Appendix 10M

Negative Binomial PMF

For a negative binomial distribution, let p be the probability of failure. Then, the probability of having n failures before the r^{th} success is

$$p_n = \binom{n+r-1}{r-1} p^n (1-p)^r, \qquad n = 0,\, 1,\, 2,\, \ldots$$

Let $p = \dfrac{\beta}{1+\beta}$.

$$\begin{aligned} p_n &= \frac{(n+r-1)!}{(r-1)!\,n!} \left(\frac{\beta}{1+\beta}\right)^n \left(\frac{1}{1+\beta}\right)^r \\ &= \begin{cases} \dfrac{1}{(1+\beta)^r}, & n = 0 \\ \dfrac{r\,(r+1)\ldots(n+r-1)}{n!} \cdot \dfrac{\beta^n}{(1+\beta)^{n+r}}, & n = 1,\, 2,\, \ldots \end{cases} \end{aligned}$$

In this parameterization, $\beta = \dfrac{p}{1-p}$ is the odds of failure.

Poisson-Gamma Mixture

Let:

- $(X \mid \Lambda = \lambda) \sim \text{Poisson } (\lambda)$

$$\Pr(X = x \mid \Lambda = \lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

- $\Lambda \sim \text{Gamma } (\alpha,\, \theta)$

$$f_{\Lambda}(\lambda) = \frac{(\lambda/\theta)^{\alpha} e^{-\lambda/\theta}}{\lambda \cdot \Gamma(\alpha)} = \frac{\lambda^{\alpha-1} e^{-\lambda/\theta}}{\theta^{\alpha} \cdot \Gamma(\alpha)}$$

Derive the unconditional probability of $X = x$:

$$\begin{aligned}
\Pr(X = x) &= \mathrm{E}_{\Lambda}[\Pr(X = x \mid \Lambda)] \\
&= \int_0^{\infty} \Pr(X = x \mid \Lambda = \lambda) \cdot f_{\Lambda}(\lambda)\, \mathrm{d}\lambda \\
&= \int_0^{\infty} \frac{e^{-\lambda}\lambda^{x}}{x!} \cdot \frac{\lambda^{\alpha-1} e^{-\lambda/\theta}}{\theta^{\alpha} \cdot \Gamma(\alpha)}\, \mathrm{d}\lambda \\
&= \frac{1}{x!} \cdot \frac{1}{\theta^{\alpha} \cdot \Gamma(\alpha)} \int_0^{\infty} \lambda^{x+\alpha-1} e^{-\lambda[(1+\theta)/\theta]}\, \mathrm{d}\lambda
\end{aligned}$$

Focus on the integral:

$$\begin{aligned}
\int_0^{\infty} \lambda^{x+\alpha-1} e^{-\lambda[(1+\theta)/\theta]}\, \mathrm{d}\lambda &= \int_0^{\infty} \frac{\lambda^{x+\alpha} e^{-\lambda/[\theta/(1+\theta)]}}{\lambda} \cdot \frac{[\theta/(1+\theta)]^{x+\alpha}}{[\theta/(1+\theta)]^{x+\alpha}} \cdot \\
&= [\theta/(1+\theta)]^{x+\alpha} \cdot \Gamma(x+\alpha) \int_0^{\infty} \frac{\lambda^{x+\alpha} e^{-}}{\lambda \cdot [\theta/(1+\theta}} \\
&= [\theta/(1+\theta)]^{x+\alpha} \cdot \Gamma(x+\alpha) \cdot 1 \\
&= \frac{\theta^{x+\alpha}}{(1+\theta)^{x+\alpha}} \cdot \Gamma(x+\alpha)
\end{aligned}$$

Note that $\displaystyle\int_0^\infty \frac{\lambda^{x+\alpha} e^{-\lambda / [\theta / (1+\theta)]}}{\lambda \cdot [\theta/(1+\theta)]^{x+\alpha} \cdot \Gamma(x+\alpha)}\, \mathrm{d}\lambda = 1$ because the integrand is a gamma PDF with parameters $x+\alpha$ and $[\theta/(1+\theta)]$ integrated from 0 to infinity.

Therefore,

$$\begin{aligned}\Pr(X=x) &= \frac{1}{x!} \cdot \frac{1}{\theta^{\alpha} \cdot \Gamma(\alpha)} \cdot \frac{\theta^{x+\alpha}}{(1+\theta)^{x+\alpha}} \cdot \Gamma(x+\alpha) \\ &= \frac{1}{x!} \cdot \frac{1}{(\alpha-1)!} \cdot \frac{\theta^{x}}{(1+\theta)^{x+\alpha}} \cdot (x+\alpha-1)! \\ &= \frac{(x+\alpha-1)!}{x! \cdot (\alpha-1)!} \cdot \left(\frac{\theta}{1+\theta}\right)^{x} \left(\frac{1}{1+\theta}\right)^{\alpha} \\ &= \binom{x+\alpha-1}{x} \cdot \left(\frac{\theta}{1+\theta}\right)^{x} \left(\frac{1}{1+\theta}\right)^{\alpha}\end{aligned}$$

Notice that the expression above has the same form as a negative binomial PMF:

$$\Pr(X=x) = \binom{x+r-1}{x} \left(\frac{\beta}{1+\beta}\right)^{x} \left(\frac{1}{1+\beta}\right)^{r}$$

Therefore,

$$X \sim \text{Negative Binomial}\ (r=\alpha,\ \beta=\theta)$$