Some Results on FX-related Options with Dual Interest Rates

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1 Abstract

In this paper, we consider foreign exchange options with the presence of both a domestic and a foreign interest rate. This set-up also allows us to interpret the underlying asset as a stock with dividends, a commodity with storage costs, etc. We investigate the existence of exercise boundary values for various option derivatives, beyond which the option should be optimally exercised. Since the option price is exactly its intrinsic value beyond the boundary, we also derive pricing formulas for the value of the option before it is to be optimally exercised. We consider option prices under both binomial and perpetual models and show their equivalence. Our results reveal an alternative method to smooth pasting of pricing options as well as interesting behavior at certain boundaries for some interesting options.

2 Introduction

The central question when one holds an option is whether and when to exercise it. Intuitively, given a non-dividend paying American call option on a stock, under a positive domestic interest rate environment r_d, the option will never be exercised. This is because by exercising the call option early, the holder is paying more for the time value of the strike price (Kwok, 2008) [1]. However, when there is a constant dividend yield, which we will call r_f, one might speculate the existence of an upper boundary φ such that beyond that point, the dividends received from the high stock price more than compensate for the higher value of the strike price paid to exercise the option.

The analogous situation in FX options is clear. For instance, one might hold a call option to buy Great British Pound (GBP), the asset currency, using United States Dollars (USD), the numéraire. When the option is not exercised, the holder is earning risk-free interest at a domestic rate r_d on USD, presumably at an American financial institution. However, once the option is exercised, the holder is now earning interest at the foreign rate r_f on the newly owned GBP instead.

The stationary Black-Scholes Ordinary Differential Equation (ODE) is a popular model for studying such options in perpetual time, given by

$$\frac{1}{2}\sigma^2 S^2 V''(S) + (r_d - r_f) \cdot S \cdot V'(S) - r_d \cdot V(S) = 0 \tag{1}$$

Other authors have previously studied perpetual cases based on this equation (see Gerber and Shiu (1994) [2], Grossinho et al [3] etc). Our addition to the literature is three-fold. First, the common approach to deriving an upper boundary φ and a lower boundary θ is to first assume that a boundary exists, using intuition similar to that described above, and then finding it by a method called *smooth-pasting.* This method works for continuous intrinsic value functions $g(S)$, but in a later section on what we call tent options, where the intrinsic value is not everywhere differentiable, we shall see that this logic is less robust. Our first contribution is to formulate the option price as a function of both the underlying price S and the applicable boundary φ. We consider φ to be a variable and proceed with a first-order maximization of $V(S, \varphi)$ across all applicable φ. Each φ under consideration can be understood as a barrier option that pays $g(\varphi)$ the first time the stock moves from K to φ.

Our second contribution is to broaden the scope of discussion in option derivatives. We started with ordinary American puts and calls that have constant volatility. We then branched out to the Constant Elasticity of Variance (CEV) model, where $\sigma(s) = \gamma\sqrt{s}$. Next, we considered *restricted straddles*, given by $g(S) = |S - K|$, and *power options*, with a non-zero exponent q. Lastly, we examined a *restricted American Butterfly* with positive width d, otherwise known as the *tent option* due to the shape of its intrinsic value function. We deduced formulas relating φ and θ to K, r_d, r_f, σ and $V(S)$ for all of the above options, as well as a more general Theorem for perpetual options. Below are the payoffs of the aforementioned options:

$$g(S) = \begin{cases} (S-K)^q, & S > K \\ 0, & 0 \leq S \leq K \end{cases} \qquad \text{for a power call,}$$

$$g(S) = \begin{cases} 0, & S \geq K \\ (K-S)^q, & 0 \leq S \leq K \end{cases} \qquad \text{for a power put,}$$

$$g(S) = \begin{cases} (|S-K|)^q, & S \geq 0 \end{cases} \qquad \text{for a power straddle,}$$

$$g(S) = \begin{cases} 0, & S \geq K + d \\ (K+d) - S, & K + d > S \geq K \\ S - (K-d), & K > S > K - d \\ 0, & K - d \geq S \end{cases} \qquad \text{for a tent option.}$$

Our third contribution is the exploitation of numerical techniques to approximate problems in perpetual time where analytical solutions are not readily available. In almost all cases, a solution of the binomial model tends to the perpetual solution when time to maturity $T \to \infty$. This helped us postulate ϕ and θ for a straddle option with positive interest rates, which we then substituted into (1) to apply Euler's Method. The main exception was with negative interest rates, which is possible in finite time but gives rise to possible arbitrage opportunities in perpetual options. In those cases, we state and prove results that are only valid in finite time. Lastly, we will discuss identities that are time-independent even in a finite time setting.

The structure of the paper is as follows: we first outline the maximization problem that motivates a study of φ and θ in Section 3. American "pure" puts and calls (with power $q = 1$) are looked at in Section 4, followed by power calls, power puts and straddles in Section 5. Section 6 states and proves the general Theorem, Section 7 turns to a Binomial model, and Section 8 tackles the tent option.

3 The Maximization Problem

In a discrete binomial model, we are able to succinctly describe this decision-making process in binomial time. To do so, we assume a risk-neutral measure $\widetilde{\mathbb{E}}$. Given values for r_d, r_f, u, d, the risk-neutral probabilities are $\tilde{p} = \frac{\frac{1+r_d}{1+r_f} - d}{u-d}$ and $\tilde{q} = \frac{u - \frac{1+r_d}{1+r_f}}{u-d}$ (Shreve, 2012) [4]. Since the option is finite-time and discrete, the option price $V_T(S_T)$ at a time just before expiry is exactly its intrinsic value $g(S_T)$. By the process of backward induction, we can determine $V_{T-1}(S_{T-1})$ by taking the *maximum* of the value of exercising the option, $g(S_{T-1})$, and the value of holding the option, $\frac{1}{1+r_d}\widetilde{\mathbb{E}}_{T-1}[g(S_T)] = \frac{1}{1+r_d}[\tilde{p} \cdot g(u \cdot S_{T-1}) + \tilde{q} \cdot g(d \cdot S_{T-1})]$. Written another way, an option should not be exercised if $\frac{1}{1+r_d}\widetilde{\mathbb{E}}[S_{t+1}] \geq g(S_t)$, which implies that the value of holding is at least as large as the exercise value. Hence, before the exercise boundary is reached, $V(S_t) = \frac{1}{1+r_d}\widetilde{\mathbb{E}}[S_{t+1}]$. Once the boundary is reached, $V(S) = g(S)$.

To go from a discrete binomial setting to continuous time setting, we consider the step size $h = \frac{T}{N}$, where T is the maturity of the option, and N is the number of periods in this model. We then take a limit of $V(S_t) = \frac{1}{1+r_d}\widetilde{\mathbb{E}}[S_{t+1}]$ as $h \to 0$ and assume that the perpetual option's price is time independent to get Equation (1) above. We guess that at least one solution to this equation will have the form s^p, and by substitution, we obtain the following equation for the exponent p:

$$\frac{1}{2}\sigma^2 p^2 + (r_d - r_f - \frac{1}{2}\sigma^2)p - r_d = 0 \tag{2}$$

Before proceeding to manipulate equation 2, we shall state and prove the following lemma that will guide our intuition for perpetual options:

Lemma 3.1. *Given a* ***finite*** *call-like option with non-decreasing and convex payoff function g s.t. $g(S) = 0\ \forall\ 0 \leq S \leq K$ (with $K > 0$), , under $1 > r_d > 0$ and $r_f = 0$ it is always optimal to hold the option to maturity rather than exercise it.*

Proof. Since g is convex, we have $g(t \cdot k_1 + (1-t) \cdot k_2) \leq t \cdot g(k_1) + (1-t) \cdot g(k_2)\ \ \forall\ 0 \leq t \leq 1, k_1, k_2 \geq 0$. Now let $t = \frac{1}{1+r_d}$. Since $1 > r_d > 0$, $0 < t < 1$. Choose $k_2 = 0$. Since $g(0) = 0$, g being convex means that $g(t \cdot k_1) \leq t \cdot g(k_1)$. Making the substitution $t = \frac{1}{1+r_d}$, we know $g(\frac{k_1}{1+r}) \leq \frac{1}{1+r} g(k_1)$. Bearing in mind Jensen's Inequality for convex functions ϕ, i.e. $\phi(\mathbb{E}(Y_n)) \leq \mathbb{E}(\phi(Y_n))$, and setting $k_1 = \widetilde{\mathbb{E}}[S_{t+1}]$, we see that

$$\begin{aligned} g(S_t) &= g(\frac{1}{1+r}\widetilde{\mathbb{E}}[S_{t+1}]) \\ &\leq \frac{1}{1+r} g(\widetilde{\mathbb{E}}[S_{t+1}]) \quad \text{(from above)} \\ &\leq \frac{1}{1+r}\widetilde{\mathbb{E}}[g(S_{t+1})] \quad \text{(Jensen's Inequality)} \end{aligned}$$

$\square$

Clearly, A non-dividend paying ($r_f = 0$) **finite** vanilla call option will never be exercised early. We also have the following corollary, which we revisit in Section 6:

Corollary 3.1.1. *A non-dividend paying* ***perpetual*** *call option has no optimal exercise boundary φ.*

4 American Perpetual Calls & Puts

4.1 Constant Volatility

When $r_d > 0$, $r_f > 0$, by solving Equation (2), we can get that $p = \frac{\frac{1}{2}\sigma^2 - r_d + r_f \pm \sqrt{(r_d - r_f - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r_d}}{\sigma^2}$. For the two solutions, WLOG let $p_1 \geq p_2$, then $p_1 > 0$ and $p_2 < 0$ here. So the value of option has the form $V(S) = c_1 S^{p_1} + c_2 S^{p_2}$.

(*a*) Call Option (Let φ represent the optimal exercise value for a Call):

When $S \to 0$, $V(S) \to 0$, but the term $c_2 S^{p_2}$ approaches infinity because $p_2 < 0$. So to avoid the term to blow up, we get that $c_2 = 0$.

By solving the equation $V(\varphi) = c_1 \varphi^{p_1} = \varphi - K$,

we get $c_1 = \frac{\varphi - k}{\varphi^{p_1}}$. Define $f(\varphi) = c_1 = \frac{\varphi - k}{\varphi^{p_1}}$

Now $V(S) = \frac{\varphi - k}{\varphi^{p_1}} S^{p_1}$, and we want to optimize φ in order to maximize the option value for a fixed and arbitrary S, so we only consider the coefficient c_1 here. We first consider the critical points where $f'(\varphi) = 0$.

$f'(\varphi) = \frac{\varphi^{p_1} - (\varphi - K) p_1 \varphi^{p_1 - 1}}{\varphi^{2p_1}} = \frac{1 - \frac{(\varphi - K)p_1}{\varphi}}{\varphi^{p_1}} = 0$

Since $\varphi^{p_1} \neq 0$, we get $\frac{(\varphi - K)p_1}{\varphi} = 1$, so $\varphi(p_1 - 1) = K p_1$, $\varphi = \frac{K p_1}{p_1 - 1} = \frac{K}{1 - \frac{1}{p_1}}$.

$$V(S) = \begin{cases} S - K, S > \varphi \\ (\varphi - K)(\frac{S}{\varphi})^{p_1}, S \leq \varphi \end{cases}$$

(*b*) Put Option (Let θ represent the optimal exercise value for a put):

When $S \to \infty$, $V(S)$ should stay bounded, but the term $c_1 S^{p_1}$ approaches infinity because $p_1 > 0$. So we set $c_1 = 0$.

By solving the equation $V(\theta) = c_2 \theta^{p_2} = K - \theta$, we get $c_2 = \frac{K-\theta}{\theta^{p_2}}$. Define $h(\theta) = c_2 = \frac{K-\theta}{\theta^{p_2}}$.

$V(S) = \frac{K-\theta}{\theta^{p_2}} S^{p_2}$, and we want to optimize θ in order to maximize the option value for a fixed and arbitrary S, so we only consider the coefficient c_2 here. We first consider the critical points where $h'(\theta) = 0$.

$h'(\theta) = \frac{-\theta^{p_2} - (K-\theta)p_2\theta^{p_2-1}}{\theta^{2p_2}} = \frac{-1-\frac{(K-\theta)p_2}{\theta}}{\theta^{p_2}} = 0$

Since $\theta^{p_2} \neq 0$, we get that $\frac{(\theta-K)p_2}{\theta} = 1$, so $\theta(p_2 - 1) = Kp_2$, $\theta = \frac{K}{1-\frac{1}{p_2}}$.

$V(S) = \begin{cases} K - S, S < \theta \\ (K-\theta)(\frac{S}{\theta})^{p_2}, S \geq \theta \end{cases}$

Here we have relied on maximization over φ and θ, rather than the traditional smooth-pasting method. Had we used smooth-pasting on a differentiable $V(S)$ we would have $V(\varphi) = \varphi - K = c_1\varphi^{p_1}$ and $V'(\varphi) = c_1 \cdot p_1 \cdot \varphi^{p_1-1} = 1$. Then $c_1 \cdot \varphi^{p_1} \cdot \frac{p_1}{\varphi} = (\varphi - K) \cdot \frac{p_1}{\varphi} = 1$. Solving for φ, $\varphi(p_1 - 1) = K \cdot p_1$ so $\varphi = \frac{K}{1-\frac{1}{p_1}}$, as we had found in part *(a)*. Thus, the two methods are equivalent wherever $V(S)$ is differentiable. Because of its greater flexibility, we will henceforth use the maximization method over smooth-pasting in our derivations whenever possible.

4.2 Relation between φ and θ

In subsection 4.1, we have shown that $\varphi = \frac{K}{1-\frac{1}{p_1}}$ and $\theta = \frac{K}{1-\frac{1}{p_2}}$. Since both p_1 and p_2 satisfy Equation (2), we can derive the following relation between the two numbers:

$$\varphi\theta = \frac{K^2}{1 - \frac{1}{p_1} - \frac{1}{p_2} + \frac{1}{p_1 p_2}} = \frac{K^2}{1 - \frac{p_1+p_2-1}{p_1p_2}} = \frac{k^2}{1 + \frac{\frac{\frac{1}{2}\sigma^2 - r_d + r_f}{\frac{1}{2}\sigma^2} - 1}{\frac{r_d}{\frac{1}{2}\sigma^2}}} = \frac{K^2}{1 + \frac{r_f - r_d}{r_d}} = K^2\frac{r_d}{r_f}$$

(We used Vieta's fomula to relate the sum and product of p_1 and p_2 to the equation's coefficients : if a quadratic equation $ap^2 + bp + c = 0$ has roots p_1, p_2 with $p_1 \geq p_2$, then $p_1 + p_2 = -\frac{b}{a}$ and $p_1p_2 = \frac{c}{a}$).

4.3 Constant Elasticity of Variance (CEV)

When $\sigma(S) = \gamma\sqrt{S}$, $\gamma \in \mathbb{R}^+$ constant, Equation (1) is generally hard to solve analytically. In this subsection, we will consider the special case $r_f = 0$, and in the following subsection the case $r_d = r_f$.

When $r_f = 0$ (no dividends), $r_d = r > 0$, $\sigma^2(S) = \gamma^2 S$, we have

$$\frac{1}{2}V''(S)\sigma^2S^3 + rSV'(S) - rV = 0$$

Clearly, $V_1(S) = S$ is a particular solution to this equation. To find a second linearly independent solution, we use the method of *reduction of order* by guessing $V_2(S) = V_1(S) \cdot w(S)$ for some function $w(S)$ to be determined. Then

$$\frac{1}{2}\gamma^2(Sw''(S)+2w'(S))S^3+rS(Sw'(S)+w(S))-rSw(S)=0$$
$$\frac{1}{2}\gamma^2 s^4 w''(S)+(\gamma^2S^3)w'(S)+rs^2w'(S)=0$$
$$\frac{1}{2}s^2w''(S)=-(s\gamma^2+r)w'(S)$$
$$\frac{w''(S)}{w'(S)}=\frac{-(s\gamma^2+r)}{\frac{1}{2}\gamma^2s^2}=-\frac{2}{S}-\frac{(\frac{2r}{\gamma^2})}{S^2}$$
Integrating both sides with respect to S,
$$ln(w'(S))=-2ln(S)+\frac{(\frac{2r}{\sigma^2})}{S^2}$$
$$w'(S)=\frac{C}{S^2}e^{\frac{2r}{\gamma^2S}}$$
$$w(S)=Ce^{\frac{2r}{\gamma^2s}}+D$$

Hence, $V_2(S)=S\cdot e^{\frac{2r}{\gamma^2S}}$, and $V(S)=c_1Se^{\frac{2r}{S}}+c_2S$. Now, note that with the substitution $y=\frac{1}{x}$, we can show that

$$\lim_{x\to 0} xe^{\frac{1}{x}}=\lim_{y\to\infty}\frac{e^y}{y}\geq\lim_{y\to\infty}\frac{\frac{y^2}{2}}{y}=\infty$$

Therefore, $\lim_{S\to 0}Se^{\frac{2r}{S}}=\infty$. We conclude that with a CEV *Call Option*, $c_1=0$ and $V(S)=c_2S$. So

$$V_{call}(S)=\begin{cases}S-K, S>\varphi\\ (\varphi-K)(\frac{S}{\varphi})^{p_1}, S\leq\varphi\end{cases}$$

With a CEV *Put Option*, the relevant boundary condition is that $V(\infty)$ remains bounded. Notice that $\lim_{S\to\infty}Se^{\frac{2r}{S}}=S\cdot e^0=S$, so the only way to keep $V(S)$ bounded is for the terms '$c_1Se^{\frac{2r}{S}}$' and 'c_2S' to cancel each other out, i.e. $c_1+c_2=0$. Hence $c_2=-c_1$, and $V(S)=c_1S(e^{\frac{2r}{S}}-1)$. Since $V(\theta)=K-\theta$,

$$K-\theta=c_1\theta(e^{\frac{2r}{\gamma^2\theta}}-1)$$
$$c_1=\frac{K-\theta}{\theta(e^{\frac{2r}{\gamma^2\theta}}-1)}$$

$$\therefore V_{put}(S)=\begin{cases}K-S, & S<\theta\\ \frac{K-\theta}{\theta(e^{\frac{2r}{\gamma^2\theta}}-1)}S(e^{\frac{2r}{S}}-1)=(K-\theta)\left[\frac{S(e^{\frac{2r}{\gamma^2S}}-1}{\theta(e^{\frac{2r}{\gamma^2\theta}}-1)}\right], & S\geq\theta\end{cases}$$

Notice the similarity of $V_{put}(S)$ to that found in 4.1. We will now proceed to solve for θ by maximizing $V(S,\theta)$ over θ:

$$V(S,\theta) = (K-\theta)\Big[\frac{S(e^{\frac{2r}{\gamma^2 S}}-1)}{\theta(e^{\frac{2r}{\gamma^2 \theta}}-1)}\Big]$$

Since we are taking derivatives w.r.t θ, we can write $S(e^{\frac{2r}{\gamma^2 S}}-1)$ as a constant term C:

$$V'(\theta) = C\big[-(\frac{1}{\theta(e^{\frac{2r}{\gamma^2\theta}}-1)}) + (K-\theta)(-\theta^{-2}(e^{\frac{2r}{\gamma^2\theta}}-1)^{-1} + \theta^{-1}(-(e^{\frac{2r}{\gamma^2\theta}}-1)^{-2})(-\frac{2r}{\gamma^2\theta^2}e^{\frac{2r}{\gamma^2\theta}}))\big]$$

$$V'(\theta) = C\big[-\frac{1}{\theta(e^{\frac{2r}{\gamma^2\theta}}-1)} + \frac{K-\theta}{\theta(e^{\frac{2r}{\gamma^2\theta}}-1)}(-\frac{1}{\theta} + \frac{2r}{\gamma^2\theta^2}\frac{e^{\frac{2r}{\gamma^2\theta}}}{e^{\frac{2r}{\gamma^2\theta}}-1})\big]$$

$$V'(\theta) = 0 \iff -\frac{K}{\theta} + \frac{2r(K-\theta)}{\gamma^2\theta^2}\frac{e^{\frac{2r}{\gamma^2\theta}}}{e^{\frac{2r}{\gamma^2\theta}}-1} = 0$$

Substituting $K = 10, \gamma = 1$, $r = 0.04$, we get $\theta = 0.619$ and $V(S) \approx 109.823 \cdot S(e^{\frac{0.08}{s}} - 1)$. Of note is that V(S) tends to a non-zero finite value as $S \to \infty$. Using a Taylor Series expansion,

$$\begin{aligned}\lim_{S\to\infty} V_{put}(S) &\approx \lim_{S\to\infty} 109.823 \cdot S(e^{\frac{2r}{s}} - 1) \\ &= \lim_{S\to\infty} 109.823 \cdot S\big[(\frac{0.08}{S}) + (\frac{1}{2!})(\frac{0.08}{S})^2 + (\frac{1}{3!})(\frac{0.08}{S})^3 + \dots\big] = 109.823 \cdot 0.08 = 8.786.\end{aligned}$$

4.4 Bessel Solutions

If we instead assume that $\sigma(S) = \sqrt{S}$, and $r_d = r_f = r > 0$, we have the following form of Equation (1):

$$\frac{1}{2}(S)(S^2)V''(S) + (0)SV'(S) - rV(S) = 0$$

$$\frac{1}{2}S^3V''(S) - rV(S) = 0$$

$$\therefore \frac{1}{2}S^2V''(S) - \frac{r}{S}V(S) = 0 \tag{3}$$

The solution to this equation can be expressed in terms of Modified Bessel functions. If $r_d = r_f = 0.04$, we get via `Mathematica` (`BesselI` is the modified Bessel Function of the First Kind, `BesselK` is the modified Bessel Function of the Second Kind):

$$V(S) = c_1 \cdot \texttt{BesselI}[1, \frac{2\sqrt{2}}{5\sqrt{S}}]\sqrt{S} + c_2 \cdot \texttt{BesselK}[1, \frac{2\sqrt{2}}{5\sqrt{S}}]\sqrt{S}$$

By a similar logic to that in Section 4.1, we can eliminate one of the two constants to derive the following forms of V(S) for the pure call and pure put respectively:

$$\texttt{Call:}\quad V(S) = c_2 \cdot \texttt{BesselK}[1, \frac{2\sqrt{2}}{5\sqrt{S}}]\sqrt{S}$$

$$V(\varphi) = c_2 \cdot \texttt{BesselK}[1, \frac{2\sqrt{2}}{5\sqrt{\varphi}}]\sqrt{\varphi} = \varphi - K$$

$$c_2 = \frac{\varphi - K}{\texttt{BesselK}[1, \frac{2\sqrt{2}}{5\sqrt{\varphi}}]\sqrt{\varphi}}$$

$$\therefore V(S,\varphi) = (\varphi - K)\frac{\texttt{BesselK}[1, \frac{2\sqrt{2}}{5\sqrt{S}}]\sqrt{S}}{\texttt{BesselK}[1, \frac{2\sqrt{2}}{5\sqrt{\varphi}}]\sqrt{\varphi}} \quad \textit{(for } S < \varphi\textit{)} \tag{4}$$

$$\texttt{Put:}\quad V(S) = c_1 \cdot \texttt{BesselI}[1, \frac{2\sqrt{2}}{5\sqrt{S}}]\sqrt{S}$$
$$V(\theta) = c_1 \cdot \texttt{BesselI}[1, \frac{2\sqrt{2}}{5\sqrt{\theta}}]\sqrt{\theta}$$
$$c_1 = \frac{K-\theta}{\texttt{BesselI}[1, \frac{2\sqrt{2}}{5\sqrt{\theta}}]\sqrt{\theta}}$$
$$\therefore V(S,\theta) = (K-\theta)\frac{\texttt{BesselI}[1, \frac{2\sqrt{2}}{5\sqrt{S}}]\sqrt{S}}{\texttt{BesselI}[1, \frac{2\sqrt{2}}{5\sqrt{\theta}}]\sqrt{\theta}} \quad \textit{(for } S > \theta\textit{)} \tag{5}$$

Let the function $V(S)$ of the call be f. Using `Mathematica`, we note that its derivative f' is strictly increasing but does so very slowly. If we were to think in terms of smooth-pasting, this means that in order for $V(S)$ to hit $g(S)$, $f'(S)$ must start very near 1 and increase very slowly to hit 1. Thus $\varphi \gg K$. Since smooth-pasting is equivalent to maximization over φ for call and put options, we should expect the same results using the maximization method, which we plot in the following table of φ against K. Indeed, it is clear that as K gets even slightly larger, φ increases by several orders of magnitude:

K	φ
0.302531	10.3555
1	68095.8
2	8.60077 $\times 10^7$

With puts, we find that θ is, in a similar fashion, much smaller than K. The only difference is that θ is bound at the bottom by 0 (since we cannot have a negative price):

K	θ
100	1.97353
10	0.60642
7.53452	0.523041

From these results, we can see that as the strike price increases, the take-profit point gets further and further away from K.

5 American Perpetual Power Calls & Puts

5.1 Constant Volatility

As a continuation of section 4.1, we start off with $r_d > 0, r_f > 0$. Following the Black-Scholes equation, the value of the option before the exercise boundary is $V(S) = c_1 S^{p_1} + c_2 S^{p_2}$.

Now consider an American call option with a payoff function $(S-K)^q$, where $q \in \mathbb{R}^+$. q is positive such that $g(S)$ has a positive relationship with S.

We use a similar approach to get the equations as follows:

$$V(\varphi) = c_1 \varphi^{p1} = (\varphi - K)^q$$
$$c_1 = \frac{(\varphi-K)^q}{\varphi^{p1}}$$
$$V(S) = \frac{(\varphi-K)^q}{\varphi^{p1}} S^{p1}$$

Maximizing V(S) with respect to φ, we get:

$$V_\varphi = \frac{\varphi^{p_1} q(\varphi-K)^{q-1} - (\varphi-K)^q p_1 \varphi^{p_1-1}}{\varphi^{2p_1}} \times S^{p_1}$$
$$= (\varphi-K)^q(\frac{q}{\varphi-K} - \frac{p_1}{\varphi})(\frac{S}{\varphi})^{p_1}$$

To solve the equation: $V_\varphi = 0$, we get either $\varphi = K$ (which we reject) or

$$\frac{q}{\varphi - K} - \frac{p_1}{\varphi} = 0$$
$$\varphi = \frac{K}{1 - \frac{q}{p_1}}$$

Therefore we conclude that:

$$V(S) = \begin{cases} (S-K)^q, S > \varphi \\ (\varphi - K)^q (\frac{S}{\varphi})^{p_1}, S \leq \varphi \end{cases}$$

The procedure of getting the pricing function of American puts will be similar:

$$V(S) = \begin{cases} (K-S)^q, S < \theta \\ (K-\theta)^q (\frac{S}{\theta})^{p_2}, S \geq \theta \end{cases}$$

5.2 Relation between φ and θ

Given the φ values and θ values we calculated in section 5.1, we obtain the product using Vieta's formula:

$$\varphi\theta = \frac{K^2(-r_d)}{\frac{1}{2}\sigma^2 q^2 + (r_d - r_f - \frac{1}{2})q - r_d}$$

where $p_2 < q < p_1$, when $r_d, r_f > 0$.

6 Theorem for Pricing American Perpetual Options

In this section, we generalize our arguments from Section 4 & 5, consider the American Perpetual "call-like" and "put-like" options, and derive a general pricing formula for those options. Detailed specification about the definition of "call-like" and "put-like" options can be found in the following theorems.

6.1 Theorem

Theorem 6.1. *Let $r_d > 0$. An American Perpetual Option with payoff function* $g(S) = \begin{cases} 0, & S < K \\ f(S), & S \geq K \end{cases}$ *satisfying:*

1. $g(0) = 0$
2. *g is continuous and increasing*
3. $\lim\limits_{S\to\infty} \frac{g(S)}{S^{p_1}} = 0$

will have its value given by:

$$V(S) = g(\varphi_0)(\frac{S}{\varphi_0})^{p_1}$$

where φ_0 is the optimal exercise upper barrier, and $p_1 = \frac{\frac{1}{2}\sigma^2 + r_f - r_d + \sqrt{(r_d - r_f - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r_d}}{\sigma^2}$.

Proof. We know that the solution to the Black-Scholes ODE $r_d V(S) = (r_d - r_f)SV^{'}(S) + \frac{1}{2}\sigma^2 S^2 V^{''}(S)$ has the form: $V(S) = c_1 S^{p_1} + c_2 S^{p_2}$, where p_1, p_2 are the two roots to Equation (2). WLOG, assume $p_1 \geq p_2$.

Since $r_d > 0$, we get that

$$p_1 = \frac{\frac{1}{2}\sigma^2 + r_f - r_d + \sqrt{(r_d - r_f - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r_d}}{\sigma^2} > 0,$$

$$p_2 = \frac{\frac{1}{2}\sigma^2 + r_f - r_d - \sqrt{(r_d - r_f - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r_d}}{\sigma^2} < 0.$$

Since g(0) = 0, we know that $\lim_{S\to 0} V(S) = 0$, so $c_2 = 0$. (otherwise the S^{p_2} term would blow up near 0)

Now, $V(S) = c_1 S^{p_1}$. Let φ be an exercise barrier s.t. we would exercise the option if $S \geq \varphi$. This gives $V(\varphi) = g(\varphi)$.

$$V(\varphi) = c_1 \varphi^{p_1} = g(\varphi)$$

$$c_1 = \frac{g(\varphi)}{\varphi^{p_1}} \implies V(S) = \frac{g(\varphi)}{\varphi^{p_1}} S^{p_1}$$

Given a fixed and arbitrary S, we want to choose an optimal φ that maximizes c_1.

If $\varphi = K,\ c_1 = \frac{g(K)}{K^{p_1}} = 0$.

By the limiting condition, $\lim_{S\to\infty} c_1 = \lim_{S\to\infty} \frac{g(\varphi)}{\varphi^{p_1}} = 0$.

Thus, $\exists\ \varphi_0 \in [K, \infty)$ s.t. c_1 achieves a global maximum over $[K, \infty)$. We optimally choose $\varphi = \varphi_0$.

So $V(S) = g(\varphi_0) \cdot (\frac{S}{\varphi_0})^{p_1}$ for $S < \varphi_0$.

□

Theorem 6.2. *let $r_d > 0$. An American Perpetual Option with payoff function* $h(S) = \begin{cases} f(S), & S < K \\ 0, & S \geq K \end{cases}$ *satisfying:*

1. $h(\infty) < \infty$
2. *h is continuous and decreasing*
3. $\lim_{S\to 0} \frac{h(S)}{S^{p_2}} = 0$

will have its value given by:

$$V(S) = h(\theta_0)(\frac{S}{\theta_0})^{p_2}$$

where θ_0 is the optimal exercise lower barrier, and $p_2 = \frac{\frac{1}{2}\sigma^2 + r_f - r_d - \sqrt{(r_d - r_f - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r_d}}{\sigma^2}$.

Proof. We know that the solution to the Black-Scholes ODE $r_d V(S) = (r_d - r_f) S V'(S) + \frac{1}{2}\sigma^2 S^2 V''(S)$ has the form: $V(S) = c_1 S^{p_1} + c_2 S^{p_2}$, where p_1, p_2 are the two roots to Equation (2). WLOG, $p_1 \geq p_2$.

Since $r_d > 0$, we get that

$$p_1 = \frac{\frac{1}{2}\sigma^2 + r_f - r_d + \sqrt{(r_d - r_f - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r_d}}{\sigma^2} > 0,$$

$$p_2 = \frac{\frac{1}{2}\sigma^2 + r_f - r_d - \sqrt{(r_d - r_f - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r_d}}{\sigma^2} < 0.$$

Since h(∞) < ∞, we know that $\lim_{S\to\infty} V(S) < \infty$, so $c_1 = 0$. (otherwise the S^{p_1} term would blow up near ∞)

Now, $V(S) = c_2 S^{p_2}$. Let θ be an exercise barrier s.t. we would exercise the option if $S \leq \theta$. This gives $V(\theta) = h(\theta)$.

$$V(\theta) = c_2 \theta^{p_2} = h(\theta)$$

$$c_2 = \frac{h(\theta)}{\theta^{p_2}} \implies V(S) = \frac{h(\theta)}{\theta^{p_2}} S^{p_2}$$

Given a fixed and arbitrary S, we want to choose an optimal θ that maximizes c_2.

If $\theta = K,\ c_2 = \frac{h(K)}{K^{p_2}} = 0$.

By the limiting condition, $\lim_{S\to 0} c_2 = \lim_{S\to 0} \frac{h(\theta)}{\theta^{p_2}} = 0$.

Thus, $\exists\ \theta_0 \in (0, K]$ s.t. c_2 achieves a global maximum over $(0, K]$. We optimally choose $\theta = \theta_0$.

So $V(S) = h(\theta_0) \cdot (\frac{S}{\theta_0})^{p_2}$ for $S > \theta_0$.

□

6.2 Counterexample & Interpretation

In Corollary 3.1.1, we stated that a non-dividend paying perpetual call option ($r_f = 0$) has no optimal exercise boundary φ. We now consider this option in the setting of Thm 6.1.

Notice that when $r_f = 0$, we have $p_1 = 1$. In this case, $\lim_{S\to\infty} \frac{g(S)}{S^{p_1}} = \lim_{S\to\infty} \frac{S-K}{S^1} = 1 \neq 0$. The limiting condition in Thm 6.1 is not met, so our theorem does not guarantee an optimal exercise barrier φ. Indeed, if we follow the similar derivation, when we try to maximize $c_1 = \frac{\varphi - K}{\varphi} = 1 - \frac{K}{\varphi}$, we would get that $\varphi \to \infty$, which essentially means that there is no optimal exercise barrier.

To help us understand how things change when dividends are introduced, we shall state and prove the following lemma that reveals the relationship between r_f and the roots to Equation (2):

Lemma 6.3. *In the static Black-Scholes ODE setting (Equation (1)),*

$$r_f > 0 \iff p_1 = \frac{\frac{1}{2}\sigma^2 + r_f - r_d + \sqrt{(r_d - r_f - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r_d}}{\sigma^2} > 1. \tag{6}$$

Proof.

$$\begin{aligned}
&p_1 = \frac{\frac{1}{2}\sigma^2 + r_f - r_d + \sqrt{(r_d - r_f - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r_d}}{\sigma^2} > 1 \\
&\iff \sqrt{(r_d - r_f - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r_d} \;>\; r_d - r_f + \frac{1}{2}\sigma^2 \\
&\iff (r_d - r_f - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r_d \;>\; (r_d - r_f + \frac{1}{2}\sigma^2)^2 \\
&\iff 2\sigma^2(r_f - r_d) + 2\sigma^2 r_d > 0 \\
&\iff 2\sigma^2 r_f > 0 \\
&\iff r_f > 0.
\end{aligned}$$

□

When $r_f > 0$, by Lemma 6.3, we have $p_1 > 1$. Now, $\lim_{S\to\infty} \frac{g(S)}{S^{p_1}} = \lim_{S\to\infty} \frac{S-K}{S^{p_1}} = 0$ as $p_1 > 1$.

This verifies Thm 6.1: there exists an optimal exercise barrier φ for a dividend paying perpetual call.

7 Multi-Period Binomial Models with $T \to \infty$

We now turn to modelling binomial models numerically. This verifies results in perpetual time, and suggests results for perpetual options that we have yet to prove, such as the Perpetual Straddle.

The way we coded our binomial model is most applicable to options on foreign exchange. Since exchange rates frequently display a "reversion to the mean", we assumed that up-factors and down-factors have a product of 1. with $u \cdot d = 1$, our binomial tree consists of many repeated paths which simplifies the modeling process. At time $i \geq 0$ there are simply $i+1$ possible values for the underlying exchange rate. Hence, the tree is represented as a 2-D list in Python, with each element of index i in the outer list being a list with length $i+1$. Each sub-list represents the option price at the same time but with different values for the underlying exchange rate. By the principle of backward induction, we fill the last sub-list first with the intrinsic values and then the preceding lists successively, using the maximization idea from section 3. Sample Python code is provided in the appendix. The same algorithm is rewritten and run instead in C when N gets large.

7.1 Negative Interest Rates

Before trying to approximate perpetual options, we want to present an interesting result regarding negative interest rates. Recall the earlier note that it is problematic to consider negative rates in perpetual time. Now let $r_f < 0$ in finite time. While it is difficult to interpret this for equities, we can conceive it as *storage costs* for commodities such as Gold. In an FX model, $r_f < 0$ simply means depositing money in a foreign country with a negative interest rate environment. Now suppose r_d is *less negative* than r_f (either $r_d \geq 0$ or $0 > r_d > r_f$) over some finite time period, and let E_n be the relevant exchange rate at time n, and K be the strike price. Then for a finite pure call option,

$$\begin{aligned}
\text{Value of Waiting} &= \frac{1}{1+r_d}\widetilde{\mathbb{E}}(E_{n+1}-K) \\
&= \frac{1}{1+r_d}\Big[\frac{\frac{1+r_d}{1+r_f}-d}{u-d}\cdot u\cdot E_n + \frac{u-\frac{1+r_d}{1+r_f}}{u-d}\cdot d\cdot E_n - K\Big] \\
&= \frac{1}{1+r_d}\Big[\frac{u\cdot E_n - d\cdot u\cdot E_n + u\cdot d\cdot E_n - \frac{1+r_d}{1+r_f}\cdot d\cdot E_n}{u-d} - K\Big] \\
&= \frac{1}{1+r_d}\Big[\frac{u\cdot E_n - \frac{1+r_d}{1+r_f}\cdot d\cdot E_n}{u-d} - K\Big] \\
&= \frac{E_n}{1+r_f} - \frac{K}{1+r_d} \\
&= \frac{1}{1+r_f}(E_n - \frac{1+r_f}{1+r_d}\cdot K) \\
&> E_n - \frac{1+r_f}{1+r_d}\cdot K && (\frac{1}{1+r_f} > 1) \\
&> E_n - K \qquad\qquad = \text{Exercise Value} && (\frac{1+r_f}{1+r_d} < 1)
\end{aligned}$$

With a similar reasoning for the put, we obtain the following pair of theorems:

Theorem 7.1. *When $r_f < 0$, $r_d > r_f$, a finite pure call option should not be exercised early.*

Theorem 7.2. *When $r_d < 0$, $r_f > r_d$, a finite pure put option should not be exercised early.*

We first formulated these theorems by observing plots of $V(S)$ against S in our model. To be precise, we are talking about $V(S,t)$ at $t=0$, plotting the start of the binomial tree for various starting prices S_0. From first principles, we can approximate the derivative at each point along the graph to determine where φ and/or θ are. By setting T to be large (e.g. 100), and N to be even larger (say 1000 or 2000), we obtain a large T and reasonably small h (0.1 or 0.05), which approximates the behavior of the Black-Scholes solution very well when we plot $V(S)$ against S. Since we are working in finite time, we can also plot $\varphi(S,t)$ and $\phi(S,t)$ against t, $0 \leq t \leq T$, for a fixed initial value S_0. This represents how the exercise boundaries change with time. One would expect $\varphi \to K^+$ and $\theta \to K^-$ as $t \to T$ (see subsection 7.4 for details).

7.2 Constant $\varphi \cdot \theta$ over time

Carr and Chesney (1999) [5] point out a "Put-Call Symmetry" between dividend-paying pure call and put options that is time-independent. Their original idea can be written in the following notation:

$$\sqrt{\varphi_c(K_c, r_d, r_f, t)\cdot\theta_p(K_p, r_f, r_d, t)} = \sqrt{K_c\cdot K_p},\ \text{where}\ \frac{S_c}{K_c} = \frac{K_p}{S_p}$$

The authors proved this result for a call and put option on two different stocks that have different strike prices but the same time to maturity and "moneyness" (as in the second line). In our case, we are considering FX puts and calls on the same underlying exchange rate so we can drop the subscripts on φ and θ. Suppose now that the options have the same strike price. Note that a call on a currency pair can be seen as a put on the same currency pair, except with the domestic and foreign interest rates reversed. An appreciation of one currency pair is a depreciation of the opposite currency pair, so the "moneyness" assumption is preserved (see Shreve [4]). Lastly, we make the reasonable assumption that both the call and put options are denominated in the same currency. Then we should expect, at least for a vanilla call and put FX pair, that

$$\sqrt{\varphi(K, r_d, r_f, t)\cdot\theta(K, r_f, r_d, t)} = K \tag{7}$$

If Equation (7) is true, then once we know one of the exercise boundaries at any point in time, we can immediately find the other boundary at the same point in time. This seems to match up very well with numerical evidence. For an example, suppose $S_0 = K = 10, r_d = 0.1, r_f = 0.2, \sigma = 0.06$. Then we have in the left figure:

Figure 1: $\varphi \cdot \theta$: Pure Call and Put

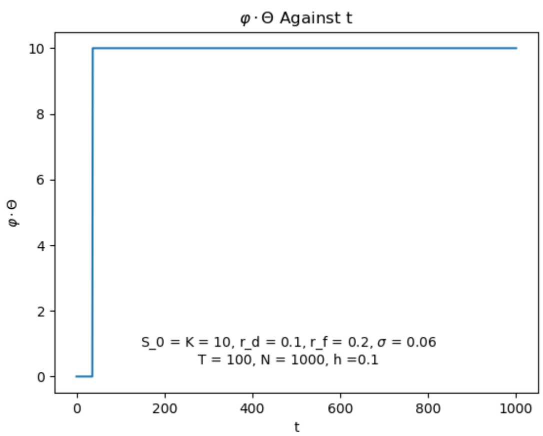
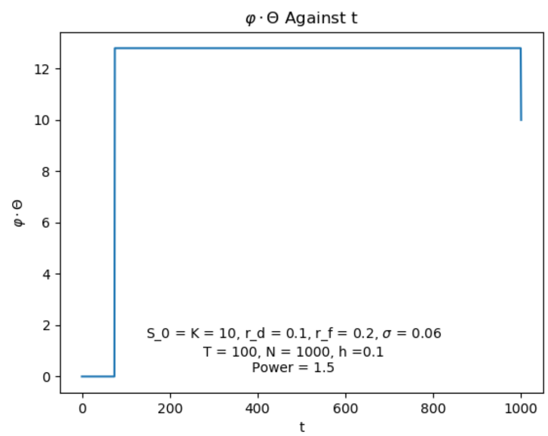


Figure 2: $\varphi \cdot \theta$: Power = 1.5 Call and Put



(For an explanation of the left end, note that at $t = 0$, neither option is worth anything since $S_0 = K_0$, so $\varphi = \theta = 0$. In general, for small t neither option should be exercised because the value of the underlying has not moved very far from S_0.)

Does the same idea work for power calls? Based on our numerical analysis, $\varphi \cdot \theta$ does remain constant, but the relationship between this value and the power term remains to be seen. With the same parameters but power $q = 1.5$, $\varphi \cdot \theta \approx 12.8$. See the above figures for comparison.

7.3 "Power Put-Call Asymmetry"

Carr and Chesney (1999) [5] also show the existence of a linear relationship between the t_0 prices of a pure ($q = 0$) call or put that satisfy the "moneyness" assumption:

$$\frac{C(S_C, T; K_c, r_f, r_d)}{\sqrt{S_c K_c}} = \frac{P(S_p, T; K_p, r_d, r_f)}{\sqrt{S_p K_p}}$$
$$\frac{S_c}{K_c} = \frac{K_p}{S_p}$$

Again letting $K_c = K_p = K$, and rearranging the "moneyness" assumption into $S_p = \frac{K^2}{S_c}$, we expect

$$C(S_c, K, r_f, r_d) = P(\frac{K^2}{S_c}, K, r_d, r_f)(\frac{S_c}{K})$$

In other words, there should be a linear relationship between the ratio of Call over Put price and the stock price. Carr and Chesney's original proof demonstrated the equivalence of two boundary value problems. However, when we used a similar approach we failed to get above equation. Our attempted proof is attached in the Appendix.

Although we did not obtain our expected result, we did get "piecewise linearity" between the ratio and S^q, with the split occurring where $S = 10^q$:

Figure 3: Piecewise Linearity (Power = 0.5)

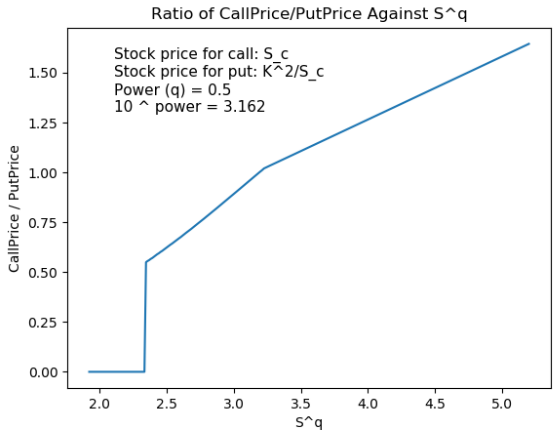
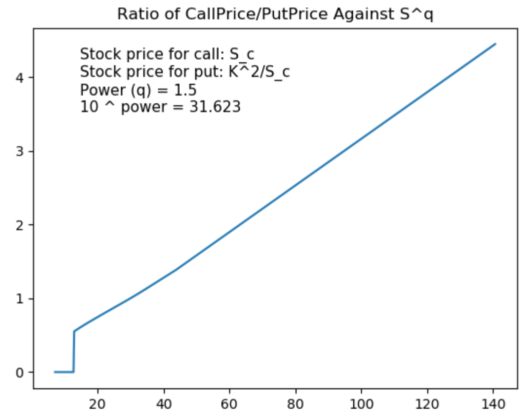


Figure 4: Piecewise Linearity (Power = 1.5)

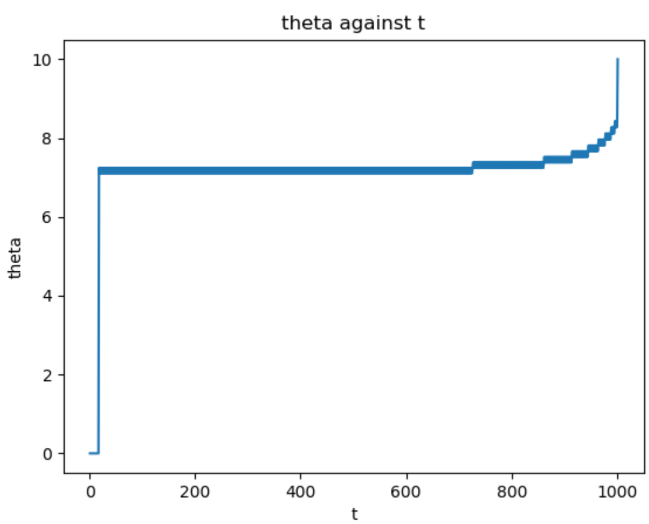


This is an interesting observation that deserves future study.

7.4 Loss of Concavity over time

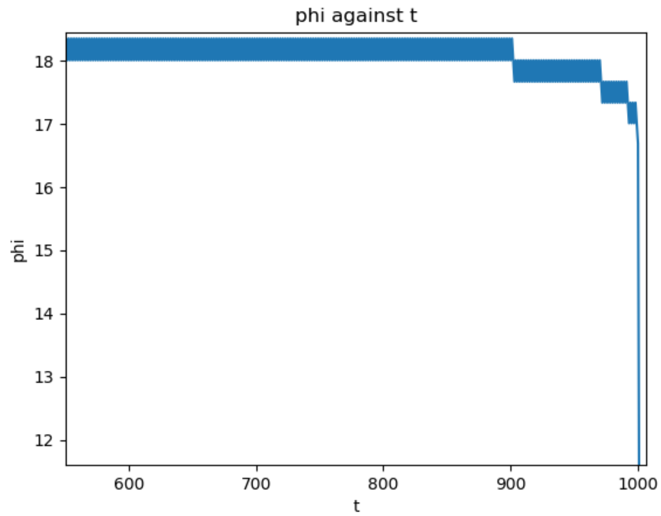
There are other subtleties in how φ and θ approach K across time, as observed by Chen et al (2013) [6]. Their numerical experiments suggest a "breakdown of the convexity" of the early exercise American put boundary as the continuous dividend yield (r_f) increases past the risk-free interest rate, r_d. Using stochastic calculus, they prove this result when $0 < r_f - r_d \ll 1$. Indeed, we see a "spike" in the θ values in the following graph:

Figure 5: Plot of θ against t when $r_f = 0.035, r_d = 0.03, \sigma = 0.06, N = 1000, T = 100$



With the call, too, numerical analysis reveals unusual behavior when $0 < r_d - r_f \ll 1$:

Figure 6: Plot of φ against t when $r_f = 0.03, r_d = 0.05, \sigma = 0.06, N = 1000, T = 100$



7.5 "Shooting Method"

In this subsection we briefly discuss the process of approximating the $V(S)$ of an perpetual straddle (with $q = 1$), using the corresponding finite straddle option under a Binomial model as reference. The reason we cannot solve the perpetual straddle explicitly is that $\lim_{S\to\infty} V(S) = lim_{S\to 0}V(S) = \infty$, which prevents us from eliminating either of the constants. Since the Binomial results suggest that both φ and θ exist, we turn to the numerical method known as the "Shooting Method". This method works best when we let $r_d = r_f$ to avoid complications relating to the loss of concavity.

Having obtained φ and θ for the binomial straddle, we can guess the φ and θ for the perpetual straddle. Intuitively, the perpetual option's boundaries should be further from K than the finite option's boundaries because the holder stands to profit from a swing of the stock price in either direction. Therefore, he will be less willing to exercise the option early. Having fixed φ and θ, we then proceed with Euler's Method, starting from the two boundaries and moving toward K. Based on the shape of the binomial straddle, we want to adjust the perpetual boundaries until the shape of $V(S)$ is smooth in the region around K.

Our use of Euler's Method essentially relies on the following approximations and boundary conditions:

$$\left.\begin{aligned}
V'(x) &= \frac{V(x+h) - V(x)}{h} \\
\therefore V(x+h) &= h \cdot V'(x) + V(x) \\
V''(x) &= \frac{V'(x+h) - V'(x)}{h} \\
&= \frac{\frac{V(x+2h)-V(x+h)}{h} - \frac{V(x+h)-V(x)}{h}}{h} \\
&= \frac{V(x+2h) - 2V(x+h) + V(x)}{h^2} \\
V(\theta) &= K - \theta \\
V'(\theta) &= -1 \\
V(\varphi) &= \varphi - K \\
V'(\varphi) &= 1
\end{aligned}\right\}$$

On the put side, first we find $V(\theta)$ and $V(\theta + h)$. Note that h must be positive as we are increasing x

from θ to K. To increase accuracy, we chose $h = 0.0001$. Then, from Equation (1),

$$\frac{1}{2}\sigma^2 x^2 V^{''}(x) + (r_d - r_f)xV^{'}(x) - r_d V(x) = 0$$
$$r_d V(x) = (r_d - r_f) \cdot x \cdot (\frac{V(x+h) - V(x)}{h}) + \frac{1}{2}\sigma^2 x^2 (\frac{V(x+2h) - 2V(x+h) + V(x)}{h^2})$$
$$h^2 \cdot r_d \cdot V(x) = h(r_d - r_f) \cdot x \cdot [V(x+h) - V(x)] + \frac{1}{2}\sigma^2 x^3 (V(x+2h) - 2V(x+h) + V(x))$$
$$V(x+2h) = 2V(x+h) - V(x) + \frac{h^2 r_d V(x) - h(r_d - r_f) \cdot x \cdot [V(x_h) - V(x)]}{\frac{1}{2}\sigma^2 x^2}$$

This allows us to find $V(\theta + 2h)$ from $V(\theta)$ and $V(\theta + h)$. By repeated application of this formula, we are able to generate all subsequent values of $V(S)$. The same procedure is used on the call side but with a negative h ($h = -0.0001$).

The following figures show the $V(S)$ of the Binomial Straddle as well as the approximated $V(S)$ of the Perpetual Straddle:

Figure 7: Binomial Straddle

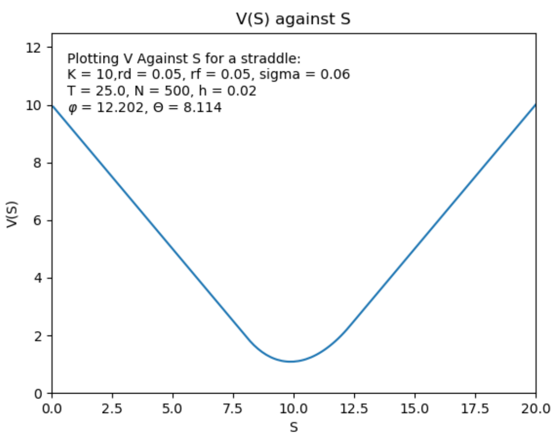
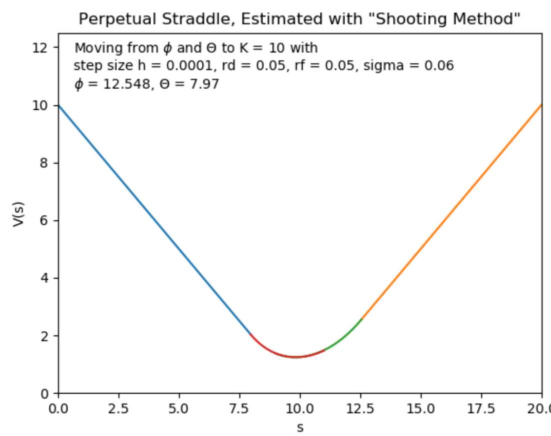


Figure 8: Perpetual Straddle



8 Tent Options

Figure 9: No early exercise on the call side

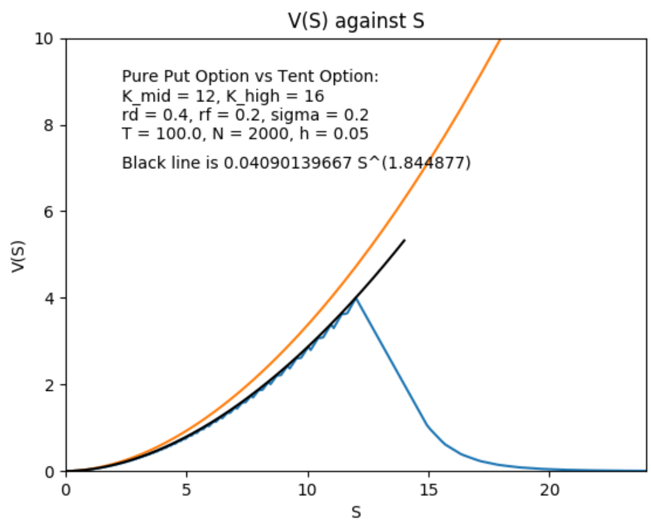
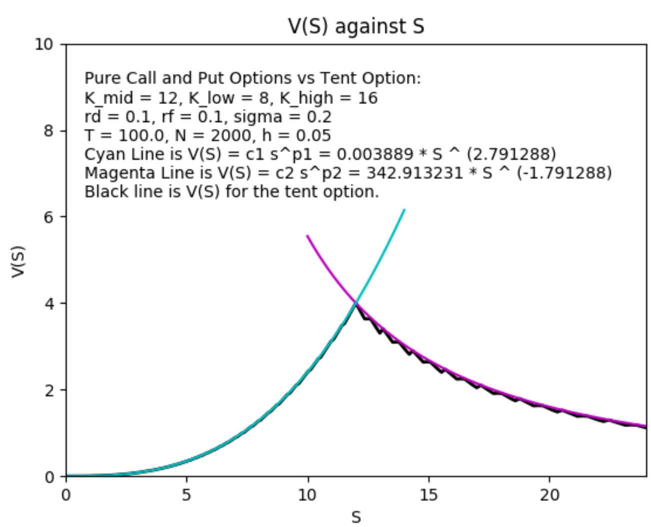


Figure 10: No early exercise on either call or put side



9 Other Diagrams

10 Appendix 1: Proof of American Power Put-Call Asymmetry

For an alive American call-like option $C(S, \tau; K_c, \delta, r)$ and its exercise boundary φ should satisfy the following p.d.e:

$$\frac{\sigma^2}{2}S^2\frac{\partial^2 C}{\partial S^2}(S,\tau) + (r-\delta)S\frac{\partial C}{\partial S}(S,\tau) - rC(S,\tau) = \frac{\partial C}{\partial \delta}(S,\tau)$$

where $S \in (0, \varphi), \tau \in (0, T]$. The following boundary conditions also hold:

$$\lim_{S \to 0} C$$

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