Homework 10

Math 167 / CS 142: Complexity Theory

1. Consider the tree underlying a logical formula; that is, a tree in which every internal vertex has degree one or two; and regard each edge as being directed toward the root. Recall that the *pebble number* of such a tree is the smallest number of pebbles sufficient to allow a pebble to be placed on the root, with the following rules: 1. a pebble may be placed on a vertex provided all immediate predecessors of that vertex currently have pebbles on them, and 2. a pebble may be removed from a vertex at any time. Show that there is a polynomial-time algorithm for determining the pebble number of such a tree.

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2. Show that in the unbounded-fan-in model for circuits (gates with fan-in m contribute m to the size, but only 1 to the depth), the parity function $f(x_1, \ldots, x_n) = x_1 \oplus \cdots \oplus x_n$ can be computed by circuits with polynomial size and depth $\mathcal{O}(\log n / \log \log n)$.

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3. Show that the majority function

$$f(x_1, \ldots, x_{2m+1}) = \begin{cases} 1, & \text{if at least } m+1 \text{ of the inputs are 1s,} \\ 0, & \text{otherwise,} \end{cases}$$

can be computed by circuits (ordinary circuits, with fan-in two) of depth $\mathcal{O}(\log(2m+1))$. (Hint: Let $2m+1 = 2^k - 1$; divide and conquer!)
