Tail-Value-at-Risk

20M

The *Tail-Value-at-Risk (TVaR)* or the *conditional tail expectation (CTE)* of a continuous random variable X at the $100p\%$ security level is its conditional expected value given that X exceeds its VaR at the $100p\%$ level.

$$\begin{aligned}\mathrm{TVaR}_p(X) &= \mathrm{E}[X \mid X > \mathrm{VaR}_p(X)] \\ &= \frac{\displaystyle\int_{\mathrm{VaR}_p(X)}^{\infty} x \cdot f(x)\,\mathrm{d}x}{\Pr[X > \mathrm{VaR}_p(X)]}\end{aligned} \qquad (\mathrm{S2.6.2.1})$$

From the equation above, notice the numerator is expressed as an integral. To avoid integration, express the TVaR in terms of the mean excess loss introduced in Section S2.3:

$$\begin{aligned}\mathrm{TVaR}_p(X) &= \mathrm{VaR}_p(X) + \mathrm{E}[X - \mathrm{VaR}_p(X) \mid X > \mathrm{VaR}_p(X)] \\ &= \mathrm{VaR}_p(X) + e[\mathrm{VaR}_p(X)]\end{aligned}$$

$(\mathrm{S2.6.2.2})$ will often come in handy because the VaR formula for most distributions is included in the exam table.

Similar to VaR, the TVaR helps actuaries to study extreme losses. For example, if the TVaR at the 95% security level is 5,000, it means that for losses greater than its 95% VaR, the average loss amount is 5,000.

Let's work on a few examples.

Example S2.6.2.1

Suppose X is a random variable that has the following density function:

$$f(x) = \frac{2}{3}x, \qquad 1 < x < 2$$

Determine

1. the Value-at-Risk of X at the 75% security level.
2. the Tail-Value-at-Risk of X at the 75% security level.

Solution to (1)

Let $\pi_{0.75}$ be the VaR of X at the 75% level.

$$\begin{aligned} 0.75 &= \int_1^{\pi_{0.75}} \frac{2}{3} x \, \mathrm{d}x \\ &= \left[\frac{1}{3} x^2 \right]_1^{\pi_{0.75}} \\ &= \frac{\pi_{0.75}^2 - 1}{3} \end{aligned}$$

$$\begin{aligned} \pi_{0.75} &= \sqrt{1 + 3(0.75)} \\ &= \mathbf{1.8028} \end{aligned}$$

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Solution to (2)

To determine $\mathrm{TVaR}_{0.75}(X)$, use $(\mathrm{S2.6.2.1})$.

$$\begin{aligned} \mathrm{TVaR}_{0.75}(X) &= \mathrm{E}[X \mid X > \pi_{0.75}] \\ &= \mathrm{E}[X \mid X > 1.8028] \end{aligned}$$

$$\mathrm{E}[X \mid X > 1.8028] = \frac{\displaystyle\int_{1.8028}^{2} x \cdot (2x\,/\,3)\,\mathrm{d}x}{\Pr(X > 1.8028)}$$

Recall that 1.8028 is the 75th percentile of X. This means $\Pr(X \le 1.8028) = 0.75$. Therefore, its complement, which is $\Pr(X > 1.8028)$, is equal to 0.25.

$$\begin{aligned}\mathrm{E}[X \mid X > 1.8028] &= \frac{\displaystyle\int_{1.8028}^{2} 2x^2\,/\,3\,\mathrm{d}x}{0.25} \\ &= \frac{\left[2x^3\,/\,9\right]_{1.8028}^{2}}{0.25} \\ &= \mathbf{1.9031}\end{aligned}$$

Coach's Remarks

The result above makes intuitive sense. Since X has an upper bound of 2 and is conditioned to be greater than 1.8028, its TVaR at the 75% level should be a value between 1.8028 and 2. If the result falls outside of this range, check the calculations, including the limits of integration. This will help catch mistakes and eliminate bad answer choices.

Example S2.6.2.2

Loss amount X follows a continuous distribution that has the following evaluated

CDFs and limited expected values:

x	$F(x)$	$\mathrm{E}[X \wedge x]$
500	0.44	375
600	0.49	429
1,800	0.79	818
2,415	0.85	925
3,000	0.89	1,000
4,350	0.93	1,115
∞	1.00	1,500

Calculate the TVaR of X at the 85% level.

Solution

The question does not provide the full distribution function of X, so $(\mathbf{S2.6.2.1})$ cannot be used because it requires the PDF of X. Apply $(\mathbf{S2.6.2.2})$ instead.

$$\begin{aligned}\mathrm{TVaR}_{0.85}(X) &= \mathrm{VaR}_{0.85}(X) + e[\mathrm{VaR}_{0.85}(X)] \\ &= \mathrm{VaR}_{0.85}(X) + \frac{\mathrm{E}[X] - \mathrm{E}[X \wedge \mathrm{VaR}_{0.85}(X)]}{1 - F[\mathrm{VaR}_{0.85}(X)]}\end{aligned}$$

Recall that $\mathrm{VaR}_{0.85}(X)$ is the value of X that produces $F(x) = 0.85$.

$$\begin{aligned}\mathrm{VaR}_{0.85}(X) &= F^{-1}(0.85) \\ &= 2{,}415\end{aligned}$$

Now, compute

$$\mathrm{TVaR}_{0.85}(X) = 2{,}415 + \frac{\mathrm{E}[X] - \mathrm{E}[X \wedge 2{,}415]}{1 - F(2{,}415)}$$

- $\mathrm{E}[X] = \mathrm{E}[X \wedge \infty] = 1{,}500$
- $\mathrm{E}[X \wedge 2{,}415] = 925$
- $F(2{,}415) = 0.85$

Therefore,

$$\begin{aligned}\mathrm{TVaR}_{0.85}(X) &= 2{,}415 + \frac{1{,}500 - 925}{1 - 0.85} \\ &= \mathbf{6{,}248.33}\end{aligned}$$

As mentioned in Section S2.2.2, the exam tables only provide the TVaR formula for certain distributions, such as Pareto, exponential, and single-parameter Pareto. However, exam questions could ask about the TVaR of a lognormal distribution, which is not listed on the table. Using $(\mathbf{S2.6.2.2})$, we can derive a shortcut for the TVaR for a lognormal random variable.

If X is lognormal with parameters μ and σ^2, then

$$\mathrm{TVaR}_p(X) = \mathrm{E}[X] \cdot \left[\frac{\Phi(\sigma - z_p)}{1 - p}\right] \qquad (\mathbf{S2.6.2.3})$$

The derivation is provided in the appendix at the end of this section.

Coach's Remarks

The advantage of using $(\mathbf{S2.6.2.3})$ is that this formula does not require the calculation of $\mathrm{VaR}_p(X)$. Therefore, while $(\mathbf{S2.6.2.2})$ can still be used, we recommend memorizing $(\mathbf{S2.6.2.3})$ to speed up calculations.

TVaR questions involving normal distributions are less common on the exam, but it is a good idea to be prepared. The formula below should be memorized because it is impractical to use $(\mathrm{S2.6.2.1})$ or $(\mathrm{S2.6.2.2})$ for the normal distribution under exam conditions.

If X is normal with parameters μ and σ^2, then

$$\mathrm{TVaR}_p(X) = \mu + \sigma\left[\frac{\phi(z_p)}{1-p}\right] \qquad (\mathrm{S2.6.2.4})$$

where $\phi(x)$ is the PDF of a standard normal distribution.

$$\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

Do not confuse $\phi(x)$ with $\Phi(x)$, which is the CDF of a standard normal.

The derivation is provided in the appendix at the end of this section.

Example S2.6.2.3

X follows a distribution with the density function:

$$f(x) = \frac{1}{x \cdot 3\sqrt{2\pi}}e^{-(\ln x - 2)^2/[2(9)]}, \qquad x > 0$$

Calculate $\mathrm{TVaR}_{0.85}(X)$.

Solution

Based on the PDF, X follows a lognormal distribution with parameters $\mu = 2$ and $\sigma^2 = 9$. Applying $(\mathrm{S2.6.2.3})$:

$$\mathrm{TVaR}_{0.85}(X) = \mathrm{E}[X] \cdot \left[\frac{\Phi(\sigma - z_{0.85})}{1 - 0.85}\right]$$

- $\mathrm{E}[X] = e^{2+(9/2)} = 665.142$
- $z_{0.85} = \Phi^{-1}(0.85) = 1.03643$

Calculate the TVaR:

$$\begin{aligned}\mathrm{TVaR}_{0.85}(X) &= 665.142 \cdot \left[\frac{\Phi(3 - 1.03643)}{1 - 0.85}\right] \\ &= 665.142\left(\frac{0.97521}{0.15}\right) \\ &= \mathbf{4{,}324.352}\end{aligned}$$

Alternative Solution

As previously mentioned, this question can be solved using $(\mathrm{S2.6.2.2})$.

$$\begin{aligned}\mathrm{TVaR}_{0.85}(X) &= \mathrm{VaR}_{0.85}(X) + e[\mathrm{VaR}_{0.85}(X)] \\ &= \mathrm{VaR}_{0.85}(X) + \frac{\mathrm{E}[X] - \mathrm{E}[X \wedge \mathrm{VaR}_{0.85}(X)]}{1 - F[\mathrm{VaR}_{0.85}(X)]}\end{aligned}$$

First, solve for $\mathrm{VaR}_{0.85}(X)$.

$$0.85 = \Phi\left(\frac{\ln\left[\mathrm{VaR}_{0.85}(X)\right] - 2}{\sqrt{9}}\right)$$

$$\begin{aligned}\ln\left[\mathrm{VaR}_{0.85}(X)\right] &= 2 + \Phi^{-1}(0.85) \cdot \sqrt{9} \\ &= 5.109\end{aligned}$$

$$\begin{aligned}\mathrm{VaR}_{0.85}(X) &= e^{5.109} \\ &= 165.553\end{aligned}$$

Thus, the goal is

$$\mathrm{TVaR}_{0.85}(X) = 165.553 + \frac{\mathrm{E}[X] - \mathrm{E}[X \wedge 165.553]}{1 - F(165.553)}$$

Compute each component using the exam table formulas:

- As calculated in the first solution, $\mathrm{E}[X] = 665.142$.
- To calculate $\mathrm{E}[X \wedge 165.553]$,

$$\begin{aligned}\mathrm{E}[X \wedge 165.553] &= e^{2+(9/2)} \cdot \Phi\left[\frac{\ln(165.553) - 2 - 9}{\sqrt{9}}\right] + 165.553\,[1 - \\ &= e^{6.5} \cdot \Phi(-1.96357) + 165.553\,(0.15) \\ &= 665.142\,(0.02479) + 24.80 \\ &= 41.322\end{aligned}$$

- Since 165.553 is the 85th percentile of X, we know $F(165.553) = 0.85$.

Therefore,

$$\begin{aligned}\mathrm{TVaR}_{0.85}(X) &= 165.553 + \frac{665.142 - 41.322}{1 - 0.85} \\ &= \mathbf{4{,}324.352}\end{aligned}$$