Allocating Emission Permits Efficiently via Uniform Linear Mechanisms

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Abstract

We study how to initially allocate emission permits in an emissions trading system where firms also compete in a production market a la Cournot. We provide efficiency guarantee of simple uniform linear allocation mechanisms in the broad class of component-wise concave mechanisms in which the popular Cap-and-Trade approach is a special case. Classic economic theory states that, under standard assumptions, the equilibrium consumer surplus and social welfare are independent of the initial allocation of emission permits, and that Cap-and-Trade and Carbon Tax systems are equivalent. However, the initial allocation mechanisms previously considered in the literature were largely restricted to lump sum ones that do not take the firms' current production quantities as input. We show that, by allowing more general allocation mechanisms that are component-wise concave in the firm's production decision, which capture many realistic allocation rules including lump sum allocations (such as grandfathering), output-based allocations (either top-down or bottom-up), etc., the system's equilibrium outcome will no longer be independent of the initial allocations. In particular, for N firms operating under Cournot competition that differs only in their abatement abilities, uniform linear permit allocation mechanisms achieve Pareto efficiency in consumer surplus and total pollution, which also achieve efficiency in total emission reduction. With this result, the regulator's infinite-dimensional policy design question can be reduced to a single-dimensional one, and the original N-firm system can be equivalently represented by a monopoly so that no firm's private information is required. Numerical experiments show that the benefit of uniform linear mechanisms compared to lump sum ones can be large. We also explain when the efficiency of uniform linear allocation mechanisms might fail, and give managerial insight into the design of allocation coefficient in practically used output-based allocation methods.

1 Introduction

Climate change has become a major challenge to humanity and the rest of the world. Among the key contributors is the excessive emissions of greenhouse gas, primarily carbon dioxide, from industrial activities. It has been estimated that emissions need to be cut half by 2030 if the global temperature goal set by the Paris Agreement, which over 190 parties around the globe signed as of 2023, is to be met [40].

Cap-and-Trade (CAT) is a popular approach to regulate polluters ([30]). In a CAT system, the total amount of allowed pollution generated by regulated entities is capped by a fixed amount of tradable permits, where each permit represents one unit of pollution (e.g., one ton of carbon dioxide). In practice, compared with pollution taxes, CAT systems are often considered as more practically feasible and flexible, hence have been widely implemented, such as the US SO2 trading system ([2]), the US NOx trade system ([9]), the European Union Emissions Trading Scheme ([8]), etc..

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One major design question for CAT systems is how to distribute permits initially. Although auction is considered to promote price discovery, encourage innovation, improve adminstrative transparency and fairness ([16],[43], [20], [5],[34]), it is, however, often in limited use due to technical challenges and strong opposition from the industry, as this will significantly increase their costs. Therefore, free distributions of permits are still prevalent in practice. For example, in the US SO2 trading system, permits of two billion dollar value were distributed to coal-fired power plants gratis. The European Union mandated that, in the first compliance cycles (2005 – 2007), its members should distribute at least 95% permits for free.

Gratis allocation methods used in practice are usually simple and straightforward. For example, the commonly used grandfathering method allocates permits to entities according to historical data, such as emissions or market shares ([21]). Once determined, the allocation quantity is a lump sum that does not accommodate situations of the current compliance cycle. [10] and [26] introduce cases using grandfathering allocation method. Another popular allocation method is output-based allocation (OBA), which allocates permits in proportion to the firm's current outputs ([39]). In its simplest form, OBA allocates permits according to a uniform multiplier across firms with different green technology ([33]). For example, it may cover various sources of power generation such as wind, hydro, solar or nuclear. Also, some jurisdictions choose technology-specific multipliers that differ from coal-fired plants to gas-fired plants.

Given the various ways to allocate permits, what can be said regarding the efficiency of the allocation? Is there any superiority guarantee of one allocation method over the others, and if so, what are the key drivers of such optimality? We aim to investigate these questions in this paper by considering a regulator's market design problem in terms of allocating emission permits.

Before we go into further details of our setup, at this point, it is worthwhile to mention one celebrated result in the previous literature: under suitable conditions, how permits are initially distributed does not affect the equilibrium social outcome provided that there is a secondary trading market ([13], [17],[32]). While this independence property seems to bring a conclusion to the above raised questions, it requires a few important restrictions that deviate from reality. Firstly, in [32] and subsequent literature, the initial allocations of permits are treated as lump sum subsidies that do not depend on the firms' production level, which usually is not the case in practice (e.g., see the above discussion of OBA). In fact, as we show in this paper, lump sum subsidies are strictly suboptimal in the general policy space that we consider. Also, previous work typically assumes that the regulated entities operate in a large market, if not a monopoly, such that the impact of regional emissions regulation on the overall production market (especially the price of the produced good) is negligible. Furthermore, earlier work mostly focuses on efficiency in terms of firms' profits, overlooking consumer surplus, even though the latter is a major component of social welfare. Contrary to these, we explicitly model the impact of pollution regulation on the original production market, and characterize efficiency in terms of consumer surplus (which is also equivalent to maximally incentivizing emission reductions). Social welfare is analyzed through numerical studies.

The regulator's policy space that we optimize over is general. Note that the above-mentioned gratis allocation methods can all be viewed as special functions of the firms' production outputs. In particular, let $\Phi_i(\vec{q})$ denote the number of permits firm i receives given the production output vector $\vec{q} = (q_1, ..., q_N)$ of the N firms under regulation. Lump sum allocations correspond to constant mechanisms where $\Phi_i(\vec{q}) = K_i$ for some constant $K_i \geq 0$. OBA with an output-adjusted cap can be represented by $\Phi_i(\vec{q}) = \alpha_i q_i$ with some firm-specific multiplier $\alpha_i \geq 0$. The policy space we consider only requires that $\Phi_i(\vec{q})$ is component-wise concave in q_i for all $i = 1, ..., N$. It hence also allows more complex forms with arbitrary cross-firm dependency structures.

Under moderate assumptions, we are able to show, without actually solving the underlying equilibrium, that in the broad class of component-wise concave allocation mechanisms described above, simple uniform linear allocation mechanisms $\Phi_i(\vec{q}) = \alpha q_i$ achieve Pareto efficiency (in terms of traditional consumer surplus versus pollution cost), and that the potential benefit compared to lump sum allocations can be large. The class of uniform linear allocation mechanisms, in which efficiency lies, resembles the OBA method with a uniform benchmark multiplier used in California, Québec, etc. ([29]). We discuss when the benchmark should and should not be uniform across heterogeneous firms. In particular, the benchmark should be uniform if the firms differ only in their abatement abilities (reflected by their heterogeneous abatement costs). On the other hand, if the firms differ in their production costs, then the regulator should compensate firms with higher marginal production costs by choosing larger allocation coefficients to them.

Model. We consider, in a single period, N firms operating in a homogeneous product market under Cournot competition, with emission as a byproduct. Each firm must comply with environmental regulation by submitting enough permits to cover its own emission. It can achieve this by reducing its production q_i (hence emission), doing emission abatement, and trading permits. The firms differ in their abatement abilities, captured by a nondecreasing convex cost function $f_i(\cdot)$ of the abatement quantity. The regulator determines some permit allocation mechanism $\Phi_i(\vec{q})$, $i = 1, ..., N$, which is common knowledge to the firms. We consider the class of component-wise concave allocation mechanisms, i.e., $\Phi_i(\vec{q})$ is concave in q_i for all $i = 1, ..., N$, and aims to maximize the resulting equilibrium augmented consumer surplus, which is the traditional consumer surplus less the cost of pollution. In an equilibrium outcome induced by $\Phi_i(\vec{q})$, each firm i produces q_i^*, emits x_i^* (hence abates $q_i^* - x_i^*$ after nomalization), receives $\Phi_i(\vec{q}^*)$ free permits and buys (sells) $|x_i^* - \Phi_i(\vec{q}^*)|$ permits at market-clearing price τ^* to maximize its own profit. The resulting equilibrium pollution level (total emission) is $K^* := \sum_{i=1}^N x_i^*$. Under Cournot competition, for any fixed pollution level K^*, consumer surplus is a nondecreasing function of the total production output $Q^* := \sum_{i=1}^N q_i^*$. Therefore, we can equivalently consider the alternative objective of maximizing equilibrium Q^* given any equilibrium pollution K^*, and we want to achieve this by choosing an appropriate permit allocation mechanism $\Phi_i(\vec{q})$ in the class of component-wise concave mechanisms.

Main results and contributions. When the firms compete a la Cournot and only differ in their abatement cost structures, we show that within the class of component-wise concave allocation mechanisms, simple uniform linear allocation mechanisms $\Phi_i(\vec{q}) = \alpha q_i$ must induce the largest aggregate equilibrium production quantity Q^* for any fixed equilibrium pollution level K^*, hence maximize augmented consumer surplus. This also implies that uniform linear allocation mechanisms achieve the maximal incentive for emission reductions (since given Q^*, K^* is minimized). Therefore, the regulator's infinite-dimensional problem of designing a $\mathbb{R}^N \to \mathbb{R}^N$ mapping can be reduced to searching for a single coefficient α. Instead of solving for the equilibrium of the original N-firm CAT system, we can define an equivalent monopoly's problem to solve for the equilibrium aggregate pollution level K^* and production output Q^* under any allocation coefficient α. Numerical experiments show that the potential benefit of uniform linear allocation mechanisms compared to lump sum ones can be large. Note that the assumption that all firms are identical in their marginal production profit structures and only differ in their abatement cost structures is crucial to our findings. When this assumption is violated, for example, if the CAT system covers multiple sectors that differ in their marginal production profits, we show that a sector-specific linear allocation mechanism outperforms uniform allocation.

The main contributions of this work can be summarized as follows.

- We employ a general framework to model multiple firms' production-pollution-abatement-trading decisions under a CAT system where the impact of the regulator's permit allocation mechanism on consumer surplus (and emission reductions) can be analyzed without solving the system equilibrium.

- We establish the Pareto efficiency of uniform linear mechanisms in the broad class of component-wise allocation mechanisms when the firms only differ in their abatement abilities. This reduces the regulator's policy space from infinite-dimensional to single-dimensional. Moreover, we can solve the $N-$firm system equilibrium using an equivalent monopoly's problem, without requiring individual firm's private information.

- Our results provide an example of the violation of the independence property, i.e., the initial distribution of permits having no impact on the social outcome. In this work, the violation of the independence property is not due to uncertainty or trading frictions. Rather, by enlarging the policy space from lump sum allocations to general component-wise concave functions of the production vector, the firms endogenize their actions' impact on the number of permits they can receive.

- Due to its simple form, linear permit allocation schemes (e.g., OBA) are widely adopted in practice. However, little is known regarding how well it performs compared to other allocation schemes. It is also unclear whether one should customize the allocation coefficients at the individual level. In this regard, this work is the first to discuss the efficiency of uniform linear permit allocation mechanisms in a general class of allocation mechanisms.

- We explain when the efficiency of uniform linear mechanisms might fail. In particular, if the firms are heterogeneous in their production costs, then nonuniform linear mechanisms outperform uniform linear ones. This gives managerial insight into the design of the practical OBA method. For example, if the firms' production technologies are identical but abatement technologies are not, then the allocation coefficient should be uniform across the firms. On the other hand, if the firms' production technologies are different, then the allocation coefficient should be larger for firms with a higher production cost.

- This work opens up more potential research directions. For example, our efficiency guarantee of uniform linear allocation mechanisms requires that the firms only differ in their abatement abilities. This applies to situations where the CAT system covers a single sector. When multiple sectors are under the same CAT system, we may imagine that linear allocation mechanisms with sector-specific multipliers should prevail. We may also extend the current framework to include multiple compliance cycles to take into account temporal effects.

- Our results shed new light on the previously established equivalency between CAT and Taxes. While we broaden the policy space for CAT and show that efficiency can be improved, the implication for Taxes is also generalized.

Related work. The initial allocation of emission permits has long been a central issue in operating tradable emission permit markets. [32] put forward that the initial allocation of permits won't affect market efficiency, and that emission tax and tradable permits are equivalent in a deterministic setting. Later, [41] show that this isomorphism breaks down when costs and benefits are uncertain. Also, [24] show that the initial distribution of permits can lead to inefficiencies if some firms can exert market power on the permit market. [37] establishes that in the presence of transaction costs, the system's equilibrium outcome is sensitive to the initial permit distributions. Different from the previous work, in this paper, we show that different modes of allocating permits initially impact consumer surplus when firms endogenize their actions' impact on the number of permits they can receive.

There is a growing body of research studying multi-unit auctions with production costs (i.e., permit auction with the social cost of pollution). [6] and [27] took a mechanism design approach to develop allocation and payment rules in pursuit of social welfare maximization outcomes. More recently, [22] studies uniform-price auctions with a price floor and a price ceiling in a system with imperfect competition and private information, and gives Price of Anarchy guarantee of such auctions. [35] proposes a simple auction mechanism for selling carbon permits — True-Cost Pay as Bid, which does not need to know in detail the bidders' abatement costs. We do not consider auction in this work and focus on the design of gratis allocation mechanisms. We also look at consumer surplus instead of social welfare. And we do not take a mechanism design approach.

[19] compares lump sum, OBA, and auctioning methods of allocating permits, and find that OBA based on historical emissions reduces carbon leakage, but at higher welfare costs. In contrast, the allocation mechanisms we consider is a broader class that includes both lump sum and OBA. We show that simple OBA with a uniform multiplier maximizes consumer surplus (under suitable conditions) and can increase social welfare. [7] study the dynamic grandfathering allocation method in both closed and open emission trading systems. They show that grandfathering based on past emission level is efficient in closed systems. However, this creates a perverse incentive by granting fewer permits if a firm reduces its emissions. In our work, the allocation method is based on the current production output, which resolves the perverse incentive. In fact, as we show, firms with better abatement ability would benefit more.

In the operations management literature, there is a growing body of research in sustainable operations. See [28] and [4] for an overview. For example, the influential paper of [11] study in green product development, how to choose costly green technology and environmental friendly materials concerning ordinary and green customers, and show that green product development might not benefit the environment. [1] and [12] show that firms can effectively reduce carbon emissions through operational adjustments to the supply chain. [23] study a manufacturer's optimal production planning under stochastic demand when it can buy or sell emission allowances in an outside market to comply to the regulation, and show the optimality of a target interval emission trading policy combined with a base-stock production policy. [18] study the impact of emissions tax and CAT regulation on a firm's technology choice and capacity decisions, and show that price uncertainty under CAT yields higher profit compared with a constant emission tax. Compared with these papers that take environmental regulation as an input and focus on the firm's operational questions, we ask the regulator's design question of how to optimally allocate permits in a CAT system. The increase in operational cost associated with emission abatement is reflected in the firm's integrated production-pollution-abatement model.

A closely related work to ours is [3] that employs the integrated production–pollution–abatement model as in the current paper. They compare the effect of different pollution regulations, including both CAT and tax systems, on social outcomes. Their main result is that under competition, well-chosen regulation can simultaneously improve consumer surplus and firms' profits compared with laissez-faire (i.e., no regulation). The CAT system they consider allocates permits only lump sum, and the equilibrium outcome is independent of how the permits are initially distributed, which instantiates the Coase theorem ([14]). In contrast, we focus on the choice of efficient permit allocation schemes in the CAT system, and provide efficiency guarantee of simple OBA with a uniform multiplier. The independence property, i.e., initial allocation does not affect social welfare and consumer surplus, no longer holds.

Organization of the paper. Section 2 introduces the base model where the CAT system covers a single sector. We define the firms' production-pollution-abatement problem and the regulator's problem of choosing permit allocation mechanisms. Section 3 establishes the efficiency of uniform linear allocation mechanisms in the base model. Section 5 discusses an extension of the base model where the CAT system covers multiple sectors. We conclude in Section 6.

2 Model

In the base model, we consider a single period Cap-and-Trade (CAT) system of N firms competing in a homogeneous product market. To comply with the regulation, each firm must submit enough permits to cover its emission (one permit accounts for one ton of carbon dioxide and others measured in equivalent terms). We extend the competing firms' production-pollution-abatement model in [3] to incorporate the regulator's problem of choosing permit allocation mechanisms. The sequence of events is as follows.

1. The regulator decides and announces how the emission permits will be allocated to the firms as functions of their production vector, represented by $\Phi_i(\vec{q}) : \mathbb{R}_+^N \to \mathbb{R}_+$, $i = 1, ..., N$. This becomes common knowledge to the firms in the second stage.

2. Fully acknowledging the regulation announced in the previous stage, firms simultaneously decide their production output $\vec{q}$, emission level $\vec{x}$, and trade permits if necessary to comply with the regulation and maximize their profits.

The regulator's choice of permit allocation mechanism in the first stage can be general. We impose mild restrictions on $\Phi_i(\vec{q})$ in Section 2.2 and discuss the various practical allocation schemes that are covered by our definition. In the second stage, firms make production and pollution (hence abatement and trading) decisions to maximize their profits while complying with the regulation. The price of the permits traded will be the equilibrium market-clearing price that equates supply with demand. Note that each firm receives a supply of permits via the initial allocation mechanism $\Phi_i(\vec{q})$ determined by the regulator in the first stage.

Remark (permit allocation mechanisms). We assume that the initial distribution of permits $\Phi_i(\cdot)$ only possibly depends on the firms' production outputs $\vec{q}$. This captures most practically relevant allocation schemes including lump sum allocations (such as grandfathering, etc.) and output-based allocations. We also assume that all firms indeed comply with the regulation in the second stage. We do not consider the issue of private information, i.e., the possibility of firms falsely reporting their production outputs or pollution levels, and hence do not take the mechanism design approach.

In the remainder of this section, we first introduce the firms' production-pollution-abatement model in the second stage under a specific permit allocation mechanism $\Phi_i(\vec{q})$, and then define the regulator's problem of choosing $\Phi_i(\vec{q})$ in the first stage.

2.1 The Firms' Production-pollution-abatement Model

In the second stage, given full knowledge of the allocation mechanism $\vec{\Phi}_i(\cdot)$, firm i's problem is to maximize its profit during this compliance cycle by choosing its production output q_i and emission level x_i. Specifically, we consider a stylized model where firm i's profit is composed of four parts: the sales revenue from the homogeneous product market, the production cost of producing q_i units, the abatement cost of reducing emission to x_i units, and the permit trading cost (revenue) in the emissions trading market. We now formally define these four terms one by one.

Sales revenue under Cournot competition. We assume that the N firms covered by the regulator's CAT system operate in a homogeneous product market under Cournot competition (for example, Cournot competition is commonly used to model electricity markets [42], which are major sources of Carbon emissions). Let q_i be firm i's production output. Then the market price of the perfect substitute homogeneous product is determined by $p\left(q_i + \sum_{i\neq j} q_j\right)$. We assume $p(\cdot)$ to be a decreasing and concave function of the firms' aggregate production quantity. This includes, for example, the linear inverse demand curve $p(q_i + \sum_{i\neq j} q_j) = b - a(q_i + \sum_{i\neq j} q_j)$ [36], where $a, b > 0$ are known parameters. Given other firms' production quantities $\vec{q}_{-i}$, the sales revenue of firm i for producing q_i units of product is hence

$$R_i(q_i, \vec{q}_{-i}) = p\left(q_i + \sum_{i\neq j} q_j\right) q_i.$$

Common marginal production cost. In the base model, we consider the firms to only differ in their abatement abilities and to bear a common marginal production cost h. That is, producing q_i units of products will incur a total production cost of hq_i, and this is true for any firm $i = 1, ..., N$. Therefore, by redefining $p(\cdot) := p(\cdot) - h$, we can normalize $h = 0$ without loss of generality.

In Section 5, we consider the case where the firms' marginal production costs are different from each other. For example, when the CAT system covers multiple sectors, each under Cournot competition with a sector-specific production cost, we show that linear allocation schemes with sector-specific multipliers outperform uniform linear allocation schemes.

Heterogeneous convex abatement cost. Without any abatement efforts, all firms will produce η units of emission as a byproduct of unit production. We normalize η to one without loss of generality. Therefore, producing q_i units of products and generating x_i units of emission corresponds to an abatement of $q_i - x_i$ units. Note that x_i takes value between 0 and q_i. We assume the firms' abatement abilities are heterogeneous. In particular, firm i incurs a cost of $f_i(q_i - x_i)$ for reducing $q_i - x_i$ units of emission. We assume that $f_i(\cdot)$ is strictly convex and non-decreasing for all $i = 1, ..., N$. This form of abatement cost is commonly used in the literature, for example, see [31], [38], etc., and is supported by the estimates in [25] studying air pollutants from 100,000 U.S. manufacturing firms across 37 industries. Convex costs in the abatement quantity indicate that reducing pollution may be easy to start with, but becomes increasingly difficult as the quantity grows, since it may require untrivial modification of the production process and technological innovations.

We do not restrict $f_i(\cdot)$ to take similar forms, e.g., quadratic with different coefficients as in [3]. Later we show that we can appropriately aggregate the N firms' abatement costs into a single function, using which we can define a monopoly's problem that is equivalent to the original CAT system.

Permit trading cost (revenue). To comply with the regulation, each firm needs to submit enough permits to cover its emission x_i. In particular, if firm i receives $\Phi_i(\vec{q})$ number of permits and emits x_i units of pollution, it needs to buy (sell if negative) $x_i - \Phi_i(\vec{q})$ additional permits for this compliance cycle. This trading behavior incurs for firm i an additional cost (revenue if this value is negative) of $\tau(x_i - \Phi_i(\vec{q}))$, where τ is the equilibrium market-clearing unit price for permits in the emissions trading market. Note that while the market-clearing price τ is endogenously determined by the equilibrium, we assume that each firm considers τ as a fixed price in its own profit-maximization problem. That is, each firm considers itself to have no market power in the emissions trading system[1]. This assumption is realistic especially if the CAT system covers multiple sectors (see Section 5).

Remark (market power). We assume that firms have market power in the goods market but have no market power in the permits market. To be specific, the price for goods enters into firms' objective as a function of firms' production decisions, while the market price for permits is considered as a fixed quantity (whose value is determined in equilibrium via market clearing conditions). This assumption is reasonable in the following sense. In reality, it's often the case that multiple sectors are covered in the same cap-and-trade system, where firms from one sector do not compete in the goods market with firms from another sector. Therefore when the cap-and-trade system covers more sectors, each firm should have diminishing market power in the permits market, while still maintaining their market power in their own goods market. We consider the extension of CAT system covering multiple sectors in Section 5.

In summary of the above, firm i's total profit under production outputs $\vec{q}$ and emission levels $\vec{x}$

[1] In fact, our results still hold if firms treat τ as a function of their decisions.

is given by

$$p\left(q_i + \sum_{j\neq i} q_j\right) q_i - f_i(q_i - x_i) - \tau\left(x_i - \Phi_i(\vec{q})\right).$$

Therefore, in the second stage (after the allocation mechanism $\Phi_i(\vec{q})$ is announced), firm i solves the following problem:

$$\max_{q_i\geq 0, x_i\in[0,q_i]} p\left(q_i + \sum_{j\neq i} q_j\right) q_i - f_i(q_i - x_i) - \tau\left(x_i - \Phi_i(\vec{q})\right). \tag{1}$$

To facilitate the analysis, we can equivalently rewrite program (1) as a two-stage optimization problem:

$$\max_{q_i\geq 0} \quad p\left(q_i + \sum_{j\neq i} q_j\right) q_i - g(q_i), \tag{2}$$

where

$$g(q_i) = \min_{x_i\in[0,q_i]} f_i(q_i - x_i) + \tau\left(x_i - \Phi_i(\vec{q})\right) \tag{3}$$

is the minimum total cost of compliance under regulation that can be achieved by optimally conducting pollution abatement and permit trading given production quantity q_i.

The allocation mechanism $\vec{\Phi}(\cdot)$, considered as common knowledge at this stage, is assumed to be component-wise concave, i.e., $\Phi_i(\vec{q})$ is concave in q_i, $i = 1, ..., N$. We defer the formal definition of the regulator's set of admissible allocation mechanisms in Section 2.2. We also summarize and complete the requirements on $p(\cdot)$ and $f_i(\cdot)$ in the following assumptions.

Assumption 1. *The demand function $p(\cdot) \in \mathcal{C}^1$ is concave and strictly decreasing. Moreover, $p(Q) \geq 0$ if $Q \geq 0$.*

The monotonicity in Assumption 1 describes the inverse correlation between price and demand according to the law of diminishing marginal utility. The concavity is assumed in order to guarantee that firms' profit-maximization problems are well defined and have tractable solution. The demand curve used in many literature can be found to satisfy the assumption, as shown in the examples below.

Example 1 (Linear demand curve [36])**.** *Consider $p(Q) = b - aQ$, where $a, b > 0$ are known parameters. The linear form can be interpreted as demands arising from the utility-maximizing behavior of consumers with quadratic, additively separable utility functions. Here, a measures the inverse price elasticity of demand, and b is proportional to the market size.*

Example 2 (Monomial demand curve [15])**.** *Consider $p(Q) = b - aQ^n$, where $a, b > 0, n \in \mathbb{N}^+$ are known parameters.*

Assumption 2. *The abatement cost functions $f_i(\cdot) \in \mathcal{C}^1$, $i = 1, ..., N$, are strictly convex and non-decreasing. Moreover, $f_i(x) \geq 0, \forall x \geq 0$, and $f_i(0) = 0, f_i'(0) = 0$.*

Assumption 2 is reasonable as in practice it's getting more expensive to reduce emissions with the development of pollution-reduction technologies, resulting in the increasing marginal abatement cost. Furthermore, the abatement cost and the marginal abatement cost should be zero when there is no emission reduction to make.

We conclude Section 2.1 with the following lemma.

Lemma 1. *Fix a component-wise concave allocation mechanism $\vec{\Phi}(\cdot) : \mathbb{R}_+^N \to \mathbb{R}_+^N$. Under Assumption 1–2, the optimization problem in* (2) *is a concave program.*

The proof of Lemma 1 is straightforward and is in Appendix A.

2.2 The Regulator's Objective and Admissible Permit Allocation Mechanisms

Before we formally introduce the regulator's problem, we first define the market equilibria in the second stage given an allocation mechanism $\vec{\Phi}(\vec{q})$.

Definition 1 (Market equilibrium)**.** *Fix an allocation scheme $\vec{\Phi}(\cdot) : \mathbb{R}_+^N \to \mathbb{R}_+^N$. A triplet $(\vec{q}^*, \vec{x}^*, \tau^*) \in \mathbb{R}_+^N \times \mathbb{R}_+^N \times \mathbb{R}_+$ is called a market equilibrium under this allocation scheme if for all $i = 1, ..., N$,*

$$(q_i^*, x_i^*) \in \underset{q_i \geq 0, x_i \in [0, q_i]}{\arg\max} \left\{ p\left(q_i + \sum_{j \neq i} q_j^* \right) q_i - f_i(q_i - x_i) - \tau^* \left(x_i - \Phi_i(q_i, q_{-i}^*) \right) \right\}, \tag{4}$$

and

$$\sum_{i=1}^N x_i^* = \sum_{i=1}^N \Phi_i(\vec{q}^*). \tag{5}$$

Furthermore, if $0 < x_i^* < q_i^*$ for all $i = 1, ..., N$, then it is called an interior market equilibrium.

Note that (4) states that in a market equilibrium, each firm i chooses (q_i^*, x_i^*) to maximize its own profit, provided the other firms' decisions and the permit's price are fixed (at the equilibrium value). Eq. (5) is the market-clearing condition for the permit's price τ^*.

Remark (interior market equilibrium). To capture realistic situations where gratis allocation of permits is used for moderate regulation, we want to exclude corner cases and focus on interior market equilibria that are interior solutions in (4). This implies that the regulation is neither too stringent such that there is no emission ($x_i^* = 0$), nor too lax such that firms do not conduct abatement ($x_i^* = q_i^*$). Also, we want to ensure that no firms will go bankrupt due to the regulation, i.e., firm i's profit in equilibrium ≥ 0 for all $i = 1, ..., N$. For the regulation to be meaningful, we require that the total number of permits allocated does not exceed the total amount of pollution if without regulation. Later we show in Lemma 2 and Theorem 2 that these conditions are satisfied if we suitably restrict the regulator's choice set of the allocation scheme $\vec{\Phi}(\cdot)$ (see Definition 2).

Now we describe the regulator's problem. The regulator chooses the permit allocation scheme $\vec{\Phi}(\cdot)$ to maximize the adjusted consumer surplus in equilibrium, which is equivalent to maximizing the emission reductions as will be elaborated later. Specifically, the adjusted consumer surplus is composed of two parts, the traditional measure of consumer surplus minus the social cost of pollution. To ensure that the regulator's problem is well-defined, we impose a few mild conditions on $\vec{\Phi}(\cdot)$. All conditions are formally stated in the following definition of admissible allocation mechanisms.

Definition 2 (Admissible allocation mechanism)**.** *An operator $\vec{\Phi}(\cdot) = (\Phi_i(\cdot))_{i=1}^N$ on $\mathbb{R}^N$, where $\Phi_i(\cdot)$ is a function of the N firms' output vector $\vec{q}$, is called an admissible allocation mechanism if the following conditions are satisfied:*

1. *$\vec{\Phi}(\cdot)$ is component-wise concave, i.e. $\Phi_i(\vec{q})$ is concave in q_i for all $i = 1, ..., N$.*
2. *If $\vec{q} \geq 0$, then $\Phi_i(\vec{q}) \geq 0$ for all $i = 1, ..., N$.*

Under $\vec{\Phi}(\cdot)$, there exists a unique interior market equilibrium $(\vec{q}^*, \vec{x}^*, \tau^*)$ as defined in Definition 1 such that

3. $0 \leq \frac{\partial \Phi_i(\vec{q}^*)}{\partial q_i} \leq 1$ *for all* $i = 1, ..., N$.

Denote $\mathcal{A}$ the set of admissible allocation mechanisms, and $\mathcal{A}(K) \subseteq \mathcal{A}$ the set of mechanisms under which the interior equilibrium pollution vector $\vec{x}^*$ satisfies $\sum_{i=1}^{N} x_i^* = K$.

The first condition in Definition 2 requires that each firm receives a decreasing number of permits per production output as the total production increases. The second condition ensures that a firm does not receive a negative number of permits for its production. The third condition states that in equilibrium, a firm gets no less permits for more production, and that the incremental permits it gets cannot cover the incremental emission without abatement. By requiring unique interior equilibrium, an admissible allocation mechanism is moderate, i.e., it is more stringent than no regulation yet less binding than no pollution allowed.

Note that it is, in general, difficult to show any component-wise concave allocation mechanism will induce a unique interior equilibrium. To deal with this technical challenge, we start with an even broader class $\mathcal{B} \supseteq \mathcal{A}$ of allocation mechanisms that may induce multiple interior equilibria. Later we show in Section 3.1 that any interior equilibrium induced by some permit allocation mechanism in this new set is Pareto dominated by the unique interior equilibrium under an admissible allocation mechanism in $\mathcal{A}$.

Definition 3. *Denote $\mathcal{B}$ the set of allocation mechanisms satisfying all conditions in Definition 2 except for the uniqueness of interior equilibria.*

There exists an interior market equilibrium under any $\vec{\Phi}(\cdot) \in \mathcal{B}$. Furthermore, the interior market equilibrium is unique under any $\vec{\Phi}(\cdot) \in \mathcal{A}$. Thus $\mathcal{A} \subseteq \mathcal{B}$. An efficient (we define efficiency later) allocation mechanism in $\mathcal{B}$ must also be efficient in $\mathcal{A}$ if it is indeed in $\mathcal{A}$.

As discussed next, the conditions in Definition 3 capture most realistic permit allocation mechanisms in practice.

Example 3 (Lump sum allocation). *Consider $\Phi_i(\vec{q}) = K_i$ for some $\vec{K} \geq 0$. Then the first three conditions in Definition 3 are trivially satisfied. The existence of interior equilibrium can also be satisfied under appropriately chosen $\vec{K}$. This allocation scheme can describe the commonly used grandfathering method and others that do not base the allocation on the firms' current actions.*

Example 4 (Top-down Output-based allocation). *Consider $\Phi_i(\vec{q}) = \frac{\alpha_i q_i}{\sum_{i=1}^{N} \alpha_i q_i} K$ for some $K, \vec{\alpha} \geq 0$. Then one can verify that the first two conditions in Definition 3 are satisfied. The existence of interior equilibrium can be satisfied for appropriately chosen $\vec{\alpha}$ and K. This allocation scheme corresponds to a situation where the total cap K on pollution is fixed and distributed proportionally (with weights $\vec{\alpha}$) to the firms' market shares.*

Example 5 (Bottom-up Output-based allocation). *Consider $\Phi_i(\vec{q}) = \alpha_i q_i$ for some $0 \leq \vec{\alpha} \leq 1$. The first three conditions in Definition 3 are trivially satisfied. The existence of interior equilibrium can also be satisfied if $\vec{\alpha}$ is appropriately chosen. This allocation scheme corresponds to the simple linear allocation rule where the total cap is not fixed a priori, but depends on the equilibrium outcome of the system. We show, in Section 3, that the uniform case ($\alpha_i \equiv \alpha$ for some $\alpha \geq 0$) achieves the highest (adjusted) consumer surplus within the admissible set $\mathcal{A}$.*

Both Example 3 and Example 5 are special cases of linear allocation mechanisms: $\Phi_i(\vec{q}) = \alpha_i q_i + K_i$. In Section 3.1 we show that it is, without loss of generality, to only consider linear allocation mechanisms.

We are now ready to formally state the regulator's problem. Denote $Q^*(\vec{\Phi})$ the (interior) equilibrium aggregate production output under an admissible allocation $\vec{\Phi}(\cdot) \in \mathcal{A}$. Similarly, denote $K^*(\vec{\Phi})$ the corresponding (interior) equilibrium total emission. The regulator solves

$$\max_{\vec{\Phi}(\cdot)\in\mathcal{A}} \mathrm{ACS}(\vec{\Phi}) = \mathrm{CS}(Q^*(\vec{\Phi})) - S(K^*(\vec{\Phi})), \tag{6}$$

where $\mathrm{CS}(Q^*(\vec{\Phi}))$ is the traditional consumer surplus and $S(K^*(\vec{\Phi}))$ the social cost of pollution.

Remark (Infinite-dimension program). The regulator's problem in (6) is an infinite-dimensional optimization problem, which is hard to solve in general. But as we show in Section 3, the regulator's policy search space can be reduced to finite-dimensional and even one-dimensional under mild conditions.

We highlight that, as will become clear in Section 3, the exact forms of $\mathrm{CS}(\cdot)$ and $S(\cdot)$ do not affect our main results, given the following monotonicity assumption holds.

Assumption 3. *The traditional consumer surplus* $\mathrm{CS}(Q)$ *is strictly increasing in the aggregate production quantity* Q*. Also, the social cost of pollution* $\mathrm{S}(K)$ *is strictly increasing in the total emissions* K*.*

As specified by the Cournot competition of the homogeneous product market, the linear inverse demand curve $p(Q)$ corresponds to $\mathrm{CS}(Q) = \int_0^Q p(x)dx - Qp(Q)$, which is nondecreasing in Q under Assumption 1. Also note that we need not impose any assumptions on $\mathrm{S}(\cdot)$ apart from it being a strictly increasing function. In fact, in Section 3.1 we establish that the entire Pareto frontier (in terms of $\mathrm{CS}(\cdot)$ and $\mathrm{S}(\cdot)$) can be achieved by uniform linear allocation mechanisms. In other words, within each nonempty set $\mathcal{A}(K)$, any $\vec{\Phi}(\cdot) \in \mathcal{A}(K)$ incurs the same pollution cost $S(K)$, and there must exist a $\vec{\Phi}^*(\vec{q}) = \alpha\vec{q}$ for some $\alpha > 0$ such that $\vec{\Phi}^* \in \arg\max_{\vec{\Phi}\in\mathcal{A}(K)} \mathrm{ACS}(\vec{\Phi})$.

Remark (Maximize emission reductions). We argue that under Assumption 3, the regulator's problem in (6) is equivalent to an alternative problem of maximizing the equilibrium emission reductions

$$\max_{\vec{\Phi}(\cdot)\in\mathcal{A}} Q^*(\vec{\Phi}) - K^*(\vec{\Phi}), \tag{7}$$

in the sense that they achieve the same Pareto frontier of the two competing terms. The equivalence can be explained as follows. Within any nonempty set $\mathcal{A}(K)$, since the pollution costs are the same for all $\vec{\Phi}(\cdot) \in \mathcal{A}(K)$, (6) reduces to maximizing the aggregate production output $Q^*(\vec{\Phi})$ under Assumption 3, thus also maximizing the emission reductions (7) in $\mathcal{A}(K)$. Conversely, denote $\mathcal{A}(Q) \subseteq \mathcal{A}$ the set of mechanisms under which the interior equilibrium production quantity satisfies $\sum_{i=1}^N q_i^* = Q$. Then within any nonempty set $\mathcal{A}(Q)$, (7) reduces to minimizing the total emissions $K^*(\vec{\Phi})$, thus also maximizing the adjusted consumer surplus (6) under Assumption 3. In fact, the equilibrium permit price will also be maximized under the optimal allocation mechanism in problem (7).

In the remaining part, unless otherwise specified, the equilibrium under an allocation mechanism refers to the interior market equilibrium.

3 Efficiency of Uniform Linear Allocation Mechanisms

In this section, we discuss admissible permit allocation mechanisms in Definition 2, and establish the efficiency of simple uniform linear allocation mechanisms $\vec{\Phi}(\vec{q}) = \alpha\vec{q}$.

As a benchmark, we first characterize the market equilibrium for no regulation, i.e., $\vec{\Phi}(\vec{q}) = \vec{q}$.

Theorem 1. *Without regulation, under Assumption 1–2, firm i's equilibrium production output* $q^*_{i,NR}$ *and pollution level* $x^*_{i,NR}$ *satisfy*

$$q^*_{i,NR} = x^*_{i,NR} = \frac{Q^*_{NR}}{N}$$

for all $i = 1, ..., N$*. Moreover, the equilibrium aggregate production output* Q^*_{NR} *is the unique solution of*

$$p'(Q^*_{NR})Q^*_{NR} + Np(Q^*_{NR}) = 0,$$

and the total pollution $K^*_{NR} = Q^*_{NR}$*.*

The proof of Theorem 1 is in Appendix B.

Next we show that the admissible set $\mathcal{A}$ is nonempty.

Lemma 2. *Suppose Assumption 1–2 hold. There exist* $0 < K_{LB} < K_{UB}$ *such that* $\mathcal{A}(K^*) \neq \emptyset$ *if* $K_{LB} < K^* < K_{UB}$*. Moreover,* $K_{UB} = Q^*_{NR}$*, where* Q^*_{NR} *is the equilibrium aggregate production output without regulation described in Theorem 1.*

Specifically, the condition on K^* in Lemma 2 is required for the existence of an admissible uniform linear policy in $\mathcal{A}(K^*)$. The upper bound ensures that regulation is more stringent than no regulation, while the lower bound means that regulation is less binding than no pollution allowed.

3.1 Pareto Frontier

Recall that in Section 2.2 we stated the regulator's problem of maximizing the adjusted consumer surplus $\mathrm{CS}(Q^*(\vec{\Phi})) - S(K^*(\vec{\Phi}))$ in equilibrium by choosing $\vec{\Phi}(\cdot) \in \mathcal{A}$ (see Eq. (6)). In this section, we establish a stronger result: the entire Pareto frontier of $\mathrm{CS}(Q^*(\vec{\Phi}))$ versus $S(K^*(\vec{\Phi}))$ can be achieved by uniform linear allocation mechanisms $\vec{\Phi}(\vec{q}) = \alpha\vec{q}$ for appropriately chosen $\alpha \in (0, 1)$, which also achieve the Pareto frontier of emission reductions versus total production. Therefore, the regulator can restrict her search space $\mathcal{A}$ of operators on $\mathbb{R}^N$ to only a one-dimensional scalar $\alpha \in (0, 1)$.

Before going into detail about our main results, we first give a partial characterization of any interior market equilibrium under an allocation mechanism $\vec{\Phi}(\cdot) \in \mathcal{B}$ in the next theorem.

Theorem 2. *Under Assumption 1–2 and consider any allocation mechanism* $\vec{\Phi}(\cdot) \in \mathcal{B}$*. Denote* $(\vec{q}^*, \vec{x}^*, \tau^*)$ *any corresponding interior market equilibrium as in Definition 1. Let* $Q^* := \sum_{i=1}^N q^*_i$ *be the equilibrium aggregate production quantity, then we have*

1. *all firms achieve positive profits,*
2. $x^*_i = q^*_i - (f'_i)^{-1}(\tau^*)$ *for all* $i = 1, 2, ..., N$*, and*
3. $\tau^* = h^{-1}\left(Q^* - \sum_{i=1}^N \Phi_i(\vec{q}^*)\right)$*, where* $h(\cdot) := \sum_{i=1}^N (f'_i)^{-1}(\cdot)$ *is a strictly increasing function.*

The proof of Theorem 2 is in Appendix B.

Theorem 2 states that at any interior equilibrium, all firms' marginal abatement costs $f'_i(q^*_i - x^*_i)$ are equal, and are identical to the permit's price τ^*. This is intuitive since if not true, then firm i can always save some compliance costs by buying more (selling less) permits and doing less abatement (if $\tau^* < f'_i(q^*_i - x^*_i)$), or buying less (selling more) permits and doing more abatement (if $\tau^* > f'_i(q^*_i - x^*_i)$). Moreover, we can also express τ^* as some function $h^{-1}(\cdot)$ of the overproduction quantity $Q^* - \sum_{i=1}^N \Phi_i(\vec{q}^*)$, or equivalently, the aggregate abatement quantity. The economic interpretation of function $h^{-1}(Q^* - \sum_{i=1}^N \Phi_i(\vec{q}^*))$, in brief terms, is the marginal abatement cost for each firm, set equal, such that the N firms, in their aggregate effort, abate the overproduction quantity $Q^* - \sum_{i=1}^N \Phi_i(\vec{q}^*)$ to comply with the regulation.

We now establish that it is without loss of generality to consider only linear allocation mechanisms.

Theorem 3 (Search Space Collapse)**.** *Suppose Assumption 1–2 hold. Consider any allocation mechanism $\vec{\Phi}(\cdot) \in \mathcal{B}$ and any corresponding interior equilibrium $(\vec{q}^*, \vec{x}^*, \tau^*)$. Then there must exist a linear allocation mechanism $\tilde{\vec{\Phi}}(\cdot) = (\alpha_i q_i + K_i)_{i=1}^N \in \mathcal{B}$ for some $\alpha_1, ..., \alpha_N \in [0,1]$ and $K_1, ..., K_N \geq 0$, such that $(\vec{q}^*, \vec{x}^*, \tau^*)$ is also an interior equilibrium under $\tilde{\vec{\Phi}}(\cdot)$.*

Denote $\mathcal{B}_0 = \left\{(\alpha_i q_i + K_i)_{i=1}^N : \alpha_i \in [0,1], K_i \geq 0, i = 1, ..., N\right\} \subseteq \mathcal{B}$ the class of linear allocation mechanisms. Theorem 3 states that for any general nonlinear allocation mechanism $\vec{\Phi}(\cdot) \in \mathcal{B}$ and any interior equilibrium induced by it, we can always find an "equivalent" linear allocation policy in the class $\mathcal{B}_0$, which achieves an identical equilibrium outcome as $\vec{\Phi}(\cdot)$, thus also the same total output, total emissions, consumer surplus, adjusted consumer surplus, social welfare, and firms' profits. Furthermore, the result implies that the regulator's infinite-dimensional search space for permit allocation mechanisms can be reduced to a finite dimensional space of linear allocation mechanisms. The next theorem shows that within the subset of linear allocation mechanisms, if we consider all the corresponding interior equilibria, Pareto optimal ones can be achieved by uniform linear mechanisms that are admissible.

Theorem 4 (Efficiency of Uniform Linear Allocation Mechanisms)**.** *Suppose Assumption 1–3 hold. Consider a general nonuniform linear allocation mechanism $\tilde{\vec{\Phi}}(\cdot) \in \mathcal{B}_0$ and any (interior) market equilibrium $\left(\tilde{\vec{q}}^*, \tilde{\vec{x}}^*, \tilde{\tau}^*\right)$ under $\tilde{\vec{\Phi}}(\cdot)$. Denote $\tilde{Q}^*, \tilde{K}^*$ the corresponding equilibrium aggregate production output and total emissions. Then there must exist a uniform linear allocation mechanism $\vec{\Phi}^*(\vec{q}) = \alpha^* \vec{q} \in \mathcal{A}(\tilde{K}^*)$ for some $\alpha^* \in (0,1)$, under which the unique (interior) market equilibrium $(\vec{q}^*, \vec{x}^*, \tau^*)$ satisfy $K^* = \tilde{K}^*$ and $CS(Q^*) > CS(\tilde{Q}^*)$. Furthermore, α^* is the unique solution of*

$$p'\left(\frac{\tilde{K}^*}{\alpha^*}\right)\frac{\tilde{K}^*}{\alpha^*} + Np\left(\frac{\tilde{K}^*}{\alpha^*}\right) - h^{-1}\left(\tilde{K}^*\left(\frac{1}{\alpha^*} - 1\right)\right)(N - N\alpha^*) = 0$$

where $h(\cdot) := \sum_{i=1}^N (f_i')^{-1}(\cdot)$ is a strictly increasing function.

In Theorem 4, we show that uniform linear policies outperform other linear policies, which together with Theorem 3 suggests that the Pareto frontier of the adjusted consumer surplus, or equivalently, the total emission reductions versus pollution, is achieved by uniform linear policies. This implies that the regulator can further reduce the search space to a single-dimensional one, and admissibility is guaranteed.

The complete proof of Theorem 4 is in Appendix B. Here we briefly demonstrate the key ideas behind the proof.

Key ideas behind the proof of Theorem 3–4. Instead of directly solving for the value of an equilibrium Q^* and compare across different $\vec{\Phi}(\cdot) \in \mathcal{B}$, we apply a more elegant indirect approach of comparing the first order conditions characterizing Q^* for different choices of $\vec{\Phi}(\cdot) \in \mathcal{B}$. We explain our approach in the next key lemma.

Lemma 3. *Suppose Assumption 1–2 holds. Consider any allocation mechanism $\vec{\Phi}(\cdot) \in \mathcal{B}$ where $\mathcal{B}$ is as defined in Definition 2. Let Q^* and K^* be a corresponding interior equilibrium aggregate production and pollution levels, and $\vec{q}^*$ the equilibrium production vector. Then Q^*, K^* and $\vec{q}^*$ satisfy*

$$p'(Q^*)Q^* + Np(Q^*) - h^{-1}(Q^* - K^*)\left(N - \sum_{i=1}^N \frac{\partial \Phi_i(\vec{q}^*)}{\partial q_i}\right) = 0. \tag{8}$$

Observe that the LHS of Eq. (8) is strictly decreasing in Q^* and strictly increasing in $\sum_{i=1}^{N} \frac{\partial \Phi_i(\vec{q}^*)}{\partial q_i}$. Therefore for choices $\vec{\Phi}(\cdot), \tilde{\vec{\Phi}}(\cdot) \in \mathcal{B}$ yielding an identical total equilibrium pollution level K^*, the Q^* corresponding to the allocation mechanism that generates a larger value of $\sum_{i=1}^{N} \frac{\partial \Phi_i(\vec{q}^*)}{\partial q_i}$ must also be larger. From here, one can already see that any lump sum allocation $\vec{\Phi}(\cdot) \in \mathcal{A}(K^*)$ must be the least efficient since $\sum_{i=1}^{N} \frac{\partial \Phi_i(\vec{q}^*)}{\partial q_i} = 0$. Also, any lump sum allocation $\vec{\Phi}(\cdot) \in \mathcal{A}(K^*)$ achieves the same Q^* regardless of how K^* is divided among the firms. This instantiates the independence property from the Coase theorem [14] and the abundant work thereafter studying deterministic CAT systems where the permits are distributed lump sum.

However, for other general $\vec{\Phi}(\cdot) \in \mathcal{B}$ yielding an equilibrium total emissions K^*, it is less obvious what the value of $\sum_{i=1}^{N} \frac{\partial \Phi_i(\vec{q}^*)}{\partial q_i}$ is. Since the equilibrium pollution level being K^* is an implicit requirement, it is not even clear what mechanisms the set $\mathcal{A}(K^*)$ contains. Therefore proving the efficiency of uniform linear allocation mechanisms is a nontrivial task. We are able to establish this result by first restricting to linear allocation mechanisms, utilizing the fact that an equilibrium under some allocation mechanism only concerns the allocation function's local partial derivative and its value at the equilibrium, from where a linear allocation function can be constructed. Furthermore, in the space of linear allocation mechanisms, the superiority of uniform compared to nonuniform multipliers for linear allocation mechanisms can be shown by referring to the following two lemmas.

Lemma 4. *If, under an allocation mechanism $\vec{\Phi}(\cdot) \in \mathcal{B}$, an interior market equilibrium $(\vec{q}^*, \vec{x}^*, \tau^*)$ satisfies $\frac{\partial \Phi_1(\vec{q}^*)}{\partial q_1} \geq \frac{\partial \Phi_2(\vec{q}^*)}{\partial q_2} \geq .... \geq \frac{\partial \Phi_N(\vec{q}^*)}{\partial q_N}$, then it must also satisfy $q_1^* \geq q_2^* \geq ... \geq q_N^*$.*

Lemma 5. *Assume $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_N \geq 0,\ q_1 \geq q_2 \geq ... \geq q_N \geq 0$, and $\Sigma_{i=1}^{N} q_i = 1$. Then, we have $\Sigma_{i=1}^{N} q_i \alpha_i \geq \frac{1}{N} \Sigma_{i=1}^{N} \alpha_i$.*

The proofs of Lemma 4–5 are in Appendix B.

Implications of Theorem 3–4. Theorem 4 has several interesting implications. Although the firms differ in their abatement abilities $f_i(\cdot)$, the most efficient (in terms of consumer surplus and emission reductions) permit allocation mechanism ignores this fact and distributes permits according to a common multiplier α in front of the firms' production outputs q_i. Moreover, any allocation rule that differs across different firms performs worse than the uniform linear allocation mechanism, i.e., the former incurs a lower traditional consumer surplus for the same pollution level, or equivalently, generates more pollution for the same production output. In Section 5 we show that this no longer holds if the firms differ in their marginal production profits. In particular, if N firms compete a la Cournot in a homogeneous product market but have different marginal production costs, then a nonuniform linear allocation mechanism outperforms the uniform one.

An important thing to note is that while firms are allocated according to a homogeneous factor α and have the same output level q under uniform linear allocation mechanisms, different firms actually yield different outcomes. Particularly, compared with firms with low abatement abilities, firms with high abatement abilities (low marginal abatement cost $f_i'(\cdot)$) abate more, earn more in the permits market, and get higher profits in total.

3.2 Equivalent monopoly's problem

Under uniform linear allocation mechanisms, we can define an equivalent monopoly's problem which enables us to solve for the equilibrium outcome of the original CAT system easily.

First observe that under an admissible allocation mechanism $\vec{\Phi}(\vec{q}) = \alpha\vec{q}$, we have $K^*(\vec{\Phi}) = \alpha Q^*(\vec{\Phi})$. Therefore Eq. (8) in Lemma 3 can be rewritten as

$$p'(Q)Q + Np(Q) - h^{-1}\left(Q(1-\alpha)\right)(1-\alpha) \cdot N = 0.$$

Define $R(y) = \frac{yp(y)}{N} + \frac{N-1}{N}\int_0^y p(t)dt$ and $H(y) = \int_0^y h^{-1}(t)dt$. Then the above equation can be rewritten as

$$R'(Q) - (1-\alpha)H'(Q(1-\alpha)) = 0,$$

which is the first order condition of the following concave program:

$$\max_{Q\geq 0} R(Q) - H(Q(1-\alpha)). \tag{9}$$

The above optimization problem can be viewed as a monopoly's problem, where $R(Q)$ and $H(Q(1-\alpha))$ correspond to the sales revenue and abatement cost, respectively, under production quantity Q and an environmental regulation that demands a $(1-\alpha)$ percentage reduction of emission. Note that there is no permit trading market for this monopoly firm. The optimal solution of problem (9) can be regarded as a function of α, i.e. $Q^* = Q^*(\alpha)$. We can then easily solve, for any given $\alpha \in (0,1)$, the corresponding maximizer Q^* of program (9), and hence the corresponding $K^* = \alpha Q^*$ as well. We present the monotonicity result of Q^* and K^* in α in Section 4.

Remark (No private information required). Note that the monopoly's problem can be solved without knowing each individual firm's abatement costs. Instead, only the aggregate information $R(\cdot), H(\cdot)$ are needed. Moreover, both $R(\cdot)$ and $H(\cdot)$ can be inferred from the market information. Recall from Theorem 2 that $\tau^* = h^{-1}(Q^* - K^*)$, hence $h(\cdot)$ can be inferred from the relationship between the permit price and the industry's aggregate abatement.

3.3 Equivalent Tax Regime

In this section, we investigate a Tax regime that is equivalent to uniform linear allocation mechanisms in CAT. Carbon Tax is an alternative approach to CAT widely discussed in practice. Under Taxes, firms pay for each unit of their emissions. Specifically, after the regulator announces a tax rate δ in the first stage, firms choose the production level $\vec{q}$ and emission level $\vec{x}$ to maximize their profits, which consist of sales revenue, production cost, abatement cost and pollution tax. If firm i decides to emit x_i units of pollution, it needs to pay a tax δx_i. Thus, firm i's problem is

$$\max_{q_i\geq 0, x_i\in[0,q_i]} p\left(q_i + \sum_{j\neq i} q_j\right) q_i - f_i(q_i - x_i) - \delta x_i.$$

As we'll show in the next theorem, an admissible uniform linear allocation $\vec{\Phi}(\vec{q}) = \alpha\vec{q}$ for some $\alpha > 0$ is equivalent to a tax regime where the tax rate δ is a specific functional form with respect to firms' decisions q_i, x_i, that is, $\delta = \delta(q_i, x_i)$.

Theorem 5. *Assume that the tax rate δ is given by $\delta = \delta(q_i, x_i) = \frac{-Aq_i + B}{x_i} + C$, where A, B, C are positive numbers and B is sufficiently large such that the tax rate is positive. Consider an admissible uniform linear allocation $\vec{\Phi}(\vec{q}) = \alpha\vec{q} \in \mathcal{A}$ for some $\alpha > 0$. Denote firms' equilibrium output and emissions under $\vec{\Phi}(\cdot)$ by q_i^{UL}, x_i^{UL}. Then we have*

1. *a Nash equilibrium exists under Taxes, such that the equilibrium is identical to the market equilibrium (see Definition 1) under the uniform linear allocation $\vec{\Phi}$. That is, $q_i^{Tax} = q_i^{UL}, x_i^{Tax} = x_i^{UL}$, where q_i^{Tax}, x_i^{Tax} denote firms' equilibrium output and emissions under Taxes. Furthermore, the parameter A, C in the tax regime $\delta(\cdot)$ satisfy $A = \alpha\tau^*, C = \tau^*$, where τ^* is the equilibrium market clearing price under $\vec{\Phi}(\cdot)$ in the CAT system.*

2. *The aggregate production output, total emissions, consumer surplus, and adjusted consumer surplus (consumer surplus net pollution cost) are identical under tax $\delta(\cdot)$ identified in part (1) and under $\vec{\Phi}(\cdot)$ in the CAT system.*

3. *The only difference is the firms' profits. Specifically, all firms make more profit under $\vec{\Phi}(\cdot)$ in the CAT system compared with under tax $\delta(\cdot)$ identified in part (1).*

Theorem 5 proves that uniform linear allocations in a CAT system and a carbon Tax menu are equivalent. Under Taxes, firms that are greener (with larger $\frac{q_i}{x_i}$) will be levied with a lower tax rate, which in turn encourages green production. Though an identical market equilibrium can be achieved by both uniform linear allocation in CAT and a Tax menu, such a Tax menu is harder to implement since it requires additional information on the firm's pollution x_i.

4 Comparative statistics and numerics

In this section, we provide comparative statistics and numerics of the uniform linear allocation mechanism.

4.1 Monotonicity of uniform linear mechanisms

In this section, we give monotonicity results of uniform linear mechanisms, specifically, how the equilibrium production and pollution change with the varying allocation coefficient.

Theorem 6. *Suppose Assumption 1–3 hold. Furthermore assume $p(\cdot), f_i(\cdot), i = 1, .., N \in \mathcal{C}^2$. Consider an admissible uniform linear allocation mechanism $\vec{\Phi}(\vec{q}) = \alpha^* \vec{q} \in \mathcal{A}$ for some $\alpha^* \in (0, 1)$. Denote by Q^*, K^* the equilibrium aggregate output and total emissions under $\vec{\Phi}(\cdot)$. Then*

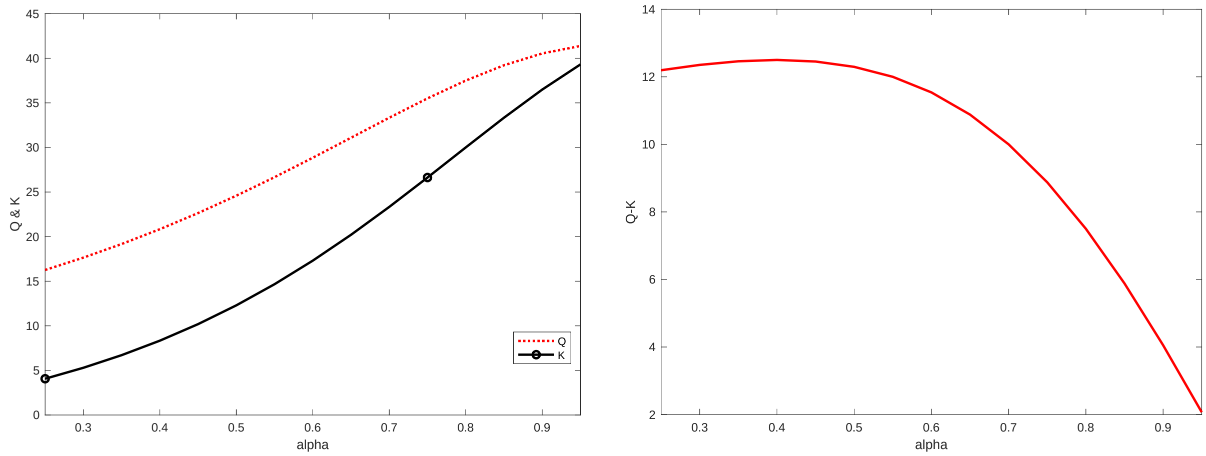
1. *Q^* is strictly increasing in α^*.*
2. *K^* is strictly increasing in α^*.*

Moreover, if the inverse demand is specified to be linear, i.e., $p(Q) = b - aQ$ for some positive a, b, then we further have

3. *there exists a threshold α_{thres} such that if $K^*(\alpha_{thres}) \in (K_{LB}, K_{UB})$ as stated in Lemma 2, then $Q^* - K^*$ is strictly increasing when $\alpha^* < \alpha_{thres}$ and strictly decreasing when $\alpha^* \geq \alpha_{thres}$. Otherwise, $Q^* - K^*$ is strictly decreasing in α^*. The equilibrium market-clearing price $\tau^* = h^{-1}\left(Q^* - \sum_{i=1}^N \Phi_i(\vec{q}^*)\right)$ shows the same monotonicity.*

Theorem 6 implies that the equilibrium emission level under uniform linear allocation comes down as the allocation factor decreases. This result gives us an insight into gradual emission reduction through dynamically adjusting the regulation. To be specific, one can view our single-compliance-cycle problem as part of a dynamic allocation problem, where banking and borrowing from period to period are not allowed. Under such a setting, the dynamic allocation problem of T periods can be decomposed into T single-compliance-cycle problems, each can be solved via our framework. By implementing uniform linear allocation mechanisms $\vec{\Phi}^t = \alpha_t \vec{q}, t = 1, ..., T$ within each time period with a decreasing allocation factor $\alpha_1 > ... > \alpha_T$, we can then reduce emissions little by little.

We also numerically illustrate the monotonicity of equilibrium aggregate production output, emissions and abatement under uniform linear allocation with varying factor α. In the experiment, we set linear inverse demand and quadratic abatement costs as specified in Figure 1. Both aggregate production Q and total emission K are increasing in α, while $Q - K$ is first increasing then decreasing in α.



(a) Aggregate production output and total emissions **(b)** Total abatement

Figure 1: Equilibrium aggregate production output, total emissions and total abatement under uniform linear allocation for various values of allocation factor α. Parameter values: $N = 2; p(Q) = b - aQ, a = 2, b = 100; f_i(y) = \frac{y^2}{2c_i}, c_1 = 0.4, c_2 = 0.8$.

4.2 Uniform Linear vs. Lump sum Allocation Mechanisms

To better illustrate the benefit of uniform linear allocations, we compare the equilibrium aggregate production outputs under uniform linear allocations and lump sum allocations. With a slight abuse of notations, for a fixed $K > 0$, denote $Q^*_{\text{uniform linear}}(K)$ the equilibrium aggregate production output under the unique uniform linear allocation mechanism in $\mathcal{A}(K)$ (see Theorem 4), and $Q^*_{\vec{\Phi}}(K)$ the equilibrium aggregate production output under some other allocation mechanism $\vec{\Phi}(\cdot) \in \mathcal{A}(K)$. We define the relative benefit of the uniform linear allocation in $\mathcal{A}(K)$ over $\vec{\Phi}(\cdot)$ as the ratio:

$$\text{relative benefit} = \frac{\left(Q^*_{\text{uniform linear}}(K) - Q^*_{\vec{\Phi}}(K)\right) \times 100}{Q^*_{\vec{\Phi}}(K)}\%.$$

Note that the above metric is always non-negative by Theorem 4.

We focus on the comparison with lump sum allocations since they are most commonly used in practice and referred to in previous work. Recall that a lump sum allocation distributes to each firm a constant number of permits, i.e., $\Phi_i(\vec{q}) = K_i, i = 1, ..., N$ for some $\vec{K} > 0$. Lump sum allocations have the independence property that the division of K among the firms does not affect the equilibrium production output. Therefore in our experiments, WLOG, we set $K_i = K/N$ for all $i = 1, ..., N$.

For the sake of convenience, we fix $N = 2$ and set the parameters $a = 0.8, b = 100$. We use linear demand curve $p(Q) = b - aQ$ and quadratic abatement cost functions $f_i(y) = \frac{y^2}{2c_i}, \forall i = 1, ..., N$, where bigger c_i suggests greater abatement ability.

Figure 2 illustrates the relative benefit of uniform linear allocation compared with lump sum allocation for various values of the total pollution K. As can be seen from Figure 2, the relative benefit increases at first and then decreases as K increases. For $K = 27.5$, the optimal allocation mechanism can improve the aggregate output Q for about 20.4% compared with lump sum allocation, suggesting a huge benefit from using uniform linear allocation schemes.

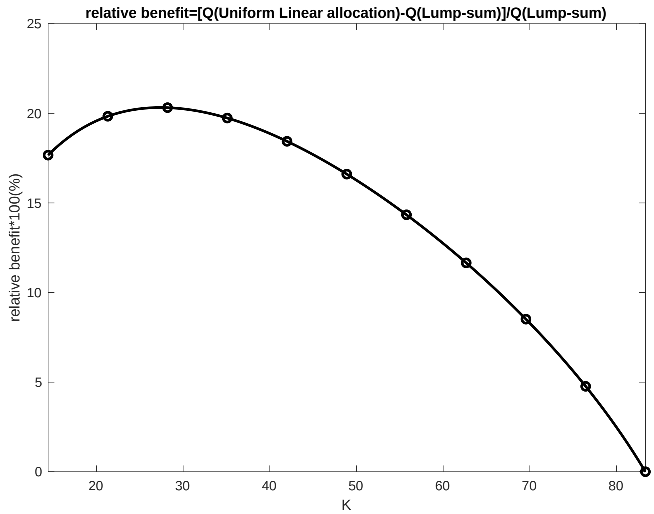


Figure 2: Relative benefit of uniform linear allocation compared with lump sum allocation. Parameter values: $N = 2; p(Q) = b - aQ, a = 0.8, b = 100; f_i(y) = \frac{y^2}{2c_i}, c_1 = 0.1, c_2 = 0.3; K_i = K/N, i = 1, 2$.

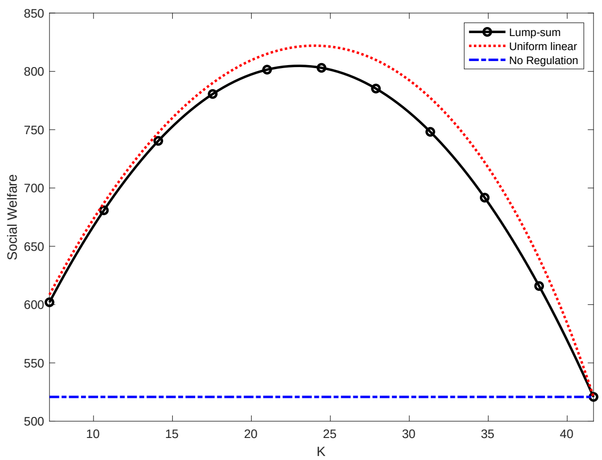
4.3 Firms' Profits and Social Welfare

We have shown, in Theorem 4, that the uniform linear allocation mechanism achieves the Pareto frontier of consumer surplus versus pollution cost, hence also maximizes the overall adjusted consumer surplus. Regarding other performance indicators, such as the firms' profits and social welfare, in this section, we evaluate the performance of this allocation mechanism numerically.

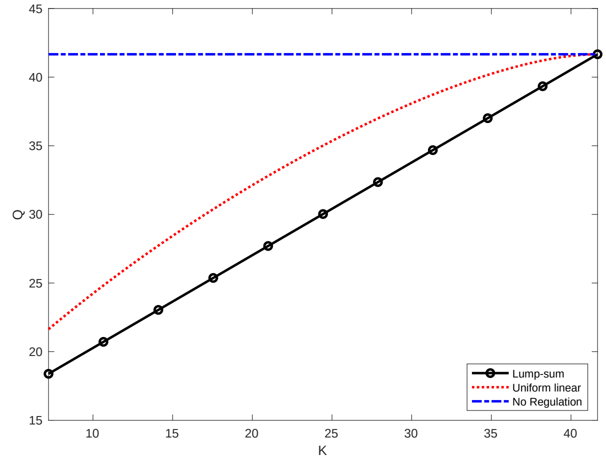
Particularly, we show that other than consumer surplus, the uniform linear allocation also achieves higher social welfare compared with lump sum allocation. Moreover, similar to the main finding of [3], the firms' profits under moderate regulation can increase compared with no regulation.

In this experiment, we describe the equilibrium results, specifically the social welfare, the firms' profits and their aggregate production output, under uniform linear allocation, lump sum allocation and no regulation. Note that the social welfare is composed of two parts, the adjusted consumer surplus described in Section 2.2 and the firms' total profits. We use similar parameters settings as in Section 4.2. We fix $N = 2$ and set the parameters $a = 0.8, b = 100$. We set a quadratic pollution cost function $S(K) = sK^2$ where $s = 0.5$. We also set linear demand curve $p(Q) = b - aQ$ and quadratic abatement cost functions $f_i(y) = \frac{y^2}{2c_i}, \forall i = 1, ..., N$, More details are described in the legend of Figure 3.

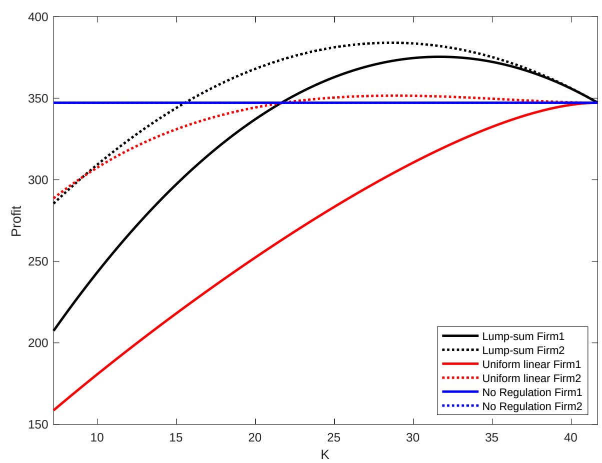
Figure 3 illustrates the social welfare, total production quantity and firms' profits under market equilibrium as functions of K. Denote the equilibrium aggregate production output under uniform linear allocation, lump sum allocation and market without regulation by $Q^*_{UL}(K), Q^*_{LS}(K), Q^*_{NR}(K)$, respectively. Figure 3b shows that $Q^*_{NR}(K) \geq Q^*_{UL}(K) \geq Q^*_{LS}(K)$, which is consistent with Theorem 4. In particular, when $K \approx 13.8$, the uniform linear allocation can improve the aggregate output Q for about 20.4% compared with lump sum allocation. Also, Figure 3a illustrates that implementing uniform linear allocation also achieves a greater social welfare, compared with lump sum allocation and no regulation. Note that this comparison is not affected by the choice of pollution cost s, since fixing K, the pollution costs are identical. For $K \approx 35.5$, the uniform linear allocation can improve social welfare for about 4.4% compared with lump sum allocation. Furthermore, we can see from Figure 3c that for appropriately chosen regulation regimes, the firms' profits may increase compared with under no regulation. This is consistent with the findings in [3].



(a) Social welfare



(b) Total production output



(c) Firms' profits

Figure 3: Equilibrium social welfare, total production output and firms' profits under lump sum allocation, uniform linear allocation and no regulation for various values of equilibrium pollution level K. Parameter values: $N = 2; p(Q) = b - aQ, a = 0.8, b = 50; S(y) = sy^2, s = 0.5; f_i(y) = \frac{y^2}{2c_i}, c_1 = 0.1, c_2 = 0.3; K_i = K/N, i = 1, 2.$

5 CAT Covering Multiple Sectors

In the previous sections, we focus on the case where the firms only differ in their abatement costs. We prove in Theorem 4 that the uniform linear allocation achieves the largest consumer surplus. In this section, we show that when the firms differ in their marginal production costs, the uniform linear allocation is no longer efficient, and is strictly outperformed by a nonuniform linear allocation that distributes more permits (per unit of production) to firms with higher marginal production cost.

Denote firm i's marginal production cost by b_i, which may be different from each other. Then a larger value of b_i indicates a larger marginal production cost. As Theorem 4 shows, the uniform linear allocation is optimal when b_i are identical. We highlight that the assumption of identical marginal production costs b_i is crucial to the efficiency of the uniform linear allocation. In fact, the next theorem shows that any uniform linear allocation is suboptimal when b_i are heterogeneous.

Theorem 7. *Suppose Assumption 1–3 hold. Assume that $b_1 \geq b_2 \geq \ldots \geq b_N$ and $\exists\ i \neq j$ such that $b_i \neq b_j$. Consider a uniform linear allocation mechanism $\vec{\Phi}(\cdot) \in \mathcal{A}(K)$ with multiplier α, i.e., $\vec{\Phi}(\vec{q}) = \alpha\vec{q}$. Then, there must exist another linear allocation mechanism $\vec{\Phi}^*(\cdot) \in \mathcal{B}$ with multipliers $\alpha_1, \ldots, \alpha_N$ i.e. $\Phi_i^*(\vec{q}) = \alpha_i q_i$, where $\alpha_1 \geq \ldots \geq \alpha_N$ are not all equal, and a corresponding interior market equilibrium with $Q^*(\vec{\Phi}^*)$ and $K^*(\vec{\Phi}^*)$, such that $K^*(\vec{\Phi}^*) = K^*(\vec{\Phi})$ and $CS(Q^*(\vec{\Phi}^*)) > CS(Q^*(\vec{\Phi}))$.*

The proof of Theorem 7 is in Appendix D.

Theorem 7 states that when the firms' production costs are heterogeneous, the simple uniform linear allocation may no longer be optimal, and we can find a strict improvement in the set $\mathcal{B}_0$ that generates an identical level of pollution. This superior linear allocation mechanism uses nonuniform multipliers. Moreover, the allocation multiplier is larger for firms with larger marginal production cost. That is, we can strictly do better by giving firms that have higher production costs more permits per unit of production.

CAT covering multiple sectors. In the base model in Section 2, we consider a CAT system that covers a single sector, where all firms are assumed to be identical in their marginal production profit structures and only differ in their abatement cost structures. We can extend this model to cover multiple, say M, sectors that differ in their marginal production profit, and firms in the same sector bear homogeneous marginal production profit structures. Then similar to the construction of the monopoly's problem that is equivalent to the base model (see the end of Section 3.1), this M-sector CAT system can be remodeled as a M-monopoly system that achieves the same equilibrium outcome and is easy to solve.

We first introduce a CAT system covering M sectors. Denote by $\mathcal{J} = \{1, \ldots, M\}$ the set of sectors and $\mathcal{I}_j = \{1, \ldots, N_j\}$ the set of firms in sector j. Firms in the same sector produce a homogeneous product and compete a la Cournot, while firms from different sectors only possibly interact in the permit trading market. We assume that the firms' marginal production costs are homogeneous within each sector, and heterogeneous across sectors. Denote the production output and actual emissions of firm i in sector j by q_i^j and x_i^j, respectively. Given an admissible allocation mechanism $\vec{\Phi}(\vec{q}) = (\Phi_i^j(\vec{q}))_{\forall i,j}$, where $\vec{q} = (q_i^j)_{\forall i,j}$ denotes the firms' production output vector and $\Phi_i^j(\cdot)$ denotes the number of permits to allocate to firm i in sector j, then firm i in sector j aims to maximize its profit:

$$p_j\left(q_i^j + \sum_{k\neq i, k\in\mathcal{I}_j} q_k^j\right) q_i^j - f_i^j(q_i^j - x_i^j) - \tau\left(x_i^j - \Phi_i^j(\vec{q})\right)$$

by choosing $q_i^j \geq 0$ and $x_i^j \in [0, q_i^j]$. Here $p_j(\cdot)$ are sector-specific inverse demand functions satisfying Assumption 1, and $f_i^j(\cdot)$ is the firm's abatement cost function that satisfies Assumption 2. Note that firms from other sectors have no impact on the first two terms (product market and abatement) above, but affect the last permit trading term through $\Phi_i^j(\vec{q})$ and τ implicitly, where, at equilibrium, τ is the market clearing price for the permit trading market.

Similar to Definition 1, we can define a market equilibrium $(\vec{q}^*, \vec{x}^*, \tau^*)$ to satisfy

$$(q_i^{j*}, x_i^{j*}) \in \underset{q_i^j \geq 0, x_i^j \in [0, q_i^j]}{\arg\max} \left\{ p_j \left(q_i^j + \sum_{k \neq i, k \in \mathcal{I}_j} q_k^{j*} \right) q_i^j - f_i^j(q_i^j - x_i^j) - \tau^* \left(x_i^j - \Phi_i^j(q_i^j, \vec{q}_{-i}^*) \right) \right\}$$

for all $i \in \mathcal{I}_j, j \in \mathcal{J}$, and

$$\sum_{i \in \mathcal{I}_j, j \in \mathcal{J}} x_i^{j*} = \sum_{i \in \mathcal{I}_j, j \in \mathcal{J}} \Phi_i^j(\vec{q}^*).$$

Denote by $Q_j^* = \sum_{i=1}^{N_j} q_i^{j*}$ and $K_j^* = \sum_{i=1}^{N_j} \Phi_i^j(\vec{q}^*) = \sum_{i=1}^{N_j} x_i^{j*}$ the total production output and the total emissions of sector j under market equilibrium, respectively. Let $Q^* = \sum_{j=1}^{M} Q_j^*$ and $K^* = \sum_{j=1}^{M} K_j^*$. Then similar to Theorem 2, the equilibrium market-clearing permit price τ^* satisfies:

$$x_i^{j*} = q_i^{j*} - \left(\left(f_i^j \right)' \right)^{-1} (\tau^*), \forall i \in \mathcal{I}_j, j \in \mathcal{J},$$

and

$$Q_j^* - K_j^* = \sum_{i=1}^{N_j} \left(\left(f_i^j \right)' \right)^{-1} (\tau^*), \forall j \in \mathcal{J}.$$

Furthermore,

$$\tau^* = h^{-1}(Q^* - K^*) = h_j^{-1}(Q_j^* - K_j^*), \forall j \in \mathcal{J},$$

where $h(\cdot) = \sum_{j=1}^{M} \sum_{i=1}^{N_j} \left(\left(f_i^j \right)' \right)^{-1} (\cdot)$ and $h_j(\cdot) = \sum_{i=1}^{N_j} \left(\left(f_i^j \right)' \right)^{-1} (\cdot), \forall j \in \mathcal{J}$ are strictly increasing functions.

Similar to Lemma 3, the first order conditions characterizing sector-specific total productions Q_j^* under allocation mechanism $\vec{\Phi}(\cdot)$ are given by

$$p_j'(Q_j^*)Q_j^* + N_j p(Q_j^*) - h_j^{-1}\left(Q_j^* - K_j^*\right) \left(N_j - \sum_{i \in \mathcal{I}_j} \frac{\partial \Phi_i^j(\vec{q}^*)}{\partial q_i^j} \right) = 0, \forall j \in \mathcal{J}.$$

We focus on the sector-specific linear allocation mechanism $\Phi_i^j(\vec{q}) = \alpha_j q_i^j$. That is, we implement uniform linear allocation scheme within each sector. Then the above first order condition reduces to

$$p_j'(Q_j^*)Q_j^*/N_j + p(Q_j^*) - h_j^{-1}\left(Q_j^* - K_j^*\right)(1 - \alpha_j) = 0, \forall j \in \mathcal{J}. \tag{10}$$

Next we show that the original M-sector CAT system can be reformulated as an equivalent M-monopoly system.

An equivalent M-monopoly system. Specifically, we view the N_j firms in each sector j as a single monopoly in that sector, called entity j, with a marginal production profit of $R_j(Q_j) = \frac{Q_j p_j(Q_j)}{N_j} + \frac{N_j - 1}{N_j} \int_0^{Q_j} p_j(t)dt$ given production quantity Q_j. Denote by K_j the emission level of entity j. The marginal abatement cost for entity j reducing $Q_j - K_j$ units of emission is $F_j(Q_j - K_j) := \int_0^{Q_j-K_j} h_j^{-1}(x)dx$. The permit allocation mechanism in the M-monopoly system is $\Phi_j(\vec{Q}) = \alpha_j Q_j, j = 1, ..., M$, where $\vec{Q} = (Q_1, ..., Q_M)$ is the M entities' production output vector. Then to comply with the regulation, firm j also need to purchase an additional (sell if negative) $K_j - \Phi_j(\vec{Q})$ number of permits at price τ to cover its overall emission. In summary, entity j's profit-maximization problem is

$$\max_{Q_j \geq 0, K_j \in [0, Q_j]} R_j(Q_j) - F_j(Q_j - K_j) - \tau \left(K_j - \alpha_j Q_j\right), \tag{11}$$

where τ, at equilibrium, is the market-clearing permit price such that $\sum_{j=1}^M K_j^* = \sum_{j=1}^M \alpha_j Q_j^*$.

One can verify that the first order condition of program (11) is exactly Eq. (10). Moreover, given $\alpha_j, j = 1, ..., M$, we can solve the equilibrium $(\vec{Q}^*, \vec{K}^*)$ via the following system of $2M$ equations

$$\begin{cases} p_j'(Q_j^*)Q_j^*/N_j + p(Q_j^*) - h_j^{-1}\left(Q_j^* - K_j^*\right)(1-\alpha_j) = 0, \forall j = 1, ...M, \\ F_1(Q_1 - K_1) = F_j(Q_j - K_j), \forall j = 2, ..., M, \\ \sum_{j=1}^M \alpha_j Q_j = \sum_{j=1}^M K_j. \end{cases}$$

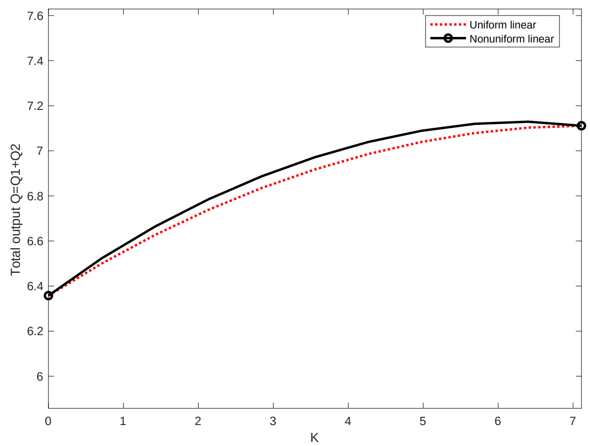
Numerical experiments. We now numerically illustrate the performance of sector-specific uniform linear allocation in the CAT system covering multiple sectors. The uniform linear allocation, which allocates each sector j with a homogeneous factor α, is a special case of the sector-specific linear allocation described above. To compare different allocation mechanisms, we fix the equilibrium total pollution K and compare the equilibrium aggregate output Q^* under uniform linear allocation and a sector-specific uniform linear allocations.

For simplicity, we set $M = 2$ and $N_1 = N_2 = 1$ in the experiment. Figure 4 illustrates the variation of the production quantities with K under different linear allocations. We compare the results of uniform linear allocation with multiplier α and linear allocation with nonuniform multipliers α_j. We assume linear inverse demand functions $p_j(Q_j) = b_j - a_j Q_j$, where we set $a_1 = 1.5, a_2 = 2.5, b_1 = 20, b_2 = 20$. We define the abatement cost functions to be $f_1^j(y) = \frac{y^2}{2c_j}, j = 1, 2$, with parameters $c_1 = 1, c_2 = 2$. The detailed calculations of the two linear allocation mechanisms are omitted here and can be found in Appendix E.

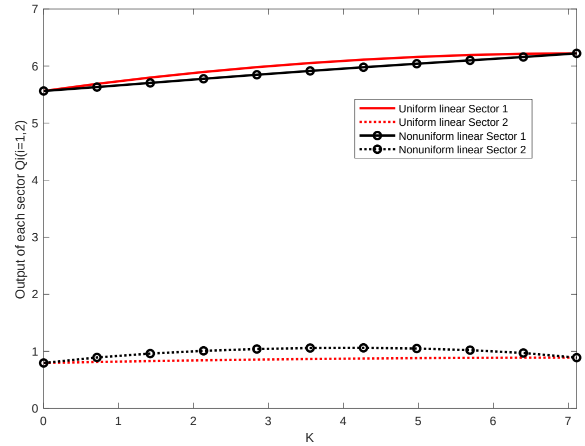
As can be seen in Figure 4a, the sector-specific linear allocation with nonuniform α_j achieves greater aggregate outputs than uniform linear allocation. Figure 4b shows the production quantities by sector. Moving from uniform linear to sector-specific uniform linear allocations, sector 1's production decreases while sector 2's production increases. These observations, together with Theorem 7, implies that the firms operating in a homogeneous product market and incurring identical production costs is crucial to the efficiency of uniform linear allocation mechanisms (see Theorem 4).

6 Discussion

In this paper, we study how to distribute emission permits to N regulated firms competing in a homogeneous product market in a cap and trade system. We focus on the broad class of component-wise concave allocation schemes, which include the commonly used allocation schemes in practice such as lump sum and output-based allocations. Interestingly, we find that despite its simple form, uniform linear allocation mechanisms can achieve the entire Pareto frontier of traditional consumer surplus versus pollution cost, hence maximizes the adjusted consumer surplus. This is in contrast to previous works that only focus on lump sum allocations under which the independence property (the initial distribution of permits do not matter) holds. We numerically demonstrate that the benefit



(a) Aggregate production output $Q^* = Q_1^* + Q_2^*$



(b) Separate production output of each sector Q_1^*, Q_2^*

Figure 4: Equilibrium production quantities under different linear allocation mechanisms. Parameter values: $M = 2, N_1 = N_2 = 1; p_j(Q_j) = b_j - a_j Q_j, j = 1, 2; a_1 = 1.5, a_2 = 2.5, b_1 = 20, b_2 = 20, s = 0.5; f_1^1(y) = \frac{y^2}{2c_1}, f_1^2(y) = \frac{y^2}{2c_2}, c_1 = 1, c_2 = 2.$

of uniform linear allocations compared with lump sum ones can be large. Our result indicates that the regulator can ignore the firms' heterogeneity in their abatement abilities, and can reduce the search space of operators on $\mathbb{R}^N$ to only a single scalar $\alpha \in (0, 1)$. The system equilibrium outcome for any choice of α can be easily solved using an equivalent monopoly's problem.

The efficiency of uniform linear allocation schemes crucially requires that the firms only differ in their abatement cost structures. When the firms have different production costs, we show that any uniform linear allocation is suboptimal and can be dominated by appropriately chosen linear allocations with nonuniform multipliers. We extend our results from single-sector (where firms only differ in their abatement costs) to multi-sector settings, and construct equivalent M-monopoly systems that are easy to solve. We numerically show that sector-specific uniform linear allocation mechanisms outperforms uniform allocations.

Discussion of model assumptions. Indeed, our main results established depend on the specific modeling choices we made so far, e.g., a homogoneous product market under Cournot competition with a linear inverse demand curve. The modeling choices are made to simplify our analysis and generate clear insights. A natural question to ask is, to what extent can we generalize the model without affecting our main findings? In fact, the efficiency of uniform linear allocation mechanisms will be not affected as long as firms only differ in their abatement costs. The specific forms of the marginal production profit need not be Cournot, if identical across firms.

Future directions. This work opens ups a number of interesting future research directions. We show that endogenizing the effect of firms' decisions on the number of permits they receive induces a different equilibrium outcome and breaks the initial permit distribution's independence property. More work along this line would be beneficial. For example, one particularly interesting direction to pursue next would be the efficiency of sector-specific uniform linear allocation mechanisms in a multi-sector CAT system. One can also extend the current setting to incorporate multiple compliance cycles, the effect of uncertainty, etc.. Another interesting direction is mechanism design when the firms have private information.

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A Proofs of Results in Section 2

In this appendix, we give the detailed proofs of Lemma 1.

Proof of Lemma 1. Consider firm i's problem for fixed $\vec{q}_{-i} > 0$ and $\tau > 0$, the objective of the second stage problem stated in Eq. (3) is strictly convex by Assumption 2. Thus the optimal solution of the second stage problem (3) exists and is unique for a fixed q_i. Let the derivative of the objective in (3) be zero, we have

$$\hat{x}_i = q_i - \left(f_i'\right)^{-1}(\tau),$$

where the inverse function of the derivative function f_i', i.e. $\left(f_i'\right)^{-1}$, exists as f_i' is strictly increasing by Assumption 2. Denote the optimal solution of (3) by x_i^*, then x_i^* satisfies

$$x_i^* = \max(\min(\hat{x}_i, q_i), 0) = \max\left(q_i - \left(f_i'\right)^{-1}(\tau), 0\right)$$

note that $\left(f_i'\right)^{-1}(\tau) \geq \left(f_i'\right)^{-1}(0) = 0$ by Assumption 2. Therefore, we have

$$g_i(q_i) = \begin{cases} f_i(q_i) - \tau\Phi_i(\vec{q}) \text{ if } q_i \leq \left(f_i'\right)^{-1}(\tau) \\ f_i\left[\left(f_i'\right)^{-1}(\tau)\right] + \tau\left(q_i - \left(f_i'\right)^{-1}(\tau) - \Phi_i(\vec{q})\right) \text{ if } q_i > \left(f_i'\right)^{-1}(\tau) \end{cases}.$$

Since $\Phi_i(\vec{q})$ is concave in q_i by our assumption, $g(q_i)$ is piecewise convex. Moreover, note that

$$\lim_{y \uparrow \left(f_i'\right)^{-1}(\tau)} g_i(y) = \lim_{z \downarrow \left(f_i'\right)^{-1}(\tau)} g_i(z) = f_i\left[\left(f_i'\right)^{-1}(\tau)\right] - \tau\Phi_i\left(\left(f_i'\right)^{-1}(\tau), \vec{q}_{-i}\right)$$

and

$$\lim_{y \uparrow \left(f_i'\right)^{-1}(\tau)} g_i'(y) = \lim_{z \downarrow \left(f_i'\right)^{-1}(\tau)} g_i'(z) = \tau\left(1 - \frac{\partial \Phi_i}{\partial q_i}\left(\left(f_i'\right)^{-1}(\tau), \vec{q}_{-i}\right)\right).$$

Thus $g_i(q_i)$ is convex in $[0, \infty)$.

Moreover, we claim that $p\left(q_i + \sum_{j \neq i} q_j\right) q_i$ is strictly concave in q_i. Specifically, for all $q_i, \tilde{q}_i > 0$ such that $q_i \neq \tilde{q}_i$ and $\lambda \in (0,1)$, we have

$$\begin{aligned}
&p\left(\lambda q_i + (1-\lambda)\tilde{q}_i + \sum_{j \neq i} q_j\right)(\lambda q_i + (1-\lambda)\tilde{q}_i) - \left[\lambda p\left(q_i + \sum_{j \neq i} q_j\right) q_i + (1-\lambda) p\left(\tilde{q}_i + \sum_{j \neq i} q_j\right)\tilde{q}_i\right] \\
=&\lambda q_i\left[p\left(\lambda q_i + (1-\lambda)\tilde{q}_i + \sum_{j \neq i} q_j\right) - p\left(q_i + \sum_{j \neq i} q_j\right)\right] \\
&+ (1-\lambda)\tilde{q}_i\left[p\left(\lambda q_i + (1-\lambda)\tilde{q}_i + \sum_{j \neq i} q_j\right) - p\left(\tilde{q}_i + \sum_{j \neq i} q_j\right)\right] \\
\geq&\lambda(1-\lambda) q_i\left[p\left(\tilde{q}_i + \sum_{j \neq i} q_j\right) - p\left(q_i + \sum_{j \neq i} q_j\right)\right] + \lambda(1-\lambda)\tilde{q}_i\left[p\left(q_i + \sum_{j \neq i} q_j\right) - p\left(\tilde{q}_i + \sum_{j \neq i} q_j\right)\right] \\
=&\lambda(1-\lambda)\left(q_i - \tilde{q}_i\right)\left[p\left(\tilde{q}_i + \sum_{j \neq i} q_j\right) - p\left(q_i + \sum_{j \neq i} q_j\right)\right]
\end{aligned}$$

>0,

where the first inequality follows from the concavity of $p(\cdot)$ and the second inequality follows from the monotonicity of $p(\cdot)$ in Assumption 1.

Thus the objective of the first stage problem (2) is strictly concave.

In addition, note that

$$\begin{aligned}
&\lim_{q_i\to\infty}\frac{d}{dq_i}\left(p\left(q_i+\sum_{j\neq i}q_j\right)q_i-g_i(q_i)\right)\\
=&\lim_{q_i\to\infty}p'\left(q_i+\sum_{j\neq i}q_j\right)q_i+p\left(q_i+\sum_{j\neq i}q_j\right)-g_i'(q_i)\\
=&-\infty
\end{aligned}$$

by Assumption 1–2. Thus the solution to the first stage problem (2) exists and is unique. □

B Proofs of Results in Section 3

In this appendix, we give the detailed proofs of Lemma 2, 3, 4, 5 and Theorem 1, 2, 3, 4, 5.

Proof of Theorem 1. Under the situation without regulation, firm i's problem reduces to

$$\max_{q_i}\ p\left(q_i+\sum_{j\neq i}q_j\right)q_i,$$

since firms can emit carbon dioxide as much as their production quantity i.e. $x_i=q_i$. We can also define the corresponding market equilibrium as follows:

$$q^*_{i,NR}\in\arg\max_{q_i}\left\{p\left(q_i+\sum_{j\neq i}q^*_{j,NR}\right)q_i\right\}.$$

Note that the objective of the above problem is strictly concave in q_i by Lemma 1 and

$$\begin{aligned}
&\lim_{q_i\to 0}\frac{d}{dq_i}\left(p\left(q_i+\sum_{j\neq i}q^*_{j,NR}\right)q_i\right)\geq 0,\\
&\lim_{q_i\to\infty}\frac{d}{dq_i}\left(p\left(q_i+\sum_{j\neq i}q^*_{j,NR}\right)q_i\right)=-\infty
\end{aligned}$$

by Assumption 1. Thus the optimal solution satisfies the first order condition, that is,

$$p'(Q^*_{NR})q^*_{i,NR}+p(Q^*_{NR})=0, \tag{12}$$

where $Q^*_{NR}=\sum_{i=1}^N q^*_{i,NR}$ denotes the equilibrium aggregate output. By summing over i, we can further derive the necessary condition of Q^*_{NR}:

$$p'(Q^*_{NR})Q^*_{NR}+Np(Q^*_{NR})=0. \tag{13}$$

Define a function $H(Q) = p'(Q)Q + Np(Q)$, which is strictly decreasing by Assumption 1. Since $H(0) > 0$ and $H(\infty) < 0$, the aggregate equilibrium output $Q^*_{NR} > 0$ without regulation is the unique root of $H(\cdot)$ (or Eq. (13)). Moreover, note that $p'(Q^*_{NR})q_i + p(Q^*_{NR})$ is strictly decreasing in q_i, and has a non-negative value when $q_i = 0$ and approaches minus infinity when $q_i \to \infty$. Therefore, $q^*_{i,NR}$ is the unique solution of Eq. (12) for fixed Q^*_{NR}. Thus $q^*_{i,NR} = \frac{Q^*_{NR}}{N}$ for all $i = 1, ..., N$. □

Proof of Lemma 2. First we give a partial characterization of any interior market equilibrium under an allocation mechanism $\vec{\Phi}(\cdot) \in \mathcal{B}$.

Consider an allocation mechanism $\vec{\Phi}(\cdot)$ in $\mathcal{B}$, an interior market equilibrium $(\vec{q}^*, \vec{x}^*, \tau^*)$ exists by Definition 3. In the following, we establish a relation that $(\vec{q}^*, \vec{x}^*, \tau^*)$ has to satisfy. Note that $q^*_i > x^*_i > 0$ holds for all i, which implies (q^*_i, x^*_i) is an interior solution of (4), thus we can describe the market equilibrium using the first order conditions by Lemma 1. The first order condition of the second stage problem (3) gives that

$$f'_i(q^*_i - x^*_i) - \tau^* = 0, \forall i = 1, ..., N,$$

which implies

$$x^*_i = q^*_i - \left(f'_i\right)^{-1}(\tau^*), \forall i = 1, ..., N. \tag{14}$$

Sum the above equations over i, we have

$$\sum_{i=1}^{N} \Phi_i(q^*) = \sum_{i=1}^{N} x^*_i = Q^* - \left[\sum_{i=1}^{N} \left(f'_i\right)^{-1}\right](\tau^*),$$

where the first equality is by the market clearing condition (5) in Definition 1. Denote $h = \sum_{i=1}^{N} \left(f'_i\right)^{-1}$, then we have

$$\tau^* = h^{-1}\left(Q^* - \sum_{i=1}^{N} \Phi_i(\vec{q}^*)\right), \tag{15}$$

where the inverse function h^{-1} exists and is strictly increasing as f'_i is strictly increasing by Assumption 2.

Next, we give the implicit expression of K_{LB} in detail. For a fixed K^* such that $0 \le K^* < Q^*_{NR}$, consider a $Q^* > 0$ satisfying the following equation:

$$p'(Q^*)Q^* + Np(Q^*) - h^{-1}(Q^* - K^*)\left(N - N\frac{K^*}{Q^*}\right) = 0. \tag{16}$$

Note that LHS of the above equation is strictly decreasing in Q^*. By Theorem 1 and the assumption $K^* < Q^*_{NR}$, we also have

$$\begin{aligned} LHS(Q^*)|_{Q^*=K^*} &= p'(K^*)K^* + Np(K^*) \\ &> p'(Q^*_{NR})Q^*_{NR} + Np(Q^*_{NR}) = 0, \end{aligned}$$

where the inequality holds since $p'(Q)Q + Np(Q) - l(Q)$ is strictly decreasing in Q, and

$$LHS(Q^*)|_{Q^*=Q^*_{NR}} = p'(Q^*_{NR})Q^*_{NR} + Np(Q^*_{NR}) - h^{-1}(Q^*_{NR} - K^*)\left(N - N\frac{K^*}{Q^*_{NR}}\right) < 0.$$

Therefore, the Q^* satisfying Eq. (16) exists and is unique, and satisfies $K^* < Q* < Q^*_{NR}$. By the **implicit function theorem**, there exists a function $Q^*(\cdot) \in \mathcal{C}^1$ such that $Q^* = Q^*(K^*)$.

Moreover, $Q^*(K^*)$ is strictly increasing in K^*. Specifically, assume $0 \leq K_1 < K_2 < Q^*_{NR}$ while $Q_1 \triangleq Q^*(K_1) \geq Q^*(K_2) \triangleq Q_2$. Then we have

$$\begin{aligned} 0 &= p'(Q_1)Q_1 + Np(Q_1) - h^{-1}(Q_1 - K_1)\left(N - N\frac{K_1}{Q_1}\right) \\ &< p'(Q_2)Q_2 + Np(Q_2) - h^{-1}(Q_2 - K_2)\left(N - N\frac{K_2}{Q_2}\right) = 0, \end{aligned}$$

which is a contradiction. Thus we must have $Q_1 < Q_2$, which implies that $Q^*(K^*)$ is strictly increasing in K^*.

Define a function in terms of K^* as follows:

$$F(K^*) = \min_{i=1,...,N} f_i'\left(\frac{Q^*(K^*)}{N}\right) - h^{-1}\left(Q^*(K^*) - K^*\right).$$

Then $F(K^*)$ is continuous in K^* by Assumption 2. Since $Q^*(K^*)|_{K^*=Q^*_{NR}} = Q^*_{NR}$ by Theorem 1, $F(K^*)|_{K^*=Q^*_{NR}} = \min_{i=1,...,N} f_i'\left(\frac{Q^*_{NR}}{N}\right) > 0$. In addition, since

$$p'(Q^*)Q^* + Np(Q^*) - h^{-1}(Q^*)\,|_{Q^*=Q^*_{NR}} < 0,$$

and

$$p'(Q^*)Q^* + Np(Q^*) - h^{-1}(Q^*)\,|_{Q^*=0} > 0,$$

we have $0 < Q^*(0) < Q^*_{NR}$. Denote $f_i'\left(\frac{Q^*(0)}{N}\right)$ by a_i for all i. Then we have

$$Q^*(0) = N\left(f_i'\right)^{-1}(a_i), i = 1, ..., N. \tag{17}$$

Denote $a_k = \min_{i=1,...,N} a_i$. Then $a_i > a_k > 0$ for all $i \neq k$. By Assumption 2, it holds that

$$\left(f_i'\right)^{-1}(a_i) > \left(f_i'\right)^{-1}(a_k), i \neq k. \tag{18}$$

Combining Eq. (17) and (18), we have

$$Q^*(0) \geq N\left(f_i'\right)^{-1}(a_k), i = 1, ..., N,$$

and the equality holds only if $i = k$. Summing the above inequality over i gives that

$$Q^*(0) > \sum_{i=1}^{N}\left(f_i'\right)^{-1}(a_k) = h(a_k),$$

where the equality follows by the definition of $h(\cdot)$. Thus $h^{-1}(Q^*(0)) - a_k > 0$, which implies that $F(K^*)|_{K^*=0} < 0$.

Now we are ready to give a description of K_{LB}. Define

$$K_{LB} = \sup\{0 \leq K \leq Q^*_{NR} | F(K) = 0\},$$

which exists and satisfies $0 < K_{LB} < Q^*_{NR}$ by the above arguments.

Next, we claim that if $K_{LB} < K^* < K_{UB} = Q^*_{NR}$, then $\mathcal{A}(K^*) \neq \emptyset$. Specifically, we prove the existence of an admissible uniform linear allocation mechanism in $\mathcal{A}(K^*)$ for K^* satisfying $K_{LB} < K^* < K_{UB}$.

For a fixed K^* in (K_{LB}, K_{UB}), define $\alpha = \frac{K^*}{Q^*(K^*)}$, which is a constant in $(0,1)$ by the above arguments. Denote $\vec{\Phi}(\vec{q}) = \alpha\vec{q}$, we show that $\vec{\Phi} \in \mathcal{A}(K^*)$.

Denote $\tau^* = h^{-1}(Q^*(K^*) - K^*)$ and $q_i^* = \frac{Q^*(K^*)}{N}, x_i^* = q_i^* - (f_i')^{-1}(\tau^*)$ for all $i = 1, \ldots, N$. Then $\tau^* > 0$ since $Q^*(K^*) > K^*$. In addition, by the construction of K_{LB}, we have $F(K^*) > 0$. Thus $f_i'(q_i^*) > \tau^*$ for all $i = 1, \ldots, N$, which implies that $q_i^* > x_i^* > 0$ for all $i = 1, \ldots, N$. Next we show that $(\vec{q}^*, \vec{x}^*, \tau^*)$ is an interior market equilibrium under $\vec{\Phi}$. By Eq. (16) and the above notation, $(\vec{q}^*, \vec{x}^*, \tau^*)$ satisfies

$$p'(Q^*)q_i^* + p(Q^*) - \tau^*(1-\alpha) = 0, i = 1, ..., N, \tag{19}$$

$$x_i^* = q_i^* - \left(f_i'\right)^{-1}(\tau^*), i = 1, ..., N, \tag{20}$$

which are the first order conditions of the firms' two stage problem. Since the first order conditions have an interior solution, that is, $x_i^* \in (0, q_i^*)$ for all $i = 1, \ldots, N$, we have

$$(q_i^*, x_i^*) \in \underset{q_i \geq 0, x_i \in [0, q_i]}{\arg\max} \left\{ p\left(q_i + \sum_{j \neq i} q_j^*\right) q_i - f_i(q_i - x_i) - \tau^*(x_i - \alpha q_i) \right\}.$$

Moreover, we have

$$K^* = \alpha Q^* = \sum_{i=1}^{N} \Phi_i(\vec{q}^*)$$

by the definition of α and $\vec{\Phi}$, and

$$\sum_{i=1}^{N} x_i^* = Q^* - h(\tau^*) = Q^* - (Q^* - K^*) = K^*$$

by the definition of x_i^* and τ^*. Combining the above two equations gives that

$$K^* = \sum_{i=1}^{N} x_i^* = \sum_{i=1}^{N} \Phi_i(\vec{q}^*),$$

which is the market clearing condition. Therefore, $(\vec{q}^*, \vec{x}^*, \tau^*)$ is an interior market equilibrium under $\vec{\Phi}$, and the corresponding equilibrium pollution level is K^*.

Next, we show that $(\vec{q}^*, \vec{x}^*, \tau^*)$ is also a unique interior market equilibrium.

Assume that $\left(\hat{\vec{q}}^*, \hat{\vec{x}}^*, \hat{\tau}^*\right)$ is another interior market equilibrium. Then $(\vec{q}^*, \vec{x}^*, \tau^*), \left(\hat{\vec{q}}^*, \hat{\vec{x}}^*, \hat{\tau}^*\right)$ satisfy the first order conditions (27) by Theorem 2. That is,

$$p'(Q^*)q_i^* + p(Q^*) - \tau^*(1-\alpha) = 0, i = 1, ..., N \tag{21}$$

$$p'\left(\hat{Q}^*\right)\hat{q}_i^* + p\left(\hat{Q}^*\right) - \hat{\tau}^*(1-\alpha) = 0, i = 1, ..., N, \tag{22}$$

where $\tau^* = h^{-1}(Q^*(1-\alpha)), \hat{\tau}^* = h^{-1}\left(\hat{Q}^*(1-\alpha)\right)$ by Theorem 2. The summation of the first order conditions gives that

$$p'(Q^*)Q^* + Np(Q^*) - h^{-1}(Q^*(1-\alpha))(N - N\alpha) = 0, \tag{23}$$

$$p'\left(\hat{Q}^*\right)\hat{Q}^* + Np\left(\hat{Q}^*\right) - h^{-1}\left(\hat{Q}^*(1-\alpha)\right)(N - N\alpha) = 0. \tag{24}$$

Note that the LHS of the above two equations, that is, $p'(Q)\,Q+Np(Q)-h^{-1}(Q(1-\alpha))\,(N-N\alpha)$ is strictly decreasing in Q. We must have $Q^*=\hat{Q}^*$ by Eq. (23)–(24). Thus $\tau^*=\hat{\tau}^*$. Moreover, since LHS of Eq. (21) is strictly decreasing in q_i^*, we must have $q_i^*=q_j^*$ for all $i\neq j$. Thus $q_i^*=\frac{Q^*}{N}, i=1,...,N$. Similarly, $\hat{q}_i^*=\frac{\hat{Q}^*}{N}, i=1,...,N$. Therefore $\vec{q}^*=\hat{\vec{q}}^*$. It then follows that $\vec{x}^*=\hat{\vec{x}}^*$ by the first order conditions of the second stage problem (14).

Above all, $\vec{\Phi}(\cdot)$ is admissible, and $\vec{\Phi}(\cdot)\in\mathcal{A}(K^*)$. □

Proof of Theorem 2. Under an allocation mechanism $\vec{\Phi}(\cdot)$ in $\mathcal{B}$, an interior market equilibrium $(\vec{q}^*,\vec{x}^*,\tau^*)$ exists by Definition 2.

In the following, we establish a relation that $(\vec{q}^*,\vec{x}^*,\tau^*)$ has to satisfy. The points 2 and 3 of Theorem 2 follows directly from the proof of Lemma 2 (See Eq. (14) and (15)).

It remains to prove that all firms achieve positive profits. Substitute the relation (14) into firm i's problem (4), it then becomes

$$q_i^*\in\arg\max_{q_i^*}\left\{p\left(q_i+\sum_{j\neq i}q_j^*\right)q_i-f_i\left(\left(f_i'\right)^{-1}(\tau^*)\right)-\tau^*\left(q_i-\left(f_i'\right)^{-1}(\tau^*)-\Phi_i(q_i,q_{-i}^*)\right)\right\}. \quad (25)$$

Note that the objective value of the above optimization problem under the decision $q_i=0$ is positive. To be specific, the objective value corresponding to the decision of not producing is given by

$$-f_i\left(\left(f_i'\right)^{-1}(\tau^*)\right)+\tau^*\left(f_i'\right)^{-1}(\tau^*)+\tau^*\Phi_i(0,q_{-i}^*), \quad (26)$$

where the summation of first two terms is positive since

$$-f_i\left(\left(f_i'\right)^{-1}(\tau^*)\right)+\tau^*\left(f_i'\right)^{-1}(\tau^*)$$
$$y=\left(f_i'\right)^{-1}(\tau^*)-f_i(y)+f_i'(y)y>0$$

by Assumption 2 and $\tau^*>0$ (see Definition 1). Moreover, since $\Phi_i(0,q_{-i}^*)\geq 0$ by Definition 2, the third term is non-negative. Hence the objective value when there is no production is positive. Then the optimal value of the optimization problem must also be positive, hence firm i's profit.

Besides, we can also describe q_i^*'s as follows. As we mentioned above, since $q_i^*>0$ is an interior solution of the concave program (25), it has to satisfy the first order condition

$$p'(Q^*)q_i^*+p(Q^*)-\tau^*\left(1-\frac{\partial\Phi_i(\vec{q}^*)}{q_i}\right)=0, \quad (27)$$

and it holds for all $i=1,...,N$. The above first order conditions will help assist subsequent analysis. □

Proof of Theorem 3. Consider a general non-linear allocation mechanism $\vec{\Phi}(\cdot)\in\mathcal{B}$ and a corresponding interior equilibrium $(\vec{q}^*,\vec{x}^*,\tau^*)$, that is, there exists at least one $i\in\{1,...,N\}$ such that $\frac{\Phi_i}{q_i}(q_i,\vec{q}_{-i})\not\equiv\frac{\partial\Phi_i}{\partial q_i}(q_i,\vec{q}_{-i})$ for $q_i\in[0,\infty)$. We show that there must exist a linear allocation mechanism $\tilde{\Phi}_i(\vec{q})=\alpha_i q_i+K_i\in\mathcal{B}$ for some $\alpha_1,...,\alpha_N\in[0,1]$ and $K_1,...,K_N\geq 0$ which can achieve an identical interior market equilibrium as under $\vec{\Phi}(\cdot)$. The proof is constructive.

The interior equilibrium $(\vec{q}^*,\vec{x}^*,\tau^*)$ under $\vec{\Phi}(\cdot)$ can be described by the following system by Theorem 2: $\mathrm{p}'(Q^*)q_i^*+p(Q^*)-\tau^*\left(1-\frac{\partial\Phi_i(\vec{q}^*)}{q_i}\right)=0, i=1,...,N,$
$x_i^*=q_i^*-\left(f_i'\right)^{-1}(\tau^*), i=1,...,N,$
$\sum_{i=1}^N\Phi_i(\vec{q}^*)=\sum_{i=1}^N x_i^*,$

$\tau^* = h^{-1}\left(Q^* - \sum_{i=1}^{N}\Phi_i(\vec{q}^*)\right)$, where Eq. (B) are the first order conditions (27) of firms' first stage problems, Eq. (B) are the first order conditions (14) of firms' second stage problems, Eq. (B) is the market-clearing condition (See Eq. (5)), and Eq. (B) follows from Theorem 2. Denote

$$\alpha_i \triangleq \frac{\partial \Phi_i(\vec{q}^*)}{\partial q_i}, i = 1, ..., N$$

the equilibrium partial derivatives, and

$$K_i \triangleq \Phi_i(\vec{q}^*) - \frac{\partial \Phi_i(\vec{q}^*)}{\partial q_i} q_i^*.$$

Note that $\alpha_i, K_i, i = 1, ..., N$ are constants, where $\alpha_i \in [0, 1]$ by Definition 2 (See Condition 3 in Definition 2). Moreover, we have $K_i \geq 0$ by concavity and non-negativity of Φ_i in Definition 2. Specifically, let

$$F_{i,\vec{q}^*_{-i}}(q_i) = \Phi_i(q_i, \vec{q}^*_{-i})$$

denote a single-variable function with respect to q_i, where the value of the firms'(except for firm i) output vector $\vec{q}$ is fixed at $\vec{q}^*_{-i} = (q_j^*)_{j \neq i}$. Note that

$$\begin{aligned}
&F_{i,\vec{q}^*_{-i}}(q_i^*) = \Phi_i(\vec{q}^*), \\
&\left.\frac{dF_{i,\vec{q}^*_{-i}}}{dq_i}\right|_{q_i = q_i^*} = \frac{\partial \Phi_i(\vec{q}^*)}{\partial q_i}, \\
&\frac{d^2 F_{i,\vec{q}^*_{-i}}}{d{q_i}^2} = \left.\frac{\partial^2 \Phi_i(\vec{q})}{\partial q_i^2}\right|_{\vec{q}_{-i} = \vec{q}^*_{-i}} \leq 0.
\end{aligned}$$

The last inequality is by the concavity of $\vec{\Phi}$ in Definition 2. Thus $F_{i,\vec{q}^*_{-i}}$ is concave with respect to q_i. By the property of concave functions, we know that the slope of the secant lines

$$\frac{F_{i,\vec{q}^*_{-i}}(q_i^*) - F_{i,\vec{q}^*_{-i}}(x)}{q_i^* - x}$$

is non-increasing in x. Therefore, we must have

$$\frac{\Phi_i(\vec{q}^*) - \Phi_i(q_i = 0, \vec{q}^*_{-i})}{q_i^*} = \frac{F_{i,\vec{q}^*_{-i}}(q_i^*) - F_{i,\vec{q}^*_{-i}}(0)}{q_i^* - 0} \geq \frac{F_{i,\vec{q}^*_{-i}}(q_i^*) - F_{i,\vec{q}^*_{-i}}(x)}{q_i^* - x}$$

for $\forall x \in [0, q_i^*)$. Let x tend to q_i^* from below, the right hand side of the above inequality becomes $\left.\frac{dF_{i,\vec{q}^*_{-i}}}{dq_i}\right|_{q_i = q_i^*}$, which is $\frac{\partial \Phi_i(\vec{q}^*)}{\partial q_i}$. Hence

$$\Phi_i(\vec{q}^*) \geq \Phi_i(q_i = 0, \vec{q}^*_{-i}) + \frac{\partial \Phi_i(\vec{q}^*)}{\partial q_i} q_i^* \geq \frac{\partial \Phi_i(\vec{q}^*)}{\partial q_i} q_i^*,$$

the last inequality is by $\Phi_i(q_i = 0, \vec{q}^*_{-i}) \geq 0$ by Definition 2. Equality holds if and only if $\Phi_i(\vec{q}) = \beta_i q_i$ for some β_i. Therefore, $K_i \geq 0$.

Define $\tilde{\Phi}_i(\vec{q}) = \alpha_i q_i + K_i$. Next we show that $(\vec{q}^*, \vec{x}^*, \tau^*)$ is an interior market equilibrium under $\tilde{\vec{\Phi}}$.

Observe that $(\vec{q}^*, \vec{x}^*, \tau^*)$ satisfies the following equation system $\mathrm{p}'(Q^*) q_i^* + p(Q^*) - \tau^* (1 - \alpha_i) = 0, i = 1, ..., N,$

$x_i^* = q_i^* - (f_i')^{-1}(\tau^*), i = 1, ..., N,$
$\sum_{i=1}^N (\alpha_i q_i^* + K_i) = \sum_{i=1}^N \Phi_i(\vec{q}^*),$
$\tau^* = h^{-1}\left(Q^* - \sum_{i=1}^N \Phi_i(\vec{q}^*)\right)$, by Eq. (B)–(B) and the definition of $\alpha_i, K_i, i = 1, ..., N$. Note that Eq. (B)–(B) are the first order conditions of the two-stage concave program (2) under $\tilde{\vec{\Phi}}$. Since $0 < x_i^* < q_i^*$, the first order conditions (B)–(B) have an interior solution, which implies that

$$(q_i^*, x_i^*) \in \underset{q_i \geq 0, x_i \in [0, q_i]}{\arg\max} \left\{ p\left(q_i + \sum_{j \neq i} q_j^*\right) q_i - f_i(q_i - x_i) - \tau^*\left(x_i - \tilde{\Phi}_i(q_i, q_{-i}^*)\right) \right\}.$$

Moreover, we also have

$$\sum_{i=1}^N x_i^* = \sum_{i=1}^N \Phi_i(\vec{q}^*) = \sum_{i=1}^N (\alpha_i q_i^* + K_i) = \sum_{i=1}^N \tilde{\Phi}_i(\vec{q}^*), \tag{28}$$

where the first equality is the market-clearing condition under $\vec{\Phi}(\cdot)$, and the last two equalities follow from the definition of α_i, K_i and $\tilde{\Phi}_i(\cdot)$ for all i. Therefore, $(\vec{q}^*, \vec{x}^*, \tau^*)$ is an interior market equilibrium under $\tilde{\vec{\Phi}}(\cdot)$.

Thus by the formulation of $\tilde{\vec{\Phi}}$, we have that $\tilde{\vec{\Phi}} \in \mathcal{B}$.

□

Proof of Lemma 3. When restricted to a single sector, the first order conditions of firms' profit-maximizing problem stated in the proof of Theorem 2 (See Eq. (27)) become

$$p'(Q^*)q_i^* + p(Q^*) - \tau^*\left(1 - \frac{\partial \Phi_i(\vec{q}^*)}{q_i}\right) = 0, i = 1, .., N. \tag{29}$$

Summing up the above equations for N firms, we get

$$p'(Q^*)Q^* + Np(Q^*) - \tau^*\left(N - \sum_{i=1}^N \frac{\partial \Phi_i(\vec{q}^*)}{\partial q_i}\right) = 0. \tag{30}$$

Moreover, we have $\sum_{i=1}^N \Phi_i(\vec{q}^*) = \sum_{i=1}^N x_i^* = K^*$ by the market clearing condition (5), thus Eq. (30) becomes

$$p'(Q^*)Q^* + Np(Q^*) - h^{-1}(Q^* - K^*)\left(N - \sum_{i=1}^N \frac{\partial \Phi_i(\vec{q}^*)}{\partial q_i}\right) = 0.$$

By calculating $\sum_{i=1}^N \frac{\partial \Phi_i(\vec{q}^*)}{\partial q_i}$ for different allocation schemes in $\mathcal{A}(K^*)$, we can compare the corresponding aggregate production quantities Q^*, thus the adjusted consumer surplus and the relative emissions reductions as we mentioned in Section 2.2. □

Proof of Lemma 4. Consider the first order condition for determining q_i^* for firm i in Eq. (29):

$$p'(Q^*)q_i^* + p(Q^*) - \tau^*\left(1 - \frac{\partial \Phi_i(\vec{q}^*)}{q_i}\right) = 0, i = 1, .., N,$$

where $\tau^* > 0$ is the equilibrium permit price under this allocation mechanism. Assume that $\frac{\partial \Phi_i(\vec{q}^*)}{\partial q_i} \geq \frac{\partial \Phi_j(\vec{q}^*)}{\partial q_j}$, then we have

$$p'(Q^*)q_i^* + p(Q^*) \leq p'(Q^*)q_j^* + p(Q^*),$$

where the equality holds if and only if $\frac{\partial \Phi_i(\vec{q}^*)}{\partial q_i} = \frac{\partial \Phi_j(\vec{q}^*)}{\partial q_j}$. It then follows that $q_i^* \geq q_j^*$ note that $p'(Q^*)q + p(Q^*)$ is strictly decreasing in q. Moreover, the equality $q_i^* = q_j^*$ holds if and only if $\frac{\partial \Phi_i(\vec{q}^*)}{\partial q_i} = \frac{\partial \Phi_j(\vec{q}^*)}{\partial q_j}$. □

Proof of Lemma 5. Since $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_N \geq 0, q_1 \geq q_2 \geq ... \geq q_N \geq 0$, and $\Sigma_{i=1}^N q_i = 1$, by **Chebyshev's sum inequality**, we have

$$\Sigma_{i=1}^N q_i \alpha_i \geq \frac{1}{N} \cdot \left(\sum_{i=1}^N q_i \right) \cdot \left(\sum_{i=1}^N \alpha_i \right) = \frac{1}{N} \cdot \left(\sum_{i=1}^N \alpha_i \right)$$

Hence, $\Sigma_{i=1}^N q_i \alpha_i \geq \Sigma_{i=1}^N \frac{1}{N} \alpha_i$, which is the desired result. Moreover, when α_i is strictly decreasing, the equality holds if and only if $q_1 = ... = q_N$. □

Proof of Theorem 4. For $\tilde{K}^*$ satisfying the condition in Lemma 2, there exists an admissible uniform linear allocation $\vec{\Phi}^*(\vec{q}) = \alpha^* \vec{q}$ in $\mathcal{A}\left(\tilde{K}^*\right)$ for some $\alpha^* \in (0,1)$ by the proof of Lemma 2. Now consider a general nonuniform linear allocation mechanism $\tilde{\vec{\Phi}}(\cdot) = (\alpha_i q_i + K_i)_{i=1}^N$ for some $\alpha_1, ..., \alpha_N, \alpha^* \in [0,1]$ and $K_1, ..., K_N \geq 0$, and an interior market equilibrium $\left(\tilde{\vec{q}}^*, \tilde{\vec{x}}^*, \tilde{\tau}^*\right)$ yielding the emission level $\tilde{K}^*$, where $\tilde{\vec{\Phi}}$ is not uniform linear, i.e. there doesn't exist an $\tilde{\alpha}$ such that $\tilde{\vec{\Phi}}(\vec{q}) = \tilde{\alpha}\vec{q}$. Denote firms' equilibrium production output vector under the uniform linear allocation $\vec{\Phi}^*(\cdot)$ by $\vec{q}^*$, and aggregate production in equilibrium under the two mechanisms by $Q^*, \tilde{Q}^*$ respectively. Firms' permits allocated in equilibrium are then given by $\tilde{\Phi}_i(\tilde{\vec{q}}^*), \Phi_i^*(\vec{q}^*)$.

Summation of firms' first order conditions , i.e. Eq. (8), can be written as

$$p'(\tilde{Q}^*)\tilde{Q}^* + Np(\tilde{Q}^*) - h^{-1}\left(\tilde{Q}^* - \tilde{K}^*\right)\left(N - \sum_{i=1}^N \alpha_i\right) = 0, \tag{31}$$

$$p'(Q^*)Q^* + Np(Q^*) - h^{-1}\left(Q^* - \tilde{K}^*\right)(N - N\alpha^*) = 0. \tag{32}$$

Furthermore, as the total emissions are the same in equilibrium, the following relationship holds by the market clearing condition:

$$\sum_{i=1}^N (\alpha_i \tilde{q}_i{}^* + K_i) = \sum_{i=1}^N \alpha^* q_i^* = \tilde{K}^*,$$

from which we further have

$$\sum_{i=1}^N \alpha_i \tilde{q}_i{}^* + \sum_{i=1}^N K_i = \alpha^* Q^*. \tag{33}$$

Firstly, we claim that

$$\text{if } \sum_{i=1}^N \alpha_i < N\alpha^*, \text{then } \tilde{Q}^* < Q^*. \tag{34}$$

Otherwise if $\tilde{Q}^* \geq Q^*$, then under Assumption 1–2, Eq. (32) together with Eq. (31) gives that

$$0 = p'(\tilde{Q}^*)\tilde{Q}^* + Np(\tilde{Q}^*) - h^{-1}\left(\tilde{Q}^* - \tilde{K}^*\right)\left(N - \sum_{i=1}^N \alpha_i\right)$$

$$< p'(Q^*)Q^* + Np(Q^*) - h^{-1}\left(Q^* - \tilde{K}^*\right)(N - N\alpha^*) = 0,$$

where $h^{-1}(\tilde{Q}^* - \tilde{K}^*) \geq h^{-1}(Q^* - \tilde{K}^*) > 0$ since h^{-1} is a strictly increasing function, $h^{-1}(0) = 0$ by Theorem 2, and also $Q^*, \tilde{Q}^* > \tilde{K}^*$ by Definition 2. This is a contradiction, thus the claim (34) holds. Therefore, it suffices to show that $\sum_{i=1}^{N} \alpha_i < N\alpha^*$.

Secondly, we'll prove by contradiction. Suppose $\tilde{Q}^* \geq Q^*$. We next prove $\sum_{i=1}^{N} \alpha_i < N\alpha^*$ holds, thus reaching a contradiction.

Notice that by Lemma 4 if $\alpha_i \leq \alpha_j$, then $\tilde{q}_i^* \leq \tilde{q}_j^*$. And $\tilde{q}_i^* = \tilde{q}_j^*$ holds if and only $\alpha_i = \alpha_j$. We discuss the specific form of $\tilde{\vec{\Phi}}(\cdot)$ as follows.

If $K_i = 0$ for all $i = 1, .., .N$, then α_i's are not all equal by our assumption that $\tilde{\vec{\Phi}}(\cdot)$ is not uniform linear. WLOG we can assume that $\alpha_1 > ... > \alpha_N$, then $\tilde{q}_1^* > ... > \tilde{q}_N^*$ by Lemma 4. We then have

$$\frac{\sum_{i=1}^{N} \alpha_i}{N} < \sum_{i=1}^{N} \alpha_i \frac{\tilde{q}_i^*}{\tilde{Q}^*} \leq \sum_{i=1}^{N} \alpha_i \frac{\tilde{q}_i^*}{Q^*} = \alpha^*,$$

where the first inequality holds by Lemma 5, the second inequality is by our assumption that $\tilde{Q}^* \geq Q^*$, and the last equality follows from Eq. (33). Otherwise, if there exists i such that $K_i > 0$. Then $\sum_{i=1}^{N} K_i > 0$, and we further have

$$\frac{\sum_{i=1}^{N} \alpha_i}{N} \leq \sum_{i=1}^{N} \alpha_i \frac{\tilde{q}_i^*}{\tilde{Q}^*} \leq \sum_{i=1}^{N} \alpha_i \frac{\tilde{q}_i^*}{Q^*} < \alpha^*,$$

where the first inequality is by Lemma 5, and the last equality follows from Eq. (33). Above all we can conclude that $\sum_{i=1}^{N} \alpha_i < N\alpha^*$. Thus $\tilde{Q}^* < Q^*$ by claim (34), which is contradictory to our assumption. Now we have found a uniform linear allocation $\vec{\Phi}^*$, such that $Q^*(\vec{\Phi}^*) > Q^*(\tilde{\vec{\Phi}})$, and thus CS($Q^*(\vec{\Phi}^*)$) >CS($Q^*(\tilde{\vec{\Phi}})$) by Assumption 3.

Next, we show the uniqueness of uniform linear allocation. Assume that there is another uniform linear allocation mechanism $\hat{\vec{\Phi}}(\cdot)$ such that it can achieve an interior market equilibrium with $\hat{K}^* = \tilde{K}^*$ and CS($\hat{Q}^*$) >CS($\tilde{Q}^*$). By Lemma 2, for any uniform linear allocation mechanism, if there exists an interior market equilibrium, then it must also be a unique interior equilibrium. Thus we must have $\hat{\vec{\Phi}}(\cdot) \in \mathcal{A}\left(\tilde{K}^*\right)$. Therefore it's sufficient to prove the uniqueness of uniform linear allocation in $\mathcal{A}\left(\tilde{K}^*\right)$.

Consider $\Phi_i^*(\vec{q}) = \alpha^* q_i \in \mathcal{A}\left(\tilde{K}^*\right)$, denote firms' equilibrium output vector by $\vec{q}_{UL}^*$ and the equilibrium aggregate output by Q_{UL}^*. Put the first order conditions and the emission relation together, we have

$$\begin{aligned} &p'(Q_{UL}^*)Q_{UL}^* + Np(Q_{UL}^*) - h^{-1}\left(Q_{UL}^* - \tilde{K}^*\right)(N - N\alpha^*) = 0, \\ &\sum_{i=1}^{N} \Phi_i(\vec{q}_{UL}^*) = \sum_{i=1}^{N} \alpha^* q_{i,UL}^* = \alpha^* Q_{UL}^* = \tilde{K}^*. \end{aligned}$$

It then follows that

$$p'\left(\frac{\tilde{K}^*}{\alpha^*}\right)\frac{\tilde{K}^*}{\alpha^*} + Np\left(\frac{\tilde{K}^*}{\alpha^*}\right) - h^{-1}\left(\tilde{K}^*\left(\frac{1}{\alpha^*} - 1\right)\right)(N - N\alpha^*) = 0, \tag{35}$$

which is a necessary condition for the coefficient α^*. Define a function as follows:

$$LHS(\beta) = \tilde{K}^* p'\left(\tilde{K}^*\beta\right)\beta + Np\left(\tilde{K}^*\beta\right) - h^{-1}\left(\tilde{K}^*\left(\beta-1\right)\right)\left(N - \frac{N}{\beta}\right), \beta \in (1,\infty),$$

which is strictly decreasing in β. Also note that $LHS(\infty) = -\infty$ and

$$\begin{aligned} LHS(1) &= \tilde{K}^* p'\left(\tilde{K}^*\right) + Np\left(\tilde{K}^*\right) \\ &> Q^*_{NR} p'\left(Q^*_{NR}\right) + Np\left(Q^*_{NR}\right) = 0, \end{aligned}$$

where the inequality follows from the condition on $\tilde{K}^*$ in Lemma 2 and the last equality follows from Theorem 1. Hence the root of $LHS(\beta)$ exists and is unique, and we denote it by β^*. Then $\alpha^* = \frac{1}{\beta^*}$ is the unique solution of Eq. (35). □

Proof of Theorem 5. We consider the tax regime with tax rate given by $\delta(q_i, x_i) = \frac{-\alpha\tau^* q_i + B}{x_i} + \tau^*$, where τ^* is the equilibrium market clearing price under the uniform linear allocation mechanism $\vec{\Phi}$ in the cap-and-trade system. The parameter B is chosen that the firms' profits and the tax rates are positive under Nash equilibrium. Then firm i's problem, given by

$$\max_{q_i \geq 0, x_i \in [0,q_i]} p\left(q_i + \sum_{j\neq i} q_j\right) q_i - f_i(q_i - x_i) - \delta(q_i, x_i)\cdot x_i.$$

, can be rewritten as a two-stage optimization problem:

$$\max_{q_i \geq 0} \quad p\left(q_i + \sum_{j\neq i} q_j\right) q_i - g(q_i), \tag{36}$$

where

$$g(q_i) = \min_{x_i \in [0,q_i]} f_i(q_i - x_i) + \delta(q_i, x_i)\cdot x_i. \tag{37}$$

Note that the minimum of (37) is convex in x_i. The first order condition of (37) on x_i yields

$$\begin{aligned} 0 &= -f_i'(q_i^{Tax} - x_i^{Tax}) + \frac{\partial\delta(q_i^{Tax}, x_i^{Tax})}{x_i}\cdot x_i^{Tax} + \delta(q_i^{Tax}, x_i^{Tax}) \\ &= -f_i'(q_i^{Tax} - x_i^{Tax}) - \frac{-\alpha\tau^* q_i^{Tax} + B}{x_i^{Tax}} + \frac{-\alpha\tau^* q_i^{Tax} + B}{x_i^{Tax}} + \tau^* \\ &= -f_i'(q_i^{Tax} - x_i^{Tax}) + \tau^*, \end{aligned}$$

which implies

$$x_i^{Tax} = q_i^{Tax} - \left(f_i'\right)^{-1}(\tau^*).$$

Substitute the expression of x_i^{Tax} to Eq. (36), we can rewrite the firm i's first stage problem as

$$q_i^{Tax} \in \arg\max_{q_i^{Tax}} \left\{ p\left(q_i + \sum_{j\neq i} q_j^{Tax}\right) q_i - f_i\left[\left(f_i'\right)^{-1}(\tau^*)\right] - \tau^*(1-\alpha)q_i - B + \tau^*\cdot\left(f_i'\right)^{-1}(\tau^*)\right\}. \tag{38}$$

The first order condition of the above convex optimization program yields

$$p'\left(Q^{Tax}\right)q_i^{Tax} + p\left(Q^{Tax}\right) - \tau^*(1-\alpha) = 0,$$

which is exactly the same as the first order condition under the uniform linear allocation $\vec{\Phi}(\vec{q}) = \alpha\vec{q}$. Thus the first order conditions under taxes have an interior solution $\left(\vec{q}^{UL}, \vec{x}^{UL}\right)$, which is also a unique interior solution by admissibility of $\vec{\Phi}$. This implies that there exists a unique interior market equilibrium under taxes, and the equilibrium satisfies $q_i^{Tax} = q_i^{UL}, x_i^{Tax} = x_i^{UL}$.

Result 2 then directly follows since the equilibrium aggregate output $Q^I = \sum_{i=1}^N q_i^I$, the total emissions $K^I = \sum_{i=1}^N x_i^I$ and the adjusted consumer surplus $ACS = CS(Q^I) - S(K^I)$ is a function of Q^I and K^I for $I \in \{Tax, UL\}$.

The difference of firm i's profit under taxes (Eq. (38)) and under uniform linear allocation (Eq. (25)) is

$$\text{Profit}_i^{UL} - \text{Profit}_i^{Tax} = B > 0,$$

which implies result 3.

□

C Proofs of Results in Section 4

In this appendix, we give the detailed proofs of Theorem 6.

Proof of Theorem 6. Consider an admissible uniform linear allocation $\vec{\Phi}(\vec{q}) = \alpha^*\vec{q}$, note that $\sum_{i=1}^N \Phi_i(\vec{q}^*) = \sum_{i=1}^N \alpha^* q_i^* = \alpha^* Q^* = K^*$ by the market-clearing condition, thus the first order condition in Lemma 3 becomes,

$$p'(Q^*)Q^* + Np(Q^*) - h^{-1}\left(Q^*(1-\alpha^*)\right)(N - N\alpha^*) = 0, \tag{39}$$

where $\vec{q}^*, Q^*, K^*$ denote the firms' output vector, the equilibrium aggregate production quantity and the total emissions under $\vec{\Phi}(\cdot)$.

Next we'll use the **implicit function theorem** to show the existence of $Q^*(\cdot)$ and $K^*(\cdot)$. Specifically, let $F(Q,\alpha)$ be a function of two variables Q and α, where

$$F(Q,\alpha) = p'(Q)Q + Np(Q) - h^{-1}\left(Q(1-\alpha)\right)(N - N\alpha).$$

Then we have $F(Q^*, \alpha^*) = 0$ by Eq.(39). Since $f_i \in C^2$, the function $h^{-1} \in C^1$ and the partial derivatives are then given by

$$\frac{\partial F(Q,\alpha)}{\partial Q} = p''(Q)Q + (N+1)p'(Q) - N\left(h^{-1}\right)'\left(Q(1-\alpha)\right)(1-\alpha)^2, \tag{40}$$

$$\frac{\partial F(Q,\alpha)}{\partial \alpha} = Nh^{-1}\left(Q(1-\alpha)\right) + N\left(h^{-1}\right)'\left(Q(1-\alpha)\right)Q\,(1-\alpha). \tag{41}$$

It follows from Eq. (40) that

$$\frac{\partial F(Q^*,\alpha^*)}{\partial Q} = p''(Q^*)Q^* + (N+1)p'(Q^*) - N\left(h^{-1}\right)'\left(Q^*(1-\alpha^*)\right)(1-\alpha^*)^2 < 0$$

note that h^{-1} is strictly increasing on $[0,\infty)$ by Theorem 2 and $\alpha^* \in (0,1)$. Hence there exists a function $Q = Q^*(\alpha) \in C^1$ defined in a neighborhood of α^* such that $Q^*(\alpha^*) = Q^*$ by the **implicit**

function theorem. Let $K^*(\alpha) = \alpha Q^*(\alpha)$, then $K^*(\cdot)$ is also a C^1 function in a neighborhood of α^* and it satisfies $K^*(\alpha^*) = \alpha^* Q^*(\alpha^*) = \alpha^* Q^* = K^*$.

Next we'll move on to the monotonicity properties of functions $Q^*(\cdot)$ and $K^*(\cdot)$. By the **implicit function theorem**, we further have

$$\begin{aligned}\frac{dQ^*(\alpha^*)}{d\alpha} &= -\frac{\partial F(Q^*(\alpha^*),\alpha^*)}{\partial \alpha}\Bigg/\frac{\partial F(Q^*(\alpha^*),\alpha^*)}{\partial Q}\\ &= \frac{Nh^{-1}\left(Q^*(1-\alpha^*)\right) + N\left(h^{-1}\right)'\left(Q^*(1-\alpha^*)\right)Q^*\left(1-\alpha^*\right)}{-p''(Q^*)Q^* - (N+1)p'(Q^*) + N\left(h^{-1}\right)'\left(Q^*(1-\alpha^*)\right)\left(1-\alpha^*\right)^2} > 0 \end{aligned} \tag{42}$$

note that $h^{-1}\left(Q^*(1-\alpha^*)\right) > h^{-1}(0) = 0$ and $\left(h^{-1}\right)'(\cdot) \geq 0$. Thus $Q^* = Q^*(\alpha^*)$ is strictly increasing in α^*, which is Property 1 in Theorem 6.

By our definition of $K^*(\cdot)$, we then have by Eq. (42) that

$$\frac{dK^*(\alpha^*)}{d\alpha} = Q^*(\alpha^*) + \alpha^* \cdot \frac{dQ^*(\alpha^*)}{d\alpha} > 0.$$

Thus $K^* = K^*(\alpha^*)$ is also strictly increasing in α^*, which is Property 2 in Theorem 6.

Now assume linear inverse demand $p(Q) = b - aQ$, we discuss the monotonicity of $Q^* - K^*$. Note that

$$\begin{aligned}\frac{d\left(Q^*(\alpha^*) - K^*(\alpha^*)\right)}{d\alpha} &= (1-\alpha^*)\cdot\frac{dQ^*(\alpha^*)}{d\alpha} - Q^*(\alpha^*)\\ &= \frac{Nh^{-1}\left(Q^*(1-\alpha^*)\right)(1-\alpha^*) + p''(Q^*)Q^{*2} + (N+1)p'(Q^*)Q^*}{-p''(Q^*)Q^* - (N+1)p'(Q^*) + N\left(h^{-1}\right)'\left(Q^*(1-\alpha^*)\right)\left(1-\alpha^*\right)^2}\\ &= \frac{p''(Q^*)Q^{*2} + (N+2)p'(Q^*)Q^* + Np(Q^*)}{-p''(Q^*)Q^* - (N+1)p'(Q^*) + N\left(h^{-1}\right)'\left(Q^*(1-\alpha^*)\right)\left(1-\alpha^*\right)^2},\\ &= \frac{Nb - 2a(N+1)Q^*}{a(N+1) + N\left(h^{-1}\right)'\left(Q^*(1-\alpha^*)\right)\left(1-\alpha^*\right)^2}\end{aligned}$$

where the third equality follows from Eq. (39). Then $\frac{d(Q^*(\alpha^*)-K^*(\alpha^*))}{d\alpha} > 0$ if $Q^* < \frac{Nb}{2a(N+1)}$ and $\frac{d(Q^*(\alpha^*)-K^*(\alpha^*))}{d\alpha} \leq 0$ if $Q^* \geq \frac{Nb}{2a(N+1)}$. Thus it suffices to compare the value of Q^* and $\frac{Nb}{2a(N+1)}$.

- Case 1: Assume $h^{-1}\left(\frac{Nb}{2a(N+1)}\right) > \frac{b}{2}$. Define

  $$g(\alpha) = \frac{Nb}{2} - Nh^{-1}\left(\frac{Nb}{2a(N+1)}(1-\alpha)\right)(1-\alpha), \alpha \in [0,1].$$

  Then $g(\alpha)$ is strictly increasing in α. Denote by $\alpha_{thres} \in (0,1)$ the root of $g(\cdot)$, i.e.

  $$\frac{Nb}{2} - Nh^{-1}\left(\frac{Nb}{2a(N+1)}(1-\alpha_{thres})\right)(1-\alpha_{thres}) = 0,$$

  which exists by the monotonicity of $g(\cdot)$ and

  $$\begin{aligned}g(0) &= \frac{Nb}{2} - Nh^{-1}\left(\frac{Nb}{2a(N+1)}\right) < 0,\\ g(1) &= \frac{Nb}{2} > 0,\end{aligned}$$

  where the first inequality follows from our assumption.

If $\alpha_{thres} \in (\alpha_{LB}, 1)$ as we mentioned at the beginning, then we have $\frac{Nb}{2a(N+1)} = Q^*(\alpha_{thres})$ note that the first order condition Eq. (39) holds for $Q^* = \frac{Nb}{2a(N+1)}$ and $\alpha^* = \alpha_{thres}$. That it, the equilibrium total output under uniform linear allocation $\vec{\Phi}(\vec{q}) = \alpha_{thres} \cdot \vec{q}$ is $\frac{Nb}{2a(N+1)}$. By the monotonicity of $Q^*(\alpha^*)$ in Property 1, for $\alpha^* \in (\alpha_{thres}, 1)$ we have $Q^*(\alpha^*) > \frac{Nb}{2a(N+1)}$, and for $\alpha^* \in (\alpha_{LB}, \alpha_{thres})$ we have $Q^*(\alpha^*) < \frac{Nb}{2a(N+1)}$. Therefore, $Q^*(\alpha^*) - K^*(\alpha^*)$ is strictly increasing in $(\alpha_{LB}, \alpha_{thres})$ and strictly decreasing in $[\alpha_{thres}, 1)$.

Otherwise if $\alpha_{thres} \leq \alpha_{LB}$, then for any $\alpha^* \in (\alpha_{LB}, 1)$, we have

$$g(\alpha^*) = \frac{Nb}{2} - Nh^{-1}\left(\frac{Nb}{2a(N+1)}(1-\alpha^*)\right)(1-\alpha^*) > 0.$$

Define

$$l(Q) = Nb - a(N+1)Q - h^{-1}\left(Q(1-\alpha^*)\right)(N - N\alpha^*).$$

Note that $l(\cdot)$ is strictly decreasing in Q and $l(Q^*(\alpha^*)) = 0, l\left(\frac{Nb}{2a(N+1)}\right) = g(\alpha^*) > 0$. Thus we have $Q^*(\alpha^*) > \frac{Nb}{2a(N+1)}$. Thus $Q^*(\alpha^*) - K^*(\alpha^*)$ is strictly decreasing in α^*.

- Case 2: Assume $h^{-1}\left(\frac{Nb}{2a(N+1)}\right) \leq \frac{b}{2}$. Let $g(\cdot), l(\cdot)$ be the same functions defined above. Note that $g(0) \geq 0$ by our assumption. Thus $g(\alpha) > 0$ for any $\alpha \in (0,1)$.

  For any $\alpha^* \in (\alpha_{LB}, 1)$, we have $l(Q^*(\alpha^*)) = 0$ and $l\left(\frac{Nb}{2a(N+1)}\right) = g(\alpha^*) > 0$. It then follows that $Q^*(\alpha^*) > \frac{Nb}{2a(N+1)}$ by the monotonicity of $l(\cdot)$. Thus $Q^*(\alpha^*) - K^*(\alpha^*)$ is strictly decreasing in α^*.

From the above discussion we can conclude Property 3. □

D Proofs of Results in Section 5

In this appendix, we give the detailed proofs of Theorem 7.

Proof of Theorem 7. Assume $b_1 \geq b_2 \geq ... \geq b_N$ and $\exists\, i \neq j$ such that $b_i \neq b_j$, then we can show that any uniform linear allocation $\Phi_i(\vec{q}) = \alpha q_i$ for some $\alpha \geq 0$ is suboptimal by constructing a superior linear allocation $\Phi_i^*(\vec{q}) = \hat{\alpha}_i q_i$ with ununiform $\hat{\alpha}_i$. Consider the uniform linear allocation with homogeneous weight α. Denote the total output quantity under uniform case by Q^*.

For a general linear allocation $\tilde{\vec{\Phi}}$ in $\mathcal{B}$ with allocation parameters $\alpha_1, ...\alpha_N$, that is, $\tilde{\Phi}_i(\vec{q}) = \alpha_i q_i$, which has an interior market equilibrium $\left(\tilde{\vec{q}}^*, \tilde{\vec{x}}^*, \tilde{\tau}^*\right)$ with $\sum_{i=1}^N \tilde{x}_i^* = K$. Then firms' first-order conditions (See Eq. (29)) are as follows:

$$p'(\tilde{Q}^*)\tilde{q}_i^* + p(\tilde{Q}^*) - b_i - \tilde{\tau}^*(1-\alpha_i) = 0, i = 1, .., N.$$

It follows that

$$\tilde{q}_i^* = \frac{p(\tilde{Q}^*) - b_i - \tilde{\tau}^*(1-\alpha_i)}{-p'(\tilde{Q}^*)}. \tag{43}$$

Let heterogeneous $\hat{\alpha}_1 \geq \hat{\alpha}_2 ... \geq \hat{\alpha}_N$ be such that the output quantity under market equilibrium with parameters $\hat{\alpha}_1, ..., \hat{\alpha}_N$ satisfy $\hat{q}_1^* \leq \hat{q}_2^* ... \leq \hat{q}_N^*$ and $\hat{q}_1^* \leq \hat{q}_2^* ... \leq \hat{q}_N^*$ are not uniform. Such $\hat{\alpha}_1, ..., \hat{\alpha}_N$ exist given $b_1 \geq b_2 \geq ... \geq b_N$ and by Eq. (43). Now consider the linear allocation $\vec{\Phi}^*$

given by $\Phi_i^*(\vec{q}) = \hat{\alpha}_i q_i$, and a corresponding interior market equilibrium denoted by $\left(\hat{\vec{q}}^*, \hat{\vec{x}}^*, \hat{\tau}^*\right)$, where $\sum_{i=1}^N \hat{x}_i^* = K$.

Suppose that $Q^* \geq \hat{Q}^*$. Since $\hat{q}_1^* \leq \hat{q}_2^* ... \leq \hat{q}_N^*$, there exists $s \in \mathbb{N}$ such that $s = \max\{i \in \{1,...,N\} : \frac{\hat{q}_i^*}{\hat{Q}^*} \leq \frac{1}{N}\}$. Then,

$$\begin{aligned}
&\sum_{i=1}^N \hat{\alpha}_i \left(\frac{\hat{q}_i^*}{\hat{Q}^*} - \frac{1}{N}\right) \\
&= \sum_{i=1}^s \hat{\alpha}_i \left(\frac{\hat{q}_i^*}{\hat{Q}^*} - \frac{1}{N}\right) + \sum_{i=s+1}^N \hat{\alpha}_i \left(\frac{\hat{q}_i^*}{\hat{Q}^*} - \frac{1}{N}\right) \\
&< \sum_{i=1}^s \hat{\alpha}_s \left(\frac{\hat{q}_i^*}{\hat{Q}^*} - \frac{1}{N}\right) + \sum_{i=s+1}^N \hat{\alpha}_s \left(\frac{\hat{q}_i^*}{\hat{Q}^*} - \frac{1}{N}\right) \\
&= \hat{\alpha}_s \left(\frac{\sum_{i=1}^N \hat{q}_i^*}{\hat{Q}^*} - 1\right) = 0.
\end{aligned}$$

Thus, we have

$$\sum_{i=1}^N \hat{\alpha}_i \frac{\hat{q}_i^*}{Q^*} \leq \sum_{i=1}^N \hat{\alpha}_i \frac{\hat{q}_i^*}{\hat{Q}^*} < \sum_{i=1}^N \hat{\alpha}_i \frac{1}{N},$$

where the first inequality is due to our assumption $Q^* \geq \hat{Q}^*$. Since $\sum_{i=1}^N \hat{\alpha}_i \hat{q}_i^* = \alpha Q^* = K$, we have

$$\alpha = \sum_{i=1}^N \frac{\hat{q}_i^*}{Q^*} \hat{\alpha}_i.$$

Hence

$$\alpha < \sum_{i=1}^N \hat{\alpha}_i \frac{1}{N}$$

by the above two expressions. The summation of firms' first order conditions under $\vec{\Phi}$ and $\vec{\Phi}^*$ are as follows:

$$\text{Uniform } \vec{\Phi}\text{: } p'(Q^*)Q^* + Np(Q^*) - \sum_{i=1}^N b_i - h^{-1}\left(Q^* - K\right)\left(N - N\alpha\right) = 0;$$

$$\text{Nonuniform } \vec{\Phi}^*\text{: } p'(\hat{Q}^*)\hat{Q}^* + Np(\hat{Q}^*) - \sum_{i=1}^N b_i - h^{-1}\left(\hat{Q}^* - K\right)\left(N - \sum_{i=1}^N \hat{\alpha}_i\right) = 0. \qquad (44)$$

Combine the above three expressions, we have

$$p'(Q^*)Q^* + Np(Q^*) - \sum_{i=1}^N b_i - h^{-1}\left(Q^* - K\right)\left(N - \sum_{i=1}^N \hat{\alpha}_i\right) > 0.$$

Also observe that the LHS above is strictly decreasing function in Q^*. Therefore, this together with Eq. (44) give $Q^* < \hat{Q}^*$, which contradicts the previous assumption of $Q^* \geq \hat{Q}^*$. Thus we have $Q^* < \hat{Q}^*$. Furthermore, we have CS(Q^*) <CS($\hat{Q}^*$) by Assumption 3.

By now, we have constructed a linear allocation $\Phi_i^*(\vec{q}) = \hat{\alpha}_i q_i$ with ununiform $\hat{\alpha}_1 \geq ... \geq \hat{\alpha}_N$ which is superior than the uniform linear allocation with homogeneous weight α. Thus a uniform linear allocation must be suboptimal given that $b_1, ..., b_N$ are heterogeneous. Combine the above results and Theorem 4 together, we can conclude that homogeneous production costs $h_1(\cdot), ..., h_N(\cdot)$ is sufficient and necessary for uniform linear allocation with some uniform weight α to achieve the maximal total output quantity, hence the consumer surplus among allocation mechanisms in $\mathcal{A}(K)$. □

E Calculations of the two linear allocation mechanisms in the numerical experiments of Section 5

Specifically, consider the linear allocation with uniform allocation parameter α. For a fixed cap K, the first-order condition for firm i in sector j is given by

$$-a_j q_i^{j^*} + b_j - a_j Q_j^* - \tau^* (1 - \alpha) = 0.$$

which can be further reduced to

$$-a_j q_i^{j^*} + b_j - a_j Q_j^* - h^{-1}(Q^* - K)(1 - \alpha) = 0.$$

Also note that α should be chosen such that $Q^* > K$ and

$$\sum_{j=1}^{M} \sum_{i=1}^{N_j} \Phi_i^j(\vec{q}^*) = \sum_{j=1}^{M} \sum_{i=1}^{N_j} \alpha q_i^{j^*} = \alpha Q^* = K.$$

Next we'll construct a sector-specific linear allocation $\tilde{\vec{\Phi}}(\cdot)$ with ununiform parameters α_j. Denote the corresponding equilibrium output vector by $\tilde{\vec{q}}^*$ and let $\tilde{Q}^* = \sum_{j=1}^{M} \sum_{i=1}^{N_j} \tilde{q}_i^{j^*} = \sum_{j=1}^{M} \tilde{Q}_j^*$. Let $\alpha_1 = 0.8\alpha$ and α_2 has to satisfy $\tilde{Q}^* > K$ and

$$\sum_{j=1}^{M} \sum_{i=1}^{N_j} \tilde{\Phi}_i^j(\tilde{\vec{q}}^*) = \sum_{j=1}^{M} \sum_{i=1}^{N_j} \alpha_j \tilde{q}_i^{j^*} = \sum_{j=1}^{M} \alpha_j \tilde{Q}_j^* = K.$$

Then $Q^*, \tilde{Q}^*$ under such choice of α and α_1, α_2 can be computed respectively.