Pareto 15M

Pareto

On the exam, "Pareto" refers to *two-parameter Pareto* unless specified otherwise.

Let X follow a Pareto distribution with parameters α and θ, i.e.

$$X \sim \text{Pareto}\,(\alpha,\,\theta)$$

Then, X has the following PDF:

$$f(x) = \frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}}, \qquad x > 0$$

A Pareto distribution is recognizable from its PDF's denominator having the term "x plus a constant" raised to a positive power, and its numerator being a constant.

$$f(x) = \frac{c}{(x+\theta)^{\alpha+1}}$$

The expected value is given in the exam table via the moments formula evaluated at $k = 1$. The variance can be calculated using basic principles from the first and second moments, or using this shortcut.

$$\mathrm{E}[X] = \frac{\theta}{\alpha - 1}$$

$$\mathrm{Var}[X] = \mathrm{E}[X]^2 \left(\frac{\alpha}{\alpha - 2} \right)$$

Note that a Pareto distribution only has k moments where $k < \alpha$. For example, when $\alpha \leq 2$, the 2nd moment does not exist, and neither does the variance because the variance depends on the second moment.

The Pareto distribution has a special property that is worth remembering. If $X \sim \text{Pareto}\ (\alpha,\ \theta)$ and $Y = X - d \mid X > d$, then

$$Y \sim \text{Pareto}\ (\alpha,\ \theta + d)$$

The derivation is included in the appendix at the end of this section.

Example S2.2.3.1

For each accident, loss size follows a Pareto distribution with parameters $\alpha = 3$ and $\theta = 500$.

Insurance only covers the portion of loss in excess of 200.

Calculate the probability that the insurer pays more than 500 for a loss, given the loss is greater than 200.

Solution

Let X represent loss size.

$$X \sim \text{Pareto}\ (3,\ 500)$$

The insurance payment is $X - 200$ if X exceeds 200. Thus, the desired probability is

$$\Pr\left(X-200>500\mid X>200\right)$$

Apply the Pareto's special property.

$$X-d\mid X>d\sim\text{Pareto}\left(\alpha,\theta+d\right)$$

$$X-200\mid X>200\sim\text{Pareto}\left(3,700\right)$$

Therefore,

$$\begin{aligned}\Pr(X-200>500\mid X>200)&=\left(\frac{700}{500+700}\right)^{3}\\&=\mathbf{0.1985}\end{aligned}$$

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Alternative Solution

$$\begin{aligned}\Pr(X-200>500\mid X>200)&=\Pr(X>700\mid X>200)\\&=\frac{\Pr(X>700)}{\Pr(X>200)}\\&=\frac{\left(\frac{500}{700+500}\right)^{3}}{\left(\frac{500}{200+500}\right)^{3}}\\&=\frac{0.0723}{0.3644}\\&=\mathbf{0.1985}\end{aligned}$$

Inverse Pareto

As the name suggests, an ***inverse Pareto*** distribution is an inverted Pareto distribution.

Specifically, if $Y \sim \text{Pareto}\left(\alpha,\, \theta^*\right)$ and $X = Y^{-1}$, then

$$X \sim \text{Inverse Pareto}\left(\tau = \alpha,\, \theta = \theta^{*-1}\right)$$

with the PDF

$$f(x) = \frac{\tau\theta x^{\tau-1}}{(x+\theta)^{\tau+1}}, \qquad x > 0$$

The proof of the inversion is included in the appendix at the end of this section.

An inverse Pareto's PDF resembles a Pareto's PDF, except the inverse Pareto's PDF has an $x^{\tau-1}$ term in its numerator.

$$f(x) = \frac{c \cdot x^{\tau-1}}{(x+\theta)^{\tau+1}}$$

Coach's Remarks

The same inversion methodology applies to all distributions on the exam table that have an inverse counterpart. The parameter θ is parameterized such that when X follows a certain distribution, X^{-1} follows the corresponding inverted distribution with the same parameters except theta becomes θ^{-1}. Another example would be the gamma and inverse gamma distributions.

Single-Parameter Pareto

Let X follow a *single-parameter Pareto* distribution with parameters α and θ, i.e.

$$X \sim \text{S-P Pareto}\ (\alpha,\ \theta)$$

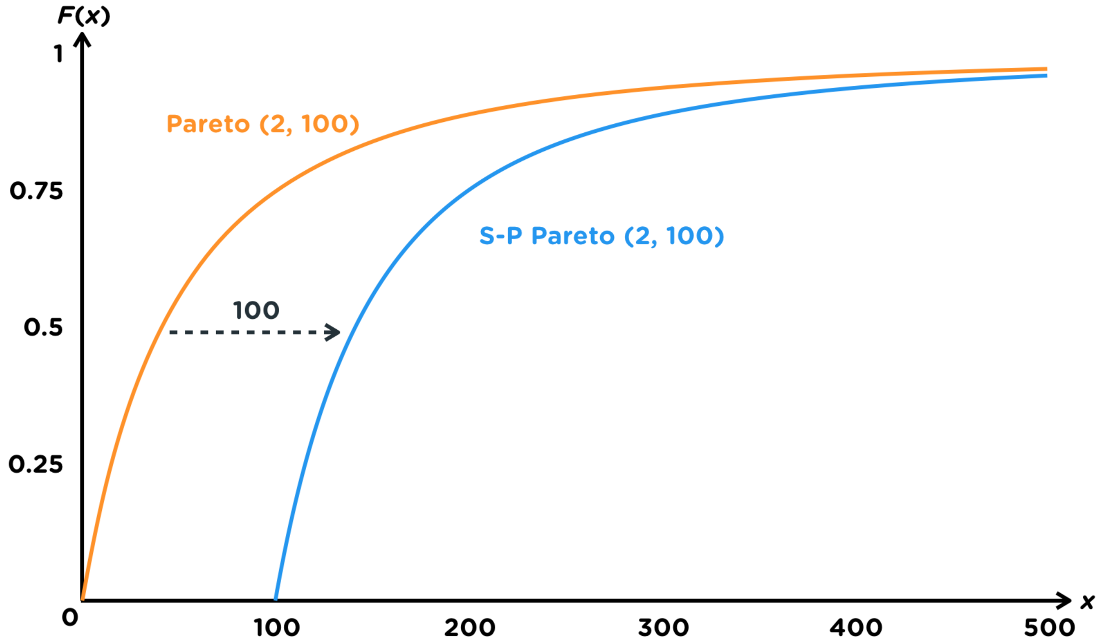
Then, X has the following PDF:

$$f(x) = \frac{\alpha\theta^{\alpha}}{x^{\alpha+1}}, \qquad x > \theta$$

A single-parameter Pareto distribution is recognizable from its domain, which is from a positive constant to infinity, whereas most other distributions have domains from 0 to infinity.

A single-parameter Pareto distribution is a Pareto distribution shifted by θ. For $Y \sim \text{Pareto}\ (\alpha,\ \theta)$ and $X = Y + \theta$,

$$X \sim \text{S-P Pareto}\ (\alpha,\ \theta)$$



Thus, the single-parameter Pareto mean is greater than the Pareto mean with the same parameters by θ

$$\begin{aligned}\mathrm{E}[X] &= \mathrm{E}[Y+\theta] \\ &= \mathrm{E}[Y]+\theta \\ &= \frac{\theta}{\alpha-1}+\theta \\ &= \frac{\alpha\theta}{\alpha-1}\end{aligned}$$

and its variance is equal to the Pareto variance.

$$\begin{aligned}\mathrm{Var}[X] &= \mathrm{Var}[Y+\theta] \\ &= \mathrm{Var}[Y] \\ &= \mathrm{E}[Y]^2\left(\frac{\alpha}{\alpha-2}\right) \\ &= \left(\frac{\theta}{\alpha-1}\right)^2\left(\frac{\alpha}{\alpha-2}\right)\end{aligned}$$

Coach's Remarks

θ is the amount by which the Pareto distribution is shifted. Thus, it is not a true parameter and must be determined in advance. The only true parameter is α, hence the name single-parameter Pareto.

For $X \sim \text{S-P Pareto}\ (\alpha,\ \theta)$ and $d > \theta$, then

$$X \mid X > d \sim \text{S-P Pareto}\ (\alpha,\ d)$$