Deductibles 40M

The deductible is the amount of each claim that the **policyholder** is responsible for paying before the insurer will pay a claim. In this section, we will focus on two types of deductibles: ordinary deductibles and franchise deductibles. As a reminder, if a question only says "deductible", it means **ordinary deductible**.

Ordinary Deductibles

Let X be the loss variable. For a policy with an ordinary deductible d, the **policyholder** (not insurer) pays the loss up to d.

$$X \wedge d = \begin{cases} X, & X < d \\ d, & X \geq d \end{cases}$$

Coach's Remarks

Yes, this is equivalent to what an **insurer** would pay for a policy with a policy limit d.

Thus, the **insurer** will be responsible for covering the remaining amount:

$$(X-d)_+ = \begin{cases} 0, & X \leq d \\ X-d, & X > d \end{cases}$$

Naturally, the contributions of the policyholder and insurer must add up to the full loss amount.

$$(X \wedge d) + (X-d)_+ = X$$

Coach's Remarks

Recall in Section S1.2.1, we expressed the insurer's payment

- for a policy with a policy limit u as $\min(X,\, u)$, and
- for a policy with a deductible d as $\max(X - d,\, 0)$.

In this section, we express them as $(X \wedge u)$ and $(X - d)_+$, respectively. The two sets of expressions are equal.

$$(X \wedge u) = \min(X,\, u) = \begin{cases} X, & X \le u \\ u, & X > u \end{cases}$$

$$(X - d)_+ = \max(X - d,\, 0) = \begin{cases} 0, & X \le d \\ X - d, & X > d \end{cases}$$

Let's see why the relationship before this remarks holds by using the min and max expressions.

$$\begin{array}{ccccc} \min(X,\, d) & + & \max(X,\, d) & = & X + d \\ \min(X,\, d) & + & \max(X,\, d) - d & = & X \\ \min(X,\, d) & + & \max(X - d,\, 0) & = & X \\ (X \wedge d) & + & (X - d)_+ & = & X \end{array}$$

The first line is true because the minimum of any two quantities plus the maximum of the same two quantities will always equal the sum of the quantities.

Based on the above relationship, the following is also true.

$$\mathrm{E}[X \wedge d] + \mathrm{E}\big[(X - d)_+\big] = \mathrm{E}[X]$$

Thus, the expected insurance payment can be calculated as

$$\mathrm{E}\left[(X-d)_+\right] = \mathrm{E}[X] - \mathrm{E}[X \wedge d] \qquad \text{(S2.3.2.1)}$$

This formula is **recommended** under most circumstances. That is because the mean and limited expectation formulas for almost all distributions are provided in the exam table.

However, the formula does **not** apply to moments other than the first. Take the second moment for example, as seen below.

	$\boldsymbol{X^2}$	$\boldsymbol{(X \wedge d)^2}$	$\boldsymbol{X^2 - (X \wedge d)^2}$	$\boldsymbol{(X-d)_+^2}$
$\boldsymbol{X \le d}$	X^2	X^2	0	0
$\boldsymbol{X > d}$	X^2	d^2	$X^2 - d^2$	$X^2 - 2dX + d^2$

Notice that the values in column 4 are not equal to the values in column 5. Therefore,

$$\mathrm{E}\left[(X-d)_+^2\right] \ne \mathrm{E}\left[X^2\right] - \mathrm{E}\left[(X \wedge d)^2\right]$$

In cases where higher moments are needed, such as when calculating the variance, apply first principles instead, as shown below.

For losses **below** the deductible, i.e. $X \le d$, the insurer pays nothing, and the average payment amount for this portion is:

$$\int_0^d 0 \cdot f(x)\,\mathrm{d}x = 0$$

For the losses **above** the deductible, i.e. $X > d$, the insurance pays the loss amount minus the deductible, and the average payment amount for this portion is:

$$\int_d^\infty (x-d) f(x)\,\mathrm{d}x$$

Thus, the average payment amount is:

$$\mathrm{E}\left[(X-d)_+\right] = \int_d^\infty (x-d) f(x)\,\mathrm{d}x$$

This can be extended to the k^{th} moment of the payment variable:

$$\mathrm{E}\left[(X-d)_+^k\right] = \int_d^\infty (x-d)^k f(x)\,\mathrm{d}x \qquad \text{(S2.3.2.2)}$$

Likewise, using the survival function method,

$$\mathrm{E}\left[(X-d)_+\right] = \int_d^\infty S(x)\,\mathrm{d}x$$

$$\mathrm{E}\left[(X-d)_+^k\right] = \int_d^\infty k(x-d)^{k-1} S(x)\,\mathrm{d}x \qquad \text{(S2.3.2.3)}$$

The derivation of the survival function method is provided at the end of this section.

Example S2.3.2.1

For an insurance policy:

- Losses follow an exponential distribution with mean 500.
- The policy has an ordinary deductible of 100 per loss.

Calculate the expected insurance payment per loss.

Solution

Define X to be the loss.

$$X \sim \text{Exponential}\ (500)$$

$$\text{E}\left[(X-d)_+\right] = \text{E}[X] - \text{E}[X \wedge d]$$

The limited expected value of the exponential distribution is provided in the exam table.

$$\begin{aligned} \text{E}[X \wedge 100] &= 500\left(1 - e^{-100\,/\,500}\right) \\ &= 90.63 \end{aligned}$$

Then, the expected insurance payment per loss is

$$\begin{aligned} \text{E}\left[(X-100)_+\right] &= 500 - 90.63 \\ &= \mathbf{409.37} \end{aligned}$$

Loss Elimination Ratio

The ***loss elimination ratio (LER)*** measures how much the insurer saves by imposing an ordinary deductible.

So what does the insurer no longer pay, on average, after imposing an ordinary deductible? It is the portion of the loss that the policyholder has to pay, on average, i.e.,

$$\text{E}[X \wedge d]$$

Divide that by the average full loss amount, $\mathrm{E}[X]$, to compute the LER.

$$LER = \frac{\mathrm{E}[X \wedge d]}{\mathrm{E}[X]} \tag{S2.3.2.4}$$

Example S2.3.2.2

In the year 2017, claim amounts have the density function

$$f(x) = \frac{1}{10}, \qquad 0 < x < 10$$

In 2018, assume claims will follow the same distribution and an ordinary deductible of 2 will be implemented.

Calculate the loss elimination ratio in 2018.

Solution

Since the PDF is constant, we know it is a uniform distribution.

$$X \sim \text{Uniform}\ (0,\ 10)$$

Calculate the mean and limited expected value:

$$\begin{aligned}\mathrm{E}[X] &= \frac{0+10}{2} \\ &= 5\end{aligned}$$

$$\begin{aligned}\mathrm{E}[X \wedge 2] &= \int_0^2 \frac{x}{10}\,\mathrm{d}x + 2 \cdot S(2) \\ &= \left[\frac{x^2}{20}\right]_0^2 + 2 \cdot \frac{10-2}{10} \\ &= 1.8\end{aligned}$$

The loss elimination ratio is

$$\begin{aligned}LER &= \frac{1.8}{5} \\ &= \mathbf{0.36}\end{aligned}$$

Coach's Remarks

Although the textbook defines the LER as "***the ratio of the decrease in the expected payment with an ordinary deductible to the expected payment without the deductible***", there have been old released questions asking for the LER of a policy with a **franchise** deductible.

To prepare students for coverage modifications other than just an ordinary deductible, including policy limits and coinsurance, here is a generalized formula for LER:

$$LER = \frac{\mathrm{E}[X] - \mathrm{E}[Y]}{\mathrm{E}[X]}$$

where $\mathrm{E}[X]$ is the expected loss and $\mathrm{E}[Y]$ is the expected insurance payment.

For a policy with an ordinary deductible d, the expected payment is

$$\mathrm{E}[Y] = \mathrm{E}\left[(X-d)_+\right]$$

Thus, the LER is

$$LER = \frac{\mathrm{E}[X] - \mathrm{E}\left[(X-d)_+\right]}{\mathrm{E}[X]} = \frac{\mathrm{E}[X \wedge d]}{\mathrm{E}[X]}$$

which is consistent with $(\mathrm{S2.3.2.4})$.

The **eliminated loss** is simply the average amount the insurer no longer needs to pay due to the implementation of coverage modification(s).

Payment per Loss vs. Payment per Payment

Up to this point, the expected payment we have been calculating is the ***expected payment per loss***.

Most of the time, losses below the deductible are not reported since the policyholder would not be reimbursed anyway. Therefore, insurance companies often do not have sufficient information about losses below the deductible.

Therefore, insurance companies are sometimes more interested in the ***expected payment per payment***. To illustrate the difference between these two terms, consider the following example.

Example S2.3.2.3

You are given the following losses:

$$2 \qquad 3 \qquad 7 \qquad 9 \qquad 14$$

An insurance policy has an ordinary deductible of 5.

Calculate

1. the average payment **per loss**.

2. the average payment **per payment**.

Solution

The average payment per **loss** is simply the total payments divided by the number of **losses**.

$$\begin{aligned}\frac{\text{Total payment}}{\text{Number of losses}} &= \frac{0+0+2+4+9}{5} \\ &= \mathbf{3}\end{aligned}$$

Using the same logic, the average payment per **payment** is the total payments divided by the number of **payments**.

There are no payments for the first two losses since they are below the deductible. Thus, the number of payments is 3.

$$\begin{aligned}\frac{\text{Total payment}}{\text{Number of payments}} &= \frac{0+0+2+4+9}{3} \\ &= \mathbf{5}\end{aligned}$$

■

Coach's Remarks

Intuitively speaking, the former is the average payment **per loss incurred**, and the latter is the average payment **per payment made**. Since the number of payments will never exceed the number of losses, the average payment per payment will always be greater than or equal to the average payment per loss.

$$\mathrm{E}\left[Y^P\right] \geq \mathrm{E}\left[Y^L\right]$$

In general, we use Y^L to represent the payment per loss variable, and Y^P for the payment per payment variable. As a refresher, for a policy with an ordinary deductible, the expected payment per loss is

$$\mathrm{E}\left[Y^L\right] = \mathrm{E}\left[(X-d)_+\right]$$

As shown in the previous example, the expected payment per payment is the expected payment **given there is actually a payment**, which occurs when the loss is greater than the deductible. Therefore,

$$\begin{aligned} \mathrm{E}\left[Y^P\right] &= \mathrm{E}\left[Y^L \mid Y^L > 0\right] \\ &= \mathrm{E}[X-d \mid X > d] \end{aligned}$$

Recall from Section S2.1.5 that the conditional PDF is

$$f(x \mid X > d) = \frac{f(x)}{S(d)}, \qquad x > d$$

Using first principles, the expected payment per payment can be derived as follows:

$$\begin{aligned} \mathrm{E}\left[Y^P\right] &= \mathrm{E}[X-d \mid X > d] \\ &= \int_d^\infty (x-d) f(x \mid X > d)\,\mathrm{d}x \\ &= \int_d^\infty (x-d) \frac{f(x)}{S(d)}\,\mathrm{d}x \\ &= \frac{1}{S(d)} \cdot \int_d^\infty (x-d) f(x)\,\mathrm{d}x \\ &= \frac{\mathrm{E}\left[(X-d)_+\right]}{S(d)} \\ &= \frac{\mathrm{E}\left[Y^L\right]}{S(d)} \end{aligned}$$

Coach's Remarks

Many students mistake the following as the expected payment per payment.

$$\int_d^\infty (x-d)f(x)\,\mathrm{d}x = \mathrm{E}\left[Y^L\right] \neq \mathrm{E}\left[Y^P\right]$$

Remember that in order to calculate the expected payment per payment, we need to use the conditional PDF.

$$\int_d^\infty (x-d)\frac{f(x)}{S(d)}\,\mathrm{d}x = \mathrm{E}\left[Y^P\right]$$

In conclusion, an easier way to switch between the expected payment per payment and the expected payment per loss is to use the following equations.

$$\mathrm{E}\left[Y^P\right] = \frac{\mathrm{E}\left[Y^L\right]}{S(d)} \qquad \Leftrightarrow \qquad \mathrm{E}\left[Y^L\right] = \mathrm{E}\left[Y^P\right]\cdot S(d)$$

To remember this relationship, note that $\mathrm{E}\left[Y^P\right]$ will always be greater than or equal to $\mathrm{E}\left[Y^L\right]$. Because $S(d) \le 1$, in order to convert $\mathrm{E}\left[Y^L\right]$ to $\mathrm{E}\left[Y^P\right]$, we would need to divide by $S(d)$ to get a larger value.

This relationship is applicable to all higher-order moments.

$$\mathrm{E}\left[\left(Y^P\right)^k\right] = \frac{\mathrm{E}\left[\left(Y^L\right)^k\right]}{S(d)}$$

⇳

$$\mathrm{E}\left[\left(Y^L\right)^k\right] = \mathrm{E}\left[\left(Y^P\right)^k\right] \cdot S(d) \tag{S2.3.2.5}$$

Coach's Remarks

Note that Y^L is the general notation for the payment per loss variable. Y^L is not always $(X-d)_+$. It is only the case if the policy has an ordinary deductible and no other coverage modifications.

For example, for a policy that has a policy limit u,

$$Y^L = X \wedge u$$

And for a policy that has a deductible d and a policy limit u,

$$Y^L = (X \wedge m) - (X \wedge d)$$

where m, which will be introduced shortly, is the maximum covered loss.

The same can be said about Y^P. Its expression depends on the coverage modifications.

Also, note that when there is no deductible, $Y^P = Y^L$. This is because all losses will result in payments.

Example S2.3.2.4

For an insurance policy:

- Losses follow an exponential distribution with mean 500.
- The policy has an ordinary deductible of 100 per loss.
- Y^P is the claim payment per payment random variable.

Calculate $\mathrm{Var}\left[Y^P\right]$.

Solution

Let X be the loss. Define the variables:

$$X \sim \textbf{Exponential}\ (500)$$

$$Y^L = (X - 100)_+$$

$$Y^P = X - 100 \mid X > 100$$

Our goal is to calculate

$$\mathrm{Var}\left[Y^P\right] = \mathrm{E}\left[\left(Y^P\right)^2\right] - \mathrm{E}\left[Y^P\right]^2$$

Apply $(\mathrm{S2.3.2.5})$ to calculate the payment per payment moments.

$$\begin{aligned} \mathrm{E}\left[Y^P\right] &= \frac{\mathrm{E}\left[Y^L\right]}{S(d)} \\ &= \frac{409.37}{0.8187} \\ &= 500 \end{aligned}$$

Recall that the expected payment per loss was calculated in Example S2.3.2.1 as 409.37, and that $S(100) = e^{-100\,/\,500} = 0.8187$.

Next,

$$\begin{aligned}\mathrm{E}\left[\left(Y^L\right)^2\right] &= \int_d^\infty 2(x-d)\,S(x)\,\mathrm{d}x \\ &= \int_{100}^\infty (2x-200)e^{-x\,/\,500}\,\mathrm{d}x \\ &= 2\left[x\left(-500e^{-x\,/\,500}\right) - 500^2e^{-x\,/\,500}\right]_{100}^{\infty} - 200\left[-500e^{-x\,/\right. \\ &= 409{,}365.3765\end{aligned}$$

$$\begin{aligned}\mathrm{E}\left[\left(Y^P\right)^2\right] &= \frac{\mathrm{E}\left[\left(Y^L\right)^2\right]}{S(d)} \\ &= 500{,}000\end{aligned}$$

Thus, the variance is

$$\begin{aligned}\mathrm{Var}\left[Y^P\right] &= 500{,}000 - 500^2 \\ &= \mathbf{250{,}000}\end{aligned}$$

Alternative Solution

A simpler way of solving this problem is by recalling the memoryless property of the exponential distribution.

For $X \sim \text{Exponential}\ (\theta)$ and $d > 0$,

$$X - d \mid X > d \sim \text{Exponential}\,(\theta)$$

Thus, in this case,

$$Y^P \sim \text{Exponential}\,(500)$$

$$\begin{aligned}\text{Var}\left[Y^P\right] &= 500^2 \\ &= \mathbf{250{,}000}\end{aligned}$$

$(X \wedge d)$ is called the limited loss variable, while $(X - d \mid X > d)$ is called the ***excess loss variable***. Its mean, denoted by $e(d)$, is the ***mean excess loss function***, also called the ***mean residual life function***.

$$e(d) = \text{E}[X - d \mid X > d]$$

Sometimes, we will use $e_X(d)$ to indicate the loss variable in the subscript.

Here are a few special properties discussed in Section S2.3 that might be helpful when working with excess loss variables:

Loss, $\boldsymbol{X}$	**Mean,** $\mathbf{E}[\boldsymbol{X}]$	**Excess Loss,** $\boldsymbol{X - d \mid X > d}$	**Mean Excess Loss,** $\mathbf{E}[\boldsymbol{X - d \mid X > d}]$
Exponential (θ)	θ	Exponential (θ)	θ
Uniform (a, b)	$\dfrac{a+b}{2}$	Uniform $(0, b-d)$	$\dfrac{b-d}{2}$
Pareto (α, θ)	$\dfrac{\theta}{\alpha-1}$	Pareto $(\alpha, \theta+d)$	$\dfrac{\theta+d}{\alpha-1}$
S-P Pareto (α, θ)	$\dfrac{\alpha\theta}{\alpha-1}$	Pareto (α, d)	$\dfrac{d}{\alpha-1}$

Franchise Deductibles

Recall from Section 1.2.1, when a loss is greater than the deductible, a policy with a franchise deductible will pay the **full** amount of the loss.

$$Y^L = \begin{cases} 0, & X \le d \\ X, & X > d \end{cases}$$

The expected payment of a policy with a franchise deductible can be calculated by taking the expected payment of a policy with an ordinary deductible and adding back the deductible when a payment is made.

$$\begin{aligned} \mathrm{E}\left[Y^L\right] &= \int_d^\infty x\, f(x)\, \mathrm{d}x \\ &= \int_d^\infty x\, f(x)\, \mathrm{d}x - \int_d^\infty d\, f(x)\, \mathrm{d}x + \int_d^\infty d\, f(x)\, \mathrm{d}x \\ &= \int_d^\infty (x-d) f(x)\, \mathrm{d}x + d \cdot \int_d^\infty f(x)\, \mathrm{d}x \\ &= \mathrm{E}\left[(X-d)_+\right] + d \cdot S(d) \end{aligned}$$

The expected payment per payment is obtained by dividing the expected payment per loss by the survival function evaluated at the deductible, i.e., by applying the relationship in $(\mathrm{S2.3.2.5})$.

$$\begin{aligned} \mathrm{E}\left[Y^P\right] &= \frac{\mathrm{E}\left[(X-d)_+\right] + d \cdot S(d)}{S(d)} \\ &= e(d) + d \end{aligned}$$

Example S2.3.2.5

Loss amounts have a distribution with survival function

$$S(x) = \sqrt{\left(\frac{100}{x+100}\right)^5}, \qquad x > 0$$

An insurance coverage for these losses has a franchise deductible of 50.

Calculate the expected payment per payment.

Solution

$$S(x) = \sqrt{\left(\frac{100}{x+100}\right)^5} = \left(\frac{100}{x+100}\right)^{2.5} = \left(\frac{\theta}{x+\theta}\right)^{\alpha}$$
$$\Downarrow$$
$$X \sim \text{Pareto}\ (2.5,\ 100)$$

For a policy with a franchise deductible, the expected payment per payment is

$$\mathrm{E}\left[Y^P\right] = e(d) + d$$

Recall that for $X \sim \text{Pareto}\ (\alpha,\ \theta)$ and $d > 0$,

$$X - d \mid X > d \sim \text{Pareto}\ (\alpha,\ \theta + d)$$

Thus,

$$\begin{aligned} e(d) &= \mathrm{E}[X - d \mid X > d] \\ &= \frac{\theta + d}{\alpha - 1} \end{aligned}$$

Calculate the expected payment per payment:

$$\begin{aligned}\mathrm{E}\left[Y^P\right] &= \frac{100+50}{2.5-1}+50 \\ &= \mathbf{150}\end{aligned}$$