Aggregate Payments 25M

Coach's Remarks

Up until now, we have been using S to represent aggregate **losses**. In this section, we will reuse S to represent aggregate **payments**. In general, S is the notation for the aggregate model.

An insurance company not only needs to model the aggregate loss but also the aggregate payment to the policyholders. There are two ways we can model the aggregate payment.

The first way is to consider all losses and sum the resulting payment amounts, even if the payment amounts are 0. Let N be the number of **losses** and Y^L be the payment **per loss**.

$$S = \sum_{i=1}^{N} Y_i^L, \qquad N = 1,\, 2,\, \ldots$$

The second way is to only consider losses that result in positive payments and sum the payment amounts. Let N' be the number of **payments** and Y^P be the payment **per payment**.

$$S = \sum_{i=1}^{N'} Y_i^P, \qquad N' = 1,\, 2,\, \ldots$$

The key is to make sure the frequency and severity are consistent: either both in terms of losses or both in terms of payments.

The aggregate payment model is no different than the aggregate loss model. Instead of summing the losses, we sum the payments. Therefore, the compound mean and variance formulas still hold.

$$\begin{aligned}\mathrm{E}[S] &= \mathrm{E}[N]\mathrm{E}\left[Y^L\right] \\ &= \mathrm{E}\left[N'\right]\mathrm{E}\left[Y^P\right]\end{aligned}$$

$$\begin{aligned}\mathrm{Var}[S] &= \mathrm{E}[N]\mathrm{Var}\left[Y^L\right] + \mathrm{Var}[N]\mathrm{E}\left[Y^L\right]^2 \\ &= \mathrm{E}\left[N'\right]\mathrm{Var}\left[Y^P\right] + \mathrm{Var}\left[N'\right]\mathrm{E}\left[Y^P\right]^2\end{aligned}$$

Example S2.5.5.1

A policyholder has a 30% chance to incur a loss each year. In the event of a loss, the amount has the following distribution:

$\boldsymbol{x}$	$\mathbf{Pr}(\boldsymbol{X}=\boldsymbol{x})$
100	0.4
200	0.3
500	0.3

Each loss is subject to a deductible of 150.

The number of losses and the loss amounts are mutually independent.

Calculate the variance of the amount the insurer pays the policyholder in a year.

Solution

We can calculate the mean and variance of the aggregate payments using the number of **losses** and payment **per loss** approach.

$$\text{Var}[S] = \text{E}[N]\,\text{Var}\left[Y^L\right] + \text{Var}[N]\,\text{E}\left[Y^L\right]^2$$

Start with the mean and variance of the number of losses:

$$N \sim \text{Bernoulli }(0.3)$$

$$\text{E}[N] = 0.3$$

$$\begin{aligned}\text{Var}[N] &= 0.3(1-0.3)\\ &= 0.21\end{aligned}$$

Then, calculate the mean and variance of the payment per loss:

y	$\Pr(Y^L = y)$
0	0.4
50	0.3
350	0.3

$$\begin{aligned}\text{E}\left[Y^L\right] &= 0.4(0) + 0.3(50) + 0.3(350)\\ &= 120\end{aligned}$$

$$\begin{aligned}\text{E}\left[\left(Y^L\right)^2\right] &= 0.4\left(0^2\right) + 0.3\left(50^2\right) + 0.3\left(350^2\right)\\ &= 37{,}500\end{aligned}$$

$$\begin{aligned}\text{Var}\left[Y^L\right] &= 37{,}500 - 120^2\\ &= 23{,}100\end{aligned}$$

Putting them together, the variance of aggregate payments is

$$\begin{aligned}\mathrm{Var}\left[S\right] &= 0.3\left(23{,}100\right) + 0.21\left(120^2\right) \\ &= \mathbf{9{,}954}\end{aligned}$$

Alternative Solution

We can also calculate the mean and variance of the aggregate payments using the number of **payments** and payment **per payment** approach.

$$\mathrm{Var}\left[S\right] = \mathrm{E}\left[N'\right]\mathrm{Var}\left[Y^P\right] + \mathrm{Var}\left[N'\right]\mathrm{E}\left[Y^P\right]^2$$

Construct the number of payments distribution:

- There will be no payment when
  - there is no loss, or
  - there is a loss and it is less than the deductible.

$$\begin{aligned}\Pr\left(N' = 0\right) &= p_N(0) + p_N(1) \cdot F_X(150) \\ &= 0.7 + 0.3(0.4) \\ &= 0.82\end{aligned}$$

- There will be 1 payment when there is a loss and it exceeds the deductible.

$$\begin{aligned}\Pr\left(N' = 1\right) &= p_N(1) \cdot S_X(150) \\ &= 0.3(0.6) \\ &= 0.18\end{aligned}$$

In conclusion, $N' \sim \text{Bernoulli}\ (0.18)$. Calculate the mean and variance of the number of payments:

$$\mathrm{E}\left[N'\right] = 0.18$$

$$\begin{aligned}\mathrm{Var}\left[N'\right] &= 0.18(1 - 0.18) \\ &= 0.1476\end{aligned}$$

Then, construct the payment per payment distribution:

$$\begin{aligned}\Pr\left(Y^P = 50\right) &= \Pr\left(Y^L = 50 \mid Y^L > 0\right) \\ &= \Pr(X - 150 = 50 \mid X > 150) \\ &= \frac{\Pr(X = 200)}{\Pr(X > 150)} \\ &= \frac{0.3}{1 - 0.4} \\ &= 0.5\end{aligned}$$

$$\begin{aligned}\Pr\left(Y^P = 350\right) &= \frac{\Pr(X = 500)}{\Pr(X > 150)} \\ &= \frac{0.3}{1 - 0.4} \\ &= 0.5\end{aligned}$$

y	$\Pr(Y^P = y)$
50	0.5
350	0.5

Calculate the mean and variance of the payment per payment:

$$\begin{aligned} \mathrm{E}\left[Y^P\right] &= 0.5(50) + 0.5(350) \\ &= 200 \end{aligned}$$

$$\begin{aligned} \mathrm{E}\left[\left(Y^P\right)^2\right] &= 0.5\left(50^2\right) + 0.5\left(350^2\right) \\ &= 62{,}500 \end{aligned}$$

$$\begin{aligned} \mathrm{Var}\left[Y^P\right] &= 62{,}500 - 200^2 \\ &= 22{,}500 \end{aligned}$$

Finally, put them together to calculate the variance of aggregate payments:

$$\begin{aligned} \mathrm{Var}\left[S\right] &= 0.18\left(22{,}500\right) + 0.1476\left(200^2\right) \\ &= \mathbf{9{,}954} \end{aligned}$$

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Example S2.5.5.2

For an auto insurance policy:

- The number of claims follows a Poisson distribution with mean 25.
- Each claim size is uniformly distributed on the interval from 5 to 55.

- The number of claims and claim sizes are mutually independent.

A per-claim deductible of 20 is imposed.

Using the normal approximation, determine the probability that an aggregate payment does not exceed 110% of the expected aggregate payment.

Solution

We can calculate the mean and variance of the aggregate payments using the number of **losses** and payment **per loss** approach.

$$\mathrm{E}[S] = \mathrm{E}[N]\mathrm{E}\left[Y^L\right]$$

$$\mathrm{Var}[S] = \mathrm{E}[N]\mathrm{Var}\left[Y^L\right] + \mathrm{Var}[N]\mathrm{E}\left[Y^L\right]^2$$

For a Poisson distribution, the mean and variance are equal.

$$\mathrm{E}[N] = 25 = \mathrm{Var}[N]$$

Calculate the severity mean and variance:

$$\begin{aligned}\mathrm{E}\left[Y^L\right] &= \mathrm{E}\left[(X-20)_+\right] \\ &= \int_{20}^{55} (x-20)\frac{1}{50}\,\mathrm{d}x \\ &= \frac{1}{50}\left[\frac{(x-20)^2}{2}\right]_{20}^{55} \\ &= 12.25\end{aligned}$$

$$\begin{aligned}
\mathrm{E}\left[\left(Y^L\right)^2\right] &= \mathrm{E}\left[(X-20)_+^2\right] \\
&= \int_{20}^{55} (x-20)^2 \frac{1}{50}\, \mathrm{d}x \\
&= \frac{1}{50}\left[\frac{(x-20)^3}{3}\right]_{20}^{55} \\
&= 285.8333
\end{aligned}$$

$$\mathrm{Var}\left[Y^L\right] = 285.8333 - 12.25^2 = 135.7708$$

Substitute into the formulas to calculate the mean and variance of the aggregate payment.

$$\begin{aligned}
\mathrm{E}[S] &= \mathrm{E}[N]\mathrm{E}\left[Y^L\right] \\
&= 25(12.25) \\
&= 306.25
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}[S] &= \mathrm{E}[N]\mathrm{Var}\left[Y^L\right] + \mathrm{Var}[N]\mathrm{E}\left[Y^L\right]^2 \\
&= 25(135.7708) + 25(12.25)^2 \\
&= 7{,}145.8333
\end{aligned}$$

Finally, calculate the desired probability.

$$\begin{aligned}
\Pr(S \le 1.1 \cdot \mathrm{E}[S]) &= \Pr\left(Z \le \frac{1.1\,(306.25) - 306.25}{\sqrt{7{,}145.8333}}\right) \\
&= \Phi(0.36228) \\
&= \mathbf{0.64143}
\end{aligned}$$

Alternative Solution

We can also calculate the mean and variance of the aggregate payments using the number of **payment** and payment **per payment** approach.

$$\mathrm{E}[S] = \mathrm{E}\left[N'\right]\mathrm{E}\left[Y^P\right]$$

$$\mathrm{Var}[S] = \mathrm{E}\left[N'\right]\mathrm{Var}\left[Y^P\right] + \mathrm{Var}\left[N'\right]\mathrm{E}\left[Y^P\right]^2$$

Recall, in the previous section, we learned that the sum of independent Poisson random variables is a Poisson random variable. The reverse is also true. In this example, we can split the claims into two independent categories: those that exceed the deductible and those that don't. We do that by scaling the Poisson mean by the corresponding probability.

- The number of claims that don't exceed the deductible follows a Poisson distribution with mean

$$25 \cdot \Pr(X \leq 20) = 25 \cdot \frac{20-5}{55-5} = 7.5$$

- The number of claims that exceed the deductible, i.e., number of payments, or N', follows a Poisson distribution with mean

$$25 \cdot \Pr(X > 20) = 25 \cdot \frac{55-20}{55-5} = 17.5$$

Now, we have our payment frequency mean and variance.

$$\mathrm{E}\left[N'\right] = 17.5 = \mathrm{Var}\left[N'\right]$$

Then, using the property discussed in Section S2.2.5, the payment per payment also follows a uniform distribution.

$$X \sim \text{Uniform}\ (5,\ 55)$$

$$Y^P = (X - 20 \mid X > 20) \sim \text{Uniform}\ (0,\ 35)$$

Calculate the mean and variance of the payment per payment.

$$\mathrm{E}\left[Y^P\right] = \frac{0 + 35}{2} = 17.5$$

$$\mathrm{Var}\left[Y^P\right] = \frac{(35 - 0)^2}{12} = 102.0833$$

Substitute into the formulas to calculate the mean and variance of the aggregate payment.

$$\begin{aligned} \mathrm{E}[S] &= \mathrm{E}\left[N'\right]\mathrm{E}\left[Y^P\right] \\ &= 17.5(17.5) \\ &= 306.25 \end{aligned}$$

$$\begin{aligned} \mathrm{Var}[S] &= \mathrm{E}\left[N'\right]\mathrm{Var}\left[Y^P\right] + \mathrm{Var}\left[N'\right]\mathrm{E}\left[Y^P\right]^2 \\ &= 17.5(102.0833) + 17.5(17.5)^2 \\ &= 7{,}145.8333 \end{aligned}$$

Note that the mean and variance are identical to those calculated using the previous approach.

Finally, calculate the desired probability.

$$\begin{aligned}\Pr(S \leq 1.1 \cdot \mathrm{E}[S]) &= \Pr\left(Z \leq \frac{1.1\,(306.25) - 306.25}{\sqrt{7{,}145.8333}}\right) \\ &= \Phi(0.36228) \\ &= \mathbf{0.64143}\end{aligned}$$

Coach's Remarks

The concept of splitting the Poisson distribution is not applicable to the other $(a, b, 0)$ class distributions. Thus, if this example had a binomial or a negative binomial loss frequency, our only option would be to use the number of losses and payment per loss approach.

Example S2.5.5.3

For an auto insurance policy, the insurance company receives, on average, 300 claims per year.

For each claim, the claim size has the following probability density function:

$$f(x) = \frac{24{,}000}{(x+20)^4}, \qquad x > 0$$

Each claim is subject to a policy limit of 10.

The number of claims and amount of each claim are mutually independent.

Calculate the expected insurance payment in a year.

Solution

We are given the frequency mean is 300.

$$\mathrm{E}[N] = 300$$

Let X be the claim amount. Observing the PDF, X has a domain from 0 to infinity and does not have an e term, so it is likely to be Pareto or some related distribution. Comparing the PDFs, conclude that

$$X \sim \text{Pareto}\ (3,\ 20)$$

Look up the limited moments formula from the exam tables. The expected payment per claim is

$$\begin{aligned}\mathrm{E}[X \wedge 10] &= \frac{20}{3-1}\left[1 - \left(\frac{20}{10+20}\right)^{3-1}\right] \\ &= 5.556\end{aligned}$$

Let S be the aggregate insurance payment. The expected insurance payment in a year is

$$\begin{aligned}\mathrm{E}[S] &= \mathrm{E}[N]\mathrm{E}[X \wedge 10] \\ &= 300(5.5556) \\ &= \mathbf{1{,}667}\end{aligned}$$

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Coach's Remarks