

21-355

3/5/2021

Continuity

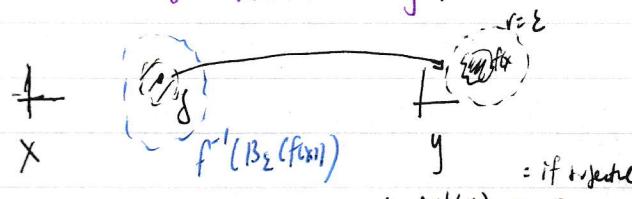
$f: X \rightarrow Y$. let (X, d) , (Y, ρ) be metric spaces.

(1) \Leftrightarrow (2): f is continuous at $x \in X$ iff

$$\text{① } \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$$

② $\forall \varepsilon > 0, \exists \delta > 0$ s.t. if $|x-y| < \delta \Rightarrow |f(y)-f(x)| < \varepsilon$.

\downarrow
 $= \delta(\varepsilon, x, f)$ 3 things!

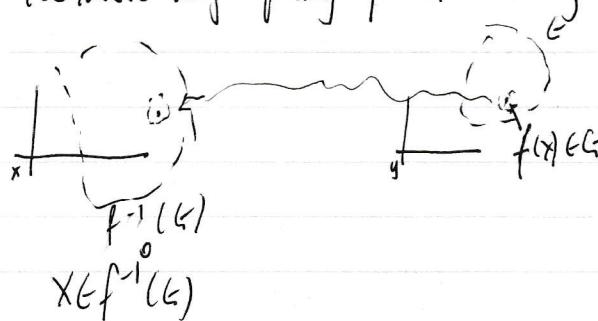


$$\text{③ } \begin{array}{l} \forall \varepsilon > 0 \\ \exists \delta > 0 \end{array} \left\{ \begin{array}{l} \text{principle!} \\ f^{-1}(B_\varepsilon(f(x))) \supseteq B_\delta(x) \end{array} \right. \begin{array}{l} f \circ f^{-1}(B) \subseteq B \\ f^{-1} \circ f(B) \supseteq B \end{array}$$

= if injective

④ The inverse image of any ball centered at $f(x)$ contains a ball around x .

⑤ The inverse image of any open set containing $f(x)$ contains x as an interior point.



Prove ⑤ \Rightarrow ① let $\varepsilon > 0$. Then define $B_\varepsilon = B_\varepsilon(f(x)) \Rightarrow x \in f^{-1}(B_\varepsilon)$, x is an interior point of $f^{-1}(B_\varepsilon) \Rightarrow \exists \delta > 0$ s.t. $B_\delta(x) \subseteq f^{-1}(B_\varepsilon)$
 $\Rightarrow f(B_\delta(x)) \subseteq f(f^{-1}(B_\varepsilon)) = B_\varepsilon$.

Sequence Criterion.

f is continuous at $x \Leftrightarrow (x_n \rightarrow x \Rightarrow f(x_n) \rightarrow f(x))$

$\Rightarrow f$ is cont. at $x, x_n \rightarrow x$. wts $f(x_n) \rightarrow f(x)$

let $\varepsilon > 0$, $\exists \delta > 0$ s.t. $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$

$\exists N = N(\delta)$ s.t. $\forall n \geq N : x_n \in B_\delta(x)$

$\Rightarrow f(x_n) \in B_\varepsilon(f(x))$

$N(\delta) = N(\delta(\varepsilon)) \Leftrightarrow \forall n \geq N : |f(x_n) - f(x)| < \varepsilon$.

$= N(\varepsilon)$

(\Leftarrow) Ex. Assume f not continuous at $x \Rightarrow \exists \varepsilon > 0, \forall \delta > 0, f(B_\delta(x)) \not\subseteq B_\varepsilon(f(x))$

Take $\delta = \frac{\varepsilon}{2} \Rightarrow \exists x_n \in B_\delta(x)$ s.t. $f(x_n) \notin B_\varepsilon(f(x)) \rightarrow$ construct the bad sequence

$f(B_\delta(x)) \not\subseteq B_\varepsilon(f(x))$ Then $x_n \rightarrow x$ but $f(x_n) \not\rightarrow f(x)$.

Def. $f: X \rightarrow Y$ is continuous iff it is continuous everywhere.

Notation $f \in C(X, Y)$, or $f \in C$.

Lemmas TFAE:

(1) $f \in C$

(2) $G \subseteq Y$ open $\Rightarrow f^{-1}(G)$ open

(3) $F \subseteq Y$ closed $\Rightarrow f^{-1}(F)$ closed.

21-355

3/5/2021

$$\textcircled{1} \Rightarrow \textcircled{2} \quad f^{-1}(U) \xrightarrow{x} \{f(x)\} \in U$$

Criterion 5: If x , f is continuous at f .

(iii) $x \in f^{-1}(U)$ is an interior point of $f^{-1}(U)$

$\textcircled{2} \xrightarrow{\text{?}} \textcircled{1}$ (Let $x \in X$. $U := B_\delta(f(x))$ is open and $f(x) \in U$

$\Rightarrow f^{-1}(B_\delta(f(x)))$ contains $B_\delta(x)$ for some $\delta > 0$. (Criterion 4)

$\Rightarrow f(B_\delta(x)) \subseteq B_\delta(f(x)) \Rightarrow f$ continuous at x .

$\textcircled{2} \Rightarrow \textcircled{3}$ $F \subseteq Y$ closed $\Rightarrow F^c$ open

$\Rightarrow f^{-1}(F^c)$ open (from $\textcircled{2}$)

$\Rightarrow [f^{-1}(F^c)]^c = f^{-1}(F)$ closed.

let $f: X \rightarrow Y$

f is continuous (everywhere) $\Leftrightarrow \varepsilon-\delta$ definition?

$$\forall x, (\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \begin{cases} f(B_\delta(x)) \subseteq B_\varepsilon(f(x)) \\ d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon \end{cases})$$

Theorem $f: X \rightarrow Y, f \in \mathcal{C}$

1) $K \subseteq X$ compact $\Rightarrow f(K)$ compact

2) $A \subseteq X$ connected $\Rightarrow f(A)$ connected

Proof: HWS

Corollary. 1) $f \in \mathcal{C}(X, \mathbb{R}), K \subseteq X$ compact

$$f(x) \in \mathbb{R} \quad s := \sup_{x \in K} f(x), \quad t := \inf_{x \in K} f(x)$$

Then $s, t \in \mathbb{R}$ and $\exists x, x' \in K$ s.t. $f(x) = s := \max_K f$
 $f(x') = t := \min_K f$

Proof sketch. $f(K)$ compact in $\mathbb{R} \Rightarrow$ bounded and closed.

$$\sup_{f(k)} < \infty$$

WTS $\text{sg } f(k) \dots ?$

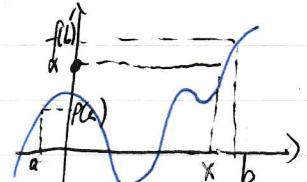
$$\sup A \in \bar{A} = A \text{ closed}$$

"intermediate value theorem" 2) $f \in \mathcal{C}(\mathbb{R}), f(a) \leq d \leq f(b) \quad (a \leq b)$

$\Rightarrow \exists x \in [a, b] \text{ s.t. } f(x) = d$.

Proof. $f([a, b])$ connected $\stackrel{\text{P.31}}{\Rightarrow} f([a, b])$ is an interval.

which by definition contains every $y \in [f(a), f(b)]$,
 in particular $y = d$.



21-355

3/8/2021

Theorem

let $f: (K, d) \rightarrow (Y, P)$ be continuous and K compact.
 If f bijective $\Rightarrow f^{-1}$ is continuous.

'continuous'

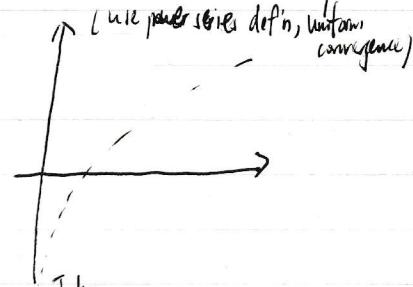
Proof. $K \xleftarrow{f} Y$

$\xrightarrow{\text{inverse function}}$ f^{-1} continuous $\Leftrightarrow (f^{-1})^{-1}(F)$ is closed $\forall F$ closed $\subseteq K$.

But $(f^{-1})^{-1}(F) = f(F)$ is compact, and therefore closed.
 \Leftrightarrow closed subset
 of compact K

Application: $\exp: X \mapsto e^x, \mathbb{R} \rightarrow \mathbb{R}^+$ is continuous + bijective. We assume that this has been proven.

wts. $\exp^{-1} := \log, \log: \mathbb{R}^+ \rightarrow \mathbb{R}$ is continuous
 \uparrow
 not compact



$\exp: [-\ln n, \ln n] \rightarrow \mathbb{R}$

\uparrow
 continuous + compact, $\ln [-\ln n, n]$ is continuous. Take $n \rightarrow \infty$

Ex. (1) Continuous functions: identity $f = id_X: X \rightarrow X$ is continuous
 $x \mapsto x$

In particular $X \mapsto X$ etc

$\mathbb{Z} \mapsto \mathbb{Z}$ etc

(2) $f, g \in C(X, \mathbb{R})$ or $C(X, \mathbb{C}) \Rightarrow$

1) $f + g \in C$

2) $\alpha \cdot f \in C$

3) $f \cdot g \in C$

4) $\forall x, f(x) \neq 0, \frac{1}{f}$ continuous.

} Prove with sequence criteria.

Proof that $f, g \in \mathcal{C}$. Use sequence criterion:

$$x_n \rightarrow x \Rightarrow (fg)(x_n) = f(x_n) \cdot g(x_n) \xrightarrow{?} f(x) \cdot g(x) = (fg)(x)$$

\downarrow
 $f(x)$ $g(x)$

$(f(x) \pm \varepsilon), (g(x) \pm \varepsilon), \dots$

Lemma $X \xrightarrow{f} Y \xrightarrow{g} Z$

① f, g continuous, $\Rightarrow g \circ f$ continuous.

If $C \subseteq Z$ open $\Rightarrow (g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$ open in X .

② If f continuous at $x \in X$ and g continuous at $y = f(x)$
 $\Rightarrow g \circ f$ continuous at x .

Sequence criterion: $x_n \rightarrow x, \quad g(f(x_n)) \rightarrow g(f(x))$

\downarrow
 $f(x)$

Ex. All (complex) polynomials are continuous.

$$p(z) = \sum_{k=0 \dots n} c_k \cdot z^k \in \mathcal{C}(\mathbb{C})$$

$$p(x) \in \mathcal{C}(\mathbb{R})$$

Defined as the limit of
 Transcendental functions \rightarrow Power series (e.g. $\exp(z)$),

21-355

3/10/2021

let $(f_n)_{n \geq 1}$ seq of functions, $f_n: X \rightarrow Y$.

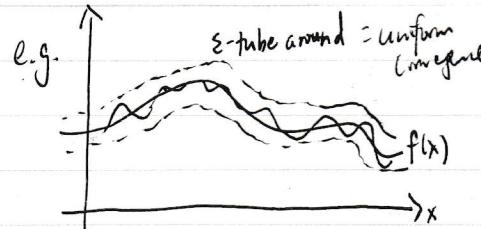
We say $f_n \rightarrow f$ pointwise iff $\forall x: f_n(x) \rightarrow f(x)$

Formally,

$$f_n \rightarrow f \Leftrightarrow \forall x, \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N \quad p(f_n(x), f(x)) < \varepsilon$$

Uniform convergence

$f_n \rightarrow f$ uniformly iff $\forall \varepsilon \exists N = N(\varepsilon) \text{ s.t. } \forall n \geq N \quad \forall x \in X: p(f_n(x), f(x)) < \varepsilon$



Uniform convergence \Rightarrow pointwise convergence

'closer convergence' at some points \rightarrow pointwise

Theorem:

let (f_n) be a sequence of continuous functions converging to f uniformly
 $\Rightarrow f$ is continuous itself.

Proof (Sequence Criterion)

Pick $x \in X$. Let $x_n \rightarrow x$, w.t.s $f(x_n) \rightarrow f(x)$

Let $\varepsilon > 0$ w.t.s $\exists k_0 = k_0(\varepsilon)$ s.t.

$\forall k \geq k_0 : d(f(x_k), f(x)) < \varepsilon$.

* $f_n: X \rightarrow Y$
 $(X, d_X) \text{ } (Y, d_Y)$

$$d(f(x_k), f(x)) \leq d(f(x_k), f_n(x_k)) + d(f_n(x_k), f_n(x)) + d(f_n(x), f(x))$$



- specify n, k $\overset{\leftarrow}{<} \frac{\varepsilon}{3}$ specify f_n
 ① choose $n \geq N(\frac{\varepsilon}{3})$ ② choose $k \geq k_0(\frac{\varepsilon}{3})$ same choose
 \rightarrow valid $\forall x$!
 uniform converge $= k_0(\varepsilon)$ ③

Ex: $f(z) = e^z$ is continuous on \mathbb{C} .

$\left(\sum_{n=0}^{\infty} \frac{z^n}{n!} \right)$ Weierstrass M-test, test
 for uniform absolute convergence

Def $f: X \rightarrow Y$ is uniformly continuous on $A \subseteq X$

✓

if $\forall \varepsilon > 0, \exists \delta_{\varepsilon} \text{ s.t. } d(x, y) < \delta \Rightarrow p(f(x), f(y)) < \varepsilon.$

$\forall x, \forall \varepsilon > 0 \exists \delta = \delta(\varepsilon, x) \text{ s.t.}$

$|x - y| < \delta \Rightarrow p(f(x), f(y)) < \varepsilon.$

($\forall x, y!$) Hws.

Theorem. $f: X \rightarrow Y, f \in \mathcal{C}, X$ compact $\rightarrow f$ is uniformly continuous.

Pf. let $\varepsilon > 0$ and $x \in X \Rightarrow \exists \delta_x > 0$ s.t. $d(x, y) < \delta_x \Rightarrow p(f(x), f(y)) < \varepsilon/2$.

$\rightarrow (B_{\delta_x}(x))_{x \in X}$ is an open cover of X ,

and since X is compact, $\exists x_1, x_2, \dots, x_n$ s.t. $(B_{\delta_{x_k}}(x_k))_{k=1, 2, \dots, n}$

covers X . Set $\delta := \min_{k=1, \dots, n} \delta_{x_k} > 0$

\rightarrow pick $x, y \in X$ s.t. $d(x, y) < \delta \Rightarrow \exists k$ s.t. $x \in B_{\delta_{x_k}}(x_k)$.

$y \in B_{\delta}(x) \subseteq B_{\delta_{x_k}}(x_k) \quad \delta + \delta_{x_k} \leq 2\delta_{x_k} \cdot \text{So } y \in B_{2\delta_{x_k}}(x_k).$

Then $p(f(y), f(x_k)) < \varepsilon/2$.

$p(f(x), f(x_k)) < \varepsilon/2$.

$p(f(x), f(y)) < \varepsilon/2 + \varepsilon/2 < \varepsilon$.

\triangle -inequality

21-355

3/12/2021

Sequences in \mathbb{R} (\mathbb{C})

Theorem $a_n, b_n \in \mathbb{R}$ (or \mathbb{C}) $a_n \rightarrow a, b_n \rightarrow b \rightarrow |(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b|$

$$1) (a_n + b_n) \rightarrow a + b$$

$$2) c \in \mathbb{R} (\mathbb{C}), \text{ then } (c a_n) \rightarrow c a.$$

$$3) a_n \cdot b_n \rightarrow a \cdot b \rightarrow |c_n b_n - ab| = |a_n(b_n - b) + b(a_n - a)| \leq |a_n| |b_n - b| + |b| |a_n - a|$$

$$4) |a_n| \rightarrow |a| . \text{ Define } \psi(a_n) = |a_n| \leq M \cos \frac{\varepsilon}{2M} M' \leq \frac{\varepsilon}{2M}$$

$$5) \left| \frac{1}{a_n} \right| \rightarrow \frac{1}{a} \text{ if } a \neq 0, \text{ s.t. } a_n \neq 0 \forall n.$$

$$\text{Inequality: } ||a| - |b|| \leq |a - b|. \text{ wlog } |a| \geq |b| \Rightarrow |a - b| + |b - 0| \geq |a - 0|$$

$$\text{Let } \varepsilon > 0: ||a_n| - |a|| \leq |a_n - a| \leq \varepsilon \text{ q.e.d}$$

$$\text{for } n \geq n_0 = n_0(\varepsilon)$$

$$|a| \leq |b| \Rightarrow |b - a| + |a| \geq |b|$$

$$|b - a| \geq |b| - |a|$$

$$|a - b| \geq \frac{1}{2}(|a| + |b|)$$

For every n sufficiently large, $s_n \neq 0$. since $s_n \rightarrow s$.

$$\left| \frac{1}{s_n} - \frac{1}{s} \right| = \left| \frac{s - s_n}{s \cdot s_n} \right| \text{ choose } n_0 \text{ s.t. } \forall n \geq n_0, |s_n| \geq \frac{1}{2} |s|.$$

$$\frac{|s - s_n|}{|s| + |s_n|} \stackrel{n \geq n_0}{\leq} 2 \cdot \frac{|s - s_n|}{|s|^2} \xrightarrow[n \geq n_0]{\uparrow} \frac{\varepsilon}{2|s|} \leq \varepsilon.$$

$$\text{Choose } n_1 \text{ s.t. } \forall n \geq n_1, |s - s_n| \leq \frac{\varepsilon |s|^2}{2}.$$

Lemma: $\vec{x}_n \rightarrow \vec{x}$ in $\mathbb{R}^d \Leftrightarrow \forall k=1\dots d \quad (\vec{x}_n)_k \rightarrow (\vec{x})_k$.

$$d=2 \text{ case: } d(\vec{x}, \vec{x}_n) \leq \sum_{k=1,2} |x_k - (x_n)_k| \xrightarrow{k \rightarrow 0}$$

Lemma: If $(x_n) \nearrow$ (monotone increasing) i.e. $x_n \leq x_{n+1} \dots$
 Then $\lim_n x_n = \begin{cases} \infty, & \text{if } x_n \text{ unbounded} \\ \sup_n x_n := s, & \text{if } (x_n) \text{ bounded.} \end{cases}$ 'divergent'

Proof. $x_n \rightarrow \infty$
 $\uparrow \downarrow \text{def.}$

$$\frac{1}{\delta} \longrightarrow \left(\frac{\infty}{\delta} \right) \quad \forall M: \exists n_0 = n_0(M) \text{ s.t. } \forall n \geq n_0: a_n > M$$

$$(a) \quad \text{"Open ball around } \infty \text{"} \Rightarrow B_M(\infty) := \{y \mid y > M\}$$

WTS x_n unbounded $\Leftrightarrow \lim_n x_n = \infty$.

$$\begin{aligned} (\Rightarrow) \quad & \forall M, \exists n \text{ s.t. } a_n \geq M, \forall k \geq n, a_k > a_n \therefore a_k > M \\ (\Leftarrow) \quad & \forall n \geq n_0, a_n > M \dots \text{ find } n_0(M) \end{aligned}$$

$$(b) \quad \forall n: x_n \leq b < \infty, \quad \sup_n x_n = s \leq b$$

let $\varepsilon > 0 \Rightarrow \exists n_0 \text{ s.t. } x_{n_0} > s - \varepsilon$

Since $x_n \nearrow, \forall n \geq n_0, x_n > s - \varepsilon$

$$\begin{aligned} \forall n \geq n_0, s - \varepsilon \leq x_n \leq s + \varepsilon \\ \Rightarrow x_n \in B_\varepsilon(s) \end{aligned}$$

21-355

3/17/2021

Comparison Test

1) If $0 \leq x_n \leq c_n$ and $c_n \rightarrow 0 \Rightarrow x_n \rightarrow 0$

2) $b_n \leq x_n \leq a_n$ and, $a_n \rightarrow a$, $b_n \rightarrow a$ $\Rightarrow x_n \rightarrow a$
 "policeman"

$\forall \varepsilon > 0$, $\exists n_0$ s.t. $n \geq n_0$,

$0 \leq c_n < \varepsilon$.

$\Rightarrow \frac{\sqrt{b_n - n_0}}{\sqrt{c_n + n_0}} < \varepsilon$,
 $\Rightarrow x_n \rightarrow a$.

$0 \leq x_n - b_n \leq c_n - b_n$, use part (1)

$$\begin{aligned} |a_n - b_n| &= |a_n - a + a - b_n| \\ &\leq |a_n - a| + |b_n - a| \\ &\leq \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

$x_n - b_n \rightarrow 0$, $x_n \rightarrow b_n$
 $x_n \rightarrow a$.

Examples.

1) $\left| p > 0 : \frac{1}{n^p} \rightarrow 0 \right|$ ($p \in \mathbb{Q}$)

$$n^p \leq (n+1)^p$$

$$\therefore \left(\frac{1}{n^p} \right) \searrow, \text{ and } 0 \leq \frac{1}{n^p}$$

Also, $\forall \varepsilon > 0 \Rightarrow \frac{1}{n^p} \leq \varepsilon$, for $\exists n \quad \therefore \lim_{n \rightarrow \infty} \left(\frac{1}{n^p} \right) = 0$.

2) $\left| \sqrt[n]{a} \rightarrow 1 \quad \forall a > 0. \right|$

$$\text{Let } a > 1: \quad a = (1 + (\sqrt[n]{a} - 1))^n = 1^n + \binom{n}{1} (\sqrt[n]{a} - 1) + \dots$$

$$\geq 1 + n(\sqrt[n]{a} - 1)$$

$$\text{WTS } \sqrt[n]{a} - 1 \rightarrow 0$$

$$\text{but } 0 \leq (\sqrt[n]{a} - 1) \leq \frac{a-1}{n} \quad \downarrow \text{log.-e.d.}$$

from here

$$\text{let } a = 1/\varepsilon$$

$$\text{let } a < 1 \Rightarrow \left(\frac{1}{a} \right) > 1 \Rightarrow \left(\frac{1}{a} \right)^{\frac{1}{n}} \rightarrow 1 \Rightarrow a^{\frac{1}{n}} \rightarrow \frac{1}{1} = 1.$$

* 3) $\left| \sqrt[n]{n} \rightarrow 1. \right|$

$$\text{set } x_n := \sqrt[n]{n} - 1 \geq 0 \quad \text{proof by contradiction} \quad (\sqrt[n]{n} < 1 \Rightarrow \frac{n}{\infty} < x_n^{\infty})$$

$$n = (nx_n)^n \stackrel{n \geq 2}{\geq} (\sqrt[2]{x_n})^2 \Rightarrow 0 \leq x_n \leq \left(\frac{2n}{n(n-1)} \right)^{\frac{1}{2}}$$

One of the terms

$$= \sqrt{\frac{2}{n-1}} \searrow 0$$

* 4) $a > 0, \alpha \in \mathbb{R} : \left(\frac{n^\alpha}{(1+a)^n} \right) \rightarrow 0$

choose $k \in \mathbb{N} > \alpha$ and let $n > 2k$:

$$\begin{aligned} (1+a)^n &> \binom{n}{k} a^k = \frac{n(n-1)\dots(n-k+1)}{k!} a^k \\ n &> \frac{\left(\frac{n}{2}\right)^k a^k}{k!} \end{aligned}$$

$$\begin{aligned} \frac{n^\alpha}{(1+a)^n} &= \frac{n^k \cdot n^{\alpha-k}}{(1+a)^n} \leq \frac{n^k \cdot n^{\alpha-k}}{\left(\frac{n}{2}\right)^k a^k / k!} = \frac{n^{-(k-\alpha)}}{0} \cdot \underbrace{\frac{2^k \cdot k!}{a^k}}_{\text{constant}} \\ &\rightarrow 0. \end{aligned}$$

5) $|x| < 1 \Rightarrow x^n \rightarrow 0$

$$x^n = \frac{1}{(1/x)^n} \quad \text{use 4) with } \alpha = 0,$$

Liminf and Limsup

Let $(a_n) \subseteq \mathbb{R}$ sequence \rightarrow Liminf and Limsup always exist!
 \rightarrow proves convergence

$$\limsup_n a_n = \overline{\lim_n} a_n = \lim_n \bigvee \sup_{k \geq n} a_k := \inf_n \sup_{k \geq n} a_k = \lim_n \bigvee b_n \in \{-\infty\} \cup \mathbb{R}$$

$$\liminf_n a_n = \underline{\lim_n} a_n = \lim_n \bigwedge \inf_{k \geq n} a_k = \sup_n \inf_{k \geq n} a_k = \lim_n \bigwedge b_n \in \mathbb{R} \cup \{+\infty\}$$

21-355

3/17/2021

Theoremlet (a_n) be a sequence. $\overline{\lim} a_n$ is the largest point of accumulation, $\underline{\lim} a_n$ is the smallest "

$$\underline{\lim} a_n = \overline{\lim} a_n \quad \text{if } \forall \varepsilon > 0, \exists \text{ only finitely many points } a_n > \overline{\lim} + \varepsilon.$$

Lemma let $(a_n) \subseteq \mathbb{R}$, $\ell := \overline{\lim} a_n \in \mathbb{R}$ (1) $\forall \varepsilon > 0$ only finitely many a_n 's are $\stackrel{(1)}{>} \ell + \varepsilon$.(2) $\forall \varepsilon > 0$ ∞ -many a_n 's are $\stackrel{(2)}{>} \ell - \varepsilon$.(3) $\underline{\lim} a_n = \overline{\lim} a_n := \ell \iff a_n \rightarrow \ell$.1) AFSCC $n_1, n_2, \dots, n_j, \dots \in \infty$ -many indices, $a_{n_k} > \ell + \varepsilon$
 $\Rightarrow \forall k: (\sup_{n \geq k} a_n) > \ell + \varepsilon \Rightarrow \inf_k \sup_{n \geq k} a_n > \ell + \varepsilon.$

this option is important

2) AFSCC if $\exists j < \infty$ and $n_1 < n_2, \dots < n_j$ s.t. $k=1 \dots j$, $a_n > \ell - \varepsilon$
and all other a_n 's $\leq \ell - \varepsilon$.
 $\Rightarrow (\sup_{n \geq n_j+1} a_n) \leq \ell - \varepsilon \Rightarrow \ell - \varepsilon \geq \inf_{k \geq n_j+1} (\sup_{n \geq k} a_n) = \ell$
3) \Rightarrow Use (1) twice. $\underline{\lim} a_n = -\overline{\lim}(-a_n)$ $\overline{\lim}(-a_n) = -\ell \Rightarrow$ finitely many a_n 's
 $\sup_{n \geq k} a_n > -\ell + \varepsilon$.
 $a_n \leq \ell - \varepsilon$.In particular, if $a_n \rightarrow a$, $a = \overline{\lim} a_n = \underline{\lim} a_n$

$$\rightarrow \underline{\lim} g_n \leq \overline{\lim} a_n, \quad \underline{\lim} g_n = -\overline{\lim} (-a_n)$$

Hw2 $\rightarrow \overline{\lim} (a_n + b_n) \leq \overline{\lim} a_n + \overline{\lim} b_n$

$$\underline{\lim} (a_n + b_n) \geq \underline{\lim} a_n + \underline{\lim} b_n$$

$$\rightarrow a_n \leq b_n \Rightarrow \overline{\lim} a_n \leq \overline{\lim} b_n$$

$$\underline{\lim} a_n \leq \underline{\lim} b_n$$

$$\rightarrow \underline{\lim}, \overline{\lim} \in [-\infty, \infty]$$

*not true
for inf/sup Lemma: Removing finitely many elements of the sequence does not change $\overline{\lim}$ or $\underline{\lim}$.

$$\text{Ex: } a_n \rightarrow a, b_n \rightarrow b \Rightarrow a_n + b_n \rightarrow a+b$$

Proof $\underline{\lim} (a_n + b_n) \leq \overline{\lim} (a_n + b_n) \leq \overline{\lim} a_n + \overline{\lim} b_n$

$$\leq \underline{\lim} a_n + \underline{\lim} b_n$$

$$\begin{aligned} a_n &\rightarrow a: & a+b &\leq \underline{\lim} (a_n + b_n) \leq \overline{\lim} (a_n + b_n) \leq a+b \\ b_n &\rightarrow b \end{aligned}$$

$$\therefore \underline{\lim} (a_n + b_n) = \overline{\lim} (a_n + b_n) = a+b$$

$$\therefore a_n + b_n \rightarrow a+b \quad (\text{by (3)})$$

21-355

3/24/2021

Series $(a_n) \subseteq \mathbb{R}$ or \mathbb{C} .
 $s_n := \sum_{k=1}^n a_k$ are the partial sums, and form a series.

$$\begin{aligned} \left(\sum_{k=1}^{\infty} a_k \right) &= \sum_{k \geq 1} a_k := \lim_{n \rightarrow \infty} \sum_{k=1 \dots n} a_k \\ &= \lim s_n = s \end{aligned}$$

$\nearrow \mathbb{R} (= \mathbb{R} \cup (\infty, +\infty))$
 $\searrow \mathbb{C}$

if this increasing limit exists. (counterexample, $\sum_{n=1}^{\infty} (-1)^n$)

we say that the series is divergent if $s \notin \mathbb{R}, \mathbb{C}$

convergent if $s \in \mathbb{R}$

Theorem (Cauchy Criterion)

$s_n = \sum_{k=1}^n a_k$ is convergent $\Leftrightarrow s_n$ is Cauchy
 $\Leftrightarrow \forall \varepsilon, \exists N$ s.t. $\forall n \geq m \geq N$,

$$\left| \sum_{k=m+1 \dots n} a_k \right| \leq \varepsilon.$$

Corollary If $\sum a_k$ is convergent $\Rightarrow |a_n| \leq \varepsilon \Rightarrow a_n \rightarrow 0$,
take $n \rightarrow \infty$

$$\cancel{\Leftrightarrow} \quad \sum a_k$$

Theorem If $|c_n| < c_n \quad \forall n \geq N_0, \quad \sum c_n < \infty \Rightarrow \sum a_n$ is convergent.

$\therefore \sum |a_n| < \infty, \quad \Leftrightarrow \sum a_n$ is an absolutely convergent series.

Proof: $\left| \sum_{k=n+1}^m a_k \right| \leq \sum_{k=n+1}^m |a_k| \leq \sum_{k=n+1}^m c_k < \varepsilon \Rightarrow$ Cauchy Criterion
 $\forall m, n \geq N(\varepsilon)$

Examples (A) $0 \leq x < 1 \Rightarrow \left| \sum_{k \geq 0} x^k = \frac{1}{1-x} \right| (\text{ER})$

$$(1-x)(1+x+\dots+x^n) = 1-x^{n+1}$$

$$\underbrace{\quad}_{S_n}$$

$$S_n = \frac{1-x^{n+1}}{1-x} \xrightarrow{n \rightarrow \infty} \frac{1}{1-x} \text{ f.d.}$$

rem. 1) $x \geq 1$ series diverges

$$2) z \in \mathbb{C}, 0 \leq |z| < 1, \sum z^k = \frac{1}{1-z}$$

$z^n \rightarrow 0$
argument, magnitude

Theorem Assume $(a_n) \subseteq \mathbb{R}^+, a_n \searrow 0$. Then $\sum_{k \geq 1} a_k$ converges $\Leftrightarrow \sum_{k \geq 0} 2^k a_{2^k} < \infty$.

'sampling'

$$\text{Define } t_m := \sum_{k=0}^m 2^k a_{2^k}.$$

Claim 1 : $n < 2^k$ for some $k \Rightarrow s_n < t_k$

$$\begin{aligned} (\Leftarrow) \quad \text{e.g. } n=2^k-1: \quad s_n &= a_1 + \underbrace{a_2+a_3}_{\leq 2a_2} + \underbrace{a_4+a_5+a_6+a_7}_{\leq 4a_4} \\ &+ a_8 \dots a_{15} + \underbrace{(a_{2^k-1} + a_{2^k+1} + \dots + a_{2^{k+1}-1})}_{\leq 2^{k+1} a_{2^k-1}} \end{aligned}$$

non-dec + bounded = convergent!

Claim 2 : $n \geq 2^k \Rightarrow s_n \geq \frac{1}{2} t_k$

$$\begin{aligned} (\Rightarrow) \quad \text{e.g. } n=2^k. \quad s_n &= a_1 + a_2 + (a_3+a_4) + \underbrace{(a_5+\dots+a_8)}_{\geq \frac{1}{2}(a_5+a_6)} + \dots + (a_{2^k-1}) \\ &\geq \frac{1}{2}(a_1+2a_2+4a_4+8a_8+\dots+2^k a_{2^k}) \end{aligned}$$

If RHS diverges, LHS also $\rightarrow \infty$

21-355

3/24/2021

Theorem

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$
 converges for $p > 1$ and diverges for $p \leq 1$

Observe that $\sum_{k=0}^{\infty} 2^k \frac{1}{2^{kp}} = \sum_{k=0}^{\infty} 2^{k(1-p)}$
 $(= 2^{(1-p)k})$ geometric series!

If we want convergence, $g = 2^{1-p} < 1 \Rightarrow p > 1$.
 from previous theorem
 \Rightarrow Original series converges iff $p > 1$.

Theoremwe know $\frac{1}{n^{1/p}}$ is convergent,what about ζ functionthat grows slower than $n^{-\epsilon}$?

$$\begin{cases} \sum_{n=2}^{\infty} \frac{1}{n(\log n)^p} < \infty & p > 1 \\ & = \infty & p \leq 1 \end{cases}$$

Since $\log n$ is increasing, $\frac{1}{n(\log n)^p}$ is decreasing.

$$\sum_{k=1}^{\infty} 2^k \frac{1}{2^{k(\log 2)^p}} = \sum_{k=1}^{\infty} \frac{1}{(\log 2)^p} \cdot \frac{1}{k^p}$$

$$= \frac{1}{(\log 2)^p} \sum_{k=1}^{\infty} \frac{1}{k^p} \begin{cases} < \infty & (p > 1) \\ = \infty & (p \leq 1) \end{cases}$$

Note that

$$\sum_n \frac{1}{n(\log n)(\log \log n)}$$
 diverges,

$$\sum_n \frac{1}{n(\log n)(\log \log n)}^p$$
 converges

... in general no "sharp" boundary

Theorem

$$1) e := \sum_{n \geq 0} \frac{1}{n!} \quad (\text{convergent}) < \infty$$

$$2) e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

Proof.

$$1) s_n = 1 + 1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{2 \cdot 3 \cdots n}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2 \cdot 2} + \dots + \frac{1}{2^{n-1}}$$

$$\text{So we have } \frac{s_n}{2^n} < 1 + \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 1 + \frac{1}{1-\frac{1}{2}} = 3$$

$$2) s_n = \sum_{k=1}^n \frac{1}{k!}, \text{ and define } t_n := \left(1 + \frac{1}{n}\right)^n$$

$$t_n \stackrel{\substack{\text{binomial} \\ \text{theorem}}}{=} 1 + \binom{n}{1} \left(\frac{1}{n}\right)^1 + \binom{n}{2} \left(\frac{1}{n}\right)^2 + \dots + \binom{n}{n} \left(\frac{1}{n}\right)^n$$

$$= 1 + 1 + \frac{n(n-1)}{2! n^2} + \frac{n(n-1)(n-2)}{3! n^3} + \dots + \frac{n(n-1) \dots (n-m+1)}{n! n^n}$$

$$= 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \frac{1}{3!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) +$$

$$\dots + \frac{1}{n!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \dots \left(1 - \frac{n-1}{n}\right)$$

\lim

$$\text{(clearly } s_n \geq t_n \Rightarrow \liminf_n s_n \geq \liminf_n t_n\text{.)}$$

\lim

For the other inequality, let m be fixed

$$\forall n \geq m, t_n \geq 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \dots + \frac{1}{m!} \left(1 - \frac{1}{n}\right) \dots \left(1 - \frac{m-1}{n}\right) + \dots \text{ (use above)} \quad \text{at here!}$$

$$\lim_n \frac{t_n}{n} \geq \lim_n \left(1 + \frac{1}{2!} + \dots + \frac{1}{m!}\right) \stackrel{\text{for fixed } m}{=} 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{m!}$$

Since this is valid for all m , we see that

$$\liminf_n t_n \geq \sup_m \left(1 + 1 + \frac{1}{2!} + \dots + \frac{1}{m!}\right) = e$$

$$\text{So } e = \limsup_n s_n \geq \liminf_n t_n \geq \lim_n t_n \geq e$$

$$\therefore \lim_n t_n = e.$$

21355

3/26/2021

Remark (1) $s_n \rightarrow 0$ very fast:

$$e - s_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots$$

$$\leq \frac{1}{(n+1)!} \left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \dots \right)$$

$$\leq \frac{1}{(n+1)!} \left(1 + \frac{1}{n+1} \right)$$

$$\leq \frac{1}{n(n!)}$$

(2)

$e \notin \mathbb{Q}$, e is not algebraic (not root of any polynomial)

Other Convergence Tests

Theorem (Root Test) Given $\sum a_n$, $a_n \in \mathbb{C}$

$$r := \limsup \sqrt[n]{|a_n|}$$

1) if $r < 1 \Rightarrow$ series convergent (in particular, series converges absolutely)

2) if $r > 1 \Rightarrow$ not convergent (not necessarily diverges... under what this means for (1))

3) $r = 1 \Rightarrow$ can go both ways

Proof • $r < 1$: choose $\varepsilon > 0$ s.t. $r + \varepsilon < 1$. Then $\forall n \geq n(\varepsilon)$,

$$\sqrt[n]{|a_n|} \leq r + \varepsilon \quad \text{by definition of limsup}$$

$$\Rightarrow |a_n| \leq (r + \varepsilon)^n \text{ at the limit}$$

convergent geometric series

• $r > 1$: choose $\varepsilon > 0$ s.t. $r - \varepsilon > 1$.

$$\Rightarrow \exists (n_k) \text{ subsequence s.t. } \sqrt[n_k]{|a_{n_k}|} \geq r - \varepsilon$$

$$\Rightarrow |a_{n_k}| \geq (r - \varepsilon)^{n_k}$$

Then $a_n \not\rightarrow 0$, so by contrapositive $\sum a_n$ cannot be convergent. 55

$r = 1$: Can go both ways.

$$(1) \text{ (divergent)} \quad \sum \frac{1}{n} : r = \lim \sqrt[n]{\sum_{k=1}^n 1} = 1.$$

$$(2) \text{ (convergent)} \quad \sum \frac{1}{n^2} : r = \lim \sqrt[n]{\sum_{k=1}^n \frac{1}{k^2}} = (\lim \sqrt[n]{k})^2 = 1 \cancel{\rightarrow 0}.$$

Theorem (Ratio Test)

For a series $\sum a_n$,

1) $\lim \left| \frac{a_{n+1}}{a_n} \right| < 1 \Rightarrow$ convergent (in particular, absolutely convergent)

2) $\exists n_0$ s.t. $\forall n \geq n_0$, $\left| \frac{a_{n+1}}{a_n} \right| \geq 1 \wedge a_n \neq 0 \Rightarrow$ not convergent

Proof 1) $\exists \beta < 1$ and $N = N(\beta)$ s.t. $\forall n \geq N$:

$$\left| \frac{a_{n+1}}{a_n} \right| < \beta \Rightarrow |a_{n+1}| < \beta |a_n|$$

$$\therefore \forall k \geq 1, |a_{N+k}| < \beta |a_{N+k-1}| < \beta^2 |a_{N+k-2}| \dots$$

$$\Rightarrow \sum |a_n| \leq \sum_{k=1}^N |a_k| + \underbrace{\sum_{k=N+1}^{\infty} |a_{N+k}|}_{\leq \text{ geometric series of}}$$

$$|a_{N+k}| (1 + \beta + \beta^2 + \dots)$$

$\leftarrow \infty$.

2) If $\forall n \geq n_0$, $|a_{n+1}| \geq |a_n|$

$$\Rightarrow |a_{n_0}| \leq |a_{n_0+1}| \leq \dots \leq |a_{n_0+k}|$$

$\neq 0$.

$\Rightarrow a_n \not\rightarrow 0$ so series is not convergent.

21-335

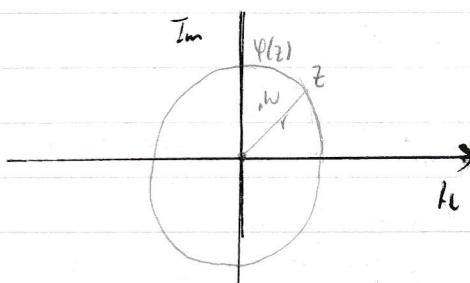
Def. $z \in \mathbb{C}, \varphi(z) := \sum_{n=0}^{\infty} c_n z^n$ $\ell \subset (\text{usually } \mathbb{R})$
is called the power series associated with (c_n) .

Ex. 1) $c_n := \frac{1}{n!} \Rightarrow \varphi(z) := \sum \frac{1}{n!} z^n := e^z$
(complex exponential function)

$$2) c_n = \begin{cases} \frac{(-1)^{n+1}}{n!} & n = 2k+1 \\ 0 & n = 2k \end{cases} \quad k \geq 0$$

$$\sum c_n z^n := \sin(z)$$

Note: If $\sum c_n z^n$ is absolutely convergent ($\sum c_n |z|^n < \infty$),
 $\Rightarrow \sum c_n w^n$ is absolutely convergent for $|w| \leq |z|$



$\rho := \sup \{ r \mid \sum c_n w^n \text{ is absolutely convergent in } |w| < r \}$
is the radius of absolute convergence
 \rightarrow everyly in the open ball is convergent

Theorem

$$\text{Let } R := (\overline{\lim}_{n \rightarrow \infty} |c_n|)^{-1} \in [0, \infty]$$

$\Rightarrow \sum_{n=0}^{\infty} c_n z^n$ is absolutely convergent for $|z| < R$, and
not convergent for $|z| > R$
 $\Rightarrow \rho = R$.

Proof. def $|c_n| = |(c_n z^n)| \Rightarrow$ The root test has $\sqrt[n]{|c_n|} = |z| \sqrt[n]{|c_n|} \geq |z| < \frac{1}{\sqrt[n]{|c_n|}} = R$
 $\sqrt[n]{|z| \sqrt[n]{|c_n|}} = |z| \sqrt[n]{|c_n|} < 1 \Rightarrow$ abs. convergent
 $> 1 \Rightarrow$ not convergent.

Theorem

$\sum \frac{1}{n!} z^n$ is absolutely convergent $\forall z \in \mathbb{C}$.

By the ratio test, $\frac{z^{n+1}}{z^n} \cdot \frac{n!}{(n+1)!} = |z| \cdot \frac{1}{n+1} \rightarrow 0$
 $\lim (\cdot) = 0 < 1$

- . If $\varphi: \mathbb{C} \rightarrow \mathbb{C}$ or $\varphi: \mathbb{R} \rightarrow \mathbb{R}$, then we can do term-by-term operations.

Lemma

$$A := \sum_n a_n, B := \sum_n b_n$$

$$\Rightarrow 1) \sum (a_n + b_n) = A + B \quad (\text{linearity})$$

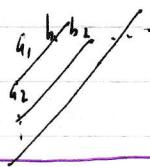
$$2) \sum \gamma \cdot a_n = \gamma A$$

* $\sum a_n b_n \neq A \cdot B$ (we get many more terms in RHS)

Def Given $\sum_{n \geq 0} a_n, \sum_{n \geq 0} b_n$, we set:

$$(\text{for } n \geq 0) \quad c_n := \sum_{k=0 \dots n} a_k b_{n-k} = \sum_{i+j=n} a_i b_j = a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0$$

(Motivation: $(\sum a_i \sum b_j = \sum_{i+j=n} a_i b_j = \sum_{n \geq 0} \sum_{i+j=n} a_i b_j)$)



Theorem

Assume

1) $A := \sum a_n$ is absolutely convergent

2) $B := \sum b_n$ (just convergent)

Then $A \cdot B = \left(\sum_{n \geq 0} a_n \right) \left(\sum_{m \geq 0} b_m \right) = \sum c_n$

i.e. $\sum c_n$ is convergent with $\sum c_n = A \cdot B$.

2/355

3/29/2021

Proof. Set $A_n = \sum_{k=0 \dots n} a_k$, $B_n = \sum_{k=0 \dots n} b_k$, $C_n = \sum_{k=0}^n c_k$

WTS $C_n \rightarrow A \cdot B$.

$$\text{write } C_n = a_0 b_0 + (a_0 b_1 + a_1 b_0) + \dots + (a_0 b_n + \dots + a_n b_0)$$

$$= a_0 B_n + a_1 B_{n-1} + a_2 B_{n-2} + \dots + a_n B_0$$

$$\Rightarrow = a_0 (\underbrace{B + (B_n - B)}_{\substack{\text{use convergence} \\ \text{of } A \text{ by} \\ \text{tail sum}}}) + a_1 (B + (B_{n-1} - B)) + \dots + a_n (B + (B_0 - B))$$

$$\text{adding & factoring} = B \cdot A_n + \underbrace{a_0 B_n + a_1 B_{n-1} + \dots + a_n B_0}_{\substack{\text{add} \\ \text{of } B \cdot A \\ \text{=: } \gamma_n \rightarrow 0}}$$

So $C_n \rightarrow BA$ iff $\gamma_n \rightarrow 0$.

Note that $\beta_n := B_n - B \rightarrow 0$

$$\Rightarrow \forall \varepsilon > 0, \exists N, \forall n \geq N |\beta_n| < \varepsilon.$$

$$\begin{aligned} |\gamma_n| &\leq |a_0 \beta_n + a_1 \beta_{n-1} + \dots + a_{n-N} \beta_N| + |a_{n-N+1} \beta_{n-1} + \dots + a_n \beta_0| \\ &\quad ? (\leq \varepsilon |A_{n-N}|) + |\beta_{n-N+1} a_{n-1} + \dots + \beta_{n-1} a_{n-N+1}| \\ &\leq \varepsilon |A_{n-N}| + \varepsilon \sum_{k=0}^{n-N} |a_k| \\ &= \varepsilon a_n + \varepsilon a_{n-1} + \dots + \varepsilon a_{n-N} \\ &\quad a_n \beta_n \quad \text{absolute convergence} \leq \varepsilon \sum_{k=0}^{n-N} |a_k| \quad \sum_{k=n-N}^n |a_k| \xrightarrow[n \rightarrow \infty]{} 0 \\ &\leq \varepsilon (n-N) \cdot \sum_{k=n-N}^n |a_k| \quad \sum_{k=0}^{n-N} |a_k| \xrightarrow[n \rightarrow \infty]{} 0 \\ &\leq \varepsilon \sum_{k=n-N}^n |a_k| \quad \xrightarrow[n \rightarrow \infty]{} 0 \end{aligned}$$

$$\Rightarrow \lim_{n \rightarrow \infty} |\gamma_n| \leq \varepsilon \cdot d + 0$$

Since this is valid for $\forall \varepsilon$,

$$\Rightarrow \lim_{n \rightarrow \infty} (\gamma_n) = 0.$$

Absolutely convergent series = finite sums

Altinately series - Reversing the summing order allows the sum to change (?)

Rearrangements

Let $\varphi: \mathbb{N} \rightarrow \mathbb{N}$ bijective, $(a_n) \subseteq \mathbb{C}$.

$$a'_n := a_{\varphi(n)}, s'_n = \sum_{k=0 \dots n} a'_k.$$

Theorem If $\sum a_n$ is absolutely convergent

$\Rightarrow \sum a'_n$ is also abs. convergent and $\sum a'_n = \sum a_k = s$.

WLOG $a_n \geq 0 \forall n$. (else 1.1 at the end)

For given n , set $M := \max_{k \leq n} \{|\varphi(k)|\} = M_n$.

Then $0 \leq s'_n = \sum_{k=1 \dots n} a_{\varphi(k)} \leq \sum_{j=1 \dots M_n} a_j = s_M \leq s$ (the original sum)

$\Rightarrow \lim s'_n = s' \leq s$ and (a'_n) is also absolutely convergent.

Inversing the roles of (a_n) and (a'_n) , we have $s \leq s' \Rightarrow \boxed{s = s'}$
(possible because (a'_n) is abs. conv.)

21-355

3/31/2021

FIVE STAR. / ★★★★
 locally Given by a
 Convergent Power Series \rightarrow Analytic function

Theorem

$$f(z) = \sum_{k \geq 0} c_k z^k \quad \text{power series with}$$

$$R = \sup \{ r > 0 \mid f(z) \text{ is absolutely convergent in } \overline{B_r(0)} \}$$

$$= \left(\lim_{n \rightarrow \infty} \sqrt[n]{|c_n|} \right)^{-1}$$

Then $f_n(z) := \sum_{k \geq 0} c_k z^k \xrightarrow[n \rightarrow \infty]{\text{uniformly}} f(z)$ on $\overline{B_R(0)}$ for every $r < R$.
 (closed ball)

In particular $f(z)$ is continuous on $B_R(0)$ (open ball)

Proof. Let $r < R \Rightarrow \sum |c_k|r^k$ is convergent. $\Rightarrow \lim_{n \rightarrow \infty} \sum_{k \geq n} |c_k|r^k = 0$.

$$\text{Let } z \in \overline{B_r(0)} \Rightarrow |f(z) - f_n(z)| = \left| \sum_{k > n} c_k z^k \right|$$

$$\leq \sum_{k > n} |c_k| |z|^k$$

$$\leq \sum_{k > n} |c_k| r^k \xrightarrow{n \rightarrow \infty} 0$$

If $z \notin B_R(0) \Rightarrow z \in B_r(0)$ for some $r < R$,

f is continuous at $z \in B_r(0)$. (see proof on P43.)

Application

 $f(g)$ is a power series

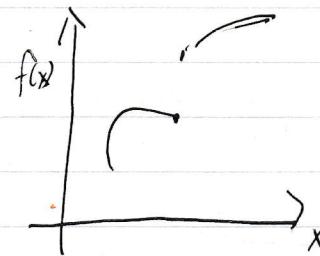
$f \cdot g(z) = \sum_k c_k z^k$ is also absolutely convergent in $B_r(0)$

$\left(\sum_{j=0..k} a_j b_{k-j} \right)$ if $r < R$ (calculate $(\lim_{n \rightarrow \infty} \sqrt[n]{|c_n|})^{-1}$)

Right and Left Continuity

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

Def. f is right continuous $\Leftrightarrow \forall \varepsilon > 0 \exists \delta > 0$ s.t. $\forall y > x$ with $|y-x| < \delta$
 $\Rightarrow |f(x) - f(y)| \leq \varepsilon$



$$\Leftrightarrow f(B_{\delta}^+(x)) \subseteq B_{\varepsilon}(f(x))$$

\downarrow
if $y > x$ and $|y-x| < \delta \Rightarrow f(y) \in f(x) \cap [x, \infty)$

left continuity : $f(B_{\delta}^-(x)) \subseteq B_{\varepsilon}(f(x))$
 \Downarrow
 $B_{\delta}(x) \cap (-\infty, x]$

f is r.c. at $x \Leftrightarrow f_x: [x, \infty) \rightarrow \mathbb{R}$ is continuous at x
 w.r.t $x \mapsto f(x)$

We can check with the sequence criterion \rightarrow check sequences that are greater than x

$$f: \mathbb{R} \rightarrow \mathbb{R} \text{ continuous} \Leftrightarrow \boxed{L \text{ (cont)}} + \boxed{R \text{ (cont)}}$$

21-355

3/31/2021

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

Def $\overline{\lim}_{h \downarrow 0} f(x+h) := \left\{ \begin{array}{l} \lim_{h \downarrow 0} \sup_{y \in B_h^+(x)} \{f(y)\} \\ h \downarrow 0 \end{array} \right.$

Similarly, $\underline{\lim}_{h \downarrow 0} f(x+h) := \lim_{h \downarrow 0} \inf_{y \in B_h^+(x)} \{f(y)\}$

If $\alpha := \overline{\lim}_{h \downarrow 0} f(x+h) = \underline{\lim}_{h \downarrow 0} f(x+h)$ then

$\lim_{h \downarrow 0} f(x+h)$ exists and $= \alpha$

Also $f(x^+) := \lim_{x \rightarrow x^+} f(x) = \alpha$ is the right limit.

| If is right continuous at $x \Leftrightarrow f(x^+) = \lim_{x \rightarrow x^+} f(x)$ exists, and $= f(x)$.

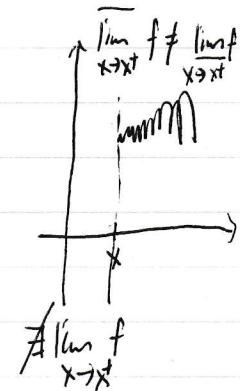
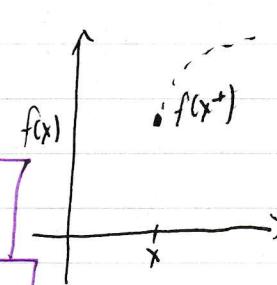
Ex. (1) $f(x) = \begin{cases} \sin \frac{1}{x} & x > 0 \\ -1 & x = 0 \end{cases}$

$f: [0, \infty) \rightarrow \mathbb{R}$

$\overline{\lim}_{x \rightarrow 0^+} f(x) = \overline{\lim}_{x \downarrow 0} f(x) = 1$

so $f(0^+)$ does not exist.

$\underline{\lim}_{x \downarrow 0} f(x) = -1$



Limit \Rightarrow Right limit = Left limit $\in \mathbb{R}$

Monotone Functions

Assume $f: I \rightarrow \mathbb{R}$, $I \subseteq \mathbb{R}$ an interval. (e.g. $I = \mathbb{R}$)

f is monotone $\nearrow \Leftrightarrow x \leq y \Leftrightarrow f(x) \leq f(y)$

(1) $\forall x$ $f(x^+)$ and $f(x^-)$ exist,

$$f(x^+) \geq f(x) \geq f(x^-)$$

(2) Set $D = \{x \mid f(x) < f(x^+)\}$ to be the set of (jump) discontinuities.

Then D is countable.

Pf. (1) $x := \inf\{y \mid y > x\} \geq f(x)$

Then $\lim_{n \downarrow 0} f(x_n) = x$ (sequence criterion)

$$f(x^-) = \lim_{h \downarrow 0} f(x-h) \dots$$

(2) Case 1 $I = [a, b]$

(finite interval) Define $p(x) := f(x^+) - f(x^-)$,

$$p(x) > 0 \Leftrightarrow x \in D$$

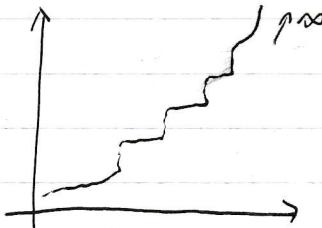
$$\text{let } D_n := \{x \mid p(x) > \frac{1}{n}\}$$



Note. $\forall n: D_n$ is finite, each jump takes up $\frac{1}{n}$ space, and $f(b) - f(a) < \infty$

$$\text{So } D = \bigcup_n D_n \quad (p(x) > 0 \Rightarrow \exists n \text{ s.t. } p(x) > \frac{1}{n})$$

Counterexample for open intervals



21-355

4/2/2021

$$\text{Case 2 } I = (a, b) \Rightarrow I = \bigcup_{n \geq 0} [a + \frac{1}{n}, b - \frac{1}{n}]$$

$A_n := D_n \cap [a + \frac{1}{n}, b - \frac{1}{n}]$ is countable.

$$A = \bigcup_n A_n \text{ is countable}$$

Case 3 $(-\infty, \cdot) \cup (\cdot, \infty)$

$$\text{Take } A_n := D_n \cap [a + \frac{1}{n}, n], D_n \cap [-n, b - \frac{1}{n}]$$

$(-\infty, \infty)$, take $D_n \cap [-n, n] \dots$ countable union...

Application - a function can be discontinuous at every rational number.
(e.g. modified Dirichlet)

Differentiation

$f: I \rightarrow \mathbb{R}$, $I = [a, b]$, (a, b) , (a, ∞) etc.

Def. $f'(x) := \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x}$ if RHS exists and $\in \mathbb{R}$. At the endpoints, e.g. $f'(a) := \lim_{t \rightarrow a^+} \frac{f(t) - f(a)}{t - a}$

"local linearization" $f(x+t) = f(x) + f'(x) \cdot t \dots$ $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = ax + b$ (Technically $b \neq 0$ is affine not linear)
take $b=0$, $f(x)=ax$, $f(x+c-y) = f(x) + f(y)$

$$\text{if } b \neq 0, f(x) + f(y) = ax + b + ay + b = a(x+y) + 2b \neq f(x+y)$$

Theorem: if f is diff at $x \Rightarrow$ cont at x

$$\text{If. } f(t) - f(x) = \frac{f(t) - f(x)}{t - x} (t - x) \xrightarrow[t \rightarrow x]{} 0$$

(w.t.s $f(B_r(x)) \subseteq f(B_\delta(x))$) $\xrightarrow[t \rightarrow x]{} f'(x) \in \mathbb{R}$
fixed

(sequence criterion)

Theorem

f, g diff at $x \in I \Rightarrow$

1) $f+g$ diff, $(f+g)' = f'+g'$

add and subtract

2) $(fg)' = f'g + gf'$

$$(fg)(t) - (fg)(x) = f(t) \frac{g(t) - g(x)}{t-x} + g(x) \left(\frac{f(t) - f(x)}{t-x} \right)$$

3) $\left(\frac{f}{g}\right)' = \frac{gf' - g'f}{g^2}$ if $g \neq 0$.

Let $f \in I$, prove $\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$ then use product rule

$$\frac{\frac{1}{g(t)} - \frac{1}{g(x)}}{t-x} = \frac{g(x) - g(t)}{g(t)g(x)(t-x)} = -\frac{1}{g(t)g(x)} \left(\frac{g(t) - g(x)}{t-x} \right)$$

Apply these to polynomials, rationals...

Theorem (Chain Rule)

$$I \xrightarrow{f} J \xrightarrow{g} \mathbb{R}$$

"special case of Taylor Expansion"

1) f continuous, diff at $x \in I$ everywhere

2) $f(I) \subseteq J$, g diff at $f(x)$

* g need not be continuous anywhere

$\Rightarrow g \circ f$ is diff at x , $(g \circ f)'(x) = g'(f(x))f'(x)$

If. $\frac{f(t) - f(x)}{t-x} - f'(x) =: \delta(t) \xrightarrow[t \rightarrow x]{} 0$

$\Rightarrow f(t) - f(x) = (t-x)f'(x) + \delta(t)(t-x)$

$$g(s) - g(y) = (s-y)g'(y) + \varepsilon(s)(s-y)$$

$$= f'(x) \cdot g'(f(x))$$

$$g(f(t)) - g(f(x)) = [f(t) - p(x)] [g'(y) + \varepsilon(s)]$$

$$\stackrel{\text{def. } s=y}{=} (t-x)f'(x) + f(t)(t-x) [g'(y) + \varepsilon(s)]$$

Dividing both sides
by $t-x$,

$$\frac{g(f(t)) - g(f(x))}{t-x} = \left(f'(x) + f(t) \right) \left[g'(y) + \varepsilon(f(t)) \right]$$

continuity $\Rightarrow f(x) = y$

21-355'

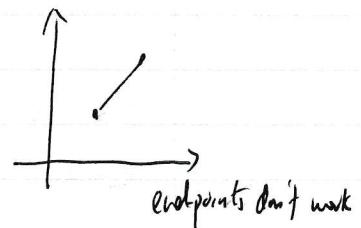
4/7/2021

Def. $f: X \rightarrow \mathbb{R}$ has local max at x $\Leftrightarrow \exists \delta \text{ s.t. } \forall y \in B_\delta(x), f(y) \leq f(x)$

Theorem

let $f: [a, b] \rightarrow \mathbb{R}$, $x \in (a, b)$ ^{Interior point} local maximum,

If $f'(x)$ exists, then $f'(x)=0$.



endpoints don't work

$$f'(\bar{x}) = \lim_{t \nearrow \bar{x}} \frac{f(t) - f(\bar{x})}{t - \bar{x}} \stackrel{<0}{\geq 0}$$

$$f'(\bar{x}^+) = \lim_{t \searrow \bar{x}} \frac{f(t) - f(\bar{x})}{t - \bar{x}} \stackrel{>0}{\leq 0}$$

$$\Rightarrow 0 \leq f'(\bar{x}) = f'(\bar{x}) = f'(\bar{x}^+) \leq 0$$

since $f'(x)$ exists

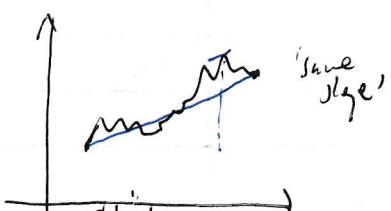
Theorem (Mean Value Theorem)

$f \in C([a, b], \mathbb{R})$, diff in (a, b)

control of derivative

control of function

$\Rightarrow \exists x \in (a, b)$ s.t. $f(b) - f(a) = (b-a)f'(x)$



Bounded linear
Helper function

case (1) $f(b) - f(a) = 0$

if f is constant, $f' = 0$.

else $\exists f(t) > f(a) \Rightarrow \text{let } x \in [a, b] \text{ s.t. } f(x) = \max_{[a, b]} f \Rightarrow x \in (a, b)$

(2) $f(b) - f(a) \neq 0$. Def $h(t) := f(t) - \frac{f(b)-f(a)}{b-a}(t-a)$ "linear prediction of a from t "

$h(t)$ satisfies (1) ... $h(b) = h(a) = f(a)$

$\exists x \in (a, b), h'(x) = 0$. $0 = h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a} = 0$.

Remark

$f, g \in C([a,b], \mathbb{R})$, diff on (a,b) . \Rightarrow

$$\exists x \in (a,b) \text{ s.t. } [f(b)-f(a)]g'(x) = [g(b)-g(a)]f'(x)$$

$$\left(\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x)}{g'(x)} \right)$$

Proof let $h(t) := (f(t)-f(a))(g(t)-g(a))$. $h(a)=h(b)=0$
 $- (g(b)-g(a))(f(t)-f(a)) \Rightarrow \exists x \in (a,b), h'(x)=0 \Rightarrow (f(b)-f(a))g'(x) = (g(b)-g(a))f'(x)$

Corollary $f: [a,b] \rightarrow \mathbb{R}$ diff on $[a,b]$ and $f'(a) < \lambda < f'(b)$
 $\Rightarrow \exists x \in (a,b) \text{ s.t. } f'(x) = \lambda$

Proof if f' is continuous \rightarrow intermediate value theorem

(else (not obvious!))

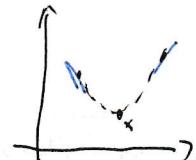
Def. $g(t) := f(t) - \lambda t$ $\Rightarrow g'(a) = f'(a) - \lambda < 0$
 $g'(b) > 0$

$\Rightarrow \exists t_1 \in (a, a+\delta)$ s.t. $g(t_1) < g(a)$

and $t_2 \in (b-\delta, b)$ s.t. $g(t_2) < g(b)$ def. derivative.

$\Rightarrow g$ attains its minimum at $x \in (a,b)$

$\Rightarrow g'(\lambda) = 0 = f'(x) - \lambda$ Q.E.D.



Corollary If f is diff on $[a,b] \Rightarrow$ jump discontinuity

Pf. (IVT)

If we have $\begin{array}{c} \text{---} \\ | \\ a \\ | \\ b \\ | \\ c \end{array}$ then we cannot have $a < y < b$ with $f(y) = L$, where L is chosen to be different from $\lim_{x \rightarrow a^+} f(x)$, $\lim_{x \rightarrow b^-} f(x)$, and $f(c)$.

(MVT)

choose b and s sufficiently close...

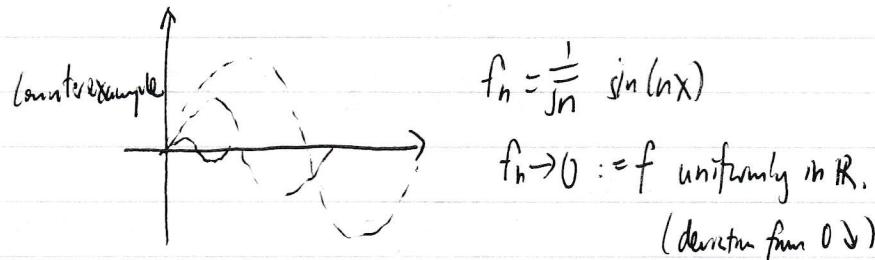
$f(c)$.

21-355

4/7/2021

$f_n \rightarrow f$, f_n differentiable. Does $f'_n \rightarrow f'$?
pointwise?
uniform?

No Even if $f_n \rightarrow f$ uniformly.



Note that $f'(0) = 0$.

Derivatives do not converge

$$\text{But } f'_n(0) = \frac{1}{j_n} j_n \cos(j_n x) \xrightarrow{n \rightarrow \infty} \pm \infty$$

limiting function can be
nowhere differentiable

Weierstrass $f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x),$
 $0 < a < 1, b > 0 \text{ odd}, ab > 1 + \frac{3}{2}\pi$

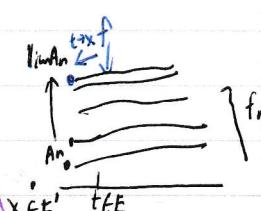
$f'_n(x)$ doesn't converge for typical x (fluctuates between $\pm \infty$)
⇒ We need stronger conditions for convergence of f_n — see p70.

Theorem $f_n, f : E \rightarrow \mathbb{R}$ (not necessarily continuous)
 $f_n \rightarrow f$ uniformly on $E \subseteq X$

Let $x \in E'$. Assuming $\lim_{E \ni t \rightarrow x} f_n =: A_n \in \mathbb{R}$

$f(x)$ need not be defined
 $\Rightarrow \lim_n A_n$ and $\lim_{t \rightarrow x} f(t)$ exist and are equal, i.e.

$$\lim_n \left[\lim_{t \rightarrow x} f_n(t) \right] = \lim_{t \rightarrow x} \left[\lim_n f_n(t) \right]$$



Proof. 1) Since we don't know what the limit is, we show that A_n is Cauchy.

Let $\epsilon > 0$: Since $f_n \rightarrow f$ uniformly in E ,

$$\forall t \in E: |f_n(t) - f_m(t)| < \epsilon \text{ given } n, m \text{ large}$$

$$|A_n - A_m| \stackrel{\substack{\downarrow t \rightarrow x \\ \text{completeness}}}{\leq} \epsilon \Rightarrow A_n \rightarrow A$$

2) It suffices to show $A = \underline{\lim_{t \rightarrow x} f(t)}$

(more clearly, $A = A_0 + A_n - f_n(t) + f_n(t) - f(t)$)

$$|A - f(t)| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|$$

For $\epsilon > 0$, pick n : (fixed!)

After picking n , choose t : Define $\delta = \delta(\epsilon, n) = \delta(\epsilon)$ s.t.

$$|f_n(t) - A_n| < \epsilon/3 \quad \forall t \in B_\delta(x)$$

and then pick any such t .

Theorem

$\forall n, f_n : [a, b] \rightarrow \mathbb{R}$ differentiable, $\exists x_0 \in [a, b]$ s.t. $f_n(x_0) \xrightarrow{n \rightarrow \infty} f(x_0)$ (same limit)

Assume f'_n conv. uniformly on $[a, b]$.

Then (1) $f_n \rightarrow f$ uniformly on $[a, b]$ and

(2) $f'(x) = \lim_{n \rightarrow \infty} f'_n(x) \quad \forall x \in [a, b]$ the derivative converges

Key idea: Mean Value Theorem
Know Derivatives

$$\textcircled{1} \quad |f_n(x) - f_m(x)| \leq \dots \leq \epsilon$$

$$|(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)|$$

$$|f(b) - f(a)| \sqrt{m \cdot v \cdot 1} < \epsilon/2 \text{ for } m \geq N_1$$

$$= |(f'_n(\xi) - f'_m(\xi))(x - x_0)| \stackrel{n, m \geq N_2}{\leq} |x - x_0| \cdot \left(\frac{\epsilon}{2(b-a)} \right) \leq \epsilon/2$$

21355

4/9/2021

② We have established the pointwise limit $f := \lim_n f_n$

Let $x \in [a, b]$ fixed and

$$\forall t \neq x: q_n(t) = \frac{f_n(t) - f_n(x)}{t - x} \xrightarrow{n \rightarrow \infty} \frac{f(t) - f(x)}{t - x} (= g(t))$$

Furthermore, $q_n(t) \xrightarrow{t \rightarrow x} f'_n(x)$ (we used differentiability here)

$$\text{S. } |q_n(t) - q_m(t)| = \left| \frac{(f_n(t) - f_m(t)) - (f_n(x) - f_m(x))}{|t - x|} \right|$$

$$\stackrel{\text{Since } n, m \geq N(\varepsilon)}{\leq} \frac{\varepsilon}{2|b-a|} \quad \forall n, m \geq N(\varepsilon)$$

\therefore As $n \rightarrow \infty$, $q_n(t)$ converges uniformly in $t \in [a, b]$ to $g(t)$

Then $\forall x \in [a, b]$:

$$\lim_{t \rightarrow x} \left(\lim_n q_n(t) \right) = \underbrace{\lim_{n \rightarrow \infty} \left[\lim_{t \rightarrow x} q_n(t) \right]}_{\substack{\text{w/ prev} \\ \text{theorem}}} = \underbrace{\lim_{n \rightarrow \infty} f'_n(x)}_{f'(x)}$$

Since x was arbitrary, we conclude that $\forall x \in [a, b], f'(x) = \lim_{n \rightarrow \infty} f'_n(x)$ as required.

Recall, for $g(x) := \sum_{k \geq 0} c_k x^k$ (power series) (restricted to \mathbb{R}),

$R := (\overline{\lim_n} |c_n|)^{-1}$ is the radius of convergence.

i.e. if $|x| < R$, $\sum c_n x^n$ is absolutely convergent.

(Claim: (Termwise Differentiation))

$h(x) := \sum_{k \geq 1} k \cdot c_k x^{k-1}$ has the same R as $g(x)$.

Proof $\overline{\lim_n} |c_n| = \overline{\lim_n} (|\overline{f_n}| \cdot \overline{f'_n})$

original coeffs derivative coeffs

\Rightarrow In every compact subset of $(-R, R)$ (such as the closed interval $[-R+\varepsilon, R-\varepsilon]$)

$h(x) = g'(x)$, furthermore every power series is infinitely differentiable.

to show

Note we used the previous theorem, that the derivative converges uniformly (p 61), and to the 'actual' derivative (p 70).

21.355

4/12/2021

$f \in C^{n-1}$: $n-1$ times continuously differentiable

(\Rightarrow the n^{th} derivative need not be continuous.)
but interior

Theorem (Taylor)

$f \in C([a, b], \mathbb{R})$, $n \in \mathbb{N}^+$, $f^{(n-1)} \in C[a, b]$ and diff. on (a, b) .

Let $a \leq x \leq b$.

$$P(t) := \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!} (t-x)^k \quad (\text{"Expansion" around } x) \quad \text{depends on } \beta$$

$$\Rightarrow \exists \xi \in (a, \beta) \text{ or } (\beta, b) \text{ s.t. } f(\beta) = P(\beta) + \frac{f^{(n)}(\xi)}{n!} (\beta-x)^n$$

Proof. (certainly $\exists! M$ s.t. $f(\beta) = P(\beta) + M(\beta-x)^n$).

WTS: $M = \frac{1}{n!} f^{(n)}(\xi)$ for some $\xi \in (a, \beta)$

Set the 'difference' $g(t) := f(t) - P(t) - M(t-x)^n$, $t \in [a, b]$

$$\Rightarrow g^{(n)}(t) = f^{(n)}(t) - 0 - Mn!$$

So we need to find $\xi \in (a, \beta)$ s.t. $g^{(n)}(\xi) = 0$ ($\Rightarrow f^{(n)}(\xi) = Mn!$)

Recall MVT: $g(\alpha) - g(\beta) = (\alpha - \beta) g'(\xi_1)$

Notice that $g(\beta) = 0$ (that's how we chose $M!$), and [we have $g^{(k)}(x) = 0 \quad \forall k=0 \dots n-1$]
automatically

Then $g'(\alpha) = g'(\xi_1) = 0 \Rightarrow \exists \xi_2 \in (\alpha, \xi_1) \text{ s.t. } g''(\xi_2) = 0 \text{ etc...}$

Theorem (L'Hopital)

See p68.
Recall that

$$\left[\begin{array}{l} f, g \in C^1([a, b], \mathbb{R}), \text{ diff. on } (a, b) \\ \exists x \in (a, b) \text{ s.t. } [f(b) - f(a)] g'(x) = [g(b) - g(a)] f'(x) \\ \left(\frac{|f(b) - f(a)|}{|g(b) - g(a)|} = \frac{f'(x)}{g'(x)} \right) \end{array} \right]$$

Let $-\infty < a, b < \infty$, f, g diff. in (a, b) , $g'(x) \neq 0$ for $x \in (a, b)$.

Assume $\frac{f'(x)}{g'(x)} \xrightarrow{x \rightarrow a} A \in [-\infty, \infty]$

and (1) $f(x) \rightarrow 0$ and $g(x) \rightarrow 0$ (as $x \rightarrow a$)

OR

(2) $f, g \rightarrow \pm\infty$ as $x \rightarrow a$

Then $\frac{f(x)}{g(x)} \rightarrow A$ (as $x \rightarrow a$)

Proof of (1) let $-\infty \leq A < \infty$

(choose $r \in \mathbb{R}$, $A < r$. By L^t), $\exists c \in (a, b)$ st. $\frac{f''(x)}{g'(x)} < r \forall x \in (a, c)$

let $a < x < y < c$, by , $\exists z \in (x, y)$ st.

$$\frac{f(y) - f(x)}{g(y) - g(x)} = \frac{f'(z)}{g'(z)} < r . \text{ As } x \rightarrow a, \text{ by (1) we have (and sequence criterion)}$$

$$\frac{f(y)}{g(y)} \leq r, \forall y \in (a, c).$$

$$\therefore \lim_{y \rightarrow a} \frac{f(y)}{g(y)} \leq r \quad \forall r > A$$

$$\therefore (\quad \downarrow \quad) \leq A$$

21-355

4/14/2021

$$\text{If } A = \infty, \lim_{y \rightarrow \infty} (\cdot) \leq A \quad \checkmark$$

$\frac{f(y)}{g(y)} \geq r'$ let $r' < A$

$A > \infty$ (i.e. $A \in \mathbb{R}$) we show similarly that $\lim_{y \rightarrow \infty} \frac{f(y)}{g(y)} \geq A$

$$\exists C, G(a, b) \text{ s.t. } \frac{f'(x)}{g'(x)} \geq r' \forall x \in (a, b)$$

$$A \leq \lim_{y \rightarrow \infty} \frac{f(y)}{g(y)} \leq \lim_{y \rightarrow \infty} \frac{f(y)}{g(y)} \leq A$$

$$\frac{f(y)-f(x)}{g(y)-g(x)} = \frac{f'(z)}{g'(z)} >$$

$$\therefore A = \lim_{y \rightarrow \infty} \frac{f(y)}{g(y)}$$

The Riemann Integral

"more ordered/heatsy"
than differentiation

$f: [a, b] \rightarrow \mathbb{R}$, bounded : $|f| \leq M < \infty$ but necessarily continuous!

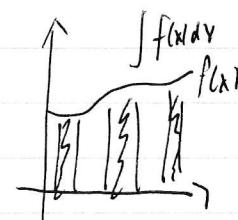
Define the partition

$$P = \{a = p_0 < p_1 < p_2 \dots < p_n = b\}$$

Define the upper Riemann sum:

$$\rightarrow U(P, f) := \sum_{k=0, \dots, n-1} \Delta_k p \cdot \sup_{\substack{x \in [p_k, p_{k+1}] \\ p_{k+1} - p_k := \Delta_k}} f(x)$$

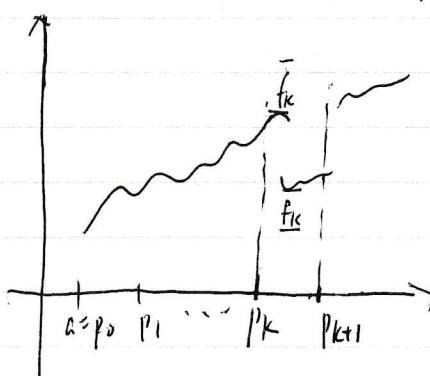
(1) $f_n \rightarrow f$
 $\int f_n \rightarrow \int f$



$$\int f(x) \mu(dx)$$

Lebesgue
mass/probability
strength

$$L(P, f) = \sum_k \Delta_k p \cdot \inf_{x \in [p_k, p_{k+1}]} f(x)$$



Remarks lower bound upper bound.

(1) $m \leq f \leq M$:

$$m(b-a) \leq L(P) \leq U(P) \leq M(b-a)$$

(2) $P \subseteq Q \Rightarrow U(Q) \leq U(P)$

e.g. add one point to
the partition

po $c_{p_1} \dots c_{p_k} < q < c_{p_{k+1}} \dots$

Proof. $U(P) - U(Q) = (p_{k+1} - p_k) \sup_{[p_k, p_{k+1}]} f - (p_{k+1} - q) \sup_{[q, p_{k+1}]} f - (q - p_k) \sup_{[p_k, q]} f$

$$\geq \underline{c}_{kp} \cdot \bar{f}_k - \bar{f}_k (p_{k+1} - q - p_k) = 0$$

(3) $P \subseteq Q \Rightarrow L(Q) \geq L(P)$

(3) $H(p, p') \Rightarrow L(P) \leq U(P')$

Proof. $Q := P \cup P'$ (common refinement)

$$\Rightarrow L(P) \leq L(Q) \leq U(Q) \leq U(P')$$

$$\stackrel{P \subseteq Q}{\text{↑}} \quad \stackrel{P' \subseteq Q}{\text{↑}} \quad Q \geq P'$$