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Recall def: Metric Space

distance must be finite

let X be a non-empty set and $d: \underline{X \times X \to [0,\infty)}$ ("distance function")

"Holy Trinity"

iff

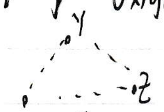
existential universal

$\subseteq \quad \supseteq / < >$

(X,d) is a metric space if (i) $d(x,x)=0 \Leftrightarrow x=y$ $\left(\begin{smallmatrix}\Rightarrow (a)\\ \Leftarrow (b)\end{smallmatrix}\right)$

(ii) $(\forall x,y \in X)$ $d(x,y)=d(y,x)$

(iii) (Δ-inequality) $\forall x,y,z \in X$ $d(x,z) \leq d(x,y)+d(y,z)$



there might be no points here! No algebraic properties! $(+,-,\ldots)$

$1(b)+(2)+(3)$ = 'pseudo-metric'

Example: 1. $(\mathbb{R}^d, d_2)$ $X=(x_1\ldots x_d)^T$, $y=(y_1\ldots y_d)^T$

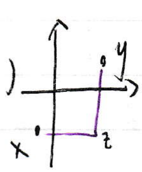
$d_2(x,y) = \sqrt{(x_1-y_1)^2+\ldots (x_d-y_d)^2}$

Prove that $(\mathbb{R}^d,d_2)$ is a metric (Th 1.35 in Rudin)

$d_2(x,y) \leq d_2(x,z)+d_2(z,y)$

For higher dimensions, just focus on the two dimensions spanned by the 3 points.
Alternatively, derive the result from Cauchy-Schwarz.

2. (Manhattan Distance)



$d_1(x,y) = |y_1-x_1|+\ldots |y_d-x_d|$

(i) ✓ use $|a| \geq 0$

(ii) $|a| = |-a|$

(iii) $|z_1-x_1| \leq |z_1-y_1|+|y_1-x_1|$

$d_1(x,y) =$ $\leq |z_1-y_1| + |y_1-x_1| + |z_2-y_2| + |y_2-x_2|$

$\overline{d_1(x,z) + d_1(y,z)}$

3. $X = \mathbb{R}^d$, $d_\infty(x,y) := \max_{i=1\dots d} |y_i - x_i|$

Claim (X, d_∞) is a metric spaces.

(i), (ii)

(iii) $d(x,z) = \max_{i=1\dots d} |z_i - x_i| = |z_j - x_j|$ for some j

$\leq |z_j - y_j| + |y_j - x_j|$

$\leq \max_{i=1\dots d} |z_i - y_i| + \max_{i=1\dots d} |y_i - x_i|$

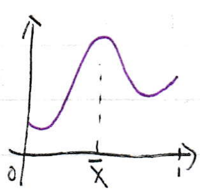
$= d_\infty(y,z) + d_\infty(x,y)$

Remark: The notion of convergence is the same for ①, ② and ③. Results can be proved just for topological spaces (X, T) ← $\emptyset \in T$, $X \in T$, intersection of finite ✓ arbitrary union

Further examples of metric spaces:

4. $X = C([0,1], \mathbb{R}) = \{f : [0,1] \to \mathbb{R},\ f \text{ continuous}\}$

$d_\infty(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|$ ← exists from compactness, 355?



5. $X = C_b((a,b), \mathbb{R})$ $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$

$= \{f \in C((a,b), \mathbb{R})$ and f - bounded$\}$

f is bounded if $\exists M \in \mathbb{R}$ such that $\forall x \in (a,b)$ $|f(x)| \leq M$.

$d_\infty(f,g) = \sup_{x \in (a,b)} |f(x) - g(x)|$

Claim: d_∞ is a metric (~~Check~~ Ensured that $d_\infty : X \times X \to [0, \infty)$)

(i) $\sup(\cdot) = 0 \Leftrightarrow f = g$ (ii) $|\cdot|$ is symmetric

(iii) needs some work:

let $f,g,h \in X$, and consider arbitrary $\varepsilon > 0$. There exists $x \in (a,b)$ $= x(\varepsilon)$ such that

$$\varepsilon + |f(x) - h(x)| > d_\infty(f,h).$$

Using the Δ-inequality,
$$\begin{aligned} d_\infty(f,h) &< \varepsilon + |f(x_\varepsilon) - h(x_\varepsilon)| \\ &\le \varepsilon + |f(x_\varepsilon) - g(x_\varepsilon)| + |g(x_\varepsilon) - h(x_\varepsilon)| \\ &\le \varepsilon + d_\infty(f,g) + d_\infty(g,h) \end{aligned}$$

$$\inf_{\varepsilon>0} d_\infty(f,h) \le \inf_{\varepsilon>0} \varepsilon + d_\infty(f,g) + d_\infty(g,h)$$

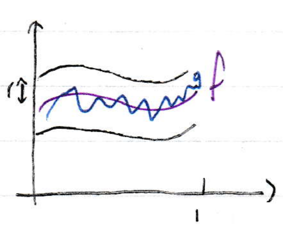
$$d_\infty(f,h) \le d_\infty(f,g) + d_\infty(g,h)$$

6. (Exercise) $X = C([0,1],\mathbb{R})$, $d_1(f,g) = \int_0^1 |f(x) - g(x)|\,dx$

d_∞ and d_1 are different in topology/convergence, as in functional analysis. ① is more "interesting".

7. ~~(X,d) metric space~~ Some definitions

- A (open) <u>ball</u> in a metric space (X,d), for $x \in X, r>0$ is
$$B(x,r) = \{y \in X, d(x,y) < r\}$$
Take $X = C([0,1],\mathbb{R})$, d_∞ to be the metric.



$$B(f,r) = \{g \in X, \max_{x\in[0,1]} |f(x) - g(x)| < r\}$$
$$= \{g \in X : \forall x \in [0,1]\ |f(x) - g(x)| < r\}$$

- Diameter of a set: Given (X,d) metric space, $A \subseteq X$,
$$\operatorname{diam}(A) = \sup\{d(x,y) : x \in A, y \in A\} \in [0,\infty]$$

subset of metric space is also "

- $A \subseteq X$ is <u>bounded</u> if $\operatorname{diam}(A) < \infty$. X in ④ is not bounded, but $Y = C([0,1],[-7,7])$ has diam $= 14$.

(7) X- nonempty, $d_{discrete}(x,y) = \begin{cases} 0 & x=y \\ 1 & \text{else} \end{cases}$

$X = \mathbb{R}$, $B(0,1) = \{0\}$

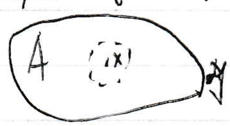
$B(0, 1.0001) = \mathbb{R}$

Topology of Metric Spaces

(every metric space is a topological space)

Def Let (X,d) be a metric space.

(i) $x \in A$ is an interior point of A if $\exists r>0$, $B(x,r) \subseteq A$



x ✓

y X

(ii) A is an open set if $\forall a \in A$, a is an interior point.

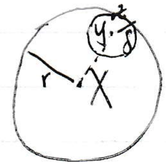
(iii) Point $p \in X$ is a limit point (i.e. accumulation point) of A if $\forall r>0$,

$(B(p,r) \setminus \{p\}) \cap A \neq \emptyset$. (interior point $\not\Rightarrow$ limit point)

(iv) A is closed if it contains all of its limit points $\Leftrightarrow A' \subseteq A$.

Lemma. Let (X,d) be metric space. Then $\forall x \in X$, $\forall r>0$, $B(x,r)$ is an open set.

Pf. Let $y \in B(x,r)$, w.T.S y is an interior point.



$d(x,y) < r$, $\delta = r - d(x,y)$

(Claim: $B(y,\delta) \subseteq B(x,r)$) // let $z \in B(y,\delta)$,

$d(x,z) \leq d(x,y) + d(y,z) < d(x,y) + \delta = d(x,y) + r - d(x,y) = r$.

So $z \in B(x,r)$, and $B(y,\delta) \subseteq B(x,r)$

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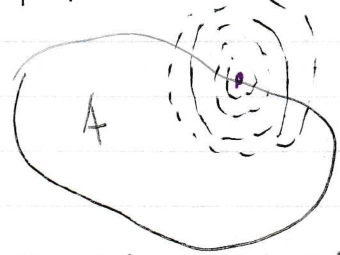
Exercise Let (X,d) be metric space, $A \subseteq X$, $p \in X$. Assume $\{r_n\}_{n=1,\dots}$, $r_n > 0$, $r_n \to 0$ as $n \to \infty$. If $\forall n \in \mathbb{N}$, $B(p, r_n) \setminus \{p\} \cap A \neq \emptyset$, then p is a limit point of A.



Proof Let $r > 0$, WTS $B(p,r) \setminus \{p\} \cap A \neq \emptyset$

(Can find even smaller ball) Since $r_n \to 0$, as $n \to \infty$, $\exists n_0$ s.t. $\forall n \geq n_0$, $r_n < r$.

In particular, $(B(p,r) \setminus \{p\}) \cap A \supseteq (B(p, r_{n_0}) \setminus \{p\}) \cap A \neq \emptyset$

Theorem (X,d) is metric space.

(i) $\emptyset$ is open and X is open. (by definition)

(ii) Let $A_i \subseteq X$ for all $i \in I$; $\forall i \in I$, A_i is open. Then $\bigcup_{i \in I} A_i$ is open.

(iii) $n \in \mathbb{N}$, $A_1 \dots A_n$ open subsets of X. Then $A_1 \cap A_2 \dots A_n$ is open.

Pf (ii) $\bigcup_{i \in I} A_i = \{x \in X : (\exists i \in I) x \in A_i\}$, $\bigcap_{i \in I} A_i = \{x \in X : (\forall i \in I) x \in A_i\}$

Say $x \in \bigcup_{i \in I} A_i$. Then $\exists j \in I$, $x \in A_j$.

Since A_j is open, $\exists r > 0$ s.t. $B(x,r) \subseteq A_j$.

$A_j \subseteq \bigcup_{i \in I} A_i \Rightarrow B(x,r) \subseteq \bigcup_{i \in I} A_i$. So x is an interior point of $\bigcup_{i \in I} A_i$.

(iii) Let $A_1 \dots A_n$ be open, $x \in A_1 \cap A_2 \dots A_n$.

$\forall i = 1, \dots, n$, $x \in A_i$. Thus $\exists r_i > 0$, $B(x, r_i) \subseteq A_i$.

Let $r = \min\{r_1, r_2, \dots, r_n\} > 0$. Claim: $B(x,r) \subseteq A_1 \cap \dots A_n$.

Then $\forall i = 1 \dots n$, $B(x,r) \subseteq B(x, r_i) \subseteq A_i$.

$B(x,r) = \bigcap_{i=1}^{n} B(x,r) \subseteq \bigcap_{i=1}^{n} A_i$. So is an interior point of $\bigcap_{i \in I} A_i$.

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Lemma: (X,d) metric space, $K \subseteq X$. K is closed iff $X \setminus K$ is open.

Proof. ($\Rightarrow$) Assume K is closed, i.e. $K = K \cup K'$. Let $p \in X \setminus K$.

$p \notin K$, and $p \notin K' \Rightarrow \exists r > 0$, $(B(p,r) \setminus \{p\}) \cap K = \emptyset$.

$\Rightarrow B(p,r) \cap K = \emptyset$

$B(p,r) \subseteq X \setminus K$

p is an interior point of $X \setminus K \Rightarrow X \setminus K$ is open.

($\Leftarrow$) Assume $X \setminus K$ is open.

Let $p \in K'$, and assume $p \notin K$ ($p \in X \setminus K$) for sake of contradiction.

Then $\exists r > 0$, $B(p,r) \subset X \setminus K$.

$B(p,r) \cap K = \emptyset$

$(B(p,r) \setminus \{p\}) \cap K = \emptyset$

$\therefore p \notin K'$. Contradiction.

We have similarly the following theorems on closed sets, by the previous lemma and De Morgan's.

(i) $\emptyset, X$ are closed. ($\emptyset = X \setminus X$)

(ii) C_i, $i \in I$ are closed. Then $\bigcap_{i \in I} C_i$ is closed.

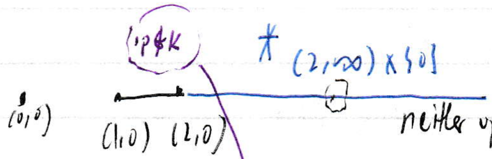
(iii) $C_1 \dots C_n$ closed. Then $C_1 \cup C_2 \dots C_n$ is closed.

Example: $(\mathbb{R}, |\cdot|)$, $d(x,y) = |x-y|$

(i) $(0,\infty)$ is open. $\bigcup_{x>0} B(x,x) = \bigcup_{x>0} (x-x, x+x) = \bigcup_{x>0} (0,2x)$

(ii) $K = [a,b]$ is closed. $\mathbb{R} \setminus K = (-\infty, a) \cup (b, \infty)$ is open!

(iii) $(\mathbb{R}^2, d_2)$ $[1,2]\times\{0\} =: K$ is closed.



neither open nor closed! $(2,0)\in A'$ but $(2,0)\notin A$

In $\mathbb{R}^2$ ($\mathbb{R}^n$) open sets are the arbitrary unions of open balls

Pf. Let $p \notin K$, where $p=(p_1,p_2)$.

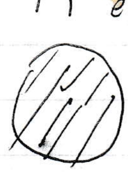
Then $B(p, p_2) \subseteq X-K$.

(i) $1 = p_1 - 2$ etc.

There are 4 cases to tediously consider...

(iv) $(\mathbb{R}^2, d_{\text{discrete}})$ $d(x,y) = \begin{cases} 0 & x=y \\ 1 & \text{else} \end{cases}$

$A = B_{d_2}((0,0),1)$



Is A open? Yes. Taking $p \in A$, $B(p,1)=\{p\} \subseteq A$.

So every set $(W \subseteq \mathbb{R}^2)$ is open in the discrete metric. Then every set is also closed.

<u>Topological Space</u> $(X, \mathcal{T})$ 'Define all the open sets'

$\Rightarrow \mathcal{T} \subseteq \mathcal{P}(X)$ (power set) is called a topology.

$\Rightarrow$ (i) $\emptyset, X \in \mathcal{T}$

(ii) $A_i \in \mathcal{T}$ for all $i \in I \Rightarrow \bigcup_{i\in I} A_i \in \mathcal{T}$

(iii) $A_1, \dots, A_n \in \mathcal{T} \Rightarrow \bigcap_{i\in I} A_i \in \mathcal{T}$.

Sets in $\mathcal{T}$ are called open sets,

- So we have for each metric space (X,d), the collection $\mathcal{T} = \{S \in X \mid S \text{ open wrt } d\}$ and the induced topological space $(X,\mathcal{T})$.
- $(X, \{\emptyset, X\})$ is a topological space.
- Some come from metric spaces, some don't.

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Let (X,d) be metric space, $A \subseteq X$. The closure of A is $\bar{A} = A \cup A'$. (Recall A' are limit points of A.)

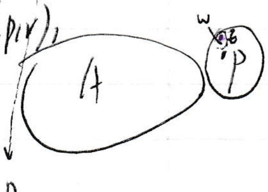
Lemma (i) $\bar{A}$ is closed set.

(ii) If $A_1 \subseteq A_2$ then $A_1' \subseteq A_2'$ and thus $\bar{A}_1 \subseteq \bar{A}_2$.

(iii) A is closed iff $A = \bar{A}$ ($\Leftarrow$ by definition and (i)
$\Rightarrow$ $A' \subseteq A \Rightarrow A \cup A' = \bar{A} \subseteq A$
$A \subseteq \bar{A}$ clearly)

(iv) $\bar{A}$ is equal to the intersection of all closed sets containing A.

Proofs. (i) We need to show that $X \setminus \bar{A}$ is open. Let $p \in X \setminus \bar{A}$.
Assume p is not an interior point of $X \setminus \bar{A}$. Then $\forall B(p,r)$,
$B(p,r) \cap \bar{A} \neq \emptyset$. We claim $B(p,r) \cap A \neq \emptyset$.



↳Pf. $\exists z \in \bar{A}$, $z \in B(p,r)$, $z \neq p$.
$d(z,p) < r$, so we take $\delta = r - d(z,p)$ and we have

$\begin{cases} B(z,\delta) \subseteq B(p,r) \\ z \in \bar{A} = A \cup A' \end{cases}$

Case ① $z \in A$, $z \in B(p,r) \cap A$
② $z \in A'$, $B(z,\delta) \cap A \neq \emptyset$
$\exists x \in B(z,\delta) \cap A$.
$x \in B(p,r) \cap A$ □

$p \notin \bar{A} \Rightarrow p \notin A$
$\Rightarrow (B(p,r) \setminus \{p\}) \cap A \neq \emptyset$.
Thus $p \in A' \subseteq \bar{A}$ contradiction!

(iv) Let K be a closed set such that $A \subseteq K$. Then $A' \subseteq K'$ and $\bar{A} \subseteq \bar{K} = K$.
$\therefore \bar{A} \subseteq \bigcap \{K : A \subseteq K, K\text{-closed}\} \subseteq \bar{A}$
Since $A \subseteq \bar{A}$, $\bar{A}$ closed, $\bar{A}$ belongs here

Exercise Show that

(i) A° (all interior points of A) is open
(ii) If $A_1 \subseteq A_2$ then $A_1^\circ \subseteq A_2^\circ$
(iii) A is open iff $A = A^\circ$
(iv) $A^\circ = \bigcup \{E : E \subseteq A, E\text{-open}\}$
(v) $(A^\circ)^\circ = A^\circ$ tr...
(vi) $X \setminus \bar{A} = (X \setminus A)^\circ$
If $E := X \setminus A$, $E^\circ = X \setminus (\overline{X \setminus E})$

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Boundary of a set : $\partial A = \bar{A} \setminus A^\circ$

Exercise. Show that $x \in \partial A \iff \forall r>0$ $B(x,r) \cap A \neq \emptyset$
$B(x,r) \cap (X \setminus A) \neq \emptyset$.

Dense subset (X,d) metric space, $A \subseteq X$. A is dense in X if $\bar{A} = X$.

Example: $\mathbb{Q}$ is dense in $\mathbb{R}$. Show any $x \in \mathbb{R}$ (e.g. $x \notin \mathbb{Q}$) is a limit point.
... $\forall r>0$, $\exists$ rational $q \in (x-r, x+r) \cap \mathbb{Q}$

Separable metric space (X,d) is separable if $\exists A \subseteq X$, A is at most countable and dense.

e.g. $(\mathbb{R}, d_2)$ is separable.

$(\mathbb{R}, d_{discrete})$ is not separable. $A \subseteq \mathbb{R}$, $\bar{A} = A$. So only dense subset is $\mathbb{R}$.

'You can get as close as you like'

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Let $\ell^p = \{ \{a_n\}_{n=1,2,\ldots} : a_n \in \mathbb{R}, \sum_{n=1}^{\infty} |a_n|^p < \infty \}$, $p \in [1,\infty)$

$\ell^\infty = \{ \{a_n\}_{n=1,2,\ldots} : a_n \in \mathbb{R}, \sup_{n \in \mathbb{N}} |a_n| < \infty \}$

(ℓ^p, d_p) is metric space (Minkowski's Inequality) $d_p(\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}) = \left(\sum_{n=1}^{\infty} |a_n - b_n|^p\right)^{1/p}$, $d_\infty(\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}) = \sup_{n\in\mathbb{N}} |a_n - b_n|$

Lemma: For $p \in [1,\infty)$, (ℓ^p, d_p) is separable.

$(\cong \mathbb{Q}^k)$ $A_k = \{ \{a_n\}_{n\in\mathbb{N}} : a_n \in \mathbb{Q}, \forall n > k \ a_n = 0 \}$, $A = \bigcup_{k=1}^{\infty} A_k$

Claim: $\bar{A} = \ell^p$. Let $\{x_n\}_{n=1,\ldots}$ belong to ℓ^p. If $\{x_n\}_{n=1,2,\ldots} \in A$, done.

Else we show it is an accumulation point.

W.T.S, in other words, $\forall r>0\ \exists \{a_n\}_{n\ldots} \in A$ s.t. $d_p(\{x_n\}_n, \{a_n\}_n) < r$.

By assumption, $\sum_{n=1}^{\infty} |x_n|^p < \infty$. Thus $\exists k$ s.t. $\sum_{n=k+1}^{\infty} |x_n|^p < \frac{r^p}{2}$. (Cauchy Criterion)

could be anything actually

$\exists a_1 \ldots a_k \in \mathbb{Q}$ ($a_n = 0\ \forall n > k$), $\sum_{i=1}^{k} |x_i - a_i|^p < \frac{r^p}{2}$. This will be our sequence.

$\left(\lim_{k\to\infty} \sum_{n=1}^{k} |x_n|^p = \sum_{n=1}^{\infty} |x_n|^p \quad \lim_{k\to\infty} \sum_{n=1}^{k} |x_n|^p - \sum_{n=1}^{\infty} |x_n|^p = 0 \right)$

$$\left[d_p(\{x_n\}_n, \{a_n\}_n)\right]^p = \sum_{n=1}^{k} |x_n - a_n|^p + \sum_{n=k+1}^{\infty} |x_n - \overbrace{a_n}^{=0}|^p$$

$$< \frac{r^p}{2} + \frac{r^p}{2} = r^p.$$

<u>lemma</u> (l^∞, d_∞) is not separable.

proof. Assume A is a <u>countable</u> dense subset

let $\alpha^k = \{a_n^k\}_{n=1,\ldots}$, $A = \{\alpha^k, k \in \mathbb{N}\}$.

enumeration of sequences

Want to construct $\{x_n\}_{n\ldots}$ s.t. $d_\infty(\{x_n\}_{n=1,\ldots}, \{a_n^k\}_{n=1,\ldots\infty}) \geq 1, \forall k$.

$\geq |x_n - a_n^k|\ \forall n.$

$\geq |x_k - a_k^k|$

'diagonalization' $x_n \overset{def}{:=} \begin{cases} -1 & \text{if } a_n^n \geq 0 \\ 1 & \text{if } a_n^n < 0 \end{cases} \Rightarrow |x_n - a_n^n| \geq 1$, $d_\infty(\{x_n\}_{n=1,\ldots}, \{a_n^k\}_{n=1}) \geq |x_k - a_k^k| \geq 1$.

'packing number' — compactness!

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Compactness "Convergence on compact sets"

Let (X, d) be a metric space.

- Open cover of X is any collection of open sets $\{A_i, i \in I\}$ such that $\bigcup_{i \in I} A_i = X$
- Given a cover $\{A_i : i \in I\}$, $\{E_j : j \in J\}$ is a subcover if $(\forall j \in J, \exists i \in I, E_j = A_i)$ and $\{E_j : j \in J\}$ is a cover.

Def (X, d) is compact if every open cover has a finite subcover.

Examples
- $(\mathbb{R}, |\cdot|)$ is not compact. E.g. $E_n = (-n, n)$, $\bigcup_{n=1}^{\infty} E_n = \mathbb{R}$. Taking $E_{n_1} \dots E_{n_k}$, we order them such that $n_1 < n_2 < \dots < n_k$ $\bigcup_{i=1}^{k} E_{n_i} = E_{n_k}$.
- $((0,1), |\cdot|)$ ✗ $E_n = (\frac{1}{n+2}, \frac{1}{n})$ e.g. $E_1 = (\frac{1}{3}, 1)$, $E_2 = (\frac{1}{4}, \frac{1}{2})$ ...

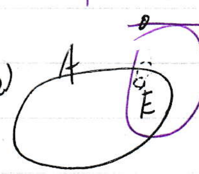
  For $E_{n_1}, \dots E_{n_k}$, $n_1 < n_2 < \dots < n_k$, $x \in (0, \frac{1}{n_k+2}) \Rightarrow x \notin \bigcup_{i=1}^{k} E_{n_i}$
- $([0,1], |\cdot|)$ compact! Note that $[0,1]$ is open here. $B(0, \frac{1}{2}) = \{x \in [0,1] : |x - 0| < \frac{1}{2}\} = [0, \frac{1}{2})$

Theorem (Heine-Borel)

Consider $(\mathbb{R}^d, |\cdot|)$. $K \subseteq \mathbb{R}^d$ is compact $\Leftrightarrow$ K is closed and bounded.

def $\hookrightarrow$ $(K, |\cdot|)$ is a compact metric space

Subspace topology $\boxed{E \subseteq A \text{ is open in } (A, d) \Leftrightarrow \exists\, O \subseteq X \text{ open in } (X, d) \text{ such that } E = A \cap O.}$



$(\Leftarrow)$ Assume O open $= \bigcup_{e \in O} B_X(e, r_e)$

$A \cap B_X(e, r_e) \subseteq E = A \cap O$
$\underbrace{}_{\text{open ball in } A.}$

$(\Rightarrow)$ Pf. Assume E is open in (A, d). $\forall e \in E, \exists r_e > 0, B_A(e, r_e) \subseteq E$.

Then $E = \bigcup_{e \in E} B_A(e, r_e)$ $\subseteq$ clear, $\supseteq$ by openness

$O = \bigcup_{e \in E} B_X(e, r_e)$. $E \subseteq O \cap A$

Let $a \in A \cap O$. $\exists e \in E, a \in B_X(e, r_e) \cap A = B_A(e, r_e) \subseteq E$
$\Rightarrow A \cap O \subseteq E$ //

e.g. $[0,\frac{1}{2}) = \underbrace{[0,1]}_{A} \cap \underbrace{(-1,\frac{1}{2})}_{\text{open in } \mathbb{R}}$

So $[0,\frac{1}{2})$ is open in $([0,1], |\cdot|)$

Another lemma: let (X,d) be metric space, $K \subseteq X$.

(K,d) is compact $\Leftrightarrow$ for every collection of open sets O_i in X, $i \in I$, s.t. $K \subseteq \bigcup_{i \in I} O_i$, $\exists J \subseteq I$, J finite, $K \subseteq \bigcup_{j \in J} O_j$.

Proof. $(\Rightarrow)$ we previous lemma $(\Leftarrow)$ to define open sets in K

$(\Leftarrow)$ 〃 $(\Rightarrow)$

Lemma: (X,d) metric space.

$K \subseteq X$ is compact $\Rightarrow$ K is bounded $(\exists x \in X, r > 0, K \subseteq B(x,r))$

Proof. Assume the claim does not hold, K-compact but not bounded.

Take $x \in K$, define $O_i = B(x,i)$

$$K \subseteq \bigcup_{i=1}^{\infty} O_i \quad \left[z \in K,\ d(x,z) < i \text{ for some } i \Rightarrow z \in B(x,i). \right]$$

Assume $O_{i_1}, \dots O_{i_m}$ - finite subcover, $i_1 < \dots < i_m$

$K \subseteq O_{i_m} \Rightarrow K$ bounded. Contradiction!

Lemma If K is compact, then K is closed.

Proof from 21-355

Show K^c is open $\Leftrightarrow \exists r>0, B_r(p) \cap K = \emptyset$.

Proof. Assume K is not closed, which means $\exists y \in K' \setminus K$.

Let $r_n = \frac{1}{n}$. $O_n = \{x \in X : d(x,y) > \frac{1}{n}\}$ is open ← Triangle inequality

$$\bigcup_{n \geq 1}^{\infty} O_n = X \setminus \{y\} \supseteq K$$

Assume $O_{n_1}, \dots O_{n_m}$ finite subcover, with $\bigcup_{i=1}^{m} O_{n_i} = O_{n_m}$.

If $K \subseteq O_{n_m} \Rightarrow K \cap B(y, \frac{1}{n_m}) = \emptyset$. But y is a limit point! Contradiction.

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⚠ In general, Closed and bounded $\not\Rightarrow$ compact.

(Let 2 P'S) gives a counterexample. Here is another, from HW1.

let $(\mathbb{R}, d)$, where $d(x,y) = |x-y|$. $f(x) = x/(1+x) = 1 - \frac{1}{1+x}$, $f'(x) = \frac{1}{(1+x)^2}$, $f''(x) = \frac{-2}{(1+x)^3}$, $x \geq 0$.

$f(0) = 0$, $f'(0) > 0$, $f'' < 0$

$\bar{d}(x,y) := f(|x-y|)$ induces another metric space $(\mathbb{R}, \bar{d})$.

~~let $r \in 1$.~~ $B_E(x,r) = \{y : |x-y| < r\}$

$$B_{new}(x,s) = \left\{y : \frac{|x-y|}{1+|x-y|} < s\right\} = B_E\left(x, \frac{s}{1-s}\right) \quad (s < 1)$$

$|x-y| < s(1+|x-y|)$

$|x-y| < s/(1-s)$

* If $s > 1$, $B_{new}(x,s)$ is not directly translatable to open ball under E. It is the whole space, a union of open subsets around X

(Claim/exercise) $\mathcal{T}_d = \mathcal{T}_{\bar{d}}$. (Similarly, d_2, d_∞ ... give the same topology!)

$\mathrm{diam}_{\bar{d}}(\mathbb{R}) = 1$ is a bounded space!

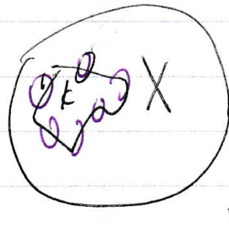
In $(\mathbb{R}, \bar{d})$, $\mathbb{R}$ is bounded and closed! But $\mathbb{R}$ is not compact.

⚠ Compactness is a topological concept! Does not depend on metric.

f makes every metric bounded.

Lemma: (X,d) compact metric space, $K \subseteq X$, K closed. Then K is compact.

Proof



Let $O_i \in \tau_X$. $K \subseteq \bigcup_{i\in I} O_i$ (O_i open cover of K)

$\{O_i : i \in I\} \cup \{X \setminus K\}$ is an open cover of X. (open! $\in \tau_X$)

X is compact $\Rightarrow$

$O_{i_1}, \dots, O_{i_n}, X\setminus K$ is an open subcover.

Then $O_{i_1}, \dots, O_{i_n}$ is open cover of K.

Lemma. (X,d) metric space, $K \subseteq X$ compact. "compactness creates object"
Assume $A \subseteq K$, A infinite. Then $A' \neq \emptyset$.

Proof. Assume $A' = \emptyset$, A closed. ($\bar{A} = A \cup A' = A$.) Thus A is compact.

Note that for all $a \in A$, $\exists r_a > 0$, $B(a, r_a) \cap A = \{a\}$ by assumption.

$\{B(a, r_a) : a \in A\}$ is an infinite open cover of A which contains no finite subcover.

Since $\forall a \in A$ it is contained in exactly one set in the cover, there are no strict subcovers.

Def. (X,d) is sequentially compact if every sequence in X has a convergent subsequence.

Def. $\{a_n\}_{n=1\dots}$ in X converges to $\alpha \in X$ if $(\forall \varepsilon > 0)(\exists n_0)(n \geq n_0)\ d(a_n, \alpha) < \varepsilon$.

Theorem. If (X,d) is compact, then it is sequentially compact.

Proof. Let $\{a_n\}_{n=1\dots}$ sequence in X. Let $A = \{a_n : n \in \mathbb{N}\}$

If A is finite, at least 1 element is repeated infinitely times. Then take $a_{n_1} = a_{n_2} = \dots = \alpha$.

Otherwise, by the lemma there is $\alpha \in A'$: for all $r > 0$, $B(\alpha, r) \cap A$ is infinite.

$\exists n_1 \quad d(a_{n_1}, \alpha) < 1$

$\exists n_2 > n_1 \quad d(a_{n_2}, \alpha) < 1/2$

$\exists n_{k+1} > n_k \quad d(a_{n_k}, \alpha) < 1/2^{k-1}$

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$\boxed{\text{Def. } (X,d) \text{ is complete if every Cauchy sequence converges (to a point in } X)}$

Remark: Convergent $\Rightarrow$ Cauchy always.

Cauchy $\not\Rightarrow$ Convergent (e.g. rationals)

Every metric space can be completed using Cauchy sequences (see notes)

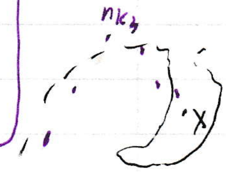
Peano Existence Theorem

$X' = f(X)$, f continuous, not Lipschitz (Not Pic-Lindelof)

$X(0) = a \to$ Might not converge to unique solution

But compact $\Rightarrow$ convergent subsequence! This allows to prove that a local solution exists.

Def. (X,d) is totally bounded if $(\forall \varepsilon > 0)\ \exists x_1, \dots x_m \in X$, $X = \bigcup_{i=1}^{m} B(x_i, \varepsilon)$



TFAE
(i) (X,d) is sequentially compact
(ii) (X,d) is totally bounded and complete
(iii) (X,d) is compact

T_1 spaces (iii) $\Rightarrow$ (i)
In general (iii) $\not\Leftrightarrow$ (i)

Proof. (iii) $\Rightarrow$ (i) As above.

(i) $\Rightarrow$ (ii) Assume (X,d) is seq. compact. Let $\{x_n\}_{n=1\dots}$ be a Cauchy Sequence in X.

(ii) $\Rightarrow$ (i)

(i)+(ii) $\Rightarrow$ (iii)

Since (X,d) is seq cpt, $\exists (x_{n_k}) \to x$ as $k \to \infty$. Claim: $x_n \to x$ as $n \to \infty$

Let $\varepsilon > 0$. $\exists n_0\ \forall m, n \geq n_0$. $d(x_n, x_m) < \varepsilon/2$

$\exists k_0\ \forall k \geq k_0\ d(x_{n_k}, x) < \varepsilon/2$.

Fix n_0 as above. $k \to \infty \Rightarrow n_k \to \infty$, Take $\bar{k} \geq k_0$ s.t. $n_{\bar{k}} \geq n_0$.

Let $\bar{n} = n_{\bar{k}}$. For $m \geq \bar{n}$, $d(x_m, x) \leq d(x_m, x_{n_{\bar{k}}}) + d(x_{n_{\bar{k}}}, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

For (i)⇒(ii), we also want to show seq cpt ⇒ totally bounded.

x_3 x_2 (ε, x_1)

Assume (X,d) is not totally bounded. Thus there exists $\varepsilon > 0$ such that there is no finite collection of points $x_1 \dots x_m$ such that $X = \bigcup_{i=1}^{m} B(x_i, \varepsilon)$.

let x_1 be arbitrary

$\exists x_2 \in X \setminus B(x_1, \varepsilon)$ (otherwise this single ball contains everything)

$\exists x_3 \in X \setminus (B(x_1, \varepsilon) \cup B(x_2, \varepsilon))$, iterate...

We get $\{x_n\}_{n=1,\dots}$ $d(x_i, x_j) \geq \varepsilon$ $\forall i \neq j$. Thus this sequence does not have a convergent subsequence.

(ii) ⇒ (i) Assume (X,d) is totally bounded and complete.

Let $\{x_n\}_n$ be a sequence in X. let $\mathcal{B}_k$ be the a finite collection of balls of radius $\frac{1}{2^k}$ such that $X = \bigcup \mathcal{B}_k$ [notation: $\bigcup \{A_i : i \in I\} = \bigcup_{i \in I} A_i = \bigcup \mathcal{A}$]

Suppose $\{x_n\}_n$ is infinite. when $k=1$, $\exists B_1 \in \mathcal{B}_1$, $A := \{x_n : n \in \mathbb{N}\}$, $A \cap B_1$ is infinite.

'Heine-Borel' By the same reasoning, $\exists B_2 \in \mathcal{B}_2$, $(A \cap B_1) \cap B_2$ infinite.

$B_k \in \mathcal{B}_k, \dots$ $A \cap (B_1 \cap \dots \cap B_k) \dots$

$\exists n_1$, $x_{n_1} \in B_1$. Then $\exists n_2 > n_1$, by its infinite cardinality, such that $x_{n_2} \in B_1 \cap B_2$.

... $\exists n_k > n_{k-1}$, $x_{n_k} \in B_1 \cap B_2 \dots B_k$.

Claim: $\{x_{n_k}\}_{k=1,2,\dots}$ is Cauchy.

$\forall j \geq k$. $x_{n_j} \in B_k \Rightarrow d(x_{n_j}, x_{n_l}) < 2/2^k$ if $j, l \geq k$.

l_2: $B(0,1)$ is not totally bounded.

Since (X,d) is complete, x_{n_k} converges, so we have our convergent subsequence.

(i)+(ii) ⇒ (iii) See next page.

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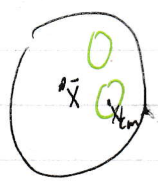
We have to show that [Sequentially compact] + [Totally Bounded and Complete] $\Rightarrow$ compact

Define $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ is open cover. $\mathcal{B}_k = \{B(x_i, \frac{1}{2^k}), i = 1, \dots m_k\}$ $\bigcup \mathcal{B}_k = X$.

Claim: $\exists k$ such that $(\forall B_i \in \mathcal{B}_k)(\exists \alpha \in A)\ B_i \subseteq U_\alpha$.
"sufficiently small balls that all fit into an element of the cover"

Assume this were not the case: $(\forall k)(\exists B_{i_k} \in \mathcal{B}_k)(\forall \alpha \in A)\ B_{i_k} \setminus U_\alpha \neq \emptyset$.

Let x_k be the center of B_{i_k}. $\{x_k\}_{k=1\dots}$ has a convergent subsequence, by seq cpct, so $x_{k_m} \to \bar{x}$ as $m \to \infty$. But $\bar{x}$ must belong to some part of the cover: $\exists \bar{\alpha} \in A$, $\bar{x} \in U_{\bar{\alpha}}$.
$\exists r > 0 : B(\bar{x}, r) \subseteq U_{\bar{\alpha}}$. Then $(\exists m_0)(\forall m \geq m_0)\ d(\bar{x}, x_{k_m}) < r/2$.



$\exists m_1$, $\frac{1}{2^{k_{m_1}}} < r/2$.

Take $m = \max\{m_0, m_1\}$.

$$B(x_{k_m}, \frac{1}{2^{k_m}}) \subseteq B(\bar{x}, r) \subseteq U_{\bar{\alpha}}.$$
contradiction.

Let $\mathcal{B}_k$ be as in the claim.
$\mathcal{B}_k = \{B(y_j, \frac{1}{2^k}) : j = 1 \dots l_k\}$
Let $\alpha_j \in A$ such that $B(y_j, \frac{1}{2^k}) \subseteq U_{\alpha_j}$
Then $\{U_{\alpha_j}, j = 1 \dots l_k\}$ is a finite subcover of $\{U_\alpha : \alpha \in A\}$
$X = \bigcup_{j=1}^{l_k} B(y_j, \frac{1}{2^k}) \subseteq \bigcup_{j=1}^{l_k} \alpha_j \subseteq X$
$\Rightarrow \bigcup_{j=1}^{l_k} U_{\alpha_j} = X$.

Arzela-Ascoli - Compact sets in Continuous Functions

$$\mathcal{C}([0,1], \mathbb{R}), \quad d(f, g) = \max_{x \in [0,1]} |f(x) - g(x)|$$

Connectedness Def: (X, d) is connected if the only subsets of X which are closed and open (i.e. clopen) are $\emptyset$ and X.

For subsets: use restricted ~~metric space~~

$A = [0,1] \cup [2,3]$, $A \cap \underbrace{(-\frac{1}{2}, \frac{3}{2})}_{\text{open in } \mathbb{R}} = [0,1]$ open in (A, d)

$[2,3]$ - open $\Rightarrow A \setminus [2,3] = [0,1]$ is closed.
Conclusion: $[0,1]$ is clopen. So A is not connected.

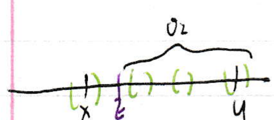
referring to the space

Remark: (X,d) is disconnected iff $\exists\, O_1, O_2$ open, disjoint, nonempty and such that $O_1 \cup O_2 = X$.
Such O_1, O_2 are said to 'separate' X. Connected $\Leftrightarrow$ not separated

Theorem (2.47) Consider $(\mathbb{R}, |\cdot|)$. $A \subseteq \mathbb{R}$ is connected iff A is an interval (it may be infinite, half-open...)

Proof. A is an interval $\Leftrightarrow \forall a, b \in A,\ a<b,\ a<c<b \Rightarrow c \in A$.

1. Such set A (satisfying (*)) is connected. Otherwise, $\exists\, O_1, O_2$ which separate A. Let $x \in O_1$, $y \in O_2$. WLOG $x<y$. Let $z = \inf O_2 \cap [x,y]$. $\exists\, \delta>0,\ (x, x+\delta) \subset O_1$. Since O_1 and O_2 are disjoint, $\forall w \in O_2 \cap [x,y],\ w \geq x+\delta$.



$\Rightarrow z = \inf O_2 \cap [x,y] \geq x+\delta > x.$

Likewise, $\exists\, \varepsilon,\ (y-\varepsilon, y) \subset O_2 \ \therefore\ z \leq y-\varepsilon$ $\Big]\ z \in (x,y)$

By assumption, $z \in O_1 \cup O_2$. Case 1: $z \in O_2$. Then there is a ball around z, $z-r\ (<z) \in O_2 \cap (x,y) \Rightarrow z \neq \inf O_2 \cap (x,y)$

2: $z \in O_1$. $\forall \varepsilon>0,\ [z, z+\varepsilon) \cap O_2 \neq \emptyset.$
$\hookrightarrow \exists\, r>0,\ (z-r, z+r) \subseteq O_1.$ $\Big]\ O_1 \cap O_2 \neq \emptyset.$

2. Suppose A is connected. We want to show then $\forall a, b \in A,\ a<b,\ a<c<b \Rightarrow c \in A$.

Assume otherwise. Then the division $-(\ )\cdot(\ )-$ separates A.

- Is there a quick way to show that a line segment in $\mathbb{R}^2$ is connected?
  - projection
  - ~~$\emptyset$~~ must be closed?! open

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What does it mean that two metric spaces are indistinguishable? We use the concept of isometry.

Take (X, d_X), (Y, d_Y).

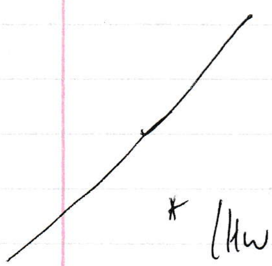
$\varphi: X \to Y$ is an isometry if φ is a bijection and $\forall x_1, x_2 \in X$, $d_X(x_1, x_2) = d_Y(\varphi(x_1), \varphi(x_2))$

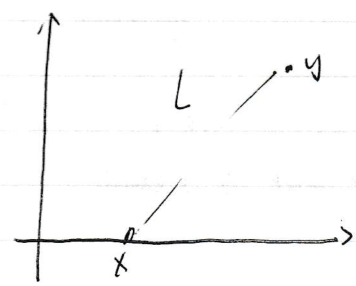
φ^{-1} is also isometry

Example: fold paper (2D) into 3D space, and require that we travel across the paper.

open sets are mapped to open sets...
compact $\to$ compact
connected $\to$ connected

Consider the previous example.





$L = \{x(1-t) + yt : t \in [0,1]\}$

This is already a mapping $[0,1] \to L$...

... but not isometry! Because $d_2(x,y) \neq 1$. $[0, |y-x|]$ distances in which fall within

So we define $\varphi(s) = x\left(1 - \frac{s}{|y-x|}\right) + y\frac{s}{|y-x|}$

$= x + \frac{(y-x)s}{|y-x|}$

for $s \in [0, |y-x|]$.

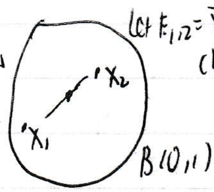
Since L is isometric to

* (HW Problem) A_1, A_2 connected
  $A_1 \cap A_2 \neq \emptyset$
  $\Rightarrow A_1 \cup A_2$ is connected

* Intermediate Value Theorem applies to and only to connected sets

Lemma: (X, d) is connected iff $\forall x_1, x_2 \in X$, $\exists E_{1,2} \subseteq X$, $E_{1,2}$ connected and $x_1 \in E_{1,2}$, $x_2 \in E_{1,2}$.

Instantiations

– 

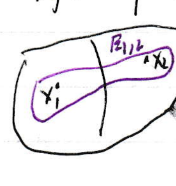
Let $E_{1,2} = \overline{x_1 x_2}$ (line)

$(\Rightarrow)$ $x_1, x_2 \in X$, $E_{1,2} = X$

$(\Leftarrow)$ Assume X is not connected $\Leftrightarrow \exists O_1, O_2$ which separate X, $x_1 \in O_1$, $x_2 \in O_2$.

By assumption $\exists E_{1,2}$ connected, $x_1, x_2 \in E_{1,2}$.

– Any convex subset of $\mathbb{R}^d$



$A_1 := E_{1,2} \cap O_1$, $A_2 := E_{1,2} \cap O_2$, $A_1 \cap A_2 = \emptyset$.
open in $E_{1,2}$

$A_1 \cup A_2 = (E_{1,2} \cap O_1) \cup (E_{1,2} \cap O_2)$
$= E_{1,2} \cap (O_1 \cup O_2) = E_{1,2}$ contradiction! //

Continuity — ... $X \to Y$ ... definition omitted

Recall

(i) $f^{-1}\left(\bigcup_{i \in I} E_i\right) = \bigcup_{i \in I} f^{-1}(E_i)$ (iii) $f\left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} f(A_i)$

(ii) $f^{-1}\left(\bigcap_{i \in I} E_i\right) = \bigcap_{i \in I} f^{-1}(E_i)$ (iv) $f\left(\bigcap_{i \in I} A_i\right) \subseteq \bigcap_{i \in I} f(A_i)$

e.g. $f(x) = x^2$, $A_1 = \{-2\}$, $A_2 = \{2\}$

Lemma: f is continuous iff $(\forall p \in X)(\forall \varepsilon > 0)$, p is an interior point of $f^{-1}(B_Y(f(p), \varepsilon))$

where $f^{-1}(B_Y(f(p), \varepsilon)) = \{x \in X : f(x) \in B_Y(f(p), \varepsilon)\}$

$(\Rightarrow)$ Let $p \in X$, $\varepsilon > 0$. By continuity, $\exists \delta > 0$. $\forall x \in B_X(p, \delta)$, $f(x) \in B_Y(f(p), \varepsilon)$

$B_X(p, \delta) \subseteq f^{-1}(B_Y(f(p), \varepsilon))$

So p is an interior point of $f^{-1}(B_Y(f(p), \varepsilon))$

$(\Leftarrow)$ p interior point $\Rightarrow$ δ exists...

Examples: $\varphi : \mathbb{R} \to \mathbb{R}$ continuous

$A \subseteq \mathbb{R}$ open e.g. $\varphi(x) = 0$

$\varphi(A)$ not open!

$C \subseteq \mathbb{R}$ closed, $\varphi(C)$ not closed! $\varphi(x) = 1/x$, $C = [1, \infty)$, $\varphi(C) = (0, 1]$

Alternatively,

- $(\forall p \in X)(\forall \varepsilon > 0)$ p is an interior point of $f^{-1}(B_Y(f(p), \varepsilon))$

$\downarrow$

- $\boxed{f \text{ is continuous} \Leftrightarrow \forall O \subseteq Y,\ O \text{ is open (w.r.t } d_Y) \Rightarrow f^{-1}(O) \text{ is open w.r.t } d_X}$ (1)

("Topological definition")

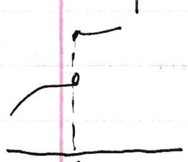
$(\Leftarrow)$ Take $O = B_Y(f(p), \varepsilon)$ and apply the previous definition.

$(\Rightarrow)$ Let $O \subseteq Y$ open. Let $p \in f^{-1}(O)$.

Since O is open, $f(p)$ is an interior point of O.

$\Leftrightarrow (\exists \varepsilon > 0)\ B_Y(f(p), \varepsilon) \subseteq O$

For discontinuous functions,



$O = (-\infty, y)$

$f^{-1}(O) = (-\infty, x]$

By continuity, $B_X(p, \delta) \subseteq f^{-1}(B_Y(f(p), \varepsilon)) \subseteq f^{-1}(O)$

So p is an interior point of $f^{-1}(O)$.

(continuous $\Leftarrow$)

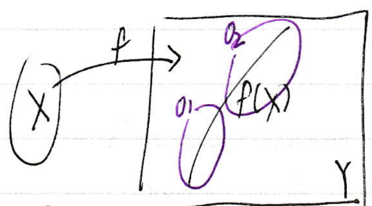
Corollary: $\forall C \subseteq Y$ closed $\Rightarrow f^{-1}(C) \subseteq X$ closed.

$\overline{C}$ open $\Rightarrow f^{-1}(\overline{C})$ open, $f^{-1}(\overline{C}) = \overline{f^{-1}(C)}$ open...

<u>Lemma</u>. Continuity of composition. $X \to Y \to Z$. $f \in C(X,Y)$, $g \in C(Y,Z) \Rightarrow h = g \circ f \in C(X,Z)$

<u>Lemma</u> $f \in C(X, \mathbb{R}^d)$ iff $\forall n = 1 \dots d$, $\overset{\text{(projection)}}{\pi_n} \circ f \in C(X, \mathbb{R})$

<u>Lemma</u> $f \in C(X,Y)$ (X, d_X) connected. Then $f(X)$ is connected.



Proof. Assume $f(X)$ disconnected. There exist O_1, O_2 open disjoint, $O_1 \cap f(X) \neq \emptyset$.
O_2 ''
$f(X) \subseteq O_1 \cup O_2$.

$V_1 = f^{-1}(O_1)$, $V_2 = f^{-1}(O_2)$ are open subsets of X.

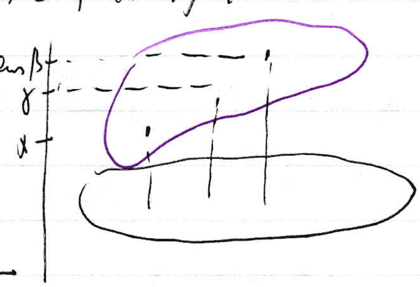
$f^{-1}(O_1 \cap f(X)) = f^{-1}(O_1) \cap X = f^{-1}(O_1) \neq \emptyset$

$f^{-1}(O_2) \neq \emptyset$

$f^{-1}(O_1) \cap f^{-1}(O_2) = \emptyset$.

$f^{-1}(O_1) \cup f^{-1}(O_2) = f^{-1}(O_1 \cup O_2) \supseteq f^{-1}(f(X)) = X$.

Consequence: Intermediate Value Theorem



Another definition:

(2) $(\forall y \in Y)(\forall r > 0)$ $f^{-1}(B_Y(y,r)) \in \mathcal{T}_X$. Then (1) $\Leftrightarrow$ (2)

(2) $\Rightarrow$ (1) $\mathcal{O} = \bigcup_{y \in \mathcal{O}} B_Y(y, r_y)$

$f^{-1}(\mathcal{O}) = \bigcup_{y \in \mathcal{O}} \underbrace{f^{-1}(B_Y(y,r))}_{\in \mathcal{T}_X}$

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Special case $Y = \mathbb{R}$.

(2) $\forall a<b,\ f^{-1}((a,b)) \in \tau_X$

(3) $\forall a \in \mathbb{R}\ \ f^{-1}((a,\infty)) \in \tau_X$

$\forall b \in \mathbb{R}\ \ f^{-1}((-\infty,b)) \in \tau_X$

$(3) \overset{?}{\Rightarrow} (2)$ $a<b$, $f^{-1}((a,b)) = f^{-1}((-\infty,b) \cap (a,\infty))$

$= \underbrace{f^{-1}((-\infty,b))}_{\in \tau_X} \cap \underbrace{f^{-1}((a,\infty))}_{\in \tau_X} \in \tau_X.$

Lemma: $f: X \to Y$ continuous. If X is compact, then $f(X)$ is compact
(metric space (X, d_X)) $(f(X), d_Y)$

let $V_i = f^{-1}(O_i) \in \tau_X$, where $f(X) \subset \bigcup_{i \in I} O_i$
$^{\uparrow} \in \tau_Y$ (whole Y)

$$X \supseteq \bigcup_{i \in I} V_i = \bigcup_{i \in I} f^{-1}(O_i) = f^{-1}\left(\bigcup_{i \in I} O_i\right) \supseteq f^{-1}(f(X)) = X.$$

Since X is compact, there is finite subcover $V_{i_1} \dots V_{i_n}$.

$$f(X) = f(V_{i_1} \cup V_{i_2} \dots \cup V_{i_n}) = f(V_{i_1}) \cup \dots f(V_{i_n})$$
$$= O_{i_1} \cup \dots \cup O_{i_n}.$$

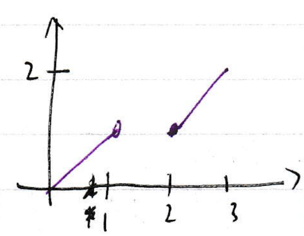
Q. $f: A \to E$, $A \subseteq \mathbb{R}, E \subseteq \mathbb{R}$, f is a bijection. Find function f which is continuous but whose inverse f^{-1} is not.

Eg. 1 $A = (0,1) \cup [2,3)$

$E = (0,2)$

$$f(x) = \begin{cases} x & \text{on } (0,1) \\ x-1 & \text{on } [2,3) \end{cases}$$

$$f^{-1}(y) = \begin{cases} y & \text{on } (0,1) \\ y+1 & [1,2) \end{cases}$$



Eg. 2 $X = (\mathbb{R}, \text{discrete})$
$Y = (\mathbb{R}, \text{standard})$
$f(x) = x.$

Note $\tau_Y \subseteq \tau_X$.
'finer topology'

f is continuous - use same single gen set!
$f^{-1}(y) = y$ let $O = \{3\} \in \tau_X$.
$(f^{-1})^{-1}(O) = f(O) = O = \{3\} \notin \tau_Y$

Theorem Let $f: X \to Y$ be a continuous bijection. If X is <u>compact</u> then f^{-1} is continuous.

Proof. Let $g = f^{-1}$, $C \subseteq X$ closed. We have to show that $g^{-1}(C)$ is closed in Y.

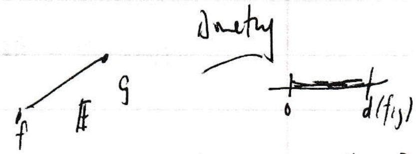
But $(f^{-1})^{-1}(C) = f(C)$, and we know that $f(C)$ is compact $\Rightarrow$ closed.

We do not get the same result if X is merely connected.

E.g. $Y = X = C([0,1], [0,1])$, $\Phi(f) = f$.

$d_X(f,g) = \max |f(x) - g(x)|$

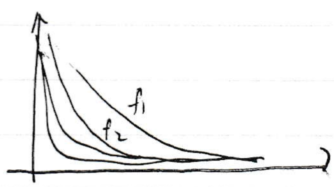
$d_Y(f,g) = \int_0^1 |f(x) - g(x)|\, dx$



Show (X, d_X) is connected (Pathwise connected)

$E = \{(1-s)f + sg : s \in [0,1]\}$

$d_Y(f,g) \le d_X(f,g)$



$f_n \xrightarrow{d_Y} 0$

$d_X(f_n, 0) = 1$ for all n!

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Def. X is totally bounded if $\forall r>0$, $\exists x_1 \dots x_n$ s.t. $X = \bigcup_{i=1}^{n} B(x_i, r)$.

let $\ell^2 = \{ \{a_n\}_{n=1,2,\dots} : \sum a_n^2 < \infty \}$, $d_2^2(\{a_n\}, \{b_n\}) = \sum (a_n - b_n)^2$;

let $e_n = \{\underbrace{0, \dots, 0}_{n-1}, 1, 0, \dots 0\}$

$d_2(e_n, e_m) = \sqrt{2}$ $(m \neq n)$, $X = \overline{B}(\{0\}, 1)$.

This shows that Bounded $\not\Rightarrow$ Totally Bounded. If we choose $r = 1/2$, then for each z we have $x, y \in B(z, \frac{1}{2}) \Rightarrow d(x,y) < 1$.

- can also use discrete metric

So each z can only contain at most 1 e_n.

(21-355 Hw3) Compact $\Rightarrow$ Totally Bounded but Totally Bounded $\not\Rightarrow$ Compactness.

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Theorem (Weierstrass) Let $f \in C(X, \mathbb{R})$. X compact. Then f reaches its maximum and minimum in X.

Proof. $A = f(X)$ is compact. Since $A \in \mathbb{R}$, Heine-Borel implies A is closed and bounded.

* A is not necessarily connected $\Rightarrow$ does not have to be an interval

Easy lemma: $M = \sup A \Rightarrow M \in A'$. Since A is closed, $M \in A$. So there exists $x_1 \in X$ such that $f(x_1) = M$. (x_1 need not be unique)

Uniform Continuity | Let $f: X \to Y$.

f is continuous $(\forall x \in X)(\forall \varepsilon > 0)(\exists \delta_x > 0)$ If $d_X(x,z) < \delta_x$ then $d_Y(f(x), f(z)) < \varepsilon$.

uniformly cont. $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in X)$ If $d_X(x,z) < \delta$ then $d_Y(f(x), f(z)) < \varepsilon$

Remark. To show that f is not uniformly cont, it suffices to find two sequences $\{x_n\}_n, \{z_n\}_n$ in X (iff) such that $d_X(x_n, z_n) \to 0$ and $d_Y(f(x_n), f(z_n)) \not\to 0$ as $n \to \infty$.

Explanation: By $\exists \varepsilon > 0, \forall n \in \mathbb{N}, \exists k \geq n, \ d_Y(f(x_k), f(z_k)) \geq \varepsilon$.

For this ε, if f were in fact u.c., then we can find $\delta > 0$ such that $d_X(x,z) < \delta \Rightarrow d_Y(f(x), f(z)) < \varepsilon$.

contradiction

Since $d_X(x_n, z_n) \to 0$, $\exists n_0, \forall n \geq n_0 \ d_X(x_n, z_n) < \delta \Rightarrow d_Y(f(x), f(z)) < \varepsilon$.

Ex. $f(x) = x^2$ is not uniformly continuous.

① Let $x_n = n$, $x_n^2 = n^2$ $\qquad |x_n - z_n| = \frac{1}{n}$

$z_n = n + \frac{1}{n}$ $\quad z_n^2 = n^2 + 2 + \frac{1}{n^2}$, $\quad |f(x_n) - f(z_n)| = 2 + \frac{1}{n^2} > 1$.

② $f(x) = \ln x$ on $(0, \infty)$ is not u.c.

$x_n = e^{-n}$ $\qquad \ln x_n = -n$

$z_n = e^{-(n+1)}$ $\qquad \ln z_n = -(n+1)$

Note: on $(1, \infty)$ $\ln x$ is u.c.

- Lipschitz $\Rightarrow$ Uniformly continuous.
- $d_Y(f(x_1), f(x_2)) \leq L \underbrace{d_X(x_1, x_2)}_{\delta = \varepsilon / L}$
- Bounded derivative $\Rightarrow$ Lipschitz cont.

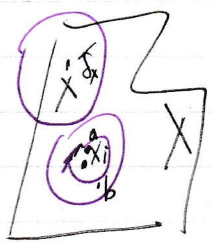
But Lipschitz $\not\Rightarrow$ differentiable (eg. $f(x) = |x|$)

(3) $f(x)=\sqrt{x}$ on $[0,\infty)$ ✓ u.c.! $f'(x)=\frac{1}{2\sqrt{x}}$ proves $[1,\infty)$ case

The following theorem proves $[0,2]$ case.

(Glue them!... If $x\in[1,2]$ take minimum δ...

Theorem. Suppose $f\in C(X,Y)$, X is compact. Then f is U.C.



Let $\varepsilon>0$. By continuity, $\forall x\in X$, $\exists \delta_x>0$, $d_X(x,z)<\delta_x \Rightarrow d_Y(f(x),f(z))<\varepsilon/2$.

$\{B_X(x,\frac{\delta_x}{2}) : x\in X\}$ is an open cover, so by compactness we have finite subcover $\{B_X(x_i,\frac{\delta_{x_i}}{2}),\ i=1\ldots n\}$.

Take $\delta=\frac{1}{2}\min\{\delta x_i : i=1,2,\ldots n\}$

Let $a,b\in X$, $d_X(a,b)<\delta$. So $\exists\, i\in\{1\ldots n\}$, $a\in B(x_i,\frac{\delta_{x_i}}{2})$.

$$d(x_i,b)\le d(x_i,a)+d(a,b)<\frac{\delta_{x_i}}{2}+\delta\le\frac{\delta_{x_i}}{2}+\frac{\delta_{x_i}}{2}=\delta_{x_i}$$

$$b\in B(x_i,\delta_{x_i}).$$

$\therefore$ By Δ-ineq, $d_Y(f(a),f(b))\le d_Y(f(a),f(x_i))+d_Y(f(x_i),f(b))$
$$<\varepsilon/2+\varepsilon/2=\varepsilon$$

Let $f_n: X\to\mathbb{R}$, $n=1,2,\ldots$, $f: X\to\mathbb{R}$

Def. $\{f_n\}_{n=1\ldots}$ converges <u>pointwise</u> to function f iff $(\forall x\in X)$, $\{f_n(x)\}_{n=1,2,\ldots}\to f(x)$.

$\{f_n\}_{n=1\ldots}$ converges <u>uniformly</u> if $\sup_{x\in X}|f_n(x)-f(x)|\to 0$ as $n\to\infty$. $[\,f_n\rightrightarrows f\,]$

Remark. If $f_n, f\in C_b(X,\mathbb{R})$ (bounded), $d_\infty(f,g)=\sup_{x\in X}|f(x)-g(x)|$ $(<\infty)$.

$f_n\rightrightarrows f$ as $n\to\infty \iff d_\infty(f_n,f)\to 0$.

Consider the space $C_b([0,1], \mathbb{R})$.

Find family $f_n \xrightarrow{\text{pointwise}} f \in C_b$ but $f_n \not\rightrightarrows f$

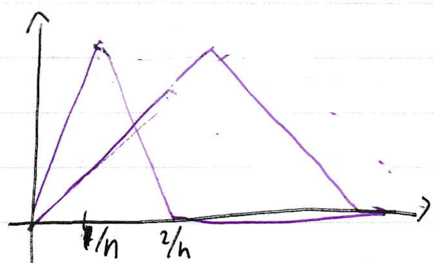
Suppose $f=0$.

Midterm 1 Post Mortem

$f(A') \not\subseteq f(A)'$ in general!

E.g. $f(x)=2$, $A=[0,1] \Rightarrow A'=[0,1]$

$f(A')=2$, $[f(A)]' = (\{2\})' = \emptyset$ (no limit points $\Rightarrow$ every singleton set is trivially closed)

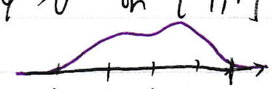


$$f_n(x) = \begin{cases} nx & 0 \le x \le \frac{1}{n} \\ -nx+2 & \frac{1}{n} < x < \frac{2}{n} \\ 0 & \text{else} \end{cases}$$

$$d_\infty(f_n, f) = \sup_{x\in[0,1]} |f_n(x) - f(x)| \ge |f_n(\tfrac{1}{n}) - f(\tfrac{1}{n})| = 1-0=1.$$

Is it possible to have a smooth function that has a compact support?
(has all derivatives.

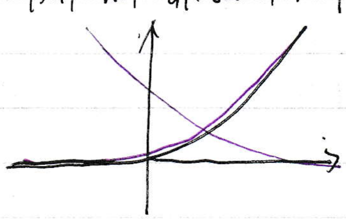
i.e. $\varphi > 0$ on $[-1,1]$, $\varphi(x)=0$, $|x|\ge 2$. If f is analytic,



'bump'

$$f(x) = f(3) + f'(3)x + \dots$$
$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!} x^n = 0$$

But not all smooth functions are analytic.



$$\psi(x) = \begin{cases} 0 & x \le 0 \\ e^{-1/x} & x > 0 \end{cases} \qquad \psi'(x) = e^{-1/x} \cdot \frac{1}{x^2}, \ x>0$$

$\lim_{x\to 0} (\cdot) = 0 \dots$

$\psi(1-x)$

Then we can define $\varphi(x) \sim \psi(x)\psi(1-x)\dots$

Theorem (7.15 in Rudin)

Let (X,d) be a metric space. Then $(C_b(X,\mathbb{R}), d_\infty)$ is a complete metric space.

Proof: Let $\{f_n\}_n$ be a Cauchy sequence in d_∞. $\forall x \in X$, $\{f_n(x)\}_n$ is a Cauchy sequence in $\mathbb{R}$ since $|f_n(x) - f_m(x)| \le d_\infty(f_n, f_m)$. Since $\mathbb{R}$ is complete, $\{f_n(x)\}_n$ converges.

Let $f(x) = \lim_{n\to\infty} f_n(x)$, pointwise limit. But we need to show uniform convergence ① and continuity ② of $f(x)$.

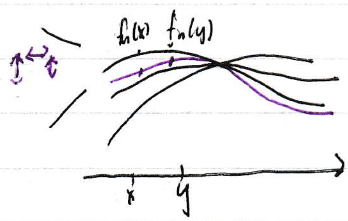
① Note for u.c. we don't need continuity.

$(\forall \varepsilon > 0)(\exists n_0)(\forall m,n \ge n_0)(\forall x \in X)\ |f_n(x) - f_m(x)| < \varepsilon/2$.

$$\lim_{m\to\infty} |f_n(x) - f_m(x)| = |f_n(x) - f(x)| \le \varepsilon/2 < \varepsilon.$$

So $(\forall \varepsilon > 0)(\exists n_0)(\forall n \ge n_0)(\forall x \in X)$, $|f_n(x) - f(x)| < \varepsilon$.

② $f \in C_b(X,\mathbb{R})$. Bounded is left as an exercise (limit point... A'...)



Use '$\varepsilon/3$' argument to show continuity. $\forall n$, $\forall x,y \in X$,

$$|f(x) - f(y)| = \underbrace{|f(x) - f_n(x)|}_{<\varepsilon/3 \text{ for } n \text{ large } ⓐ} + \underbrace{|f_n(x) - f_n(y)|}_{ⓑ} + \underbrace{|f_n(y) - f(y)|}_{<\varepsilon/3 \text{ for } n \text{ large } ⓐ}$$

Let $\varepsilon > 0$, $x \in X$

By ① (u.c.), certainly there exists n s.t. $\forall z \in X$ $|f_n(z) - f(z)| < \varepsilon/3$ ⓐ
$= n_0$ e.g.

Furthermore $(\exists \delta > 0)(\forall y \in B_X(x,\delta))$ $|f_n(x) - f_n(y)| < \varepsilon/3$. ⓑ
As in (X,d)

The target space is complete, but X could be rationals etc.

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Using the same ideas, we can prove the following corollary:

Let $\{f_n\}_{n=1,2,3\ldots}$ be a sequence of continuous functions on X, with values in $\mathbb{R}$. If f_n's converge uniformly to f, then f must be continuous.

Def. Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f: X \to Y$ is Hölder continuous with exponent $\gamma \in (0,1]$ if
$$|f|_\gamma := \sup_{\{x_1, x_2 \in X,\, x_1 \neq x_2\}} \frac{d_Y(f(x_2), f(x_1))}{d_X(x_2, x_1)^\gamma} < \infty$$

The special case $\gamma = 1$ is known as Lipschitz continuity.

The set of Hölder continuous functions is denoted by $C^{0,\gamma}(X)$.

Remarks. · Every Lipschitz continuous function is uniformly continuous.

· If $0 < \alpha_2 < \alpha_1 \leq 1$ then (f is α_1-Hölder continuous $\Rightarrow$ f is α_2-Hölder continuous).

case on $x := d_X(\cdot,\cdot)$. $x > 1$. $\alpha_2 < \alpha_1 \Rightarrow \frac{1}{(x)^{\alpha_1}} < \frac{1}{(x)^{\alpha_2}} < 1$. $x \leq 1$: $\frac{1}{x^{\alpha_2}} < \frac{1}{x^{\alpha_1}}$ also ok.

but $d_Y(\cdot) < \infty$ so okay.

Theorem (7.16 in Rudin). Assume $f_n: [a,b] \to \mathbb{R}$ are Riemann-integrable, $f_n \rightrightarrows f$ on $[a,b]$ as $n \to \infty$. (not necessarily continuous)
Then f is R-intg and $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$. 'exchange limits'

Proof sketch. Recall $\left| \int_a^b g(x)\,dx \right| \leq \int_a^b |g(x)|\,dx \leq |b-a| \sup_{x \in [a,b]} |g(x)|$

$$\left| \int_a^b f_n(x) - f(x)\,dx \right| \leq |b-a| \cdot \sup_{x\in[a,b]} |f_n(x) - f(x)| = |b-a|\, d_\infty(f_n, f) \to 0$$

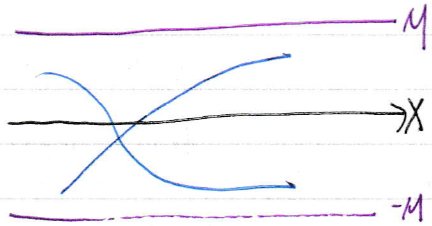
* Interchanging limits in differentiation does not work.

Compactness in $C_b(X,\mathbb{R})$ (Arzela-Ascoli Theorem)

Let (X,d) metric space, $\mathcal{F} \subseteq C(X,\mathbb{R})$. $\mathcal{F}$ is <u>uniformly bounded</u> if $\exists M \in \mathbb{R}$,

$(\forall f \in \mathcal{F})\ (\forall x \in X)\ (|f(x)| \leq M)$

Picture:



in $\mathbb{R}$: $|f(x_1) - f(x_2)|$

$\mathcal{F}$ is <u>equicontinuous</u> if $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall f \in \mathcal{F})(\forall x_1, x_2 \in X)\ d_X(x_1,x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \varepsilon$

without ✓, we merely have uniform continuity.

Examples:

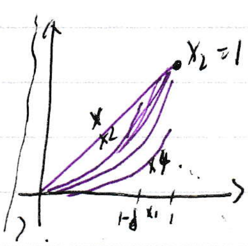
① Let $Lip(L) := \{ f \in C(X,\mathbb{R}), f \text{ is } L\text{-Lipschitz cont} \}$

$|f(x_1) - f(x_2)| \leq L\ d_X(x_1, x_2)$

Then $\mathcal{F} = Lip(L)$ is equicontinuous.

Proof. let $\varepsilon > 0$, $\delta = \varepsilon / L$.

② $\mathcal{F} = \{ x^n : n \in \mathbb{N}, x \in [0,1] \}$ is NOT equicontinuous.



Let $\varepsilon = 1/2$. AFSOC there is δ satisfying ( ).

Take $x_1 \in (1-\delta, 1)$. There exists n such that $x_1^n < \frac{1}{2} \Rightarrow |x_1^n - x_2^n| > 1/2$.

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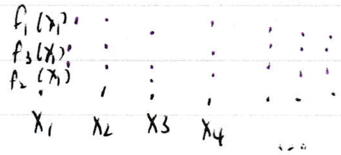
Theorem (~7.23 in Rudin) [Cantor's Diagonal Argument]

Let E be a countable set. Let $\{f_n : n \in \mathbb{N}\}$ be uniformly bounded functions (not necessarily continuous). Then there exists a subsequence $\{f_{n_k}\}_{k=1,2,\dots}$ such that $\forall x \in E$, $\{f_{n_k}(x)\}_{k=1,2,\dots}$ converges.

Proof.



$E = \{x_i : i \in \mathbb{N}\}$

- For every x_i we can find a convergent subsequence, but can we do it simultaneously?
  [compact $\Rightarrow$ seq cpct ...]

Doing it iteratively doesn't work (we have to start later and later...)

- What we have so far: Note that $\{f_n(x_1)\}_{n=1,\dots}$ is a bounded seq in $\mathbb{R}$. Thus by compactness we know there exists a ~~subsequence~~ convergent subsequence $\{f_{n_j}\}_{j=1,2,\dots}$ such that $f_{n_j}(x_1)$ converges as $j \to \infty$.

  $1 \quad 2 \quad \underset{n_1}{\boxed{3}} \quad 4 \quad \underset{n_2}{\boxed{5}} \quad \underset{n_3}{\boxed{6}} \quad 7 \quad \underset{n_4}{\boxed{8}} \quad \boxed{9} \quad 10$

  In other words, there exists a strictly increasing function $\sigma_1 : \mathbb{N} \to \mathbb{N}$, $n_j = \sigma_1(j)$.
  $\to \{f_{\sigma_1(n)}\}_{n=1,2,\dots}$ $f_{\sigma_1(n)}(x_1)$ converges as $n \to \infty$.

  Again by compactness, $\{f_{\sigma_1(n)}(x_2)\}_{n=1,2,\dots}$ has a convergent subsequence. Thus, $\exists\, \sigma_2 : \mathbb{N} \to \mathbb{N}$ increasing, $\{f_{\sigma_1(\sigma_2(n))}(x_2)\}_{n=1,2,\dots}$ converges likewise.

  Recall that $\{f_{\sigma_1(\sigma_2(n))}(x_1)\}_{n=1,2,\dots}$ still converges as $n \to \infty$.

  We can define therefore $\{f_{\sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_k(n)}\}_{n=1,2,\dots}$. Then $\{f_{\sigma_1 \circ \sigma_2 \cdots \sigma_k(n)}(x_j)\}_n$ converges as $n \to \infty$ for all $j \leq k$ (only n)

  * $\lim_{k\to\infty} \sigma_1(\sigma_2(\dots \sigma_k(n))\dots)$ may not exist e.g. $\sigma_k(i) = i+1$

- $\boxed{\text{Cantor's Idea}}$: Define the desired subsequence as follows.

  $\sigma(k) = \sigma_1 \circ \sigma_2 \cdots \sigma_k(k)$

  f_{σ_1}
  $f_{\sigma_1 \circ \sigma_2}$
  $f_{\sigma_1 \circ \sigma_2 \circ \sigma_3}$

The subsequence we consider is $\{f_{\sigma(k)}\}_{k=1,\dots}$

Let $x_j \in E$, why does $f_{r(k)}(x_j)$ converge as $k \to \infty$?

Note that by definition the tail (high indices of k) $\{f_{\sigma(k)}(x_j)\}_{k=j,j+1,\dots}$ is a subsequence of $\{f_{r_1 \circ \dots \circ \sigma_j(k)}(x_j)\}_{k=j,j+1,\dots}$ which converges by construction.

Lemma (Theorem 7.24 in Rudin)

Assume (K,d) is compact. Let $f_n \in C(K,\mathbb{R})$, $n=1,2,\dots$ If $\{f_n\}_{n=1,2,\dots}$ converges uniformly then $\{f_n : n \in \mathbb{N}\}$ is equicontinuous.

Proof. Let $\varepsilon > 0$. Since $\{f_n\}_{n=1,2,\dots}$ is Cauchy w.r.t d_∞,

(1) $(\exists n_0)(\forall n \geq n_0)\ \sup_{x \in K} |f_n(x) - f_{n_0}(x)| < \varepsilon/3$

So for $i = 1,2,\dots n_0$, there is δ_i such that $d(x,y) < \delta_i \Rightarrow |f_i(x) - f_i(y)| < \varepsilon/3$

Define $\delta := \min\{\delta_1, \dots, \delta_{n_0}\}$

(2) For $n \geq n_0$, $d(x,y) < \delta$

$|f_n(x) - f_n(y)| \leq \underbrace{|f_n(x) - f_{n_0}(x)|}_{<\varepsilon/3} + \underbrace{|f_{n_0}(x) - f_{n_0}(y)|}_{<\varepsilon/3} + \underbrace{|f_{n_0}(y) - f_n(y)|}_{\varepsilon/3}$

For $n \leq n_0$, the argument already works by

Theorem (Arzela-Ascoli) Conditions for compactness in $C_b(K)$

Assume (K,d) is compact. If $\{f_n\}_{n\in\mathbb{N}}$ is pointwise bounded and equicontinuous, then
(i) $\{f_n : n\in\mathbb{N}\}$ is uniformly bounded
(ii) $\{f_n\}_{n=1,2,\dots}$ has a uniformly convergent subsequence.

Proof. (i) Let $\varepsilon=1$. $\exists\, \delta>0$ (from compactness) $(\forall n)(\forall x,y)\ d(x,y)<\delta \Rightarrow |f_n(x)-f_n(y)|<1$.
Since K is totally bounded, there exists $x_1,\dots x_j$ such that $K\subseteq \bigcup_{i=1}^{j} B(x_i,\delta)$

Let $M = \max_{i=1\dots j} \Big(\underbrace{\sup_{n\in\mathbb{N}} f_n(x_i)}_{<\infty \text{ by pointwise boundedness}} \Big)$

<u>Claim</u> $\forall x\in K, \forall n\in\mathbb{N}$, $|f_n(x)|\le M+1$

Proof. $\exists\, i\in\{1,\dots j\}$ $x\in B(x_i,\delta)$.

$|f_n(x)|<|f_n(x_i)|+1\le M+1$. So the f_n's are uniformly bounded.

(ii) K is compact and thus separable $\Rightarrow$ there exists a countable dense subset E.
By Cantor's diagonal argument, $\exists$ subsequence $\{f_{n_i}\}_{i=1\dots}$ such that $\forall x\in E$, $\{f_{n_i}(x)\}_{i=1,2,\dots}$ converges as $i\to\infty$. Let $g_i = f_{n_i}$.

Since we have shown $(C(K), d_\infty)$ is complete, we want to show $\{g_i\}$ is Cauchy. Let $\varepsilon>0$, then
(equicontinuity) $\exists\,\delta>0$ $(\forall i)(\forall x,y)$ $d(x,y)<\delta \Rightarrow |g_i(x)-g_i(y)|<\varepsilon/3$. By compactness, since $\{B(y,\delta): y\in E\}$ is an open cover of K, there is a finite subcover: $\{B(y_j,\delta),\ j=1\dots l\}$.

$\{g_i(y_j)\}_{i=1\dots}$ is Cauchy for all $j=1\dots l$, because in fact they are all convergent (the point is to show they are jointly convergent ...)
So $\exists\, n_j$. $\forall m,n\ge n_j$, $|g_n(y_j)-g_m(y_j)|<\varepsilon/3$. Let $\bar{n}=\max\{n_j : j=1\dots l\}$.

Let $y \in K$. Using the subcover, $\exists j$ $d(y, y_j) < \delta$. So for $m, n \geq n_1$

$$|g_m(y)-g_n(y)| \le \underbrace{|g_m(y)-g_m(y_j)|}_{<\varepsilon/3 \text{ by equicontinuity}} + |g_m(y_j)-g_n(y_j)| + \underbrace{|g_n(y_j)-g_n(y)|}_{<\varepsilon/3 \text{ by eq. cont.}} \le \varepsilon$$

Thus $\{g_i\}$'s are Cauchy, and by completeness they are also convergent in d_∞ ($\Leftrightarrow$ uniformly convergent)

Corollary. Let (K,d) compact. Let $F \subseteq C(K,\mathbb{R})$. F is compact iff

(i) F is closed (w.r.t. d_∞)

(ii) F is uniformly bounded

(iii) F is equicontinuous

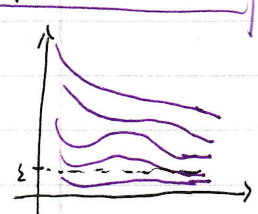
Proof. ($\Leftarrow$) Recall that compact $\Leftrightarrow$ every sequence has a convergent subsequence. Let $\{f_n\}$ be a sequence in F. Then by Arzelà-Ascoli, there is a uniformly convergent subsequence $f_{n_i} \rightrightarrows f$ as $i \to \infty$. Since F is closed, $f \in F$.

($\Rightarrow$) Exercise. compact $\Rightarrow$ closed (fact)
$\Rightarrow$ totally bounded i.e. $B_{d_\infty}(f_i, \delta)$ can cover, $0 \le i \le n$
$\Rightarrow$ uniformly bounded
$\Rightarrow$ equicontinuous

Theorem (Dini) Assume (K,d) is compact, $f \in C(K)$ and
(i) $f_n \in C_K$
(ii) $\forall x \in K$, $f_n(x) \to f(x)$ as $n \to \infty$ (pointwise)
(iii) $(\forall n \in \mathbb{N})(\forall x \in K)$ $f_n(x) \le f_{n+1}(x)$.

Then $f_n \rightrightarrows f$ as $n \to \infty$. $(\forall n, f \ge f_n)$

Proof. Let $g_n = f - f_n$. g_n is continuous, $g_{n+1} \le g_n$ and $g_n \to 0$ pointwise.



We want to show that $(\forall \varepsilon > 0)(\exists n_0)(\forall n \ge n_0)(\forall x)$ $g_n(x) < \varepsilon$.

Let $\varepsilon > 0$. Let $K_n = \{x : g_n(x) \ge \varepsilon\}$ 'bad set'. Since $g_{n+1} \le g_n$, $K_{n+1} \subseteq K_n$ (bad sets get smaller)

$\bigcap_{n=1}^{\infty} K_n = \emptyset$ by pointwise convergence. By a homework problem, there exists n_0, $\bigcap_{n=1}^{n_0} K_n = \emptyset$ $(=: K_{n_0})$.

$K_{n_0+1}, K_{n_0+2}, \dots$ are all empty.

Stone-Weierstrass Theorem

Weierstrass: One can approximate in d_∞ continuous functions by polynomials on bounded interval.

Stone: Algebras of functions can approximate continuous functions (see next page)

Exercise (HW) $p_1(x) := 0$ on $[-1,1]$, and recursively define

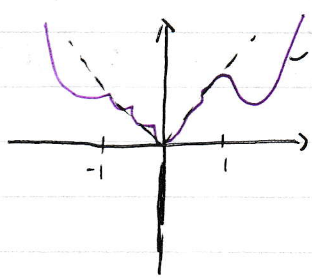
$$p_{n+1}(x) = p_n(x) + \frac{x^2 - (p_n(x))^2}{2} \quad \text{for } x \in [-1,1]$$

Claims. (i) by n $0 \le p_n(x) \le p_{n+1}(x) \le |x|$ for all n, all $x \in [-1,1]$

(ii) $p_n(x) \rightrightarrows |x|$ on $[-1,1]$, $|x| - p_{n+1}(x) \le \lambda\left(|x| - p_n(x)\right)$

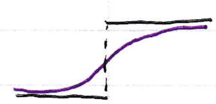
Consider operator $Tf(x) := f(x) + \frac{x^2 - f(x)^2}{2}$. If $f(x) = |x|$, $Tf(x) = |x| + \frac{x^2 - |x|^2}{2} = |x|$ *

So the abs. function is a <u>fixed point</u> of T. (Second-order convergence p40.)



Cannot approximate discontinuous functions by polynomials in d_∞

Kolmogorov: Fourier series converges to midway of a jump discontinuity



(any compact space)

(Stone) Let $I = [a,b]$ $\quad$ $\mathcal{F}$ is an algebra if $\forall f,g \in \mathcal{F}$, $\forall t \in \mathbb{R}$ $f+g, f\cdot g, tf$ belong to $\mathcal{F}$.

$\mathcal{F} \subseteq C(I,\mathbb{R})$. $\mathcal{F}$ separates points if $\forall x,y \in I$, $\exists f \in \mathcal{F}$ $f(x) \neq f(y)$.

e.g. Even polynomials don't separate -1 and 1. → this family cannot approximate $f(x)=x$.

$\{f(x)=x\}$ separates points.

Theorem (Stone) Let (K,d) be a compact metric space.

Consider $(C(K,\mathbb{R}), d_\infty)$ (metric) Let $\mathcal{F} \subseteq C(K,\mathbb{R})$.

If (i) $\mathcal{F}$ is an algebra

(ii) $\mathcal{F}$ separates points

(iii) $\mathcal{F}$ contains constant functions.

Then $\mathcal{F}$ is dense in $(C(K,\mathbb{R}), d_\infty)$.

$(f \in (C(K,\mathbb{R}), d_\infty) \Leftrightarrow f \in \overline{\mathcal{F}})$

(Stone ⇒ Weierstrass) Note that $\mathcal{F} = \{\text{polynomials}\}$ on $[a,b]$ satisfies (i)-(iii).

$\boxed{\text{Step 1.}}$ Claim: $\overline{\mathcal{F}}$ is an algebra.

Let $f,g \in \overline{\mathcal{F}}$. Then $\exists$ $(f_n) \rightrightarrows f$, $(g_n) \rightrightarrows g$, $f_n, g_n \in \mathcal{F}$

So $f_n + g_n \in \mathcal{F}$.

$f_n + g_n \rightrightarrows f+g \in \overline{\mathcal{F}}$ (limit of sequence is limit point)

We use approximations to verify the other properties of an algebra.

$\boxed{\text{Step 2.}}$ Claim: If $f \in \bar{\mathcal{F}}$ then $|f| \in \bar{\mathcal{F}}$.

Proof. Let $M = \max_{x \in K} |f(x)|$. Let $g(x) = \frac{|f(x)|}{M}$ 'normalization' ($\Rightarrow g(x) \in [-1,1]$ for all $x \in K$)

(compact $\Rightarrow$ attains max)

By exercise, $\exists\, p_n$ - polynomial, $p_n(z) \rightrightarrows |z|$ as $n \to \infty$ on $[-1,1]$.

$$\max_{x \in K} \left| p_n\left(\frac{f(x)}{M}\right) - \left|\frac{f(x)}{M}\right| \right| \le \max_{z \in [-1,1]} \left| p_n(z) - |z| \right| \to 0 \text{ as } n \to \infty$$

So $p_n\left(\frac{f(x)}{M}\right) \rightrightarrows \frac{|f(x)|}{M}$ (squeeze theorem)

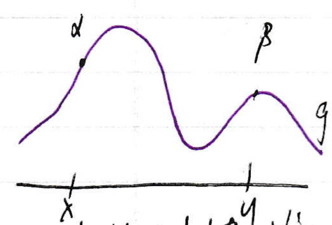
$$\underbrace{M p_n\left(\frac{f(x)}{M}\right)}_{\in \bar{\mathcal{F}}} \rightrightarrows |f(x)|, \text{ so } |f(x)| \in \bar{\mathcal{F}}.$$

* since $\bar{\mathcal{F}}$ is algebra (composition, multiplying by constant)

$\boxed{\text{Step 3.}}$ Note $\max\{f,g\}(x) = \frac{f(x)+g(x)}{2} + \frac{1}{2}|f(x)-g(x)|$

$\min\{f,g\}(x) = \frac{f(x)-g(x)}{2} - \frac{1}{2}|f(x)-g(x)|$

So if $f,g \in \bar{\mathcal{F}}$, then $\min\{f,g\}, \max\{f,g\} \in \bar{\mathcal{F}}$.



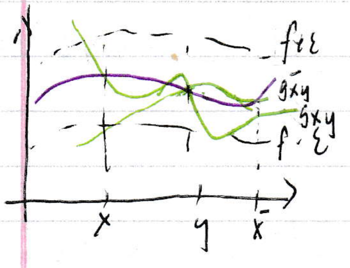
$\boxed{\text{Step 4.}}$ $\forall x,y \in K$, $\forall \alpha, \beta \in \mathbb{R}$, $\exists g \in \bar{\mathcal{F}}$, $g(x) = \alpha$, $g(y) = \beta$.

Proof. By separation (property ii), $\exists f \in \mathcal{F}$ s.t. $f(x) \ne f(y)$. We can stretch and shift this to get g.

Let $g(z) = \dfrac{\alpha(f(z) - f(y)) + \beta(f(x) - f(z))}{f(x) - f(y)}$

$\forall f \in C(K,\mathbb{R}), \forall \varepsilon > 0, \exists g \in \mathcal{F}, d_\infty(f,g) < \varepsilon.$

$\boxed{\text{Step 5}}$ ("magic"): Finding the approximating function. Let $x, y \in K$ be given.
We can find $g_{xy} \in \mathcal{F}$ such that $g_{xy}(x) = f(x)$, $g_{xy}(y) = f(y)$.



Define $U_{xy} := \{z \in K : g_{xy}(z) < f(z) + \varepsilon\}$ – open! $= \{z \mid g_{xy}(z) - f(z) < \varepsilon\} = \underbrace{(g_{xy} - f)}_{\text{continuous}}^{-1}(\underbrace{(-\infty, \varepsilon)}_{\text{open}})$

$V_{xy} := \{z \in K : g_{xy}(z) > f(x) - \varepsilon\}$ – open!

Fix y. $\{U_{xy} \mid x \in K\}$ is an open cover of K $(x \in U_{xy})$. Since K is compact, $\exists$ finite subcover $U_{x_1,y}, \dots U_{x_n,y}$

Let $g_y := \min\{g_{x_1 y}, \dots g_{x_n y}\}$, and note that g_y is continuous (take minimum of g's).

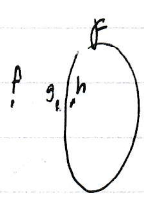
Claim. $\forall z \in K$, $g_y(z) < f(z) + \varepsilon$.

pf. Take $z \in K$. $\exists i$ $z \in U_{x_i,y}$, $g_y(z) \le g_{x_i,y}(z) < f(z) + \varepsilon$.

Furthermore, $g_y(z) > f(z) - \varepsilon$ for all $z \in V_y := \bigcap_{i=1}^{n} V_{x_i,y}$ $(g_i > f(x) - \varepsilon \Rightarrow \min g_i > f(x) - \varepsilon)$

Since V_y is open and contains y, the family $\{V_y\}_{y \in X}$ is an open cover of X. By compactness, there exist the subcover $\{V_{y_i} : 1 \le i \le n\}$, $\bigcup_{i=1}^{n} V_{y_i} = X$. Define $g := \max\{g_{y_1}, g_{y_2} \dots g_{y_n}\}$.

- g was defined by taking repeated max and min's of g_{xy}'s, which are in $\mathcal{F}$, so by step 3 $g \in \bar{\mathcal{F}}$.
- Furthermore, $f(z) - \varepsilon < g(z) < f(z) + \varepsilon$ for all $z \in X$ by construction.



So $g \in \bar{\mathcal{F}}$ ($\Rightarrow$ there exists $h \in \mathcal{F}$ such that $\max_{z \in X} |h(z) - g(z)| \le \varepsilon$). By the Δ-inequality, $\max_{z \in X} |f(z) - h(z)| \le 2\varepsilon$. These h's approximate f arbitrarily well.

w.T.s $\int_0^1 M f(y)\,dy \leq \int_0^1 M\cdot N\,dy$ pick $\delta \leq \varepsilon/MN$.

Tf is unif. continuous.

21-356 Notes and Comments about HW5

3/16/2022

1. $\mathcal{F}$ equicontinuous, $\mathcal{F}$ pointwise bounded

let x be fixed $|Tf_n(x)| = \left|\int_0^1 k(x-y)\, f_n(y)\,dy\right|$

$\leq \int_0^1 |k(x-y) f_n(y)|\,dy$

$= \int_0^1 |k(x-y)|\,|f_n(y)|\,dy$

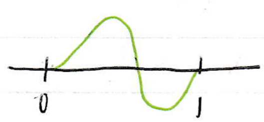
$\leq \int_0^1 |k(x-y)|\,dy$

$[-1,1]$ compact ...

let ε be given. $Tf_n(x+\delta) = \int_0^1 k(x+\delta-y)\, f(y)\,dy$

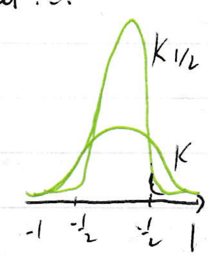
$Tf_n(x+\delta) - Tf_n(x) = \int_0^1 [\underbrace{k(x+\delta-y) - k(x-y)}]\, f(y)\,dy$

cont. on compact set $\Rightarrow$ unif. cont.



$f: X \to \mathbb{R}$

$\operatorname{supp} f = \overline{\{x : f(x) \neq 0\}}$



$\int_{\mathbb{R}} K_\varepsilon(x)\,dx = \frac{1}{\varepsilon}\int K(x/\varepsilon)\,dx$

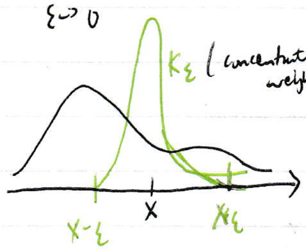
$s = x/\varepsilon \quad = \int K(s)\,ds = 1$

$ds = dx/\varepsilon$

$\int_{\mathbb{R}} K(x)\,dx = 1,\ K > 0,\quad K_\varepsilon(x) = \frac{1}{\varepsilon} K(x/\varepsilon)$

$\operatorname{supp} K = [-1,1]$

w.T.s $\lim_{\varepsilon\to 0} f_\varepsilon(x) = f(x)$



$f_\varepsilon(x) = \int_{-\infty}^{\infty} K_\varepsilon(x-z) f(z)\,dz$

$s = x-z$
$ds = -dz$

$= -\int_{\infty}^{-\infty} K_\varepsilon(s) f(x-s)\,ds$

$= \int_{-\infty}^{\infty} K_\varepsilon(s) f(x-s)\,ds$

convolution is symmetric

$= K_\varepsilon * f(x)$

$= f * K_\varepsilon(x)$

Hint. $f(x) - f_\varepsilon(x) = f(x)\int_{\mathbb{R}} K_\varepsilon(x-z)\,dz - \int_{\mathbb{R}} K_\varepsilon(x-z) f(z)\,dz$

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let $p(x) = ax^2 + bx + c$. Let K above be this polynomial (Rudin's proof of Weierstrass)

$f * p(x) = \int f(z) p(x-z)\,dz = \int f(z)\,(a(x-z)^2 + b(x-z) + c)\,dz$

$= \int f(z)\,(ax^2 - 2azx + az^2 + bx - bz + c)\,dz$

$= ax^2 \int f(z)\,dz + x\int f(z)(-2az + b)\,dz \ldots$ is another polynomial!

Q4a... Why do these approximations work?

$p_{n+1}(x) = p_n(x) + \frac{1}{2}(x - p_n^2(x))$ ($\sqrt{x}$ is the fixed point)

$|\sqrt{x} - p_{n+1}(x)| \le C(\sqrt{x} - p_n(x))^2$ quadratic convergence!

$g'(x) = 1 - f(x)\left(-\frac{f''(x)}{(f'(x))^2}\right) - \frac{f'(x)}{f'(x)} = \frac{f(x)f''(x)}{f'(x)f'(x)}$

→ Boils down to Newton's method: $g(x) = x - \frac{f(x)}{f'(x)}$ should be differentiable on $[x_0-\varepsilon, x_0+\varepsilon]$, $|g'(x)| \le \alpha < 1$ on $[x_0-\varepsilon, x_0+\varepsilon]$

Alternatively, we need $\left|\frac{f''(x)}{2f'(y)}\right| \le M$, $x, y \in [x_0-\varepsilon, x_0+\varepsilon]$ and $M\varepsilon < 1$ for quad. convergence. 3/18/2022

$0 = f(x_0) = f(x) + f'(x)(x_0 - x) + \frac{f''(\xi)}{2}(x_0 - x)^2$

$= \frac{f(x)}{f'(x)} + (x_0 - x) + \frac{f''(\xi)}{2f'(x)}(x_0 - x)^2$

$= x_0 - g(x) + \frac{f''(\xi)}{2f'(x)}(x_0 - x)^2$

POWER SERIES

$f(x) = \sum_{n=0}^{\infty} a_n x^n$

Recall: series' definition: $\sum_{n=0}^{\infty} c_n = \lim_{k\to\infty} s_k$, $s_k = \sum_{n=0}^{k} c_k$

- $\sum_{n=1}^{\infty} \frac{1}{n^p}$, converges iff $p > 1$, $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$ (complex analysis...)
- $\sum_{n=0}^{\infty} c_n < \infty \Rightarrow c_n \to 0$ as $n \to \infty$
- $\limsup_{n\to\infty} \sqrt[n]{|c_n|} < 1$ series converges (geometric series)
- $\limsup_{n\to\infty} \sqrt[n]{|c_n|} > 1$ series diverges. ($|c_n| \not\to 0$)

So $\sum_{n=0}^{\infty} a_n x^n$ converges when $\limsup_{n\to\infty} \sqrt[n]{|a_n||x|^n} = |x| \underbrace{\limsup \sqrt[n]{|c_n|}}_{\alpha} < 1$

So $R = \frac{1}{\alpha} = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|c_n|}}$ is the radius of convergence

($\frac{1}{\infty} = 0$)

If $|x| > R$: series diverges.

4 cases at the boundary