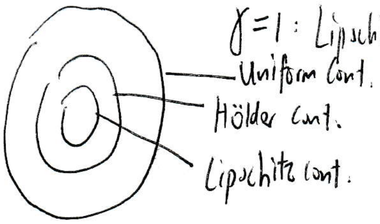
21-356 Midterm 2 (+)

- (X,d) is metric space $\Rightarrow$ $(C_b(X,\mathbb{R}), d_\infty)$ is complete. (uniform convergence)

- Let $\{f_n\}_{n=1,2,\dots}: X \to \mathbb{R}$. If $\{f_n\} \rightrightarrows f$, then f is continuous. ["pointwise limit is continuous"] (Also, continuity on compact set = uniform continuity) function achieves min and max on compact set

- Hölder Continuity: $\exists \gamma \in [0,1]$, $|f|_\gamma := \sup_{\{x_1, x_2 \in X, x_1 \neq x_2\}} \dfrac{d_Y(f(x_1), f(x_2))}{d_X(x_1,x_2)^\gamma} < \infty$

$\gamma = 1$: Lipschitz continuity



- [Exchange limits under integration] $\{f_n\}: [a,b] \to \mathbb{R}$ R-intg, $f_n \rightrightarrows f$ on $[a,b]$ $\Rightarrow$ f R-intg, $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$

* - Let (K,d) compact, $\{f_n\} \in C(K,\mathbb{R})$, $\{f_n\} \rightrightarrows f \Rightarrow \{f_n : n \in \mathbb{N}\}$ is equicontinuous $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall f \in \mathcal{F})(\forall x_1, x_2 \in X)$ $d_X(x_1,x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \varepsilon$.

- (Arzela-Ascoli) Assume (K,d) is compact. If $\{f_n\}_{n=1,2,\dots}$ is pointwise bounded and equicontinuous, then
  (i) $\{f_n : n \in \mathbb{N}\}$ is uniformly bounded.
  (ii) $\{f_n\}$ has a convergent subsequence in d_∞ (= uniformly convergent)

COROLLARY. Let (K,d) compact, $\mathcal{F} \subseteq C(K,\mathbb{R})$. $\mathcal{F}$ compact $\iff$
(i) $\mathcal{F}$ is closed (w.r.t. d_∞)
(ii) $\mathcal{F}$ is uniformly bounded
(iii) $\mathcal{F}$ is equicontinuous.

- (Dini) Assume (K,d) is compact, $f \in C(K,-)$ and
  (i) $f_n \in C(K,-)$
  (ii) $\forall x \in K$, $f_n(x) \to f(x)$ [pointwise limit]
  (iii) $(\forall n \in \mathbb{N})(\forall x \in K)$ $f_n(x) \leq f_{n+1}(x)$ [monotonicity]

  Then $f_n \rightrightarrows f$ as $n \to \infty$

- (Stone-Weierstrass) Weierstrass: $\mathcal{P}[0,1]$ (set of polynomials) is dense in $(C[0,1], d_\infty)$ e.g. $\{B_{f,n}(x) = \sum_{k=1\dots n} f(\frac{k}{n})\binom{n}{k} x^k (1-x)^{n-k}\} \to f$ as $n \to \infty$. ("Bernstein polynomials")
  $[a,b]$ $\quad$ $[a,b]$ by translation.

  Stone (Generalization) Let $\mathcal{F} \subseteq (C(K,\mathbb{R}), d_\infty)$.
  (i) $\mathcal{F}$ is an algebra ($f,g \in \mathcal{F} \Rightarrow f+g, f\cdot g, tf, tg \in \mathcal{F}$)
  (ii) $\mathcal{F}$ separates points ($\forall x,y \in K, \exists f, f(x) \neq f(y)$)
  (iii) $\mathcal{F}$ contains constant functions

  $\Rightarrow \mathcal{F}$ is dense in $(C(K,\mathbb{R}), d_\infty)$

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POWER SERIES

· (Ratio test) $\limsup_{n\to\infty} \sqrt[n]{|c_n|} \begin{cases} <1 & \Rightarrow \sum_{n\geq 0}^{\infty} c_n \text{ converges} \\ >1 & \Rightarrow \quad " \quad \text{diverges} \end{cases}$

(Harmonic series) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p>1$.

Cauchy Schwarz: $\left(\sum_{i=1}^n u_i v_i\right)^2 \leq \left(\sum_{i=1}^n u_i^2\right)\left(\sum_{i=1}^n v_i^2\right)$

$\|Ax\| \leq \|A\|_{op} \|x\|$ by definition of $\|A\|_{op}$ (same norm)

$\|AB\| \leq \|A\| \cdot \|B\|$

$X \mapsto \|X\|$ is continuous

A Norm must satisfy (i) $\|v\|=0 \Rightarrow v=\vec{0}$
$V \to [0,\infty)$
(ii) $\forall \alpha \in \mathbb{R}$, $\|\alpha V\| = |\alpha| \|V\|$
(iii) $\|V+w\| \leq \|V\| + \|w\|$

$\sum_{n=0}^{\infty} a_n x^n$ converges when $\limsup_{n\to\infty} \sqrt[n]{|a_n||x|^n} = |x| \underbrace{\limsup \sqrt[n]{|a_n|}}_{=:\alpha} < 1$.

radius of convergence

$\Rightarrow$ So we want $x \in (-R, R) = \left(-\frac{1}{\limsup_{n\to\infty}\sqrt[n]{|a_n|}}, \frac{1}{\limsup_{n\to\infty}\sqrt[n]{|a_n|}}\right)$ to guarantee convergence.

$\Rightarrow$ In fact, $r<R \Rightarrow \sum_{n=0}^{k} a_n x^n \rightrightarrows f(x)$ on $[-r,r]$, using absolute convergence.

(Termwise Integration) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Then $F(x) := \int_0^x f(z)\,dz$ exists, and $F(x) = \sum_{n=1}^{\infty} \frac{a_n}{n+1} x^{n+1}$ on $(-R,R)$ if R is the radius of convergence of f. R is also the radius of convergence of F.

(Termwise Differentiation) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ have radius of convergence R. Then $\sum_{n=1}^{\infty} n a_n x^{n-1}$ also has radius of convergence R and $f'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1}$.

with constant term a_0

Using FTOC, $\frac{d}{dx}\left[\int_0^x \sum_{n=1}^{\infty} a_n n s^{n-1}\,ds\right] = \frac{d}{dx}\left[\sum_{n=1}^{\infty} \int_0^x a_n n s^{n-1}\,ds\right] = \frac{d}{dx} \sum_{n=1}^{\infty} a_n x^n = \frac{d}{dx} f(x) = f'(x)$.

(Abel's Theorem) If $\sum c_n$ converges, and $f(x) = \sum_{n=0}^{\infty} c_n x^n$ has $R \geq 1$ (in particular we have equality when $-1<x<1$), then $\lim_{x\nearrow 1} f(x) = \sum_{n=0}^{\infty} c_n$. Corollary: if $c_n = a_n z^n$ for $z \in [-R,R]$, then $\lim_{x\uparrow 1} g(xz) = \sum_{n=0}^{\infty} a_n z^n$.

(Taylor's Theorem) If $f(x) := \sum_{n=0}^{\infty} c_n x^n$ converges in $|x|<R$, and $a \in (-R,R)$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 \dots \text{ when } |x-a| < R-|a|.$$

(Necessary condition for non-triviality) Let R be the radius of convergence of $f(x) = \sum_{n=0}^{\infty} c_n x^n$, $E = \{x : f(x) = 0\}$. If there exists $a \in E' \cap (-R,R)$, then $c_n = 0\ \forall n$ ($\Leftrightarrow f \equiv 0$).
Sufficient condition for triviality

BANACH CONTRACTION. Let (X,d) be a complete m.s. and $\phi: X \to X$ be a contraction $((\exists \lambda<1)(\forall x,y \in X)(d(\phi(x),\phi(y)) \leq \lambda d(x,y)))$.
Then there exists a unique fixed point $\bar{x} \in X$ such that $\phi(\bar{x}) = \bar{x}$.

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A linear mapping $L: X \to Y$ is continuous iff L is bounded. We let $\mathcal{L}(X,Y)$ be the set of all bounded linear mappings from X to Y. $\mathcal{L}(X,Y)$ is equipped with the operator norm, $\|L\|_{\mathcal{L}} := \sup_{\|x\|_X \le 1} \|Lx\|_Y = \sup_{x \neq 0} \frac{\|Lx\|_Y}{\|x\|_X}$.

The first derivative [Frechét] (DF) is a bounded linear operator $L \in \mathcal{L}(X,Y)$ (e.g. $\mathbb{R}^n \to \mathbb{R}$, $\mathbb{R}^n \to \mathbb{R}^m$) for some $F: \underset{\text{open}}{U} \subseteq X \to Y$ at x_0 such that $\lim_{x \to x_0} \frac{\|F(x) - F(x_0) - L(x - x_0)\|_Y}{\|x - x_0\|_X} = 0$. For $f: \mathbb{R}^n \to \mathbb{R}$, if all partial derivatives exist and are continuous, then

$$DF_{x_0}: \mathbb{R}^n \to \mathbb{R}, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \sum_{i=1}^n \underbrace{\frac{\partial F}{\partial x_i}(x_0)}_{\text{number!}} x_i, \text{ in other words } DF_{x_0} = \left[\frac{\partial F}{\partial x_1}(x_0) \frac{\partial F}{\partial x_1}(x_0) \dots \frac{\partial F}{\partial x_n}(x_0)\right]$$

F is continuously differentiable if the map $\mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, \mathbb{R})$, $x_0 \mapsto DF_{x_0}$ is continuous. Equivalent, $F \in C^1$ if all the partial derivatives $\frac{\partial F}{\partial x_i}: \mathbb{R}^n \to \mathbb{R}$ are continuous. A weaker notion is the directional derivative $\left(D_v F|_{x_0} = \lim_{t \to 0} \frac{F(x+tv) - F(x)}{t} \right)$ Note $\frac{\partial F}{\partial x_i}\Big|_{x_0} := D_{e_i} F|_{x_0}$

(Chain Rule) Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, $(Z, \|\cdot\|_Z)$ be Banach spaces. If $F: X \to Y$, $G: Y \to Z$, F is differentiable at x_0, and G is differentiable at y_0, then $G \circ F$ is differentiable at x_0 and $D(G \circ F)|_{x_0} = DG|_{y_0} \circ DF|_{x_0}$

(Inverse Function Theorem) Suppose X and Y are Banach spaces and U is an open subset of X. Let $F \in C^1(U, Y)$ and assume that for some $x_0 \in U$, $DF(x_0)$ is invertible. Then there exists $r > 0$ such that $B = B(x_0, r) \subseteq U$. Furthermore, $f|_B$ is an open mapping from B to $V := f|_B(B)$ and injective, so its inverse $g := (f|_B)^{-1}: V \to B$ exists. Lastly, $g \in C^1(V, B)$ and $Dg(y) = (Df|_{g(y)})^{-1}$.

(Implicit Function Theorem) Let $f \in C^1(\Omega, \mathbb{R}^n)$ for some open set $\Omega \subseteq \mathbb{R}^{n+m}$, and $f(a,b) = 0$ for some $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. Let $A = Df(a,b) = [A_x \ A_y]$ and assume A_x is invertible. Then there exists open set $U \subseteq \mathbb{R}^{n+m}$ containing (a,b), $W \subseteq \mathbb{R}^m$ containing b, such that for all $y \in W$, there exists a unique $(x,y) \in U$ such that $f(x,y) = 0$. In other words, there is $g: W \to \mathbb{R}^n$ such that $f(\underbrace{g(y)}_{x = g(y)}, y) = 0 \ \forall y \in W$. Furthermore, $g \in C^1(W, \mathbb{R}^n)$ and $Dg(b) = -(A_x)^{-1} A_y = -(D_x f)^{-1}(D_y f)$.

(Leibniz Integral Rule) Let $f \in C^1([a,b] \times [c,d])$ (with $x \in [a,b]$, $t \in [c,d]$), $g(t) := \int_a^b f(x,t)\,dx$ for $c \le t \le d$. Then for all $t \in [c,d]$, $g'(t) = \int_a^b \frac{\partial f}{\partial t}(x,t)\,dx$.
More generally, $\frac{d}{dt}\left(\int_{a(t)}^{b(t)} f(x,t)\,dx\right) = f(x, b(t)) \cdot \frac{d}{dt} b(t) - f(x, a(t)) \cdot \frac{d}{dt} a(t) + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(x,t)\,dx$ if $a(t), b(t) \in C^1[c,d]$

(2D MVT, Clairaut) Notation: $D_{ij} f = D_i(D_j f)$. If $f \in C^2(E)$, then $D_{21} f = D_{12} f$ on E.

(Change of variables) The Jacobian of a function $\varphi: \underset{\text{open}}{E} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ at x, if $\varphi(x_1 \dots x_n) = (y_1 \dots y_n)$, is $\det \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & & & \\ \vdots & & & \\ \frac{\partial y_n}{\partial x_1} & & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix}$
Suppose E is also connected, φ is C^1 and injective, and $J(\varphi) \neq 0$ on E. Let $f: \mathbb{R}^n \to \mathbb{R}$ be continuous with support contained in $\varphi(E)$. Then $\boxed{\int_{\mathbb{R}^n} f(\varphi(x)) |J(\varphi)|\,dx_1 \dots dx_n = \int_{\mathbb{R}^n} f(y)\,dy_1 \dots dy_n}$

Weaker version (lcx). Let f be continuously with compact support K, $\varphi(x) = x$ whenever $|x| \ge R$ (smooth). Then the same formula (usually) applies; we don't need 1:1

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$M \subseteq \mathbb{R}^n$ is a (differentiable) manifold of dimension $d \Leftrightarrow \forall p \in M$, $\exists \Omega$ ball around p in M such that we have a continuous bijection (with continuous inverse) $\varphi: U \to \Omega$ for $U \subseteq \mathbb{R}^d$

$\underset{\subseteq M}{\varphi^{-1}: \Omega} \to \underset{\subseteq \mathbb{R}^d}{U}$ is smooth if there is open set $W \subseteq \mathbb{R}^n$ such that $p \in W$, and for some smooth function $\in: W \to \mathbb{R}^d$,

$\in(\xi) = \varphi^{-1}(\xi)$ for all $\xi \in W \cap \Omega$. (U, Ω) is the local chart (or coordinate chart). $T_pM = \mathrm{Range}(D\varphi(\varphi^{-1}(p))) = \mathrm{Range}(D\psi(\psi^{-1}(p)))$ is a vector field of dimension d.

Tangent space: Space of vectors! In Euclidean space this is the "gradient vector" $b^j = \sum_i \frac{\partial y_j}{\partial x_i} a^i$ for $V = \sum_{j=1}^d b^j \frac{\partial \psi}{\partial y_j}$

Cotangent space: Set of linear mappings! $T_pM^*: \{T_pM \to \mathbb{R}\}$ " "derivative" $\beta_j = \sum_i \alpha_i \frac{\partial x_i}{\partial y_j}$ for $W = \sum_{j=1}^d \beta_j \, dx_j$ basis in (V, ψ)

(V, φ), (W, ψ)

$\{\frac{\partial \varphi}{\partial x_1} \dots \frac{\partial \varphi}{\partial x_d}\}$ $\{\frac{\partial \psi}{\partial x_1} \dots \frac{\partial \psi}{\partial x_d}\}$

We denote $\partial y_j / \partial x_i := \frac{\partial (\psi^{-1} \circ \varphi)^j}{\partial x_i}$. Then $V_i = \frac{\partial \varphi}{\partial x_i} = \frac{\partial \psi \circ (\psi^{-1} \circ \varphi)}{\partial x_i} = \sum_j \frac{\partial \psi}{\partial y_j} \frac{\partial (\psi^{-1} \circ \varphi)_j}{\partial x_i} = \sum \frac{\partial y_j}{\partial x_i} \frac{\partial \psi}{\partial y_j}$

$dx_i := d\varphi_i^{-1}$, where $d\varphi_i^{-1}(V_j) = \frac{\partial (\varphi^{-1} \circ \varphi)_i}{\partial x_j} = \delta_i^j$ (Kronecker Delta)

$\therefore V = \sum a^i V_i = \sum a^i (\sum \cdot) = \sum_{j=1}^d \underbrace{\left(\sum_{i=1}^d \alpha_i \frac{\partial (\psi^{-1} \circ \varphi)^j}{\partial x_i}(x)\right)}_{b^j} \frac{\partial \psi}{\partial y_j}$

Let $f: M \to \mathbb{R}$, $df[V] = \nabla f \cdot V = \sum_{i=1}^d a^i \frac{\partial f \circ \psi}{\partial x_i} = \sum_{j=1}^d a^j \sum_{j=1}^d \alpha_j dx_j(V_j)$ $= \sum_{j=1}^d a^j \sum_{i=1}^d \beta_i dy_i(V_j) = \sum_{j=1}^d a^j \beta_j$

Assume $V = \sum a^i \frac{\partial \psi}{\partial x_i}$, $f = \underbrace{\alpha_1 dx_1 + \dots \alpha_n dx_n}_{\text{in } (U, \varphi)} = \underbrace{\beta_1 dx_1 + \dots \beta_n dx_n}_{\text{in } (V, \psi)}$

if $f = \psi^{-1}$ (special case)

stuff supplementing Real I

A topological space T is connected iff there is no proper subset of S clopen in T ... (i.e. not disconnected) isometry ✓ $\forall x_1, x_2 \in X$, $E_{1,2} \subseteq X$ connected, $x_1 \in$, $x_2 \in E_{1,2}$

$\ell^p = \{\{a_n\}_{n=1,2,\dots} \mid a_n \in \mathbb{R}, \sum_{n=1}^{\infty} |a_n|^p < \infty\}$ for $p \in [1, \infty) \Rightarrow (\ell^p, d_p)$ is separable (ℓ^2, d_2 is not compact)

$\ell^\infty = \{\{a_n\}_{n=1,2,\dots} \mid a_n \in \mathbb{R}, \sup_{n \in \mathbb{N}} |a_n| < \infty\}$ $\Rightarrow (\ell^\infty, d_\infty)$ is not separable ($\Rightarrow$ not compact)

Continuous functions map compact to compact, connected to connected.

$f: X \to Y$ cont bijection, X compact $\Rightarrow f^{-1}$ continuous

$h^{-1}((a,b)) = h^{-1}((-\infty, b) \cap (a, \infty)) = h^{-1}((-\infty, b)) \cap h^{-1}((a, \infty))$

boundary of a set Ω is $\partial\Omega = \bar{\Omega} \setminus \Omega^\circ$. It is the only place an indicator function on a compact convex set could be discontinuous.

A non-limit point can only be outside Ω (compact $\Rightarrow$ closed), and $d(x, \Omega) > 0$. An interior point is clearly continuous.

$\partial\Omega = \bar{\Omega} \cap \overline{\Omega^c} = \partial(X \setminus \Omega) = \{x \in X : (\forall U \text{ open}) \text{ If } x \in U, \text{ then } \underset{\text{'not too far away'}}{\Omega \cap U \neq \emptyset} \text{ and } \underset{\text{'interior'}}{U \setminus \Omega \neq \emptyset}\} = \{x \in X : (\forall U \text{ open}) \text{ if } x \in U \text{ then } E \cap U \neq \emptyset \text{ and } U \setminus E \neq \emptyset\}$

Let $X = C([0,2], [0,1])$, $d(f,g) = \int_0^2 |f(x) - g(x)| dx$. Then (X, d) is not complete.

$x \mapsto \mathrm{dist}(x, E) = \inf_{e \in E} d(x, e)$ is Lipschitz continuous with constant 1.

Let (X, d) be given, $f_n: K \to \mathbb{R}$ continuous functions on compact set $K \subset X$. $\{f_n\}$ equicontinuous, pointwise convergence $\Rightarrow f_n \to f$ uniformly.