Gamma 15M

Gamma

Let X follow a *gamma* distribution with parameters α and θ, i.e.

$$X \sim \text{Gamma}\,(\alpha,\, \theta)$$

Then, X has the following PDF:

$$f(x) = \frac{(x\,/\,\theta)^{\alpha} e^{-x\,/\,\theta}}{x \cdot \Gamma(\alpha)}, \qquad x > 0$$

Note, the gamma function is interpreted as $\Gamma(n) = (n-1)!$ for integer values of n.

A gamma distribution is commonly recognizable from its PDF having the term e raised to a negative multiple of x, multiplied by x raised to a positive power.

$$f(x) = c \cdot x^{\alpha-1} e^{-x\,/\,\theta}$$

The mean and variance are

$$\mathrm{E}[X] = \alpha\theta$$

$$\mathrm{Var}[X] = \alpha\theta^2$$

Because the expression of the gamma CDF changes based on the value of α, the CDF is expressed in a general manner using the incomplete gamma function, which is defined at the beginning of Appendix A of the exam table.

$$F(x) = \Gamma\left(\alpha;\ \frac{x}{\theta}\right)$$

Coach's Remarks

Students tend to confuse the gamma function, $\Gamma(n)$, with the incomplete gamma function, $\Gamma(\alpha;\ x)$. Note that the latter has two inputs while the former only has one.

Evaluating an incomplete gamma function using the integral provided by the exam table is messy and algebraically intensive.

$$\Gamma(\alpha;\ x) = \frac{1}{\Gamma(\alpha)} \int_0^x t^{\alpha-1} e^{-t}\,\mathrm{d}t, \qquad \alpha > 0,\ x > 0$$

Fortunately, there is a shortcut when α is a **positive integer**.

To evaluate an incomplete gamma function $\Gamma(\alpha;\ x)$, let N be a Poisson random variable with mean x, i.e. the second input of the function. Then

$$\Gamma(\alpha;\ x) = 1 - \Pr(N < \alpha)$$

The derivation of the Poisson shortcut is included at the end of this section.

Coach's Remarks

The Poisson distribution is an $(a,\, b,\, 0)$ class distribution and will be discussed in the next section. At this point, you only need to know the PMF of the Poisson random variable.

$$p(n) = \frac{e^{-\lambda}\lambda^n}{n!}, \qquad n = 0,\, 1,\, 2, \ldots$$

The shortcut is better illustrated with an example.

Example S2.2.4.1

X follows a gamma distribution with parameters $\alpha = 2$ and $\theta = 5$.

$$f(x) = \frac{x\, e^{-x\,/\,5}}{25}, \qquad x > 0$$

Calculate $F(20)$.

Solution

$F_X(20) = \Gamma\left(2;\, \dfrac{20}{5}\right) = \Gamma(2;\, 4)$. So, let $N \sim \text{Poisson }(\lambda = 4)$.
Therefore,

$$\begin{aligned} F_X(20) &= \Gamma(2;\, 4) \\ &= 1 - \Pr(N < 2) \\ &= 1 - [p_N(0) + p_N(1)] \\ &= 1 - \left(e^{-4} + 4\, e^{-4}\right) \\ &= \mathbf{0.9084} \end{aligned}$$

Alternative Solution

Alternatively, using first principles,

$$\begin{aligned} F_X(20) &= \int_0^{20} \frac{x\, e^{-x\,/\,5}}{25}\, \mathrm{d}x \\ &= \frac{1}{25} \cdot \left[-x \cdot 5e^{-x\,/\,5} - 25e^{-x\,/\,5} \right]_0^{20} \\ &= \mathbf{0.9084} \end{aligned}$$

The sum of **independent** gamma random variables, given that they have the **same** value of θ, is also a new gamma random variable.

Assume we have n gamma random variables that are independent of each other.

$$X_1 \sim \text{Gamma}\ (\alpha_1,\, \theta)$$

$$X_2 \sim \text{Gamma}\ (\alpha_2,\, \theta)$$

$$\vdots$$

$$X_n \sim \text{Gamma}\ (\alpha_n,\, \theta)$$

Let X be the sum of the gamma random variables.

$$X = \sum_{i=1}^{n} X_i$$

Then, X follows a gamma distribution with α equal to the sum of the parameters α_i for each of the n gamma random variables and with θ remaining the same.

$$X \sim \text{Gamma}\left(\sum_{i=1}^{n} \alpha_i,\, \theta\right)$$

Exponential

The ***exponential*** distribution is a special case of the gamma distribution, i.e. it is a gamma distribution with $\alpha = 1$. The PDF, mean, and variance simplify to:

$$f(x) = \frac{e^{-x/\theta}}{\theta}, \qquad x > 0$$

$$\text{E}[X] = \theta$$

$$\text{Var}[X] = \theta^2$$

The exponential distribution has the ***memoryless property***, which loosely implies that what happened in the past does not affect what will happen in the future. For example, assume the time until the next event is exponentially distributed with mean θ (in minutes). Then, regardless of how much time elapses before the next event, the average wait time is still another θ minutes.

From a mathematical standpoint, the memoryless property means $(X - d \mid X > d)$ will follow the same distribution as X, where d is a positive constant.

Thus, for $X \sim \text{Exponential}\ (\theta)$,

$$X - d \mid X > d \sim \text{Exponential}\ (\theta)$$

Based on the memoryless property, the following are also true:

- $\text{E}[X - d \mid X > d] = \text{E}[X] = \theta$
- $\text{Var}[X - d \mid X > d] = \text{Var}[X] = \theta^2$
- $\Pr(X - d \leq a \mid X > d) = \Pr(X \leq a)$

Coach's Remarks

Only two distributions have the memoryless property: ***exponential*** and ***geometric***. The geometric distribution will be covered in the next section.

The sum of n **independent and identically distributed (i.i.d.)** exponential random variables follows a gamma distribution with parameters $\alpha = n$ and θ.

$$X_1,\ X_2,\ \ldots,\ X_n \sim \text{Exponential}\ (\theta)$$

$$Y = X_1 + X_2 + \ldots + X_n$$

$$Y \sim \text{Gamma}\ (n, \theta)$$

Coach's Remarks

This is an application of the property for the sum of gamma random variables with the same θ. Think of the sum of n exponential random variables as the sum of n gamma random variables, all with parameters $\alpha = 1$ and θ.

Weibull

The ***Weibull*** distribution is a transformed exponential distribution. For $Y \sim \text{Exponential}\ (\mu)$ and $X = Y^{1/\tau}$,

$$X \sim \text{Weibull}\left(\theta = \mu^{1/\tau},\ \tau\right)$$

and has PDF

$$f(x) = \frac{\tau(x/\theta)^{\tau} e^{-(x/\theta)^{\tau}}}{x}, \qquad x > 0$$

Thus, when $\tau = 1$, the Weibull distribution is equivalent to the exponential distribution.

The Weibull PDF is similar to the gamma PDF, except the x/θ in the exponent is raised to a positive power. Use this fact to differentiate between a Weibull PDF and a gamma PDF.

$$f(x) = c \cdot x^{\tau-1} e^{-(x/\theta)^{\tau}}$$

Inverse Counterparts

The inverted counterparts of these distributions can be derived based on the same logic used to derive inverse Pareto from Pareto.

Inverse Gamma

For $Y \sim \text{Gamma}\,(\alpha,\, \theta)$ and $X = Y^{-1}$,

$$X \sim \text{Inverse Gamma}\,\left(\alpha,\, \theta^{-1}\right)$$

Inverse Exponential

For $Y \sim \text{Exponential}\,(\theta)$ and $X = Y^{-1}$,

$$X \sim \text{Inverse Exponential}\,\left(\theta^{-1}\right)$$

Inverse Weibull

For $Y \sim \text{Weibull}\,(\theta,\, \tau)$ and $X = Y^{-1}$,

$$X \sim \text{Inverse Weibull}\,\left(\theta^{-1},\, \tau\right)$$

The proofs of the inversions are included at the end of this section.