Continuity

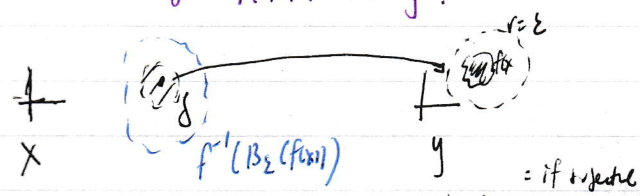
$f: X \to Y$. let (X,d), (Y,ρ) be metric spaces.

① ⟺ ②: f is $\boxed{\text{continuous at } x \in X}$ iff

** ① $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$

② $\forall \varepsilon > 0, \exists \delta > 0$ s.t. if $|x-y| < \delta \Rightarrow |f(y) - f(x)| < \varepsilon$.

$= \delta(\varepsilon, x, f)$ 3 things!



$f \circ f^{-1}(B) \subseteq B$ = if surjective

$f^{-1} \circ f(B) \supseteq B$ = if injective

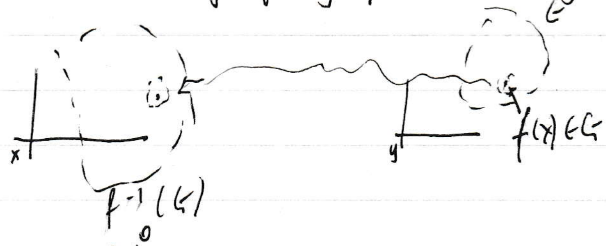
③ $\forall \varepsilon > 0$ $\exists \delta > 0 \ni f^{-1}(B_\varepsilon(f(x))) \supseteq B_\delta(x)$ (preimage!)

⇓

* ④ The inverse image of any ball centered at $f(x)$ contains a ball around x.

⇓

⑤ The inverse image of any open set containing $f(x)$ contains x as an interior point.



$x \in f^{-1}(G)^{\circ}$

Prove ⑤ ⇒ ① let $\varepsilon > 0$. Then define $B := B_\varepsilon(f(x)) \Rightarrow x \in f^{-1}(B)$, x is an interior point of $f^{-1}(B) \Rightarrow \exists \delta > 0$ s.t. $B_\delta(x) \subseteq f^{-1}(B)$

$\Rightarrow f(B_\delta(x)) \subseteq f(f^{-1}(B)) = B$.

Sequence Criterion.

f is continuous at $x \iff (x_n \to x \Rightarrow f(x_n) \to f(x))$

($\Rightarrow$) f is cont. at x, $x_n \to x$. WTS $f(x_n) \to f(x)$

let $\varepsilon > 0$, $\exists \overset{=\delta(\varepsilon)}{\delta} > 0$ s.t. $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$

$\exists N = N(\delta)$ s.t. $\forall n \geq N$: $x_n \in B_\delta(x)$

$\Rightarrow f(x_n) \in B_\varepsilon(f(x))$

$N(\delta) = N(\delta(\varepsilon)) = N(\varepsilon)$ $\iff \forall n \geq N$: $|f(x_n) - f(x)| < \varepsilon$.

($\Leftarrow$) Exc. Assume f not continuous at $x \Rightarrow \exists \varepsilon > 0, \forall \delta > 0, f(B_\delta(x)) \not\subseteq B_\varepsilon(f(x))$

Take $\delta = \frac{1}{n} \downarrow$ $\therefore \exists x_n \in B_\delta(x)$ s.t. $f(x_n) \notin B_\varepsilon(f(x)) \to$ construct the bad sequence

$\cancel{f(B_\delta(x)) \subseteq B_\varepsilon(f(x))}$ Then $x_n \to x$ but $f(x_n) \not\to f(x)$.

Def. $f: X \to Y$ is $\boxed{\text{continuous}}$ iff it is continuous everywhere.

Notation $f \in \mathcal{C}(X,Y)$, or $f \in \mathcal{C}$.

Lemma TFAE:

1. $f \in \mathcal{C}$
2. $G \subseteq Y$ open $\Rightarrow f^{-1}(G)$ open
3. $F \subseteq Y$ closed $\Rightarrow f^{-1}(F)$ closed.

① ⇒ ②

Criterion 5: $\forall x$, f is continuous at f.
(P37) $x \in f^{-1}(E)$ is an interior point of $f^{-1}(E)$

② ⇒ ① Let $x \in X$. $E := B_\varepsilon(f(x))$ is open and $f(x) \in E$

$\Rightarrow f^{-1}(B_\varepsilon(f(x)))$ contains $B_\delta(x)$ for some $\delta > 0$. (Criterion 4)

$\Rightarrow f(B_\delta(x)) \subseteq B_\varepsilon(f(x)) \Rightarrow f$ continuous at x.

② ⇒ ③ $F \subseteq Y$ closed $\Rightarrow F^c$ open

$\Rightarrow f^{-1}(F^c)$ open (from ②)

$\Rightarrow [f^{-1}(F^c)]^c = f^{-1}(F)$ closed.

Let $f : X \to Y$

f is continuous (everywhere) $\Leftrightarrow$ ε-δ definition?

$$\forall x,\ (\forall \varepsilon > 0,\ \exists\ \underbrace{\delta}_{=\delta(\varepsilon, x)} > 0 \text{ s.t. } f(B_\delta(x)) \subseteq B_\varepsilon(f(x)))$$

$$\big|\ d(x,y) < \delta \Rightarrow \rho(f(x), f(y)) < \varepsilon\ \big|$$

Theorem $f: X \to Y$, $f \in \mathcal{C}$

1) $K \subseteq X$ compact $\Rightarrow f(K)$ compact

2) $A \subseteq X$ connected $\Rightarrow f(A)$ connected

Proof: HW5

Corollary.

$f(x) = x^2$, $x \in (0,1)$

1) $f \in \mathcal{C}(X, \mathbb{R})$, $K \subseteq X$ compact

$s := \sup f(K)$, $t := \inf f(K)$

Then $s, t \in \mathbb{R}$ and $\exists\, x, x' \in K$ s.t. $f(x) = s := \max_K f$

$f(x') = t := \min_K f$

Proof sketch: $f(K)$ compact in $\mathbb{R}$ $\Rightarrow$ bounded and closed.

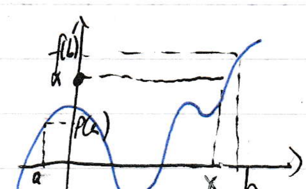
$\Rightarrow \sup_{f(K)} < \infty$

WTS $s \in f(K)$ ... ?

$\to \sup A \in \bar{A} = A$ closed

"intermediate value theorem" 2) $f \in \mathcal{C}(\mathbb{R})$, $f(a) \le \alpha \le f(b)$ $(a \le b)$

$\Rightarrow \exists\, x \in [a,b]$ s.t. $f(x) = \alpha$.



Proof. $f([a,b])$ connected $\overset{p.31}{\Rightarrow}$ $f([a,b])$ is an interval, which by definition contains every $y \in [f(a), f(b)]$, in particular $y = \alpha$.

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Theorem Let $f: (K,d) \to (Y,\rho)$ be continuous and K compact. If f bijective $\Rightarrow f^{-1}$ is continuous.

'group isomorphism'

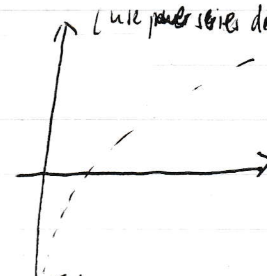
Proof. $K \underset{f}{\overset{f^{-1}}{\rightleftarrows}} Y$

inverse function f^{-1} continuous $\Leftrightarrow (f^{-1})^{-1}(F)$ is closed $\forall F$ closed $\subseteq K$.

But $(f^{-1})^{-1}(F) = f(F)$ is compact, and therefore closed.

closed subset of compact K

Application: $\exp: x \mapsto e^x$, $\mathbb{R} \to \mathbb{R}^+$ is continuous + bijective. We assume that this has been proven. (use power series def'n, uniform convergence)



wts. $\exp^{-1} =: \log$, $\log: \mathbb{R}^+ \to \mathbb{R}$ is continuous

not compact

$\exp|_{\cdot}: [-\log n, \log n] \to \mathbb{R}$

continuous + compact, $\log[1/n, n]$ is continuous. Take $n \to \infty$

Ex. ① Continuous functions: identity $f = id_X : X \to X$, $x \mapsto x$ is continuous

in particular $x \mapsto x \in \mathbb{R}$

$z \mapsto z \in \mathbb{C}$

② $f, g \in \mathcal{C}(X, \mathbb{R})$ or $\mathcal{C}(X, \mathbb{C}) \Rightarrow$

1) $f + g \in \mathcal{C}$

2) $\alpha \cdot f \in \mathcal{C}$

3) $f \cdot g \in \mathcal{C}$

4 $\forall x, f(x) \neq 0$, $1/f$ continuous.

Prove with Sequence Criterion.

Proof that $f \cdot g \in \mathcal{C}$. Use sequence criterion:

$$x_n \to x \Rightarrow (fg)(x_n) = \underset{\substack{\downarrow \\ f(x)}}{f(x_n)} \cdot \underset{\substack{\downarrow \\ g(x)}}{g(x_n)} \xrightarrow{?} f(x) \cdot g(x) = (fg)(x)$$

$(f(x) \pm \varepsilon)(g(x) \pm \varepsilon) \ldots$

<u>Lemma</u> $X \xrightarrow{f} Y \xrightarrow{g} Z$

① f, g continuous, $\Rightarrow g \circ f$ continuous.

Pf. $C \subseteq Z$ open $\Rightarrow (g \circ f)^{-1}(C) = f^{-1}(\underbrace{g^{-1}(C)}_{\text{open}})$ open in X.

② If f continuous at $x \in X$ and g continuous at $y = f(x)$

$\Rightarrow g \circ f$ continuous at x.

Sequence Criterion: $x_n \to x$, $g(\underset{\substack{\downarrow \\ f(x)}}{f(x_n)}) \to g(f(x))$

Ex. All (complex) polynomials are continuous.

$$p(z) = \sum_{k=0 \ldots n} c_k \cdot z^k \in \mathcal{C}(\mathbb{C})$$

$$p(x) \in \mathcal{C}(\mathbb{R})$$

Transcendental functions $\to$ Defined as the limit of Power Series (e.g. $\exp(\cdot)$),

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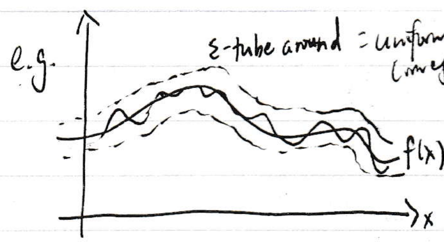
let $(f_n)_{n\geq 1}$ seq of functions, $f_n: X \to Y$.

$\boxed{\text{we say } f_n \to f \text{ pointwise iff } \forall x: f_n(x) \to f(x)}$ → Formally, $f_n \to f \iff \forall x, \forall \varepsilon > 0, \exists N = N(x, \varepsilon)$ $\rho(f_n(x), f(x)) \leq \varepsilon$

Uniform Convergence

$\boxed{f_n \to f \text{ uniformly}^{(\text{in } X)} \text{ iff } \forall \varepsilon\ \exists N = N(\varepsilon) \text{ s.t. } \forall n \geq N\ \forall x \in X: \rho(f_n(x), f(x)) \leq \varepsilon}$

e.g.



'closer convergence' at some points → pointwise

Uniform Convergence ⇒ Pointwise Convergence

Theorem: let (f_n) be a sequence of continuous functions converging to f uniformly ⇒ f is continuous itself.

Proof (Sequence Criterion)

Pick $x \in X$. let $x_n \to x$, w.t.s $f(x_n) \to f(x)$

let $\varepsilon > 0$ w.t.s $\exists k_0 = k_0(\varepsilon)$ s.t.

$\forall k \geq k_0: d(f(x_k), f(x)) < \varepsilon$.

* $f_n: X \to Y$
(X, ρ) (Y, d)

$$d(f(x_k), f(x)) \leq \underbrace{d(f(x_k), f_n(x_k))}_{< \varepsilon/3} + d(f_n(x_k), f_n(x)) + d(f_n(x), f(x))$$

– Specify n, k

① choose $n \geq N(\varepsilon/3)$ → valid $\forall x$! uniform convergence

② choose $k \geq k_0(\varepsilon, n)$ specify fn $= k_0(\varepsilon)$

same choice as ①

Ex: $f(z) = e^z$ is continuous on $\mathbb{C}$. $\left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)$ Weierstrass M-test, test for uniform/absolute convergence

Def $f: X \to Y$ is <u>uniformly continuous on $A \subseteq X$</u>
iff $\forall \varepsilon > 0, \exists \delta_{=\delta(\varepsilon)}$ s.t. $d(x,y) \le \delta \Rightarrow \rho(f(x), f(y)) < \varepsilon$.
($\forall x, y!$) HW5.

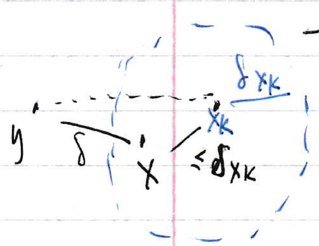
f continuous everywhere
~~$\forall x, \forall \varepsilon > 0\ \exists \delta = \delta(\varepsilon, x)$ s.t.~~
$(x-y) < \delta \Rightarrow \rho(f(x), f(y)) < \varepsilon$.

Theorem. $f: X \to Y$, $f \in \mathcal{C}$, X compact $\to$ f is uniformly continuous.

Pf. Let $\varepsilon > 0$ and $x \in X \Rightarrow \exists\ \delta_x > 0$ s.t. $d(x,y) < 2\delta_x \Rightarrow \rho(f(x), f(y)) < \varepsilon/2$.

$\to (B_{\delta_x}(x))_{x \in X}$ is an open cover of X,
and since X is compact, $\exists\ x_1, x_2 \ldots x_n$ s.t. $(B_{\delta_{x_k}}(x_k))_{k=1,2,\ldots,n}$
covers X. Set $\delta := \min\limits_{k=1\ldots n} \delta_{x_k} > 0$

$\to$ pick $x, y \in X$ s.t. $d(x,y) < \delta \Rightarrow \exists\ k$ s.t. $x \in B_{\delta_{x_k}}(x_k)$.



$\delta + \delta_{x_k} \le 2\delta_{x_k}$. So $y \in B_{2\delta_{x_k}}(x_k)$.

Then $\rho(f(y), f(x_k)) < \varepsilon/2$.

$\rho(f(x), f(x_k)) < \varepsilon/2$.

$\rho(f(x), f(y)) < \varepsilon/2 + \varepsilon/2 < \varepsilon$.
$\triangle$-inequality

Sequences in $\mathbb{R}$ ($\mathbb{C}$)

Theorem $a_n, b_n \in \mathbb{R}$ (or $\mathbb{C}$) $a_n \to a, b_n \to b$

1) $(a_n + b_n) \to a+b$ $\quad \to \left| (a_n+b_n) - (a+b) \right| \le \underbrace{|a_n - a|}_{<\varepsilon/2} + \underbrace{|b_n - b|}_{<\varepsilon/2}$

2) $c \in \mathbb{R}(\mathbb{C})$, then $ca_n \to ca$.

3) $a_n \cdot b_n \to a \cdot b$ $\quad \to |a_n b_n - ab| = |a_n(b_n - b) + b(a_n - a)|$

$\le \underbrace{|a_n|}_{\le M < \infty} \underbrace{|b_n - b|}_{<\varepsilon/2M} + \underbrace{|b|}_{M'} \underbrace{|a_n - a|}_{\le \varepsilon/2M'}$

4) $|\alpha_n| \to |\alpha|$. Define $\varphi(a_n) = |a_n|$

5) $\frac{1}{a_n} \to \frac{1}{a}$ if $a \neq 0$, $a_n \neq 0$ $\forall n$.

Inequality: $\big| |a| - |b| \big| \le |a - b|$. WLOG $|a| \ge |b| \Rightarrow |a-b| + |b - 0| \ge |a - 0|$

$|a-b| \ge |a| - |b| \ge 0$.

$|a| \le |b| \Rightarrow |b-a| + |a| \ge |b|$

$|b-a| \ge |b| - |a|$

$|a-b| \ge \big| |a| - |b| \big| \ge 0$

Let $\varepsilon > 0$: $\big| |a_n| - |a| \big| \le |a_n - a| \le \varepsilon$ q.e.d

for $n \ge n_0 = n_0(\varepsilon)$

For every n sufficiently large, $s_n \neq 0$. since $s_n \to s$.

$\left| \frac{1}{s_n} - \frac{1}{s} \right| = \left| \frac{s - s_n}{s * s_n} \right|$ $\quad$ Choose n_0 s.t. $\forall n \ge n_0$, $|s_n| \ge \frac{1}{2}|s|$.

$\frac{|s - s_n|}{|s| * |s_n|} \overset{n \ge n_0}{\le} 2 \cdot \frac{|s - s_n|}{|s|^2} < \varepsilon$.

$\forall n \ge n_1$

Choose n_1 s.t. $\forall n \ge n_1$, $|s - s_n| \le \frac{\varepsilon |s|^2}{2}$.

Lemma: $\vec{x}_n \to \vec{x}$ in $\mathbb{R}^d \iff \forall k = 1 \dots d \quad (\vec{x}_n)_k \to (\vec{x})_k$.

$d=2$ case: $$d(\vec{x}, \vec{x}_n) \le \sum_{k=1,2} \overset{\to 0}{|x_k - (x_n)_k|}$$

Lemma: If $(x_n) \nearrow$ (monotone increasing) i.e. $\dots \le x_n \le x_{n+1} \le \dots$

Then $\lim_n x_n = \begin{cases} \infty; & \text{if } x_n \text{ unbounded} \quad \text{'divergent'} \\ \sup_n x_n =: s, & \text{if } (x_n) \text{ bounded.} \end{cases}$

Proof. $x_n \to \infty$

$\Updownarrow$ def.

$\forall M: \exists n_0 = n_0(M)$ s.t. $\forall n \ge n_0: a_n > M$

(a) "Open ball around ∞" $\Rightarrow B_M(\infty) := \{y \mid y > M\}$

WTS x_n unbounded $\iff \lim_n x_n = \infty$.

($\Rightarrow$) $\forall M, \exists n$ s.t. $a_n \ge M$, $\forall k \ge n$, $a_k \ge a_n \therefore a_k \ge M$

($\Leftarrow$) $\forall n \ge n_0, a_n > M \dots$ find $n_0(M)$

(b) $\forall n: x_n \le b < \infty$, $\sup_n x_n = s \le b$

let $\varepsilon > 0 \Rightarrow \exists n_0$ s.t. $x_{n_0} > s - \varepsilon$

Since $x_n \nearrow$, $\forall n \ge n_0$, $x_n > s - \varepsilon$

$\forall n \ge n_0$, $s - \varepsilon \le x_n \le s < s + \varepsilon$
$\Rightarrow x_n \in B_\varepsilon(s)$

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Comparison Test

1) If $0 \le x_n \le c_n$ and $c_n \to 0 \Rightarrow x_n \to 0$

2) $b_n \le x_n \le a_n$ and, $a_n \to a$, $b_n \to a \Rightarrow x_n \to a$
"policeman"

$\forall \varepsilon > 0$, $\exists n_0$ s.t. $\forall n \ge n_0$, $0 \le c_n < \varepsilon$.
$\Rightarrow$ ~~$\forall \varepsilon > 0, \exists n_0$~~ $\forall n \ge n_0$, $0 \le x_n < \varepsilon$.
$\Rightarrow x_n \to 0$.

$0 \le x_n - b_n \le a_n - b_n$, use part (1)
$|a_n - b_n| = |a_n - a + a - b_n|$
$\le |a_n - a| + |b_n - a|$
$\le \varepsilon + \varepsilon = 2\varepsilon$.
$\therefore x_n - b_n \to 0$, $x_n \to b_n$
$x_n \to a$.

Examples.

1) $\boxed{p > 0 : \frac{1}{n^p} \to 0}$ $(p \in \mathbb{Q})$

$n^p \le (n+1)^p \nearrow$

$\therefore \left(\frac{1}{n^p}\right) \searrow$, and $0 \le \frac{1}{n^p}$

Also, $\forall \varepsilon > 0 \Rightarrow \frac{1}{n^p} \le \varepsilon$, for $\varepsilon^{(-\frac{1}{p})} \le n$ $\therefore$ ~~$\inf\left(\frac{1}{n^p}\right)$~~ $= 0$.

2) $\boxed{\sqrt[n]{a} \to 1 \ \forall a > 0.}$

Let $a > 1$: $a = (1 + (\sqrt[n]{a} - 1))^n = 1^n + 1^{n-1}\binom{n}{1}(\sqrt[n]{a} - 1) + \dots$
$\ge 1 + n(\sqrt[n]{a} - 1)$

* Binomial Theorem

WTS $\sqrt[n]{a} - 1 \to 0$

but $0 \le (\sqrt[n]{a} - 1) \le \frac{a-1}{n} \searrow 0$ q.e.d
from here

Let $a = 1$ ✓

Let $a < 1 \Rightarrow \left(\frac{1}{a}\right) > 1 \Rightarrow \left(\frac{1}{a}\right)^{\frac{1}{n}} \to 1 \Rightarrow a^{\frac{1}{n}} \to \frac{1}{1} = 1$.

* 3) $\boxed{\sqrt[n]{n} \to 1.}$

proof by contradiction $\left(\sqrt[n]{n} < 1 \Rightarrow \underset{\infty}{n} < \underset{\to 0}{x^n}\right)$

Let $x_n := \sqrt[n]{n} - 1 \ge 0$

$n = (1 + x_n)^n \overset{n \ge 2}{\ge} \binom{n}{2} x_n^2 \Rightarrow 0 \le x_n \le \left(\frac{2n}{n(n-1)}\right)^{1/2}$
One of the terms
$= \sqrt{\frac{2}{n-1}} \searrow 0$

*4) $\boxed{a>0,\quad \alpha\in\mathbb{R}: \frac{n^\alpha}{(1+a)^n} \to 0}$

Choose $k\in\mathbb{N} > \alpha$ and let $n>2k$:

$$(1+a)^n \underset{n\geq k}{>} \binom{n}{k} a^k = \frac{n(n-1)\dots(n-k+1)}{k!} a^k$$

$$> \frac{\left(\frac{n}{2}\right)^k a^k}{k!}$$

$$\frac{n^\alpha}{(1+a)^n} = \frac{n^k \cdot n^{\alpha-k}}{(1+a)^n} \leq \frac{n^k \cdot n^{\alpha-k}}{\left(\left(\frac{n}{2}\right)^k a^k/k!\right)} = \underset{\downarrow\, 0}{n^{-(k-\alpha)}} \cdot \underbrace{\frac{2^k \cdot k!}{a^k}}_{\text{constant}}$$

$$\to 0.$$

5) $\boxed{|x|<1 \Rightarrow x^n \to 0}$

$x^n = \frac{1}{\left(\frac{1}{x}\right)^n}$ Use 4) with $\alpha=0$.

Liminf and Limsup

Let $(a_n)\subseteq\mathbb{R}$ sequence $\to$ liminf and limsup always exist!
$\to$ proves convergence

$$\limsup_n a_n = \overline{\lim_n}\, a_n = \lim_n \downarrow \sup_{k\geq n} a_k := \inf_n \sup_{k\geq n} a_k = \lim_n \downarrow b_n \in \{-\infty\}\cup\mathbb{R}$$

$$\liminf_n a_n = \underline{\lim_n}\, a_n = \lim_n \uparrow \inf_{k\geq n} a_n = \sup_n \inf_{k\geq n} a_k = \lim_n \uparrow b_n \in \mathbb{R}\cup\{+\infty\}$$

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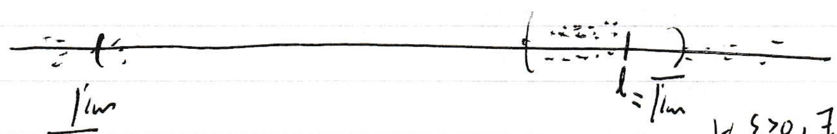
Theorem Let (a_n) be a sequence.
$\overline{\lim}\, a_n$ is the largest point of accumulation,

$\underline{\lim}\, a_n$ is the smallest "



$\forall \varepsilon > 0$, $\exists$ only finitely many points $\geq \overline{\lim} + \varepsilon$.

Lemma Let $(a_n) \subseteq \mathbb{R}$, $l := \overline{\lim}\, a_n \in \mathbb{R}$

(1) $\forall \varepsilon > 0$ only finitely many a_n-s are $\overset{(>)}{\geq} l+\varepsilon$.

(2) $\forall \varepsilon > 0$ ∞-many a_n's are $\overset{(\geq)}{>} l-\varepsilon$.

(3) $\underline{\lim}\, a_n = \overline{\lim}\, a_n =: a \iff a_n \to a$.

1) AFSOC $n_1, n_2 < \dots n_j < \dots$ ∞-many indices, $a_{n_k} > l+\varepsilon$

$\Rightarrow \forall k: (\sup_{n \geq k} a_n) > l + \varepsilon \Rightarrow \inf_k \sup_{n\geq k} a_n \geq l + \varepsilon$.

this epsilon is important

2) AFSOC if $\exists j < \infty$ and $n_1 < n_2, \dots < n_j$ s.t. $k = 1 \dots j$, $a_n > l - \varepsilon$
and all other a_n's $\leq l - \varepsilon$.

$\Rightarrow (\sup_{n \geq n_j+1} a_n) \leq l - \varepsilon \Rightarrow l - \varepsilon \geq \inf_{k \geq n_j+1} (\sup_{n\geq k} a_n) = l$

3) HW $(\Rightarrow)$ Use (1) twice. $\underline{\lim}\, a_n = -\overline{\lim}(-a_n)$ $\quad \overline{\lim}(-a_n) = -a \Rightarrow$ finitely many $-a_n$'s $> -a + \varepsilon$
sup/inf
a_n's $< a - \varepsilon$.

In particular, if $a_n \to a$, $a = \overline{\lim} = \underline{\lim}$

$\underline{\lim}\, a_n \le \overline{\lim}\, a_n$, $\underline{\lim}\, a_n = -\overline{\lim}\,(-a_n)$

HW2 $\overline{\lim}\,(a_n+b_n) \le \overline{\lim}\, a_n + \overline{\lim}\, b_n$

$\underline{\lim}\,(a_n+b_n) \ge \underline{\lim}\, a_n + \underline{\lim}\, b_n$

$a_n \le b_n \Rightarrow \overline{\lim}\, a_n \le \overline{\lim}\, b_n$

$\underline{\lim}\, a_n \le \underline{\lim}\, b_n$

$\underline{\lim}\,,\ \overline{\lim} \in [-\infty, \infty]$

*not true for inf/sup

Lemma: Removing finitely many elements of the sequence does not change $\overline{\lim}$ or $\underline{\lim}$.

EX. $a_n \to a,\ b_n \to b \Rightarrow a_n+b_n \to a+b$

Proof $\underline{\lim}\,(a_n+b_n) \le \overline{\lim}\,(a_n+b_n) \le \overline{\lim}\, a_n + \overline{\lim}\, b_n$

$\le$

$\underline{\lim}\, a_n + \underline{\lim}\, b_n$

$a_n \to a$: $b_n \to b$ $\quad a+b \le \underline{\lim}\,(a_n+b_n) \le \overline{\lim}\,(a_n+b_n) \le a+b$

$\therefore \underline{\lim}\,(a_n+b_n) = \overline{\lim}\,(a_n+b_n) = a+b$

$\therefore a_n+b_n \to a+b$ (by 3)

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Series

$(a_n) \subseteq \mathbb{R}$ or $\mathbb{C}$.

$S_n := \sum_{k=1}^{n} a_k$ are the partial sums, and form a series.

$$"\sum_{k=1}^{\infty} a_k" = \sum_{k\geq 1} a_k := \lim_{n\to\infty} \sum_{k=1\ldots n} a_k$$

$$= \lim S_n = S \begin{cases} \in \bar{\mathbb{R}} \ (= \mathbb{R} \cup \{-\infty, +\infty\}) \\ \in \mathbb{C} \end{cases}$$

if this increasing limit exists. (counterexample, $\sum_{n\geq 1} (-1)^n$)

we say that the series is divergent if $S \in \{-\infty, \infty\}$

convergent if $S \in \mathbb{R}$

Theorem (Cauchy Criterion)

$S_n = \sum_{k=1}^{n} a_k$ is convergent $\Leftrightarrow S_n$ is Cauchy

$\Leftrightarrow \forall \varepsilon, \exists N$ s.t. $\forall n \geq m \geq N$,

$$\left|\sum_{k=n,\ldots,m} a_k\right| \leq \varepsilon.$$

Corollary If $\sum a_k$ is convergent $\underset{\text{take } n=m}{\Rightarrow} |a_n| \leq \varepsilon \Rightarrow a_n \to 0$.

$\not\Leftarrow \quad \sum \frac{1}{n}$

Theorem If $|a_n| < c_n \ \forall n \geq N_0$, $\sum_n c_n < \infty \Rightarrow \sum a_n$ is convergent.

So $\sum |a_n| < \infty$, $\Longleftrightarrow$ $\sum a_n$ is an absolutely convergent series.

Proof: $\left|\sum_{k=n,\ldots,m} a_k\right| \leq \sum_{k=n,\ldots,m} |a_k| \leq \sum_{k=n,\ldots,m} c_k < \varepsilon \Rightarrow$ Cauchy Criterion

$\forall m, n \geq N(\varepsilon)$

Examples (A) $0 \le x < 1 \Rightarrow \boxed{\sum_{k\ge 0} x^k = \frac{1}{1-x}}$ (GR)

$(1-x)(\underbrace{1+x+\dots}_{s_n} x^n) = 1-x^{n+1}$

$s_n = \frac{1-x^{n+1}}{1-x} \xrightarrow{n\to\infty} \frac{1}{1-x}$ qed.

rem. 1) $x \ge 1$ series diverges

2) $z \in \mathbb{C}$, $0 \le |z| < 1$, $\sum z^k = \frac{1}{1-z}$

$z^n \to 0$

argument, magnitude

Theorem $\boxed{\text{Assume } (a_n) \subseteq \mathbb{R}^+,\ a_n \searrow 0. \text{ Then } \sum_{k\ge 1} a_k \text{ converges} \Leftrightarrow \sum_{k\ge 0} 2^k a_{2^k} < \infty.}$

'control'

'sampling'

Define $t_m := \sum_{k=0}^{m} 2^k a_{2^k}$.

Claim 1 : $n < 2^k$ for some $k \Rightarrow s_n < t_k$

($\Leftarrow$) e.g. $n = 2^k - 1$: $s_n = a_1 + \underbrace{a_2 + a_3}_{\le 2a_2} + \underbrace{a_4 + a_5 + a_6 + a_7}_{\le 4a_4}$

$+ a_8 \dots a_{15} + \underbrace{(a_{2^{k-1}} + a_{2^{k-1}+1}, \dots + a_{2^k - 1})}_{\le 2^{k-1} a_{2^{k-1}}}$

monotone + bounded = convergent!

Claim 2 : $n \ge 2^k \Rightarrow s_n \ge \frac{1}{2} t_k$

($\Rightarrow$) e.g. $n = 2^k$. $s_n = a_1 + a_2 + (a_3 + a_4) + (a_5 + \dots a_8) + \dots (\dots a_{2^k})$

$a\ (\ge \frac{1}{2} a_1)\ (\ge a_2)\ \ (\ge 2a_4)\ \ \ge 4a_8$

$\ge \frac{1}{2}(a_1 + \underbrace{2a_2 + 4a_4 + 8a_8 + \dots 2^k a_k}_{t_k})$

If RHS diverges, LHS also $\to \infty$.

Theorem $\boxed{\sum_{n\geq 1} \frac{1}{n^p} \text{ converges for } p>1 \text{ and diverges for } p\leq 1}$

Observe that $\sum_{k\geq 0} 2^k \frac{1}{2^{kp}} = \sum_{k\geq 0} 2^{k(1-p)}$ $(= (2^{1-p})^k)$ geometric series!

from previous theorem

If we want convergence, $\rho = 2^{1-p} < 1 \iff p > 1$.

$\Rightarrow$ Original series converges iff $p > 1$.

Theorem

we know $\frac{1}{n^{1+\varepsilon}}$ is convergent, what about a function that grows slower than n^{ε}?

$$\boxed{\sum_{n\geq 2} \frac{1}{n(\log n)^p} \begin{cases} < \infty & p > 1 \\ = \infty & p \leq 1 \end{cases}}$$

Since $\log n$ is increasing, $\frac{1}{n(\log n)^p}$ is decreasing.

$$\sum_{k\geq 1} 2^k \frac{1}{2^k(\log 2^k)^p} = \sum_{k\geq 1} \frac{1}{(\log 2)^p} \cdot \frac{1}{k^p}$$

$$= \frac{1}{(\log 2)^p} \sum_{k\geq 1} \frac{1}{k^p} \begin{cases} < \infty & (p > 1) \\ = \infty & (p \leq 1) \end{cases}$$

Note that $\boxed{\sum_n \frac{1}{n(\log n)(\log\log n)} \text{ diverges,}}$

$\boxed{\sum_n \frac{1}{n(\log n)(\log\log n)^2} \text{ converges}}$

... in general no "sharp" boundary

Theorem

$$\boxed{\begin{array}{l} 1)\ e := \sum_{n\geq 0} \frac{1}{n!} < \infty \quad \text{convergent} \\ \Updownarrow \\ 2)\ e = \lim_{n\to\infty} (1+\frac{1}{n})^n \end{array}}$$

def.

Proof. 1) $s_n = 1 + 1 + \frac{1}{2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{2\cdot 3\cdots n}$

$< 1 + 1 + \frac{1}{2} + \frac{1}{2\cdot 2} + \dots + \frac{1}{2^{n-1}}$

So we have $s_n < 1 + \sum_{k\geq 0} (\frac{1}{2})^k = 1 + \frac{1}{1-\frac{1}{2}} = 3$

2) $s_n = \sum_{k=1}^{n} \frac{1}{k!}$, and define $t_n := (1+\frac{1}{n})^n$

$t_n \overset{\text{binomial}}{\underset{\text{theorem}}{=}} 1 + \binom{n}{1}(\frac{1}{n})^1 + \binom{n}{2}(\frac{1}{n})^2 + \dots + \binom{n}{n}(\frac{1}{n})^n$

$= 1 + 1 + \frac{n(n-1)}{2!\,n^2} + \frac{n(n-1)(n-2)}{3!\,n^3} + \dots + \frac{n(n-1)\dots(n-n+1)}{n!\ n^n}$

$= 1 + 1 + \frac{1}{2!}(1-\frac{1}{n}) + \frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n}) +$

$\dots + \frac{1}{n!}(1-\frac{1}{n})(1-\frac{2}{n})\dots(1-\frac{n-1}{n})$

$\overline{\lim}$, $\underline{\lim}$

Clearly $s_n \geq t_n \Rightarrow \underbrace{\overline{\lim_n}\ s_n}_{=e} \geq \overline{\lim}\ t_n$.

For the other inequality, let m be fixed

$\forall n \geq m$, $t_n \geq 1 + 1 + \frac{1}{2!}(1-\frac{1}{n}) + \dots + \frac{1}{m!}(1-\frac{1}{n})\dots(1-\frac{m-1}{n}) + \dots$ ~~cut~~ cut here!

$\underline{\lim_n}\ t_n \geq \underline{\lim_n}\ (\ \downarrow\ ) \overset{\text{for fixed } m}{=} 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{m!}$

Since this is valid for all m, we see that

$\underline{\lim}\ t_n \geq \sup_m (1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots \frac{1}{m!}) = e$

So $e = \overline{\lim}\ s_n \geq \overline{\lim}\ t_n \geq \underline{\lim}\ t_n \geq e$

$\therefore \lim t_n = e$.

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Remark ① $S_n \to e$ very fast:

$$e - S_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots$$

$$\leq \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \dots\right)$$

$$\leq \frac{1}{(n+1)!}\left(\frac{1}{1-\frac{1}{n+1}}\right)$$

$$\leq \frac{1}{n(n!)}$$

② $e \notin \mathbb{Q}$, e is not algebraic (not root of any polynomial)

Other Convergence Tests

Theorem (Root Test) Given $\sum a_n$, $a_n \in \mathbb{C}$

$$r := \overline{\lim} \sqrt[n]{|a_n|}$$

1) if $r < 1 \Rightarrow$ series convergent (in particular, series converges absolutely)

2) if $r > 1 \Rightarrow$ not convergent (not necessarily diverges... unclear what this means for $\mathbb{C}$)

3) $r = 1 \Rightarrow$ can go both ways

Proof • $r<1$: Choose $\varepsilon > 0$ s.t. $r + \varepsilon < 1$. Then $\forall n \geq n(\varepsilon)$, by definition of limsup

$$\sqrt[n]{|a_n|} \leq r + \varepsilon$$

$\Rightarrow |a_n| \leq (r+\varepsilon)^n$ at the limit

convergent geometric series

• $r > 1$: Choose $\varepsilon > 0$ s.t. $r - \varepsilon > 1$.

$\Rightarrow \exists (n_k)$ subsequence s.t. $\sqrt[n_k]{|a_{n_k}|} \geq r - \varepsilon$

$\Leftrightarrow |a_{n_k}| \geq \underbrace{(r-\varepsilon)^{n_k}}_{\geq 1}$

Then $a_n \not\to 0$, so by contrapositive $\sum a_n$ cannot be convergent.

$r = 1$: Can go both ways.

① (divergent) $\sum \frac{1}{n}$: $r = \overline{\lim} \sqrt[n]{\frac{1}{n}} = 1$.

② (convergent) $\sum \frac{1}{n^2}$: $r = \overline{\lim} \sqrt[n]{\frac{1}{n^2}} = \left(\lim \sqrt[n]{\frac{1}{n}}\right)^2 = 1$ too.

Theorem (Ratio Test)

For a series $\sum a_n$,

1) $\overline{\lim} \left| \frac{a_{n+1}}{a_n} \right| < 1 \Rightarrow$ convergent (in particular, absolutely convergent)

2) $\exists n_0$ s.t. $\forall n \geq n_0$, $\left| \frac{a_{n+1}}{a_n} \right| \geq 1 \wedge a_n \neq 0 \Rightarrow$ not convergent

Proof 1) $\exists \beta < 1$ (between $\overline{\lim}(\cdot)$ and 1) and $N = N(\beta)$ s.t. $\forall n \geq N$:

$$\left| \frac{a_{n+1}}{a_n} \right| < \beta \Rightarrow |a_{n+1}| < \beta |a_n|$$

$$\therefore \forall k \geq 1,\ |a_{N+k}| < \beta |a_{N+k-1}| < \beta^2 |a_{N+k-2}| \dots$$

$$\Rightarrow \sum |a_n| \leq \sum_{k=1}^{N} |a_k| + \sum_{k \geq 1} \underbrace{|a_{N+k}|}$$

$\leq$ geometric series of $|a_{N+k}| (1 + \beta^2 + \beta^3 + \dots)$

$< \infty$.

2) If $\forall n \geq n_0$, $|a_{n+1}| \geq |a_n|$

$$\Rightarrow |a_{n_0}| \leq |a_{n_0+1}| \leq \dots \leq |a_{n_0+k}|$$

$\neq 0$. $\Rightarrow a_n \not\to 0$ so series is not convergent.

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Def. $z \in \mathbb{C}$, $\varphi(z) := \sum_{n \geq 0} c_n z^n$ ($\in \mathbb{C}$ (usually $\mathbb{R}$))

is called the <u>power series</u> associated with (c_n).

Ex. 1) $c_n := \frac{1}{n!} \Rightarrow \varphi(z) := \sum \frac{1}{n!} z^n := e^z$

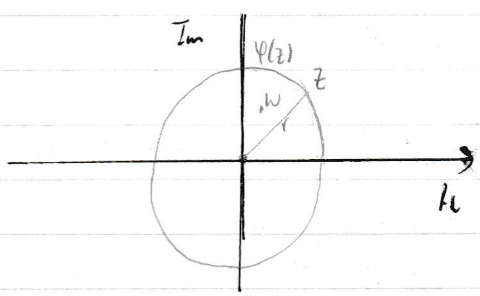
(complex exponential function)

2) $c_n = \begin{cases} \frac{(-1)^{n+1}}{n!} & n = 2k+1 \quad k \geq 0 \\ 0 & n = 2k \quad k \geq 0 \end{cases}$

$\sum c_n z^n := \sin(z)$

Note: If $\sum c_n z^n$ is <u>absolutely convergent</u> ($\sum c_n |z|^n < \infty$),

$\Rightarrow \sum c_n w^n$ is absolutely convergent for $|w| \leq |z|$



$\rho := \sup \{ r \mid \sum c_n w^n$ is absolutely convergent in $|w| < r \}$

is the <u>radius of absolute convergence</u>

$\rightarrow$ everything in the open ball is convergent

<u>Theorem</u> let $R := (\overline{\lim} \sqrt[n]{|c_n|})^{-1} \in [0, \infty]$

$\Rightarrow \sum_{n \geq 0} c_n z^n$ is absolutely convergent for $|z| < R$, and

not convergent for $|z| > R$

$\Rightarrow \rho = R$.

Proof. $|a_n| \overset{def}{=} |c_n z^n| \Rightarrow$ The root test has $\sqrt[n]{|a_n|} = |z| \sqrt[n]{|c_n|}$ $\Leftrightarrow$ $|z| < \frac{1}{\overline{\lim} \sqrt[n]{|c_n|}} = R$

$\overline{\lim} |z| \sqrt[n]{|c_n|} = |z| \overline{\lim} \sqrt[n]{|c_n|} < 1 \Rightarrow$ abs. convergent

$> 1 \Rightarrow$ not convergent.

Theorem

$\sum \frac{1}{n!} z^n$ is absolutely convergent $\forall z \in \mathbb{C}$.

By the ratio test, $\frac{z^{n+1}}{z^n} \cdot \frac{n!}{(n+1)!} = |z| \cdot \underbrace{\frac{1}{n+1}} \to 0$

$\lim(\cdot) = 0 < 1$

• If $\underset{\text{power series}}{\varphi}: \mathbb{C} \to \mathbb{C}$ or $\varphi: \mathbb{R} \to \mathbb{R}$, then we can do term-by-term operations.

Lemma $A := \sum_n a_n$, $B := \sum_n b_n$

$\Rightarrow$ 1) $\sum (a_n + b_n) = A + B$ (linearity)

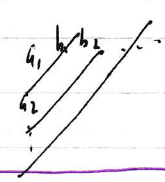
2) $\sum \gamma \cdot a_n = \gamma A$

* $\sum a_n b_n \neq A \cdot B$ (we get many more terms on RHS)

Def Given $\sum_{n \geq 0} a_n$, $\sum_{n \geq 0} b_n$, we set:

(for $n \geq 0$) $c_n := \sum_{k=0,\dots,n} a_k b_{n-k} = \sum_{i+j=n} a_i b_j = a_0 b_n + a_1 b_{n-1} + \dots a_n b_0$

(Motivation: $\left(\sum a_i \sum b_j \overset{\text{"formally" (not necessarily correct)}}{=} \sum_{i,j} a_i b_j = \sum_{n \geq 0} \underbrace{\sum_{j+i=n} a_i b_j}_{c_n}\right)$



Theorem Assume

1) $A := \sum a_n$ is absolutely* convergent

2) $B := \sum_n b_n$ (just convergent)

Then $A \cdot B = \left(\sum_{n \geq 0} a_n\right)\left(\sum_{m \geq 0} b_m\right) = \sum c_n$

i.e. $\sum c_n$ is convergent with $\sum c_n = AB$.

Proof. Set $A_n = \sum_{k=0,\dots,n} a_n$, $B_n = \sum_{k=0\dots n} b_n$, $C_n = \sum^n c_k$

WTS $C_n \to A \cdot B$.

write $C_n = \overset{c_0}{a_0 b_0} + \overset{c_1}{(a_0 b_1 + a_1 b_0)} + \dots + \overset{c_n}{(a_0 b_n + \dots a_n b_0)}$

$= a_0 B_n + a_1 B_{n-1} + a_2 B_{n-2} + \dots + a_n B_0$

$= a_0 (B + \underbrace{(B_n - B)}_{\beta_n \text{ (tail sum)}}) + a_1 (B + (B_{n-1} - B)) + \dots a_n (B + (B_0 - B))$

use convergence of A by adding & subtracting

$= B \cdot \underbrace{A_n}_{\xrightarrow[n\to\infty]{} B \cdot A} + \underbrace{a_0 \beta_n + a_1 \beta_{n-1} + \dots + a_n \beta_0}_{=: \gamma_n \underset{?}{\to} 0}$

So $C_n \to BA$ iff $\gamma_n \to 0$.

Note that $\beta_n := B_n - B \to 0$

$\Rightarrow \forall \varepsilon > 0, \exists N, \forall n \geq N \ |\beta_n| < \varepsilon$.

$|\gamma_n| \underset{(n \geq N)}{\leq} |a_0 \beta_n + a_1 \beta_{n-1} + \dots + a_{n-N} \beta_N| + |a_{n-N+1} \beta_{N-1} + \dots + a_n \beta_0|$

$?(\leq \varepsilon |A_{n-N}|)$ $\quad + |\beta_0 a_n + \beta_1 a_{n-1} + \dots + \beta_{N-1} a_{n-N+1}|$

$\varepsilon |A_{n-N}|$

$= |\varepsilon a_0 + \varepsilon a_1 + \dots \varepsilon a_{n-N}|$

$a_0 \beta_n$

$\leq \varepsilon \sum_{k \geq 0}^{n-N} |a_k|$ $\quad \leq (\max_{k \leq N-1} |\beta_k|) \cdot \sum_{k \geq n-N}^{n} |a_k|$, $\max < \infty$

absolute convergence $\leq \varepsilon \underbrace{\sum_{k \geq 0} |a_k|}_{< \infty}$ $\quad \leq \sum_{k \geq n-N} |a_k| \to 0 \ (n \to \infty)$

$\xrightarrow[n\to\infty]{} 0$

$\Rightarrow \overline{\lim_n} |\gamma_n| \leq \varepsilon \cdot \alpha + 0$, $\forall \varepsilon$

Since this is valid for $\forall \varepsilon$,

$\Rightarrow \overline{\lim}(\gamma_n) = 0$.

Absolutely Convergent Series "=" finite Sums

Alternating series – Rearranging the summing order allows the sum to change (??)

Rearrangements

Let $\varphi : \mathbb{N} \to \mathbb{N}$ bijective, $(a_n) \subseteq \mathbb{C}$.
$a'_n := a_{\varphi(n)}$, $s'_n = \sum_{k=0\ldots n} a'_k$.

$\boxed{\text{Theorem If } \sum a_n \text{ is absolutely convergent} \Rightarrow \sum a'_n \text{ is also abs. convergent and } \sum a'_n = \sum_k a_k =: s.}$

WLOG $a_n \geq 0 \ \forall n$. (else $|\cdot|$ at the end)
For given n, set $M := \max_{k \leq n} \{\varphi(k)\} = M_n$.

Then $0 \leq s'_n = \sum_{k=1\ldots n} a_{\varphi(k)} \leq \sum_{j=1\ldots M_n} a_j = s_M \leq s$ (the original sum)

$\Rightarrow \lim s'_n =: s' \underline{\underline{\leq}} s$ and (a'_n) is also absolutely convergent.
Reversing the roles of (a_n) and (a'_n), we have $s \leq s' \Rightarrow \boxed{s = s'}$
(possible because (a'_n) is abs. conv.)

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Locally Given by a Convergent Power Series — Analytic function

Theorem $f(z) = \sum_{k \geq 0} c_k z^k$ power series with

$R = \sup\{ r \geq 0 \mid f(z) \text{ is absolutely convergent in } \overline{B_r(0)} \}$

$= \left(\overline{\lim_n} \sqrt[n]{|c_n|} \right)^{-1}$

Then $\boxed{f_n(z) := \sum_{k \geq 0}^{n} c_k z^k \xrightarrow[n \to \infty]{\text{uniformly}} f(z)}$ on $\overline{B_r(0)}$ (closed ball) for every $r < R$.

In particular $f(z)$ is continuous on $B_R(0)$ (open ball)

Proof. Let $r < R \Rightarrow \sum |c_k| r^k$ is convergent. $\Rightarrow \lim_{n \to \infty} \sum_{k \geq n} |c_k| r^k = 0$.

Let $z \in \overline{B_r(0)} \Rightarrow |f(z) - f_n(z)| = \left| \sum_{k > n} c_k z^k \right|$

$$\leq \sum_{k \geq n} |c_k| |z|^k$$

$$\underset{\forall z \in \overline{B_r(0)}}{\leq} \sum_{k > n} |c_k| r^k \xrightarrow[n \to \infty]{} 0$$

If $z \in B_R(0) \Rightarrow z \in B_r(0)$ for some $r < R$, .
f is continuous at $z \in B_r(0)$. (see proof on P43.)

Application
$f(g)$, $g(z)$ power series

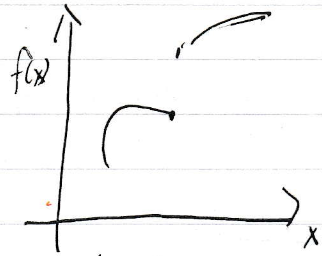
$f \cdot g(z) = \sum_k c_k z^k$ is also absolutely convergent in $B_r(0)$

$c_k = \left(\sum_{j=0..k} a_j b_{k-j} \right)$ if $r < R$ (calculate $\left(\overline{\lim_n} \sqrt[n]{|c_n|} \right)^{-1}$)

Right and Left Continuity

$f: \mathbb{R} \to \mathbb{R}$

Def. f is right continuous $\iff \forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\forall y > x$ with $|y-x| < \delta$
$\Rightarrow |f(x) - f(y)| \leq \varepsilon$



$\iff f(B^+_\delta(x)) \subseteq B_\varepsilon(f(x))$

$\{y \in \mathbb{R} \mid y > x$ and $|y-x| < \delta\} = B_\delta(x) \cap [x, \infty)$

Left continuity : $f(B^-_\delta(x)) \subseteq B_\varepsilon(f(x))$

$B_\delta(x) \cap (-\infty, x]$

f is r.c. at $x \iff f_r : [x, \infty) \to \mathbb{R}$ is continuous at x
rest $\quad x \longmapsto f(x)$

we can check with the sequence criterion $\to$ check sequences that are greater than x

$f: \mathbb{R} \to \mathbb{R}$ CONTINUOUS $\iff \boxed{\text{L CONT}} + \boxed{\text{R CONT}}$

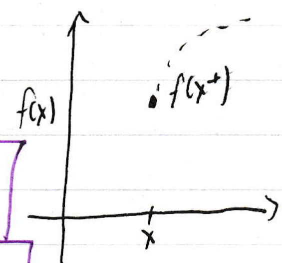
$f : \mathbb{R} \to \mathbb{R}$

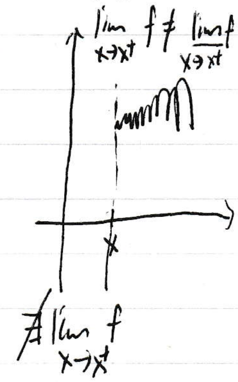
Def $\overline{\lim_{h\downarrow 0}}\, f(x+h) := \lim_{\delta \downarrow 0}\downarrow \sup_{y\in B_\delta^+(x)} \{f(y)\}$

Similarly, $\underline{\lim_{h\downarrow 0}}\, f(x+h) := \lim_{\delta \downarrow 0}\uparrow \inf_{y\in B_\delta^+(x)} \{f(y)\}$

If $\alpha := \overline{\lim_{h\downarrow 0}}\, f(x+h) = \underline{\lim_{h\downarrow 0}}\, f(x+h)$ then

$\lim_{h\downarrow 0} f(x+h)$ exists and $= \alpha$

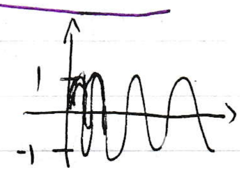




Also $\boxed{f(x^+) := \lim_{x\to x^+} f(x) = \alpha \text{ is the right limit.}}$

$\boxed{f \text{ is right continuous at } x \iff f(x^+) = \lim_{x\to x^+} f(x) \text{ exists, and } = f(x).}$

Ex. (1) $f(x) = \begin{cases} \sin\frac{1}{x} & x>0 \\ 0 & x=0 \end{cases}$



$f : [0,\infty) \to \mathbb{R}$

$\lim_{x\to 0^+} f(x) = \overline{\lim_{x\downarrow 0}}\, f(x) = 1$

$\underline{\lim_{x\downarrow 0}}\, f(x) = -1$

so $f(0^+)$ does not exist.

$\boxed{\text{Limit} \iff \text{Right Limit} = \text{Left Limit} \in \mathbb{R}}$

Monotone Functions

Assume $f: I \to \mathbb{R}$, $I \subseteq \mathbb{R}$ an interval. (e.g. $I = \mathbb{R}$)

f is monotone $\nearrow$ $\iff$ $x \leq y \iff f(x) \leq f(y)$

① $\forall x$ $f(x^+)$ and $f(x^-)$ exist,

$$f(x^+) \geq f(x) \geq f(x^-)$$

② Set $D = \{x \mid f(x^-) \underset{\neq}{\leq} f(x^+)\}$ to be the set of (jump) discontinuities.

Then D is countable.

Pf. ① $\alpha := \inf\{f(y) \mid y > x\} \underset{\text{monotone}}{\geq} f(x)$

Then $\lim_{h \downarrow 0} f(x+h) = \alpha$ (sequence criterion)

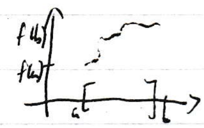
$f(x^-) = \lim_{h \downarrow 0} f(x-h) \ldots$

② Case 1 (finite interval) $I = [a, b]$

Define $p(x) := f(x^+) - f(x^-)$,

$p(x) > 0 \iff x \in D$

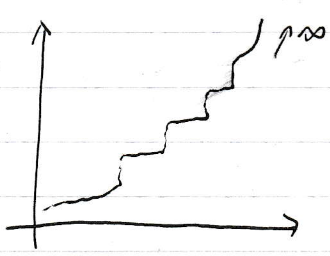
Let $D_n := \{x \mid p(x) > \frac{1}{n}\}$



Note. $\forall n$: D_n is finite, each jump takes up $\frac{1}{n}$ space, and $f(b) - f(a) < \infty$

So $D = \bigcup_n D_n$ ($p(x) > 0 \Rightarrow \exists\, n$ s.t. $p(x) > \frac{1}{n}$)

Counterexample for open intervals



Case 2 $I=(a,b) \Rightarrow I=\bigcup_{n>0}[a+\frac{1}{n}, b-\frac{1}{n}]$

$A_n := D_n \cap [a+\frac{1}{n}, b-\frac{1}{n}]$ is countable.

$A=\bigcup_n A_n$ ← countable $=$ countable

Case 3 $(-\infty, \cdot)$ or $(\cdot, \infty)$

Take $A_n := D_n \cap [a+\frac{1}{n}, n]$, $D_n \cap [-n, b-\frac{1}{n}]$

$(-\infty,\infty)$: take $D_n \cap [-n,n]$... countable union...

Application - a function can be discontinuous at every rational number. (e.g. modified Dirichlet)

Differentiation

$f: I \to \mathbb{R}$, $I=[a,b]$, (a,b), (a,∞) etc.

Def. $\boxed{f'(x) := \lim_{t\to x} \frac{f(t)-f(x)}{t-x} \text{ if RHS exists and } \in \mathbb{R}}$. At the endpoints, e.g. $f'(a) := \lim_{t\to a^+} \frac{f(t)-f(x)}{t-x}$

"local linearization"

$f(x+t)=f(x)+\underline{f'(x)}\cdot t \dots$

[$f:\mathbb{R}\to\mathbb{R}$, $f(x)=ax+b$ (technically $b\neq 0$ is affine not linear)

take $b=0$, $f(x)=ax$, $f(x+c\cdot y)=f(x)+c\,f(y)$

if $b\neq 0$, $f(x)+f(y) = ax+b+ay+b = a(x+y)+2b \neq f(x+y)$]

$\boxed{\text{Theorem: if } f \text{ is diff at } x \Rightarrow \text{cont at } x}$

Pf. $f(t)-f(x) = \underbrace{\frac{f(t)-f(x)}{t-x}}_{\to f'(x)\in\mathbb{R} \text{ fixed}} \underbrace{(t-x)}_{\to 0} \xrightarrow{t\to x} 0$

(wts $f(B_\delta(x)) \subseteq B_\varepsilon f(x)$)

(sequence criterion)

Theorem: f, g diff at $x \in I \Rightarrow$

1) $f+g$ diff, $(f+g)' = f' + g'$

add and subtract

2) $(fg)' = f'g + g'f$ $\quad \dfrac{(fg)(t) - (fg)(x)}{t-x} = f(t) \dfrac{g(t)-g(x)}{t-x} + g(x)\left(\dfrac{f(t)-f(x)}{t-x}\right)$

$\qquad\qquad f(t) \to f(x), \quad \to g'(x) + \quad \to f'(x)$

3) $\left(\frac{f}{g}\right)' = \dfrac{gf' - g'f}{g^2}$ if $g \neq 0$.
Let $f \equiv 1$, prove $\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$ then use product rule

$$\frac{1/g(t) - 1/g(x)}{t-x} = \frac{g(x) - g(t)}{g(t)g(x)(t-x)} = -\frac{1}{g(t)g(x)}\left(\frac{g(t)-g(x)}{t-x}\right)$$

Apply these to polynomials, rationals...

Theorem (Chain Rule) $\quad I \xrightarrow{f} J \xrightarrow{g} \mathbb{R}$

"special case of Taylor Expansion"

1) f continuous everywhere, diff at $x \in I$

2) $f(I) \subseteq J$, g diff at $f(x)$

* g need not be continuous anywhere

$\boxed{\Rightarrow g \circ f \text{ is diff at } x, \quad (g \circ f)'(x) = g'(f(x))f'(x)}$

Pf. $\quad \dfrac{f(t)-f(x)}{t-x} - f'(x) =: \delta(t) \xrightarrow[t \to x]{} 0$

$\Rightarrow f(t) - f(x) = (t-x)f'(x) + \delta(t)(t-x)$

$g(s) - g(y) = (s-y)g'(y) + \varepsilon(s)(s-y)$

$= f'(x) \cdot g'(f(x))$

$g(\underbrace{f(t)}_{\text{def} =: s}) - g(\underbrace{f(x)}_{=y}) = [f(t) - f(x)]\,[g'(y) + \varepsilon(s)]$

$= ((t-x)f'(x) + \delta(t)(t-x))\,(g'(y) + \varepsilon(s))$

Dividing both sides by $t-x$, $\quad \dfrac{g(f(t)) - g(f(x))}{t-x} = (f'(x) + \underbrace{\delta(t)}_{\to 0})\,[g'(y) + \underbrace{\varepsilon(\overbrace{f(t)}^{\text{continuity} \to f(x) = y})}_{\to 0}]$

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<u>Def.</u> $f: X \to \mathbb{R}$ has <u>local $\overset{(\min)}{\max}$ at x</u> $\iff$ $\exists \delta$ s.t. $\forall y \in B_\delta(x)$ ("neighborhood"), $f(y) \overset{(\geq)}{\leq} f(x)$

<u>Theorem</u>

$$\boxed{\text{let } f:[a,b] \to \mathbb{R},\ x \in (a,b) \text{ (interior point) local maximum, If } f'(x) \text{ exists, then } f'(x)=0.}$$


endpoints don't work

$$f'(x^-) = \lim_{t \uparrow x} \frac{\overbrace{f(t)-f(x)}^{\leq 0}}{\underbrace{t-x}_{<0}} \geq 0$$

$$f'(x^+) = \lim_{t \downarrow x} \frac{\overbrace{f(t)-f(x)}^{\leq 0}}{\underbrace{t-x}_{>0}} \leq 0$$

$$\Rightarrow \quad 0 \leq \underbrace{f'(x^-) = f'(x) = f'(x^+)}_{\text{since } f'(x) \text{ exists}} \leq 0$$

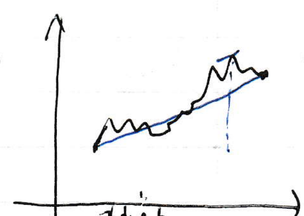
<u>Theorem</u> (Mean Value Theorem)

control of derivative
⇓
control of function

Bounded Theorem
Helper function

$f \in \mathcal{C}([a,b], \mathbb{R})$, diff in (a,b)

$\Rightarrow \exists\, x \in (a,b)$ s.t. $f(b)-f(a) = (b-a)f'(x)$


'same slope'

Case (1) $f(b)-f(a)=0$
- if f is constant, $f' \equiv 0$.
- else $\exists f(t) > f(a) \Rightarrow$ let $x \in [a,b]$ s.t. ($\exists$ due to compactness (p 40)) $f(x) = \max_{(a,b)} f \Rightarrow x \in (a,b)$

x is also a local max $\Rightarrow f'(x)=0$

(2) $f(b)-f(a) \neq 0$. Def $h(t) := f(t) - \frac{f(b)-f(a)}{b-a}(t-a)$ "linear prediction of c from t"

$h(t)$ satisfies (1) ... $h(b) = h(a) = f(a)$

$\exists\, x \in (a,b), h'(x)=0.$ $0 = h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a} = 0.$

Remark $f,g \in \mathcal{C}([a,b],\mathbb{R})$, diff on (a,b) $\Rightarrow$

$\exists\, x\in(a,b)$ s.t. $[f(b)-f(a)]\,g'(x) = [g(b)-g(a)]\,f'(x)$

$$\left(\frac{[f(b)-f(a)]}{[g(b)-g(a)]} = \frac{f'(x)}{g'(x)}\right)$$

special case, $g(b)=b$ ...

Proof let $h(t) := (f(b)-f(a))(g(t)-g(a)) - (g(b)-g(a))(f(t)-f(a))$. $h(a)=h(b)=0$

$\Rightarrow \exists\, x\in(a,b),\ h'(x)=0 \Rightarrow (f(b)-f(a))g'(x) = (g(b)-g(a))f'(x)$

Corollary $f:[a,b]\to\mathbb{R}$ diff on $[a,b]$ (including limits!) and $f'(a) < \lambda < f'(b)$

$\Rightarrow \exists\, x\in(a,b)$ s.t. $f'(x)=\lambda$

Proof. if f' is continuous $\to$ intermediate value theorem

else (not obvious!)

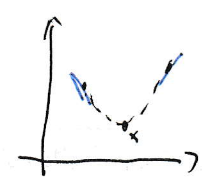
Pf. $g(t) := f(t) - \lambda t$ (differentiable!) $\Rightarrow g'(a) = f'(a) - \lambda < 0$, $g'(b) > 0$

$\Rightarrow \exists\, t_1 \in (a, a+\delta)$ s.t. $g(t_1) < g(a)$

and $t_2 \in (b-\delta, b))$ s.t. $g(t_2) < g(b)$ def. derivative

$\Rightarrow g$ attains its minimum at $x\in(a,b)$

$\Rightarrow g'(x) = 0 = f'(x) - \lambda$ QED.



Corollary If f is diff on $[a,b] \Rightarrow$ no jump discontinuities

Pf. (IVT) If we have

then we cannot have $a<y<b$ with $f'(y) = L$, where L is chosen to be different from $\lim_{x\to c^-} f(x)$, $\lim_{x\to c^+} f(x)$, and $f(c)$.

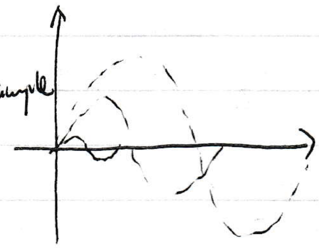
(MVT) choose b and c sufficiently close...

$f_n \to f$, f_n differentiable. Does $f_n' \to f'$?

pointwise
uniform ?

<u>No</u> Even if $f_n \to f$ uniformly.

Counterexample



$f_n = \frac{1}{\sqrt{n}} \sin(nx)$

$f_n \to 0 =: f$ uniformly in $\mathbb{R}$.

(deviation from 0 ↓)

Note that $f'(\cdot) = 0$.

But $f_n'(0) = \frac{1}{\sqrt{n}} n \cos(nx) \xrightarrow{n\to\infty} \pm\infty$

$f_n'(x)$ doesn't converge for typical x (fluctuates between $\pm\infty$)

$\Rightarrow$ We need stronger conditions for convergence of f_n — see p70.

Derivatives do not converge

limiting function can be nowhere differentiable

Weierstrass $f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$,

$0 < a < 1$, $b > 0$ odd, $ab > 1 + \frac{3}{2}\pi$

Theorem $f_n, f : E \to \mathbb{R}$ (not necessarily continuous)

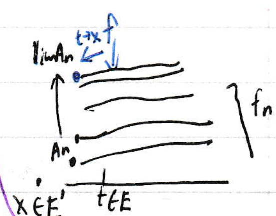
$f_n \to f$ uniformly on $E \subseteq X$

Let $x \in E'$. Assuming $\lim_{\substack{t\to x \\ \in E}} f_n =: A_n \in \mathbb{R}$

$f(x)$ need not be defined

$\Rightarrow \lim_n A_n$ and $\lim_{t\to x} f(t)$ exist and are equal, i.e.

$$\lim_n \left[\lim_{t\to x} f_n(t)\right] = \lim_{t\to x}\left[\lim_n f_n(t)\right]$$



$\overset{\checkmark}{A}$

Proof. 1) Since we don't know what the limit is, we show that $\underline{A_n \text{ is Cauchy}}$.

let $\varepsilon > 0$: Since $f_n \to f$ uniformly in E,

$\forall t \in E: |f_n(t) - f_m(t)| < \varepsilon$ given n, m large

$\downarrow t \to x$

$|A_n - A_m| \le \varepsilon \overset{\text{completeness}}{\Rightarrow} A_n \to A$

2) It suffices to show $\underline{A = \lim_{t \to x} f(t)}$

(more clearly, $A - A_n + A_n - f_n(t) + f_n(t) - f(t)$)

$|A - f(t)| \le |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|$

For $\varepsilon > 0$, pick n: (fixed!)

After picking n, choose t: Define $\delta = \delta(\varepsilon, n) = \delta(\varepsilon)$ s.t. $|f_n(t) - A_n| < \varepsilon/3 \ \forall t \in B_\delta(x)$ and then pick any such t.

☆

$\underline{\text{Theorem}}$ $\forall n$, $f_n : [a,b] \to \mathbb{R}$ differentiable, $\exists x_0 \in [a,b]$ s.t. $f_n(x_0) \xrightarrow{n \to \infty} := f(x_0)$ (some limit).

Assume f_n' conv. uniformly on $[a,b]$.

Then ① $f_n \to f$ uniformly on $[a,b]$ and

② $f'(x) = \lim_{n \to \infty} f_n'(x) \ \forall x \in [a,b]$ the derivative converges

Key idea: Mean Value Theorem (Know Derivatives)

① $|f_n(x) - f_m(x)| \le \dots \le \varepsilon$

$|[f_n(x) - f_m(x)] - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)|$

'$f(b) - f(a)$' $\downarrow$ M.V.T $\qquad < \varepsilon/2$ for $n, m \ge N_1$

$= |(f_n'(\xi) - f_m'(\xi))(x - x_0)| \overset{n,m \ge N_2}{\le} |x - x_0| \cdot \left(\frac{\varepsilon}{2|b-a|}\right) \le \varepsilon/2$

② We have established the pointwise limit $f := \lim_n f_n$

let $x \in [a,b]$ fixed and

$$\forall t \neq x: \quad g_n(t) = \frac{f_n(t) - f_n(x)}{t-x} \xrightarrow{n\to\infty} \frac{f(t)-f(x)}{t-x} \quad (= g(t))$$

Furthermore, $g_n(t) \xrightarrow{t\to x} f_n'(x)$ (we used differentiability here)

$$\text{So } |g_n(t) - g_m(t)| = \frac{|(f_n(t)-f_m(t)) - (f_n(x) - f_m(x))|}{|t-x|}$$

Same idea as $\leq \dfrac{\varepsilon}{2|b-a|} \quad \forall n,m \geq N(\varepsilon)$

$\therefore$ As $n\to\infty$, $g_n(t)$ converges uniformly in $t \in [a,b]$ to $g(t)$

Then $\forall x \in [a,b]$:

$$\underbrace{\lim_{t\to x} \big(\underbrace{\lim_n g_n(t)}_{g(t)} \big)}_{f'(x)} \underset{\substack{\text{swap limits}\\\text{w/ prev}\\\text{theorem}}}{=} \lim_{n\to\infty} \Big[\underbrace{\lim_{t\to x} g_n(t)}_{f'_n(x)} \Big]$$

Since x was arbitrary, we conclude that $\forall x \in [a,b], f'(x) = \lim_{n\to\infty} f_n'(x)$ as required.

Recall, for $g(x) := \sum_{k \geq 0} c_k x^k$ (power series) (restricted to $\mathbb{R}$),
$R := \left(\overline{\lim_n} \sqrt[n]{|c_n|}\right)^{-1}$ is the radius of convergence.
i.e. if $|r| < R$, $\sum c_n r^n$ is absolutely convergent.

Claim: (Termwise Differentiation)

$h(x) := \sum_{k \geq 1} k \cdot c_k x^{k-1}$ has the same R as $g(x)$.

Proof $\underbrace{\overline{\lim_n} \sqrt[n]{|c_n|}}_{\text{original coeffs}} = \underbrace{\overline{\lim_n} \left(\sqrt[n]{|c_n|} \cdot \sqrt[n]{n}\right)}_{\text{derivative coefficients}}$

$\Rightarrow$ In every compact subset of $(-R, R)$ (such as the closed interval $[-R+\varepsilon, R-\varepsilon]$)

$h(x) = g'(x)$, furthermore every power series is infinitely differentiable.

Note we used the previous theorems to show that the derivative converges uniformly within this subset (p 61), and to the 'actual' derivative (p 70).

$f \in \mathcal{C}^{n-1}$: $n-1$ times continuously differentiable

(the n^{th} derivative need not be continuous, but must exist)

Theorem (Taylor)

$f \in \mathcal{C}([a,b],\mathbb{R})$, $n \in \mathbb{N}^+$, $f^{(n-1)} \in \mathcal{C}[a,b]$ and diff. on (a,b).

let $a \leq \alpha \neq \beta \leq b$. $[a \; \alpha \; \beta \;]_b$ (wlog $\alpha < \beta$)

$$P(t) := \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k \quad (\text{"expansion" around } \alpha)$$

$\Rightarrow \exists \; \xi \in (\alpha,\beta)$ or (β,α) s.t. $f(\beta) = P(\beta) + \frac{f^{(n)}(\xi)}{n!}(\beta-\alpha)^n$ (depends on β)

Proof. Certainly $\exists! \; M$ s.t. $f(\beta) = P(\beta) + M(\beta-\alpha)^n$.

WTS: $M = \frac{1}{n!} f^{(n)}(\xi)$ for some $\xi \in (\alpha,\beta)$

Set the 'difference' $g(t) := f(t) - P(t) - M(t-\alpha)^n$, $t \in [a,b]$ (P: $n-1$ degree)

$\Rightarrow g^{(n)}(t) = f^{(n)}(t) - 0 - Mn!$

So we need to find $\xi \in (\alpha,\beta)$ s.t. $g^{(n)}(\xi) = 0$ ($\Leftrightarrow f^{(n)}(\xi) = Mn!$)

Recall MVT: $g(\alpha) - g(\beta) = (\alpha-\beta)\, g'(\xi_1)$

Notice that $g(\beta) = 0$ (that's how we chose M!), and [we have $g^{(k)}(\alpha) = 0 \;\; \forall k = 0 \dots n-1$] 'automatically'.

Then $g'(\alpha) = g'(\xi_1) = 0 \Rightarrow \exists \; \xi_2 \in (\alpha, \xi_1)$ s.t. $g''(\xi_2) = 0$ etc...

Theorem (L'Hôpital)

See p68.
Recall that

$f, g \in \mathcal{C}([a,b], \mathbb{R})$, diff on (a,b)

$\exists\, x \in (a,b)$ s.t. $[f(b)-f(a)]g'(x) = [g(b)-g(a)]f'(x)$

$$\left(\frac{|f(b)-f(a)|}{|g(b)-g(a)|} = \frac{f'(x)}{g'(x)} \right)$$

Let $-\infty < a, b \le \infty$, g, f diff. in (a,b), $g'(x) \neq 0$ for $x \in (a,b)$.

Assume $\dfrac{f'(x)}{g'(x)} \xrightarrow[x \to a]{(*)} A \in [-\infty, \infty]$

and ① $f(x) \to 0$ and $g(x) \to 0$ (as $x \to a$)

OR

② $f, g \to +\infty$ as $x \to a$

Then $\dfrac{f(x)}{g(x)} \to A$ (as $x \to a$)

Proof of ① Let $-\infty \le A < \infty$

Choose $r \in \mathbb{R}$, $A < r$. By $(*)$, $\exists\, c \in (a,b)$ s.t. $f'(x)/g'(x) < r$ $\forall x \in (a,c)$

Let $a < x < y < c$, by , $\exists\, \xi \in (x,y)$ s.t.

$$\frac{f(y)-f(x)}{g(y)-g(x)} = \frac{f'(\xi)}{g'(\xi)} < r.$$ As $x \to a$, by ① we have (and sequence criterion)

$$\frac{f(y)}{g(y)} \le r, \forall y \in (a,c).$$

$$\therefore \overline{\lim_{y \to a}} \frac{f(y)}{g(y)} \le r \quad \forall r > A$$

$\therefore$ ( ) $\le A$

If $A=\infty$, $\overline{\lim}_{y\to c}\,(\cdot) \le A$ ✓

let $r' < A$

$\exists\, c \in (a,b)$ s.t. $\frac{f'(x)}{g'(x)} \ge r' \;\forall x \in (a,c)$

$\frac{f(y)-f(x)}{g(y)-g(x)} = \frac{f'(\xi)}{g'(\xi)} > r'$

$\vdots$

$A > -\infty$ (i.e. $A \in \mathbb{R}$) we show similarly that $\underline{\lim}_{y\to c} \frac{f(y)}{g(y)} \ge A$

$$A \le \underline{\lim}_{y\to c} \frac{f(y)}{g(y)} \le \overline{\lim}_{y\to c} \frac{f(y)}{g(y)} \le A$$

$$\therefore A = \lim_{y\to a} \frac{f(y)}{g(y)}$$

$\boxed{\text{The Riemann Integral}}$

"more ordered/neater" than differentiation

not necessarily continuous!

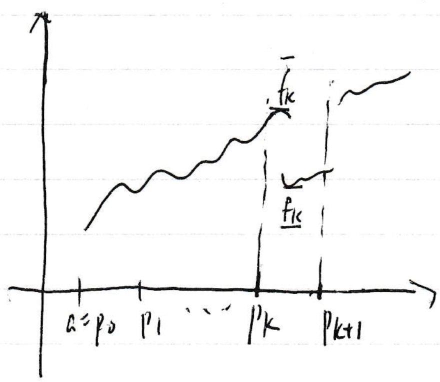
$f: [a,b] \to \mathbb{R}$, bounded: $|f| \le M < \infty$

Define the partition

$$P = \{a = p_0 < p_1 < p_2 \dots < p_n = b\}$$

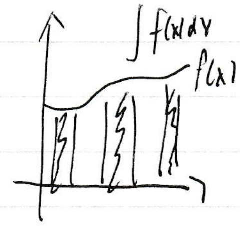
Define the upper Riemann sum:

$$\rightarrow \boxed{U(P,f)} := \sum_{k=0\dots n-1} \underbrace{\Delta_k P}_{p_{k+1}-p_k} \cdot \underbrace{\sup_{x\in[p_k,p_{k+1}]} f(x)}_{:=\bar{f}_k}$$

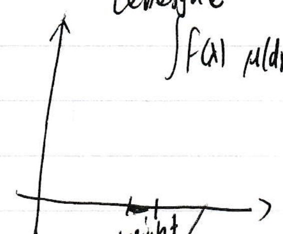


$$\rightarrow \boxed{L(P,f)} = \sum_k \Delta_k P \cdot \underbrace{\inf_{x\in[p_k,p_{k+1}]} f(x)}_{:=\underline{f}_k}$$

$(= -U(P,-f))$



Lebesgue



① $f_n \to f$

$\not\Downarrow$

$\int f_n \to \int f$

② Cannot integrate some functions e.g. Dirichlet

Remarks lower bound upper bound

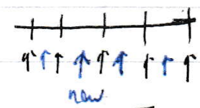
① $m \le f \le M$:

$$m(b-a) \le L(P) \le U(P) \le M(b-a)$$

② $P \subseteq Q \Rightarrow U(Q) \le U(P)$

e.g. add 1 more point to the partition

$p_0 < p_1 \dots < p_k < q < p_{k+1} \dots$



Proof. $U(P) - U(Q) = (p_{k+1}-p_k)\overbrace{\sup_{[p_k,p_{k+1}]} f}^{\bar f_k} - (p_{k+1}-q)\overbrace{\sup_{[q,p_{k+1}]} f}^{\le \bar f_k} - (q-p_k)\overbrace{\sup_{[p_k,q]} f}^{\le \bar f_k}$

$$\ge \Delta_k p \cdot \bar f_k - \bar f_k \, \cancel{(p_{k+1} - q + q - p_k)} = 0$$

②' $P \subseteq Q \Rightarrow L(Q) \ge L(P)$

③ $\forall P, P' \Rightarrow L(P) \le U(P')$

Proof. $Q := P \cup P'$ (common refinement)

$$\Rightarrow \underbrace{L(P) \le L(Q)}_{P \subseteq Q} \underbrace{\le}_{①} U(Q) \underbrace{\le}_{Q \supseteq P'} U(P')$$