1+\beta} = \frac{4}{3} \qquad \Rightarrow \qquad r = 3$$

The mean of the negative binomial distribution is

$$r\beta = 2\,(3) = \mathbf{6}$$

$\blacksquare$

Poisson-Gamma Mixture

One way to create a negative binomial distribution is by mixing Poisson distributions. It turns out that if X follows a Poisson distribution where its mean follows a gamma distribution, then the unconditional distribution of X is a negative binomial distribution:

$$\begin{array}{c}(X \mid \Lambda = \lambda) \sim \text{Poisson}\;(\lambda) \\ \Lambda \sim \text{Gamma}\;(\alpha,\,\theta)\end{array} \qquad \Rightarrow \qquad X \sim \text{Negative Binomial}\;(r = \alpha,$$

The proof of the relationship is included in the appendix at the end of this section.

Coach's Remarks

The Poisson-gamma mixture is a continuous mixture, i.e., it combines an **infinite** number of Poisson distributions. While continuous mixtures are not on the exam syllabus, this particular mixture is discussed in the prescribed reading for the negative binomial distribution. It also appears in an SOA sample question. This leads us to believe this mixture could be tested on the exam.

Let's apply the Poisson-gamma mixture in an example.

Example S2.4.1.3

The annual number of claims per insured follows a Poisson distribution with mean λ, which varies from insured to insured. The insurance company believes that λ follows an exponential distribution with mean 4.

Calculate the probability that:

1. an insured selected at random will file exactly 1 claim next year.
2. an insured, who is believed to have an average number of claims of 4 per year, will file exactly 1 claim next year.

Solution to (1)

Let N be the annual number of claims. The goal is to calculate $\Pr(N=1)$.

Using the relationship we just learned:

$$\begin{aligned} (N \mid \Lambda=\lambda) &\sim \text{Poisson}\ (\lambda) \\ \Lambda &\sim \text{Gamma}\ (1,\ 4) \end{aligned} \qquad \Rightarrow \qquad N \sim \text{Negative Binomial}\ (1,\ 4$$

Note that an exponential distribution is a special case of a gamma distribution with $\alpha=1$. Also, a negative binomial distribution with $r=1$ is equivalent to a geometric distribution.

Then, calculate the desired probability.

$$\begin{aligned} \Pr(N=1) &= \frac{4}{(1+4)^{1+1}} \\ &= \mathbf{0.16} \end{aligned}$$

Alternative Solution to (1)

We can also calculate the probability using the Law of Total Probability:

$$\begin{aligned}
\Pr(N=1) &= \mathrm{E}\left[\Pr(N=1 \mid \Lambda)\right] \\
&= \int_0^\infty \Pr(N=1 \mid \Lambda=\lambda) \cdot f_\Lambda(\lambda)\,\mathrm{d}\lambda \\
&= \int_0^\infty \frac{e^{-\lambda}\lambda^1}{1!} \cdot \frac{1}{4}e^{-\lambda/4}\,\mathrm{d}\lambda \\
&= \int_0^\infty \frac{\lambda}{4}e^{-5\lambda/4}\,\mathrm{d}\lambda \\
&= \frac{0.8}{4} \cdot \int_0^\infty \lambda \cdot \frac{1}{0.8}e^{-\lambda/0.8}\,\mathrm{d}\lambda \\
&= \frac{0.8}{4} \cdot 0.8 \\
&= \mathbf{0.16}
\end{aligned}$$

Solution to (2)

We are given the mean number of claims the insured has. Thus, the insured has a frequency model of

$$(N \mid \Lambda = 4) \sim \text{Poisson}\ (4)$$

The desired probability can be calculated directly using the Poisson distribution:

$$\begin{aligned}\Pr(N = 1 \mid \Lambda = 4) &= \frac{e^{-4}\, 4^1}{1!} \\ &= \mathbf{0.0733}\end{aligned}$$