Mean and Variance

15M

Assuming we have a **collective risk model**, where the frequency is independent of the severity and all severities are i.i.d., the compound mean and variance of the aggregate loss are

$$\mathrm{E}[S] = \mathrm{E}[N]\,\mathrm{E}[X] \tag{S2.5.2.1}$$

$$\mathrm{Var}[S] = \mathrm{E}[N]\,\mathrm{Var}[X] + \mathrm{Var}[N]\,\mathrm{E}[X]^2 \tag{S2.5.2.2}$$

The derivations of the formulas are included at the end of this section.

We often use the following notation to represent the two formulas. The subscripts indicate the frequency (N), severity (X), and aggregate loss (S) variables.

$$\mu_S = \mu_N \mu_X \tag{S2.5.2.1}$$

$$\sigma_S^2 = \mu_N \sigma_X^2 + \sigma_N^2 \mu_X^2 \tag{S2.5.2.2}$$

Coach's Remarks

When we have a compound Poisson distribution, which means a compound distribution with a Poisson frequency, then

$$\mathrm{E}[N] = \mathrm{Var}[N] = \lambda$$

Thus, the compound variance can be simplified to

$$\begin{aligned}\mathrm{Var}\,[S] &= \lambda\,\mathrm{Var}\,[X] + \lambda\,\mathrm{E}[X]^2 \\ &= \lambda\left(\mathrm{Var}\,[X] + \mathrm{E}[X]^2\right) \\ &= \lambda\,\mathrm{E}\left[X^2\right]\end{aligned}$$

Example S2.5.2.1

The number of claims reported per month at City Insurance follows a negative binomial distribution with parameters $r = 80$ and $\beta = 2$.

The distribution of claim amounts is given in this table:

Claim amount	Probability
10	0.3
30	0.4
50	0.2
100	0.1

The number of claims and individual claim amounts are independent.

Calculate the standard deviation of the aggregate loss.

Solution

Let N represent the number of claims and X represent the individual claim amount.

Recall the aggregate loss variance formula.

$$\mathrm{Var}[S] = \mathrm{E}[N]\mathrm{Var}[X] + \mathrm{Var}[N]\mathrm{E}[X]^2$$

Calculate the frequency mean and variance.

$$\mathrm{E}[N] = 80(2) = 160$$

$$\mathrm{Var}[N] = 80(2)(1+2) = 480$$

Calculate the severity mean and variance.

$$\mathrm{E}[X] = 10(0.3) + 30(0.4) + 50(0.2) + 100(0.1) = 35$$

$$\mathrm{Var}[X] = \left[10^2(0.3) + 30^2(0.4) + 50^2(0.2) + 100^2(0.1)\right] - 35^2 = 665$$

Substitute the calculated means and variances into the formula:

$$\begin{aligned}\mathrm{Var}[S] &= \mathrm{E}[N]\mathrm{Var}[X] + \mathrm{Var}[N]\mathrm{E}[X]^2 \\ &= 160(665) + 480\left(35^2\right) \\ &= 694{,}400\end{aligned}$$

The standard deviation of the aggregate loss is

$$\sqrt{\mathrm{Var}[S]} = \sqrt{694{,}400} = \mathbf{833.3067}$$

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Example S2.5.2.2

For a family of 4, the annual dental expenses for each family member has the following distribution:

Number of Visits	Probability	Payment per Visit	Probability
0	0.2		
1	0.5	25 150	0.8 0.2
2	0.3	50 200	0.5 0.5

The dental expenses for the family members are mutually independent.

Calculate the variance of the payment per year for this family.

Solution

Let X_i represent the total dental expenses for each family member.

The possible values of X_i are

- 0 when there are no visits
- 25 when there is a visit with payment 25
- 150 when there is a visit with payment 150
- 100 when there are two visits with payment 50 per visit
- 250 when there are two visits with payments 50 and 200
- 400 when there are two visits with payment 200 per visit

The probabilities are calculated in the table below.

$\boldsymbol{x}$	$\mathbf{Pr}(\boldsymbol{X_i = x})$
0	0.2
25	$0.5(0.8) = 0.4$
100	$0.3(0.5^2) = 0.075$
150	$0.5(0.2) = 0.1$
250	$0.3[2(0.5)(0.5)] = 0.15$
400	$0.3(0.5^2) = 0.075$

It is good to get into the habit of checking if the probabilities add up to 1.

Next, calculate the variance of the expenses for each family member.

$$\begin{aligned}\mathrm{E}[X_i] &= 0(0.2) + 25(0.4) + 100(0.075) + 150(0.1) + 250(0.15) + 400\\ &= 100\end{aligned}$$

$$\begin{aligned}\mathrm{E}\left[X_i^2\right] &= 0^2(0.2) + 25^2(0.4) + 100^2(0.075) + 150^2(0.1) + 250^2(0.15)\\ &= 24{,}625\end{aligned}$$

$$\mathrm{Var}[X_i] = 24{,}625 - 100^2 = 14{,}625$$

By independence, the variance of the expenses per year for the family is

$$\begin{aligned}\mathrm{Var}[S] &= \mathrm{Var}\left[\sum_{i=1}^{4} X_i\right]\\ &= \sum_{i=1}^{4} \mathrm{Var}[X_i]\\ &= 4(14{,}625)\\ &= \mathbf{58{,}500}\end{aligned}$$

Coach's Remarks

Because claim sizes are not independent of the number of claims, i.e. the payment per visit has a different distribution depending on the number of visits, $(\mathrm{S}2.5.2.2)$ cannot be used here.