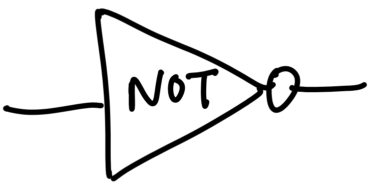
Making circuits out of stuff

KYE, 4/19/22.

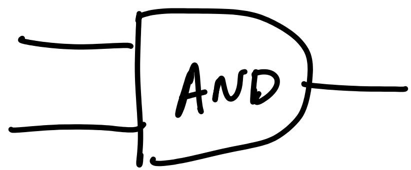
CIRCUITS

NOT $x = 1$ iff:
$x = 0$



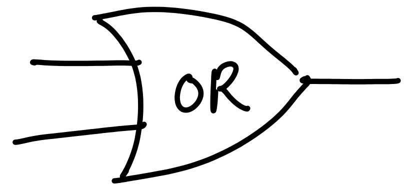
x	NOT x
0	1
1	0

x AND $y = 1$ iff:
x, y both 1



x	y	x AND y
0	0	0
0	1	0
1	0	0
1	1	1

x OR $y = 1$ iff:
at least one of $x, y = 1$



x	y	x OR y
0	0	0
0	1	1
1	0	1
1	1	1

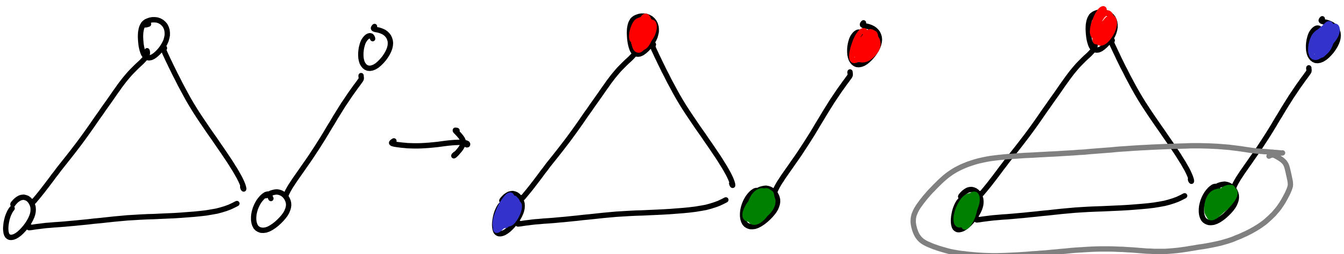
GRAPH 3-COLORINGS

Coloring: a map

κ : vertices $\to \{0, 1, 2\}$

κ is **proper** iff:

$\forall$ neighbors (u, v), $\kappa(u) \neq \kappa(v)$.



proper :)

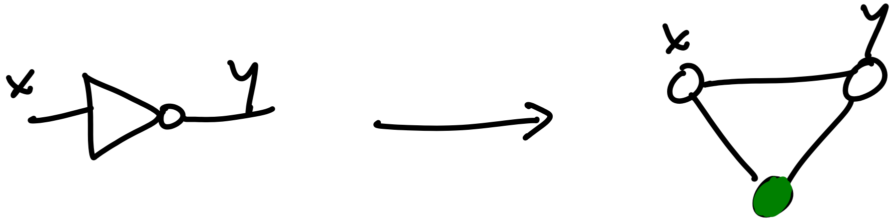
improper :/

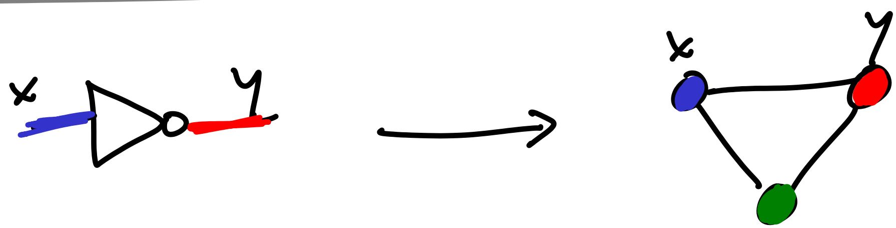
QUESTION.

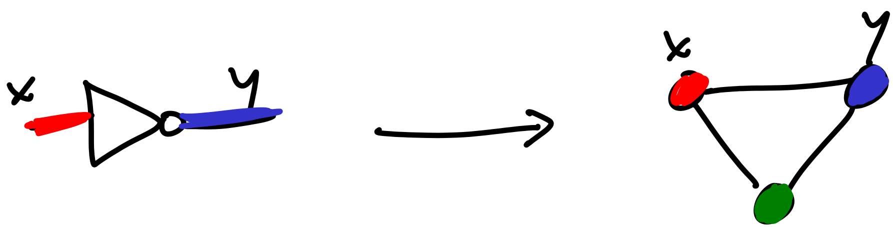
Can we emulate circuits using graph colorings?

EXAMPLE: NOT gate.

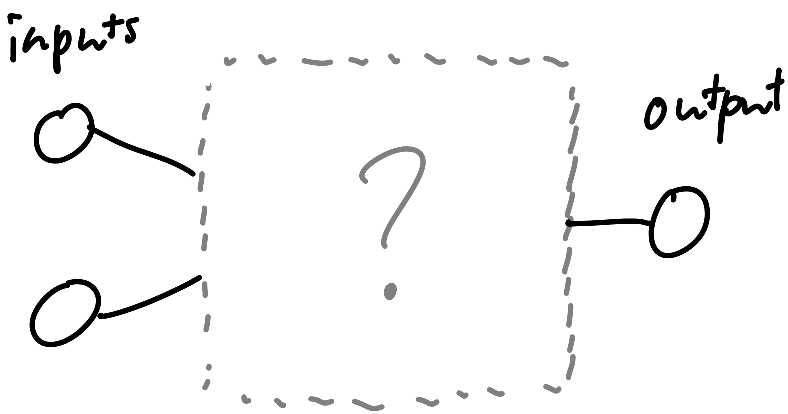
 = 0    = 1   ( is a "helper" color)







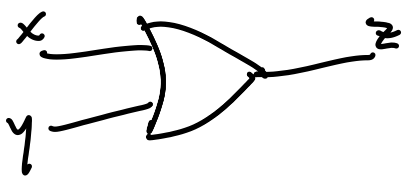
GOAL: using ● = 0 and ● = 1, construct graphs whose inputs force outputs according to boolean rules

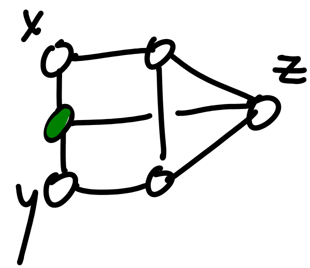


Can we make an OR gate?

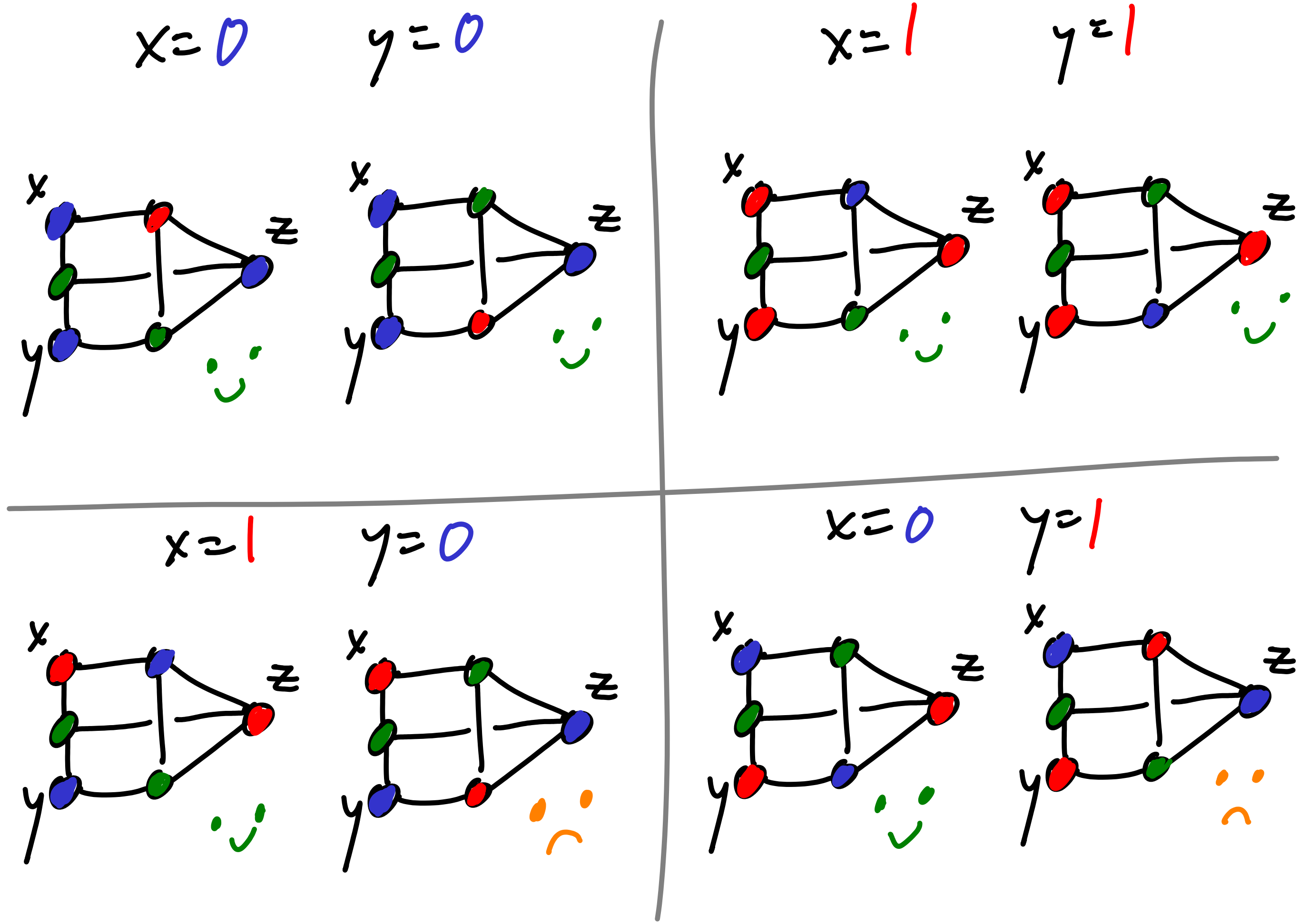
GOAL: get z's color to always be x OR y

Last time I said:



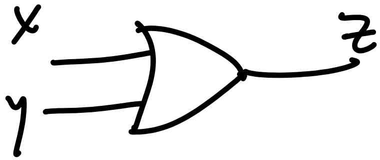


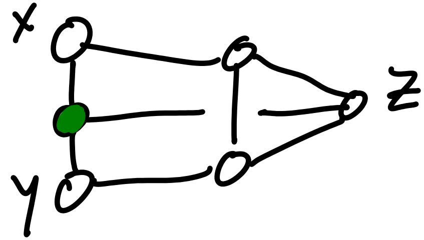
WRONG! WRONG! WRONG! Why?



x=0 y=0
x=1 y=1
x=1 y=0
x=0 y=1
x
y
z

"fake OR" gate





x	y	z
0	0	0
0	1	1
1	0	1
1	1	1

must be 1 iff x OR y

x	y	z
0	0	0
0	1	0/1
1	0	0/1
1	1	1

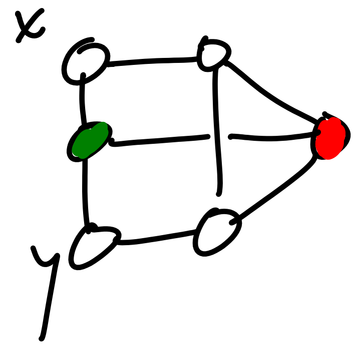
can be 1 iff x OR y

Alternate idea: use multiple fake-OR gates to constrain possible relations between input & output.

What does this do?

x	y	
~~0~~	~~0~~	~~0~~
0	1	0/1
1	0	0/1
1	1	1

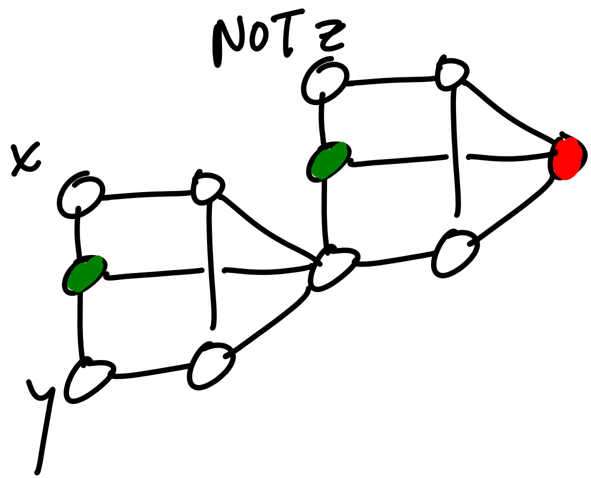
impossible!



colorable iff x OR y

Also: $x=0 \Rightarrow y=1$ $\quad$ $y=0 \Rightarrow x=1$

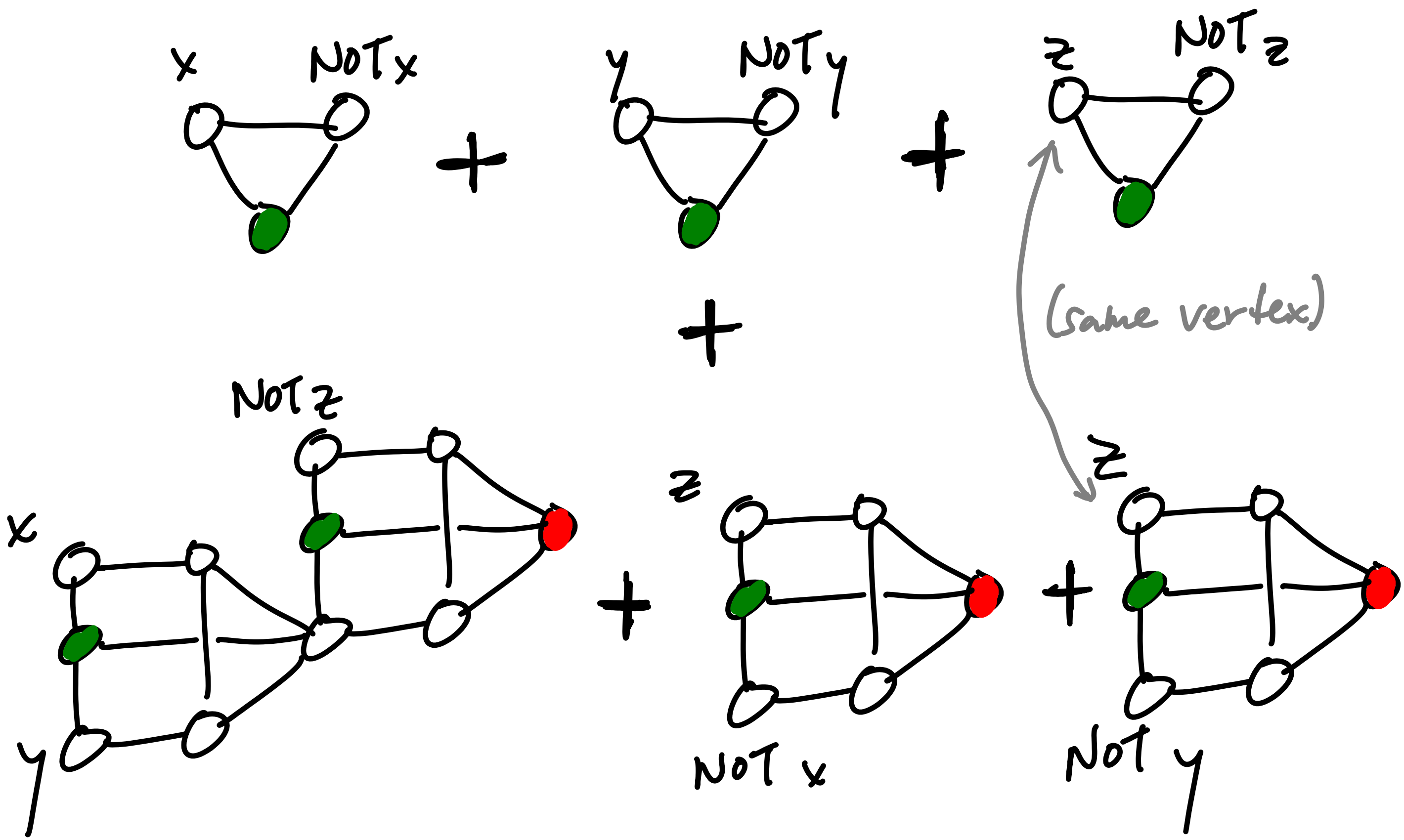
What about this?

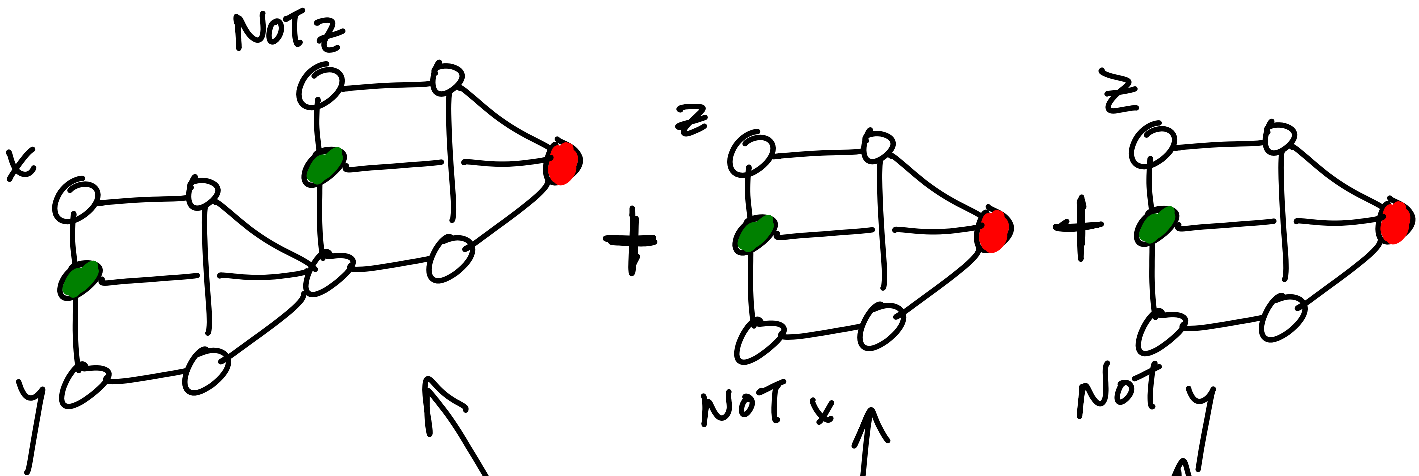


colorable iff (NOT z) OR (x OR y)

$z = 1 \Rightarrow x \text{ OR } y = 1$

The OR-gate redemption

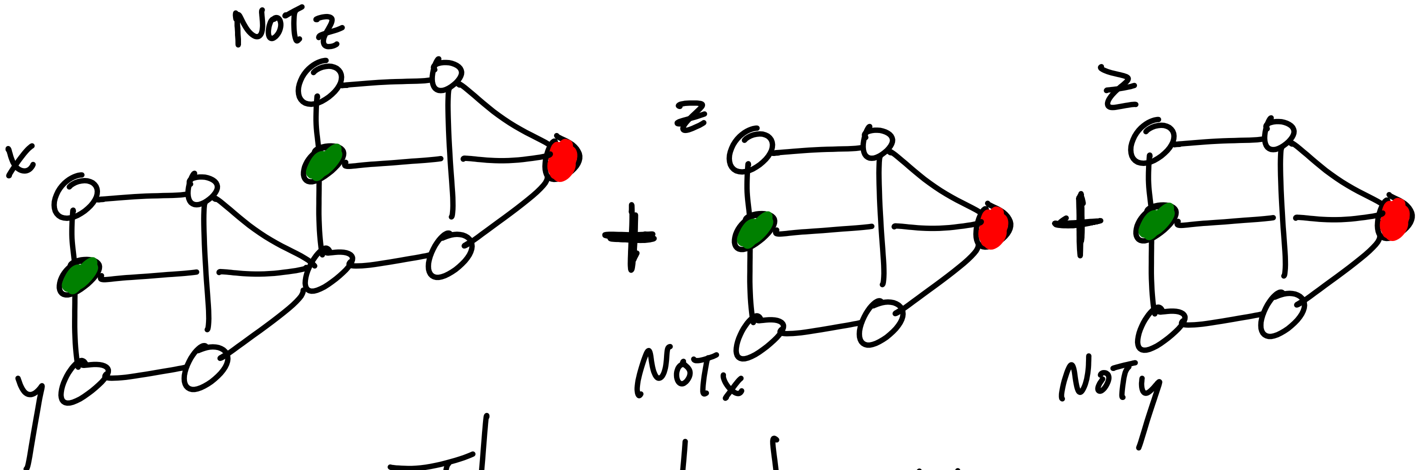




If both x, y = 0: forces z = 0

If x = 1: forces z = 1

If y = 1: forces z = 1



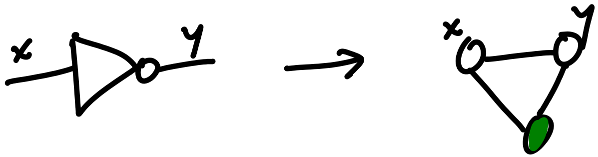
This works because:

$z = x \text{ OR } y \iff$ (NOT z OR x OR y) AND (z OR NOT x) AND (z OR NOT y)

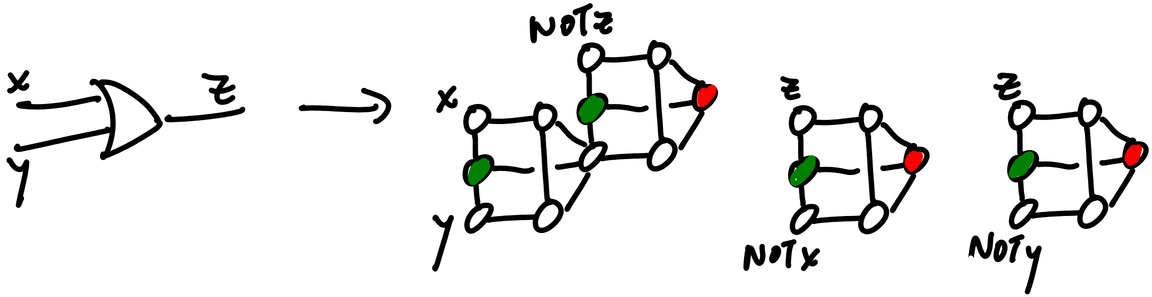
which you can derive by FOIL'ing!

(which means you could do it too :D)

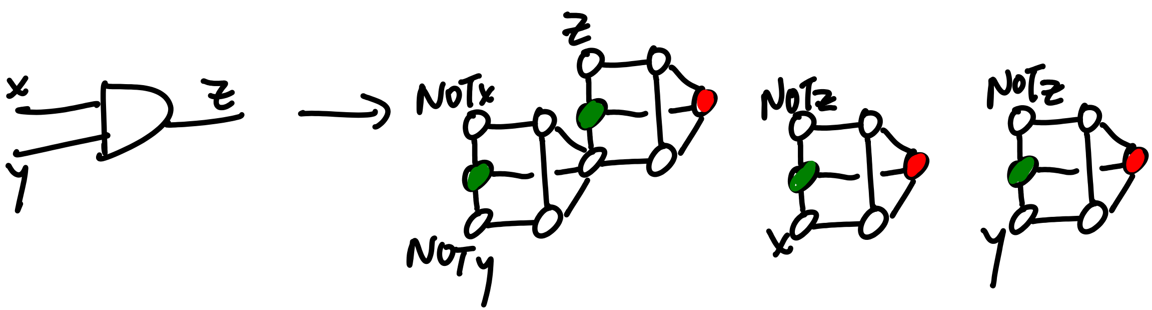
NOT:



OR:



AND:



CIRCUIT & GRAPH GAMES

Circuit game:

two players A, B take turns assigning inputs.

A wins if final output = 1, B wins if output = 0.

Graph coloring game:

two players A, B take turns coloring vertices, following properness constraints.

A wins if B has no proper moves, vice versa.

If all turns played, A wins.

PUZZLES & GAMES

k-turn circuit games: $\Sigma_k P$ - complete

$$\Downarrow$$

k-turn graph coloring games: also $\Sigma_k P$-complete!