Appendix

(L) 10M

Negative Binomial PMF

For a negative binomial distribution, let p be the probability of failure. Then, the probability of having n failures before the $r^{\rm th}$ success is

$$p_n = inom{n+r-1}{r-1} p^n (1-p)^r, \qquad n=0,\,1,\,2,\,\dots$$

Let
$$p=rac{eta}{1+eta}$$
 .

$$p_n = rac{(n+r-1)!}{(r-1)! \, n!} igg(rac{eta}{1+eta}igg)^n igg(rac{1}{1+eta}igg)^r \ = egin{cases} rac{1}{(1+eta)^r}, & n=0 \ rac{r(r+1) \dots (n+r-1)}{n!} \cdot rac{eta^n}{(1+eta)^{n+r}}, & n=1,\, 2,\, \dots \end{cases}$$

In this parameterization, $eta=rac{p}{1-p}$ is the odds of failure.

Poisson-Gamma Mixture

Let:

• $(X \mid \Lambda = \lambda) \sim \text{Poisson}(\lambda)$

$$\Pr(X = x \mid \Lambda = \lambda) = rac{e^{-\lambda} \lambda^x}{x!}$$

• $\Lambda \sim \operatorname{Gamma}\left(\alpha,\, heta
ight)$

$$f_{\Lambda}(\lambda) = rac{(\lambda/ heta)^{lpha}e^{-\lambda/ heta}}{\lambda\cdot\Gamma(lpha)} = rac{\lambda^{lpha-1}e^{-\lambda/ heta}}{ heta^{lpha}\cdot\Gamma(lpha)}$$

Derive the unconditional probability of X=x:

$$egin{aligned} \Pr(X = x) &= \mathrm{E}_{\Lambda}[\Pr(X = x \mid \Lambda)] \ &= \int_0^{\infty} \Pr(X = x \mid \Lambda = \lambda) \cdot f_{\Lambda}(\lambda) \; \mathrm{d}\lambda \ &= \int_0^{\infty} rac{e^{-\lambda} \lambda^x}{x!} \cdot rac{\lambda^{lpha - 1} e^{-\lambda/ heta}}{ heta^{lpha} \cdot \Gamma(lpha)} \; \mathrm{d}\lambda \ &= rac{1}{x!} \cdot rac{1}{ heta^{lpha} \cdot \Gamma(lpha)} \int_0^{\infty} \lambda^{x + lpha - 1} e^{-\lambda[(1 + heta)/ heta]} \; \mathrm{d}\lambda \end{aligned}$$

Focus on the integral:

$$\begin{split} \int_0^\infty \lambda^{x+\alpha-1} e^{-\lambda[(1+\theta)/\theta]} \; \mathrm{d}\lambda &= \int_0^\infty \frac{\lambda^{x+\alpha} e^{-\lambda/\left[\theta/(1+\theta)\right]}}{\lambda} \cdot \frac{\left[\theta/(1+\theta)\right]^{x+\alpha}}{\left[\theta/(1+\theta)\right]^{x+\alpha}} \cdot \frac{1}{1} \\ &= \left[\theta/(1+\theta)\right]^{x+\alpha} \cdot \Gamma(x+\alpha) \int_0^\infty \frac{\lambda^{x+\alpha} e^{-\lambda/(1+\theta)}}{\lambda \cdot \left[\theta/(1+\theta)\right]^{x+\alpha}} \\ &= \left[\theta/(1+\theta)\right]^{x+\alpha} \cdot \Gamma(x+\alpha) \cdot 1 \\ &= \frac{\theta^{x+\alpha}}{(1+\theta)^{x+\alpha}} \cdot \Gamma(x+\alpha) \end{split}$$

Note that $\int_0^\infty \frac{\lambda^{x+\alpha}e^{-\lambda/[\theta/(1+\theta)]}}{\lambda\cdot[\theta/(1+\theta)]^{x+\alpha}\cdot\Gamma(x+\alpha)}\,\mathrm{d}\lambda=1$ because the integrand is a gamma PDF with parameters $x+\alpha$ and $[\theta/(1+\theta)]$ integrated from 0 to infinity.

Therefore,

$$egin{aligned} \Pr(X = x) &= rac{1}{x!} \cdot rac{1}{ heta^{lpha} \cdot \Gamma(lpha)} \cdot rac{ heta^{x+lpha}}{(1+ heta)^{x+lpha}} \cdot \Gamma(x+lpha) \ &= rac{1}{x!} \cdot rac{1}{(lpha-1)!} \cdot rac{ heta^x}{(1+ heta)^{x+lpha}} \cdot (x+lpha-1)! \ &= rac{(x+lpha-1)!}{x! \cdot (lpha-1)!} \cdot \left(rac{ heta}{1+ heta}
ight)^x \left(rac{1}{1+ heta}
ight)^{lpha} \ &= \left(rac{x+lpha-1}{x}
ight) \cdot \left(rac{ heta}{1+ heta}
ight)^x \left(rac{1}{1+ heta}
ight)^{lpha} \end{aligned}$$

Notice that the expression above has the same form as a negative binomial PMF:

$$\Pr(X=x) = inom{x+r-1}{x}igg(rac{eta}{1+eta}igg)^xigg(rac{1}{1+eta}igg)^r$$

Therefore,

$$X \sim ext{Negative Binomial } (r = lpha, \, eta = heta)$$