(a, b, 1) Class

15M

Sometimes, special treatment is given to the probability of zero occurrences, $p(0)$, such that it deviates from what is implied by the original $(a, b, 0)$ distribution. In these cases, the $(a, b, 0)$ class distributions in their original form are inappropriate, as they do not give the desired $p(0)$.

Therefore, we adjust them and create the $(a, b, 1)$ class distributions. Note that the core difference between the $(a,\ b,\ 0)$ class and $(a,\ b,\ 1)$ class distributions is $p(0)$.

Let N be the original $(a,\ b,\ 0)$ class distribution, and p_n be its PMF.

- If $p(0)$ is set to 0, then the distribution is *zero-truncated*. We'll denote its random variable as N^T and its PMF as p_n^T.
- If $p(0)$ is set to p_0^M, where p_0^M is an arbitrary probability and $0 < p_0^M < 1$, then the distribution is *zero-modified*. We'll denote its random variable as N^M and its PMF as p_n^M.

Note that the zero-truncated or -modified distributions share the same parameters with their original distributions. The differences show up in their PMFs, which also affects the moments.

Zero-Truncated

Truncation means omitting a portion of the data. As the name suggests, the zero-truncated distributions assume $p(0) = 0$.

But by doing that, the remaining probabilities do not sum to 1.

$$\sum_{n=1}^{\infty} p_n = 1 - p_0$$

Thus, we divide each probability by $(1 - p_0)$ to scale them up so that the sum equals 1.

$$p_n^T = \frac{1}{1-p_0} \cdot p_n, \qquad n = 1,\, 2,\, \ldots$$

Coach's Remarks

Note that the p_0 in the formula is not 0. It is the probability of 0 from the original $(a,\, b,\, 0)$ class distribution.

In conclusion, for the zero-truncated PMF:

- $p_0^T = 0$
- Each p_n^T starting from $n = 1$ equals the original PMF scaled by $\frac{1}{1-p_0}$.

The mean of the zero-truncated distribution is

$$\begin{aligned}
\mathrm{E}\left[N^T\right] &= \sum_{n=1}^{\infty} n \cdot p_n^T \\
&= \sum_{n=1}^{\infty} n \cdot \frac{1}{1-p_0} \cdot p_n \\
&= \frac{1}{1-p_0} \cdot \sum_{n=1}^{\infty} n \cdot p_n \\
&= \frac{1}{1-p_0} \cdot \mathrm{E}[N]
\end{aligned}$$

This applies to all other raw moments. The k^{th} raw moment of a zero-truncated distribution can be easily calculated from the k^{th} raw moment of its original distribution.

$$\mathrm{E}\left[\left(N^T\right)^k\right] = \frac{1}{1-p_0} \cdot \mathrm{E}\left[N^k\right]$$

There is no equivalent shortcut to calculate the zero-truncated variance from the original variance. The easiest way is to calculate it from the zero-truncated first and second moments. Alternatively, you could use the formula provided in the zero-truncated subclass section of the exam table.

$$\mathrm{Var}\left[N^T\right] = \mathrm{E}\left[\left(N^T\right)^2\right] - \left(\mathrm{E}\left[N^T\right]\right)^2$$

Zero-Modified

In contrast to the zero-truncated distributions, where $p(0)$ is set to 0, zero-modified distributions assign an arbitrary $p(0) = p_0^M$, where $0 < p_0^M < 1$.

For the same purpose, to force the probabilities to sum up to 1, multiply each probability by $\frac{1-p_0^M}{1-p_0}$. Therefore,

$$p_n^M = \frac{1 - p_0^M}{1 - p_0} \cdot p_n, \qquad n = 1,\, 2,\, \ldots$$

The raw moments can be calculated by multiplying the original raw moments by the same factor.

$$\mathrm{E}\left[\left(N^M\right)^k\right] = \frac{1 - p_0^M}{1 - p_0} \cdot \mathrm{E}\left[N^k\right]$$

Similar to the zero-truncated subclass, the easiest way to calculate the zero-modified variance is from the zero-modified first and second moments.

$$\mathrm{Var}\left[N^M\right] = \mathrm{E}\left[\left(N^M\right)^2\right] - \left(\mathrm{E}\left[N^M\right]\right)^2$$

Example S2.4.3.1

The monthly number of claims, N, follows a Poisson distribution with mean 3.

1. What is the variance of the monthly number of claims, $\mathbf{Var}[N]$?
2. Assume a month with no claims is not possible. What is the variance of the monthly number of claims, $\mathbf{Var}\left[N^T\right]$?
3. Assume the probability of no claims in a month is 0.3. What is the variance of the monthly number of claims, $\mathbf{Var}\left[N^M\right]$?

Solution to (1)

$$N \sim \text{Poisson}(3)$$

For the Poisson distribution, the variance equals its mean.

$$\mathbf{Var}[N] = \lambda = \mathbf{3}$$

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Solution to (2)

To calculate the zero-truncated variance:

$$\begin{aligned}
\mathrm{E}\left[N^T\right] &= \frac{1}{1-p_0} \cdot \mathrm{E}[N] \\
&= \frac{1}{1-e^{-3}} \cdot (3) \\
&= 3.16
\end{aligned}$$

$$\begin{aligned} \mathrm{E}\left[\left(N^T\right)^2\right] &= \frac{1}{1-p_0} \cdot \mathrm{E}\left[N^2\right] \\ &= \frac{1}{1-e^{-3}} \cdot \left(\mathrm{Var}[N] + \mathrm{E}[N]^2\right) \\ &= \frac{1}{1-e^{-3}} \cdot \left(3+3^2\right) \\ &= 12.63 \end{aligned}$$

$$\begin{aligned} \mathrm{Var}\left[N^T\right] &= 12.63 - (3.16)^2 \\ &= \mathbf{2.66} \end{aligned}$$

Solution to (3)

To calculate the zero-modified variance:

$$\begin{aligned} \mathrm{E}\left[N^M\right] &= \frac{1-p_0^M}{1-p_0} \cdot \mathrm{E}[N] \\ &= \frac{1-0.3}{1-e^{-3}} \cdot (3) \\ &= 2.21 \end{aligned}$$

$$\begin{aligned} \mathrm{E}\left[\left(N^M\right)^2\right] &= \frac{1-p_0^M}{1-p_0} \cdot \mathrm{E}\left[N^2\right] \\ &= \frac{1-0.3}{1-e^{-3}} \cdot \left(3+3^2\right) \\ &= 8.84 \end{aligned}$$

$$\begin{aligned}\text{Var}\left[N^M\right] &= 8.84 - (2.21)^2 \\ &= \mathbf{3.96}\end{aligned}$$

Important Property

Note that the property we discussed at the end of Section S2.4.1 also holds for the $(a, b, 1)$ class. The only difference is the relationship only applies to probabilities starting from 1, since the probability at 0 is either truncated or modified.

$$\frac{p_n^T}{p_{n-1}^T} = a + \frac{b}{n}, \qquad n = 2,\ 3,\ \ldots$$

$$\frac{p_n^M}{p_{n-1}^M} = a + \frac{b}{n}, \qquad n = 2,\ 3,\ \ldots$$