Coherence

20M

As discussed at the beginning of this lesson, a risk measure is an expression that quantifies risks. How can one tell if a metric is a good risk measure? One method is to evaluate its *coherence*.

The risk measure of X, denoted by $\rho(X)$, is coherent if it fulfills **all** four properties below:

- **Translation invariance**

$$\rho(X+c)=\rho(X)+c$$

  for any constant c.

- **Positive homogeneity**

$$\rho(cX)=c\cdot\rho(X)$$

  for any positive constant c.

- **Subadditivity**

$$\rho(X+Y)\leq\rho(X)+\rho(Y)$$

- **Monotonicity**

$$\rho(X)\leq\rho(Y),\quad \text{if } \Pr(X\leq Y)=1$$

It is important to know both the names and the mathematical definitions of the properties.

Coach's Remarks

The exam does not require you to understand the logic and reasoning behind the properties above. However, understanding them can help you remember them.

Suppose you bought shares of a company's stock. Let X represent the loss of your investment, while $\rho(X)$ represents its risk measure, or the measure of the loss.

- **Translation invariance**
  If there is a fixed transaction cost of c during the purchase of this stock, this fixed cost is said to add the same amount of risk to the risk measure because it increases the expense and the risk exposure.
- **Positive homogeneity**
  If c units of this same stock are purchased, the risk measure is c times the original measure because you are now exposed to more risk.
- **Subadditivity**
  If investing in two different stocks, the risk measure of this portfolio must be no larger than the aggregate risk measure of the individual stocks due to diversification.
- **Monotonicity**
  If the loss of one investment is always smaller than the other, then the risk measure of this investment must be smaller because it is less risky.

As practice, let's work on the example below:

Determine if the expected value of X is a coherent risk measure.

$$\rho(X) = \mathrm{E}[X]$$

1. Test for translation invariance.

$$\begin{aligned}\rho(X+c) &= \mathrm{E}[X+c] \\ &= \mathrm{E}[X]+c \\ &= \rho(X)+c\end{aligned}$$

   This risk measure **satisfies** the translation invariance property.

2. Test for positive homogeneity.

$$\begin{aligned}\rho(cX) &= \mathrm{E}[cX] \\ &= c\cdot \mathrm{E}[X] \\ &= c\cdot \rho(X)\end{aligned}$$

   This risk measure **satisfies** the positive homogeneity property.

3. Test for subadditivity.

$$\begin{aligned}\rho(X+Y) &= \mathrm{E}[X+Y] \\ &= \mathrm{E}[X]+\mathrm{E}[Y] \\ &= \rho(X)+\rho(Y)\end{aligned}$$

   This risk measure **satisfies** the subadditivity property.

4. Test for monotonicity. If $\Pr(X \le Y)=1$, then $\Pr(X-Y \le 0)=1$. This implies $X-Y$ is always non-positive. Therefore,

$$\mathrm{E}[X-Y] \le 0$$

$$\mathrm{E}[X]-\mathrm{E}[Y] \le 0$$

$$\mathrm{E}[X] \le \mathrm{E}[Y]$$

   $\Pr(X \le Y)=1$ produces $\mathrm{E}[X] \le \mathrm{E}[Y] \Rightarrow \rho(X) \le \rho(Y)$. This risk measure **satisfies** the monotonicity property.

Since the mean satisfies all four coherent properties above, it is a coherent risk measure.

Let's work on another example.

Example S2.6.3.1

Which property or properties of coherence fail to hold for the risk measure $\rho(X) = \mathrm{E}[X] + k\,\mathrm{Var}[X]$, for $k > 0$?

1. Translation invariance
2. Positive homogeneity
3. Subadditivity
4. Monotonicity

Solution

Translation invariance

$$\begin{aligned} \rho(X+c) &= \mathrm{E}[X+c] + k\,\mathrm{Var}[X+c] \\ &= \mathrm{E}[X] + c + k\,\mathrm{Var}[X] \\ &= (\mathrm{E}[X] + k\,\mathrm{Var}[X]) + c \\ &= \rho(X) + c \end{aligned}$$

Therefore, the risk measure satisfies translation invariance.

Positive homogeneity

$$\begin{aligned} \rho(cX) &= \mathrm{E}[cX] + k\,\mathrm{Var}[cX] \\ &= c \cdot \mathrm{E}[X] + c^2 \cdot k\,\mathrm{Var}[X] \\ &= c \cdot (\mathrm{E}[X] + c \cdot k\,\mathrm{Var}[X]) \\ &\neq c \cdot \rho(X) \end{aligned}$$

Therefore, the risk measure fails to satisfy positive homogeneity.

Subadditivity

$$\begin{aligned}
\rho(X+Y) &= \mathrm{E}[X+Y] + k\,\mathrm{Var}[X+Y] \\
&= \mathrm{E}[X] + \mathrm{E}[Y] + k\,(\mathrm{Var}[X] + \mathrm{Var}[Y] + 2\mathrm{Cov}[X,\,Y]) \\
&= (\mathrm{E}[X] + k\,\mathrm{Var}[X]) + (\mathrm{E}[Y] + k\,\mathrm{Var}[Y]) + 2k\,\mathrm{Cov}[X,\,Y] \\
&= \rho(X) + \rho(Y) + 2k\,\mathrm{Cov}[X,\,Y]
\end{aligned}$$

Since $\mathrm{Cov}[X,\,Y]$ can be positive or negative, we cannot conclude that $\rho(X+Y) \le \rho(X) + \rho(Y)$. Therefore, the risk measure fails to satisfy subadditivity.

Monotonicity

Define independent variables X and Y as below:

$$\begin{aligned}
\Pr(X=0) &= 0.9 \\
\Pr(X=100) &= 0.1
\end{aligned} \qquad\qquad \Pr(Y=100) = 1$$

From the distributions of X and Y above, we know that $\Pr(X \le Y) = 1$ because X cannot be larger than Y.

$\mathrm{E}[X] = 0.9(0) + 0.1(100) = 10$	$\mathrm{E}[Y] = 100$
$\mathrm{Var}[X] = (100-0)^2(0.9)(0.1) = 900$	$\mathrm{Var}[Y] = 0$

Let $k = 1$, then

$$\begin{aligned}
\rho(X) &= 10 + 1 \cdot 900 \\
&= 910
\end{aligned}$$

$$\begin{aligned}
\rho(Y) &= 100 + 1 \cdot 0 \\
&= 100
\end{aligned}$$

In this example, $\rho(X) > \rho(Y)$, even though $\Pr(X \leq Y) = 1$. Since monotonicity does not hold under all conditions, this risk measure fails to satisfy monotonicity.

Altogether, **the risk measure fails to satisfy properties 2, 3, and 4**.

Coach's Remarks

Rather than proving the monotonicity property, it is generally easier to disprove it because we only need to provide one counterexample, as shown above.

However, this approach does not work when we need to prove a property. For a property to be satisfied, it has to hold true under all circumstances. In that case, a general proof is required, not just an example of when the property holds.

Coach's Remarks

When checking for monotonicity above, the variance of X was calculating using the Bernoulli shortcut. The Bernoulli shortcut is related to the Bernoulli distribution. It is a technique to quickly calculate the variance of a variable that has only **two** possible values.

In general, for a variable

$$X = \begin{cases} a, & \text{Probability} = q \\ b, & \text{Probability} = 1-q \end{cases}$$

the variance of X can be calculated as

$$\mathrm{Var}[X] = (a-b)^2 q(1-q)$$

This shortcut is extremely useful because the variance can be determined

The Loss Models textbook also discusses the coherence of VaR and TVaR. The result is presented in the table below:

	$\mathbf{VaR}_p(X)$	$\mathbf{TVaR}_p(X)$
Translation invariance	✓	✓
Positive homogeneity	✓	✓
Subadditivity		✓
Monotonicity	✓	✓
Coherent?	No	Yes

Keep in mind that $\mathrm{VaR}_p(X)$ is not a coherent risk measure because it does **not** satisfy subadditivity. The complete proof is provided in the appendix at the end of this section.

On the other hand, $\mathrm{TVaR}_p(X)$ is coherent, which means it satisfies all four coherent properties. The complete proof is provided in the appendix at the end of this section.

Understanding the proofs is not required for the exam, but it is important to memorize the result of the proofs, as shown in the table above.

Example S2.6.3.2

Which of the following is true?

1. $\rho(X) = \mathrm{E}\left[X^{-1}\right]$ satisfies positive homogeneity.
2. $\rho(X) = \mathrm{TVaR}_p(X)$ satisfies monotonicity.
3. $\rho(X) = \mathrm{VaR}_p(X)$ satisfies subadditivity.
4. $\rho(X) = \mathrm{Var}[X]$ satisfies translation invariance.

Solution

Statement 1
Positive homogeneity: $\rho(cX) = c \cdot \rho(X)$

For $\rho(X) = \mathrm{E}\left[X^{-1}\right]$,

$$\begin{aligned}\rho(cX) &= \mathrm{E}\left[(cX)^{-1}\right] \\ &= \mathrm{E}\left[c^{-1}X^{-1}\right] \\ &= c^{-1} \cdot \mathrm{E}\left[X^{-1}\right] \\ &= c^{-1} \cdot \rho(X)\end{aligned}$$

$$\rho(cX) \neq c \cdot \rho(X)$$

Therefore, statement 1 is **FALSE**.

Statement 2
It is known that TVaR is a coherent risk measure. Thus, it must satisfy the monotonicity property.

Therefore, statement 2 is **TRUE**.

Statement 3
As discussed above, VaR fails to satisfy the subadditivity property.

Therefore, statement 3 is **FALSE**.

Statement 4
Translation invariance: $\rho(X + c) = \rho(X) + c$

For $\rho(X) = \mathrm{Var}[X]$,

$$\begin{aligned}\rho(X+c) &= \mathrm{Var}[X+c] \\ &= \mathrm{Var}[X] \\ &= \rho(X)\end{aligned}$$

$$\rho(X+c) \neq \rho(X)+c$$

Therefore. statement 4 is **FALSE**.