Appendix 10M

Special Property of Pareto

For X following a Pareto distribution with parameters α and θ, and $Y = X - d \mid X > d$, using the CDF transformation method,

$$\begin{aligned} F_Y(y) &= \Pr(Y \le y) \\ &= \Pr(X - d \le y \mid X > d) \\ &= \Pr(X \le y + d \mid X > d) \\ &= \frac{\Pr(d < X \le y + d)}{\Pr(X > d)} \\ &= \frac{S_X(d) - S_X(y+d)}{S_X(d)} \\ &= 1 - \frac{S_X(y+d)}{S_X(d)} \\ &= 1 - \frac{\left(\frac{\theta}{y+d+\theta}\right)^{\alpha}}{\left(\frac{\theta}{d+\theta}\right)^{\alpha}} \\ &= 1 - \left(\frac{d+\theta}{y+d+\theta}\right)^{\alpha} \end{aligned}$$

which resembles the CDF of a Pareto distribution with parameters α and $\theta + d$.

In conclusion,

$$X \sim \text{Pareto}(\alpha,\, \theta) \Downarrow X - d \mid X > d \sim \text{Pareto}(\alpha,\, \theta + d)$$

Pareto and Inverse Pareto

For X following a Pareto distribution with parameters α and θ, and $Y = X^{-1}$, using the CDF transformation method,

$$\begin{aligned} F_Y(y) &= \Pr(Y \le y) \\ &= \Pr\left(X^{-1} \le y\right) \\ &= \Pr\left(X \ge y^{-1}\right) \\ &= S_X\left(y^{-1}\right) \\ &= \left(\frac{\theta}{y^{-1}+\theta}\right)^{\alpha} \\ &= \left(\frac{\theta y}{1+\theta y}\right)^{\alpha} \\ &= \left(\frac{y}{\theta^{-1}+y}\right)^{\alpha} \end{aligned}$$

which resembles the CDF of an inverse Pareto distribution with parameters α and θ^{-1}.

In conclusion,

$$X \sim \text{Pareto}\,(\alpha,\,\theta) \Downarrow X^{-1} \sim \text{Inverse Pareto}\,\left(\alpha,\,\theta^{-1}\right)$$

Pareto and Single-Parameter Pareto

For X following a Pareto distribution with parameters α and θ, and $Y = X + \theta$, using the CDF transformation method,

$$
\begin{aligned}
F_Y(y) &= \Pr(Y \leq y) \\
&= \Pr(X + \theta \leq y) \\
&= \Pr(X \leq y - \theta) \\
&= F_X(y - \theta) \\
&= 1 - \left(\frac{\theta}{y - \theta + \theta}\right)^{\alpha} \\
&= 1 - \left(\frac{\theta}{y}\right)^{\alpha}
\end{aligned}
$$

which resembles the CDF of a single-parameter Pareto distribution with parameters α and θ.

Note: The domain shifts from $x > 0$ to $y > \theta$, i.e., add θ to both sides of the inequality.

In conclusion,

$$X \sim \text{Pareto}\,(\alpha,\, \theta) \Downarrow X + \theta \sim \text{S-P Pareto}\,(\alpha,\, \theta)$$

Poisson Shortcut for Incomplete Gamma Function

For α is a positive integer,

$$
\begin{aligned}
\Gamma(\alpha;\, x) &= \frac{1}{\Gamma(\alpha)} \int_0^x t^{\alpha-1} e^{-t}\, \mathrm{d}t \\
&= \frac{1}{(\alpha-1)!} \left[-t^{\alpha-1}e^{-t} - (\alpha-1)t^{\alpha-2}e^{-t} - \ldots - (\alpha-1)!e^{-t} \right]_0^x \\
&= \frac{1}{(\alpha-1)!} \left[-x^{\alpha-1}e^{-x} - (\alpha-1)x^{\alpha-2}e^{-x} - \ldots - (\alpha-1)!e^{-x} + \right. \\
&= 1 - \left[\frac{x^{\alpha-1}e^{-x}}{(\alpha-1)!} + \frac{x^{\alpha-2}e^{-x}}{(\alpha-2)!} + \ldots + e^{-x} \right] \\
&= 1 - \Pr(N < \alpha)
\end{aligned}
$$

where $N \sim \text{Poisson}\ (\lambda = x)$.

Gamma and Inverse Gamma

For X following a gamma distribution with parameters α and θ, and $Y = X^{-1}$,

$$\begin{aligned} F_Y(y) &= \Pr(Y \le y) \\ &= \Pr\left(X^{-1} \le y\right) \\ &= \Pr\left(X \ge y^{-1}\right) \\ &= S_X\left(y^{-1}\right) \\ &= 1 - \Gamma\left(\alpha;\ \frac{1}{\theta y}\right) \\ &= 1 - \Gamma\left(\alpha;\ \frac{\theta^{-1}}{y}\right) \end{aligned}$$

which resembles the CDF of an inverse gamma distribution with parameters α and θ^{-1}.

In conclusion,

$$X \sim \text{Gamma}\ (\alpha,\ \theta) \Downarrow X^{-1} \sim \text{Inverse Gamma}\ \left(\alpha,\ \theta^{-1}\right)$$

Exponential and Inverse Exponential

For X following an exponential distribution with mean θ, and $Y = X^{-1}$,

$$
\begin{aligned}
F_Y(y) &= \Pr\left(Y \le y\right) \\
&= \Pr\left(X^{-1} \le y\right) \\
&= \Pr\left(X \ge y^{-1}\right) \\
&= S_X\left(y^{-1}\right) \\
&= e^{-y^{-1}/\theta} \\
&= e^{-\theta^{-1}/y}
\end{aligned}
$$