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Contravariance vs Covariance

A covariant functor is just another functor.

A contravariant functor reverses the direction of arrows. If F is contravariant from $\mathbb{C}$ to $\mathbb{D}$, it maps objects of $\mathbb{C}$ to objects of $\mathbb{D}$,

and arrows $f: A \to B$ in $\mathbb{C}$ to arrows $F(f): F(B) \to F(A)$ in $\mathbb{D}$.

'covariant'	'contravariant'
$F(1) = 1$	$F(1) = 1$
$F(g \circ f) = F(g) \circ F(f)$	$F(g \circ f) = F(f) \circ F(g)$
	$A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathbb{C}$
	$FA \xleftarrow{Ff} FB \xleftarrow{Fg} FC$ in $\mathbb{D}$.

Observation: A contravariant functor from $\mathbb{C}$ to $\mathbb{D}$ is just a covariant functor from $\mathbb{C}^{op}$ to $\mathbb{D}$.

Example: (Contravariant hom-functor). Given X in locally small $\mathbb{C}$, define

$$\hom_{\mathbb{C}}(-,X): \mathbb{C}^{op} \to \mathbf{Set}$$

also $\hom_{\mathbb{C}^{op}}(A,-)$
$\mathbb{C}^{op} \to \mathbf{Set}$

$$A \longmapsto \hom_{\mathbb{C}}(A,X)$$

$$(f: A \to B) \longmapsto (\hom_{\mathbb{C}}(f,X): \hom(B,X) \to \hom(A,X))$$

$h \in$

where $\hom_{\mathbb{C}}(f,X)(h) \mapsto h \circ f$

→ precomposition function by f

Notes

→ "Consider a contravariant functor $F: \mathcal{C}^{op} \to \mathbb{D}$" is confusing!

→ Dualization. covariant $\hom_{\mathcal{C}}(A,-)$

$\updownarrow$

contravariant $\hom_{\mathcal{C}}(-,X)\ldots$

$J \xrightarrow{A} \mathcal{C}_0$

$j \to A_j$

J need not be countable / a set!

Def. An (infinite) coproduct of a family $(A_j)_{j\in J}$ of objects of a category consists of:

→ An object C

→ a family $(A_j \xrightarrow{i_j} C)_{j\in J}$

$$\begin{tikzcd} & A_j \arrow[dl, "i_j"'] \arrow[dr, "f_j"] & \\ C \arrow[rr, "h"'] & & X \end{tikzcd}$$

such that for every object X and family $(A_j \xrightarrow{f_j} X)_{j\in J}$ of arrows there exists unique morphism $C \to X$ such that $\forall j \in J,\ h \circ i_j = f_j$.

In Set: The coproduct of a family $(A_j)_{j\in J}$ of sets is given by

$$\coprod_{j\in J} A_j = \bigcup_{j\in J} A_j \times \{j\}$$

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers...

with injection maps $i_j : A_j \to \bigcup_{j\in J} A_j \times \{j\}$

$a \longmapsto (a,j)$

Countable union of countable sets is countable...

Back to coequalizers ... (p 43-44)

Coequalizers in Set

$$A \overset{f}{\underset{g}{\rightrightarrows}} \underbrace{B \to B/R}$$

where R is the least equivalence relation containing $\{(f(a), g(a)) \mid a \in A\}$

for universality

What about coequalizers in Mon?

(normal subgroups...)

$$(M, \cdot, e) \overset{f}{\underset{g}{\rightrightarrows}} (N, \cdot, e) \xrightarrow{?} (N/R, ?, ?)$$

Def A congruence relation on a monoid $(N, \cdot, e)$ is a binary relation $R \subseteq N \times N$, such that

- R is an equivalence relation
- $\forall (a,b), (c,d) \in R, (a \cdot c, b \cdot d) \in R$

makes it well-defined

Observation: Given a congruence relation on $(N, \cdot, e)$, we can define

$$N/R \times N/R \xrightarrow{[\cdot]} N/R$$

is a monoid! (see proof p.48.)

$$([n], [n']) \mapsto [n \cdot n']$$

$$\begin{array}{ccc} N \times N & \xrightarrow{\cdot} & N \\ \downarrow q \times q & & \downarrow q \\ N/R \times N/R & \dashrightarrow & N/R \end{array}$$

surjective

existence → uniqueness

We have

Unit in N/R = Equivalence class of unit in original monoid

Prop Given congruence relation R on $(N, \cdot, e)$, $(N/R, [\cdot], [e])$ is a monoid.

Proof. Because identifying elements cannot break equations. ← work this out! take $a, c \in [n]$, $b, d \in [n']$. $(a,b) \sim (c,d)$ $(a,c) \sim (b,d)$

Prop. Given monoid $(N, \cdot, e)$ of binary relation $R \subseteq N \times N$, there exists a least congruence relation $S \subseteq N \times N$ containing R.

Proof. First observe that arbitrary intersections of congruence relations are congruence relations.

Define $S = \bigcap \{ T \subseteq N \times N \mid T \text{ c.rel. and } R \subseteq T \}$

Prop A coequalizer of monoid maps $f, g : (M, \cdot, e) \to (N, \cdot, e)$ is given by $(N, \cdot, e) \xrightarrow{q} (N/R, [\cdot], [e])$

where R is the least congruence relation containing

$$\{ (f(m), g(m)) \mid m \in M \}.$$

$$\begin{tikzcd}
(M, \cdot, e) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (N, \cdot, e) \arrow[r, "q"] \arrow[dr, bend right, "k"'] & (N/R, [\cdot], [e]) \arrow[d, dashed, "h"] \\
 & & X
\end{tikzcd}$$

$q(g(m)) = [gm]$

$q(f(m)) = [fm]$

$h([n]) = k(n)$

we have to show $R \subseteq \{ (n, n') \mid kn = kn' \}$ ← kernel of k

p44

congruence relation! $k(m \cdot n) = k(m) \cdot k(n)$

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Example (Group Homomorphisms)

$$G \xrightarrow{f} H$$

$$\ker(f) = \{g \mid f(g) = e_H\}$$

Always a normal subgroup. Quotient it to get G/N.

Reinterpret this as $G/N = G/R$ where $R = \{(g,h) \mid g^{-1}h \in N\}$.

In groups we (conveniently) can encode information in a set rather than as congruence relations.

$b^{-1}a \in N$
$d^{-1}c \in N \Rightarrow$
$(b^{-1}d^{-1})^{-1}(a \cdot c) \in N$

For groups, rings, vector spaces...

Congruence relations can be represented as

(normal) subgroups, (two sided) ideals (commutative), subspaces...

every subgroup of an Abelian group is normal

but the equivalence relation is more clunky yet more general.

What about Coproducts of Monoids?

$$(M, \cdot, e) + (N, \cdot, e) = (M+N, \cdot, e)$$

$\uparrow m \cdot m, n \cdot n$

"freely adding" (adjoining) new elements for non-existing products.

First, a detour to free monoids...

Free Monoids

Consider a set Σ, viewed as an 'alphabet'

'$\Sigma = \{a, b, c, \dots z\}$'

Define Σ^* to be the set of (words/strings/lists) over Σ ('Kleene star')

e.g. $abc, xyz \in \Sigma^*$

$$\Sigma^* = \underbrace{\Sigma^0}_{\{\varepsilon\} \leftarrow \text{empty list}} \cup \Sigma \cup \Sigma^2 \cup \Sigma^3 + \Sigma^4 + \dots \Sigma^n \dots$$

disjoint

'stream' / 'sequence'

Note Σ^* has a monoid structure

Multiplication is given by concatenation

$\to$ $abc \cdot cdx = abccdx$

Associative!

$\to$ unit: empty string. $abc * \varepsilon = abc$ (concat)

The monoid $(\Sigma^*, *, \varepsilon)$ has a UMP:

UMP For any monoid $(N, \cdot, e)$ and function $f : \Sigma \to N$,

$\exists!$ monoid homomorphism $\tilde{f} : (\Sigma^*, *, \varepsilon) \to (N, \cdot, e)$

such that $\tilde{f} \circ j = f$.

where $j(s) = [s]$, length-1 string

$$\begin{tikzcd} \Sigma \arrow[d, "j"] \arrow[dr, "f"] & \\ \Sigma^* \arrow[r, "\tilde{f}"] & (N, \cdot, e) \end{tikzcd}$$

$(\Sigma^*, *, \varepsilon)$

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Terminology we call $(\Sigma^*, *, \varepsilon)$ a **free monoid** (over / generated by) Σ.

(proposition)

Proof: $\tilde{f}((s_1, \ldots s_n)) = \tilde{f}((s_1) * (s_2) * (s_n))$

homomorphisms commute over multiplication $= \tilde{f}((s_1)) \cdot \tilde{f}((s_2)) \ldots \tilde{f}((s_n))$

$= \tilde{f}(j(s_1)) \ldots \tilde{f}(j(s_n))$

necessarily
$:= f(s_1) \ldots f(s_n)$ $\square$

Now we just need to check this definition works. But $\tilde{f}(j(s)) = f(s)$ by definition. (same) (letter)

→ commutation. $\tilde{f} \circ j = f$

w.t.s $\forall a \in \Sigma$, $(\tilde{f} \circ j)(a) = f(a)$

but $(\tilde{f} \circ j)(a) = \tilde{f}(j(a)) = \tilde{f}([a]) = f(a)$

→ $\tilde{f}$ is homomorphism. $\tilde{f}([a_1, \ldots a_n] \cdot [b_1 \ldots b_k])$

$= \tilde{f}([a_1 \ldots a_n, b_1 \ldots b_k])$

$= f(a_1) \ldots f(a_n) \cdot f(b_1) \ldots f(b_k)$

$= \tilde{f}([a_1, \ldots a_n]) \cdot \tilde{f}([b_1, \ldots b_k])$

→ $\tilde{f}(\varepsilon) = e$ because a 0-fold product in a monoid is the unit.

Recap (10/5/21)
(Summary)

Σ set

Σ^* set of lists / words / strings

$\Sigma^* = \overset{\Sigma^0}{\{\varepsilon\}} \cup \Sigma \cup \Sigma^2 \cup \Sigma^3 \ldots = \bigcup_{n \in \mathbb{N}} \Sigma^n$

monoid structure is given by "concatenation" $[a_1, \ldots a_n] \cdot [b_1, \ldots b_k] = [a_1 \ldots a_n, b_1, \ldots b_k]$
and "unit" ε

Def. Free monoid is a monoid $(M, \cdot, e)$ together with a function $j : S \to M$ such that

$\forall$ monoids $(N, \cdot, e)$ & functions $f : S \to N$, $\exists!$ monoid homomorphism $\tilde{f} : (M, \cdot, e) \to (N, \cdot, e)$ such that $\tilde{f} \circ j = f$

$$\begin{tikzcd} S \arrow[d, "j"'] \arrow[dr, "f"] & \\ (M, \cdot, e) \arrow[r, "\tilde{f}"'] & (N, \cdot, e) \end{tikzcd}$$

JT

Presentation by Generators and Relations

"Let M be the monoid generated by x, y, z subject to equations

$$\left.\begin{aligned} x \cdot x &= x \\ y \cdot z &= z \cdot y \end{aligned}\right\} \text{"relations"}$$

"

Example: $x^2yz \neq zyx$

$x^3yz = xzy$ can be derived

e.g. $|I| = 2$

$s_0 = [x,x], t_0 = [x]$

$s_1 = [y,z], t_1 = [z,y]$

In general, given a set Σ and a function $(s_i, t_i)_{i \in I}$ of pairs of elements of Σ^*, the monoid "presented" by generators Σ and relations $(s_i, t_i)_{i \in I}$ is given by $(\Sigma^*, \cdot, \varepsilon)/R$

where R is the smallest congruence relation on Σ^* containing all pairs (s_i, t_i).

Call a monoid "finitely presented" if it admits a presentation by finitely many operators and relations (Σ, I both finite).

* Similar to coequalizer!

$$\begin{tikzcd} I \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "i"] & \Sigma^* \arrow[r, "q"] & \Sigma^*/R \\ I^* \arrow[ur, shift left, "s^*"] \arrow[ur, shift right, "t^*"'] & & \end{tikzcd}$$

$q \circ s = q \circ t$

$s(i) = s_i$

$t(i) = t_i$

Set

Mon

(free monoid)

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Prop. If R is the least congruence relation containing all (s_i, t_i), then

$$I^* \underset{\tilde{t}}{\overset{\tilde{s}}{\rightrightarrows}} \Sigma^* \to \Sigma^*/R$$

is a coequalizer in **Mon**.

Proof. (In general, the coequalizer of (in monoids, p48)

$$M \underset{g}{\overset{f}{\rightrightarrows}} N \overset{q}{\dashrightarrow} N/R$$

is given by $N \to N/R$, where R is the <u>least</u> congruence relation containing $\{(fm, gm) \mid m \in M\}$)

① Observe $\underbrace{\{(s_i, t_i) \mid i \in I\}}_{U} \subseteq \underbrace{\{(\tilde{s}(w), \tilde{t}(w)) \mid w \in I^*\}}_{V}$ "correct containment" "lower bound" "more restrictive"

② Remains to show that V is contained in the congruence relation generated by U, where "$\bar{U}$"

We know $\bar{U} = \{(a,b) \mid qa = qb\}$ from p 48.

That means we have to check that $\forall w \in I^*$, $q(\tilde{s}(w)) = q(\tilde{t}(w))$

$\Leftrightarrow q \circ \tilde{s} = q \circ \tilde{t}$.

$km_1 = kw_1$
$km_2 = kn_2$

$q \circ \tilde{s} = q \circ \tilde{t}$ is true because both fit into the diagram

km_1m_2
unites
$= kn_1n_2$

$$\begin{tikzcd} I \arrow[r, "q \circ s = q \circ t"] \arrow[d, "i"'] & \Sigma^*/R \\ I^* \arrow[ur, "q \circ \tilde{s} = q \circ \tilde{t}"'] & \end{tikzcd}$$

'same arrow'

"Word Problem": Are two words equal in finite generated monoid?

In general, <u>Undecidable</u> !

'semidecidable'
- will stop if ok
- just generate all possibilities

<u>Coproducts of Monoids:</u>

$$(M,\cdot,e) + (N,\cdot,e) \overset{?}{=} (M+N,\cdot,e)$$

Proposition A coproduct of $(M,\cdot,e)$, $(N,\cdot,e)$ is given by

set-theoretic coproduct

$$(M,+,\cdot) \xrightarrow[j_1]{} \left((M+N)^* / R, \cdot, e \right) \xleftarrow[j_2]{} (N,+,\cdot)$$

where R is the least congruence relation on $(M+N)^*$ making j_1, j_2 homomorphisms.

''disjoint union''

"collapse the least info"

$$\begin{tikzcd}
M \arrow[r, "i_1"] \arrow[dr] \arrow[ddr, "j_1"'] \arrow[dddr, bend right] & M+N \arrow[d, "j"] & N \arrow[l, "i_2"'] \arrow[dl] \arrow[ddl, "j_2"] \arrow[dddl, bend left] \\
 & (M+N)^* \arrow[d, "q"] & \text{free monoid} \\
 & (M+N)^*/R \arrow[d] & \\
 & X &
\end{tikzcd}$$

<u>Generalizations</u>

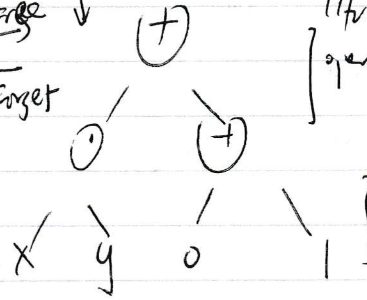
Free Monoids $\Rightarrow$ other free algebras (groups, rings ...)

$\Downarrow$ $\Downarrow$

$\boxed{\text{Free Categories}}$ $\Rightarrow$ Adjunctions

Free

$\leftarrow$ Forget



"trees", "signatures" ...

operations

generators

Free ring: binary trees

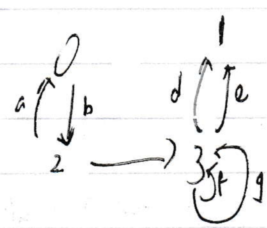
Recall Monoids $\to$ Category with 1 object

$A \underset{f}{\overset{g}{\rightleftarrows}} B \quad f \circ g = 1_A$

An example

Def A (directed, multi-) graph consists of

- a set V of vertices
- a set E of edges
- functions $s, t : E \to V$ (source, target)



$V = \{0,1,2,3\}$

$E = \{a,b,c,d,e,f,g\}$

	s	t
a	2	0
b	0	2
c	2	3
d	⋮	⋮
e		
f		
g		

Def Given graphs $(G_0 \overset{s}{\underset{t}{\leftleftarrows}} G_1)$

$(H_0 \overset{s}{\underset{t}{\leftleftarrows}} H_1)$

a morphism is a pair (f_0, f_1) of functions

$f_0 : G_0 \to H_0$, $f_1 : G_1 \to H_1$ s.t. $s \circ f_1 = f_0 \circ s$

$t \circ f_1 = f_0 \circ t$

Not a functor

no composition identity

no categories

e.g.

$$\begin{tikzcd} G_0 \arrow[d, "f_0"'] & G_1 \arrow[l, shift right, "s"'] \arrow[l, shift left, "t"] \arrow[d, "f_1"] \\ H_0 \arrow[d, "g_0"'] & H_1 \arrow[l, shift right, "s"'] \arrow[l, shift left, "t"] \arrow[d, "g_1"] \\ K_0 & K_1 \arrow[l, shift right, "s"'] \arrow[l, shift left, "t"] \end{tikzcd}$$

$f_0 \circ s = s \circ f_1$

$f_0 \circ t = t \circ f_1$

(Graph of a function $f : A \to B$

$\{(a, fa) \mid a \in A\} \subseteq A \times B$)

How is this related to categories?

Recall that for a small category $\mathbb{C}$, we have (1) $\mathbb{C}_0 \overset{dom}{\underset{cod}{\leftleftarrows}} \mathbb{C}_1$ 'arrows' (2) composition & (3) id's

Def: U: forgetful functor

$$\begin{tikzcd} \mathbf{Cat} \arrow[r] & \mathbf{Gph} \\ \mathbb{C} \arrow[r, mapsto] \arrow[d, "\text{any functor } F"'] & (\mathbb{C}_0 \overset{dom}{\underset{cod}{\leftleftarrows}} \mathbb{C}_1) \arrow[d, "(F_0, F_1)"] \\ \mathbb{D} \arrow[r, mapsto] & (\mathbb{D}_0 \leftleftarrows \mathbb{D}_1) \end{tikzcd}$$

- vertices of $U(\mathbb{C})$ are objects of $\mathbb{C}$
- edges of $U(\mathbb{C})$ are arrows
- source & target given by dom & cod

F_0 is the object part

F_1 " morphism part

Def. A <u>free category</u> on a graph $G = (G_0 \overset{s}{\underset{t}{\leftleftarrows}} G_1)$ is a <u>category</u> $\mathbb{C}$ together with a graph morphism $\underline{j = (j_0, j_1) : G \to U(\mathbb{C})}$ ~graph

such that for all cat's $\mathbb{D}$ and graph maps $f : G \to U(\mathbb{D})$, $\underline{\exists!}$ functor $F : \mathbb{C} \to \mathbb{D}$ such that $U(F) \circ j = f$

$$\begin{tikzcd} G \arrow[d, "j"'] \arrow[dr, "f"] & \\ U(\mathbb{C}) \arrow[r, "U(F)"'] & U(\mathbb{D}) \end{tikzcd}$$

$u(\mathbb{C}(G))$ "

'take paths'

compose edges... $\mathbb{C} \xrightarrow[F]{} \mathbb{D}$

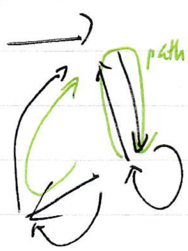
'many objects'

[for monoids:

$$\begin{tikzcd} S \arrow[d, "j"'] \arrow[dr, "f"] & \\ S^* \arrow[r, "\bar{F}"'] & M \end{tikzcd}$$

]

'take lists' 'one object'



Construction: For a graph G, define its <u>path category</u> $\mathbb{C}(G)$ as follows:

- $\mathbb{C}(G)_0 = G_0$
- $\mathbb{C}(G)_1 = $ ~~$\{(e_1 \ldots e_n) \in G_1^* \mid$ s.t. $\forall 1 \le i < n,\ t(e_i) = s(e_{i+1})$~~ $]$
- ~~$dom(e_1, \ldots e_n) = s(e_1)$~~

* . We want distinct identities for each object, and a single $\varepsilon \in G_1^*$ doesn't work

$$= \{(v_0, e_1, v_1, e_2 \ldots e_n, v_n) \mid \forall 1 \le i \le n,\ s(e_i) = v_{i-1},\ t(e_i) = v_i\}$$

Composition: $(v_0, e_1, \ldots v_n) \circ (w_0, \ldots w_k) = (w_0, \ldots w_k, e_1, v_1, \ldots v_n)$ [if $w_k = v_0$]

Identity: $1_v = (v)$

Prop: $\mathbb{C}(G)$ is a free category over G with

$$j : G \to U(\mathbb{C}(G))$$
$$v \mapsto v$$
$$e \mapsto (s(e), e, t(e))$$

Proof. As for monoids.

Def A congruence relation on a small category $\mathbb{C}$ is an equivalence relation R on $\mathbb{C}_1$ ($R \subseteq \mathbb{C}_1 \times \mathbb{C}_1$) s.t.

1. $\forall (f,g) \in R$, $\mathrm{dom}(f) = \mathrm{dom}(g)$ and $\mathrm{cod}(f) = \mathrm{cod}(g)$ "parallel arrows"
2. $\forall (f,g), (h,k) \in R$, $\mathrm{cod}(f) = \mathrm{dom}(h) \Rightarrow (h \circ f, k \circ g) \in R$

Given congruence relation R on $\mathbb{C}$, the quotient category $\mathbb{C}/R$ is given as follows:

- $(\mathbb{C}/R)_0 = \mathbb{C}_0$
- $(\mathbb{C}/R)_1 = \mathbb{C}_1/R$.

$\forall [f] \in (\mathbb{C}/R)_1$: $\mathrm{dom}([f]) = \mathrm{dom}(f)$
$\mathrm{cod}([f]) = \mathrm{cod}(f)$

- $1_C = [1_C]$
  in $\mathbb{C}/R$ ... in $\mathbb{C}$
- $[g] \circ [f] = [g \circ f]$

"homotopy"?? theory
Top $/\sim$
CW $\subseteq$ Top
CW$/\sim$ = ho ??

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Def by Generators and Relations

e.g. Let $\mathbb{C}$ be the category with objects A, B, generating arrows $f: A \to B$, $g: B \to A$

Subject to the relation $g \circ f = 1$. section-retraction pair (split mono, split epi)

$$\mathbb{C}\left(\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \end{tikzcd} \right)_1 = \left\{ \begin{array}{ll} 1)\ 1_A & 5)\ f \circ g \\ 2)\ 1_B & g \circ f \\ 3)\ f & f \circ g \circ f \quad \dots \\ 4)\ g & g \circ f \circ g \end{array} \right\}$$

$f \circ g = (B, g, A, f, B)$

In reality, 5 equivalence classes $\to$ 5 arrows in the constructed category.

> Given a graph G and a family $(f_i, g_i)_{i \in I}$ of parallel arrows in $\mathbb{C}(G)$, the category generated by G subject to relations $(f_i, g_i)_{i \in I}$ is $\underline{\mathbb{C}(G)/R}$, where R is the ~~smallest~~ least congruence containing all pairs (f_i, g_i).

HW4Q5 $\mathbb{C} \to \mathbb{C}/R$

Note: $\mathbb{C} \to \mathbb{C}/R$

always identity on ~~objects~~ arrows, cannot 'identify' objects (leaves them as they are)

e.g. $1 \overset{\text{two possible functors}}{\rightrightarrows} (0 \to 1) \to \mathbb{C} = ?$

(coequalizer doesn't work!)

we have to define the coequalizer instead by $=1$ (the number!)

$1 \rightrightarrows \downarrow f \to \circlearrowright f \quad f \circ f \circ f \dots$

$= (\mathbb{N}, +, 0) \quad f^n \circ f^k = f^{n+k}$

Pullbacks

$\to \cdot$	terminal obj	initial obj $\cdot \to$
$\to A\times B$	bin prods	bin coprod $A+B \to$
$\to E$	equalizer	coequalizers $C \to$

$\underbrace{\qquad\qquad}_{\text{Limits (pbs)}}$ $\underbrace{\qquad\qquad}_{\text{Colimits}}$

(inverse limits) (direct limits)

$\uparrow$

pullbacks

Def. A pullback of arrows $\begin{array}{c} C \\ \downarrow g \\ B \xrightarrow{f} A \end{array}$

in a category $\mathbb{C}$ is a pair of arrows $\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & C \\ B \end{tikzcd}$

such that ① $\begin{tikzcd} \cdot \arrow[r, "p_2"] \arrow[d, "p_1"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "f"'] & \cdot \end{tikzcd}$ commutes

② for all $\begin{tikzcd} Z \arrow[r, "z_2"] \arrow[d, "z_1"'] & \cdot \\ \cdot \end{tikzcd}$ such that $\begin{tikzcd} \cdot \arrow[r, "z_2"] \arrow[d, "z_1"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "f"'] & \cdot \end{tikzcd}$ commutes,

$\exists!\ h: Z \to P$ such that $p_1 \circ h = z_1$, ~~$p_2 \circ h = z_2$~~

$$\begin{tikzcd} Z \arrow[r, dotted] \arrow[rr, bend left, "z_2"] \arrow[dr, bend right, "z_1"'] & P \arrow[r, "p_2"] \arrow[d, "p_1"] & C \arrow[d, "g"] \\ & B \arrow[r, "f"'] & A \end{tikzcd}$$

"terminal in the completion to a square"

Notation

$$\begin{tikzcd} X \arrow[d, shift right, "k"'] \arrow[d, shift left, "h"] \arrow[dr, bend left, "p_2 \circ h = p_2 \circ k"] \arrow[dd, bend right, "{p_1 \circ h \atop = p_1 \circ k}"'] & \\ A \times_C B \arrow[r] \arrow[d] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

Another diagram for the sake of it...

$$\begin{tikzcd} E \arrow[r, tail, "e"] & A \times B \arrow[r, shift left, "f \circ \pi_1"] \arrow[r, shift right, "g \circ \pi_2"'] & C \end{tikzcd}$$

Observe: p_1 & p_2 are jointly monic, i.e. for

$h, k : Z \to A \times_C B$

we have $h = k$ whenever $p_1 \circ h = p_1 \circ k$ & $p_2 \circ h = p_2 \circ k$.

(by uniqueness)

Consider the morphisms

$$\begin{tikzcd} & Z \arrow[dl] \arrow[d] & \\ A \times_C B \arrow[r, tail, "\text{monomorphisms}"] & A \times B \arrow[r, shift left] \arrow[r, shift right] & {} \end{tikzcd}$$

We have $\hom_{\mathcal{C}}(Z, A \times_C B) \cong \{(z_1, z_2) \in \hom(Z, A) \times \hom(Z, B) \mid f \circ z_1 = g \circ z_2\}$ pullback maps

$\cong \{z \in \hom(Z, A \times B) \mid f \circ p_1 \circ z = g \circ p_2 \circ z\}$

Generalized element with constraints $\to$ 'equalizer'

Prop. If a $\mathcal{C}$ has equalizers and binary products, then it has pullbacks.

Proof. A pullback of $A \xrightarrow{f} C \xleftarrow{g} B$ is constructed as follows

$$\begin{tikzcd} E \arrow[r, "e"] & A \times B \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \arrow[d, "g"] \\ & A \arrow[r, "f"'] & C \end{tikzcd}$$

'proof direction' ② ① start

$\to$ e equalizer of $f \circ p_1$ and $g \circ p_2$

$\Rightarrow$ pullback is

$$\begin{tikzcd} E \arrow[r, "p_2 \circ e"] \arrow[d, "p_1 \circ e"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

Pullback lemma

$(\varphi \Rightarrow (\psi \Leftrightarrow \theta))$

Given a commutative diagram

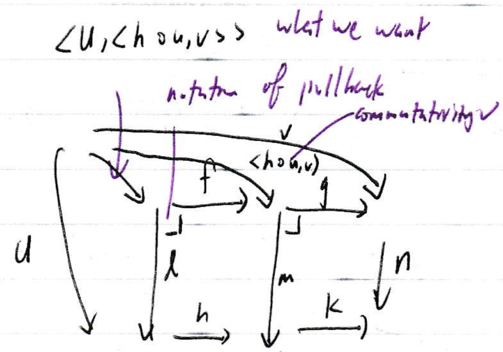
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "l"'] & B \arrow[r, "g"] \arrow[d, "m"] & N \arrow[d, "n"] \\ D \arrow[r, "h"'] & E \arrow[r, "k"'] & F \end{tikzcd}$$

In a category $\mathcal{C}$, if the right square is a pullback, then

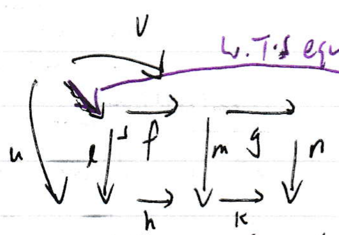
(left Rectangle is a pullback $\Leftrightarrow$ Entire rectangle is pullback)

Proof sketch.

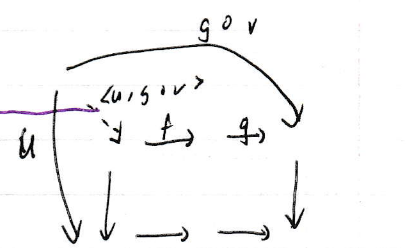
$(\Rightarrow)$



$(\Leftarrow)$

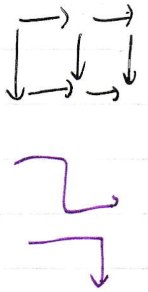


(right square is pullback)



(entire rectangle is pullback)

"diagram chase"



W.T.S equal when post-composed with m (then use jointly monic property)

$m \circ f \circ \langle u, g\circ v\rangle = m \circ v$

LHS $= h \circ l \circ \langle u, g\circ v\rangle$ RHS $= h \circ u$

$= h \circ u$

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Pullback Functors

Given an arrow $m: A \to B$ in $\mathbb{C}$ with pullbacks, construct a functor

$m^*: \mathbb{C}/B \to \mathbb{C}/A$ by

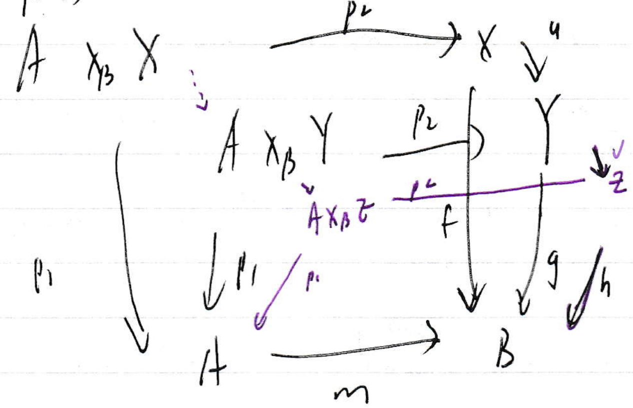
need Axiom of Choice to go function $(\mathbb{C}/B)_0 \to (\mathbb{C}/A)_0$

$$\begin{tikzcd} A \times_B X \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "f"] \\ A \arrow[r, "m"'] & B \end{tikzcd}$$

(object part) where $m^*(f: X \to B) = (p_1 : A \times_B X \to A)$

(morphism part)

recall: $\hom_{\mathbb{C}/A}((f: X \to A), (g: B \to A))$
$= \{u : X \to Y \mid g \circ u = f\}$



$m^*(u : X \to Y)$

$= \langle p_1, u \circ p_2 \rangle : \underbrace{m^* f}_{\text{obj}} \to \underbrace{m^* g}_{\text{obj}}$

Check composition: we need to show that $m^*(v) \circ m^*(u) = m^*(v \circ u)$

$$\text{LHS} = \langle p_1, v \circ p_2 \rangle \circ \langle p_1, u \circ p_2 \rangle$$
$$= \langle p_1 \circ \langle p_1, u \circ p_2 \rangle, v \circ p_2 \circ \langle p_1, u \circ p_2 \rangle \rangle$$
$$= \langle p_1, v \circ u \circ p_2 \rangle = m^*(v \circ u)$$

[use ideas similar to those from Hw3: $\langle f, g \rangle \circ h = \langle f \circ h, g \circ h \rangle$
$p_1 \circ \langle f, g \rangle = f$]

Subobjects

Def: Let $\mathbb{C}$ be a cat and A an object in $\mathbb{C}$ (i.e. $A \in \mathbb{C}_0$)

A subobject of A is a monomorphism

$$m : U \rightarrowtail A$$

(Motivation: subgroup, subset, ... subspace, subcategory)
"substructure"

In SET $U \subseteq S$, $j : U \to S$, $j(x) = x$, but also consider $f : \{0\} \to \{1,2\}$, $f(0) = 1$ another injective map!

Also from set theory,

$$\begin{tikzcd} U \arrow[rr, "\cong"] \arrow[dr, "f"'] & & \mathrm{Im}(m) = \{m(x) \mid x \in U\} \arrow[dl, "j"] \\ & S & \end{tikzcd}$$

isomorphic in $\mathrm{Set}/_S$

So j and f are 'the same' anyway.

Principle: Definitions in category theory should be iso-invariant.

(in general: between two subgroups, isomorphism??)

Notation $\mathrm{Sub}_{\mathbb{C}}(A)$ is the full subcategory (p12.) of $\mathbb{C}/A$ (slice) on monomorphisms/subobjects

$$\mathrm{Sub}_{\mathbb{C}}(A) \subseteq \mathbb{C}/A.$$

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Observe: $Sub_{\mathcal{C}}(A)$ is a preorder category.
('posetal' 'thin' 'unique arrow')

these are the morphisms! in the slice category

$$\begin{tikzcd} \bullet \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[dr, tail, "m"'] & \bullet \arrow[d, tail, "h"] \\ & A \end{tikzcd}$$

$\left.\begin{array}{l} h \circ f = m \\ h \circ g = m \end{array}\right\} \Rightarrow h \circ f = h \circ g \Rightarrow f = g$

Notation
→ write $m \leq n$ if $\exists f : m \to n$ in $Sub(A)$
→ write $m \equiv n$ if $m \leq n$ and $n \leq m$
(Say "m equivalent n" i.e. isomorphism in $Sub(A)$)

Observe: Equivalence is an equivalence relation!

Furthermore, $(Sub(A)/\equiv, \leq)$ is a poset when ordered by representatives
$([m] \leq [n]$ iff $m \leq n)$

In **Set**,

$$(P(A), \subseteq) \xrightarrow[\text{inclusion map}]{\text{injective}} (Sub_{Set}(A), \leq) \xrightarrow{\text{surjective}} (Sub_{Set}(A)/\equiv, \leq)$$

$u \subseteq A$
$u \hookrightarrow A$

Quotiented subcategory of Set/A

every monomorphism is isomorphic to a subset inclusion morphism

isomorphism of posets (can verify same cardinality...)

In **Set**, subsets give canonical representatives of subobjects.

Proposition: If m mono in pullback, then n also.

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "n"'] \arrow[dr, phantom, "\lrcorner"] & \cdot \arrow[d, "m"] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

Proof: Homework.

Let $\mathbb{C}$ be cat with pullbacks

$$\begin{tikzcd} A \arrow[d, "m"'] & \mathbb{C}/A & \mathrm{Sub}_{\mathbb{C}}(A) \arrow[l, hook', "\text{(inclusion)}"'] \\ B \arrow[d, "n"'] & \mathbb{C}/B \arrow[u, "m^* \text{ pullback functor}"] & \mathrm{Sub}_{\mathbb{C}}(B) \arrow[l, hook', "\text{(inclusion)}"] \arrow[u, "m^{-1}"'] \\ C & \mathbb{C}/C \arrow[u, "n^*"] & \end{tikzcd}$$

$\Rightarrow$

:= restriction of PB functor to subobjects

this works because m^* 'converts' mono $\rightarrowtail B$ to $\rightarrowtail A$

we know $(\ )^*$ is a monomorphism by the prop above.

$$\begin{tikzcd} \cdot \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d] \\ \cdot \arrow[r, tail] & B \end{tikzcd}$$

Question: Functoriality of $\Rightarrow$?

i.e. Do we have functors? No!! $\underbrace{\mathbb{C}^{op} \to \mathcal{C}at}$ $\qquad$ $\overbrace{\mathbb{C}^{op} \to \mathrm{Preord}}$

$A \mapsto \mathbb{C}/A \qquad A \mapsto \mathrm{Sub}(A)$

$m \downarrow \ \mapsto \ \uparrow f^* \qquad m\downarrow \ \mapsto \ \uparrow m^{-1}$

$B \mapsto \mathbb{C}/B \qquad B \mapsto \mathrm{Sub}(B)$

$$\begin{tikzcd} (X \times_C B) \times_B A \arrow[r] \arrow[d] & A \arrow[d, "m"] \\ X \times_C B \arrow[r] \arrow[d] & B \arrow[d, "g"] \\ X \arrow[r, "f"'] & C \end{tikzcd} \qquad \begin{tikzcd} X \times_C A \arrow[r] \arrow[dd] & A \arrow[d] \\ & B \arrow[d] \\ X \arrow[r] & C \end{tikzcd}$$

might not get equality! Choose wrong object? (They're only unique up to isomorphism)

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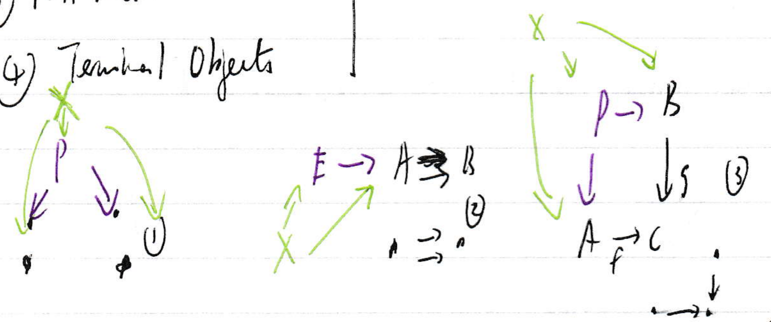
Limits (first appeared on p60)

(1) Products
(2) Equalizers
(3) Pull backs
(4) Terminal Objects

Examples of limits



(4) empty: $[\ ]$ empty graph/cat

Every free category has an underlying graph (p56)

Def: Let $\mathbb{J}, \mathbb{C}$ be categories. A diagram of type $\mathbb{J}$ in $\mathbb{C}$ is a functor

$$D : \mathbb{J} \to \mathbb{C} \qquad (\mathbb{J} \text{ is the index category})$$

e.g. (1) $\mathbb{J}$ is the discrete cat $\underline{2}$ with two objects.

(2) $\mathbb{J}$ is the free cat on the graph $(\cdot \rightrightarrows \cdot)$

(3)
$$\begin{tikzcd} & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

Let
$$E = \left(\begin{tikzcd} & 2 \arrow[d, "f"] \\ 0 \arrow[r, "e"'] & 1 \end{tikzcd} \right) = \left(\{e, f\} \underset{t}{\overset{s}{\rightrightarrows}} \{0,1,2\} \right)$$

$$\begin{tikzcd} E \arrow[d] \arrow[dr, "f"] & \\ U(\mathbb{C}(E)) \arrow[r] & U(\mathbb{C}) \end{tikzcd}$$

forgetful functor from cat to graph

$\underline{\text{Cat}}$ $\quad \mathbb{C}(E) \xrightarrow[\exists!]{} \mathbb{C}$

To define a diagram of type $\mathbb{C}(E)$, it suffices to give a graph morphism $E \to U(\mathbb{C})$.

A graph morphism from $\begin{tikzcd} & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$ to $\mathbb{C}$ is the same as a diagram of type $\begin{tikzcd} & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$

(copied to prev page)

~~A graph morphism from $\cdot \to \cdot$ to $\mathbb{C}$ is the same as a diagram~~ of type $\begin{array}{c} A \\ \downarrow f \\ B \xrightarrow{g} C \end{array}$

Def: A <u>cone</u> to/over a diagram $D : \mathbb{J} \to \mathbb{C}$ ('type')
consists of an object C (<u>vertex</u>) in $\mathbb{C}$ and "projection arrows" (morphisms in $\mathbb{C}$)

$$p_j : C \to D_j \quad {}_{\in \mathbb{C}_1}$$

such that the following triangle commutes for all $\alpha : i \to j$ in $\mathbb{J}$

('special arrows in $\mathbb{C}$')

$$\begin{tikzcd} & C \arrow[dl, "p_i"'] \arrow[d, "p_j"] \\ D_i \arrow[r, "D\alpha"'] & D_j \end{tikzcd} \quad (X)$$

A morphism of cones from $(C, (p_j)_{j \in \mathbb{J}})$ to $(E, (q_j)_{j \in \mathbb{J}})$ is an arrow $\theta : C \to E$ in $\mathbb{C}$ such that

$$\begin{tikzcd} C \arrow[r, "\theta"] \arrow[dr, "p_j"'] & E \arrow[d, "q_j"] \\ & D_j \end{tikzcd}$$

commutes for all $j \in \mathbb{J}_0$

Example: $\mathbb{J} = (1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 2)$, diagram $\mathbb{J} \xrightarrow{D} \mathbb{C}$

$$\begin{tikzcd} & C \arrow[dl, "p_1"'] \arrow[d, "p_2"] \\ D_1 \arrow[r, shift left, "D\alpha"] \arrow[r, shift right, "D\beta"'] & D_2 \end{tikzcd}$$

(Equalizer)

For (X): $\underbrace{D\alpha \circ p_1 = p_2, \quad D\beta \circ p_1 = p_2}_{D\alpha \circ p_1 = D\beta \circ p_1}$

Cones to D and their morphisms form a category $\underline{\mathrm{Cone}(D)}$.

e.g. product is cone; use discrete category as diagram

Def. A $\underline{\text{limit}}$ of the diagram D is a terminal object in $\mathrm{Cone}(D)$.

A $\underline{\text{small}}$ (finite) limit is a limit of a diagram whose index category is small (finite)

$$\begin{tikzcd} & Y \arrow[dl, "f_1"'] \arrow[d, "f"] \arrow[dr, "f_2"] & \\ X_1 & X_1 \times X_2 \arrow[l, "p_1"] \arrow[r, "p_2"'] & X_2 \end{tikzcd}$$

terminal

$\uparrow D$

$\cdot\ \cdot\ \mathbb{J}$

e.g. $eq \xrightarrow{e} x \underset{g}{\overset{f}{\rightrightarrows}} y$ $\xleftarrow{D} \{x \rightrightarrows y\}$ $\mathbb{J}$

Theorem A cat $\mathbb{C}$ has (all) small/finite limits iff it has small/finite products and equalizers.

Proof ($\Rightarrow$) see explanation above

($\Leftarrow$) Assume $\mathbb{C}$ has equalizers and small products, and let $D: \mathbb{J} \to \mathbb{C}$ be a diagram.

let $\underline{f}, \underline{g}: \prod_{j \in \mathbb{J}_0} D_j \rightrightarrows \prod_{(\alpha: i \to j) \in \mathbb{J}} D_j$ ($\to p_i$, D_α)

$$\begin{tikzcd} \prod_{(\alpha: i \to j) \in \mathbb{J}} D_j \arrow[d, "p_\alpha"] \\ D_j \end{tikzcd}$$

p_j

(e.g. $\left[{}^{1}\circlearrowleft 0 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 1 \circlearrowright{}^{1}\right] = \mathbb{J}$ gives rise to

${}_{1_0}\circlearrowleft D_0 \underset{D_\beta}{\overset{D_\alpha}{\rightrightarrows}} D_1 \circlearrowright{}_1$

Then $\prod_j D_j = D_0 \times D_1$.

$\prod_{\alpha: i \to j} D_j = D_0 \times D_1 \times D_1 \times D_1$)

$\uparrow 1_0 \quad \uparrow \alpha \quad \uparrow \beta \quad \uparrow 1_1$

be the two arrows such that

$p_\alpha \circ f = p_j$ for $\alpha: i \to j$ in $\mathbb{J}$.

$p_\alpha \circ g = D_\alpha \circ p_i$ unique (definition of product!)

Let $e: E \to \prod_{j \in \mathbb{J}_0} D_j$ be the equalizer of f and g. We claim that $\underline{E \text{ is a limit}}$ of D together with projection maps

$$\begin{tikzcd} E \arrow[r, "e"] & \cdot \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "p_j"] & \cdot \\ & D_j & \end{tikzcd}$$

$\underline{\varepsilon_j: E \to D_j}$, $\varepsilon_j = p_j \circ e$

$\rightarrow$ First prove that this is a cone i.e. for all $\alpha : i \to j$,

$$\begin{tikzcd} & E \arrow[dl, "q_i"'] \arrow[d, "q_j"] \\ D_i \arrow[r, "\alpha"'] & D_j \end{tikzcd}$$ commutes (where $q_j = p_j \circ e$)

$$D\alpha \circ q_i = D\alpha \circ p_i \circ e \overset{def}{=} p_\alpha \circ \boxed{g \circ e}$$

but also $q_j = \qquad p_j \circ e = p_\alpha \circ \boxed{f \circ e}$

proj. equaliser

w.T.s Terminal in Cone (D)

$\rightarrow$ Let $(C, (r_j)_{j \in J_0})$ be an arbitrary cone to D.

We have to define cone morphism $(C, (r_j)_j) \to (E, (q_j)_j)$

To define $h : C \to E$ define first $\boxed{k : C \to \prod_{j \in J} D_j}$, $k = \langle r_j \mid j \in J_0 \rangle$

$$\begin{tikzcd} C \arrow[r, "k"] \arrow[dr, "r_j"'] & \prod_{j \in J} D_j \arrow[d, "p_j"] \\ & D_j \end{tikzcd}$$

uniquely determined by r_j

$$\begin{tikzcd} E \arrow[r, "e"] & \prod_{j \in J_0} D_j \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "p_r"] & S \arrow[d, "p_a"] \\ C \arrow[u, "h"] \arrow[ur, "k"] \arrow[r, "r_j"'] & D_j & {} \end{tikzcd}$$

To see that $k : C \to \prod_j D_j$ lifts to $h : C \to E$, we have to check that $f \circ k = g \circ k$ (then proj equaliser)

It is sufficient to check that (pointlywise) $\to$ proj equaliser

$$p_\alpha \circ f \circ k = \cancel{p_\alpha} \circ g \circ k \quad \forall \alpha : i \to j \text{ in } J$$

$LHS = p_j \circ k = r_j$

$RHS = D\alpha \circ p_i \circ k \overset{def\ D_\alpha}{=} D\alpha \circ r_i$

$$\begin{tikzcd} & C \arrow[dl, "p_i"'] \arrow[d, "r_j"] \\ D_i \arrow[r, "D\alpha"'] & D_j \end{tikzcd}$$

$\Rightarrow \exists\, h : C \to E$ with $e \circ h = k$.

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→ To see that h is cone morphism, check

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[dr, "r_j"'] & E \arrow[r, "e"] \arrow[d, "\varepsilon_j"] & \Pi D_j \arrow[dl, "p_j"] \\ & D_i & \end{tikzcd}$$

Enough to show: $p_j \circ e \circ h = r_j$
$p_j \circ e = \varepsilon_j$ } $\varepsilon_j \circ h = r_j$

→ Uniqueness by joint monicity of $(\varepsilon_j)_{j \in \mathbb{J}}$.

Prop 5.14 TFAE

(A) ~~Finite limits~~ $\mathbb{C}$ has finite limits
↕ just proved
(B) $\mathbb{C}$ has finite products and equalizers
↕ lemma p31.
(C) Binary products, terminal objects ($n=0$), and equalizers

(D) Pullbacks and a terminal object

Proof (D) ⇒ (C) Assume $\mathbb{C}$ has pullbacks and 1.
Products are given by "pb over 1"

$$\begin{tikzcd} P \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & 1 \end{tikzcd}$$

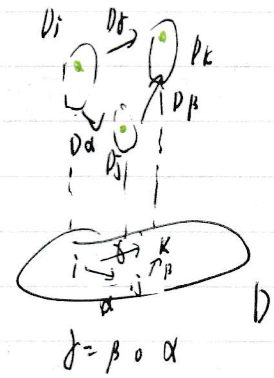
To get an equalizer of $f,g: A \to B$,

$$\begin{tikzcd} E \arrow[r, "\varepsilon"] \arrow[d, "e"'] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d, "\langle f,g \rangle"] \\ B \arrow[r, "\langle 1,1 \rangle"'] & B \times B \end{tikzcd}$$

Limits in Set:

$D: \mathbb{J} \to Set$ (small)

(equalizer) $E \overset{mono}{\rightarrowtail} \prod_j D_j \underset{g}{\overset{f}{\rightrightarrows}} \prod_{\alpha: i \to j} D_j$



a set (?)

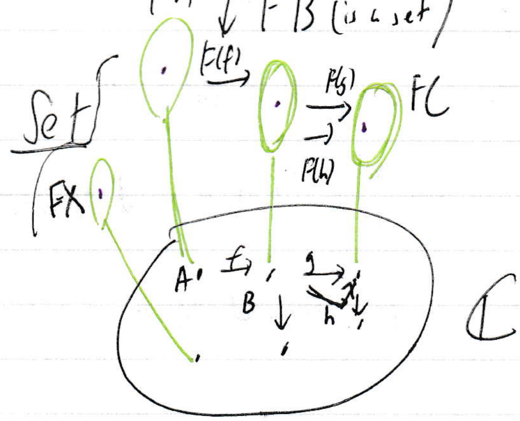
$(D_i)_{i \in \mathbb{J}}$ in E iff

? $D_\alpha(D_i) = D_j$ for all $\alpha: i \to j$

Reformulation...

(is a function)

$F: \mathbb{C} \to Set$



Choose one element of each fiber

limit of F

$$E \rightarrowtail \prod_{A \in \mathbb{C}_0} F(A) \underset{v}{\overset{u}{\rightrightarrows}}$$

$$p_f \circ u = p_B$$

$$p_f \circ v = Ff \circ p_A$$

$\lim F = E \cong \{ (x_A)_{A \in \mathbb{C}_0} \in \prod_{A \in \mathbb{C}_0} FA \mid \forall (f: A \to B) \in \mathbb{C}_1,\ F(f)(x_A) = x_B \}$

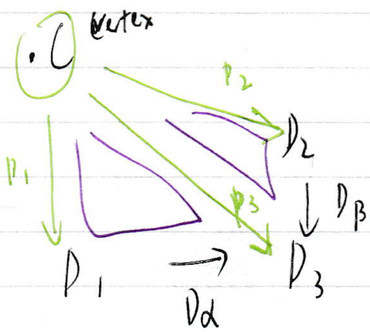
Prop. Pullbacks are Limits (OH 10/20)

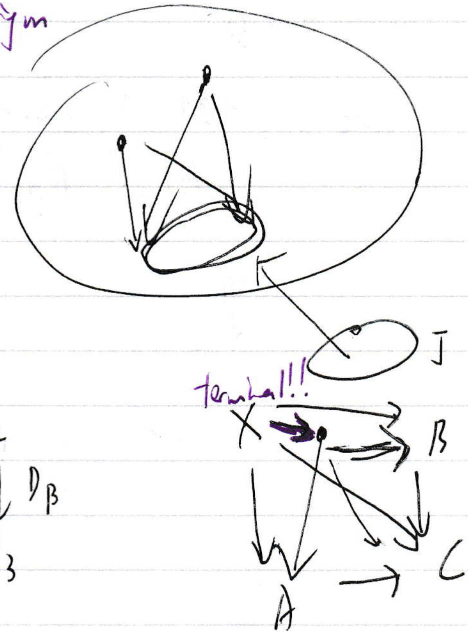
defines what we are projecting on

$$\mathbb{J} = \left(\begin{array}{ccc} & & 2 \\ & & \downarrow \beta \\ 1 & \xrightarrow{\alpha} & 3 \end{array} \right)$$

$$\downarrow D$$

$$\mathbb{C}$$





Recall:

- Diagram: $D: \mathbb{J} \to \mathbb{C}$
- Category $Cone(D)$
- Limit of D is a terminal object in $Cone(D)$
- Notation: Limits of D are denoted

$\lim(D)$ $\quad \lim\limits_{j\in\mathbb{J}} D_j$ $\quad (\varprojlim D \,/\, \varinjlim)$ colimit

$colim(\cdot)$

limits in Set $\quad D: \mathbb{J} \to Set.$ $\quad \lim D = \{ a \in \prod\limits_{j\in\mathbb{J}_0} D_j \mid \forall (\alpha: i \to j) \in \mathbb{J}_1, D_\alpha(a_i) = a_j \}$

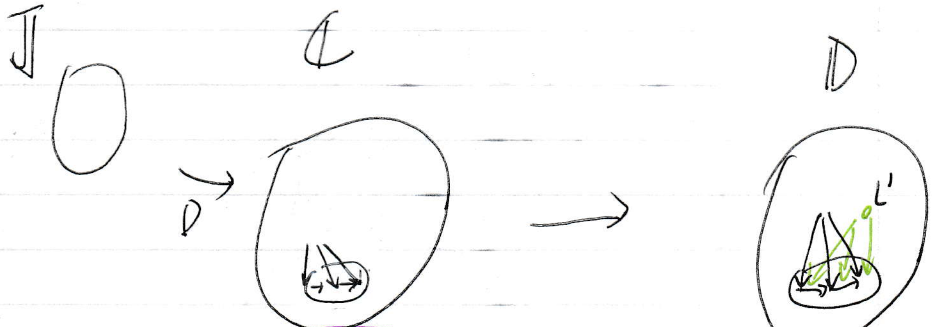
(see p72.)

$D: \mathbb{J} \xrightarrow{D} \mathbb{C} \xrightarrow{F} \mathbb{D}$, $\underbrace{\qquad}_{F \circ D}$

For Limit cone $(L, (p_j)_{j \in \mathbb{J}_0})$ in $\mathbb{C}$

cone $(FL, (Fp_j)_{j \in \mathbb{J}_0})$ in $\mathbb{D}$

We say that F preserves the limit of D if the cone $(FL, (Fp_j)_j)$ is a limit in $\mathbb{D}$.



All limit cones are isomorphic.

Prop: If $\mathbb{C}$ is locally small and $X \in \mathbb{C}_0$ then (covariant hom functor) $\hom(X, -) : \mathbb{C} \to Set$ preserves all limits that exist in $\mathbb{C}$.

$Hom(C, A \times B) \cong Hom(C, A) \times Hom(C, B)$

$$\begin{tikzcd} & C \arrow[d, "h"] \arrow[dl, dashed] \arrow[dr, dashed] & \\ A & A \times B \arrow[l, "p_1"] \arrow[r, "p_2"'] & B \end{tikzcd}$$

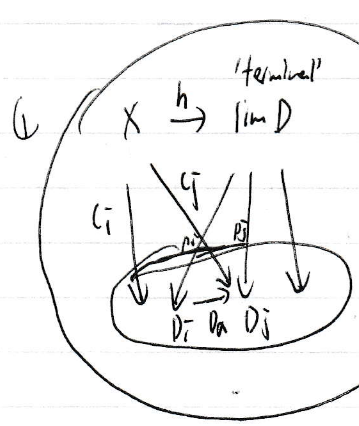
Proof. (Enough to show m is a bijection. This is true by UMP of lim(D))

$$\hom(X, \lim D) \overset{m}{\underset{SET}{\cong}} \lim_{j \in \mathbb{J}} \hom(X, D_j) = \{ c \in \prod_j \hom(X, D_j) \mid \forall (\alpha: i \to j) \in \mathbb{J},\ D\alpha \circ c_i = c_j \}$$

consistent with p72 and p73

'naive map'

$\lim (\hom(X, -) \circ D) : \mathbb{J} \to Set$ $= Cone(X, D)$ "cones with vertex X to D"



$m(h) := (p_j \circ h)_{j \in \mathbb{J}_0}$ is the mapping from unique morphism to unique morphism

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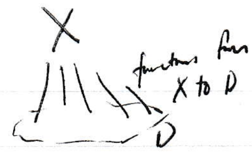
$$\begin{tikzcd} \hom(X, \lim D) \arrow[r, "\text{unique! by lims}"] \arrow[d, "p_i"'] \arrow[drr, "p_j"] & \lim_j \hom(X, D_j) \arrow[dl] \arrow[dr] & \\ \hom(X, D_i) \arrow[rr, "D\alpha"'] & & \hom(X, D_j) \end{tikzcd}$$

$\lim_j \hom(X, D_j) = \{(a_j) \in \prod_j \hom(X, D_j) \mid \forall \alpha : i \to j,\ D\alpha \circ a_i = a_j\}$



Dually, $\hom(-, X) : \mathbb{C}^{op} \Rightarrow \mathrm{Set}$ maps arbitrary colimits in $\mathbb{C}$ to limits in Set.

(limits in $\mathbb{C}^{op}$)

Colimits

Given a diagram $D : \mathbb{J} \to \mathbb{C}$, a cocone from/under D consists of an object $C \in \mathbb{C}$ and a family $(\sigma_j)_{j \in \mathbb{J}_0}$ of injection ($\leftrightarrows$ projection) maps such that

$$\begin{tikzcd} D_i \arrow[rr, "D\alpha"] \arrow[dr, "\sigma_i"'] & & D_j \arrow[dl, "\sigma_j"] \\ & C & \end{tikzcd}$$

commutes for all $\alpha : i \to j$ in $\mathbb{J}$.

A morphism of cocones from $(C, (\sigma_j)_j)$ to $(D, (\tau_j)_j)$ is an arrow $f : C \to E$ such that

$$\begin{tikzcd} & D_i \arrow[dl, "\sigma_i"'] \arrow[dr, "\tau_i"] & \\ C \arrow[rr, "f"'] & & E \end{tikzcd}$$

commutes for all $i \in \mathbb{J}_0$.

We write Cocone(D) for the category of cocones under D.

A colimit of D is an initial cocone.

↳ A category has small colimits iff it has small coproducts and small coequalizers (dual to previous theorem)

Pushouts

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "\sigma_2"] \arrow[ddr, bend left, "\tau_2"] & \\
B \arrow[r, "\sigma_1"'] \arrow[drr, bend right, "\tau_1"'] & P \arrow[dr, dashed, "h"] & \\
 & & X
\end{tikzcd}$$

alternative interpretation $\{z \in \mathbb{C} \mid |z|=1\}$

$$S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$$

$$1 \overset{0}{\underset{1}{\rightrightarrows}} [0,1] \xrightarrow{\text{epi}} S^1$$

$x \sim x' \iff x = x'$
OR $x'=1, x=0$
OR $x=1, x'=0$

'identify endpoints' =send to 1 $\rightarrow C$

$I/\sim$ is homeomorphic to S^1

Coequalize in Top
- Coequalizer in Set, and
- finest topology that makes the map continuous

'coproduct'

Ex 1.

$$\begin{tikzcd}
1+1 \arrow[r] \arrow[d] & {[0,1]} \arrow[d] \\
1 \arrow[r] & C
\end{tikzcd} \quad ??$$

$[0,1]+1$

identify extra point with both ends of the interval

Ex 2.

$$\begin{tikzcd}
8 \arrow[d] & S^1 \arrow[r, hook] \arrow[d, "z \mapsto z^2"] & B^2 = \{(x,y) \mid x^2+y^2 \le 1\} \\
O & S^1 \arrow[r] & ??
\end{tikzcd}$$

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Cartesian Closed Categories

- If A and B are sets, $\hom(A,B)$ is a set.

- If $(A, \leq)$, $(B, \leq)$ are preorders, $\hom_{Preord}((A,\leq),(B,\leq))$ is a set, but it admits a natural order

$$f, g : (A, \leq) \to (B, \leq) \text{ in Preord}$$

$$\text{Define } f \leq g \Leftrightarrow \forall a,\ fa \leq ga$$

- Given commutative monoids $(A,+,0)$, $(B,+,0)$, $\hom((A,+,0),(B,+,0))$ has a commutative monoid structure:

$$f, g : (A,+,0) \to (B,+,0)$$

$$(f+g)(a) = f(a) + g(a)$$

* 'Internal Hom'

If hom-sets in $\mathbb{C}$ can naturally be viewed as a $\mathbb{C}$-object, we speak of <u>internal homs</u>.

Given a function $f : A \times B \to C$, we can define

$$\tilde{f} : A \to \hom(B,C) = C^B$$

$$a \mapsto \tilde{f}(a) : B \to C$$

'currying' $\tilde{f}(a)(b) = f(a,b)$

For $g : A \to \hom(B,C)$, define $\bar{g} : A \times B \to C$ 'uncurrying'

$$\bar{g}(a,b) = g(a)(b)$$

* We have $\bar{\tilde{f}} = f$, $\tilde{\bar{g}} = g$.

→ Given a monotone map $f : \underbrace{(A,\le) \times (B,\le)}_{(A \times B, \le)} \to (C,\le)$

① the function $\overset{\tilde{f}(a)=}{f(a,-)} : B \to C$ is monotone for all $a \in A$.

(because $b \le b' \Rightarrow (a,b) \le (a,b') \Rightarrow f(a,b) \le f(a,b')$) (ε)

② the function $A \to \hom_{\text{preord}}((B,\le),(C,\le))$

$a \mapsto f(a,-) = \tilde{f}(a)$ is also monotone. $(A \to C^B)$

Proof. Let $a \le a'$ in $(A,\le)$

To show $f(a,-) \le f(a',-)$

$\iff \forall b,\ f(a,b) \le f(a',b)$

This follows from $(a,b) \le (a',b)$ and monotonicity of f.

→ However, for $f : (A,+,0) \times (B,+,0) \to (C,+,0)$,

$f(a,-)$ is not a homomorphism. (0 does not get mapped to 0)

Def: Let $\mathbb{C}$ be a category with binary products. An <u>exponential</u> of $B, C \in \mathbb{C}_0$ consists of an object E and an arrow $\varepsilon = \varepsilon^B_C : E \times B \to C$ such that $\forall f : A \times B \to C$, $\exists!\ \tilde{f} : A \to E$ such that $\varepsilon \circ (\tilde{f} \times 1_B) = f$

$$A \xrightarrow{\tilde{f}} E$$

$$\begin{tikzcd} A \times B \arrow[r, "\tilde{f} \times 1_B"] \arrow[dr, "f"'] & E \times B \arrow[d, "\varepsilon"] \\ & C \end{tikzcd}$$

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<u>Recall</u>

Let $\mathbb{C}$ be a category with finite products
(binary sufficient).

An <u>exponential</u> of $B, C \in \mathbb{C}_0$ is an object $C^B \in \mathbb{C}_0$ together with an arrow

$$(\varepsilon_C = \varepsilon_C^B =)\ \varepsilon : C^B \times B \to C \quad \text{"evaluation"}$$

Such that

($*$) for all objects A and arrows $f : A \times B \to C$, $\exists! \tilde{f} : A \to C^B$ s.t. $\varepsilon \circ (\tilde{f} \times 1_B) = f$.

$$\begin{tikzcd}
A \arrow[r, dashed, "\tilde{f}"] & C^B \\
A \times B \arrow[r, dashed, "\tilde{f} \times 1_B"] \arrow[dr, "f"'] & C^B \times B \arrow[d, "\varepsilon"] \\
& C
\end{tikzcd}$$

$\hookrightarrow$ "$\bar{\tilde{f}}$" (tilde then bar)

Observations

- $\tilde{f}$ is called the <u>transpose</u> of f
- For $g : A \to C^B$, write $\bar{g} = \varepsilon_C^B \circ (g \times 1_B) : A \times B \to C$
  (this is also called the <u>transpose</u>)

$$\boxed{\hom_{\mathbb{C}}(A \times B, C) \xrightarrow{\cong} \hom_{\mathbb{C}}(A, C^B)}$$
$$f \longmapsto \tilde{f}$$
$$\bar{g} \longleftarrow g$$

function from fn to fn

- ($*$) says precisely that $g \mapsto \bar{g}$ is a bijection for all A (check mutual inverse)
  right inverse exists from "left inverse"; just rename g

Def A cartesian closed category is a category which has finite products and exponentials for all $B, C \in \mathcal{C}_0$

Example 1: $\underline{Set}$ is cartesian closed.

- $C^B = \hom_{Set}(B, C) = \text{fun}(B, C)$
- $\varepsilon^B_C : C^B \times B \to C$
$(f, b) \mapsto f(b)$
- for $f: A \times B$, define
$\tilde{f} : A \to C^B$
$a \mapsto \tilde{f}(a)$
where $\tilde{f}(a) : B \to C$
$\tilde{f}(a)(b) = f(a,b)$

Check that $f \mapsto \tilde{f}$ & $g \mapsto \tilde{g}$ are mutually inverse to each other:

$\bar{\tilde{f}} = f \Leftrightarrow \forall a \in A, b \in B, \quad \bar{\tilde{f}}(a,b) = f(a,b)$

$LHS = (\varepsilon \circ (\tilde{f} \times 1_B))(a,b)$

$= \varepsilon((\tilde{f} \times 1_B)(a,b))$

$= \varepsilon(\tilde{f}(a), b)$

$= \tilde{f}(a)(b) = f(a,b)$

Conversely, check for $g : A \to C^B$ that

$\tilde{\bar{g}} = g : A \to C^B$

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W.T.S $\tilde{\tilde{g}} = g : A \to C^B$

$\iff \forall a, \; \tilde{\tilde{g}}(a) = g(a) : B \to C$

$\iff \forall a, \forall b, \; \tilde{\tilde{g}}(a)(b) = g(a)(b)$

$A \underset{g}{\overset{f}{\rightleftarrows}} B$ $\quad LHS = \varepsilon(\tilde{\tilde{g}}(a), b) = (\varepsilon \circ (\tilde{\tilde{g}} \circ 1_B))(a,b)$

suppose $g \circ f = 1$, g monomorphism

$= \bar{\tilde{\tilde{g}}}(a,b) = \bar{g}(a,b) = g(a)(b)$ by $\tilde{\tilde{f}} = f$

then f, g are inverses

since $f \circ g = 1$

$\iff g \circ f \circ g = g$ (left cancellation allowed!)

Examples of CCC's

- Pre-orders / (same construction,
- Poset need to check monotonicity) see p. 77-78

$(C, \leq)^{(B, \leq)} = (\mathrm{hom}_{\mathrm{preord}}((B,\leq),(C,\leq)), \leq)$

objects are morphisms, arrows are squares

$$\begin{array}{ccc} A & \to & C \\ \downarrow & & \downarrow \\ B & \to & D \end{array}$$

arrow category

- Arr(set)
- All presheaf cats (incl. poset) pg. 111

def comp. gen: compact Hausdorff space $C \xrightarrow{} X \to Y$ cont cont

- Compactly generated spaces (algebraic topology)

↳ Comp. gen. Hausdorff spaces

↳ comp. gen. weak Hausdorff

→ Scott Domains

↳ ω-CPOs : complete - partial order: poset $(A, \leq)$ where every increasing sequence $a_0 \leq a_1 \leq a_2 \leq \dots$ of elements has a least upper bound and $(A, \leq)$ has a least element $\bot$ 'bottom' ($\top$ = 'top')

$\{x < 2\}$: L.U.B = 2

Non-examples

, Commutative Monoids / Mon / Grp / Rings...

, Top (topological spaces)

(Domain Theory)

A continuous (Scott-continuous) function between ω-CPOs is a monotone function

$$f : (A, \leq) \to (B, \leq)$$

such that $\sup_{i \in \mathbb{N}} f(a_i) = f(\sup_{i \in \mathbb{N}} a_i)$

for all increasing sequences $(a_i)_{i \in \mathbb{N}}$.

Call ω-CPO the category of ω-CPOs & cont. functions.

$$(A, \leq) \xrightarrow{f} (A, \leq)$$

$$\bot \leq f(\bot) \leq f(f(\bot)) \leq \dots$$

get supremum(?)

least fixed point, Y-combinator...

(a) defining recursion

(see medium bookmark)

$\mathrm{fib} : \mathbb{N} \to \mathbb{N}$

better partial functions

computes

$\mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$

fixed point

Y-combinator: generates fixed points

fibonacci = fibonacciFactory (fibonacci)

→ reaching fixed point = finishing calculation

(b) any form of recursion/while loop is a fixed point in computation

'thin'

When is a preorder cat cartesian closed?

$(A, \leq)$ gives cat $\underline{(A, \leq)}$.

When does $(A, \leq)$ have finite products?

$$\begin{tikzcd} & A \\ X \arrow[ur] \arrow[r] \arrow[dr] & A \times B \arrow[u] \arrow[d] \\ & B \end{tikzcd} \qquad \begin{array}{ccc} & & a \\ & \leq & \mathrel{\rotatebox{90}{$\leq$}} \\ x & \leq & a \times b \\ & \leq & \mathrel{\rotatebox{90}{$\geq$}} \\ & & b \end{array}$$

Observation: $\underline{(A, \leq)}$ has binary products iff $(A, \leq)$ has greatest lower bounds.

$a \times b$ is the 'greatest lower bound'.

$\underline{(A, \leq)}$ has a terminal object iff $(A, \leq)$ has a greatest element.

notation: $A \to 1$, $a \leq 1$

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- Greatest lower bounds are known as infima (infimum) or <u>meets</u>

  Write $a \wedge b$ instead of $a \times b$
  'meets'

- Write T instead of 1 for terminal object.

- Dually, binary coproducts in preorder cats are l.u.b's/suprema/joins

  write $a \vee b$ instead of $a + b$.
  'joins'

Def. A Heyting algebra is a poset $(A, \leq)$ with finite meets and joins $(\vee, \wedge, \bot, T)$ such that for all elements b, c, there exists an element $(b \Rightarrow c)$ c^b s.t. Heyting implication

$$\forall a \in A, \quad \underset{\text{'product'}}{a \wedge b} \leq c \text{ iff } a \leq [b \Rightarrow c]$$

$a \wedge b \leq c$ implies $a \leq b \Rightarrow c$

Observation: $\hom(A \times B, C) \cong \hom(a, c^b)$

thin category: card $= 1$ or 0

$\bar{g} ::= \varepsilon \circ (g \times 1_B) : A \times B \to C$

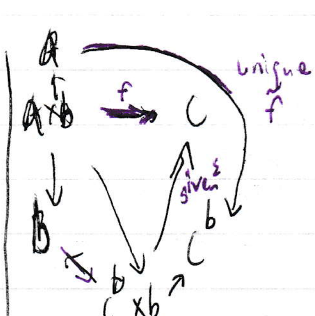
Observation: A poset $(A, \leq)$ is a Heyting algebra iff

$(A, \leq)$ is cartesian closed and has finite products.

Note: A Boolean Algebra is six-tuple $(A, \wedge, \vee, \neg, 0, 1)$ $(\bot)(T)$

Define $a \leq b$ iff $a = b \wedge a \Leftrightarrow$ equivalent $a \vee b = b$

$b^a : (\neg a \vee b)$



$(\Rightarrow)$ unique $\tilde{f}$ only

$(\Leftarrow)$ given $\tilde{f}$, can get back f.

$(X \to A)$ 10/28/2021

(Prop 6.7)

Given a fixed object X in a CCC $\mathbb{C}$, the assignment

$A \mapsto A^X$ extends to a functor
(exponent)

$E_A : \mathbb{C} \to \mathbb{C}$

Proof. The morphism part has to send arrows $f : A \to B$ to arrows

$$E_A(f) : A^X \to B^X$$

$$\hom(A^X, B^X) \cong \hom(A^X \times X, B)$$

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$$\hom(A \times B, C) \cong \hom(A, C^B)$$

$$\bar{g} := \varepsilon^B_C \circ (g \times 1_B) \leftarrow\!\shortmid \ g$$

$$f \mapsto \tilde{f}$$

$$A^X \times X \xrightarrow{\varepsilon^X_A} A \xrightarrow{f} B$$

Define $E_A(f) := \widetilde{f \circ \varepsilon^X_A}$

① $E_X(1_A) = 1_{A^X} \overset{\text{def}}{\Leftrightarrow} \widetilde{1_A \circ \varepsilon^X_A} = 1_{A^X}$

$\overset{\text{bijectivity}}{\Leftrightarrow} \quad 1_A \circ \varepsilon^X_A = \overline{1_{A^X}} \overset{\text{def}}{=} \varepsilon^X_A \circ (\underbrace{1_{A^X} \times 1_X}_{1 \text{ in } A^X \times A^X})$

② $A \xrightarrow{f} B \xrightarrow{g} C$

$$E_X(g \circ f) = E_X(g) \circ E_X(f)$$

$$\Leftrightarrow \widetilde{g \circ f \circ \varepsilon} = \widetilde{g \circ \varepsilon} \circ \widetilde{f \circ \varepsilon}$$

$$\Leftrightarrow g \circ f \circ \varepsilon = \overline{\widetilde{g \circ \varepsilon} \circ \widetilde{f \circ \varepsilon}} = \varepsilon \circ ((\widetilde{g \circ \varepsilon} \circ \widetilde{f \circ \varepsilon}) \times 1)$$

$(h \times k) \circ (f \times g) = (h \circ f) \times (k \circ g)$

Hw 3d.

$$= \varepsilon \circ ((\widetilde{g \circ \varepsilon} \times 1) \circ (\widetilde{f \circ \varepsilon} \times 1))$$

$$= g \circ \varepsilon \circ (\widetilde{f \circ \varepsilon} \times 1)$$

$$= g \circ f \circ \varepsilon = \text{LHS}! \ ]$$

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Analogously, for $X \in \mathcal{C}_0$

$\begin{cases} \hom(X,-) : \mathcal{C} \to \mathrm{Set} & \text{was the covariant hom-functor} \\ E_X : \mathcal{C} \to \mathcal{C} & \end{cases}$

Now we have 'internal hom functor'

$\begin{cases} \hom(-,X) : \mathcal{C}^{op} \to \mathrm{Set} \\ X^{(-)} = E_X' : \mathcal{C}^{op} \to \mathcal{C} \end{cases}$

$\Downarrow$

$\mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ (in contrast with $\hom_{\mathcal{C}}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Set}$)

'bifunctor lemma?'
$F : A \times B \to C$ is functor
iff F is functorial in each argument

$$\begin{array}{ccc} F(A,B) & \xrightarrow{F(A,\beta)} & F(A,B') \\ \downarrow F(\alpha,B) & & \downarrow F(\alpha,B') \\ F(A',B) & \xrightarrow{F(A',\beta)} & F(A',B') \end{array}$$

$\boxed{\lambda\text{- Calculus}}$ (strong connections to C.C.C's)

→ functional analysis
→ dual lemma

For A, B sets, we have the function $A \to B^{(B^A)}$, $a \mapsto \varphi_a$, where

$\varphi_a : B^A \to B$
$\varphi_a(f) = f(a)$

(e.g. $V \to \mathrm{Lin}(k^V, V)$)
linear maps between vector spaces

e.g. ① $a \longmapsto (\underbrace{f \mapsto f(a)}_{\text{anonymous function}})$

$\underbrace{B^{(B^{(B^A)})}} \to B^A$ $(A \to B)$

$\varphi \mapsto (a \mapsto \varphi(f \mapsto f(a)))$

$$\underbrace{A \quad \underbrace{\underbrace{B^A \quad B}_{B^{(B^A)}}}_{B}}_{B^A}$$

The ~~lambda~~ λ-calculus notation for $(a \mapsto t)$ is $(\lambda x . t)$

Let's start with <u>Untyped pure λ-calculus</u>

Terms: · countably many variables (var) $x, y, z \ldots$ (vars are terms)

· If s and t are terms, then

$(s\,t)$ is a term called application. $f(a) \Leftrightarrow (f\,a)$

λ-calcul? $(f)_a$ c'est moi?

· If t is a term and x a var,

$(\lambda x . t)$ is a term

(can add constants, but not pure anymore)

$(2, 4)$

$(3, -6)$

$\frac{E(XY) - E(X)E(Y)}{\sqrt{Var(X) Var(Y)}}$

$\|x\| = E(x)$ $\sqrt{(2^2+4^2)} \cdot \sqrt{(3^2+(6^2))}$

$x \cdot y = E(XY)$

$= \sqrt{20} \cdot \sqrt{45}$

Examples $(\lambda x . (\lambda y . \underbrace{(\lambda z . (x(yz)))}_{z \mapsto x(yz)}))$

$\forall x, y . \ x+y = y+x$ 'binding'

(Shorthand) $= \lambda xyz . \ x(yz)$

* we fill in parentheses from the left

$s\,t\,u\,v = (((st)u)v)$

* Combine successive λ's: $\lambda xyz . t = (\lambda x (\lambda y (\lambda z . t)))$

$(\lambda x . x) =_\alpha (\lambda y . y)$

$\boxed{\text{Alpha}^{\alpha}\text{-Equivalence}}$: Identify terms that differ only on names of bound variables

so we can write $(\lambda x . xy) =_\alpha (\lambda z . zy)$, $(u(\lambda x . xy)) =_\alpha (u(\lambda z . zy))$

$=_\alpha$ is the equivalence relation generated by renaming

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$\boxed{\beta\text{- reduction}}$

$$(\lambda x.t)\,u \to_\beta t[u/x]$$ 'syntactic substitution'

substitute u for x in t

e.g. $(\lambda x.\,x+x)\,3 \to_\beta 3+3$

$(\lambda x(\lambda y.\dots))$

$$(\lambda xy.\,xy)(\lambda xy.\,xy) \to_\beta (\lambda y.\,(\lambda xy.\,xy)\,y) \overset{?}{\to}_\beta (\lambda y.\,(\lambda y.\,yy))$$

Bad reduction — capturing y — different y's!

Solution: Before substituting "under a λ", rename the bound variable to a new (fresh) one.

$(\lambda xy.\,xy)(\lambda xy.\,xy)$

~~$(\lambda x\,(\lambda y.\,xy))(\lambda xy.\,xy)$~~

$(\lambda x.\,(\lambda y.\,xy))(\lambda xy.\,xy)$ "capture-free substitution"

$(\lambda x.\,(\lambda z.\,xz))(\lambda xy.\,xy) \to (\lambda z.\,(\lambda xy.\,xy)\,z) \to \lambda z.\,(\lambda y.\,zy)$

$\boxed{\beta\text{- equality}}$ $=_\beta$ is the equivalence relation generated by β-reduction. $= \lambda zy.\,zy$

$\boxed{\eta\text{ - conversion:}}$ $t =_\eta (\lambda x.\,tx)$ if x is not a free var in t

$f: A \to B$
$(x \mapsto f(x))$

β-η-equality $=_{\beta\eta}$ is the equivalence relation on terms generated by $=_\beta$ & $=_\eta$

Annoyances:

$f(f)$ $(\lambda x.\, xx)(\lambda x.\, xx) \xrightarrow[\beta]{} (\lambda x.\, xx)(\lambda x.\, xx)$

To be Turing complete, you must accept non-termination

Simply-typed λ-calculus

(a type represents sets, or more generally objects in a category)
int $x = 1$

Postulate basic type symbols $\gamma, \delta, \varepsilon \ldots$

- Every basic type is a type
- If α and β are types, then $(\alpha \to \beta)$ is a type
- If α, β are types, then $(\alpha \times \beta)$ is a type (product types)
- 1 is a type (unit type)

Convention: $\sigma_1 \to \sigma_2 \to \sigma_3 \to \sigma_4 = (\sigma_1 \to (\sigma_2 \to (\sigma_3 \to \sigma_4)))$

(Notes courtesy AJ Rausch) 11/2/2021

November 2, 2021

Recall:

Untyped λ-terms: $s, t ::= x \mid (s\ t) \mid (\lambda x. t)$

Types of simply-typed λ-terms: $\sigma, \tau ::= \underbrace{\gamma, \delta, \ldots}_{\text{base types}} \mid \underbrace{(\sigma \to \tau)}_{\text{function types}} \mid \underbrace{(\sigma \times \tau)}_{\text{product types}} \mid \underbrace{1}_{\text{unit type}}$ e.g. $()$: unit in SML

Typed λ-terms:

$$\underbrace{\underbrace{x_1 : \sigma_1, \ldots, x_n : \sigma_n}_{\substack{\text{context} \\ (\Gamma, \Delta, \ldots)}} \vdash t : \tau}_{\text{term-in-context (t-i-c)}}$$

turnstile

"t is a term of type τ in context Γ"

Well-typed t-i-c's are derived with the following rules:

$\Gamma \vdash t : \tau$

in certain context, a certain term has a certain type

$$\frac{}{x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash x_i : \sigma_i} \quad \text{for } 1 \leq i \leq n$$

$$\frac{\Gamma, x : \sigma \vdash t : \tau}{\Gamma \vdash (\lambda x : \sigma . t) : \sigma \to \tau} \quad \text{(Arrow introduction) (abstraction)}$$

implies (context changes!)

$$\frac{\Gamma \vdash s : \sigma \quad \Gamma \vdash t : \sigma \to \tau}{\Gamma \vdash (t\ s) : \tau} \quad \text{(Arrow elimination) (Application)}$$

$$\frac{\Gamma \vdash s : \sigma \quad \Gamma \vdash t : \tau}{\Gamma \vdash (s, t) : \sigma \times \tau}$$

* Observation: structure of derivations is encoded in the terms.

$$\frac{\Gamma \vdash t : \sigma \times \tau}{\Gamma \vdash \mathrm{fst}(t) : \sigma} \qquad \frac{\Gamma \vdash t : \sigma \times \tau}{\Gamma \vdash \mathrm{snd}(t) : \tau}$$

$$\frac{\text{unique axiom / term (no information)}}{\Gamma \vdash * : 1}$$

Equational rules for pairs and $*$:

$$\left.\begin{array}{lll} \mathrm{fst}(s,t) & =_\beta & s \\ \mathrm{snd}(s,t) & =_\beta & t \\ (\mathrm{fst}(t), \mathrm{snd}(t)) & =_\eta & t \\ * & =_\eta & t \end{array}\right\} \otimes \text{ (both sets of rules)}$$

There is only one value of unit type

Recall: $(\lambda x . s) t =_\beta s[t/x]$

↓ typed context

$(\lambda x : \sigma . s) t =_\beta s[t/x]$ substitute t for x in s

$t =_\eta (\lambda x : \sigma . t\ x)$

$=_{\beta\eta}$ is the family of equivalence relations on well-typed terms in context generated by the given list $\otimes$.

$\Gamma \vdash \_ : \tau$

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Interpretation in a category:

fix some "chosen ones".

Let $\mathbb{C}$ be a CCC equipped with choices of
- a terminal object 1
- a binary product $A \times B$ for all pairs $A, B \in \mathbb{C}_0$
- an exponential B^A for all pairs $A, B \in \mathbb{C}_0$

Fix an object A_γ for every base type γ.

Define the interpretation $[\![\tau]\!] \in \mathbb{C}_0$ by induction as follows:
- $[\![\gamma]\!] = A_\gamma$
- $[\![\sigma \to \tau]\!] = [\![\tau]\!]^{[\![\sigma]\!]}$
- $[\![\sigma \times \tau]\!] = [\![\sigma]\!] \times [\![\tau]\!]$
- $[\![1]\!] = 1$

$\Gamma \rightsquigarrow [\![\Gamma]\!]$

Interpretation of contexts
- $[\![x_1:\sigma_1, \dots, x_n:\sigma_n]\!] = 1 \times [\![\sigma_1]\!] \times \dots \times [\![\sigma_n]\!]$

Fix parenthesation: e.g. $1 \times A \times B \times C = (((1 \times A) \times B) \times C)$

Well-typed terms in context $\Gamma \vdash t : \tau$ should be interpreted by morphisms
$\underbrace{[\![\Gamma \vdash t:\tau]\!]}_{\text{abbreviate } [\![t]\!]} : [\![\Gamma]\!] \to [\![\tau]\!]$

$[\![\Gamma \vdash *: \underset{\text{type}}{1}]\!] : [\![\Gamma]\!] \to [\![1]\!] = \overset{\text{object}}{1}$

Example: $x : (\rho \times \sigma) \to \tau \vdash \lambda y:\rho.\ \lambda z:\sigma.\ x\,(y,z) : \tau$

$[\![\Gamma, x:\sigma \vdash t:\tau]\!] : [\![\Gamma]\!] \times [\![\sigma]\!] \to [\![\tau]\!]$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash y:\rho \quad \Gamma \vdash z:\sigma}{\Gamma \vdash (y,z):\rho\times\sigma} \qquad \Gamma \vdash x:(\rho\times\sigma)\to\tau}{\underbrace{x:(\rho\times\sigma)\to\tau,\ y:\rho,\ z:\sigma}_{\Gamma} \vdash x\,(y,z):\tau}}{x:(\rho\times\sigma)\to\tau,\ y:\rho \vdash \lambda z.\sigma\ x\,(y,z):\sigma\to\tau}}{x:(\rho\times\sigma)\to\tau \vdash \lambda y:\rho.\ \lambda z:\sigma.\ x\,(y,z):\rho\to(\sigma\to\tau)}$$

$\longleftarrow$ only one choice for each derivation going upwards

$[\![\sigma]\!]$

$[\![\Gamma \vdash (\lambda x:\sigma.t):\sigma\to\tau]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$

what is it supposed

Interpretation of well-typed t-i-c's defined by induction of terms, or equivalently, derivations.

Variable rule:

$$[\![x_1:\sigma_1, \dots, x_n:\sigma_n \vdash x_i:\sigma_i]\!] \overset{\text{interpretation}}{:} \overbrace{1 \times [\![\sigma_1]\!] \times \dots \times [\![\sigma_n]\!]}^{\text{parenthesized: actually, take } p_2 \circ p_1 \circ p_1 \circ \cdots} \to [\![\sigma_i]\!]$$

last one take p_2

$(\dots(1 \times [\![\sigma_1]\!]) \times \dots)$

Abstraction rule:

$$[\![\Gamma \vdash (\lambda x:\sigma.t):\sigma\to\tau]\!] = \widehat{[\![\Gamma, x.\sigma \vdash t.\tau]\!]} : [\![\Gamma]\!] \to [\![\tau]\!]^{[\![\sigma]\!]}$$

'curry'

$$[\![\Gamma \vdash (s\ t):\tau]\!] = \left([\![\Gamma]\!] \xrightarrow{\langle [\![\Gamma\vdash s:\sigma\to\tau]\!],\ [\![\Gamma\vdash t:\sigma]\!]\rangle} [\![\tau]\!]^{[\![\sigma]\!]} \times [\![\sigma]\!] \xrightarrow{\varepsilon} [\![\tau]\!]\right)$$

$[\![\Gamma]\!] \times [\![\sigma]\!] \to [\![\tau]\!]$

under the hood

PO

$[\![\Gamma \vdash (s,t) : \sigma \times \tau]\!] = \langle [\![\Gamma \vdash s : \sigma]\!], [\![\Gamma \vdash t : \tau]\!] \rangle : [\![\Gamma]\!] \to [\![\sigma]\!] \times [\![\tau]\!]$

$[\![\Gamma \vdash \mathrm{fst}(t) : \sigma]\!] = p_1 \circ [\![\Gamma \vdash t : \sigma \times \tau]\!]$

$[\![\Gamma \vdash \mathrm{snd}(t) : \tau]\!] = p_2 \circ [\![\Gamma \vdash t : \sigma \times \tau]\!]$

$[\![\Gamma \vdash * : 1]\!] : [\![\Gamma]\!] \xrightarrow{!_{[\![\Gamma]\!]}} 1$ (unique arrow)

there's only one way to do this...

Example: from before,

$$① \quad [\![x : (\rho \times \sigma) \to \tau \vdash \overbrace{\lambda y : \rho .\, \lambda z : \sigma .\, x\,(y, z) : \rho \to (\sigma \to \tau)}^{\text{Call this } s}]\!] : [\![\tau]\!]^{[\![\rho]\!] \times [\![\sigma]\!]} \to \left([\![\tau]\!]^{[\![\sigma]\!]}\right)^{[\![\rho]\!]}$$

Example:

$$x : \rho \to \sigma \to \tau \vdash \overbrace{(\lambda y : \rho \times \sigma .\, x\,(\mathrm{fst}\ y)(\mathrm{snd}\ y))}^{\text{Call this } t} : \rho \times \sigma \to \tau$$

$$② \quad [\![x : \rho \to \sigma \to \tau \vdash (\lambda y : \rho \times \sigma .\, x\,(\mathrm{fst}\ y)(\mathrm{snd}\ y)) : \rho \times \sigma \to \tau]\!] : \left([\![\tau]\!]^{[\![\sigma]\!]}\right)^{[\![\rho]\!]} \to [\![\tau]\!]^{[\![\rho]\!] \times [\![\sigma]\!]}$$

Claim: ① and ② are mutual inverses $\Rightarrow$ we have an isomorphism $[\![\tau]\!]^{[\![\rho]\!] \times [\![\sigma]\!]} \cong \left([\![\tau]\!]^{[\![\sigma]\!]}\right)^{[\![\rho]\!]}$

Claim: $\quad s = \lambda y : \rho .\, \lambda z : \sigma .\, x\,(y, z)$

$$x : \rho \to \sigma \to \tau \vdash \underbrace{s[t/x]}_{=_{\beta\eta}\, x\, ?} : \rho \to \sigma \to \tau$$

substitute t for x in s

Weakening lemma.

If $\Gamma, \Delta \vdash t : \tau$ is well-typed, then

$$[\![\Gamma, \overset{\text{fresh}}{x : \sigma}, \Delta \vdash t : \tau]\!] \neq [\![\Gamma, \Delta \vdash t : \tau]\!]$$

move in this order

$$\begin{tikzcd}
{[\![\Gamma, x:\sigma, \Delta]\!]} \arrow[rrd, bend left, "f"] \arrow[d, "i"', "\cong"] & & \\
{[\![\Gamma]\!] \times [\![\sigma]\!] \times [\![\Delta]\!]} \arrow[d, "\pi"] & & {[\![\tau]\!]} \\
{[\![\Gamma]\!] \times [\![\Delta]\!]} \arrow[r, "\cong", "j"'] & {[\![\Gamma, \Delta]\!]} \arrow[ur, "g"'] &
\end{tikzcd}$$

name

$[\![\Gamma, \Delta \vdash t : \tau]\!]$

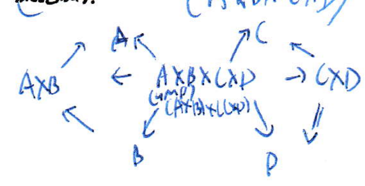
$\boxed{f = g \circ j \circ \pi \circ i.}$ Proof omitted (complicated induction).

$[\![\Gamma]\!] \times [\![\Delta]\!] \cong [\![\Gamma, \Delta]\!]$

$(((1 \times \sigma_1) \times \ldots) \times \sigma_n) \times ((1 \times \tau_1) \times \ldots \tau_k)$

$\cong ((1 \times \sigma_1) \times \ldots \sigma_n \times \tau_1 \times \tau_2 \times \ldots \tau_k)$

since $(A \times B) \times (C \times D) \overset{\text{ump}}{\cong} (A \times B \times (C \times D))$



Substitution lemma:
Given terms in context

$$\Gamma \vdash s_1 : \sigma_1 \quad \ldots \quad \Gamma \vdash s_n : \sigma_n$$

and

$$y_1 : \sigma_1, \ldots, y_n : \sigma_n \vdash t : \tau \ ?!$$

$\leftarrow$ not important

then $[\![t[s_1/x_1, \ldots, s_n/x_n] : \tau]\!] = [\![\_ \vdash t : \tau]\!] \circ \langle [\![\Gamma \vdash s_1 : \sigma_1]\!], \ldots, [\![\Gamma \vdash s_n : \sigma_n]\!] \rangle$
"substitution is composition"
Proof: By structural induction on t

Soundness theorem: given terms $\Gamma \vdash s:\tau$, $\Gamma \vdash t:\tau$ such that $s =_{\beta\eta} t$ then "interpretation of" $[\![\Gamma \vdash s:\tau]\!] = [\![\Gamma \vdash t:\tau]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$

$\boxed{\text{HW 8 Q 5 d Alt. proof}}$

$f : A \to 0$

$$\begin{tikzcd} & A \\ A \arrow[ur, "1"] \arrow[r, "\langle 1, f\rangle"] \arrow[dr, "f"'] & A \times 0 \arrow[u, "p_1"'] \arrow[d, "p_2"] \\ & 0 \end{tikzcd}$$

$\langle 1, f\rangle$ a split mono

~~Mike's Idea:~~ $\cancel{0 \times A} \to 0 \times A$

$$\begin{tikzcd} 0 \times A \arrow[dr, "1_0 \times f"'] & & 0 \times A \\ & 0 \times 0 \arrow[ur, "1_0 \times g"'] & \end{tikzcd}$$

doesn't lead to desired conclusion

$(\lambda x. t)u =_\beta t[u/x]$

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More λ-Calculus.

$\boxed{\begin{array}{l} \text{Prop: Let } A, B, C \text{ be objects of CCC } \mathbb{C}. \text{ Then} \\ \text{(i)} \quad (C^B)^A \cong C^{A\times B} \cong (C^A)^B \\ \text{(ii)} \quad (B\times C)^A \cong B^A \times C^A \end{array}}$

Proof of (i)

Define morphisms as lambda terms: Consider terms

$$[x : \alpha \to (\beta \to \gamma) \vdash \underbrace{(\lambda z : \alpha \times \beta .\ x(fst(z))(snd(z)))}_{s} : (\alpha\times\beta) \to \gamma\ ]$$

$$[\ y : (\alpha\times\beta) \to \gamma \vdash (\underbrace{\lambda u : \alpha .\ (\lambda v : \beta .\ y(u,v)}_{t})) : \alpha \to \beta \to \gamma\ ]$$

Set $A_\alpha = [[\alpha]] = A$, $A_\beta = [[\beta]] = B$, $A_\gamma = [[\gamma]] = C$

Define $f := [[x : \alpha \to \beta \to \gamma \vdash s : (\alpha\times\beta) \to \gamma]] : (C^B)^A \to C^{A\times B}$

$g := [[y : \alpha\times\beta \to \gamma \vdash t : \alpha \to \beta \to \gamma]] : C^{A\times B} \to (C^B)^A$

Enough to show that $f\circ g = 1$, $g \circ f = 1$

$$f \circ g \overset{\text{substitution}}{\underset{\text{lemma}}{=}} [[y : \alpha\times\beta \to \gamma \vdash s[t/x] : \alpha\times\beta \to \gamma]] = [[y \vdash y]] = 1$$

$[[t[s_1/x] : \tau]]$
$= [[\_ \vdash t : \tau]] \circ [[\Gamma \vdash s_1 : \sigma_1]]$

$$s[t/x] = \lambda z(\lambda u.(\lambda v.\ y(u,v))(fst(z))(snd(z)))$$
$$=_\beta \lambda z(\lambda v.\ y(fst(z), v)(snd(z)))$$
$$=_\beta \lambda z.\ y(fst(z), snd(z))$$
$$=_\beta \lambda z.\ yz$$
$$=_\beta y$$

$\left[\begin{array}{l}\text{Conversely, show that} \\ t[s/y] =_{\beta\eta} x\end{array}\right]$

(+)

Remark: If $\mathbb{C}$ has coproducts, we also have

'commuting conversions'

$$C^{A+B} \cong C^A \times C^B$$

$$(A+B)\times C \cong (A\times C) + (B\times C)$$

[Completeness of Interpretation of Simply Typed λ-calculus in Cartesian Closed Categories.]

(Theorem) Given terms $\Gamma \vdash s:\tau$, $\Gamma \vdash t:\tau$ (for base types $\alpha, \beta, \gamma, \dots$) such that $[[\Gamma \vdash s:\tau]] = [[\Gamma \vdash t:\tau]]$ for all CCCs $\mathbb{C}$, (in all models) and chosen interpretations $A_\alpha, A_\beta, A_\gamma \dots$ of base types, then $\Gamma \vdash s =_{\beta\eta} t:\tau$

(Proof) Via interpretation in syntactic category $\mathbb{C}$

→ Objects of $\mathbb{C}$: types constructed from the fixed base types

syntax lives length 1

→ Morphisms from σ to τ are β-η-equivalence classes of terms $x:\sigma \vdash t:\tau$

→ Id's $(x:\sigma \vdash x:\sigma)$

'General case'

→ Composition : Substitution

Equality in cat → $\beta\mathcal{E}$ equality in 'real-life'!!

interpretation = identity   abbreviation $[[s]] = [[t]] \Rightarrow s =_{\beta\eta} t$

If $s \neq_{\beta\eta} t$, can we find model where $[[s]] \neq [[t]]$

$[[x:\sigma, y:\tau \vdash t:\rho]]$

$= ((1\times[[\sigma]])\times[[\tau]]) \to [[\rho]]$

$$\frac{\Gamma \vdash \cancel{t}\ \sigma \Rightarrow \tau \quad \Gamma \vdash s:\sigma}{\Gamma \vdash (ts):\tau}$$ Modus Ponens

"Two rules" $$\frac{\Gamma, x:\sigma \vdash t:\tau}{\Gamma \vdash (\lambda x:\sigma.t):\sigma \Rightarrow \tau}$$

see Oxford paper

$\times$	$\wedge$
1	$\top$
$+$	$\vee$
0	$\perp$

Simply typed λ-calculus $\triangleq$ IPC (intuitionistic propositional logic)

Curry-Howard Isomorphism (Correspondence)