21-356 Remark: These theorems start by assuming that a function is analytic. 3/18/2022

$\boxed{\text{Theorem (8.1)}}$ If R is radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$, then for any $r<R$,

$\sum_{n=0}^{k} a_n x^n \rightrightarrows f(x)$ on $[-r,r]$

Proof. Recall (Theorem 7.6 in notes) If $\forall x \in A$, $|f_n(x)| \le M_n$, $\sum_{n=0}^{\infty} M_n < \infty$, then $\sum_{n=1}^{\infty} f_n$ converges uniformly.

$\rightarrow$ observe $f(x) - f_n(x) \le \sum_{j=n+1}^{\infty} M_j \to 0$ as $n \to \infty$.

$|\Sigma \cdot| \le \Sigma|\cdot|$

So if $\sum \underbrace{|a_n| r^n}_{=:M_n} < \infty$, then $\forall x \in [-r,r] \; \forall n$, $|a_n x^n| \le |a_n| r^n = M_n$, use theorem.

$\boxed{\text{Theorem}}$ (termwise integration) Assume R is radius of convergence of $\sum_{n=0}^{\infty} a_n x^n =: f(x)$. $\left({}^{*}\int_0^x a_n z^n dz = \frac{a_n}{n+1} x^{n+1} \right)$

Let $F(x) = \int_0^x f(z) dz$. Then $\sum_{n=1}^{\infty} \frac{a_n}{n+1} x^{n+1}$ has radius of convergence R, and

$\sum_{n=1}^{\infty} \frac{a_n}{n+1} x^{n+1} = F(x)$ on $(-R,R)$

$\downarrow$ not compact, otherwise converges uniformly)

Proof. (1) $j = n+1$, $n = j-1$

$\sum_{j=1}^{\infty} \frac{a_{j-1}}{j} x^j$, note $\limsup_{j\to\infty} \sqrt[j]{\frac{|a_{j-1}|}{j}} = \underbrace{\lim_{j\to\infty} \frac{1}{\sqrt[j]{j}}}_{1} \cdot \sqrt[j]{|a_{j-1}|} = \frac{1}{R}$ ✓

shift index, exercise

(2) (Theorem 8.?) $f_n \rightrightarrows f$ on $[0,x]$, then $\int_0^x f_n(s)ds \to \int_0^x f(s) ds$

Here let $f_n(s) = \sum_{j=0}^{n} a_j s^j$, $f_n \rightrightarrows f$ on $[0,x]$ for $0 \le x < R$

Then $\int_0^x \sum_{j=0}^{n} a_j s^j ds \to \int_0^x f(s) ds = F(x)$

But LHS $= \sum_{j=0}^{n} \frac{a_j}{j+1} x^{j+1}$

ADBS

Theorem Assume $\sum_{n=0}^{\infty} a_n x^n = f(x)$ has radius of convergence R.

Then $f(x)$ is differentiable on $(-R,R)$ and $f'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1}$ on $(-R,R)$

Proof. (Step 1) $\underbrace{\sum_{n=1}^{\infty} a_n n x^{n-1}}_{= g(x)}$ has radius of convergence R

(Step 2) By the previous theorem, by integrating term by term we get

$$f(x) - a_0 = \sum_{n=1}^{\infty} a_n x^n \to \int_0^x g(s)\,ds.$$

$$f(x) = a_0 + \int_0^x g(s)\,ds.$$

Then by FTOC, $f'(x) = g(x)$ (RHS is differentiable so LHS is also differentiable)

Counter-example: $\frac{1}{n^2}\sin(nx) \rightrightarrows 0$

If $\{f_n\}$ converges uniformly → integrals converge → derivatives don't

Theorem Suppose $\sum_{n=0}^{\infty} c_n$ converges and $f(x) := \sum_{n=0}^{\infty} c_n x^n$ has radius of convergence $R \geq 1$. Then

$\lim_{x \to 1} f(x) = f(1)$ (i.e. sequence is convergent at $x=1$ to $\sum_{n=0}^{\infty} c_n x^n$, $f(x)$ continuous at $x=1$)

Musings

- Defining sine only [via] power series and relating it to a circle seems easier than the other way round (What is angle? Arclength?)

Lagrange remainder of e^{-1/x^2} is e^{-1/x^2} itself!

For sine/cosine, the remainder → 0 as $n \to \infty$

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Theorem 8.4 (Rudin) If $f(x) := \sum_{n=0}^{\infty} c_n x^n$ converges in $|x| < R$, and $a \in (-R,R)$, then f can be expanded in a power series about $x=a$, converging in $|x-a| < R - |a|$, and $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$.

$$f^{(n)}(a) = \sum_{m=n}^{\infty} c_m \frac{m!}{(m-n)!} x^{m-n}$$

Pf. $f(x) = \sum_{n=0}^{\infty} c_n ((x-a)+a)^n$

$$= \sum_{n=0}^{\infty} c_n \sum_{m=0}^{n} \binom{n}{m} a^{n-m} (x-a)^m$$

want to show $\sum_{n=0}^{\infty}\sum_{m=0}^{n} \left| c_n \binom{n}{m} a^{n-m} (x-a)^m \right|$ converges and then apply lemma (Th 8.3). But $\sum_{m=0}^{n} c_n \binom{n}{m} a^{n-m} (x-a)^n = |c_n| \underbrace{(|x-a|+|a|)}_{< |x|}^n$

$$\stackrel{?}{=} \sum_{m=0}^{\infty} \left(\sum_{n=m}^{\infty} c_n \binom{n}{m} a^{n-m} \right) (x-a)^m$$

$$= \sum_{m=0}^{\infty} \frac{f^{(m)}(a)}{n!} (x-a)^m$$

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Theorem 8.3 (Rudin) Let $a_{ij} \geq 0$, $i \in \mathbb{N}$, $j \in \mathbb{N}$.

If for all i, $\sum_{j=1}^{\infty} a_{ij} =: b_i < \infty$, and $A = \sum_{i=1}^{\infty} b_i < \infty$, (absolute convergence) [Alternatively, assume $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}| < \infty$]

then $\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{ij}\right) = \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}\right) = A$.

(Counterexample: order of summation matters!

$$\begin{array}{cccccc|c} a_{11}=1 & a_{12}=-1 & a_{13}=0 & 0 & 0 & 0 \ldots & b_i = \sum_{j=1}^{\infty} a_{ij} \\ & & & & & & 0 \\ a_{21}=0 & a_{22}=1 & -1 & 0 & 0 & 0 \ldots & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \ldots & 0 \\ \vdots & \vdots & & & & & \vdots \end{array}$$

$c_1 = \sum_{i=1}^{\infty} a_{i1} = 1$.

So $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij} = 1$ but $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = 0$!)

Proof. (Rudin uses Theorem 8.2) Here's a brute-force approach.

Define $b_i = \sum_{j=1}^{\infty} |a_{ij}|$, $\beta_i = \sum_{j=1}^{\infty} a_{ij}$, $|\beta_i| \leq b_i$ by Δ-inequality.

$$\begin{array}{cccc} a_{11} & a_{12} & a_{13} & \cdots a_{1n} \\ a_{21} & a_{22} & & \\ \vdots & & & \\ a_{k1} & a_{k2} & & a_{kn} \end{array}$$

$\varepsilon/4$

γ_1 $< \varepsilon/4$

① $\varepsilon/4$

Step 1. $\forall j$, $|a_{ij}| \leq b_i$. By comparison, $\gamma_1 := \sum_{i=1}^{\infty} |a_{ij}| \leq \sum_{i=1}^{\infty} b_i < \infty$.

Step 2. $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \gamma_j$, but consider

$\sum_{j=1}^{\infty}\underbrace{\sum_{i=1}^{\infty} |a_{ij}|}_{c_j} = \sum_{j=1}^{\infty} c_j$. ($\Leftarrow$ in terms of convergence)

To show that $\sum_{j=1}^{\infty} c_j$ converges, it suffices to show that $\sum_{j=1}^{n} c_j \leq M+1$ for all n.

$\sum^{n}\Big(\underbrace{\sum^{\infty}}_{<\infty}\Big) = \sum^{\infty}\sum^{n}$ by induction: $\lim a_n + \lim b_n = \lim(a_n + b_n)$

Let $\varepsilon = \frac{1}{n}$. $\forall j$ Since $\sum_{i=1}^{\infty}|a_{ij}| = c_j$, $\exists k_j$ s.t. $\sum_{i=1}^{k_j}|a_{ij}| > c_j - \varepsilon$.

define $\bar{k} = \max\{k_1, k_2, \ldots k_n\}$.

$\forall j = 1, 2, \ldots n \quad \sum_{i=1}^{\bar{k}}|a_{ij}| > c_j - \varepsilon$.

$$\underbrace{\sum_{j=1}^{n}\sum_{i=1}^{\bar{k}}}_{\text{finite!}}|a_{ij}| > \left(\sum_{j=1}^{n} c_j\right) - n\varepsilon = \sum_{j=1}^{n} c_j - 1$$

$$1 + M \geq 1 + \sum_{i=1}^{\bar{k}} b_i \geq 1 + \sum_{i=1}^{\bar{k}}\sum_{j=1}^{n}|a_{ij}| > \sum_{j=1}^{n} c_j$$

Let $A = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} b_i$. We want to show $A = C = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$.

step 1 $\exists k_0\ (\forall \bar{k} > k_0)\ \left|A - \sum_{i=1}^{\bar{k}} b_i\right| < \varepsilon/4$.

with switching $\exists n_0\ \forall \bar{n} > n_0\ \left|\sum_{i=1}^{\bar{n}}\sum_{j=1}^{\bar{k}} a_{ij} - \sum_{j=1}^{\bar{k}} b_i\right| < \varepsilon/4$

step 2 $\exists n_1, \forall \bar{n} > n_1, \left|C - \sum_{j=1}^{\bar{n}} c_j\right| < \varepsilon/4$, ~~where $C = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$~~

step 3 $\exists k_1, \forall \bar{k} > k_1, \left|\sum_{j=1}^{\bar{n}}\sum_{i=1}^{\bar{k}} a_{ij} - \sum_{j=1}^{\bar{n}} c_j\right| < \varepsilon/4$ for $\bar{n} > \max\{n_0, n_1\}$, $\bar{k} > \max\{k_0, k_1\}$

$\vdots$

$\boxed{|A - C| < \varepsilon}$ since $|A - C| \leq \left|A - \sum_{i=1}^{\bar{k}} b_i\right| + \left|\sum_{i=1}^{\bar{k}} b_i - \quad\right| + |\quad| \cdot |$

$+ | \cdot |$

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(Rudin's proof) want to show $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|<\infty \Rightarrow \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}=\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}a_{ij}$

let $E=\{x_0, x_1, x_2, \dots\}$ countable, $x_n \to x_0$ as $n\to\infty$.

Define $f_i(x_0)=\sum_{j=1}^{\infty}a_{ij}$

$f_i(x_n)=\sum_{j=1}^{n}a_{ij}$

$g(x)=\sum_{i=1}^{\infty}f_i(x)$

$x_n\to x_0 \Rightarrow f_i(x_n)\to f_i(x_0)$ since for all i, $\sum_{j=1}^{\infty}|a_{ij}|<\infty$

so each f_i is continuous.

[Recall: If $f_n\to f$ uniformly, $\lim_{t\to\infty}f_n(t)=:A_n$, then $\lim_{t\to x}f(t)=\lim_{n\to\infty}A_n$ i.e. $\lim_{t\to x}\lim_{n\to\infty}f_n(t)=\lim_{n\to\infty}\lim_{t\to x}f_n(t)$]

Now, since for any x_k, $\sum_{i=1}^{n}f_i(x_k)=\sum_{j=1}^{\infty}a_{1j}+\sum_{j=1}^{\infty}a_{2j}+\dots+\sum_{j=1}^{\infty}a_{nj}$ missing the tail $<b_{n+1}+b_{n+2}\dots$

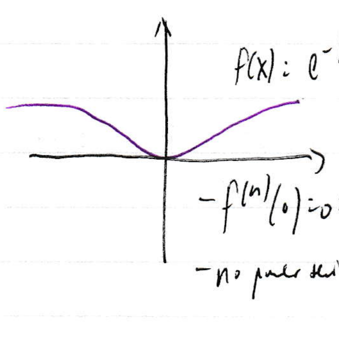
$\sum_{i=1}^{n}f_i(x)\to g$ uniformly (independent of x). $=:g(x_k)$, so $x_n\to x_0 \Rightarrow g(x_n)\to g(x_0)$.

$$\therefore \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}=\sum_{i=1}^{\infty}f_i(x_0)=g(x_0)=\lim_{n\to\infty}g(x_n)=\lim_{n\to\infty}\sum_{i=1}^{\infty}f_i(x_n)$$

$$=\lim_{n\to\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{n}a_{ij}$$

$$=\lim_{n\to\infty}\sum_{j=1}^{n}\sum_{i=1}^{\infty}a_{ij}$$

$$=\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}a_{ij}$$



- $f^{(n)}(0)=0$: Taylor expansion useless
- no power series

Theorem (8.5) Let R be the radius of convergence, $E = \{x : f(x) = 0\}$, where $f(x) = \sum_{n=0}^{\infty} c_n x^n$.
If there exists $a \in E' \cap (-R, R)$ then $c_n = 0$ for all n ($f \equiv 0$).

E': limit points

'Analytic': power series.
'More rigid' than smooth, so
extra condition ⇒ identically 0

(Alternative formulation, $S :=$ set of points where $\sum c_n x^n$ converges)
$a \in E' \cap S \ldots$

Let $a \in E' \cap (-R,R)$. WLOG $a = 0$.

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \overset{T8.4}{=} \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$g(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} z^n = f(x-a)$$

So we can consider $E_g = \{z \mid g(z) = 0\}$

$f(x) = c_0 + \sum_{n=1}^{\infty} c_n x^n$, $f(0) = c_0$. Note that by continuity E is closed since $\{0\}$ is closed.
$\Rightarrow E' \subseteq E \Rightarrow 0 \in E$. $|a-r| < \varepsilon \ \forall \varepsilon \Rightarrow a = r$.

$\therefore c_0 = 0$. Furthermore there is $x \in E$ near 0 since 0 is a limit point (by assumption)

Note $f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$

$\forall x \in E, \exists y$ between 0 and x (Rolle's Theorem) such that $f'(y) = 0$. Let $E_1 = \{x : f'(x) = 0\}$, $0 \in E_1' = E_1$ (by continuity)

$1 \cdot c_1 = 0 \Rightarrow c_1 = 0$. By continuity, we again have $0 \in E_1'$.

By induction we get $n! c_n = 0 \ \forall n$

⇒ Remark: For every non-trivial power series, its zeroes/roots within the interval of convergence $(-R, R)$ are all isolated.

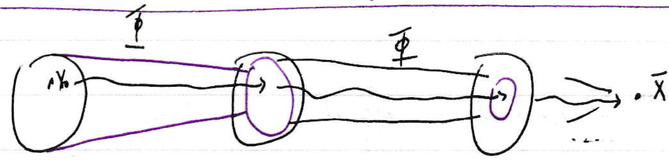
Banach Contraction Principle

Let (X,d) be complete metric space. (In applications, X is a compact or closed subset of $\mathbb{R}^d$, X closed subset of $(C(K,\mathbb{R}), d_\infty)$)

*Closed subset of complete m.s is complete

$\Phi : X \to X$ is a contraction if $\exists \lambda < 1$ s.t. $\forall x,y \in X$, $d(\phi(x),\phi(y)) \leq \lambda d(x,y)$.

Theorem Under assumptions above ((X,d) complete m.s, $\phi: X \to X$ contraction), then there exists unique $\bar{x} \in X$ such that $\phi(\bar{x}) = \bar{x}$ (known as the fixed point).



Pf. Show $\Phi^n(x_0) \to \bar{x}$.

(Existence) Let $x_0 \in X$, $x_{n+1} := \phi(x_n)$. $d(x_{n+2}, x_{n+1}) = d(\phi(x_{n+1}), \phi(x_n)) < \lambda d(x_{n+1}, x_n)$.

So by induction, $d(x_{n+1}, x_n) < \lambda^n \underbrace{d(x_1, x_0)}_{c} = c\lambda^n$

$$d(x_{n+k}, x_n) \leq d(x_{n+k}, x_{n+k-1}) + d(x_{n+k-1}, x_{n+k-2}) + \dots + d(x_{n+1}, x_n)$$
$$= c[\lambda^{n+k-1} + \lambda^{n+k-2} + \dots + \lambda^n]$$
$$< \lambda^n c \sum_{i=0}^{\infty} \lambda^i = \lambda^n \frac{c}{1-\lambda}.$$

(Uniqueness) Let $\bar{x} = \phi(\bar{x}), \bar{y} = \phi(\bar{y})$. $d(\bar{x},\bar{y}) = d(\phi(\bar{x}), \phi(\bar{y})) \leq \lambda d(\bar{x},\bar{y}) \Rightarrow \bar{x} = \bar{y}$.

Thus $\{x_n\}_n$ is Cauchy. Let $\varepsilon > 0$, $\exists n_0$, $\lambda^{n_0} \frac{c}{1-\lambda} < \varepsilon$.

So $\forall m,n \geq n_0$, $d(x_m, x_n) < \lambda^{n_0} \frac{c}{1-\lambda} < \varepsilon$. So by completeness the limit $\bar{x}$ exists.

Next ~~Lastly~~ we show that $\bar{x}$ is indeed a fixed point. ϕ is Lipschitz (hence continuous), so $\lim_{n\to\infty} \phi(x_n) = \phi(\lim_{n\to\infty} x_n) = \phi(\bar{x})$. LHS $= \bar{x} = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \phi(x_n) = \phi(\bar{x})$.

Applications

① $\phi(\bar{x}) = \bar{x} \iff \phi(\bar{x}) - \bar{x} = 0$. Convert question to fixed-point realm
e.g. root finding

② (Picard-Lindelöf) $x' = f(x,t)$, $x(0) = a \in \mathbb{R}^d$ IVP.

$\exists T > 0, \exists! \; x : (-T, T) \to \mathbb{R}^d$ $x \in C^1$ solving the IVP.

$\mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$

continuous, Lipschitz in at least 1 dimension

If $f \in C$ only, ~~the~~ a solution exists but may not be unique.

$x' = \sqrt[3]{x}$, $x(0) = 0 \Rightarrow x = 0$, $x = ct^{3/2}$

Arzela-Ascoli $\to$ compact $\to$ ~~compact~~ convergent subsequence
$\to$ 'know what it converges to' (Peano Existence, P15)

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- What happens if you cannot add?

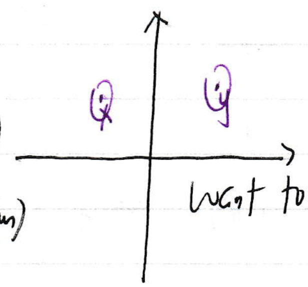
Let (X,d) be a metric space. E.g. $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$, $f: X \to \mathbb{R}$

$\boxed{\text{NORMED SPACE}}$ $(V, \|\cdot\|)$

- We want an underlying vector space V. (e.g. $\mathbb{F} = \mathbb{R}$, ~~$\mathbb{F} = \mathbb{C}$~~ (for now)) — where scalars come from
- We want a norm on the vector space.

metric vs norm

- distance between points (x, y)
- needs an origin
- everything is measured w.r.t. origin (unary dist function)



want to be able to translate (rotation about fixed point)

Def. A norm $\|\cdot\|$ is a mapping from V to $[0,\infty)$ such that

(i) If $\|v\| = 0$, then $v = \vec{0}$ (zero vector in V)

(ii) $\forall \alpha \in \mathbb{F}$, $\|\alpha v\| = |\alpha| \cdot \|v\|$

Rem. The space of metrics is huge! ($d \to \sqrt{d}$, $\tilde{d}(x,y) = f(d(x,y))$, no scaling...). Norms are much more restrictive.

(ii)+(iii) only: 'semi-norm'

(iii) $\|v + w\| \leq \|v\| + \|w\|$

Remark. If $\|\cdot\|$ is a norm on V, then $d(x,y) = \|x-y\|$ is a metric on V.

checking: (i) $d(x,y) = 0 \Rightarrow \|x-y\| = 0 \Rightarrow x - y = 0$, $x = y$

(ii) $d(x,y) = \|x-y\| = \|(-1)(y-x)\| = |-1| \cdot \|y-x\| = \|y-x\| = d(y,x)$

(iii) $d(x,z) = \|x-z\| = \|(x-y) + (y-z)\| \leq \|x-y\| + \|y-z\| = d(x,y) + d(y,z)$

→ This metric will be assumed, and it will be used to define the topology etc. (Normed space ⇒ metric space) from $\|\cdot\|$

Def. A normed space $(V, \|\cdot\|)$ is a Banach space if (V, d) is complete.

Examples: ① Euclidean space on $X \in \mathbb{R}^d$

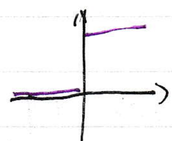
$p \ge 1$, $X \in \mathbb{R}^d$, $\|x\|_p = \sqrt[p]{\sum_{i=1}^{d} |x_i|^p}$

$\|x\|_\infty = \max_{i=1\dots d} \{x_i\}$.

② $C(K, \mathbb{R})$ $\|f\|_\infty = \sup_{x \in K} |f(x)|$
↑ compact

or $C_b(X, \mathbb{R})$

③ $C([0,1], \mathbb{R})$, $\|f\|_1 = \int_0^1 |f(x)|\,dx$ NOT Banach!



(need measure theory to complete this space)

Example (semi-norm)

Let (X, d) be a bounded metric space. $V = \{ f : X \to \mathbb{R} \mid \|f\|_\alpha < \infty \}$, where

$$\|f\|_\alpha = \sup_{\substack{x_1, x_2 \in X \\ x_1 \neq x_2}} \frac{|f(x_1) - f(x_2)|}{d(x_1, x_2)^\alpha}$$

– seminorm only
– cannot identify zero vector (constant function also have $\|f\|_\alpha = 0$ but $\neq$ zero vector!)

often $\|\cdot\|$ is replaced by $[\ ]$.

this is a norm!

Now define $\|f\|_{C^\alpha} = \boxed{\|f\|_\infty + [f]_\alpha}$. (Hölder continuous functions)

$x \overset{[0,1]}{\mapsto} \sqrt{x}$ is in $C^{1/2}$ but not in $C^{3/4}$, C^1 etc.

$\frac{\sqrt{x_1} - \sqrt{0}}{|x_1 - 0|^\alpha} = x_1^{\frac{1}{2} - \alpha}$, need $\alpha \le \frac{1}{2}$.

~~Equicont. bounded?~~

In the seminorm
- Ball is equicontinuous
- Bounded on bounded set $\Rightarrow$ Uniformly bounded.

center of ball

$d(g, f) = \sup_{x_1 \neq x_2 \in X} \frac{(g-f)(x_1) - (g-f)(x_2)}{[d(x_1, x_2)]^\alpha} < K$, but $\sup_{x_1 \neq x_2 \in X} \frac{f(x_1) - f(x_2)}{[d(x_1, x_2)]^\alpha} < L$

so $\sup \frac{g(x_1) - g(x_2)}{d(x_1, x_2)^\alpha}$

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Let V be a vector space.

$\langle \cdot, \cdot \rangle : V \times V \to F$ is an inner product if it is bilinear and <u>positive definite</u>. ($\|x\|^2 > 0$ if $x \neq 0$)

Claim: If $\langle \cdot, \cdot \rangle$ is an inner product then $\|v\| = \sqrt{\langle v, v \rangle}$ is a <u>norm</u>.
↳ gives rise to metric

A <u>Hilbert space</u> is a complete inner product space.

e.g. $\langle f, g \rangle = \int_0^1 f(x) g(x)\,dx$ is an inner product space but not complete.

Let $\underbrace{(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)}_{\text{underlying field is } \mathbb{R}}$ be normed spaces.

$L : X \to Y$ is <u>linear</u> if for $\alpha \in F$, $\forall x_1, x_2 \in X$, $L(\alpha x_1) = \alpha L(x_1)$, $L(\underset{\text{in } X}{x_1 + x_2}) = L(x_1) \underset{\text{in } Y}{+} L(x_2)$

We call such linear functions <u>operators</u>.

Def. An <u>operator</u> $L : X \to Y$ is <u>bounded</u> if $\sup_{x \neq 0} \frac{\|Lx\|_Y}{\|x\|_X} < \infty$.

Let $\mathcal{L}(X,Y)$ be the set of all bounded operators from X to Y. ($\|L\|_{op}$ makes this a normed space!)

$$x = \sum c_j u_j \quad \text{(orthonormal basis)}$$

$$\max \langle Ax, Ax \rangle = \max \langle A^T A x, x \rangle = \max \left\langle \sum_{j=1} c_j \lambda_j u_j, \sum_{j=1} c_j u_j \right\rangle$$

$$=: \|L\|_{op}$$

- operator norm (i.e. spectral norm) ($\max_{\|x\|_2 \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \max \lambda$ of $A^T A$.)
  Euclidean norm

" on matrices is lower-bounded by spectral radius (largest eigenvalue of A)

Exercise: Show that $(\mathcal{L}(X,Y), \|\cdot\|_{op})$ is a normed space.

Proof. (i) $\|L\|_{op} = 0 \Rightarrow \sup_{x \neq 0} \frac{\|Lx\|_Y}{\|x\|_X} = 0 \Rightarrow Lx = 0 \ \forall x \in X$.

(ii) $\|\alpha L\|_{op} = \sup_{x \neq 0} \frac{\|\alpha L x\|_Y}{\|x\|_X} = |\alpha| \sup_{x \neq 0} \frac{(\cdot)}{(\cdot)} = |\alpha| \, \|L\|_{op}$

(iii) $\|L_1 + L_2\|_{op} = \sup_{x \neq 0} \frac{\|(L_1 + L_2)(x)\|_Y}{\|x\|_X} \leq \sup_{x \neq 0} \frac{\|L_1 x\|}{\|x\|} + \frac{\|L_2 x\|}{\|x\|}$

$\leq \sup_{x \neq 0} \frac{\|L_1 x\|}{\|x\|} + \sup_{x \neq 0} \frac{\|L_2 x\|}{\|x\|}$

$= \|L_1\|_{op} + \|L_2\|_{op}$

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Recap. $(X, \|\cdot\|_X)$ - Vector space, normed. (+ complete $\Rightarrow$ Banach)

$d(x_1, x_2) = \|x_1 - x_2\|$

$(Y, \|\cdot\|_Y)$ - same

$L: X \to Y$ operators are <u>bounded</u> if $\sup_{\|x\|_X \le 1} \|Lx\|_Y < \infty$.

ball around 0.

Let $\mathcal{L}(X, Y)$ be the set of all bounded operators from X to Y

The operator norm is $\|L\|_{\mathcal{L}} = \sup_{\|x\|_X \le 1} \|Lx\|_Y = \sup_{x \ne 0} \frac{\|Lx\|_Y}{\|x\|_X}$

(Exercise: If Y is complete then so is $\mathcal{L}(X, Y)$) — take pointwise limit

(Exercise 2: Linear mapping $L: X \to Y$ is continuous $\Leftrightarrow$ L is bounded) Rudin Real & Complex, Th 5.4

Properties of operator norm

- $\|A + B\| \le \|A\| + \|B\|$ (as with all norms)
- If $X = Y$ (are Banach spaces) then $\|AB\| \le \|A\| \cdot \|B\|$

$$\|AB(x)\| = \|A[B(x)]\| \le \|A\| \cdot B(x) \le \|A\| \cdot \|B\| \cdot (x)$$

Example: $X = \mathbb{R}^n$, $\mathcal{L}(X, X) = \{A \in \mathbb{R}^{n \times n}\}$

(operator norm) $\|A\| = \max_{|x|=1} |Ax|$. If A is symmetric, $\|A\|$ is the largest eigenvalue. $|\lambda_1|$

all eigenvalues are real,

$p=2$ $\|A\|_2^2 = \sum_{i=1}^n \sum_{j=1}^n a_{ij}^2$, $A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{bmatrix}$, $Ax = \begin{bmatrix} a_1 \cdot x \\ \vdots \\ a_n \cdot x \end{bmatrix}$

a_i rows

$$|Ax|^2 = \sum_{j=1}^n (a_i \cdot x)^2 \overset{n \text{ times of C.S.}}{\le} \sum_{i=1}^n \|a_i\|^2 |x|^2 = \left(\sum_{j=1}^n \sum_{j=1}^n a_{ij}^2\right) |x|^2 = \|A\|_2^2 \|x\|$$

$\sup_{x \ne 0} \frac{|Ax|}{|x|} \le \|A\|_2$, so $\|A\| \le \|A\|_2$

True. Proof?

Q: $\|A\|_2 \|B\|_2 \geq \|AB\|_2$? (~~Think of matrices that serve as counterexample.~~)

Given $L \in \mathcal{L}(X,Y)$, $\ker L := \{x : Lx = 0^{\in Y}\}$

$\text{Range } L := \{Lx,\ x \in X\}$

(Rank-nullity) If $X = \mathbb{R}^n$, $\dim(\ker A) + \underbrace{\dim(\text{Range } A)}_{\text{rank } A} = n$

If A is an invertible linear operator on $\mathbb{R}^n$, $B \in L(\mathbb{R}^n)$, and $\|B-A\| \cdot \|A^{-1}\| < 1$, then B is invertible.

$IX = X \ \forall x$

Theorem 9.8 $^{*(a)}$ Let X be a Banach space, $A \in \mathcal{L}(X,X)$. If $\|A\| < 1$, then $I - A$ is invertible and $(I-A)^{-1} \in \mathcal{L}(X,X)$.

linear bounded ~~or~~ continuous

special case of Rudin's version, $A = I$, $B = A$

(ie. There exists $L \in \mathcal{L}(X,X)$ such that $L(I-A) = I = (I-A)L$)

If $A \in \Omega$ and $B \in \mathcal{L}(X,X)$ is such that $\|A - B\| \cdot \|A^{-1}\| < 1$, then $B \in \Omega$. Thus Ω is an open set

$B = BA^{-1}A$

$= \underbrace{I}_{} - \underbrace{(I - BA^{-1})}_{}$

$= (A-B)A^{-1}$

$\| \cdot \| \leq$

Note. $(1-a)^{-1} = \frac{1}{1-a} = 1 + a + a^2 + \ldots$

We hope to have $L = I + A + A^2 + \ldots = \sum_{n=0}^{\infty} A^n$?

Let $S_n = I + A + A^2 + \ldots + A^n$, ~~By the previous lemma/exercise~~ Since (X,d) is complete, $\mathcal{L}(X,X)$ is complete (exercise) so we want to show $\{S_n\}$ is Cauchy.

If $(m > n)$, $S_m - S_n = A^{n+1} + \ldots + A^m = A^{n+1}(I + A + \ldots + A^{m-n-1})$

$\|A^n\| \leq (\|A\|)^n = \alpha^n$ $\quad \|I + A + \ldots + A^k\| \leq 1 + \alpha + \alpha^2 + \ldots + \alpha^k < \frac{1}{1-\alpha}$

exercise earlier

$\|S_m - S_n\| \leq \alpha^{n+1} \frac{1}{1-\alpha}$. Thus $\{S_n\}$ is Cauchy.

[$X \to \|X\|$ is continuous]

So $\{S_n\}$ converges, and by continuity, $\|L\| \leq \frac{1}{1-\alpha}$

$\leq \|A + (B-A)\| \leq \|A\| + \|(B-A)\|$

$\|A\| - \|A-B\|$

We have to show L actually satisfies the equations.

$$S_n(I-A) = (I+A+A^2+\dots+A^n)(I-A)$$
$$= I-A+A-A^2+A^2-A^3\dots-A^{n+1}$$
$$= I-A^{n+1}$$

Taking limits, $\boxed{L(I-A)=I}$ ($I=(I-A)L$ is proved similarly.)

since $\|L(I-A)-S_n(I-A)\|$
$\leq \|L-S_n\|\cdot\|I-A\|\to 0$.

(b) Furthermore, let $\Omega \subseteq \mathbb{R}^{n\times n}$ be the set of invertible matrices. $\Phi: \Omega\to\Omega$, $\Phi(A)=A^{-1}$ is continuous.

Note: ① Ω is a dense open subset

$$\Omega=\{A: \det A\neq 0\} = \det^{-1}(\mathbb{R}\setminus\{0\})$$

$\det$: continuous; $\mathbb{R}\setminus\{0\}$: open

② Φ is not linear in general! $(a+b)^{-1}\neq \frac{1}{a}+1/b$

Pf. $A^{-1}-B^{-1} = A^{-1}(B-A)B^{-1} \Rightarrow \|A^{-1}-B^{-1}\| = \|A^{-1}(B-A)B^{-1}\|$

$\leq \|A^{-1}\|\cdot\|B-A\|\cdot\left(\frac{1}{\|A^{-1}\|}-\|B-A\|\right)^{-1}$

$\leq \frac{\beta}{\alpha(\alpha-\beta)}$, where $\alpha=\frac{1}{\|A^{-1}\|}$, $\beta=\|A-B\|$

Let $\beta=\|A-B\|<\frac{1}{\|A^{-1}\|}<\alpha$, then $\forall x\in X$,

$\alpha\|x\| = \alpha\|A^{-1}Ax\| \leq \alpha\|A^{-1}\|\cdot\|Ax\|$
$\leq \|Ax\|$
$\leq \|(A-B)x\|+\|Bx\|$
$\leq \beta\|x\|+\|Bx\| \Rightarrow (\alpha-\beta)\|x\|\leq\|Bx\|$ So $\beta\to 0$ $(B\to A) \Rightarrow B^{-1}\to A^{-1}$

$x=B^{-1}y \Rightarrow (\alpha-\beta)\|B^{-1}y\|\leq \cancel{(\alpha-\beta)\|B^{-1}A\|\|y\|}\ \|y\| \Rightarrow \frac{\|B^{-1}y\|}{\|y\|}\leq\frac{1}{\alpha-\beta}$

$|F(x)-F(y)|\geq c|x-y| \Rightarrow$ Invertible (injective)

$I-A$: Perturb the identity small enough

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(Recall definition of derivatives) Let $f: \mathbb{R} \to \mathbb{R}$. Then $f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

f is differentiable at x if the limits exists (and is finite)

Lemma. f is diff at x iff $\exists L \in \mathbb{R}$ such that $\lim_{h \to 0} \frac{|f(x+h) - f(x) - Lh|}{|h|} = 0$

$\left| \frac{f(x+h) - f(x)}{h} - L \right|$ approximate locally by linear function

Consider now $F: \mathbb{R}^d \to \mathbb{R}$ (or any open subset)

- Directional derivative. Let $v \in \mathbb{R}^d$, $D_v F = \frac{d}{dt}\Big|_{t=0} \underbrace{F(x+vt)}_{\text{function!}} = \lim_{t \to 0} \frac{F(x+tv) - F(x)}{t}$

$\frac{\partial F}{\partial x_i}(x) = D_{e_i} F(x)$

↑ basis vector

- Differential (Derivative of F)

Def $F: \mathbb{R}^d \to \mathbb{R}$ is differentiable at x if there is $L \in \mathbb{R}^d$ (row vector (co-vector)) such that

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{R}^d}} \frac{|F(x+h) - F(x) - Lh|}{|h|} = 0$$

row column

We write $\underbrace{DF(x) = L}_{\text{"derivative"}}$ (this is not the gradient!

$\nabla F(x) = L^T$)

"gradient vector"

$\nabla F(x) \cdot h$ ← this requires a dot product (Hadamard)

- tangent space

p76. - cotangent space (differential geometry)

- Def $F: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x \in \mathbb{R}^n$ if there exists a linear mapping $L \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{R}^n}} \frac{\|F(x+h) - F(x) - Lh\|}{|h|} = 0$$

$\therefore DF(x) = L$

* A derivative is by definition a linear mapping!

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We can generalize the definitions to Banach spaces.

(Fréchet)

Def. Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be Banach spaces. $F: X \to Y$ is differentiable at $x \in X$ if $\exists L \in \mathcal{L}(X,Y)$ (set of bounded linear operators) such that

finite dimensional $\Rightarrow$ everything is bounded

$$\lim_{\substack{\|h\|_X \to 0 \\ h \in X}} \frac{\|F(x+h) - F(x) - Lh\|_Y}{\|h\|_X} = 0.$$

The Fréchet derivative is denoted DF.

Directional derivatives, in turn, are called Gateaux derivatives (like 1^{st} variation)

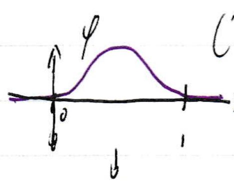
Let $v \in X$. $D_v F|_x = \left(\frac{\delta F}{\delta x}[v]\right) = \lim_{t \to 0} \underbrace{\frac{F(x+vt) - F(x)}{t}}_{\in Y}$ $= Lv$

If $v \neq 0$ and we assume Fréchet differentiability, then

$$\lim_{\substack{t \to 0 \\ h = tV}} \frac{\|F(x+tV) - F(x) - tLV\|}{|t|} = 0$$

In $\mathbb{R}^2$ we can have functions that have all directional derivatives but no derivative.

Recall bump functions (p 27.)

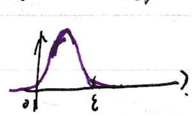


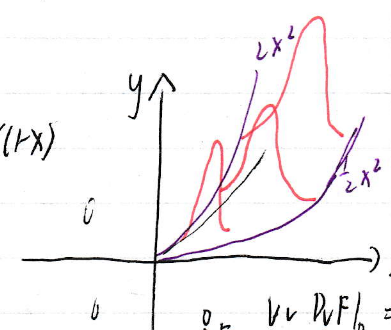
C^∞ (smooth)

$\varphi(x) = \psi(x)\psi(1-x)$

$\psi(x) = e^{-\frac{1}{x}}$

$\varphi_\varepsilon(x) = \varphi\left(\frac{x}{\varepsilon}\right)$





$\forall v \; D_v F|_0 = 0$

$\frac{\partial F}{\partial x_{\text{axis}}}(0,0) = 0.$

$\lim_{t \to 0} \frac{\partial F}{\partial x}(t, 3t^2/2) = \infty.$

$*$ If all partial derivatives exist and are continuous functions, then the function is differentiable. (in 2D)
(iff: Rudin 9.21)

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Chain Rule

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be Banach spaces.

If $F: X \to Y$, $G: Y \to Z$, $y_0 = F(x_0)$, F is diff at x_0, G is diff at y_0,

then $G \circ F$ is differentiable at x_0, and furthermore

$$D(G \circ F)|_{x_0} = DG|_{y_0} \circ DF|_{x_0}$$

Proof. We have to show

$$\lim_{h \to 0} \frac{\| G(F(x_0+h)) - G(F(x_0)) - DG|_{y_0}(DF|_{x_0} h) \|_Z}{\|h\|_X} = 0.$$

Know:

$$\lim_{\|u\|_Y \to 0} \frac{\| G(y_0+u) - G(y_0) - DG|_{y_0} u \|}{\|u\|} = 0$$

So set $\| G(y_0+u) - G(y_0) - DG|_{y_0} u \| = \varepsilon(u)$

$\| F(x_0+v) - F(x_0) - DF|_{x_0} v$

Idea from Real I:

$\frac{f(t)-f(x)}{t-x} - f'(x) = \delta(t)$ (function of t)

$\frac{g(s)-g(y)}{s-y} - g'(y) = \varepsilon(s) = \varepsilon(f(t))$

(where $y = f(x)$)

$g(f(t)) - g(f(x)) = [f(t)-f(x)][g'(y) + \varepsilon(f(t))]$

$= [(t-x)f'(x) + \delta(t)(t-x)][g'(y) + \varepsilon(f(t))]$

divide by $t-x$

Remarks: ① $DF: x \mapsto DF|_x$ is a function from a point to a linear operator. ("$f'(x)$")

② A function is continuously differentiable if DF is a continuous mapping from (X, d) to $(\mathcal{L}(X,X), \|\cdot\|)$

operator norm

INVERSE FUNCTION THEOREM

Let $(X, \|\cdot\|_X)$, $F: X \to X$.

"vector-valued fn" $f:\mathbb{R}^n \to \mathbb{R}$; $Df(\vec{x})$ is a row vector.

F is differentiable at $x \in X$ if $\exists L \in \mathcal{L}(X,X)$

$$\lim_{\substack{\|h\|_X \to 0 \\ h \in X}} \frac{F(x+h)-F(x)-Lh}{\|h\|} = 0.$$

We write $\boxed{DF(x) = L}$

Recall: $A \in \mathcal{L}(X,X)$, $\|A\|_{\mathcal{L}} < 1$ then $I-A$ "perturbation" is bijection and $(I-A)^{-1} \in \mathcal{L}(X,X)$

Lemma (40) (Perturbation of Identity) "similar" perturb by contraction

Let $B = B(0,r)$ (ball $\subseteq X$). Assume $\varphi: B \to X$ is a contraction, and let $F(x) = x - \varphi(x)$. So $F: B \to V$ where $V = F(B)$. Then F is an injection and $\forall U \subseteq B$ open, $F(U)$ is open.

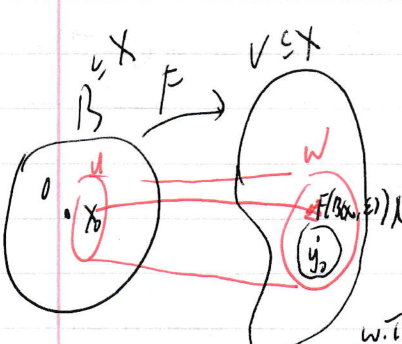
(F is an open mapping $\Rightarrow$ Thus $F^{-1}: V \to B$ exists and is continuous)

Compact set: inverse of continuous function is continuous

Pf: $\|F(x)-F(y)\| = \|x-y-(\varphi(x)-\varphi(y))\| \overset{\Delta\text{-ineq}}{\geq} \|x-y\| - \|\varphi(x)-\varphi(y)\|$

$\geq (1-\lambda)\|x-y\|$ $[\because \|\varphi(x)-\varphi(y)\| \leq \lambda\|x-y\|]$

So $x \neq y \Rightarrow F(x) \neq F(y)$ [injectivity]



Now let U be open, $W = F(U)$, and $y_0 \in W$.

w.t.s $\exists \delta > 0$, $B(y_0, \delta) \subseteq W$ i.e. $\forall y \in B(y_0,\delta)$, $\exists x \in U, F(x) = y$.

$\overset{\text{w.t.s}}{\iff}$ $x - \varphi(x) = y$ i.e. $\underbrace{y + \varphi(x)} = x$.

*Every E_y is a contraction: $\|E_y(x_1) - E_y(x_2)\| = \|\varphi(x_1)+y-(\varphi(x_2)+y)\| = \|\varphi(x_1)-\varphi(x_2)\|$

$E_y(x): B \to V$

$x_0 \in U \iff \exists \varepsilon > 0$ $B(x_0, \varepsilon) \subseteq U$. Define $\boxed{\delta := (1-\lambda)\varepsilon}$

but this is not sufficient!

closed subset of complete m.s is complete
(every complete proper subset must be closed)
else can find C.S that converges to a point outside the subset

We need to show this value of δ works:

Let $y \in B(y_0, \delta)$. WTS $E_y: \overline{B(x_0,\varepsilon)} \to \overline{B(x_0,\varepsilon)}$ [need the domain and codomain to be the same to apply Banach contraction theorem and both complete.]

i.e. $x \in \overline{B(x_0,\varepsilon)} \Rightarrow \|E_y(x) - x_0\| \leq \varepsilon$ $\qquad x_0 - \varphi(x_0) = y_0$

$$\|E_y(x) - x_0\| = \|\varphi(x) + y - x_0\| = \|\varphi(x) + y - \varphi(x_0) - y_0\|$$

$$\leq \|\varphi(x) - \varphi(x_0)\| + \|y - y_0\|$$

$$< \lambda\|x - x_0\| + (1-\lambda)\varepsilon$$

$$\leq \lambda\varepsilon + (1-\lambda)\varepsilon = \varepsilon$$

So E_y has a fixed point: $\exists x_{\in \overline{B(x_0,\varepsilon)}},\ y + \varphi(x) = x \Leftrightarrow y = x - \varphi(x)$ so F is an open mapping.

Statement of Theorem. (Inverse Function Theorem)

Consider Banach spaces $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ [If X is finite-dimensional then $X \cong Y$]

Assume U is open subset of X, $F \in C^1(U,Y) \Leftrightarrow \forall x \in U$, $DF(x)$ exists and is continuous.

Assume that for some $x_0 \in U$, $DF(x_0)_{\in \mathcal{L}(X,Y)}$ is invertible, meaning $(DF(x_0))^{-1} \in \mathcal{L}(Y,X)$.

$DF: x \mapsto DF(x)_{\in \mathcal{L}(X,Y)}$ "matrix" def
$U \to \mathcal{L}(X,Y)$ is continuous.
i.e. $\|DF(x) - DF(y)\|_{\mathcal{L}(X,Y)} \to 0$ as $y \to x$

- Then $\exists r > 0$ such that $F|_{B(x_0,r)}: B(x_0,r) \to \underbrace{F(B(x_0,r))}_{=:V}$ is bijective, and $G := (F|_{B(x_0)})^{-1}$ is continuous on V.

just need to show 'injective'

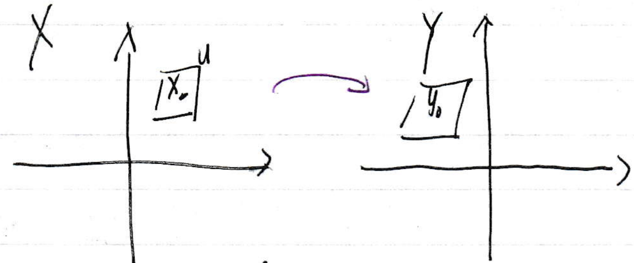
- Furthermore, G is ~~fixed~~ 'differentiable', $G \in C^1(V,B)$, $DG(y) = (DF(G(y)))^{-1}$

Remark: ① If f is a bijection, then f is an open mapping iff f^{-1} is continuous, since
$f(E) = (f^{-1})^{-1}(E)$

② Deformations do not produce 'holes'

so $\underset{\text{any set} \in X}{E}$ open $\Rightarrow f(E)^{\in Y}$ open
$f^{-1}(E')$ open $\Rightarrow (f^{-1}(E'))^{-1}$

Proof.



$y_0 = f(x_0)$. The idea is to show that

$f(x) = y_0 + A(x-x_0) + R(x)$, where

'approximately linear' $\frac{R(x)}{\|x-x_0\|} \to 0$ as $x \to x_0$.

use p.54 lemma p.59

Define $A := Df(x_0)$, where f is F on the previous page.

$F(z) := A^{-1} f(x_0 + z)$, $F \in C^1$ (chain rule)

$DF(0) = A^{-1} Df(x_0) = A^{-1}A = I$. We ~~restrict~~ translate the domain of F to

$F : \underbrace{U - x_0}_{\text{subtract } x_0 \text{ from all elements of } U \text{ (translation)}} \to X$ so that x_0 is the new origin.

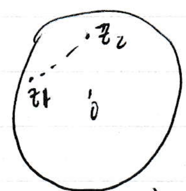
Defining $\underbrace{\varphi(z)}_{\in C^1} = z - F(z) \Leftrightarrow F(z) = z - \varphi(z) \Rightarrow D\varphi(z) = I - DF(z)$

$D\varphi(0) = I - I = 0$.

Inverse mapping of open set...

By continuity of $D\varphi$, (important!) there exists $r>0$ such that on $B(0,r)$, $\|D\varphi(z)\| \le 1/2$

(Exercise) Thus, $\forall z_1, z_2 \in B(0,r)$, $\|\varphi(z_2) - \varphi(z_1)\| \le \frac{1}{2}\|z_2 - z_1\|$



$\theta(s) = \varphi((1-s)z_1 + s z_2)$

- Proof sketch: $\|\underbrace{\theta(1)}_{\varphi(z_2)} - \underbrace{\theta(0)}_{\varphi(z_1)}\| \overset{?}{\le} \int_0^1 \|\theta'(s)\|\, ds$ (Δ-ineq)

$\le \int_0^1 1/2 \, ds = \frac{1}{2}$

See Rudin Th 9.19, 5.19 for more rigorous argument

This means that $\varphi : X \to X$ is a contraction.

$\therefore$ By lemma 40, we have that $F : B_{(0,r)} \to F(B_{(0,r)})$ is a bijection and F^{-1} is continuous. Thus AF is a bijection and so is f (since $z \mapsto x_0 + z$ is bijection). $\boxed{f : B(x_0, r) \to \underbrace{f(B(x_0,r))}_{=:V} \text{ is a bijection.}}$

It remains to show the second part (differentiability). As a further consequence of the lemma, f^{-1} is continuous.

$F(z) = A^{-1} f(x_0 + z)$

Define $G = f^{-1}$. $DF(z) = I - D\varphi(z)$, and recall $\|D\varphi(z)\| \le 1/2$ on $B(0,r)$.

So by lemma 39 (PS 4), $DF(z)$ $z \in B(0,r)$ is invertible and $(DF(z))^{-1} \in \mathcal{L}(X,X)$ → but we have to transform DF^{-1} back to f^{-1}.

(Furthermore, by corollary of L39 ($\sim$ 9.8 in Rudin), $z \mapsto DF^{-1}(z)$ is continuous.)

Goal: $DG(z) = (DF(G(z)))^{-1}$ (Exercise 2) $\forall z \in B$ $\exists\, C > 0$, $\|\underbrace{DF(z)\underset{\in X}{h}}_{\in Y}\| \ge C\|h\|_X$ (1) (follows from the existence of inverses)

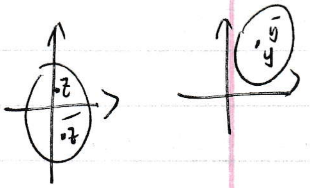
similarly, continuous (2) $\Leftrightarrow$ $\|A \cdot h\|_X < C\|h\|_X$

For (2) → (1) use $h = A^{-1}u$

Let $\bar{z} \in B(0,r)$, $\bar{y} = F(\bar{z})$. $\rho(h) := F(\bar{z} + h) - F(\bar{z}) - DF(\bar{z})h$ satisfies $\lim_{\|h\|_X \to 0} \frac{\|\rho(h)\|_X}{\|h\|_X} = 0$.

If $y = F(z)$, $z = G(y)$ is any other point,

$$\frac{\|G(y) - G(\bar{y}) - DF(\bar{z})^{-1}(y - \bar{y})\|}{\|y - \bar{y}\|} = \frac{\|G(F(z)) - G(F(\bar{z})) - DF(\bar{z})^{-1}(y - \bar{y})\|}{\|y - \bar{y}\|}$$



$\|y - \bar{y}\| \ge C\|z - \bar{z}\|$

Self: $\le \dfrac{\|DF^{-1}(\bar{z})\|\ \|(y - \bar{y}) - DF(\bar{z})(G(F(z)) - G(F(\bar{z})))\|}{\|y - \bar{y}\|}$ Class

fixed ⇒ bounded!

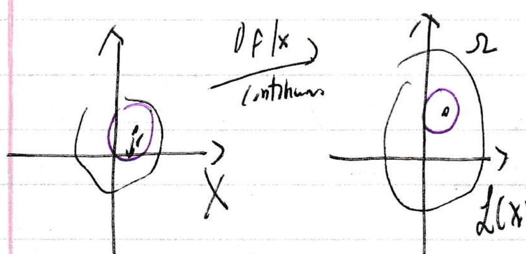
$\le \underline{\|DF^{-1}(\bar{z})\|} \cdot \dfrac{\|(y - \bar{y}) - DF(\bar{z})(z - \bar{z})\|}{\|z - \bar{z}\|} \to 0$

C

as $y \to \bar{y}$

Remark: Once we show $DG(y) = (Df|_{G(y)})^{-1}$ then $y \mapsto DG(y)$ is continuous provided r is small enough.

Note $Df|_{x_0}$ is invertible, let Ω be the set of invertible linear mappings, Ω is open (p55)



We can choose r small enough so that $Df(x)$ is invertible for all $x \in B(x_0, r)$. (in statement on p 60) Recall that the mapping $A \to A^{-1}$ on Ω is continuous. Therefore $y \mapsto (\underbrace{Df|_{G(y)}}_{\text{continuous}})^{-1}$ is continuous.

· invertibility of DF

· composition of continuous functions

$\therefore$ $\boxed{y \mapsto DG(y)$ is continuous. $V \to \mathcal{L}(X,Y)}$

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$\|x-y\| - \|DF_{\bar z}(x-y)\| \le \|x-y-DF_{\bar z}(x-y)\| \le \frac{1}{2}\|x-y\|$

$DF|_0 = I$. Thus,

- from before

If r is small, $\forall z \in B(0,r)$, $\|I - DF_z\| < \frac{1}{2}$. Therefore, ... $\|DF|_z h\| \ge \frac{1}{2}\|h\|$.

Claim: if $z = \bar z + h$ is close to $\bar z$, then $\|F(z) - F(\bar z)\| \ge \frac{1}{2}\|DF|_{\bar z}(z-\bar z)\|$

(since $\rho(h)$ satisfies $\lim_{h\to 0} \frac{\|\rho(h)\|_X}{\|h\|_X} = 0$.)

Pf.

$$\lim_{z\to\bar z} \frac{\|F(z) - F(\bar z) - DF|_{\bar z}(z-\bar z)\|}{\|z-\bar z\|} = 0,$$ thus if $\|z-\bar z\|$ is small enough,

$$\|F(z) - F(\bar z) - DF|_{\bar z}(z-\bar z)\| < \tfrac{1}{4}\|z-\bar z\| < \tfrac{1}{2}\|DF|_{\bar z}(z-\bar z)\|$$

By $\triangle$-inequality, $\|x+y\| \le \|x\| + \|y\| \Rightarrow \|x\| \ge \|x+y\| - \|y\|$

$\|x-y\| \ge \underbrace{\|x\| - \|y\|}_{\|x-y\|+\|y\| \ge \|x\|},\ \ge \|y\| - \|x\|$

$$\|F(z) - F(\bar z) - DF|_{\bar z}(z-\bar z)\| \ge -\|F(z) - F(\bar z)\| + \|DF|_{\bar z}(z-\bar z)\|$$

$$\Downarrow$$

$$\|F(z) - F(\bar z)\| \ge \tfrac{1}{2}\|DF|_{\bar z}(z-\bar z)\|$$

(recall $F(z) - F(\bar z) = DF|_{\bar z}(z-\bar z) + \rho(z-\bar z)$ by definition)

$$= \frac{\|G(F(z)) - G(F(\bar z)) - DF^{-1}|_{\bar z}(F(z) - F(\bar z))\|}{\|y-\bar y\|}$$

$$= \frac{\|\cancel{z - \bar z - (z-\bar z)} - DF^{-1}|_{\bar z}\,\rho(z-\bar z)\|}{\|z-\bar z\|} \cdot \frac{\|z-\bar z\|}{\|y-\bar y\|}$$

$$\le \frac{\|DF(\bar z)^{-1}\| \cdot \|\rho(z-\bar z)\|}{\|z-\bar z\|} \cdot \frac{\|z-\bar z\|}{\|F(z) - F(\bar z)\|}$$

bounded

$$\le \underset{\to 0}{\ldots} \cdot \frac{\|z-\bar z\|}{C\|DF_{\bar z}(z-\bar z)\|} \le \ldots \int \cdot \frac{2\|z-\bar z\|}{C\|z-\bar z\|}$$

$\to 0$ as $\vec y \to \bar y$.

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As an application of the Inverse Function Theorem,

Theorem (Implicit Function Theorem)

Let $f \in C^1(\Omega, \mathbb{R}^n)$, $\Omega \subseteq \mathbb{R}^{m+n}$, Ω is open.

Assume $f(a,b)=0$, $a\in\mathbb{R}^n$, $b\in\mathbb{R}^m$ (e.g. $f(x,y,z) = xz + x^2 + y^2 - z^2 + 1$. $m=2$, $n=1$

↳ $f([a_1, a_2, \dots, a_n, b_1, \dots, b_m]^T)$ $\quad$ $f(1,2,3)=0$

$A := Df(a,b) \in M_{n \times m+n}$ (set of matrices) $\quad$ $\begin{array}{cc} & m+n \\ n & \left[\underbrace{[n\times n]}_{A_x} \ \underbrace{[n\times m]}_{A_y} \right] \end{array}$

$= [A_x, A_y]$

Assume A_x is invertible. e.g. $A_x = \frac{\partial f}{\partial x} = z + 2x$, $\frac{\partial f}{\partial x}(1,2,3) = 3+2 = 5 \neq 0$

Then there exists open set $U \subseteq \mathbb{R}^{m+n}$, open set $W \subseteq \mathbb{R}^m$, $(a,b) \in U$ such that $\forall y \in W$, there exists a unique $(x,y) \in U$ such that $f(x,y)=0$.

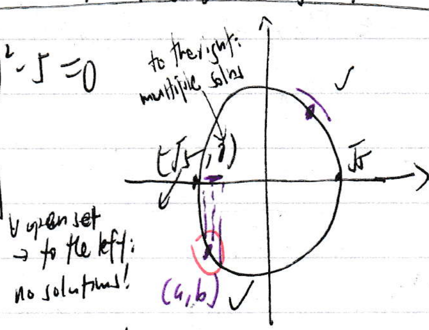
In other words, we can define $g: W \to \mathbb{R}^n$ such that $f(g(y),y)=0$ $\forall y \in W$. $[x = g(y)]$

Furthermore, $g \in C^1(W, \mathbb{R}^n)$ and $Dg(b) = -(A_x^{-1})A_y = -(D_x f)^{-1}(D_y f)$

In our simple example, there exists W a neighborhood of $(2,3)$ s.t. $\forall (y,z) \in W$, $\exists\, g(y,z) =: x$, $f(g(y,z), y, z) = 0$.

E.g. $f = x^2 + y^2 - 5 = 0$

'A_x' $\frac{\partial f}{\partial y} = 2y$



∴ for all x, exists y

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Remark: let $F(x,y) = (f(x,y), y)$, so $F: \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$
(Proof sketch)

$$DF(a,b) = \begin{array}{c} \\ n\{ \\ m\{ \end{array} \begin{array}{cc} \overbrace{}^{n} & \overbrace{}^{m} \\ \left[\begin{matrix} A_x & A_y \\ 0 & I \end{matrix}\right] \end{array}$$

This is invertible $\Rightarrow$ Invoke the Inverse Function Theorem
($\det \uparrow = \det(A_x) \cdot \det(I)$)

$\boxed{\text{Higher-order Derivatives}}$

Let $F: \mathbb{R}^n \to \mathbb{R}$. The derivative of F, $DF|_x$, is a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}$ such that
$$\lim_{\substack{h \to 0 \\ h \in \mathbb{R}^+}} \frac{|F(x+h) - F(x) - DF|_x h|}{|h|} = 0,$$ as we have seen.

Multilinear Functions: Let V be vector space, $b: V \times V \to \mathbb{R}$ is bi-linear,
if $b(\alpha_1 v_1 + \alpha_2 v_2, w) = \alpha_1 b(v_1, w) + \alpha_2 b(v_2, w)$
& $b(v, \beta_1 w_1 + \beta_2 w_2) = \beta_1 b(v, w_1) + \beta_2 b(v, w_2)$

e.g. $V = \mathbb{R}$, $b(x,y) = x+y$ is not bilinear since $b(1,2) \neq b(1,1) + b(1,1)$
$b(x,y) = cxy$ ✓

$b\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}\right) = x_i y_j$ ✓, and $b\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_i y_j$

Tensor: $\mathbb{R}^n \to \underset{\text{multilinear}}{M}(\mathbb{R}^n)$
dim $= m^n$

$\Rightarrow$ Bilinear mappings on $\mathbb{R}^n$ form a vector space.

Let $B(\mathbb{R}^n)$ be the set of bilinear functions. $\dim(B(\mathbb{R}^n)) = n^2$,
basically we choose c_{ij} (setting $x_i = y_j = 1$, others to 0 shows linear independent). But we can set $b(e_i, e_j) =: c_{ij}$, so these n^2 c_{ij}'s span $B(\mathbb{R}^n)$.

$b: V^m \to \mathbb{R}$ is 'multilinear' if b is linear w.r.t each $\underbrace{(A)}$ entry.

$b: (\mathbb{R}^n) \to \mathbb{R}^n$, $b(x^1, x^2, \dots x^n) = b\left(\begin{bmatrix} x_1^1 \\ x_2^1 \\ \vdots \\ x_n \end{bmatrix} \dots \begin{bmatrix} x^n \\ \vdots \\ x_n^n \end{bmatrix}\right) = b(A)$

E.g. $b(A) = \det(A)$. Why? Note that

(on $V = \mathbb{R}$)

$b(X_1, \dots X_m) := X_1 \cdots X_m$ is a multilinear mapping, and $\dim(M(V^m; \mathbb{R})) = (\dim(V))^m$

The determinant is the sum of products of entries from each row.

$\hookrightarrow \sum (-1)^p x_{p1}^1 \cdots x_{pn}^1$

Last time we ~~thought~~ talked about multilinear functions. Now we use them to define higher order derivatives.

Let $f: \mathbb{R}^n \to \mathbb{R}$.

① $Df|_x$ is a linear function, $Df|_x : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{\substack{h\to 0 \\ h\in\mathbb{R}^n}} \frac{f(x+h)-f(x)-Df|_x h}{|h|} = 0, \quad Df|_x[h] = \left[\frac{\partial f}{\partial x_1}(x), \dots \frac{\partial f}{\partial x_n}(x)\right]\begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$$

② Second derivative $D^2f|_x$ is a bilinear function, $D^2 f|_x : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)-Df|_x(h)-\frac{1}{2}D^2f|_x(h,h)}{|h|^2} = 0$$

Lemma If $\frac{\partial f}{\partial x_i}$ and $\frac{\partial^2 f}{\partial x_i \partial x_j}$ exist and are continuous at x for all $i=1,2,\dots n$, $j=1\dots n$, then $D^2 f|_x(h,h) = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}\Big|_x h_i h_j$ is a second derivative of f.

Pf. Define $\underset{\in\mathbb{R}\text{ 1D case!}}{\varphi(s)} := f(x+s\underset{\in\mathbb{R}^n}{V}) - f(x) - Df(x)sV$

$$\varphi'(s) = \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}(x+sV) - \frac{\partial f}{\partial x_i}(x)\right) V_i$$

$$\varphi''(s) = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_j \partial x_i}(x+sV)\cdot V_i V_j$$

$$\varphi(s) = \varphi(0) + \varphi'(0)+s + \underbrace{\tfrac{1}{2}\varphi''(\sigma s)}_{\text{Lagrange Remainder}} s^2, \text{ for some } \sigma\in(0,1)$$

We observe $\varphi(0)=\varphi'(0)=0$.

So $f(x+sV) = f(x) + Df(x)sV + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(x) V_i V_j s^2$

$$-\frac{1}{2}\sum_{i=1}^n\left(\sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(x) - \frac{\partial^2 f}{\partial x_i \partial x_j}(x+sV\sigma)\right)V_i V_j s^2$$

our hypothesis

Take $h = sv$. Then $\left|f(x+h) - f(x) - Df(x)h - \frac{1}{2}D^2 f(h,h)\right|$

$$= \frac{n^2}{2} \max_{1 \le i,j \le n} \left| \frac{\partial^2 f}{\partial x_i \partial x_j}(x) - \frac{\partial^2 f}{\partial x_i \partial x_j}(x+h\sigma) \right| |h|^2$$

$|x - (x+h\sigma)| \le |h|$

We continuity to bring $\frac{\partial^2 f}{\partial x_i \partial x_j}(\cdot)$ arbitrarily close to each other as $h \to 0$

Divide by $|h|^2$ and let $h \to 0$. This shows $\lim_{h \to 0} \frac{f(x+h) - f(x) - Df(x)(h) - \frac{1}{2}\sum\sum \frac{\partial^2 f}{\partial x_i \partial x_j} h_i h_j}{h^2} = 0$

Hessians

$$\underset{\text{'matrix'}}{\text{Hess } f} = \left[\frac{\partial^2 f}{\partial x_i \partial x_j} \right]_{\substack{i=1\ldots n \\ j=1\ldots n}} = D(\nabla f)$$

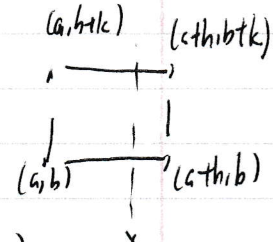
$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

matrix   function with 2 arguments

$$\boxed{\text{Hess } f \neq D^2 f.}$$

$\Rightarrow D^2 f(h,h) = h^T \text{Hess } f\, h$

Mixed Partial Derivatives ("Clairaut")



Assume $f: \mathbb{R}^2 \to \mathbb{R}$ and $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial y \partial x}$ exist.

Define $\Delta = f(a+h, b+k) - f(a+h, b) - (f(a,b+k) - f(a,b))$

$= u(a+h) - u(a)$, where $u(x) := f(x, b+k) - f(x,b)$

By MVT, $\exists\, x \in (a, a+h)$ such that

$$\Delta = u'(x) \cdot h$$
$$= h\left(\frac{\partial f}{\partial x}(x, b+k) - \frac{\partial f}{\partial x}(x, b)\right)$$
$$= hk \frac{\partial^2 f}{\partial y \partial x}(x, y)$$, so by MVT $\exists\, y \in (b, b+k)$ satisfying these conditions.

$$\Rightarrow \frac{1}{hk}\left(f(a+h, b+k) - f(a+h, b) - (f(a, b+k) - f(a,b))\right) = \frac{\partial^2 f}{\partial y \partial x}(x,y), \text{ for some } (x,y) \in R(h,k)$$

Similarly, $\Delta = f(a+h, b+k) - f(a, b+k) - (f(a+h, b) - f(a,b))$
$= v(b+k) - v(b)$
$= v'(\bar{y})k$

$$\exists \frac{1}{hk}\left(f(a+h, b+k) - f(a+h, b) - (f(a, b+k) - f(a,b))\right) = \frac{\partial^2 f}{\partial x \partial y}(\bar{x}, \bar{y}), \text{ for } (\bar{x}, \bar{y}) \in R(h,k)$$

Hence if $\frac{\partial^2 f}{\partial x \partial y}$ is continuous at (a,b) and $\frac{\partial^2 f}{\partial y \partial x}$ is also, then $\frac{\partial^2 f}{\partial x \partial y}\Big|_{(a,b)} = \frac{\partial^2 f}{\partial y \partial x}\Big|_{(a,b)}$ since we can let $h, k \to 0$.

Differentiation of Integrals

all second partial derivatives exist and are continuous

(See 21-469 p54-55 two just need C^1) **Theorem.** Let $f \in C^2([a,b] \times [c,d])$, $g(t) = \int_a^b f(x,t)\,dx$. Then $g'(t) = \int_c^b \frac{\partial f}{\partial t}(x,t)\,dx$ on $[c,d]$

(t is not in the bound! FTOC does not apply)

↳ (Lebesgue Theory: Dominated Convergence Theorem)

Proof. $$g'(t) = \lim_{h \to 0} \frac{g(t+h) - g(t)}{h} = \lim_{h \to 0} \int_c^b \frac{f(x,t+h) - f(x,t)}{h}\,dx$$

To show the claim it suffices to show $$\lim_{h \to 0} \int_c^b \left|\underbrace{\frac{f(x,t+h) - f(x,t)}{h} - \frac{\partial f}{\partial t}(x,t)}_{\text{Taylor series}}\right| dx = 0$$

$$= \left|\frac{1}{2}\frac{\partial^2 f}{\partial t^2}(x, t+\theta_x h)h\right| \le \max_{(z,s) \in [a,b]\times[c,d]} |h| \left|\frac{\partial^2 f}{\partial t^2}(z,s)\right|$$

$$0 \le \int_a^b \underbrace{\left|\frac{f(x,t+h) - f(x,t)}{h} - \frac{\partial f}{\partial t}(x,t)\right|}_{\le M|h|} dx \le M|h| \cdot (b-a)$$

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Integration

Recall (21-355): A function $f:[a,b]\to\mathbb{R}$ is Riemann-integrable if and only if

(i) f is bounded ($\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\,\Big|_0^1 = 2$ is technically an 'improper integral', defined by taking the limit $t\downarrow 0$ of $\int_t^1 \frac{1}{\sqrt{x}}\,dx$)

(ii) set of discontinuities of f has measure 0

Riemann integration is unsatisfactory because it does not behave well under convergence:

For $f\in C([0,1],\mathbb{R})$, the norm $\|f\|_1 = \int_0^1 |f(x)|\,dx$ induces a metric d_1. In a previous homework we showed that $(C([0,1],\mathbb{R}),d_1)$ is not complete. If $\mathcal{R}([0,1],\mathbb{R})$ denotes the space of Riemann-integrable functions, $(\mathcal{R}([0,1],\mathbb{R}),d_1)$ is still not complete! (Dirichlet function...)

A cell in $\mathbb{R}^n$ is some $I^n = [a_1,b_1]\times[a_2,b_2]\times\dots[a_n,b_n]$, $a_i<b_i$.

We define the iterated integral

$$L(f) = \int_{I^n} f(x)\,dx := \underbrace{\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\Big(\underbrace{\int_{a_n}^{b_n} f(x_1,\dots x_{n-1},x_n)\,dx_n}_{=:f_{n-1}(x_1,\dots x_{n-1})}\Big)\dots dx_2\Big)dx_1}_{=:f_1(x_1)}$$

'last-in first out'

- Note that every f_i is continuous w.r.t. $x_1,x_2,\dots x_i$ because f is continuous on the compact set I^n ($\Rightarrow$ uniformly continuous). Proceed by induction. So $L(f)$ is well-defined.
  uniformly convergent in $[a,b]$, $f(x_1,x_2^*\dots x_n^*)\to f(x_1^*,x_2^*\dots x_n^*)$

- Theorem: $L(f)$ as defined above does not depend on the order of integration.

Proof (by approximation!). Let $\tilde{L}(f)$ be the integral when we integrate in some other order. We want to show $\tilde{L}(f)=L(f)$.

Observation #1. If $f(\vec{x}) = f_1(x_1)f_2(x_2)\dots f_n(x_n)$, then $L(f)=\tilde{L}(f)$.

#2. If $L(f)=\tilde{L}(f)$ and $L(g)=\tilde{L}(g)$, then $L(f+g)=\tilde{L}(f+g)$ and $L(cf)=\tilde{L}(cf)$, $c\in\mathbb{R}$.

Claim III $\Rightarrow L(f)=\tilde{L}(f)$ for every polynomial $p(x_1,\dots x_n)\in\mathbb{R}[X]$.

Exercise: $|L(f)-L(f_k)| \le \sup_{x\in I^n} |f(x)-f_k(x)| \cdot \overbrace{\prod_{i=1}^{n}(b_i-a_i)}^{\mathrm{vol}\, I}$

Proof Sketch: 'Induction', 1D case is obvious, error gets multiplied over.

So we have Claim $\boxed{2}$: If $L(f_k)=\tilde{L}(f_k)$ $\forall k\in\mathbb{N}$ and $f_k \rightrightarrows f$ (e.g. equally in all dimensions) then $L(f)=\tilde{L}(f)$. $\left(L(f)=\lim_{k\to\infty}\tilde{L}(f_k)=\lim_{k\to\infty}L(\tilde{f}_k)=\tilde{L}(f)\right)$

$\boxed{1}+\boxed{2}$ together with Stone-Weierstrass imply the result.
(dense $\Rightarrow$ uniformly convergent)

Partitions of Unity

(Useful in Analysis and Differential Geometry)

* Rudin uses this to prove the change of variables formula, but Slepcev ran out of time (as expected)

Theorem (10.8 Rudin). Let $K\subseteq\mathbb{R}^n$ compact, $\{V_\alpha\}_{\alpha\in A}$ be an open cover of K. Then there exist functions $\psi_i\in C(\mathbb{R}^n,\mathbb{R})$, $i=1\ldots s$, such that

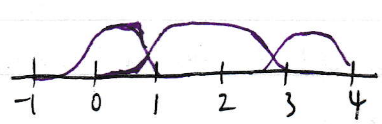
(i) $0\le\psi_i\le 1$

(ii) $(\forall i\le s)(\exists\alpha\in A)$ $\operatorname{supp}\psi_i\subseteq V_\alpha$ $\quad = \overline{\{x \mid \psi_i(x)>0\}}$ must be closed!

(iii) $\forall x\in K$, $\sum_{i=1}^{s}\psi_i(x)=1$.

"Take arguments down to V_α", prove results in V_α first

E.g. $K=[0,3]\subseteq\mathbb{R}$, $V_1=(-1,1)$, $V_2=(\frac{1}{2},3)$, $V_3=(2,4)$



Proof. Since $\{V_\alpha\}$ covers K, $\forall x\in K$, $x\in V_{\alpha(x)}$. ("choice"). So there is W_x open ball of radius r_x s.t. $x\in W\subseteq V_{\alpha(x)}$. Consider B_x with radius $r_x/2$, also centered at x.

Note that $\{B_x : x\in K\}$ is an open cover of compact K so there is a finite subcover $\{B_{x_1},\ldots,B_{x_s}\}$.

So define φ_i, $1\le i\le s$, by $\varphi_i(z)=\begin{cases}1 & z\in B_{x_i}\\ 0 & \text{otherwise}\end{cases}$. Then $\operatorname{supp}\varphi_i\subseteq W_{x_i}\subseteq V_{x_i}$.

We can define $\xi(x)=\sum_{i=1}^{s}\varphi_i(x)$, $\xi(x)>0$ on K but it might exceed 1!

Want $\operatorname{supp} \eta \subseteq \{x : \xi > 0\}$

Rudin's proof

make these continuous with the right tapering

Instead, we define ψ_i by $\psi_i = \begin{cases} \varphi_1, & i=1 \\ (1-\varphi_1)(1-\varphi_2)\cdots(1-\varphi_{i-1})\varphi_i, & i>1 \end{cases}$. (i) and (ii) are immediately satisfied.

We prove by induction that $\sum_{i=1}^{s} \psi_i(x) = 1 - \prod_{i=1}^{s}[1-\varphi_i(x)]$, which ensures (iii).

$s=1$: $\psi_1(x) = \varphi_1(x) = 1-(1-\varphi_1(x))$.

$s>1$:
$$\sum_{i=1}^{s} \psi_i(x) = \psi_s(x) + \sum_{i=1}^{s-1} \psi_i(x)$$
$$= \psi_s(x) + \left(1 - \prod_{i=1}^{s-1}[1-\varphi_i(x)]\right)$$
$$= \left(\varphi_s \prod_{i=1}^{s-1}[1-\varphi_i(x)] - \prod_{i=1}^{s-1}[1-\varphi_i(x)]\right) + 1$$
$$= 1 - \prod_{i=1}^{s}[1-\varphi_i(x)].$$

Slepcev's idea

$\psi_i = \varphi_i / \xi$ doesn't work! It is not continuous on all $\mathbb{R}^n$. Although $\sum_{i=1}^{s} \psi_i(x) = \frac{\sum \varphi_i(x)}{\xi(x)} = 1$ on K

We want to find η so if we let $\psi_i = \frac{\varphi_i \eta}{\xi}$, then $x \in K \Rightarrow \sum_{i=1}^{s} \psi_i(x) = \sum_{i=1}^{s} \psi_i(x) = \sum_{i=1}^{s} \frac{\varphi_i(x)\eta(x)}{\xi(x)} = \frac{\sum \varphi_i(x)}{\xi(x)} = 1$.

Define $\eta(z) = (1-m\,d(z,K))^+$ (positive part) for some $m>0$ (chosen later)

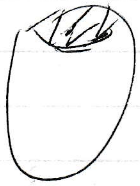
Then $\operatorname{supp} \eta \subseteq \{z : d(z,K) \leq \frac{1}{m}\}$

Define $\min_{x \in K} \xi(x) =: a > 0$. $A = \{z \in \mathbb{R}^n : \xi(z) > \frac{a}{2}\}$. A is an open set containing K. (continuity ξ is sum of open sets etc.)

Define $C = \mathbb{R}^n \setminus A$, $\inf\{d(z,k) : z \in C\} > 0$ why? $\inf\{d(z,k) : z \in C\} = \inf\{d(z,k) : z \in C, k \in K\}$
$= \inf\{d(C,k) : k \in K\}$
$= \min\{d(k,C) : k \in K\}$
$=: \beta > 0$.

minimum of positive continuous function on compact set is positive

(Choose m s.t. $\frac{1}{m} < \beta$. (Then the z's will not be in C, i.e. they are in A)
$\operatorname{supp} \eta \subseteq \{z : d(z,K) \leq \frac{1}{m}\} \subseteq \{z : d(z,k) < \beta\} \subseteq A$.



Localilize manifolds to $\mathbb{R}^2$.

$f = \sum (f\gamma)$

Change of Variables

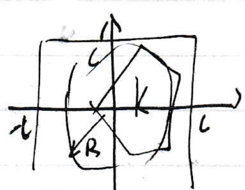
Let $f:\mathbb{R}^n \to \mathbb{R}$ be continuous (not necessarily bijective etc.), $K := \operatorname{supp} f$ is compact.

Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be smooth, where $\Phi(x) = x$ whenever $|x| > R$.

Then $$\boxed{\int_{\mathbb{R}^n} f(\Phi(x))\, J(x)\, dx_1 \dots dx_n = \int_{\mathbb{R}^n} f(y)\, dy_1 \dots dy_n}.$$

Jacobian $J(\vec{x}) = \det(D\Phi(\vec{x})) = \det\left(\left[\frac{\partial \Phi^i}{\partial x_j}\right]_{i,j=1\dots n}(\vec{x})\right)$

Let $K \subseteq [-c,c]^n$, $c > R$, $\boxed{g(y_1, y_2 \dots y_n) := \int_{-\infty}^{y_1} f(s, y_2 \dots y_n)\, ds}.$



$$\int_{\mathbb{R}^n} f(\phi(x)) \det(D\phi(x))\, dx_1 \dots dx_n = \int_{[-c,c]^n} \left(\frac{\partial g}{\partial y_1}\right)(\phi(x)) \det \overbrace{\left[\nabla\phi^1 \dots \nabla\phi^n\right]}^{=(D\phi)^T} dx_1 \dots dx_n$$

Claim 1

$$= \int_{[-c,c]^n} \det(\nabla(g\circ\phi), \nabla\phi^2 \dots \nabla\phi^n)\, dx_1 \dots dx_n$$

Cofactor expansion

$$= \int_{[-c,c]^n} \sum_{i=1}^n (-1)^{i-1} \frac{\partial(g\circ\phi)}{\partial x_i} \underbrace{M_i}_{\text{determinant of } (n-1)\times(n-1) \text{ matrix}} dx_1 \dots dx_n$$

integration by parts in each dimension

$$= \overbrace{\sum_{i=1}^n (-1)^i \int_{[-c,c]^n} g\circ\phi(\vec{x}) \frac{\partial M_i}{\partial x_i}\, dx_1 \dots dx_n}^{=0 \text{ claim 2}}$$

$$+ \int_{[-c,c]^{n-1}} g\circ\phi\, dx_2 \dots dx_n$$

More details pg 75!

$$= \boxed{\int_{[-c,c]^{n-1}} \int_{-c}^{c} f(x_1, \dots x_n)\, dx_1 \dots dx_n}$$

Claim 1: $\left(\frac{\partial g}{\partial x_1}\right)(\phi(x)) \det \left| [\nabla \phi^1 \dots \nabla \phi^n] \right| = \det(\nabla(g \circ \phi), \nabla \phi^2 \dots \nabla \phi^n)$

see p 75.

$M_i : \mathbb{R}^n \to \mathbb{R}$

Claim 2: We have to show $\sum_{i=1}^{n} g \circ \phi(\vec{x}) \frac{\partial M_i}{\partial x_i} (-1)^i = g \circ \phi(\vec{x}) \cdot \left[\sum_{i=1}^{n} \frac{\partial M_i}{\partial x_i} (-1)^i\right] = 0$.

scalar product

There are various proofs online for why $\sum_{i=1}^{n} \frac{\partial M_i}{\partial x_i} (-1)^i = 0$.

e.g. Evans PDE p 462 (Ch 8.1)

$$\left[\frac{\partial \Phi'}{\partial x_1} \;\; \frac{\partial \Phi'}{\partial x_2} \dots \;\; \frac{\partial \Phi'}{\partial x_n} \right] \to \nabla\phi' = \begin{bmatrix} \frac{\partial \Phi'}{\partial x_1} \\ \vdots \\ \frac{\partial \Phi'}{\partial x_n} \end{bmatrix}$$

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... $F \circ C$

take derivative then evaluate

$$= \dots \int_{[-c,c]^n} \left(\frac{\partial g}{\partial x_1}\right)(\phi(x)) \det \left| [\nabla\phi^1 \dots \nabla\phi^n] \right| dx_1 dx_2 \dots dx_n$$

$D(g \circ \phi)$ (by Chain Rule) is $[\;\;][\;\;]$

$\nabla(g \circ \phi)$ is the transpose:

$\nabla\phi \to \left[\nabla\phi^1 \dots \nabla\phi^n \right] \left[Dg \right]$

$$\nabla(g\circ\phi) = \nabla\phi^1 \frac{\partial g}{\partial x_1}(\phi(x)) + \nabla\phi^2 \frac{\partial g}{\partial x_2}(\phi(x)) + \dots + \nabla\phi^n \frac{\partial g}{\partial x_n}(\phi(x))$$

subtracting multiples of rows does not change the determinant

$\boxed{\text{(claim 1)}}$

$$= \int_{[-c,c]^n} \det\left(\nabla(g\circ\phi), \nabla\phi^2 \dots \nabla\phi^n\right) dx_1 \dots dx_n$$

$$= \int_{[-c,c]^n} \sum_{i=1}^n (-1)^{i-1} \frac{\partial (g\circ\phi)}{\partial x_i}(x) M_i \, dx_1 \dots dx_n$$

$\boxed{\text{integration by parts}}$

$$= \sum_{i=1}^n (-1)^{i-1} \int_{[-c,c]^{n-1}} \left[\left[g\circ\phi(\vec{x}) M_i \right]_{x_i=-c}^{x_i=c} - \int_{-c}^{c} g\circ\phi(\vec{x}) \frac{\partial M_i}{\partial x_i} dx_i \right] dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_n$$

we change order of integration here

innermost layer

$\boxed{\text{linearity}}$

$$= \underbrace{\sum_{i=1}^n (-1)^i \int_{[-c,c]^n} g\circ\phi(\vec{x}) \frac{\partial M_i}{\partial x_i} dx_1 \dots dx_n}_{} + \sum_{i=1}^n (-1)^{i-1} \int_{[-c,c]^{n-1}} \left[g\circ\phi(\vec{x}) M_i \right]_{x_i=-c}^{x_i=c} dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_n$$

$x_i \notin [-c,c] \Rightarrow |x| > R \Rightarrow \Phi(x) = 0$

$$= 0 \text{ (by claim 2)} + \cancel{\sum_{i=2}^n (-1)^{i-1} \int_{[-c,c]^{n-1}} \left[\int_{-\infty}^{x_1} f(s, \dots x_{i-1}, 0, x_{i+1}, \dots) ds - \int_{-\infty}^{x_1} f(s, \dots x_{i-1}, 0, x_{i+1}, \dots) ds \right] M_i \, dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_n}$$

$$+ \int_{[-c,c]^{n-1}} \left[\int_{-\infty}^{c_1 = \phi(c_1)} f(s, \phi(x_2) \dots \phi(x_n)) M_1 \, ds - \int_{-\infty}^{-c_1 = \phi(-c_1)} f(s, \phi(x_2) \dots \phi(x_n)) M_1 \, ds \right] dx_2 \dots dx_n$$

$$= \int_{[-c,c]^{n-1}} \underbrace{\int_{-c_1}^{c_1} f(s, \phi(x_2), \dots \phi(x_n)) ds}_{\text{we } \phi(\vec{x}) = \vec{x},\ |\vec{x}| \geq |x_1| = c} \underbrace{M_1}_{=1} dx_2 \dots dx_n$$

$$= \int_{[-c,c]^n} f(x_1, x_2, \dots x_2)\, dx_1 \dots dx_n$$

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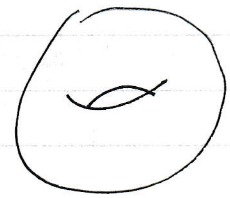
Manifolds in $\mathbb{R}^n$

- let $S^{n-1} \subseteq \mathbb{R}^n$, $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$

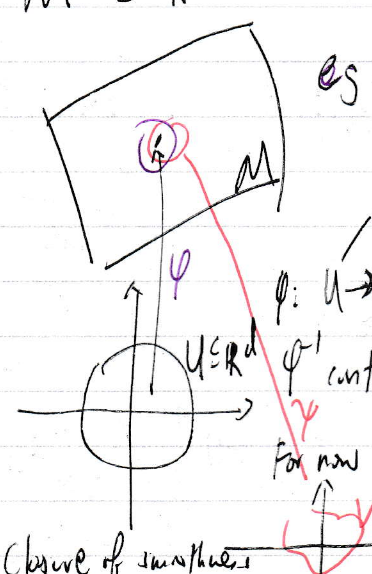
- $T = \{(x, f(x)) : x \in \mathbb{R}^n, f : \mathbb{R}^n \to \mathbb{R}^m \text{ smooth}\}$
  e.g. $f : \mathbb{R} \to \mathbb{R}^2$



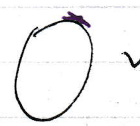
torus is a manifold

- $M \subseteq \mathbb{R}^n$ is a manifold such that at every point, some neighborhood resembles Euclidean space.

e.g. $\dim(M) = d$



is not a manifold



✓

(1) $\varphi : U \to \varphi(U) \subseteq M$ continuous bijection, U open, $U \subseteq \mathbb{R}^d$

φ^{-1} continuous, (homeomorphism)

For now we assume φ is smooth.

(2) Closure of smoothness under composition.

$\psi^{-1} \circ \varphi : \varphi^{-1}(\varphi(U) \cap \psi(V)) \to \mathbb{R}^d$ must be smooth.
$\subseteq \mathbb{R}^d$

(U, φ) is a local chart for M.

Is  $= \overline{B(0,1)}$ a manifold?

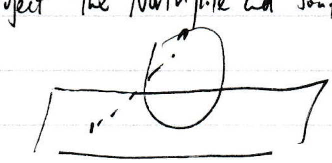
No, because x on the boundary is not homeomorphic to some $\mathbb{R}^d$.

simple case

$[0,1] \leftrightarrow (0,1)$! no bijection!

Assume otherwise. Remove $\{1\}$ and $\{f(1)\}$.
Then LHS is connected but RHS is not ("topological obstruction")
cont fn: maps connected to connected
so there is no surjective, cont map from $[0,1]$ to $[0,1)$

But the sphere is a manifold! We can project the North pole and South pole to a plane



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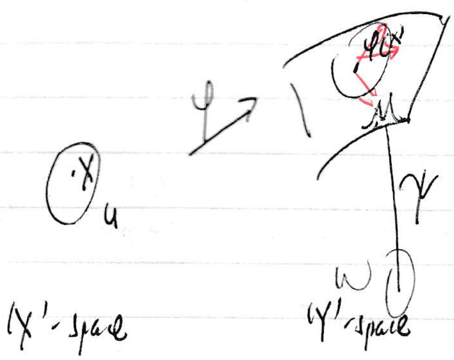
We look now at mappings between manifolds. Write $\varphi : \underset{\mathbb{R}^d}{U} \to \mathbb{R}^n$ as $\varphi = \begin{bmatrix} \varphi^1 \\ \vdots \\ \varphi^n \end{bmatrix}$



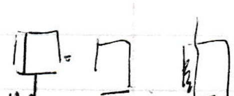
$\begin{bmatrix} \frac{\partial \varphi}{\partial x_1} \\ \vdots \\ \frac{\partial \varphi}{\partial x_d} \end{bmatrix}$ a tangent vector of M

Define $\boxed{T_{\varphi(x)} M := \operatorname{span}\left\{ \frac{\partial \varphi}{\partial x_1}(x), \dots \frac{\partial \varphi}{\partial x_d}(x) \right\}}$ 'tangent space' (vector space! but need to translate back to origin)

Claim: $\dim T_{\varphi(x)} M = d$.

Let $p = \varphi(x)$, $V \in T_p M$. We can write the same vector in 2 different local charts:

$$V = \alpha_1 \frac{\partial \varphi}{\partial x_1} + \dots \alpha_d \frac{\partial \varphi}{\partial x_d} = \beta_1 \frac{\partial \psi}{\partial y_1} \dots \beta_d \frac{\partial \psi}{\partial y_d}$$



Consider $\varphi(x) = (\psi \circ \psi^{-1}) \circ \varphi(x) = \underset{\mathbb{R}^d \to M}{\psi} \circ \underbrace{\underset{\mathbb{R}^d \to \mathbb{R}^d}{(\psi^{-1} \circ \varphi)}}_{=: T} (x)$

$$\frac{\partial \varphi}{\partial x_i} = \sum_{j=1}^{d} \frac{\partial \psi}{\partial y_j} \circ \underbrace{\frac{\partial T^j}{\partial x_i}}_{\text{number}} \text{ by the chain rule.}$$

$$V = \sum_{i=1}^{d} \alpha_i \frac{\partial \varphi}{\partial x_i} = \sum_{j=1}^{d} \sum_{i=1}^{d} \alpha_i \frac{\partial \psi}{\partial y_j} \frac{\partial T^j}{\partial x_i} = \sum_{j=1}^{d} \left(\sum_{i=1}^{d} \alpha_i \frac{\partial T^j}{\partial x_i} \right) \frac{\partial \psi}{\partial y_j}$$

must be equal! by linear independence

$$\therefore \beta_j = \sum_{i=1}^{d} \alpha_i \underbrace{\frac{\partial (\psi^{-1} \circ \varphi)^j}{\partial x_i}(x)}_{\boxed{=: \frac{\partial y_j}{\partial x_i}}} \quad (\mathbb{R}^d \to \mathbb{R}^d) \qquad = \sum_{j=1}^{d} \beta_j \frac{\partial \psi}{\partial y_j}$$

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$\Rightarrow$ $T_p M$ is a linear space, $\forall p \in M$.

<u>Example</u>: let $F : \mathbb{R}^n \to \mathbb{R}$

Assume $\forall x \in \mathbb{R}^n$, if $F(x) = 0$ then $\nabla f(x) \neq \vec{0}$. Show that $M = \{x \in \mathbb{R}^n : F(x) = 0\}$ is a manifold. (Use implicit function theorem: At every point p let $\nabla F(p)_i \neq 0$

e.g. $F(x) = |x|^2 - 25$, $M = 5\, S_{n-1}$ 'sphere' (surface only, note this is closed)

$\to$ Implicit function Theorem

$\to$ $\dim(M) = n-1$

$\to$ $B(0,1) \subseteq \mathbb{R}^n$ is not closed, but it is a manifold.

$$\boxed{b^j = \sum_{i=1}^{d} \frac{\partial y_j}{\partial x_i} a^i}$$ from the bottom of p77.

<u>Cotangent space</u>: $(T_p M)^*$ is the set of linear functions from $T_p M$ to $\mathbb{R}$

<u>Differential</u> Suppose $f : M \to \mathbb{R}$ is smooth ($\iff$ $f \circ \underset{\mathbb{R}^d \to M}{\varphi} : \mathbb{R}^d \to \mathbb{R}$ is smooth)

The differential df is defined by $$df\left(\frac{\partial \varphi}{\partial x_i}\right) := \frac{\partial f \circ \varphi}{\partial x_i}$$

$$df\left(\sum_{i=1}^{d} a^i \frac{\partial \varphi}{\partial x_i}\right) := \sum_{i=1}^{d} a^i \frac{\partial f \circ \varphi}{\partial x_i} = \sum_{i=1}^{d} a_i \nabla f \cdot \frac{\partial \varphi}{\partial x_i} = \nabla f \cdot V$$

$$\frac{\partial f \circ \varphi}{\partial x_i} = Df|_p \frac{\partial \varphi}{\partial x_i} = \nabla f|_p \cdot \frac{\partial \varphi}{\partial x_i}$$ dot product

$D(f \circ \varphi) = [\ \sim\ ] = [\ ]\,[\ ]$

Let $f = (\varphi^{-1})_i$ (sometimes denoted X_i)

$$df\left(\frac{\partial \varphi}{\partial x_j}\right) = \frac{\partial (\varphi^{-1} \circ \varphi)_i}{\partial x_j} = \begin{cases} 1 & i=j \\ 0 & \text{else} \end{cases} \quad [\delta^i_j = \text{Kronecker Delta}]$$

We can always define f to satisfy $df = \sum_{i=1}^{d} \alpha_i \underline{dx_i}$

Define $f := \sum \alpha_i x_i = (\varphi^{-1})_i$

$dx_1 \ldots dx_d$ are linearly independent.

$$\frac{\partial x_i}{\partial y_l} = \frac{\partial (\varphi^{-1} \circ \psi)^i}{\partial y_l}$$

Change of variables

evaluation

Suppose $\sum_{i=1}^{d} \alpha_i \, dx_i\left(\frac{\partial \psi}{\partial y_l}\right) = \sum_{j=1}^{d} \beta_j \, dy_j\left(\frac{\partial \psi}{\partial y_l}\right)$

$$\Rightarrow \sum_{i=1}^{d} \alpha_i \frac{\partial x_i}{\partial y_l} = \sum_{i=1}^{d} \alpha_i \frac{\partial (\varphi^{-1} \circ \psi)_i}{\partial y_l} = \sum_{j=1}^{d} \beta_j \cdot \delta^j_l = \beta_l$$

So things are completely opposite in the dual space.