Value-at-Risk

5M

The ***Value-at-Risk (VaR)*** of a random variable is just a fancy name for its percentile. The VaR of X at the $100p\%$ security level, which is its $100p^{\text{th}}$ percentile, is denoted as $\mathbf{VaR}_p(X)$ or π_p.

For $0 < p < 1$, if X is continuous, then

$$\mathbf{VaR}_p(X) = F_X^{-1}(p) \tag{S2.6.1.1}$$

Coach's Remarks

Do not mistake VaR for the variance.

- Variance: $\mathbf{Var}[X] = \mathbf{E}\left[X^2\right] - \mathbf{E}[X]^2$
- Value-at-risk: $\mathbf{VaR}_p(X) = F_X^{-1}(p)$

It is important to know the difference!

The exam table provides the VaR formulas for most distributions. That means we can efficiently calculate the VaR and not have to invert the CDF.

Let's demonstrate:

Annual loss amount, X, follows a Weibull distribution with parameters $\theta = 5{,}000$ and $\tau = 3$.

Calculate its 95th percentile, or $\mathrm{VaR}_{0.95}(X)$.

The 95th percentile of the annual loss can be calculated using the VaR formula on the exam table:

$$\begin{aligned}\mathrm{VaR}_{0.95}(X) &= 5{,}000[-\ln(1-0.95)]^{1/3}\\ &= \mathbf{7{,}207.83}\end{aligned}$$

We will get the same result by taking the inverse of the CDF. Let k represent the 95th percentile.

$$\begin{aligned}F_X(k) &= 1 - e^{-(k/5{,}000)^3}\\ &= 0.95\end{aligned}$$

$$e^{-(k/5{,}000)^3} = 0.05$$

$$\left(\frac{k}{5{,}000}\right)^3 = -\ln 0.05$$

$$\begin{aligned}k &= 5{,}000(-\ln 0.05)^{1/3}\\ &= \mathbf{7{,}207.83}\end{aligned}$$

As expected, the results match.

In real life, VaR helps actuaries understand the likelihood of extreme losses. For instance, using the example above, we calculated the 95% VaR of the annual loss to be 7,207.83. This means the chance of observing a loss greater than 7,207.83 is 5%.

Example S2.6.1.1

Let X follow a normal distribution with mean 4 and variance 0.25.

Determine:

1. the VaR of X at the 99% level.
2. the VaR of e^X at the 99% level.

Solution to (1)

$$X \sim \text{Normal}\,(4,\ 0.25)$$

Recall the VaR of X at the 99% level is the value π_X such that

$$F_X(\pi_X) = 0.99$$

$$\Phi\left(\frac{\pi_X - 4}{\sqrt{0.25}}\right) = 0.99$$

$$\begin{aligned}\frac{\pi_X - 4}{\sqrt{0.25}} &= \Phi^{-1}(0.99) \\ &= 2.32635\end{aligned}$$

$$\begin{aligned}\pi_X &= 4 + 2.32635\sqrt{0.25} \\ &= \mathbf{5.163}\end{aligned}$$

Solution to (2)

Let $Y = e^X$. Recall from Section S2.2.6 that if X is normal, then e^X is lognormal with the same parameters.

$$Y = e^X \sim \text{Lognormal}\,(4,\ 0.25)$$

Since the VaR formula for the lognormal distribution is not provided in the exam table, apply first principles.

$$F_Y(\pi_Y) = 0.99$$

$$\Phi\left(\frac{\ln[\pi_Y] - 4}{\sqrt{0.25}}\right) = 0.99$$

$$\ln\left[\pi_Y\right] = 4 + 2.32635\sqrt{0.25}$$

$$\begin{aligned}\pi_Y &= e^{4+2.32635\sqrt{0.25}} \\ &= \mathbf{174.718}\end{aligned}$$

Alternative Solution to (2)

We will arrive at the same answer by computing $\pi_Y = e^{\pi_X}$. This relationship is explained in Section S2.2.6.

$$\mathrm{VaR}_{0.99}(X) = 5.163$$

$$\begin{aligned}\mathrm{VaR}_{0.99}(Y) &= e^{5.163} \\ &= \mathbf{174.718}\end{aligned}$$