Appendix 20M

TVaR for Lognormal Distribution

Let:

$$X \sim \text{Lognormal}\,(\mu,\, \sigma)$$

Then, use the following:

$$\begin{aligned}\text{TVaR}_p(X) &= \text{E}[X \mid X > \text{VaR}_p(X)] \\ &= \text{VaR}_p(X) + \text{E}[X - \text{VaR}_p(X) \mid X > \text{VaR}_p(X)] \\ &= \text{VaR}_p(X) + \frac{\text{E}[X] - \text{E}[X \wedge \text{VaR}_p(X)]}{1-p}\end{aligned}$$

From Solution to (2) in Example S2.6.1.1, we derived the general form of the lognormal VaR to be:

$$\text{VaR}_p(X) = \exp\left(\mu + z_p\sigma\right)$$

In addition, for a lognormal random variable,

$$
\begin{aligned}
\mathrm{E}[X \wedge \mathrm{VaR}_p(X)] &= \exp\left(\mu + \frac{\sigma^2}{2}\right)\Phi\left(\frac{\ln[\mathrm{VaR}_p(X)] - \mu - \sigma^2}{\sigma}\right) + \mathrm{VaR} \\
&= \mathrm{E}[X] \cdot \Phi\left(\frac{\ln[\mathrm{VaR}_p(X)] - \mu - \sigma^2}{\sigma}\right) + \mathrm{VaR}_p(X) \cdot [1 \\
&= \mathrm{E}[X] \cdot \Phi\left(\frac{\ln[\exp(\mu + z_p\sigma)] - \mu - \sigma^2}{\sigma}\right) + \mathrm{VaR}_p(X) \\
&= \mathrm{E}[X] \cdot \Phi\left(\frac{\mu + z_p\sigma - \mu - \sigma^2}{\sigma}\right) + \mathrm{VaR}_p(X) \cdot [1 - p] \\
&= \mathrm{E}[X] \cdot \Phi(z_p - \sigma) + \mathrm{VaR}_p(X) \cdot [1 - p]
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathrm{TVaR}_p(X) &= \mathrm{VaR}_p(X) + \frac{\mathrm{E}[X] - \mathrm{E}[X \wedge \mathrm{VaR}_p(X)]}{1 - p} \\
&= \mathrm{VaR}_p(X) + \frac{\mathrm{E}[X] - (\mathrm{E}[X] \cdot \Phi(z_p - \sigma) + \mathrm{VaR}_p(X) \cdot [1 - p}{1 - p} \\
&= \mathrm{VaR}_p(X) + \mathrm{E}[X] \cdot \left[\frac{1 - \Phi(z_p - \sigma)}{1 - p}\right] - \mathrm{VaR}_p(X) \\
&= \mathrm{E}[X] \cdot \left[\frac{\Phi(\sigma - z_p)}{1 - p}\right]
\end{aligned}
$$

TVaR for Normal Distribution

Let

$$X \sim \mathrm{Normal}\ (\mu,\, \sigma)$$

Then, $Z = \dfrac{X - \mu}{\sigma}$ is the standard normal random variable, with density function

$$\phi(z)=\frac{1}{\sqrt{2\pi}}e^{-z^2/2}$$

Also, denote the 100p^{th} percentile of X and Z as x_p and z_p, respectively.

$$\begin{aligned}
\mathrm{TVaR}_p(X) &= \mathrm{E}\left[X \mid X > x_p\right] \\
&= \mathrm{E}\left[\mu+\sigma Z \mid \mu+\sigma Z > x_p\right] \\
&= \mu+\sigma\cdot\mathrm{E}\left[Z \mid Z > \frac{x_p-\mu}{\sigma}\right] \\
&= \mu+\sigma\cdot\mathrm{E}\left[Z \mid Z > z_p\right] \\
&= \mu+\sigma\cdot\int_{z_p}^{\infty} z\cdot\phi\left(z \mid Z > z_p\right)\mathrm{d}z \\
&= \mu+\sigma\cdot\int_{z_p}^{\infty} z\cdot\frac{\phi(z)}{\Pr(Z>z_p)}\,\mathrm{d}z \\
&= \mu+\sigma\cdot\int_{z_p}^{\infty} z\cdot\frac{\frac{1}{\sqrt{2\pi}}e^{-z^2/2}}{1-p}\,\mathrm{d}z \\
&= \mu+\frac{\sigma}{(1-p)\sqrt{2\pi}}\cdot\int_{z_p}^{\infty} z\cdot e^{-z^2/2}\,\mathrm{d}z \\
&= \mu+\frac{\sigma}{(1-p)\sqrt{2\pi}}\cdot e^{-z_p^2/2} \\
&= \mu+\frac{\sigma}{1-p}\cdot\frac{1}{\sqrt{2\pi}}e^{-z_p^2/2} \\
&= \mu+\frac{\sigma}{1-p}\cdot\phi\left(z_p\right)
\end{aligned}$$

The substitution method was used to evaluate the integral $\int_{z_p}^{\infty} z\cdot e^{-z^2/2}\,\mathrm{d}z$. Let $u=\frac{z^2}{2}$. Then, $\mathrm{d}u=z\cdot\mathrm{d}z$. So,

$$\begin{aligned}\int_{z_p}^{\infty} z\cdot e^{-\frac{z^2}{2}}\,\mathrm{d}z &= \int_{z_p^2\,/\,2}^{\infty} e^{-u}\,\mathrm{d}u \\ &= \left[-e^{-u}\right]_{z_p^2\,/\,2}^{\infty} \\ &= e^{-z_p^2\,/\,2}\end{aligned}$$

Disproving Subadditivity for VaR

Recall the subadditivity property:

$$\rho(X+Y) \le \rho(X) + \rho(Y)$$

For $\rho(X) = \mathrm{VaR}_p(X)$, the goal is to find out whether the following is always true:

$$\mathrm{VaR}_p(X+Y) \le \mathrm{VaR}_p(X) + \mathrm{VaR}_p(Y)$$

Define two i.i.d. random variables X and Y as follows:

$$\Pr(X=0) = \Pr(Y=0) = 0.6$$

$$\Pr(X=100) = \Pr(Y=100) = 0.4$$

Evaluate the probability function of $X+Y$.

$$\begin{aligned}\Pr(X+Y=0) &= \Pr(X=0,\, Y=0) \\ &= (0.6)(0.6) \\ &= 0.36\end{aligned}$$

$$\begin{aligned}\Pr(X+Y=100) &= \Pr(X=100,\, Y=0)+\Pr(X=0,\, Y=100) \\ &= (0.4)(0.6)+(0.6)(0.4) \\ &= 0.48\end{aligned}$$

$$\begin{aligned}\Pr(X+Y=200) &= \Pr(X=100,\, Y=100) \\ &= (0.4)(0.4) \\ &= 0.16\end{aligned}$$

If $p=0.5$, then

$$\mathrm{VaR}_{0.5}(X)=\mathrm{VaR}_{0.5}(Y)=0$$

$$\mathrm{VaR}_{0.5}(X+Y)=100$$

In this example, $\rho(X+Y)>\rho(X)+\rho(Y)$.

Since subadditivity does not hold under all conditions, VaR **fails** to satisfy the subadditivity property.

TVaR Coherence Proof

The key to the first two tests is to realize that shifting or scaling a random variable will also shift or scale its percentiles.

$$\Pr(X\le\pi)=p \quad\Rightarrow\quad \Pr(X+c\le\pi+c)=p$$

$$\Pr(X\le\pi)=p \quad\Rightarrow\quad \Pr(cX\le c\pi)=p$$

So,

- **Translation invariance**

$$\begin{aligned}\mathrm{TVaR}_p(X+c) &= \mathrm{E}\left[X+c \mid X+c > \mathrm{VaR}_p(X+c)\right] \\ &= \mathrm{E}\left[X+c \mid X > \mathrm{VaR}_p(X)\right] \\ &= \mathrm{E}\left[X \mid X > \mathrm{VaR}_p(X)\right] + c \\ &= \mathrm{TVaR}_p(X) + c\end{aligned}$$

- **Positive homogeneity**

$$\begin{aligned}\mathrm{TVaR}_p(cX) &= \mathrm{E}\left[cX \mid cX > \mathrm{VaR}_p(cX)\right] \\ &= \mathrm{E}\left[cX \mid X > \mathrm{VaR}_p(X)\right] \\ &= c \cdot \mathrm{E}\left[X \mid X > \mathrm{VaR}_p(X)\right] \\ &= c \cdot \mathrm{TVaR}_p(X)\end{aligned}$$

When we add two random variables, we get a larger pool of X (or Y) values that can result in $X+Y > \mathrm{VaR}_p(X+Y)$ than in $X > \mathrm{VaR}_p(X)$ (or $Y > \mathrm{VaR}_p(Y)$). This is because small X (or Y) values can be paired with large Y (or X) values to create a sum that exceeds $\mathrm{VaR}_p(X+Y)$, thereby increasing the size of the pool of values that meet the $X+Y > \mathrm{VaR}_p(X+Y)$ condition. So, if we compare the pool of values where $X+Y > \mathrm{VaR}_p(X+Y)$ against the pool of values where $X > \mathrm{VaR}_p(X)$ or $Y > \mathrm{VaR}_p(Y)$, the pool for $X+Y > \mathrm{VaR}_p(X+Y)$ will be larger because it contains a higher proportion of small values. Then, when we consider the expected value of values within these pools, we get the following inequalities:

$$\mathrm{E}\left[X \mid X+Y > \mathrm{VaR}_p(X+Y)\right] \ \le\ \mathrm{E}\left[X \mid X > \mathrm{VaR}_p(X)\right]$$

$$\mathrm{E}\left[Y \mid X+Y > \mathrm{VaR}_p(X+Y)\right] \ \le\ \mathrm{E}\left[Y \mid Y > \mathrm{VaR}_p(Y)\right]$$

So,

- **Subadditivity**

$$\begin{aligned}\mathrm{TVaR}_p(X+Y) &= \mathrm{E}\left[X+Y \mid X+Y > \mathrm{VaR}_p(X+Y)\right] \\ &= \mathrm{E}\left[X \mid X+Y > \mathrm{VaR}_p(X+Y)\right] + \mathrm{E}\left[Y \mid X+Y \right. \\ &\leq \mathrm{E}\left[X \mid X > \mathrm{VaR}_p(X)\right] + \mathrm{E}\left[Y \mid Y > \mathrm{VaR}_p(Y)\right] \\ &= \mathrm{TVaR}_p(X) + \mathrm{TVaR}_p(Y)\end{aligned}$$

To test **monotonicity**, let's break down the two TVaR's and study the pieces individually.

$$\mathrm{TVaR}_p(X) = \mathrm{VaR}_p(X) + \frac{\mathrm{E}\left[X\right] - \mathrm{E}\left[X \wedge \mathrm{VaR}_p(X)\right]}{1-p}$$

$$\mathrm{TVaR}_p(Y) = \mathrm{VaR}_p(Y) + \frac{\mathrm{E}\left[Y\right] - \mathrm{E}\left[Y \wedge \mathrm{VaR}_p(Y)\right]}{1-p}$$

If $\mathrm{Pr}(X \leq Y) = 1$, that means, for all p:

- $\mathrm{VaR}_p(X) \leq \mathrm{VaR}_p(Y)$
- $\mathrm{E}\left[X\right] \leq \mathrm{E}\left[Y\right]$