Notice When you submit this assignment, you are certifying therewith that you understand and accept the following policy, which applies to all assignments.

Collaboration Policy The writeup that you submit must be your own work. You are encouraged to get help from the professor and grutors. You may discuss the problems with classmates, but if you do so, it should be in groups of no more than three. You are not allowed to copy or transcribe solutions from other sources, including the work of other students, the internet, previous solution sets, and images photographed from a whiteboard or blackboard. There is to be no "partnering" where two or more students submit the same writeup. If you get help on a problem, you should say who provided the help on a per-problem basis. Blanket statements such as "worked with John and Mary" are not allowed. Detected infractions may impact your academic career.

Formatting Policy All work must be typeset in electronic media and submitted as a single pdf file, one problem on each page as shown in the following pages. Retain this header page. Handwritten and photographed or scanned work is not allowed. Do not use inverse video (light typography on dark background). Do not rotate images. You will not get credit for difficult-to-read submissions.

For JAPE proofs take a screenshot of your proof and paste it into the document. For written out proofs, just type into a copy of the Google doc master (not the pdf), or use some other method such as LaTeX if you must. You may scrape formulas and symbols directly from the doc master, or from my page full of symbols that I use throughout the course. Once your document is complete, make a pdf and submit to Gradescope.

For tableau proofs, it suggested that you use the spreadsheet method. Please remember to turn off the gridlines from the View menu before taking your screenshot for greater readability.

Once your document is complete, make a pdf and submit to Gradescope.

Please help us grade by placing your solutions in the spaces provided.

[10 points] 1. For the following entailment, use the tableau method to either prove that it is valid or construct a counterexample.

$$\forall x\,(\exists y\, A(x,y) \to \forall z\, A(z,x)) \vDash \forall x\,\forall y\,\forall z\,(A(x,y) \to A(z,x)).$$

[10 points] 2. For the following entailment, use the tableau method to either prove that it is valid or construct a counterexample. (Recall that $\exists x \top$ asserts a non-empty domain, and may be used in the tableau to provide a constant.)

$$\exists x \top \vDash \exists x\,(A(x) \to \exists y\, A(y))$$

[10 points] 3. For the following entailment, use the tableau method to either prove that it is valid or construct a counterexample.

$$\vDash \forall x\,(A(x) \to \forall y\,A(y)).$$

4 & 5. Prenex Rules are used to justify movement of quantifiers in predicate logic formulas. These have great applicability in automated reasoning and theorem proving. You should read about the prenex rules here. In this problem, we want to justify only one of the four basic rules:

$$A(x) \lor \forall y\, B(y) \implies \forall y\, (A(x) \lor B(y))$$

where $\implies$ means "replace with", and $A(x)$ is a formula not having y as a free variable. The justification for this rule involves proving the following two sequents.

4. $A(x) \lor \forall y\, B(y) \vdash \forall y\, (A(x) \lor B(y))$
5. $\forall y\, (A(x) \lor B(y)) \vdash A(x) \lor \forall y\, B(y)$

While the proof of 4 is straightforward, I found it helpful for 5 to use a tableau the first time I did it.

[5 points] 4. Give a natural deduction proof of $A(x) \lor \forall y\, B(y) \vdash \forall y\, (A(x) \lor B(y))$.

[Here x is a free variable on both sides. Treat it as if a constant.]

[15 points] 5. Give a natural deduction proof of $\forall y\,(A(x) \vee B(y)) \vdash A(x) \vee \forall y\, B(y)$.

[Here x is a free variable on both sides. Treat it as if a constant.]

[20 points] 6. As we have seen,

$$\exists x \, \forall y \, L(x, y) \vdash \forall y \, \exists x \, L(x, y),$$

but generally

$$\forall y \, \exists x \, L(x, y) \nvdash \exists x \, \forall y \, L(x, y),$$

even if we add a premise $\exists z \, \top$ to ensure a non-empty domain.

But for the special case where $L(x, y)$ can be factored as $A(x) \vee B(y)$, the second direction $\exists z \, \top, \forall y \, \exists x \, (A(x) \vee B(y)) \vdash \exists x \, \forall y (A(x) \vee B(y))$ is provable.

Show that this is the case by giving a tableau proof.

[10 points] 7. Let f and g be 1-ary function symbols. Any 1-ary function φ is called one-to-one provided that the following formula is satisfied:

$$\forall x\, \forall y\, ((\varphi(x) = \varphi(y)) \to (x = y)).$$

Give a tableau proof of the following:

If f and g are one-to-one, so is the composition of f with g. In other words, prove this entailment

$$\begin{gathered} \forall x\, \forall y\, ((f(x) = f(y)) \to (x = y)), \\ \forall x\, \forall y\, ((g(x) = g(y)) \to (x = y)) \\ \vDash \forall x\, \forall y\, ((f(g(x)) = f(g(y))) \to (x = y)). \end{gathered}$$

[20 points] 8. Recall these group axioms from the lecture notes, where $*$ (group "multiplication") is a 2-ary function symbol in infix notation and $^{-1}$ (group inverse) is a 1-ary function symbol in superscript notation, together with equality rules:

Associative Rule	GA: $\forall x\, \forall y\, \forall z\, (x * y) * z = x * (y * z)$	
Identity Rules	GLI: $\forall x\, 1 * x = x$	GRI: $\forall x\, x * 1 = x$
Inverse Rules	GLV: $\forall x\, x^{-1} * x = 1$	GRV: $\forall x\, x * x^{-1} = 1$

Recall also the following rule for equality and functions:

= Introduction Rule

$$\frac{}{t = t} \quad \text{(no antecedent)} \quad \text{where } t \text{ is any term}$$

= Elimination Rule

$$\frac{s = t \quad A(s)}{A(t)} \quad \text{where } A(x) \text{ is any formula}$$

This allows replacing a term with another term equal to the first.

Function Equality Rule (derivable from the = Elimination Rule)

$$\frac{s_1 = t_1 \quad s_2 = t_2}{s_1 * s_2 = t_1 * t_2} \quad \text{where } s_1, s_2, t_1, t_2 \text{ are any terms}$$

Prove that the inverse function -1 is one-to-one using natural deduction, i.e.

$$\forall x\, \forall y\, \left(x^{-1} = y^{-1} \to x = y\right).$$

Note: The purpose of this problem is to exercise the **form** of a natural deduction proof using axioms and equality rules, rather than the result itself, which is fairly simple. Try not to skip steps. As JAPE natural deduction does not include these function symbols, maybe use a spreadsheet or LaTeX for formatting.

[20 points] 8. Proof.