CS 81 (Spring 2018) **Assignment 9** **Name**
Computability and Logic **Due Monday, April 9 at 6:00 PM**

Notice: When you submit this assignment, you are certifying therewith that you understand and accept the following policy, which applies to all assignments.

Collaboration Policy: **The writeup that you submit must be your own work.** You are encouraged to get help from the professor and grutors. You may discuss the problems with classmates, but if you do so, it should be in groups of no more than three. You are not allowed to copy or transcribe solutions from other sources, including the work of other students, the internet, previous solution sets, and images photographed from a whiteboard or blackboard. There is to be no "partnering" where two or more students submit the same writeup. If you get help on a problem, you should say who provided the help on a per-problem basis. Blanket statements such as "worked with John and Mary" are not allowed. Detected infractions may impact your academic career.

Formatting Policy All work must be typeset in electronic media and submitted as a **single pdf file**, one problem on each page as shown in the following pages. Retain this header page. **Handwritten and photographed or scanned work is not allowed.** Do not use inverse video (light typography on dark background). Do not rotate images. **You will not get credit for difficult-to-read submissions.**

Once your document is complete, make a pdf and submit to **Gradescope**.

1. [10 points] Let N be the set of all natural numbers $\{0, 1, 2, 3, \dots\}$. Which of the following sets are *countable* and why?

   a. The set $N \times N$ of all pairs of natural numbers.

   b. The set $N^\star$ of all finite sequences of natural numbers.

   c. The set of all functions of the form $N \to N$.

   d. The set of all Turing transducers that compute total functions $\Sigma^\star \to \Sigma^\star$ (with $\Sigma^\star$ being the set of all finite strings of elements in Σ as usual) where the transducers are coded in a fixed finite alphabet. (A *total function* is a partial function that is defined for all argument values in $\Sigma^\star$. That is, it is just a function.)

   [Hint: Relate the set in question to some other set known to be countable or uncountable.]

2. [10 points] Similar to the way in which $\langle M \rangle$ represents the code for Turing machine M, $\langle M, x \rangle$ represents as a single string a **pair** consisting of $\langle M \rangle$ and an input string x to M. For example, we could use a special character, say \$, outside the alphabet for either as a divider between $\langle M \rangle$ and x, as in $\langle M \rangle \$x$. Then we can construct languages with strings representing such pairs, for example the language

$$\text{Accepts} = \left\{ \langle M, x \rangle \mid M \text{ accepts } x \right\}.$$

Is Accepts decidable, recognizable, corecognizable, or none of these? Prove your answers. [Note: It is fair to assume that a machine either accepts or diverges, as mentioned in the lecture notes.]

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3. [10 points] Similar to the above, except now we consider triples, $\langle M, q, x\rangle$ where q is a specific control state of M. For example, there is a language

$$\text{Reaches} = \left\{\langle M, q, x\rangle \mid M \text{ eventually reaches control state } q \text{ when started on } x\right\}.$$

Is Reaches decidable, recognizable, corecognizable, or none of these? Prove your answers. [Hint: Can a machine be modified to have only a single accepting state and still recognize the same language?]

4. [20 points] For any finite string x, let $|x|$ represent the length of x. We define a Turing machine M to have *prime nature* if the only strings x that M accepts are such that $|x|$ is a prime number. In other words $\{|x| \mid x \in L(M)\} \subseteq$ all prime numbers. As usual, $L(M)$ is the language of strings accepted by M.

   An example of a machine with prime nature is one that decides the language

$$\{\mathtt{1}^p \mid p \text{ is a Mersenne prime}\} = \{\mathtt{111}, \mathtt{1111111}, \dots\}.$$

   ($\mathtt{1}^p$ here representing p $\mathtt{1}$'s in a row). (A Mersenne prime has the special form $2^p - 1$ where p is prime. It is not known whether or not this language is finite, but tests for having this form and for being prime exist, so this language is decidable.)

   Show that the language

$$\mathrm{PN} = \{\langle M \rangle \mid M \text{ is a Turing machine having prime nature}\}$$

   is not recognizable.

   [Hint: Whether an element $\langle M \rangle$ in PN has prime length is not relevant.]

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5. [10 points] Referring to the previous problem, show that the complement PN^{C} is recognizable.

6. [5 points] Define a Turing machine M to have *infinite nature* provide that the language M recognizes is infinite. Show that the language

$$\mathrm{IN} = \{\langle M\rangle \mid M \text{ is a Turing machine having infinite nature}\}$$

is not corecognizable. In other words, show there is not recognizer for

$$\mathrm{IN}^{\mathsf{C}} = \{\langle M\rangle \mid M \text{ is a Turing machine such that } L(M) \text{ is finite}\}.$$

Again, $L(M)$ is the language of strings that M accepts.

[Hint: $\varnothing$ is finite and $\Sigma^\star$ is infinite.]

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7. [15 points] Using the definition of IN in the previous problem, show that

$$\text{IN} = \big\{\langle M \rangle \mid M \text{ is a Turing machine having infinite nature}\big\}$$

is not recognizable.

[Hint: Note similarity to accepting *all* strings and refer to lecture notes.]

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8. [10 points] We show in the lecture that a language is Computably Enumerable iff it is Recognizable. Define a language to be *Monotonically Computably Enumerable* (MCE) provided that the strings of L can be enumerated in increasing order, where order for strings is defined as if each string encodes a natural number in the usual fashion. That is, L is MCE, if there is a Turing transducer computing function f where

$$\forall x \in N \; \forall y \in N \; \big(x < y \text{ implies } f(x) < f(y)\big) \, .$$

Show that an infinite language is MCE iff it is decidable.

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9. [10 points] Give an informal (but nonetheless convincing) algorithm for deciding the following language:

$$\{\langle M\rangle \mid M \text{ is a Turing machine that, when started on all-blank tape, eventually prints something other than a blank}\}.$$

As usual, $\langle M\rangle$ means a string encoding M.
