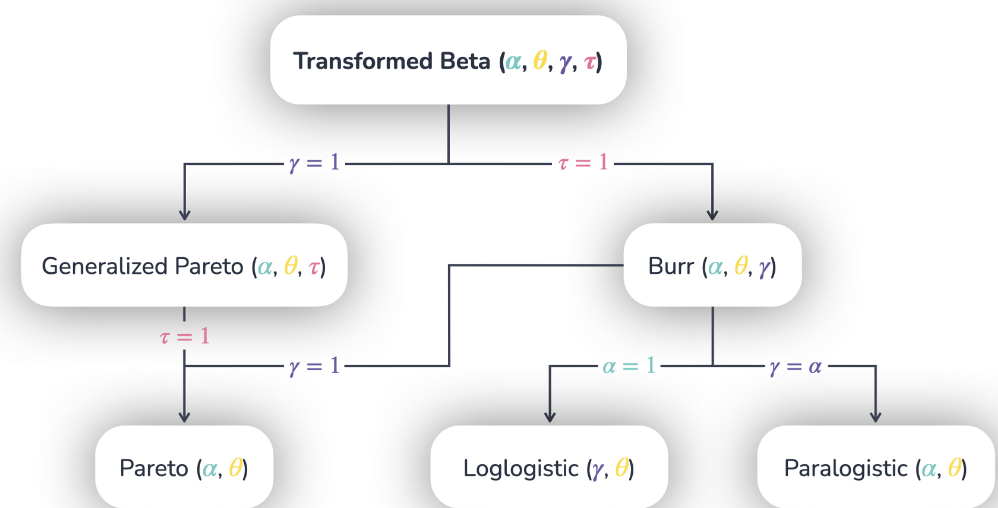
The Roles of Parameters

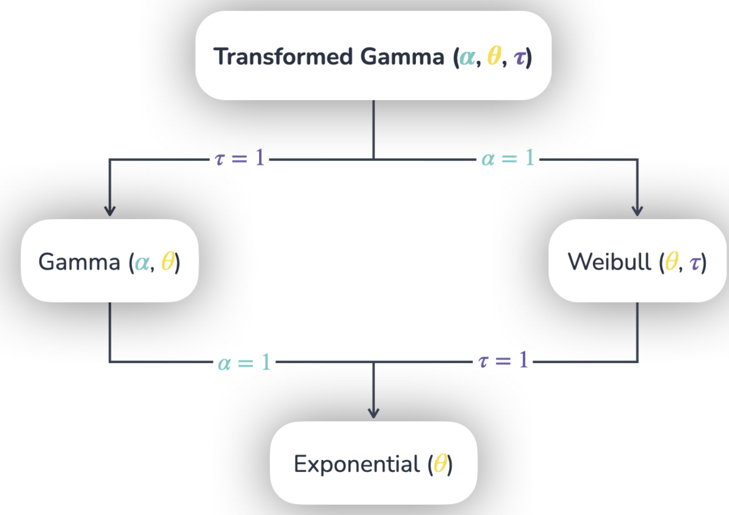
25M

In the last few sections, we discussed continuous distributions that are more commonly tested in the exam. The exam tables feature more distributions, many of which are more complex, i.e., have more parameters, than those we have discussed. While the complex distributions are less likely to be tested, in this section, we will briefly discuss the relationships between the distributions. Then, we will take a look at the roles played by the parameters.

A ***parametric distribution*** is a set of distribution functions. Each member within the set is determined by specifying one or more values called parameters. For example, the Pareto distribution is a parametric distribution. Its members include $\text{Pareto}\ (\alpha = 2,\ \theta = 100)$ and $\text{Pareto}\ (\alpha = 3,\ \theta = 200)$, among an infinite number of others.

Then, a ***parametric distribution family*** is a set of parametric distributions that are related in some meaningful way. Below are the two main parametric distribution families that are included in the exam tables:





Furthermore, a beta distribution is a generalized beta distribution where $\tau = 1$.

One parametric distribution family worth mentioning is the ***linear exponential family***.

Any distribution with a probability function that can be expressed in the form below, where the support does **not** depend on the parameter θ, belongs to the linear exponential family:

$$f(x;\theta) = \frac{p(x)e^{\,r(\theta)\cdot x}}{q(\theta)}$$

Note that $f(x;\theta)$ represents a generic probability function. The distribution can be discrete or continuous.

Coach's Remarks

The notation "$;\theta$" in "$f(x;\theta)$" signifies that θ is the natural parameter, which plays an important role in the analysis of the linear exponential family, such as in (S2.2.7.1) and (S2.2.7.2) below.

A distribution in the linear exponential family may have other parameters, but they do not play any explicit role.

For distributions in the linear exponential family, their mean and variance can be expressed as

$$\mathrm{E}\left[X\right] = \mu(\theta) = \frac{q'(\theta)}{r'(\theta)\,q(\theta)} \tag{S2.2.7.1}$$

$$\mathrm{Var}\left[X\right] = v(\theta) = \frac{\mu'(\theta)}{r'(\theta)} \tag{S2.2.7.2}$$

To illustrate these concepts better, let's walk through the proof below:

A distribution has the following probability function.

$$\Pr(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \qquad x = 0,\, 1,\, 2,\, \ldots$$

Prove that it belongs to the linear exponential family.

Note that λ plays the role of θ in this probability function. Rearrange the probability function to get:

$$\Pr(X = x; \lambda) = \frac{\left(\dfrac{1}{x!}\right) e^{\,(\ln\lambda)\cdot x}}{e^{\,\lambda}}$$

Noting that

- $\dfrac{1}{x!} = p(x)$
- $\ln \lambda = r(\lambda)$
- $e^{\lambda} = q(\lambda)$

demonstrates that the distribution is in the linear exponential family.

The table below classifies common distributions as belonging to the linear exponential family, or not:

Yes	**No**
Poisson	Beta
Binomial (with fixed m)	Lognormal
Gamma	Inverse gamma
Normal	Pareto
Negative binomial (with fixed r)	

Coach's Remarks

Note that some of the listed distributions are discrete, which we will discuss in a future section.

Also, the distribution we worked with in the motivating example is a Poisson distribution. If we were to calculate the mean and variance using (S2.2.7.1) and (S2.2.7.2), we would get λ for both, which are indeed the Poisson mean and variance as we will see in a later section.

A parametric distribution is a ***scale distribution*** if, when a random variable from that set of distributions is multiplied by a positive constant, the resulting random variable is also in the same set of distributions.

For random variables with nonnegative support, a ***scale parameter*** is a parameter for a scale distribution that meets two conditions:

1. When a member of the scale distribution is multiplied by a positive constant, the scale parameter is multiplied by the same constant.
2. When a member of the scale distribution is multiplied by a positive constant, all other parameters are unchanged.

For example, the Pareto distribution is a scale distribution with scale parameter θ, as proven below.

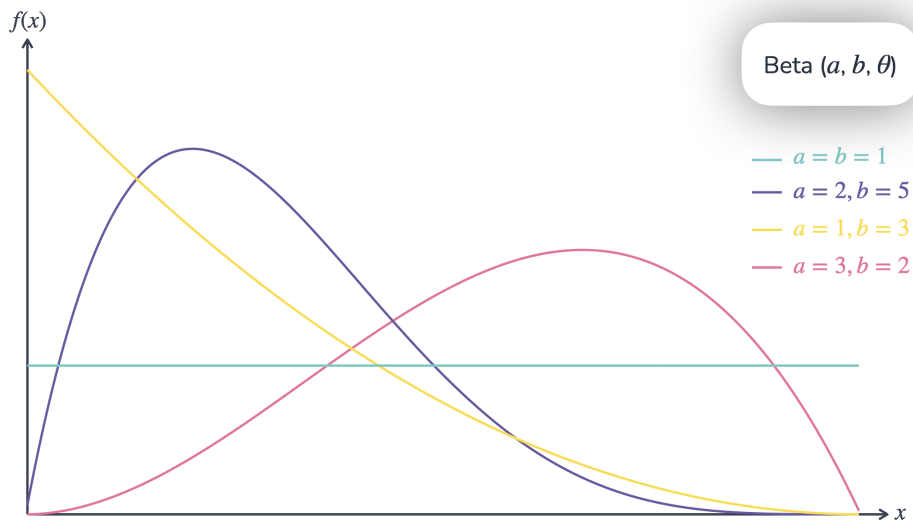
For $X \sim \text{Pareto}\ (\alpha,\ \theta)$ and $Y = cX$, where c is a positive constant,

$$\begin{aligned} F_Y(y) &= \Pr(Y \le y) \\ &= \Pr(cX \le y) \\ &= \Pr\left(X \le \frac{y}{c}\right) \\ &= 1 - \left(\frac{\theta}{\frac{y}{c} + \theta}\right)^{\alpha} \\ &= 1 - \left(\frac{c\theta}{y + c\theta}\right)^{\alpha} \end{aligned}$$

which resembles the CDF of a Pareto distribution with parameters α and $c\theta$. In conclusion, $Y \sim \text{Pareto}\ (\alpha,\ c\theta)$.

Besides the scale parameter, another type of parameter that we commonly see on this exam is the ***shape parameter***, which affects the general shape of the distribution.

The following graph shows the PDF curves of the beta distribution assuming a fixed θ and different values for a and b, which are the shape parameters:



The table below lists the shape and scale parameters of all relevant continuous distributions in the exam tables.

Distribution	**Shape Parameter(s)**	**Scale Parameter**
Generalized Pareto	α, τ	θ
Burr	α, γ	θ
Inverse Burr	τ, γ	θ
Pareto	α	θ
Inverse Pareto	τ	θ
Loglogistic	γ	θ
Paralogistic	α	θ
Inverse Paralogistic	τ	θ
Gamma	α	θ
Inverse Gamma	α	θ
Weibull	τ	θ
Inverse Weibull	τ	θ
Exponential	-	θ
Inverse Exponential	-	θ
S-P Pareto	α	θ
Beta	a, b	θ

The key takeaway from the table is that all distributions listed in the exam tables that have a θ parameter (except inverse Gaussian) are parameterized such that θ is the scale parameter.

Coach's Remarks

The choice of symbols representing the parameters is arbitrary. On the exam, the parameters might not match the ones used in the exam tables. A Pareto PDF could be given as follows, to name a couple of examples:

1. $$f(x) = \frac{\psi \tau^{\psi}}{(x+\tau)^{\psi+1}}$$
2. $$f(x) = \frac{\theta \alpha^{\theta}}{(x+\alpha)^{\theta+1}}$$

The key is recognizing the PDF is from the Pareto distribution, which allows us to distinguish between the shape and scale parameters. In the first example, ψ is the shape parameter and τ is the scale parameter. In the second example, θ is the shape parameter and α is the scale parameter.

Thus, it is important that we pay attention to the roles of the parameters, rather than the symbols used to represent them.

For teaching purposes and consistency, throughout our learning materials, we almost always list the shape parameter(s) before the scale parameter. For example, when we say $\text{Gamma}\ (2,\ 50)$, we mean a gamma distribution with a shape parameter $\alpha = 2$ and scale parameter $\theta = 50$.

Coach's Remarks

It is worth noting that there are more than one way to parameterize a distribution. For example, it is also common for the exponential distribution to be parameterized with rate λ, rather than mean θ:

$$f(x) = \lambda e^{-\lambda x} \qquad \text{vs.} \qquad f(x) = \frac{1}{\theta} e^{-x/\theta}$$

For this exam, we strictly follow the parameterizations used in the Loss Models textbook, which is the source of the exam tables student will have access to on the exam.

One application of scaling is inflation. Consider the following example:

Example S2.2.7.1

Losses in the current year follow a paralogistic distribution with parameters $\alpha = 2$ and $\theta = 100$.

Due to inflation, losses are expected to be 10% larger next year.

Calculate the probability that a loss does not exceed 55 next year.

Solution

Let X be losses in the current year.

$$X \sim \text{Paralogistic}\ (2,\ 100)$$

Let Y be losses next year. Y is 10% larger than X because of inflation. Therefore, $Y = X + 0.1X = 1.1X$.

Knowing the paralogistic distribution is a scale distribution and θ is its scale parameter, we can simply scale θ to determine the distribution of Y:

$$1.1(100) = 110$$

$$Y \sim \text{Paralogistic}\ (2, 110)$$

Finally, calculate the desired probability.

$$\begin{aligned}\Pr(Y \leq 55) &= 1 - \left(\frac{1}{1 + \left(\frac{55}{110}\right)^2}\right)^2 \\ &= 1 - \left(\frac{1}{1 + 0.5^2}\right)^2 \\ &= 1 - 0.8^2 \\ &= \mathbf{0.36}\end{aligned}$$

Alternative Solution

Alternatively, calculate the desired probability using first principles.

In general, students should remember these scaling rules:

- To scale any continuous distribution listed on the exam tables except lognormal, inverse Gaussian and log-t, multiply the parameter θ by the scaling factor.

$$\begin{gathered} X \sim \text{Distribution}\,(\text{other parameters},\ \theta) \\ \Downarrow \\ cX \sim \text{Distribution}\,(\text{other parameters},\ c\theta) \end{gathered}$$

- To scale a normal distribution, multiply the mean and the standard deviation by the scaling factor.

$$\begin{gathered} X \sim \text{Normal}\left(\mu,\ \sigma^2\right) \\ \Downarrow \\ cX \sim \text{Normal}\left(c\mu,\ (c\sigma)^2\right) \end{gathered}$$