Appendix

(L) 10M

Special Property of Pareto

For X following a Pareto distribution with parameters lpha and heta, and $Y=X-d\mid X>d$, using the CDF transformation method,

$$egin{aligned} F_Y(y) &= \Pr\left(Y \leq y
ight) \ &= \Pr\left(X - d \leq y \mid X > d
ight) \ &= \Pr\left(X \leq y + d \mid X > d
ight) \ &= rac{\Pr(d < X \leq y + d)}{\Pr(X > d)} \ &= rac{S_X(d) - S_X(y + d)}{S_X(d)} \ &= 1 - rac{S_X(y + d)}{S_X(d)} \ &= 1 - rac{\left(rac{ heta}{y + d + heta}
ight)^{lpha}}{\left(rac{ heta}{d + heta}
ight)^{lpha}} \ &= 1 - \left(rac{d + heta}{y + d + heta}
ight)^{lpha} \end{aligned}$$

which resembles the CDF of a Pareto distribution with parameters α and $\theta+d$.

In conclusion,

$$X \sim ext{Pareto} \ (lpha, \ heta) \ \! \downarrow X - d \ | \ X > d \sim ext{Pareto} \ (lpha, \ heta + d)$$

Pareto and Inverse Pareto

For X following a Pareto distribution with parameters α and θ , and $Y=X^{-1}$, using the CDF transformation method,

$$egin{aligned} F_Y(y) &= \Pr\left(Y \leq y
ight) \ &= \Pr\left(X^{-1} \leq y
ight) \ &= \Pr\left(X \geq y^{-1}
ight) \ &= S_Xig(y^{-1}ig)^lpha \ &= \left(rac{ heta}{y^{-1} + heta}
ight)^lpha \ &= \left(rac{ heta y}{1 + heta y}
ight)^lpha \ &= \left(rac{y}{ heta^{-1} + y}
ight)^lpha \end{aligned}$$

which resembles the CDF of an inverse Pareto distribution with parameters lpha and $heta^{-1}$.

In conclusion,

$$X \sim ext{Pareto} \ (lpha, \ heta) \ \! \! \downarrow X^{-1} \sim ext{Inverse Pareto} \ (lpha, \ heta^{-1})$$

Pareto and Single-Parameter Pareto

For X following a Pareto distribution with parameters α and θ , and $Y=X+\theta$, using the CDF transformation method,

$$egin{aligned} F_Y(y) &= \Pr\left(Y \leq y
ight) \ &= \Pr\left(X + heta \leq y
ight) \ &= \Pr\left(X \leq y - heta
ight) \ &= F_X(y - heta) \ &= 1 - \left(rac{ heta}{y - heta + heta}
ight)^lpha \ &= 1 - \left(rac{ heta}{y}
ight)^lpha \end{aligned}$$

which resembles the CDF of a single-parameter Pareto distribution with parameters α and θ .

Note: The domain shifts from x>0 to $y>\theta$, i.e., add θ to both sides of the inequality.

In conclusion,

$$X \sim ext{Pareto} \ (lpha, \ heta) \ \! \downarrow X + heta \sim ext{S-P Pareto} \ (lpha, \ heta)$$

Poisson Shortcut for Incomplete Gamma Function

For α is a positive integer,

$$\begin{split} \Gamma(\alpha;\,x) &= \frac{1}{\Gamma(\alpha)} \int_0^x t^{\alpha-1} e^{-t} \,\mathrm{d}t \\ &= \frac{1}{(\alpha-1)!} \Big[-t^{\alpha-1} e^{-t} - (\alpha-1) t^{\alpha-2} e^{-t} - \ldots - (\alpha-1)! e^{-t} \Big]_0^x \\ &= \frac{1}{(\alpha-1)!} \Big[-x^{\alpha-1} e^{-x} - (\alpha-1) x^{\alpha-2} e^{-x} - \ldots - (\alpha-1)! e^{-x} + 1 \Big]_0^x \\ &= 1 - \Big[\frac{x^{\alpha-1} e^{-x}}{(\alpha-1)!} + \frac{x^{\alpha-2} e^{-x}}{(\alpha-2)!} + \ldots + e^{-x} \Big]_0^x \\ &= 1 - \Pr(N < \alpha) \end{split}$$

where $N \sim \text{Poisson} \ (\lambda = x)$.

Gamma and Inverse Gamma

For X following a gamma distribution with parameters lpha and heta, and $Y=X^{-1}$,

$$egin{aligned} F_Y(y) &= \Pr\left(Y \leq y
ight) \ &= \Pr\left(X^{-1} \leq y
ight) \ &= \Pr\left(X \geq y^{-1}
ight) \ &= S_Xig(y^{-1}ig) \ &= 1 - \Gammaig(lpha; rac{1}{ heta y}ig) \ &= 1 - \Gammaig(lpha; rac{ heta^{-1}}{y}ig) \end{aligned}$$

which resembles the CDF of an inverse gamma distribution with parameters lpha and $heta^{-1}$.

In conclusion,

$$X \sim \operatorname{Gamma}\left(lpha,\, heta
ight) \Downarrow X^{-1} \sim \operatorname{Inverse \, Gamma}\left(lpha,\, heta^{-1}
ight)$$

Exponential and Inverse Exponential

For X following an exponential distribution with mean heta, and $Y=X^{-1}$,

$$egin{aligned} F_Y(y) &= \Pr\left(Y \leq y
ight) \ &= \Pr\left(X^{-1} \leq y
ight) \ &= \Pr\left(X \geq y^{-1}
ight) \ &= S_Xig(y^{-1}ig) \ &= e^{-y^{-1}/ heta} \ &= e^{- heta^{-1}/y} \end{aligned}$$

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