Appendix 10M

Limited Expected Values - Survival Function Method

$$
\begin{aligned}
\mathrm{E}\,[X \wedge u] &= \int_0^\infty \min(x,\, u) \cdot f_X(x)\,\mathrm{d}x \\
&= \int_0^u x \cdot f_X(x)\,\mathrm{d}x + \int_u^\infty u \cdot f_X(x)\,\mathrm{d}x \\
&= \left\{ [-x \cdot S_X(x)]\big|_0^u - \int_0^u 1 \cdot [-S_X(x)]\,\mathrm{d}x \right\} + u \cdot S_X(u) \\
&= \left\{ -u \cdot S_X(u) + \int_0^u S_X(x)\,\mathrm{d}x \right\} + u \cdot S_X(u) \\
&= \boldsymbol{\int_0^u S_X(x)\,\mathrm{d}x}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}\left[(X \wedge u)^k\right] &= \int_0^\infty \min\left(x^k,\, u^k\right) \cdot f_X(x)\,\mathrm{d}x \\
&= \int_0^u x^k \cdot f_X(x)\,\mathrm{d}x + \int_u^\infty u^k \cdot f_X(x)\,\mathrm{d}x \\
&= \left\{ \left[-x^k \cdot S_X(x)\right]\big|_0^u - \int_0^u kx^{k-1} \cdot [-S_X(x)]\,\mathrm{d}x \right\} + u^k \cdot S_X \\
&= \left\{ -u^k \cdot S_X(u) + \int_0^u kx^{k-1} \cdot S_X(x)\,\mathrm{d}x \right\} + u^k \cdot S_X(u) \\
&= \boldsymbol{\int_0^u kx^{k-1} \cdot S_X(x)\,\mathrm{d}x}
\end{aligned}
$$

Note: Refer to Section S2.1 Appendix for more details.

Payment Per Loss Moments for Ordinary Deductible - Survival Function Method

$$
\begin{aligned}
\mathrm{E}\left[(X-d)_+\right] &= \int_0^\infty \max(x-d,\,0)\cdot f_X(x)\,\mathrm{d}x \\
&= \int_0^d 0\cdot f_X(x)\,\mathrm{d}x + \int_d^\infty (x-d)\cdot f_X(x)\,\mathrm{d}x \\
&= 0 + \left\{\left[-(x-d)\cdot S_X(x)\right]\Big|_d^\infty - \int_d^\infty 1\cdot\left[-S_X(x)\right]\mathrm{d}x\right\} \\
&= \boldsymbol{\int_d^\infty S_X(x)\,\mathrm{d}x}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}\left[(X-d)_+^k\right] &= \int_0^\infty \max\left[(x-d)^k,\,0\right]\cdot f_X(x)\,\mathrm{d}x \\
&= \int_0^d 0\cdot f_X(x)\,\mathrm{d}x + \int_d^\infty (x-d)^k\cdot f_X(x)\,\mathrm{d}x \\
&= 0 + \left\{\left[-(x-d)^k\cdot S_X(x)\right]\Big|_d^\infty - \int_d^\infty k(x-d)^{k-1}\cdot [-S_X(\right. \\
&= \boldsymbol{\int_d^\infty k(x-d)^{k-1}\cdot S_X(x)\,\mathrm{d}x}
\end{aligned}
$$

Note: Refer to Section S2.1 Appendix for more details.

The Ultimate Formula

For a policy with a deductible d, a policy limit u, and coinsurance α, the payment per loss is

$$Y^L = \begin{cases} 0, & X \le d \\ \alpha\,(X-d), & d < X < m \\ u, & X \ge m \end{cases}$$

With a little manipulation,

$$\begin{aligned} Y^L &= \begin{cases} 0, & X \le d \\ \alpha\,(X-d), & d < X < m \\ u, & X \ge m \end{cases} \\ &= \begin{cases} \alpha\,(X-X), & X \le d \\ \alpha\,(X-d), & d < X < m \\ \alpha\,(m-d), & X \ge m \end{cases} \\ &= \alpha \cdot \begin{cases} X-X, & X \le d \\ X-d, & d < X < m \\ m-d, & X \ge m \end{cases} \\ &= \alpha \cdot \left[\begin{cases} X, & X \le d \\ X, & d < X < m \\ m, & X \ge m \end{cases} - \begin{cases} X, & X \le d \\ d, & d < X < m \\ d, & X \ge m \end{cases} \right] \\ &= \alpha \cdot [(X \wedge m) - (X \wedge d)] \end{aligned}$$

Thus, the expected payment per loss is

$$\mathrm{E}\left[Y^L\right] = \alpha \cdot \{\mathrm{E}\,[X \wedge m] - \mathrm{E}\,[X \wedge d]\}$$

The Ultimate Formula with Inflation

For a policy with a deductible d, a policy limit u, and coinsurance α, the payment per loss for the losses inflated by a factor of $(1+r)$ is

$$Y^L = \begin{cases} 0, & (1+r)X \le d \\ \alpha\,[(1+r)X - d], & d < (1+r)X < m \\ u, & (1+r)X \ge m \end{cases}$$

With a little manipulation,

$$0, \qquad (1+r)X \le d$$

Thus, the expected payment per loss is

$$\mathrm{E}\left[Y^L\right] = \alpha(1+r)\cdot\left\{\mathrm{E}\left[X \wedge \frac{m}{1+r}\right] - \mathrm{E}\left[X \wedge \frac{d}{1+r}\right]\right\}$$