Probability Function

10M

The probability function of the aggregate loss random variable can be calculated using the Law of Total Probability. If the aggregate loss is discrete, then

$$p_S(s) = \sum_{n>0}\left[p_N(n) \cdot \Pr\left(\sum_{i=1}^{n} X_i = s \mid N = n\right)\right], \qquad s > 0 \quad ($$

The probability in the brackets, $\Pr\left(\sum_{i=1}^{n} X_i = s\right)$, is called the *n-fold convolution* of the individual severity probability, $\Pr(X = s)$.

Keep in mind that S is zero when there are no losses or when all losses are zero. Thus,

$$p_S(0) = p_N(0) + \sum_{n>0}\left[p_N(n) \cdot \Pr\left(\sum_{i=1}^{n} X_i = 0 \mid N = n\right)\right]$$

The aggregate loss CDF (discrete or continuous) is

$$F_S(s) = p_N(0) + \sum_{n>0}\left[p_N(n) \cdot \Pr\left(\sum_{i=1}^{n} X_i \le s \mid N = n\right)\right]$$

Coach's Remarks

The equations above are very general. We would usually solve exam questions on a case-by-case basis using different techniques and shortcuts. Thus, it is important to understand these equations so that you don't need to memorize them.

An exam question usually limits the possible values of n to limit the number of possible combinations for $\sum_{i=1}^{n} X_i \leq s$.

Example 2.5.1.1

The number of claims in a period has a Poisson distribution with a mean of 3. Each claim amount follows a distribution such that $\Pr(X = x) = 0.25$ for $x = 1,\ 2,\ 3,\ 4$.

The number of claims and the claim amounts are independent. S is the aggregate claim amount for that period.

Calculate $\Pr(S = 3)$.

Solution

$S = 3$ can occur in three ways:

- One claim of size 3

$$\begin{aligned} p_N(1)p_X(3) &= \frac{e^{-3} \cdot 3^1}{1!}(0.25) \\ &= 0.75e^{-3} \end{aligned}$$

- Two claims of sizes 1 and 2

$$\begin{aligned} p_N(2)[p_X(1)p_X(2) + p_X(2)p_X(1)] &= p_N(2)[2\,p_X(1)p_X(2)] \\ &= \frac{e^{-3} \cdot 3^2}{2!}[2(0.25)(0.25)] \\ &= 0.5625e^{-3} \end{aligned}$$

- Three claims of size 1 each

$$\begin{aligned} p_N(3)p_X(1)^3 &= \frac{e^{-3} \cdot 3^3}{3!}(0.25^3) \\ &= 0.0703e^{-3} \end{aligned}$$

Sum the probabilities to calculate the final answer.

$$\Pr(S = 3) = 0.75e^{-3} + 0.5625e^{-3} + 0.0703e^{-3} = \mathbf{0.0688}$$

Example 2.5.1.2

For an insured car, you are given:

- The number of accidents per year has the following probability function:

$$\Pr(N = n) = \binom{2}{n} 0.2^n (0.8)^{2-n}, \qquad n = 0,\ 1,\ 2$$

- The damage per accident follows a normal distribution with mean 180 and standard deviation 100.

- The number of accidents and the damage per accident are mutually independent.

Determine the probability that the car has at most 200 in damages this year.

Solution

The damages in a year will not exceed 200 when:

- There are no claims.

$$\begin{aligned} p_N(0) &= \binom{2}{0} 0.2^0 (0.8)^{2-0} \\ &= 0.8^2 \\ &= 0.64 \end{aligned}$$

- There is one claim and the claim is not more than 200.

$$\begin{aligned} p_N(1) \cdot F_X(200) &= \left[\binom{2}{1} 0.2^1 (0.8)^{2-1} \right] \cdot \Phi\left(\frac{200-180}{100} \right) \\ &= 0.32\,(0.5793) \\ &= 0.1854 \end{aligned}$$

- There are two claims and the sum of the claims is not more than 200.

$$\begin{aligned} p_N(2) \cdot F_{X_1+X_2}(200) &= \left[\binom{2}{2} 0.2^2 (0.8)^{2-2} \right] \cdot \Phi\left(\frac{200-360}{\sqrt{20{,}000}} \right) \\ &= 0.04\,(0.1292) \\ &= 0.0052 \end{aligned}$$

Note that the sum of two independent normal distributions is a normal distribution where the mean and variance are summed.

$$\begin{aligned}\mathrm{E}[X_1+X_2]&=\mathrm{E}[X_1]+\mathrm{E}[X_2]\\&=180+180\\&=360\end{aligned}$$

$$\begin{aligned}\mathrm{Var}[X_1+X_2]&=\mathrm{Var}[X_1]+\mathrm{Var}[X_2]\\&=100^2+100^2\\&=20{,}000\end{aligned}$$

Thus,

$$X_1+X_2\sim\text{Normal}\ (360,\ 20{,}000)$$

$$F_{X_1+X_2}(200)=\Phi\left(\frac{200-360}{\sqrt{20{,}000}}\right)$$

The final answer is the sum of the three probabilities above:

$$\begin{aligned}\Pr(S\le 200)&=0.64+0.1854+0.0052\\&=\mathbf{0.8305}\end{aligned}$$

Coach's Remarks

Here is a short refresher of the mean and variance for the sum of random variables. For a sum of n independent random variables X_1, X_2, ..., X_n,

$$\mathrm{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathrm{E}\left[X_i\right]$$

$$\mathrm{Var}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathrm{Var}\left[X_i\right]$$