21-469 Cheatsheet

$E(t) = \int_0^1 (u(t,x))^2 dx$

Transport Equation (T) $\begin{cases} u_t + a u_x = 0 \\ u(0,x) = u_0(x) \end{cases}$ (or $g(x,t)$)

$a>0$: left-to-right

Assuming periodic boundary conditions, $u(t,0) = u(t,1)$, $u_x(t,0) = u_x(t,1)$

General case: $a = a(t,x)$

The trajectory of a particle depends on time and its initial position, $\eta = \eta(t, x_0)$ and has speed a units.

$\eta(t,x_0) = \int_0^t a \, ds + x_0$

solve ① $\begin{cases} \frac{d}{dt}\eta(t,x_0) = a(\eta(t,x_0),t) \\ \eta(0,x_0) = x_0 \end{cases}$

$u(t, x_0 + at) = u_0(x_0)$

and ② $\begin{cases} \frac{d}{dt} u(t,\eta(t,x_0)) = 0 \quad (\text{or } g(x,t)) \\ u(0,\eta(0,x_0)) = u(0,x_0) = u_0(x_0) \end{cases}$

$u(t,x) = u_0(x-at)$

The upwind scheme is

$$\frac{v_j^{m+1}-v_j^m}{\Delta t} + a\frac{v_{j+1}^m - v_{j-1}^m}{2\Delta x} - |a|\frac{v_{j+1}^m - 2v_j^m + v_{j-1}^m}{2\Delta x} = 0$$

with periodic B.C: $v_0^{m+1} = v_n^{m+1}$, $v_{n+1}^{m+1} = v_1^{m+1}$

$a>0$: $\frac{v_j^{m+1}-v_j^m}{\Delta t} + a\frac{v_j^m - v_{j-1}^m}{\Delta x} = 0$

$a<0$: $\frac{v_j^{m+1}-v_j^m}{\Delta t} + a\frac{v_{j+1}^m - v_j^m}{\Delta x} = 0$

Discrete Energy: $E^m := \Delta x \sum_{j=1}^n (v_j^m)^2$

$E^{m+1} \le E^m$ $\forall m$ when $\boxed{|a|\frac{\Delta t}{\Delta x} \le 1}$

($a<0$ case here, $a>0$ on pg 57)

Proof sketch:

$a\frac{v_{j+1}^m - v_j^m}{\Delta x} = a\frac{v_{j+1}^m - 2v_j^m + v_{j-1}^m}{2\Delta x} + a\frac{v_{j+1}^m - v_{j-1}^m}{2\Delta x}$

$\frac{v_{j+1}^m - v_j^m}{\Delta x} + a\frac{v_{j+1}^m - v_{j-1}^m}{2\Delta x} = -a\frac{v_{j+1}^m - 2v_j^m + v_{j-1}^m}{2\Delta x}$

$v_j \frac{v_{j+1}^m - v_j^m}{\Delta x} + a v_j\left(\frac{v_{j+1}^m - v_{j-1}^m}{2\Delta x}\right) = \frac{-a}{2\Delta x} v_j^m\left((v_{j+1}^m - v_j^m) - (v_j^m - v_{j-1}^m)\right)$

$\underbrace{\sum_{j=1}^n 2v_j(v_{j+1}^m - v_j^m)}_{E^{m+1} - E^m - \sum (v_j^{m+1} - v_j^m)^2 \text{ plug in scheme}} + \underbrace{\sum_{j=1}^n a(v_{j+1}^m - v_{j-1}^m)}_{=0 \text{ by telescoping}} = \underbrace{-a\sum_{j=1}^n v_j^m\left((v_{j+1}^m - v_j^m) - (v_j^m - v_{j-1}^m)\right)}_{\text{by parts}}$

can use to prove E uniqueness of solutions

$E(t) = \int_0^1 (\overset{u_t}{v(t,x)})^2 + (\overset{u_x}{w(t,x)})^2 dx = \int_0^1 |U(t,x)|^2 dx$, where $U(\cdot) := \binom{v}{w}$

Wave Equation (W) $\begin{cases} u_{tt} = u_{xx}, \; x \in \mathbb{R}, t>0 \\ u(0,x) = f(x) \\ u_t(0,x) = g(x) \end{cases} x \in \mathbb{R}$

$f, g \in C^1(\mathbb{R})$, compactly supported, continuously differentiable

Idea: Write $w = \partial_t u + \partial_x u$, solve $\begin{cases} \partial_t w + \partial_x w = 0 \\ w(0,x) = g(x) + f'(x) \end{cases}$

$$u(t,x) = \frac{1}{2}(f(x+t) + f(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds$$

(d'Alembert)

On a bounded domain $(u(t,0) = u(t,1) = 0)$ (Dirichlet B.C.), where

$f(x) = \sum_k a_k \sin(k\pi x)$ and $g(x) = \sum_k \pi k b_k \sin(k\pi x)$, separation of variables gives us $u_k(t,x) = (a_k \cos(k\pi t) + b_k \sin(k\pi t))\sin(k\pi x)$

$\frac{d}{dt}u_k(t,x)\big|_{x=0} = 0$

Approximation: $\frac{v_j^{m+1} - 2v_j^m + v_j^{m-1}}{\Delta t^2} = \frac{v_{j+1}^m - 2v_j^m + v_{j-1}^m}{\Delta x^2}$ $m = 1,2,3,\dots$ $j = 1,2,\dots,n$

$v_0^m = v_{n+1}^m = 0$

$v_j^0 = f(x_j)$, $v_j^1 = v_j^0 + \Delta t\, \overset{u_t(0)}{g(x_j)} + \frac{\Delta t^2}{2\Delta x^2}\overset{u_{tt}(0)}{(v_{j+1}^0 - 2v_j^0 + v_{j-1}^0)}$

Discrete Energy: $E^m := \Delta x \sum_{j=1}^n \left[\left|\frac{v_j^{m+1} - v_j^m}{\Delta t}\right|^2 + \frac{1}{2}\left(\left|\frac{v_{j+1}^{m+1} - v_j^{m+1}}{\Delta x}\right|^2 + \left|\frac{v_{j+1}^m - v_j^m}{\Delta x}\right|^2\right)\right]$

$E'(t) = 0$ can be shown in two ways

(1) Differentiation

(2) $u_t u_{tt} = u_t u_{xx}$, $\int u_t u_{tt}\,dx = \int u_t u_{xx}\,dx$

$\frac{1}{2}\frac{d}{dt}\int u_t^2\,dx = [u_t u_x]_0^1 - \int u_x u_{tx}\,dx$

$= -\frac{1}{2}\int_0^1 \frac{d}{dt} u_x^2\,dx$

Discrete Stability $\underbrace{\Delta x \sum_{j=1}^n \left[\left|\frac{v_j^{m+1} - v_j^m}{\Delta t}\right|^2 + \frac{1}{2}\left(\left|\frac{v_{j+1}^{m+1} - v_j^{m+1}}{\Delta x}\right|^2 + \left|\frac{v_{j+1}^m - v_j^m}{\Delta x}\right|^2\right)\right]}_{E^m} =$

$\underbrace{\Delta x \sum_{j=1}^n \left|\frac{v_j^1 - v_j^0}{\Delta t}\right|^2 + \frac{\Delta x}{2}\sum_{j=1}^n \left(\left|\frac{v_{j+1}^1 - v_j^1}{\Delta x}\right|^2 + \left|\frac{v_{j+1}^0 - v_j^0}{\Delta x}\right|^2\right)}_{E^0} + \frac{\Delta x}{2}\sum_{j=1}^n \left|\frac{v_{j+1}^{m+1} - v_j^{m+1} - v_{j+1}^m + v_j^m}{\Delta x}\right|^2 - \frac{\Delta x}{2}\sum_{j=1}^n \left|\frac{v_{j+1}^1 - v_j^1 - v_{j+1}^0 + v_j^0}{\Delta x}\right|^2$

Stability (a) $\|v\|_\infty$ ∞-norm, (b) $E(t)$ energy, (c) effect of perturbation

Consistency τ (truncation error) $\to 0$ as $\Delta t, \Delta x \to 0$

Convergence 'Error' $e_j^m := v_j^m - u(t^m, x_j) \to 0$ as $\Delta t, \Delta x \to 0$

(Leibniz)

$$\frac{d}{dx}\int_{a(x)}^{b(x)} F(x,y)\,dy = \int_{a(x)}^{b(x)} F_x(x,y)\,dy + F(x,b(x))b'(x) - F(x,a(x))a'(x)$$

$\hookrightarrow \frac{d}{dx}\int_0^x F(x,y)\,dy = F(x,x) + \int_0^x F_x(x,y)\,dy$

Truncation Error $= \tau_{\Delta x \Delta t}(t^m, x_j)$

→ evaluate actual solution in scheme

→ more terms around so RHS $= 0$ if we are using discrete points

(Lax Equivalence) A consistent linear numerical method of the form $v^{m+1} = Bv^m + b^m$ is convergent $\Leftrightarrow$ Lax-Richtmyer stable

$\|B^m\| \le C_T$, m satisfying $m\Delta t \le T$

($\Leftarrow$) Similar to Hw 8 Q1e.

$\|E^{m+1}\| \le \Delta t \sum_{k=0}^m \max_j \|\tau_j^k\|$

$\le m\Delta t \max_{j,k} \|\tau_j^k\|$

$\to 0$ as $\Delta t, \Delta x \to 0$.

$v^{m+1} = Bv^m + b^m$

$- u^{m+1} = Bu^m + b^m$

$E^{m+1} = BE^m - \Delta t\, \tau^m$

Heat Equation

$u_t - u_{xx} = 0$
$u(0,x) = u_0(x)$

[Bounded (optional)] $u(t,0) = u(t,1) = 0$

$E^m := \Delta x \sum_{j=1}^{n} (v_j^m)^2$

or $E(t) := \int_0^1 (u(t,x))^2 dx$ 'energy' $\dashrightarrow$ ① $2\int_0^1 u(t,x)\, u_{xx}(t,x)\, dx$ $= 2\, u(t,x) u_x(t,x)\big|_0^1 - \int_0^1 (u_x(t,x))^2 dx$

$E^{m+1} \overset{*}{\leq} E^m$ (using Dirichlet B.C.), $E'(t) \leq 0$: energy is decreasing

② $u_t - u_{xx} = 0 \Leftrightarrow u \cdot u_t = u \cdot u_{xx}$ $\Leftrightarrow \int_0^1 u \cdot u_t\, dx = \int_0^1 u \cdot u_{xx}\, dx$

$\frac{d}{dt}\left(\frac{1}{2}\int_0^1 u^2 dx\right) = u \cdot u_x\big|_0^1 - \int_0^1 u_x^2 dx$

$\Rightarrow E(t) \leq \int_0^1 (u_0(x))^2 dx$

$\int_0^1 (u(t,x) - v(t,x))^2 dx \leq \int_0^1 (u_0(x) - v_0(x))^2 dx$ Again, uniqueness follows from stability.

Exact Solutions ① (Unbounded) $u(0,x) = H(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$: $u(t,x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x/2\sqrt{t}} e^{-\theta^2} d\theta$

② (Unbounded) $u(0,x) = \phi(x)$ (General case) : $u(t,x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t} \phi(y)\, dy$ where $s(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4kt}$ is the heat kernel

③ (Bounded) $u(t,0) = u(t,1) = 0$, $u(0,x) = u_0(x) = \sum_{k \geq 1} C_k \sin(k\pi x)$

$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda \Rightarrow X(x) = \sum_k \sin(k\pi x)$, $T(t) = \sum_k C_k e^{-(k\pi)^2 t}$

Important identities/inequalities

$a^2 - b^2 = 2b(a-b) + (a-b)^2$ — Heat explicit

$(a-b)^2 \leq 2a^2 + 2b^2$

$(a+b)^2 \leq 2a^2 + 2b^2$ — HW7

$a(b-a) = \frac{b^2}{2} - \frac{a^2}{2} - \frac{1}{2}(b-a)^2$ — Transport/Wave

$b(b-a) = \frac{b^2}{2} - \frac{a^2}{2} + \frac{1}{2}(b-a)^2$

Don't forget Telescoping/Summation by parts

Finite Differences:

(Bounded, $v_j^0 = u_0(x_j)$, $v_0^m = v_{n+1}^m = 0$)

$\frac{v_j^{m+1} - v_j^m}{\Delta t} - \frac{v_{j+1}^m - 2v_j^m + v_{j-1}^m}{\Delta x^2} = 0$

① Explicit Heat $v_j^{m+1} = v_j^m + \frac{\Delta t}{\Delta x^2}(v_{j+1}^m - 2v_j^m + v_{j-1}^m)$

Energy stability/∞-norm both require $\Delta t \leq \frac{\Delta x^2}{2}$

Truncation error $O(\Delta t + \Delta x^2)$

② Implicit Heat $\frac{v_j^{m+1} - v_j^m}{\Delta t} = \frac{v_{j+1}^{m+1} - 2v_j^{m+1} + v_{j-1}^{m+1}}{\Delta x^2}$

$r := \frac{\Delta t}{\Delta x^2}$, $A v^{m+1} = v^m$ where $A = \begin{pmatrix} 1+2r & -r & & 0 \\ -r & \ddots & \ddots & \\ & \ddots & \ddots & -r \\ 0 & & -r & 1+2r \end{pmatrix}$

same as above

③ $\frac{v_j^{m+1} - v_j^m}{\Delta t} = \frac{v_{j+1}^{m+1/2} - 2v_j^{m+1/2} + v_{j-1}^{m+1/2}}{\Delta x^2}$, $v_j^{m+1/2} := \frac{1}{2}(v_j^m + v_j^{m+1})$

τ: $O(\Delta t^2 + \Delta x^2)$ $V^{m+1} = A_1^{-1} A_2 V^m$, $A_1 = \begin{pmatrix} 1+r & -r/2 & \\ -r/2 & \ddots & \ddots \\ & \ddots & -r/2 \\ & -r/2 & 1+r \end{pmatrix}$, $A_2 = \begin{pmatrix} 1-r & r/2 & \\ r/2 & \ddots & \\ & & \end{pmatrix}$

Lax-Richtmyer stability: Choose $C_T = 1$, show $\|B\|_2 \leq 1$, then $\|B^m\|_2 \leq (\|B\|_2)^m$

Explicit: $\Delta t \leq \frac{\Delta x^2}{2}$, Implicit: Unconditionally stable!

Poisson Equation (in 1D)

$-u''(x) = f(x)$, $u(0) = u(1) = 0$

$\rightarrow$ Uniqueness follows from stability

$\rightarrow$ Green's Function: $u(x) = \int_0^1 G(x,y) f(y)\, dy$ where $G(x,y) = \begin{cases} y(1-x) & 0 \leq y \leq x \leq 1 \\ x(1-y) & 0 \leq x \leq y \leq 1 \end{cases}$

$\|u\|_\infty \leq \frac{1}{8}\|f\|_\infty$

$\|V\|_{h,\infty} \leq \frac{1}{8}\|f\|_\infty$ for the numerical scheme

$A u = h^2 f$, $A = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & \ddots & \\ & & & 2 & -1 \end{bmatrix}$

$u \approx h^2 A^{-1} f$ $= h\underbrace{G^h f}_{\text{"discrete integral"}}$

$f \in C^2([0,1]) \Rightarrow \|\tau_h\|_{h,\infty} \leq \frac{\|f''\|_\infty}{12} h^2$ (consistency)

$\|u - v\|_{h,\infty} \leq \frac{\|f''\|_\infty}{96} h^2$ (convergence)

Let L be the $(-)$ (negative 2^{nd} derivative) operator.

Then we look for eigenfunction $v \in C^2([0,1])$ such that $Lv = \lambda v \Leftrightarrow v''(x) + \lambda v(x) = 0$

For $u \neq 0$,

$\langle Lu, u \rangle > 0$, $\langle u, u \rangle > 0 \Rightarrow \langle Lu, u \rangle = \lambda \underbrace{\langle u, u \rangle}_{>0} > 0$

must be positive!

So we write $\beta = \sqrt{\lambda}$, solve $u''(x) + \beta^2 u(x) = 0$, such that $u(0) = u(1) = 0$.

$u(x) = a\sin(\beta x) + \underbrace{b\cos(\beta x)}_{b=0 \text{ by B.C.}}$

$u(1) = 0 \Rightarrow \beta = k\pi$

So $v_k = \sin(k\pi x)$, $k \in \mathbb{N}$: "eigenfunctions" that are orthogonal w.r.t $\langle \cdot, \cdot \rangle$: $\int_0^1 \sin(k\pi x)\sin(m\pi x)\, dx = \begin{cases} 0 & k \neq m \\ 1/2 & k = m \end{cases}$

$\lambda_k = (k\pi)^2$

In the discrete case, $Av = \mu v$, $\frac{1}{h^2}\begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & & -1 \\ 0 & & -1 & 2 \end{pmatrix}$ $\Rightarrow v_j^k = \sin(k\pi x_j)$

$\mu_k = \frac{2}{h^2}(1 - \cos(k\pi h)) = \frac{4}{h^2}\sin^2\left(\frac{k\pi h}{2}\right)$

We can similarly show $\vec{v}^k$'s are linearly independent w.r.t. $\langle v, w \rangle_h := h\left(\frac{v_0 w_0 + v_{n+1} w_{n+1}}{2} + \sum_{j=1}^{n} v_j w_j\right)$

Furthermore $\mu_k \rightarrow \lambda_k$ as $h \rightarrow 0$ (Taylor Series)

- Positive Definite $\Rightarrow$ Non-singular

Strictly diagonally dominant $\Rightarrow$ Not singular

- Rate of convergence: $\log(e_h) = \alpha \log(h) + C$

Plug in values for h to compute α

e_h: 10^{-3}, 10^{-4}, 10^{-5}; h: 10^{-2}, 10^{-1}

$\langle u, v \rangle := \int_0^1 u(x) v(x)\, dx$ is an inner product! Can use Cauchy-Schwarz (Hölder?)

Lax Equivalence: $V^{m+1} = BV^m + b^m$ (see details in next page)

$\frac{E^{m+1} - E^m}{\Delta t} = \frac{\Delta x}{\Delta t}\sum_{j=1}^{n}[(v_j^{m+1})^2 - (v_j^m)^2]$

$= \frac{2\Delta x}{\Delta t} v_j^m (v_j^{m+1} - v_j^m) + \frac{\Delta x}{\Delta t}\sum_{j=1}^{n}(v_j^{m+1} - v_j^m)^2$

$= \frac{2\Delta x}{\Delta x^2} v_j^m (v_{j+1}^m - 2v_j^m + v_{j-1}^m)$

...

Extension: 2D Poisson: $u_{xx} + u_{yy} = f(x,y)$

$\hookrightarrow$ Project: $\partial_t u - d_u \Delta u = -uv^2 + f(1-u)$

$\partial_t v - d_v \Delta v = uv^2 - (f+k)v$

$(\Delta u = u_{xx} + u_{yy})$

$\Delta_h f_{i,j} := \frac{f_{i,j+1} + f_{i,j-1} + f_{i+1,j} + f_{i-1,j} - 4f_{i,j}}{\Delta x^2}$

Explicit (straightforward) $U_{i,j}^{m+1} = (1-4r_u) U_{i,j}^m + r_u U_{i,j+1}^m + r_u U_{i,j-1}^m + r_u U_{i+1,j}^m + r_u U_{i-1,j}^m + \Delta t \, G_u(U_{i,j}, V_{i,j})$

i.e. $\frac{U_{i,j}^{m+1} - U_{i,j}^m}{\Delta t} - d_u \Delta_h U_{i,j}^m = -U_{i,j}^m (V_{i,j}^m)^2 + f(1 - U_{i,j}^m)$

Implicit (Semi) $\left[\begin{matrix} \text{kron}(I,A) \\ + \text{kron}(S,I) \end{matrix}\right]_{n^2 \times n^2}$ Kronecker tensor $V^{m+1} = V^m + G(u,v)$ etc.

'Heat' i.e. $\frac{U_{i,j}^{m+1} - U_{i,j}^m}{\Delta t} - d_u \Delta_h U_{i,j}^{m+1/2} = -U_{i,j}^m (V_{i,j}^m)^2 + f(1 - U_{i,j}^m)$