Notice When you submit this assignment, you are certifying therewith that you understand and accept the following policy, which applies to all assignments.

Collaboration Policy The writeup that you submit must be your own work. You are encouraged to get help from the professor and grutors. You may discuss the problems with classmates, but if you do so, it should be in groups of no more than three. You are not allowed to copy or transcribe solutions from other sources, including the work of other students, the internet, previous solution sets, and images photographed from a whiteboard or blackboard. There is to be no "partnering" where two or more students submit the same writeup. If you get help on a problem, you should say who provided the help on a per-problem basis. Blanket statements such as "worked with John and Mary" are not allowed. Detected infractions may impact your academic career.

Formatting Policy All work must be typeset in electronic media and submitted as a single pdf file, one problem on each page as shown in the following pages. Retain this header page. Handwritten and photographed or scanned work is not allowed. Do not use inverse video (light typography on dark background). Do not rotate images. You will not get credit for difficult-to-read submissions.

For JAPE proofs take a screenshot of your proof and paste it into the document. For written out proofs, just type into a copy of the Google doc master (not the pdf), or use some other method such as LaTeX if you must. You may scrape formulas and symbols directly from the doc master, or from my page full of symbols that I use throughout the course. Once your document is complete, make a pdf and submit to Gradescope.

For tableau proofs, it suggested that you use the spreadsheet method. Please remember to turn off the gridlines from the View menu before taking your screenshot for greater readability.

Once your document is complete, make a pdf and submit to Gradescope.

Please help us grade by placing your solutions in the spaces provided.

These problems are formal proofs about the natural numbers. Use natural deduction and the axioms of Peano Arithmetic (PA) from class, including induction where needed. Here 0 is the constant symbol and 1 is defined as $0'$, where $'$ is the successor function.

You may use any results proved in the class slides, but you should cite anything you use by label if possible. Don't forget to use the $\forall$ Elimination rule when you plug into those axioms and lemmas. Cite what you are plugging in, e.g. $[a+b/x]$ means $a+b$ is being substituted for x.

[5 points] 1. Formally prove, using natural deduction,

$$\forall y \, y + 1 = y'.$$

[5 points] 2. Formally prove, using natural deduction,

$$\forall y\, 1 \times y = y.$$

[10 points] 3. Formally prove, using natural deduction,

$$\forall y \, y \times 1 = y.$$

[5 points] 4. (Basis only) Formally prove, using natural deduction, that $\times$ distributes over $+$, i.e.

$$\forall x\,\forall y\,\forall z\; x \times (y + z) = (x \times y) + (x \times z).$$

You may use any results proved in the slides or in previous problems as lemmas. For this problem, induction is required. So you will be showing

$$\forall x\, A(x) \quad \text{where} \quad A(x)\colon \forall y\,\forall z\; x \times (y + z) = (x \times y) + (x \times z).$$

In this problem, only show the basis of the proof. The induction step is in the next problem.

$$A(0)\colon \forall y\,\forall z\; 0 \times (y + z) = (0 \times y) + (0 \times z).$$

[25 points] 5. (Induction step only) Formally prove, using natural deduction, that $\times$ distributes over $+$, i.e.

$$\forall x\, \forall y\, \forall z\; x \times (y + z) = (x \times y) + (x \times z).$$

You may use any results proved in the slides or in previous problems as lemmas. For this problem, induction is required. So you will be showing

$$\forall x\, A(x) \quad \text{where} \quad A(x)\colon \forall y\, \forall z\; x \times (y + z) = (x \times y) + (x \times z).$$

In this problem, only show the induction step: $A(a) \vdash A(a')$, i.e.

$$\forall y\, \forall z\; a \times (y + z) = (a \times y) + (a \times z) \vdash \forall y\, \forall z\; a' \times (y + z) = (a' \times y) + (a' \times z).$$

[25 points] 6. One way to introduce a new predicate symbol representing a concept into a theory is to treat formulas using the symbol as abbreviations for other formulas. In this problem we define an abbreviation for $\leq$ (less-than-or-equal):

For any terms s and t, $s \leq t$ means $\exists w\, s + w = t$.

For example $3 + 2 \leq 4 + 3$ means $\exists w\, (3 + 2) + w = (4 + 3)$.

Give a natural deduction proof for the following derived inference rule:

$$\frac{r \leq s \quad s \leq t}{r \leq t} \quad \text{where } r, s, t \text{ are any terms.}$$

(Treat the antecedent formulas as premises, and the consequent as the conclusion. Treat r, s, and t as if constants.)

[25 points] 7. For the $\leq$ symbol defined in the previous problem, prove that $\leq$ is antisymmetric:

$$\forall x\, \forall y\, ((x \leq y) \land (y \leq x) \rightarrow x = y)$$

(It might be helpful to invent and prove a lemma.)