Transformation 5M

Transformation of random variables is not included in the syllabus but is essential to understanding the concepts in later chapters. There are times that we know the distribution of X and need to determine the distribution of Y where Y is a function of X, i.e. $Y = g(X)$.

To determine the PDF of Y in those cases, follow these steps:

1. Determine the CDF of X.
2. Restate the CDF of Y using the CDF of X.
3. Determine the PDF of Y.

Consider the following example.

Example S2.2.1.1

A random variable X follows a distribution with the following PDF:

$$f_X(x) = \frac{1}{5} e^{-x/5}, \qquad x > 0$$

You are given another random variable, Y, where

$$Y = X^{-1}$$

Determine the PDF of Y.

Solution

Step 1: Determine the CDF of X.

$$\begin{aligned} F_X(x) &= \int_0^x \frac{1}{5} \cdot e^{-t/5} \, \mathrm{d}t \\ &= \left[-e^{-t/5} \right]_0^x \\ &= 1 - e^{-x/5} \end{aligned}$$

Step 2: Restate the CDF of Y using the CDF of X.

$$\begin{aligned} F_Y(y) &= \Pr(Y \leq y) \\ &= \Pr\left(X^{-1} \leq y\right) \\ &= \Pr\left(X \geq y^{-1}\right) \\ &= S_X\left(y^{-1}\right) \\ &= 1 - F_X\left(y^{-1}\right) \\ &= e^{-1/5y} \end{aligned}$$

Step 3: Determine the PDF of Y.

$$\begin{aligned} f_Y(y) &= \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) \\ &= \frac{\mathrm{d}}{\mathrm{d}y} e^{-1/5y} \\ &= e^{-1/5y} \cdot \frac{\mathrm{d}}{\mathrm{d}y} \left(-\frac{1}{5} y^{-1} \right) \\ &= \boldsymbol{\frac{1}{5y^2} e^{-1/5y}, \quad y > 0} \end{aligned}$$

Example S2.2.1.2

A random variable X follows a distribution with the following CDF:

$$F_X(x) = \frac{x}{10}, \qquad 0 \le x \le 10$$

You are given another random variable, Y, where

$$Y = e^X$$

Determine the PDF of Y.

Solution

Skip Step 1 because we already have the CDF of X.

Step 2: Restate the CDF of Y using the CDF of X.

$$\begin{aligned} F_Y(y) &= \Pr(Y \le y) \\ &= \Pr\left(e^X \le y\right) \\ &= \Pr(X \le \ln y) \\ &= \frac{\ln y}{10} \end{aligned}$$

Step 3: Determine the PDF of Y.

$$\begin{aligned} f_Y(y) &= \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\ln y}{10}\right) \\ &= \frac{1}{10y}, \quad 1 \le y \le e^{10} \end{aligned}$$