**Appendix** 10M

## Expected Value — Survival Function Method

For a continuous, non-negative $X$, the expected value of a function $X$, i.e. $g(X)$, can be rewritten as follows using integration by parts:

$$\begin{aligned}
\mathrm{E}\left[g(X)\right] &= \int_0^\infty g(x)\cdot f_X(x)\,\mathrm{d}x \\
&= \left[-g(x)\cdot S_X(x)\right]\Big|_0^\infty - \int_0^\infty g'(x)\cdot\left[-S_X(x)\right]\mathrm{d}x \\
&= \left[-g(\infty)\cdot\underbrace{S_X(\infty)}_{0} + \underbrace{g(0)}_{0}\cdot S_X(0)\right] + \int_0^\infty g'(x)\cdot S_X(x)\,\mathrm{d}x \\
&= \int_0^\infty g'(x)\cdot S_X(x)\,\mathrm{d}x
\end{aligned}$$

Note that in order to simplify to the final line, two requirements need to be fulfilled:

1. The upper bound of the integral needs to be the upper bound of $X$, resulting in $S_X(\infty)=0$ in the equation above.
2. The function evaluated at the lower bound needs to be 0, i.e., $g(0)=0$ in the equation above.

If the two requirements are not fulfilled, we need to adjust the formula accordingly.

It follows that for the scenario where $g(X)=X$,

$$\begin{aligned}\mathrm{E}\left[X\right] &= \int_0^\infty x \cdot f_X(x)\,\mathrm{d}x \\ &= \left[-x \cdot S_X(x)\right]\Big|_0^\infty - \int_0^\infty 1 \cdot \left[-S_X(x)\right]\mathrm{d}x \\ &= \int_0^\infty S_X(x)\,\mathrm{d}x\end{aligned}$$

## 3rd Central Moment

Recall that $\mu'_k = \mathrm{E}\left[X^k\right]$. Thus,

$$\begin{aligned}\mu_3 &= \mathrm{E}\left[(X-\mu)^3\right] \\ &= \mathrm{E}\left[\binom{3}{0}X^{3-0}\mu^0 - \binom{3}{1}X^{3-1}\mu^1 + \binom{3}{2}X^{3-2}\mu^2 - \binom{3}{3}X^{3-3}\mu^3\right] \\ &= \mathrm{E}\left[X^3 - 3X^2\mu + 3X\mu^2 - \mu^3\right] \\ &= \mathrm{E}\left[X^3\right] - 3\mu\mathrm{E}\left[X^2\right] + 3\mu^2\mathrm{E}[X] - \mu^3 \\ &= \mu'_3 - 3\mu\left(\mu'_2\right) + 3\mu^2(\mu) - \mu^3 \\ &= \mu'_3 - 3\mu\left(\mu'_2\right) + 3\mu^3 - \mu^3 \\ &= \mu'_3 - 3\mu'_2\mu + 2\mu^3\end{aligned}$$

## 4th Central Moment

Recall that $\mu'_k = \mathrm{E}\left[X^k\right]$. Thus,

$$
\begin{aligned}
\mu_4 &= \mathrm{E}\left[(X-\mu)^4\right] \\
&= \mathrm{E}\left[\binom{4}{0}X^{4-0}\mu^0 - \binom{4}{1}X^{4-1}\mu^1 + \binom{4}{2}X^{4-2}\mu^2 - \binom{4}{3}X^{4-3}\mu^3 + \binom{4}{4} \right. \\
&= \mathrm{E}\left[X^4 - 4X^3\mu + 6X^2\mu^2 - 4X\mu^3 + \mu^4\right] \\
&= \mathrm{E}\left[X^4\right] - 4\mu\mathrm{E}\left[X^3\right] + 6\mu^2\mathrm{E}\left[X^2\right] - 4\mu^3\mathrm{E}[X] + \mu^4 \\
&= \mu_4' - 4\mu\left(\mu_3'\right) + 6\mu^2\left(\mu_2'\right) - 4\mu^3(\mu) + \mu^4 \\
&= \mu_4' - 4\mu\left(\mu_3'\right) + 6\mu^2\left(\mu_2'\right) - 4\mu^4 + \mu^4 \\
&= \mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 3\mu^4
\end{aligned}
$$

## The Law of Total Variance

$$\mathrm{Var}_X\left[X\right] = \mathrm{E}_X\left[X^2\right] - \mathrm{E}_X[X]^2$$

Apply the Law of Total Expectation.

$$\mathrm{E}_X\left[X\right] = \mathrm{E}_Y\left[\mathrm{E}_X\left[X \mid Y\right]\right]$$

$$
\begin{aligned}
\mathrm{E}_X\left[X^2\right] &= \mathrm{E}_Y\left[\mathrm{E}_X\left[X^2 \mid Y\right]\right] \\
&= \mathrm{E}_Y\left[\mathrm{Var}_X\left[X \mid Y\right] + \mathrm{E}_X[X \mid Y]^2\right] \\
&= \mathrm{E}_Y\left[\mathrm{Var}_X\left[X \mid Y\right]\right] + \mathrm{E}_Y\left[\mathrm{E}_X[X \mid Y]^2\right]
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathrm{Var}_X\left[X\right] &= \mathrm{E}_X\left[X^2\right] - \mathrm{E}_X[X]^2 \\
&= \mathrm{E}_Y\left[\mathrm{Var}_X\left[X \mid Y\right]\right] + \mathrm{E}_Y\left[\mathrm{E}_X[X \mid Y]^2\right] - \mathrm{E}_Y[\mathrm{E}_X\left[X \mid Y\right]]^2
\end{aligned}
$$

Recall that $\text{Var}\left[g(X)\right] = \text{E}\left[g(X)^2\right] - \text{E}[g(X)]^2$. Thus, we can translate the second half of the equation above to:

$$\text{E}_Y\left[\text{E}_X\left[X \mid Y\right]^2\right] - \text{E}_Y[\text{E}_X\left[X \mid Y\right]]^2 = \text{Var}_Y\left[\text{E}_X[X \mid Y]\right]$$

Thus,

$$\text{Var}_X\left[X\right] = \text{E}_Y\left[\text{Var}_X\left[X \mid Y\right]\right] + \text{Var}_Y\left[\text{E}_X\left[X \mid Y\right]\right]$$

## Two Forms of the Biased Sample Variance

$$\begin{aligned}
\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n} &= \frac{\sum_{i=1}^{n}\left(x_i^2 - 2\bar{x}x_i + \bar{x}^2\right)}{n} \\
&= \frac{\sum_{i=1}^{n} x_i^2 - 2\bar{x}\sum_{i=1}^{n} x_i + n \cdot \bar{x}^2}{n} \\
&= \frac{\sum_{i=1}^{n} x_i^2}{n} - 2\bar{x}^2 + \bar{x}^2 \\
&= \frac{\sum_{i=1}^{n} x_i^2}{n} - \bar{x}^2
\end{aligned}$$