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Imagine now we have

$$(H) \begin{cases} u_t - u_{xx} = \underline{g(t,x)} \quad , x \in (0,1), t > 0 \\ u(0,x) = u_0(x) \\ u(t,0) = u(t,1) = 0 \quad , t \geq 0 \end{cases}$$

'heat source/sink'
where $g : [0,\infty) \times [0,1] \to \mathbb{R}$ a given function
$g_j^m := g(t^m, x_j)$

Then $$\frac{v_j^{m+1} - v_j^m}{\Delta t} = \frac{v_{j+1}^m - 2v_j^m + v_{j-1}^m}{\Delta x^2} + \underline{g_j^m}$$

'IMPLICIT HEAT' / Backward Euler

motivation:
$$\frac{u(t+\Delta t, x) - u(t,x)}{\Delta t} = u_t(t,x) + O(\Delta t) \quad \text{(explicit heat)}$$

$\downarrow$ use this instead?

$$\frac{u(t,x) - u(t-\Delta t, x)}{\Delta t} = u_t(t,x) + O(\Delta t)$$

So now we have $$\frac{v_j^m - v_j^{m-1}}{\Delta t} = \frac{v_{j+1}^m - 2v_j^m + v_{j-1}^m}{\Delta x^2}, \quad j = 1 \dots n, \ m \geq 1$$

shifting indices in time

$v_0^m = v_{n+1}^m = 0 \quad m \geq 0$

$v_j^0 = u_0(x_j) \quad , j = 1 \dots n$

$$\frac{v_j^{m+1} - v_j^m}{\Delta t} = \frac{v_{j+1}^{m+1} - 2v_j^{m+1} + v_{j-1}^{m+1}}{\Delta x^2} \quad j = 1 \dots n, \ m \geq 0$$

* see next page

JG

Problem: The unknowns v_j^m appear on the right hand side and are coupled...

Let's try to reformulate the equation:

From $$\frac{v_j^{m+1} - v_j^m}{\Delta t} = k\frac{v_{j+1}^{m+1} - 2v_j^{m+1} + v_{j-1}^{m+1}}{\Delta x^2} \quad j=1\dots n,\ m\geq 0$$

Set $r := k\dfrac{\Delta t}{\Delta x^2}$. Then $v_j^{m+1} - r v_{j+1}^{m+1} + 2r v_j^{m+1} - r v_{j-1}^{m+1} = v_j^m$, $j=1\dots n$, $m\geq 0$

Recall Poisson Eq:
$-2v_j + v_{j+1} + v_{j-1} = h^2 f_j$.

$$(1+2r)v_j^{m+1} - r v_{j+1}^{m+1} - r v_{j-1}^{m+1} = v_j^m$$

Now define

$$v^{m+1} := (v_1^{m+1}, \dots v_n^{m+1})^T,\ v^m := (v_1^m \dots v_n^m)^T$$

$$A = \begin{pmatrix} 1+2r & -r & & 0 \\ -r & \ddots & \ddots & \\ & \ddots & \ddots & -r \\ 0 & & -r & 1+2r \end{pmatrix}$$

$$A v^{m+1} = v^m,\ m\geq 0$$

A is (strictly) diagonally dominant $\Rightarrow$ invertible $\Rightarrow$ $\boxed{v^{m+1} = A^{-1} v^m \text{ (solve this at every time level)}}$
$(r>0)$ (p18)

Simulation Details

$u_t - k u_{xx} = 0$ ← heat conductivity

$\boxed{\alpha = 1}$

implicitheat.m

>> implicitheat(1/3200, 40, ...) ✓

(1/40, 40) ✓ still works!

$dt \approx dx$

(1/1000, 1000 ...)

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More about stability... (see p74-78 later) $\rho := \max_{x \neq 0} \frac{\|Ax\|_2^2}{\|x\|_2^2}$)

$$\|v^{m+1}\|_2 \le \|A^{-1}v^m\|_2 \le \underline{\|A^{-1}\|_2} \; \|v^m\|_2$$

$\|$

* $\rho(A)\,(:= \max|\lambda_i|$ (largest eigenvalue)) when A is symmetric

* $\|A\|_2^2 = \rho(A^TA) = \rho(A^2) = (\rho(A))^2$

A better idea is to approximate u_t by a second-order method:

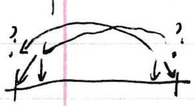
$$\frac{v_j^{m+1} - v_j^m}{\Delta t} = \frac{1}{2}\left(\frac{v_{j+1}^m - 2v_j^m + v_{j-1}^m}{\Delta x^2}\right) + \frac{1}{2}\left(\frac{v_{j+1}^{m+1} - 2v_j^{m+1} - v_{j-1}^{m+1}}{\Delta x^2}\right)$$

$$= \frac{v_{j+1}^{m+1/2} - 2v_j^{m+1/2} + v_{j-1}^{m+1/2}}{\Delta x^2}, \quad v_j^{m+1/2} := \frac{1}{2}\left(v_j^m + v_j^{m+1}\right)$$

This is stable still (effects 'cancel out'?)

and the convergence is now $\alpha = 2$ $\boxed{\text{'Crank-Nicolson'}}$ *A

(in both time and space)



let $r := \frac{\Delta t}{\Delta x^2}$

$$v_j^{m+1} - \frac{r}{2} v_{j+1}^{m+1} + r v_j^{m+1} - \frac{r}{2} v_{j-1}^{m+1} = v_j^m + \frac{r}{2} v_{j+1}^m - r v_j^m + \frac{r}{2} v_{j-1}^m \quad + \Delta t\, g(t)$$

$$(1+r)\, v_j^{m+1} - \frac{r}{2} v_{j+1}^{m+1} - \frac{r}{2} v_{j-1}^{m+1} = (1-r) v_j^m + \frac{r}{2} v_{j+1}^m + \frac{r}{2} v_{j-1}^m \quad + \Delta t\, g(t)$$

So we have $\quad A_1 v^{m+1} = A_2 v^m$

$$v^m = (v_1^m \dots v_n^m)^T, \quad v^{m+1} = (v_1^{m+1} \dots v_n^{m+1})$$

$$A_1 = \begin{pmatrix} 1+r & -r/2 & & 0 \\ -r/2 & \ddots & \ddots & \\ & \ddots & \ddots & -r/2 \\ 0 & & -r/2 & 1+r \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1-r & r/2 & & 0 \\ r/2 & & & \\ & & & r/2 \\ 0 & & r/2 & 1-r \end{pmatrix}$$

$\Rightarrow$ $\boxed{v^{m+1} = A_1^{-1} A_2 v^m}$. We need to solve a linear system to get the approximations at the next time step.

→ Go to p61.

Project ②

$$u_j^{m+1} = u_j^m + \frac{\Delta t d_u}{\Delta x^2}\left(u_{j+1}^m - 2u_j^m + u_{j-1}^m\right) - \Delta t\left(V u_j^m - f(1-u_j^m)\right)$$

$$= \Big(1 - \Delta t V - \Delta t f - 2\overbrace{\frac{\Delta t d_u}{\Delta x^2}}^{=:r}\Big) u_j^m$$

$$+ f\Delta t + \boxed{r u_{j+1}^m} + \boxed{r u_{j-1}^m}$$

$\Delta t \le \frac{d_v \Delta x^2}{2}$; $\Delta t \le \frac{d_u \Delta x^2}{2}$, $\le \frac{d_v \Delta x^2}{2}$

$(1+\frac{1}{\Delta t})(\frac{1}{\Delta x}+1) = 8000$

$\le (1-2r+r+r)\max(\cdot) + \Delta t\,(\underbrace{-V+f)}\,\cancel{\max(\dots)}\,u_j^m) + f)$

$(1+\frac{2}{d_v \Delta x^2})(\frac{1}{\Delta x}+1)$

$<0 \Rightarrow u_j^m > \frac{f}{V+f}$

But $u_j^m > 0$ so bounded by $f\Delta t$?

$$v_j^{m+1} = \Big(\underline{\Delta t\, U - \Delta t f} - \Delta t k + 1 - 2\overset{:= \frac{d_v \Delta x^2}{\Delta t}}{r}\Big) v_j^m + r v_{j+1}^m + r v_{j-1}^m$$

$\downarrow$

$$\le (1-2r+r+r)\max(\cdot) + \Delta t\,\underset{<1\dots?}{(U-f-k)}\,v_j^m$$

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Recitation / Project ①

'Periodic boundary conditions' on $[0, L]$

$u(0) = u(L)$, $u'(0) = u'(L)$

$\Leftrightarrow V_0^m = V_n^m$, $V_{n+1}^m = V_1^m$

1. $\partial_t u - d_u \Delta u = -uV + f(1-u)$

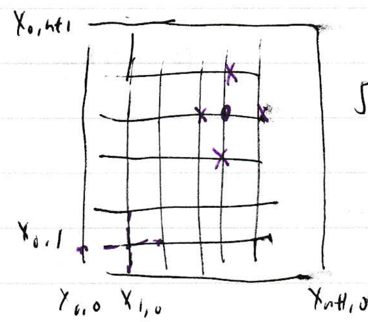
$\partial_t v - d_v \Delta v = vU - (f+k)v.$

Explicit scheme:
$$\frac{u_j^{m+1} - u_j^m}{\Delta t} = d_u \frac{u_{j+1}^m - 2u_j^m + u_{j-1}^m}{\Delta x^2} - Vu_j^m + f(1-u_j^m) \quad -(1)$$

$$\frac{v_j^{m+1} - v_j^m}{\Delta t} = d_v \frac{v_{j+1}^m - 2v_j^m + v_{j-1}^m}{\Delta x^2} + Uv_j^m - (f+k)v_j^m \quad -(2)$$

Note: u and v are decoupled...

$$\boxed{U_{xx} + U_{yy} = f(x,y)}$$



5-point stencil Laplacian

In 2D, $-\Delta u = -(\partial_{xx} u + \partial_{yy} u)$

$x_{ij} = (ih_1, jh_1)$

$$-\Delta u(x_{i,j}) = -\left(\frac{u(x_{i-1,j}) - 2u(x_{i,j}) + u(x_{i+1,j})}{h^2} + \frac{u(x_{i,j-1}) - 2u(x_{i,j}) + u(x_{i,j+1})}{h^2}\right) + O(h_1^2 + h_2^2)$$

Finite Difference
$$-\left(\frac{v_{i-1,j} - 2v_{i,j} + v_{i+1,j}}{h_1^2} + \frac{v_{i,j-1} - 2v_{i,j} + v_{i,j+1}}{h_2^2}\right) = f_{i,j}$$

set $= h^2$

Note that we have $n_1 \cdot n_2$ equations and $n_1 \cdot n_2$ unknowns (1 for each interior point).

→ Go to p 65.

Recitation 10/19

$\frac{d}{dx}\int_c^d f(x,y)\,dy = \int_c^d \frac{d}{dx} f(x,y)\,dy$ when f and $\partial_x f$ are continuous on $[a,b]\times[c,d]$.

For $f(x,y) = \begin{cases} \frac{x^2y}{(x^2+y^2)^2}, & x\neq 0, y\neq 0 \\ 0, & x=y=0 \end{cases}$, we can check that

$g(x) := \int_0^1 f(x,y)\,dy = \frac{x}{2(1+x^2)}$ and $h(x,y) := \frac{d}{dx} f(x,y) = \begin{cases} \frac{x^2y(3y^2-x^2)}{(x^2+y^2)^3} & x\neq 0, y\neq 0 \\ 0 & x=0, y=0. \end{cases}$

Let $x=0$. Then $\int_0^1 h(x,y)\,dy = \int_0^1 0\,dy = 0$.

But $g'(x) = -\frac{1}{2}x(1+x^2)^{-2}(2x) + \frac{1}{2(1+x^2)}$

$\overset{x=0}{=} \frac{1}{2}$.

$f(0,0)=0$. Along $y=x$, however, $f(x,y) = f(x,x) = \frac{x^3}{4x^4} = 1/4x$

so $f(x,y)$ is not continuous.

$\frac{d}{dx} f(x,y)$ is also not continuous. $x=y$: $h(x,y) = \left(\frac{x^3(2x^2)}{(2x^2)^3}\right)$

$= \frac{2x^5}{8x^6} = \frac{1}{4x}\ldots$

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Stencil schemes... (From P63.) $\boxed{u_{xx}+u_{yy}=f(x,y)}$

$$V = \begin{pmatrix} \left.\begin{matrix} v_{11} \\ v_{21} \\ \vdots \\ v_{n1} \end{matrix}\right\} \text{first row} \\ \left.\begin{matrix} v_{12} \\ v_{22} \\ \vdots \\ v_{n2} \end{matrix}\right\} \\ \vdots \\ \left.\begin{matrix} v_{n1} \\ v_{n2} \\ \vdots \\ v_{nn} \end{matrix}\right\} \end{pmatrix} \in \mathbb{R}^{(n^2)}, \quad F = h^2 \begin{pmatrix} \left.\begin{matrix} f_{11} \\ f_{21} \\ \vdots \\ f_{n1} \end{matrix}\right\} \\ \left.\begin{matrix} f_{12} \\ f_{22} \\ \vdots \\ f_{n2} \end{matrix}\right\} \\ \left.\begin{matrix} f_{n1} \\ f_{n2} \\ \vdots \\ f_{nn} \end{matrix}\right\} \end{pmatrix} \in \mathbb{R}^{(n^2)}$$

$$A\,v = F, \quad A \in \mathbb{R}^{(n^2)^2}.$$

$$n^2\Big(\quad\Big)_{n^2}\; n^2\Big(\;\Big)_1 = n^2\Big(\;\Big)_1$$

$$(\mathrm{kron}(I,B) + \mathrm{kron}(S,I))$$

$$/\!/$$

* $A = \begin{pmatrix} B & -I & & 0 \\ -I & B & \ddots & \\ & \ddots & \ddots & -I \\ 0 & & -I & B \end{pmatrix}$, where $B = \begin{pmatrix} 4 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & & -1 \\ 0 & & -1 & 4 \end{pmatrix}_{n\times n}$

21
12 22 32
23

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}_{n\times n}$$

What would change if we imposed $u = g \neq 0$ on $\partial\Omega$ instead?

As in the 1D case, ~~move~~ more g to the RHS. (the boundary)

$-u''(x) = f(x)$
$u(0) = \alpha,\ u(1) = \beta$

$$f = -1.25e^{(x+\frac{y}{2})} \Rightarrow u(x,y) = e^{(x+y/2)}$$

*poisson2D.m Cheat a little by setting the boundary values using the true solution (rather than setting them one by one)

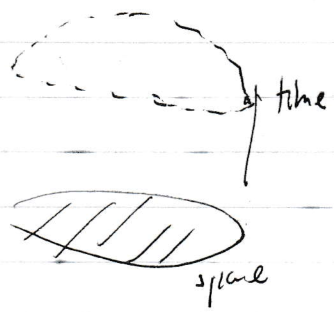
$$S = \begin{matrix} 0 & -1 & \\ -1 & 0 & -1 \\ & & \ddots \end{matrix} \qquad \mathrm{kron}(S,I) = \begin{bmatrix} 0 & -I & & & \\ -I & 0 & -I & & \\ & -I & 0 & -I & \\ & & -I & 0 & -I \\ & & & & \ddots \end{bmatrix}$$

← 1st row move to RHS...

6J

2D Heat

$$\begin{cases} \partial_t u - \Delta u = 0 \\ u(x,y,0) = f(x,y) \\ u(x,t) = 0 \end{cases}$$



Explicit Euler: $y_1 = y_0 + h \cdot f(x_0, y_0)$

→ Implicit Euler: $y_1 = y_0 + h \cdot f(x_1, y_1)$

better performance

Simulation : CNheat.m

$$\text{CNheat}\left(\tfrac{1}{10},\ 10,\ @(x)\sin(8\pi x)),\ 0.2,\ [0,1],\ 1,\ 1\right)$$

dt N $u_0(x)/f_0(x)$ k range of x T plot or not

$O(\Delta x^2 + \Delta t^2)$ err. ¼	1/20	20
	1/40	40
	1/80	80
	1/20	160 still stable!

We observe that

1. Explicit scheme
   - (+) Easy to implement
   - (+) No linear systems to solve
   - (−) Very small time step required for stability
2. Implicit scheme
   - (+) Very stable, independent of Δt
   - (−) Linear system has to be solved
   - (−) Low order of accuracy
3. Crank-Nicholson
   - (+) Very stable
   - (+) More accurate
   - (−) Two linear systems have to be solved
   - (−) More complicated to implement

Order of Accuracy

Truncation error τ : Plug the exact solution into the scheme and assume it is smooth

$$\tau = O(\Delta t^p + \Delta x^q)$$

$\Leftrightarrow$ Scheme is p^{th} order accurate in time and q^{th} order accurate in space.

Example: Explicit scheme, denoting by $u(t,x)$ the exact solution.

Truncation Error $\quad \tau_j^m = \dfrac{u(t^{m+1},x_j) - u(t^m,x_j)}{\Delta t} - \dfrac{u(t^m,x_{j+1}) - 2u(t^m,x_j) + u(t^m,x_{j-1})}{\Delta x^2}$

$\tau(t^m,x_j)$

$$= u_t(t^m,x_j) + \frac{\Delta t}{2} u_{tt}(t^m,x_j) + O(\Delta t^2)$$

$$- \left[u_{xx}(t^m,x_j) + \frac{\Delta x^2}{12} u^{(4)}(t^m,x_j) + O(\Delta x^4) \right]$$

$$= \underbrace{u_t(t^m,x_j) - u_{xx}(t^m,x_j)}_{\substack{0 \\ (u \text{ is the exact solution!})}} + \underbrace{\frac{\Delta t}{2} u_{tt}(t^m,x_j)}_{O(\Delta t)} - \overbrace{\frac{\Delta x^2}{12} u^{(4)}(t^m,x_j)}^{O(\Delta x^2)} + O(\Delta t^2 + \Delta x^4)$$

$$= O(\Delta t + \Delta x^2)$$

Explicit scheme is 1^{st} order accurate in time, and 2^{nd} order accurate in space.

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Recall...

Explicit ~~Scheme~~ method $\tau = O(\Delta t + \Delta x^2)$ 1st order accurate in time, 2nd order accurate in space

Implicit Method $\tau = O(\Delta t + \Delta x^2)$ 1st ... 2nd ...

Crank-Nicolson Method $\tau = O(\Delta t^2 + \Delta x^2)$ 2nd ... 2nd ...

Consistency: A numerical method is consistent if $\tau \to 0$ as $\Delta t, \Delta x \to 0$

Convergence: We need to show that $E(t^m, x_j) = u(t^m, x_j) - v_j^m$ goes to zero in some norm as $\Delta t, \Delta x \to 0$.

Numerics: $\max_{j=1\ldots n} |E(t^{M^{\text{fixed}}}, x_j)| = O(\Delta x^\alpha)$

'fixed refinement path $\tau(h)$'

Assume the relation between Δt and Δx is fixed ($\Delta t = c\Delta x$ in implicit / C-N scheme, and $\Delta t = c\Delta x^2$ for explicit scheme)

The numerical methods we have seen so far, we can rewrite as follows

$v^m = (v_1^m \ldots v_n^m)^T$, $v^{m+1} = Bv^m + b^m$, where

$B = B(\Delta t, \Delta x) \in \mathbb{R}^{n \times m}$, $b^m = b^m(\Delta t, \Delta x) \in \mathbb{R}^m$

Explicit scheme: $v_j^{m+1} = \frac{\Delta t}{\Delta x^2} v_{j+1}^m + (1 - 2\frac{\Delta t}{\Delta x^2}) v_j^m + \frac{\Delta t}{\Delta x^2} v_{j-1}^m + \Delta t\, g_j^m$

$v^{m+1} = Bv^m + \Delta t\, g^m$, $(g_1^m, \ldots g_n^m)^T$

$$B = \begin{bmatrix} 1-\frac{2\Delta t}{\Delta x^2} & \frac{\Delta t}{\Delta x^2} & & \\ \frac{\Delta t}{\Delta x^2} & \ddots & \ddots & \\ & \ddots & \ddots & \frac{\Delta t}{\Delta x^2} \\ 0 & & \frac{\Delta t}{\Delta x^2} & 1-\frac{2\Delta t}{\Delta x^2} \end{bmatrix}$$

* $b^m \equiv 0$ without source term

Implicit Scheme $\quad A v^{m+1} = v^m$, where $A = \begin{pmatrix} 1+\frac{2\Delta t}{\Delta x^2} & -\frac{\Delta t}{\Delta x^2} & & \\ -\frac{\Delta t}{\Delta x^2} & \ddots & \ddots & \\ & \ddots & \ddots & -\frac{\Delta t}{\Delta x^2} \\ & & -\frac{\Delta t}{\Delta x^2} & 1+\frac{2\Delta t}{\Delta x^2} \end{pmatrix}$

$v^{m+1} = A^{-1} v^m, \quad B = A^{-1}.$

$b^m = 0$ without source term,

$b^m = \Delta t\, A^{-1} g^m$

Crank-Nicolson Scheme

$$A_1 v^{m+1} = A_2 v^m \Rightarrow B = A_1^{-1} A_2$$

$$A_1 = \begin{bmatrix} 1+\frac{\Delta t}{\Delta x^2} & -\frac{\Delta t}{2\Delta x^2} & & 0 \\ -\frac{\Delta t}{2\Delta x^2} & \ddots & \ddots & \\ & \ddots & \ddots & -\frac{\Delta t}{2\Delta x^2} \\ 0 & & -\frac{\Delta t}{2\Delta x^2} & 1+\frac{\Delta t}{\Delta x^2} \end{bmatrix} \quad A_2 = \begin{bmatrix} 1-\frac{\Delta t}{\Delta x^2} & \frac{\Delta t}{2\Delta x^2} & & 0 \\ \frac{\Delta t}{2\Delta x^2} & \ddots & \ddots & \\ & \ddots & \ddots & \frac{\Delta t}{2\Delta x^2} \\ 0 & & \frac{\Delta t}{2\Delta x^2} & 1-\frac{\Delta t}{\Delta x^2} \end{bmatrix}$$

Definition: A linear method of the form (AM): $v^{m+1} = Bv^m + b^m$ is called Lax-Richtmyer stable if there is a constant $C_T > 0$ such that $\|B^m\| \le C_T$ for $\Delta t > 0$, $m \in \mathbb{N}$ such that $m\,\Delta t \le T$, and some norm $\|\cdot\|$ (i.e. stable w.r.t $\|\cdot\|$).

power

Theorem (Lax Equivalence): A consistent linear numerical method of the form (AM) is convergent iff it is Lax-Richtmyer stable.

Proof sketch (R. D. Richtmyer, K. W. Morton: Difference Methods for Initial Value Problems (1967))

($\Leftarrow$) Let $E^m = v^m - u^m$ be the global error at time m

$u^m = (u(t^m, x_1), \dots u(t^m, x_n))^T$

truncation error, $\neq E^m$!

$v^{m+1} = Bv^m + b^m, \quad u^{m+1} = Bu^m + b^m + \Delta t\, \tau^m \quad (\tau^m = (\tau_1^m, \tau_2, \dots \tau_n^m)^T)$

$\Rightarrow E^{m+1} = BE^m - \Delta t\, \tau^m$ error recursion. 'g'

$E^{m+1} = B(BE^{m-1} - \Delta t\, \tau^{m-1}) - \Delta t\, \tau^m$

$\Rightarrow E^{m+1} = B^{m+1} E^0 - \Delta t \sum_{j=0}^{m} B^{m-j} \tau^j$

$$\|E^{m+1}\| \le \|B^{m+1}E^0 - \Delta t\sum_{j=0}^{m} B^{m-j}\tau^j\|$$

Δ-inequality $\quad$ $\swarrow = 0$ since $v_j^0 = u_0(x_j)$

$$\le \|B^{m+1}E^0\| + \Delta t\sum_{j=0}^{m}\|B^{m-j}\tau^j\|$$

$$\le \Delta t\sum_{j=0}^{m}\|B^{m-j}\tau^j\|$$

$$\le \Delta t\sum_{j=0}^{m}\underbrace{\|B^{m-j}\|}_{\le C_T \text{ if } m\Delta t\le T}\|\tau^j\|$$

$$\le \Delta t\, C_T\sum_{j=0}^{m}\|\tau^j\|$$

$$\le \Delta t\, C_T\,(m+1)\max_{j=0\ldots m}\|\tau^j\|$$

$$\le C_T\cdot T\cdot \max_{j=0\ldots m}\|\tau^j\| \quad (\text{if } (m+1)\Delta t\le T)$$

$\to 0$ if $\Delta t, \Delta x \to 0$ by consistency.

10/25/2021

$u_t - k u_{xx} = g(t,x)$ $\quad x \in (0,1)$, $t > 0$

$u(0,x) = u_0(x)$

$u(t,0) = u(t,1) = 0$

'Regularity' conditions (A)

$\|g\|_\infty \le G < \infty$

$\hookrightarrow \sup_{t \ge 0, x \in [0,1]} |g(t,x)|$

$k > 0$

$$\frac{v_j^{m+1} - v_j^m}{t} - k \frac{v_{j+1}^m - 2v_j^m + v_{j-1}^m}{\Delta x^2} = g_j^m \; (= g(t^m, x_j))$$

$r := \dfrac{k \Delta t}{\Delta x^2} > 0, \; \Delta t > 0.$

We want to find sufficient conditions for stability:

$$|v_j^{m+1}| \le |(1-2r) v_j^m| + |r v_{j-1}^m| + |r v_{j+1}^m| + \Delta t |g_j^m|$$

Assuming $\boxed{1-2r \ge 0}$, then

sufficient, but might not be necessary!

$$|v_j^{m+1}| \le (1-2r)|v_j^m| + r|v_{j-1}^m| + r|v_{j+1}^m| + \Delta t |g_j^m|$$

$$\le (1-2r)\max(|v_j^m|, |v_{j-1}^m|, |v_{j+1}^m|) + r\max(|v_j^m|, |v_{j+1}^m|, |v_{j-1}^m|)$$
$$+ r\max(|v_j^m|, |v_{j+1}^m|, |v_{j-1}^m|) + \Delta t |g_j^m|$$

$$= \max(|v_j^m|, |v_{j-1}^m|, |v_{j+1}^m|) + \Delta t |g_j^m| \; (\le G \text{ (B)})$$

$$\le \max_{k=1\ldots n} |v_k^m| + \Delta t \underbrace{\max_{\substack{k=1\ldots n \\ \alpha=0,1,\ldots}} |g_k^\alpha|}_{\le \|g\|_\infty \le G}$$

$$|v_j^{m+1}| \le \max_{k=1\ldots n} |v_k^m| + \Delta t G$$

since this is independent of j, we can take max over j on LHS

$$\Rightarrow \underbrace{\max_{j=1\ldots n} |v_j^{m+1}|}_{:= \|V^{m+1}\|_{\Delta x, \infty}} \le \underbrace{\max_{k=1\ldots n} |v_j^m|}_{:= \|V^m\|_{\Delta x, \infty}} + \Delta t G$$

By induction,

$$\|v^{m+1}\|_{\Delta x,\infty} \le \underbrace{\|v^m\|_{\Delta x,\infty}}_{\le \underbrace{\|v^{m-1}\|_{\Delta x,\infty}}_{\vdots} + \Delta t E} + \Delta t E$$

$$\|v^{m+1}\|_{\Delta x,\infty} \le \underbrace{\|v^0\|_{\Delta x,\infty}}_{\le \|u_0\|_\infty \ (v_j^0 = u_0(x_j))} + (m+1)\Delta t E$$

$$\Rightarrow \quad \|v^m\|_{\Delta x,\infty} \le \|u_0\|_\infty + m\Delta t E.$$

Since $T>0$, we can define $M := \frac{T}{\Delta t}$.

Then for $m \le M$ ($m\Delta t \le T$), we have

$$\boxed{\|v^m\|_{\Delta x,\infty} \le \|u_0\|_\infty + TE}$$

$$\frac{1}{2} \ge r = \frac{k\Delta t}{\Delta x^2} \iff \frac{\Delta x^2}{2k} \ge \Delta t$$

$$V^{m+1} = Bv^m + b^m$$

$$B = B(\Delta t, \Delta x) \in \mathbb{R}^{m \times m}$$

$$b^m = b(\Delta t, \Delta x) \in \mathbb{R}^m$$

When is a scheme Lax-Richtmyer stable ??

$$\|B^m\| \le C_T, \quad T > 0 \quad \text{for all } m\Delta t \le T$$

Consider the Explicit scheme for Heat Equation

$$r = \frac{k\Delta t}{\Delta x^2}, \qquad B = \begin{pmatrix} 1-2r & r & & 0 \\ r & \ddots & \ddots & \\ & \ddots & \ddots & r \\ 0 & & r & 1-2r \end{pmatrix}$$

$\|A\|_2 = \sqrt{\rho(A^T A)}$

see top of pg 61.

B is symmetric $\Rightarrow$ 2-norm is $\underbrace{\text{spectral radius}}_{\text{eigenvalue with largest magnitude}}$

$$S = \begin{pmatrix} \alpha & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & \alpha \end{pmatrix} \Rightarrow \lambda_j = \alpha - 2\cos(j\theta) \quad j = 1 \ldots n$$

$\theta = \frac{\pi}{n+1}$ (From Hw 3)

Let $S = -\frac{1}{r}B$, $\alpha = -\frac{1-2r}{r} = 2 - \frac{1}{r}$

$\Rightarrow$ eigenvalue of $B = -r$ eigenvalue of S

$$\Rightarrow \mu_j = -r\left(2 - \frac{1}{r} - 2\cos(j\theta)\right)$$

→ Go to Pg 77

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Quiz 5 (Recitation 9)

(a) Helmholtz

$$-\Delta - c^2 u = f$$

$$u = g \text{ on } \partial\Omega = \left(\{0,1\}\times(0,1) \cup (0,1)\times\{0,1\}\right)$$

Finite difference:

$$\cancel{x_{m-1} + 2x_m - x_{m+1}}$$

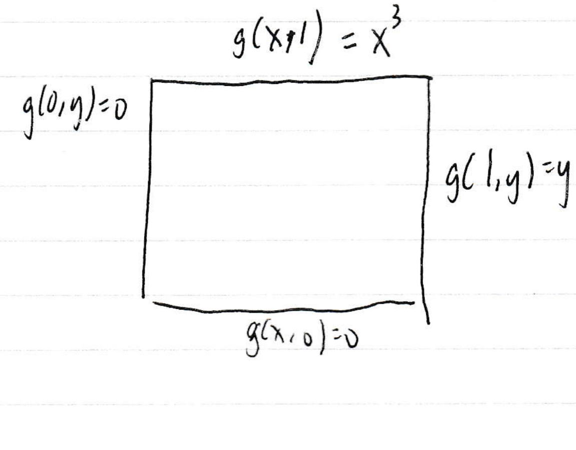
5-point Laplacian

$$(-\Delta_h - c^2 I) v_{ij} = f_{ij}$$

$$v_{0,j} = v_{1,j} = 0, \quad (0<j<n+1)$$

$$v_{i,0} = v_{i,1} = 0 \quad (0<i<m+1)$$

(b) Implement the scheme for $c=1$, $f = -6x - x^3 y$, and g satisfying



TJ

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(from pg 74)

Recalling the explicit scheme, we want to show that under some condition on r

$\|B^m\|_2 \le C_T$ $\quad \|B^m\|_2 \overset{(=)}{\le} \|B\|^m$ $\quad Ax = \lambda x \Rightarrow A^2x = \lambda^2 x$

square of symmetric matrix is symmetric $\quad \overset{\shortparallel}{\rho}(B^m) = \max |\lambda_i|$ (eigenvalues of B^m)

$\downarrow$ spectral radius of B / maximum-magnitude eigenvalue of B

We showed that the eigenvalues of B are

$$\lambda_j = -r\left(-\frac{1}{r} + 2 - 2\cos(j\theta)\right)$$

$$= 1 - 2r(1 - \cos(j\theta)) \quad , \quad j = 1 \dots n, \quad \theta = \pi/n+1$$

If we show that $\|B\|_2 \le 1$, then $\|B\|^m \le 1 \le C_T$ (so we can set $C_T = 1$, independent of Δt and Δx)

$\Rightarrow$ Show that $|\lambda_j| \le 1$ under some conditions on r.

we have $|\lambda_j| \le 1 \Leftrightarrow \left|1 - \underbrace{\underset{>0}{2r}\,(1 - \underbrace{\cos(j\theta)}_{\in[-1,1]})}_{\in[0,2]}\right| \le 1$

Need to analyze $\quad -(1 - 2r(1-\cos(j\theta))) \le 1$

$\Leftrightarrow 1 - 2r(1 - \cos(j\theta)) \ge -1$

$\Leftrightarrow 2 \ge 2r(1 - \cos(j\theta))$

$\Leftrightarrow 1 \ge r(1 - \cos(j\theta))$

worst case $j = n$, $\cos\left(\frac{n\pi}{n+1}\right) \approx -1$

$\Leftarrow 1 \ge 2r$

$\Leftrightarrow \frac{1}{2} \ge r = \frac{\Delta t}{\Delta x^2}$, $\Delta t \le \frac{\Delta x^2}{2}$

Implicit scheme (Basically restatement of P65)

$$Av^{m+1} = v^m,$$

where $A = \begin{pmatrix} 1+2r & -r & & 0 \\ -r & \ddots & \ddots & \\ & \ddots & \ddots & -r \\ 0 & & -r & 1+2r \end{pmatrix}$

Rewrite in the format of P69: $v^{m+1} = Bv^m$ ($B = A^{-1}$)

$\Rightarrow$ Analyze the eigenvalues of A^{-1} and make sure

inverse of symmetric matrix is symmetric

$(A^{-1})^T = (A^T)^{-1} = (A)^{-1}$

$\|A^{-1}\|_2 \le 1$

$\| \max|\lambda_i|$ of eigenvalues of A^{-1}

Note that if μ_i is the eigenvalue of A, then $\frac{1}{\mu_i}$ is an eigenvalue of A^{-1}.

$\Rightarrow \|A^{-1}\|_2 = \frac{1}{\min|\mu_i|}$ ← eigenvalue of A

we rewrite A as

$$A = r \begin{pmatrix} \frac{1+2r}{r} & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & \underbrace{\frac{1+2r}{r}}_{=:\alpha} \end{pmatrix}$$

(HW: $\lambda_j = \alpha - 2\cos j\theta$, $\theta = \frac{\pi}{n+1}$)

$\Rightarrow A$ has eigenvalues $r\left(\frac{1+2r}{r} - 2\cos(j\theta)\right) = 1+2r - 2r\cos(j\theta)$

$= 1+2r(1-\cos(j\theta))$ $[\ge 1]$

so long as $r \ge 0$

$\Rightarrow \min_{\lambda_i \text{ eigenvalue of } A} |\lambda_i| \ge 1$

$\Rightarrow \|A^{-1}\|_2 \le 1$ $\Rightarrow$ unconditional stability

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Periodic Boundary Conditions

$$v_0^m = v_n^m, \quad v_{n+1}^m = v_1^m$$

(Implicit case)

$$(1+2r)v_j^{m+1} - rv_{j-1}^{m+1} - rv_{j+1}^{m+1} = v_j^m, \quad j=1 \ldots n$$

(j=1) $$(1+2r)v_1^{m+1} - r\underbrace{v_0^{m+1}}_{v_n^{m+1}} - rv_2^{m+1} = v_1^m$$

$$\begin{array}{r} \text{1st row in } A \rightarrow \\ \\ \\ \\ \text{nth row in } A \rightarrow \end{array} \begin{pmatrix} 1+2r & -r & 0 & \ldots & -r \\ -r & 1+2r & -r & & \\ & & & & \\ & & & -r & 1+2r \\ -r & & & -r & 1+2r \end{pmatrix} \begin{pmatrix} v_1^{m+1} \\ \vdots \\ v_n^{m+1} \end{pmatrix}$$

(j=2) $$(1+2r)v_2^{m+1} - rv_1^{m+1} - rv_3^{m+1} = v_2^m \quad \text{(OK)}$$

$$\vdots$$

(j=n) $$(1+2r)v_n^{m+1} - rv_{n-1}^{m+1} - r\underbrace{v_{n+1}^{m+1}}_{= v_1^{m+1}}$$

Project: ~~Identify the matrix A (even for the explicit scheme?)~~

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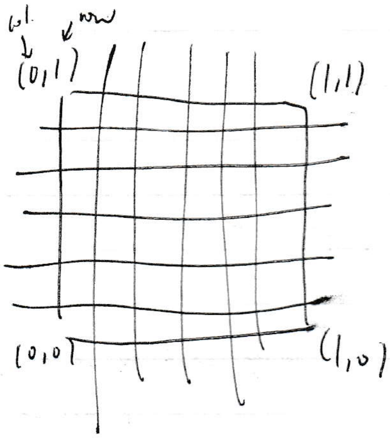
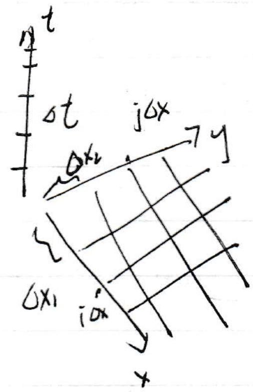
$$\begin{cases} u_t - \Delta u = 0 & , x \in (0,1)^2, t > 0 \\ u(t,x) = 0 & , t > 0, x \in \partial([0,1]^2), \text{ either } x_1 \in \{0,1\} \text{ or } x_2 \in \{0,1\} \\ u(0,x) = u_0(x) \end{cases}$$



$X = (x_1, x_2)$

$$\Delta u(x) = \partial_{x_1 x_1} u(x) + \partial_{x_2 x_2} u(x)$$



$\Delta x_1 > 0$ grid size in x_1-direction

$\Delta x_2 > 0$ " x_2-direction

Assume $\Delta x_1 = \Delta x_2 = \Delta x$, Δt is time step size

We try to find approximations

$$v_{ij}^m \approx u(t^m, x_{1,i}, x_{2,j}), \text{ where } x_{1,i} = i\Delta x, \; x_{2,j} = j\Delta x, \; t^m = m\Delta t$$

$$x_{ij} = (x_{1,i}, x_{2,j}) = (i\Delta x, j\Delta x)$$

$$\frac{V_{ij}^{m+1} - V_{ij}^m}{\Delta t} = \frac{V_{i+1,j}^m - 2V_{i,j}^m + V_{i-1,j}^m}{\Delta x^2} + \frac{V_{i,j+1}^m - 2V_{i,j}^m + V_{i,j-1}^m}{\Delta x^2}$$

$$= \frac{V_{i+1,j}^m + V_{i-1,j}^m + V_{i,j+1}^m + V_{i,j-1}^m - 4V_{i,j}^m}{\Delta x^2}$$

$4 = 2 \cdot n$ (dimension)

$$\left(r = \frac{\Delta t}{\Delta x^2}\right) = rV_{i+1,j}^m + rV_{i-1,j}^m + rV_{i,j+1}^m + rV_{i,j-1}^m + (1-4r)V_{i,j}^m$$

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Implicit Scheme

$$\frac{V_{i,j}^{m+1} - V_{i,j}^{m}}{\Delta t} = \frac{V_{i+1,j}^{m+1} - 2V_{i,j}^{m+1} + V_{i-1,j}^{m+1}}{\Delta x^2} + \frac{V_{i,j+1}^{m+1} - 2V_{i,j}^{m+1} + V_{i,j-1}^{m+1}}{\Delta x^2}$$

$$= \frac{V_{i+1,j}^{m+1} + V_{i-1,j}^{m+1} + V_{i,j+1}^{m+1} + V_{i,j-1}^{m+1} - 4V_{i,j}^{m+1}}{\Delta x^2}$$

Implicit scheme for 2D heat equation:

$r = \frac{\Delta t}{\Delta x^2}$

$$-rV_{i+1,j}^{m+1} - rV_{i-1,j}^{m+1} - rV_{i,j-1}^{m+1} - rV_{i,j+1}^{m+1} + (1+4r)V_{i,j}^{m+1} = V_{i,j}^{m}$$

$$V_{0,j}^{m} = V_{i,0}^{m} = V_{n+1,j}^{m} = V_{i,n+1}^{m} = 0.$$

$$\begin{pmatrix} 1+4r & -r & & 0 \\ -r & \ddots & & -r \\ & 0 & -1 & 1+4r \end{pmatrix} \;//\; B$$

$$A \begin{pmatrix} B & -rI & & 0 \\ -rI & B & -rI & \\ & & \ddots & -rI \\ 0 & & -rI & B \end{pmatrix}$$

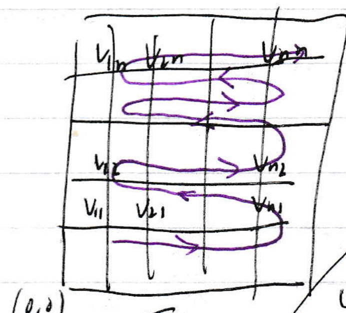
n^2 Equations/Unknowns

$A V^{m+1} = V^{m}$

$A \in \mathbb{R}^{n^2 \times n^2}$ (matrix)

$V^{m}, V^{m+1} \in \mathbb{R}^{n^2}$

$$V^{m} = \begin{pmatrix} V_{11} \\ V_{21} \\ \vdots \\ V_{n1} \\ V_{12} \\ V_{22} \\ \vdots \\ V_{n2} \\ V_{1n} \\ \vdots \\ V_{nn} \end{pmatrix}$$



$i=2, j=2$

$$-rV_{3,2} - rV_{1,2} - rV_{2,3} - r_{2,1} + (1+4r)V_{2,2}$$

Start with $i=j=1$

$$-rV_{2,1} - rV_{01} - rV_{1,2} - r\underbrace{V_{1,0}}_{=0} + (1+4r)V_{1,1}$$

$i=2, j=1$

$$-rV_{3,1} - rV_{1,1} - rV_{22} - r\underbrace{V_{2,0}}_{=0} + (1+4r)V_{2,1}$$

$i=1, j=2$

$$-rV_{22} - rV_{0,2} - rV_{13} - rV_{11} + (1+4r)V_{1,2}$$

$$A = \begin{bmatrix} (1+4r) & -r & 0 & \cdots & 0 & -r & 0 & \cdots & & 0 \\ -r & (1+4r) & -r & & & 0 & -r & & & 0 \\ & \ddots & \ddots & \ddots & & & \ddots & -r & & \\ & & & -r & (1+4r) & 0 & 0 & 0 & -r & 0 \\ -r & 0 & \cdots & & 0 & (1+4r) & -r & 0 & -r & \\ 0 & -r & \cdots & & 0 & -r & (1+4r) & -r & & -r \\ & & \ddots & & & & \ddots & & \ddots & \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{21} \\ V_{31} \\ \vdots \\ V_{n1} \\ V_{12} \\ V_{22} \\ \vdots \\ V_{n2} \\ V_{13} \\ V_{23} \end{bmatrix}$$

(n+1)

#1

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Energy Methods (in the Discrete World)

Recall: (H) $\begin{cases} u_t - u_{xx} = 0, & x \in (0,1), t > 0 \\ u(0,x) = u_0(x) & x \in [0,1] \\ u(t,0) = u(t,1) = 0, & t > 0 \end{cases}$

$$E(t) = \int_0^1 (u(t,x))^2 \, dx$$

$$E'(t) \leq 0 \Rightarrow E(t) \leq E(0) < \infty$$

now: Explicit Scheme $$\frac{v_j^{m+1} - v_j^m}{\Delta t} = \frac{v_{j+1}^m - 2v_j^m + v_{j-1}^m}{\Delta x^2}$$

$$E^m = \Delta x \sum_{j=1}^n (v_j^m)^2$$

We want to show $E^{m+1} \leq E^m, m \geq 0 \Rightarrow E^m \leq E^0 = \Delta x \sum_{j=1}^n (u_0(x_j))^2 < \infty, \forall m \geq 0$

To prove this, we will show (under some condition)

$$\frac{E^{m+1} - E^m}{\Delta t} \leq 0.$$

Note that $\frac{E^{m+1} - E^m}{\Delta t} = \frac{\Delta x}{\Delta t} \sum_{j=1}^n [(v_j^{m+1})^2 - (v_j^m)^2]$

$$\boxed{\begin{aligned} a^2 - b^2 &= (a+b)(a-b) \\ &= 2b(a-b) + (a-b)^2 \end{aligned}}$$

$$= \underbrace{\frac{2\Delta x}{\Delta t} \sum_{j=1}^n v_j^m (v_j^{m+1} - v_j^m)}_{A} + \underbrace{\frac{\Delta x}{\Delta t} \sum_{j=1}^n (v_j^{m+1} - v_j^m)^2}_{B}$$

$$(v_j^{m+1})^2 - (v_j^m)^2 = t \frac{v_{j+1}^m - 2v_j^m + v_{j-1}^m}{\Delta x^2} (v_j^{m+1} + v_j^m)$$

Alternatively,

$$= t \frac{(v_{j+1}^m - v_j^m) - (v_j^m - v_{j-1}^m)}{\Delta x^2} (v_j^{m+1} + v_j^m)$$ but this doesn't simplify... :'n

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Using 'summation by parts',

(1) $$A = 2\Delta x \sum_{j=1}^{n} v_j^m \left(\frac{v_j^{m+1}-v_j^m}{\Delta t}\right)$$

(scheme)
$$= 2\Delta x \sum_{j=1}^{n} v_j^m \left(\frac{(\overbrace{v_{j+1}^m - v_j^m}^{w_j^m}) - (\overbrace{v_j^m - v_{j-1}^m}^{w_{j-1}^m})}{\Delta x^2}\right)$$

(by parts)
$$= \frac{2}{\Delta x}\sum_{j=1}^{n} v_j^m w_j^m - \frac{2}{\Delta x}\sum_{j=0}^{n-1} v_{j+1}^m w_j^m$$

$$= \frac{2}{\Delta x}\sum_{j=1}^{n-1}(v_j^m - v_{j+1}^m) w_j^m - \frac{2}{\Delta x} v_1 w_0^m + \frac{2}{\Delta x} v_n^m w_n^m$$

can go directly

$$= -\frac{2}{\Delta x}\sum_{j=1}^{n-1}(v_{j+1}^m - v_j^m)^2 - \frac{2}{\Delta x} v_1^m (v_1^m - v_0^m) + \frac{2}{\Delta x} v_n^m (v_{n+1}^m - v_n^m)$$

add extra term $= 0$ (B.C.)

$$= \frac{-2}{\Delta x}\sum_{j=1}^{n-1}(v_{j+1}^m - v_j^m)^2 - \frac{2}{\Delta x}(v_1^m - v_0^m)(v_1^m - v_0^m) - \frac{2}{\Delta x}(v_{n+1}^m - v_n^m)(v_{n+1}^m - v_n^m)$$

$$= \frac{-2}{\Delta x}\sum_{j=0}^{n}(v_{j+1}^m - v_j^m)^2 \le 0. \quad \checkmark$$

(2) However, $B = \frac{\Delta x}{\Delta t}\sum_{j=1}^{n}(v_j^{m+1} - v_j^m)^2 \ge 0$, so we try to upper bound it by A instead.

To show $|B| \le |A|$, we have

$$B = \frac{\Delta x}{\Delta t}\sum_{j=1}^{n}\left(v_j^{m+1} - v_j^m\right)^2$$

$$= \Delta x \Delta t \sum_{j=1}^{n}\left(\frac{v_j^{m+1} - v_j^m}{\Delta t}\right)^2$$

(scheme)
$$= \Delta x \Delta t \sum_{j=1}^{n}\left(\frac{(v_{j+1}^m - v_j^m) - (v_j^m - v_{j-1}^m)}{\Delta x^2}\right)^2$$

$$= \frac{\Delta t}{\Delta x^3}\sum_{j=1}^{n}\Big(\underbrace{(v_{j+1}^m - v_j^m)}_{\text{"a"}} - \underbrace{(v_j^m - v_{j-1}^m)}_{\text{"b"}}\Big)^2$$

Using the Cauchy-Schwarz inequality(?): $(a_1b_1+a_2b_2)^2 \leq (a_1^2+a_2^2)(b_1^2+b_2^2)$

$$\Rightarrow \quad (a-b)^2 \leq 2a^2 + 2b^2 \qquad \text{(or notice } -2ab \leq a^2+b^2\text{)}$$

So $\frac{\Delta t}{\Delta x^3} \sum_{j=1}^{n} \left((v_{j+1}^m - v_j^m) - (v_j^m - v_{j-1}^m) \right)^2$

$$\leq 2\frac{\Delta t}{\Delta x^3} \left(\sum_{j=1}^{n} (v_{j+1}^m - v_j^m)^2 + \sum_{j=1}^{n} (v_j^m - v_{j-1}^m)^2 \right)$$

$$= 2\frac{\Delta t}{\Delta x^3} \left(\sum_{j=1}^{n} (v_{j+1}^m - v_j^m)^2 + \sum_{j=0}^{n-1} (v_{j+1}^m - v_j^m)^2 \right)$$

$$\leq 4\frac{\Delta t}{\Delta x^3} \sum_{j=0}^{n} (v_{j+1}^m - v_j^m)^2$$

when is $-\frac{2\Delta t}{\Delta x} + \frac{4\Delta t^2}{\Delta x^3} \leq 0$?

$$\Leftrightarrow \Delta t \leq \frac{\Delta x^2}{2} \rightarrow \text{then } E^{m+1} \leq E^m$$

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Energy stability, encore...

$$(H)\begin{cases} u_t - u_{xx} = 0 \\ u(0,x) = u_0(x) \\ u(t,0) = u(t,1) = 0 \end{cases}$$

$$(CN)\quad \frac{v_j^{m+1} - v_j^m}{\Delta t} = \frac{v_{j+1}^{m+1/2} - 2v_j^{m+1/2} + v_{j-1}^{m+1/2}}{\Delta x^2}, \quad v_j^{m+1/2} := \frac{v_j^m + v_j^{m+1}}{2}$$

$$\boxed{E^m := \Delta x \sum_{j=1}^{n} (v_j^m)^2}$$

Lemma: The Crank-Nicolson scheme (CN) satisfies

$$E^{m+1} \leq E^m, \quad \forall m \geq 0.$$

Proof. We want to show $\frac{E^{m+1} - E^m}{\Delta t} \leq 0$.

$$LHS = \frac{\Delta x}{\Delta t} \sum_{j=1}^{n} \left[(v_j^{m+1})^2 - (v_j^m)^2 \right]$$

$a^2 - b^2 = (a+b)(a-b)$

$$= \Delta x \sum_{j=1}^{n} \underbrace{(v_j^{m+1} + v_j^m)}_{=2v_j^{m+1/2}} \underbrace{\left(\frac{v_j^{m+1} - v_j^m}{\Delta t} \right)}_{\text{Plug in scheme CN}}$$

$$= 2\Delta x \sum_{j=1}^{n} v_j^{m+1/2} \frac{\left(\overbrace{v_{j+1}^{m+1/2} - v_j^{m+1/2}}^{H_j^{m+1/2}} - \overbrace{(v_j^{m+1/2} - v_{j-1}^{m+1/2})}^{H_{j-1}^{m+1/2}} \right)}{\Delta x^2}$$

$$= \frac{2}{\Delta x} \sum_{j=1}^{n} v_j^{m+1/2} (H_j^{m+1/2} - H_{j-1}^{m+1/2})$$

$$= \frac{2}{\Delta x} \sum_{j=1}^{n-1} (v_j^{m+1/2} - v_{j+1}^{m+1/2}) H_j^{m+1/2} - \frac{2}{\Delta x} v_1^{m+1/2} H_0^{m+1/2} + \frac{2}{\Delta x} v_n^{m+1/2} H_n^{m+1/2}$$

$$= -\frac{2}{\Delta x}\sum_{j=1}^{n-1}\left(V_{j+1}^{m+1/2} - V_j^{m+1/2}\right)^2 - \frac{2}{\Delta x} V_1^{m+1/2} H_0^{m+1/2} + \frac{2}{\Delta x}\underbrace{V_0}_{0 \text{ (by B.C.)}} H_0^{m+1/2}$$

$$+ \frac{2}{\Delta x} V_n^{m+1/2} H_n^{m+1/2} - \frac{2}{\Delta x}\underbrace{V_{n+1}^{m+1/2}}_{0 \text{ by B.C.}} H_n^{m+1/2}$$

$$= -\frac{2}{\Delta x}\sum_{j=1}^{n-1}\left(V_{j+1}^{m+1/2} - V_j^{m+1/2}\right)^2 - \frac{2}{\Delta x}\left(V_1^{m+1/2} - V_0^{m+1/2}\right)\underbrace{H_0^{m+1}}_{(V_1^{m+1/2}-V_0^{m+1/2})} - \frac{2}{\Delta x}\left(V_{n+1}^{m+1/2} - V_n^{m+1/2}\right)\underbrace{H_n^{m+1/2}}_{(V_{n+1}^{m+1/2}-V_n^{m+1/2})}$$

$$= -\frac{2}{\Delta x}\sum_{j=1}^{n-1}\left(V_{j+1}^{m+1/2} - V_j^{m+1/2}\right)^2 - \frac{2}{\Delta x}\left(V_1^{m+1/2} - V_0^{m+1/2}\right)^2 - \frac{2}{\Delta x}\left(V_{n+1} - V_n\right)^2$$

$$= -\frac{2}{\Delta x}\sum_{j=0}^{n}\left(V_{j+1}^{m+1/2} - V_j^{m+1/2}\right)^2 \le 0$$

Non-Linear Heat Equation

HW: $(\alpha(x)u_x)_x = f(x)$?

$$\begin{cases} u_t - (\alpha(u)u_x)_x = 0 \\ u(0,x) = u_0(x) \quad x\in(0,1) \\ u(t,0) = u(t,1) = 0 \quad t\ge 0 \end{cases}$$

Let $\alpha: \mathbb{R}\to\mathbb{R}^+$ be function depending on u:

$\alpha(u) = \frac{1+3u^2}{1+u^2} \ge 1$ e.g. bounded above and below? (Lipschitz) continuous?

Discretization: $\alpha_{j+1/2}^m = \frac{1}{2}\left(\alpha(V_{j+1}^m) + \alpha(V_j^m)\right)$

$$\frac{V_j^{m+1} - V_j^m}{\Delta t} - \frac{\alpha_{j+1/2}^m\left(V_{j+1}^{m+1/2} - V_j^{m+1/2}\right) - \alpha_{j-1/2}^m\left(V_j^{m+1/2} - V_{j-1}^{m+1/2}\right)}{\Delta x^2} = 0$$

$$\boxed{T_j^m = \underbrace{\frac{u(t^{m+1},x_j) - u(t^m,x_j)}{\Delta t}}_{u_t(t^m,x_j)+O(\Delta t)} - \frac{1}{\Delta x}\left[\frac{\alpha(u(t^m,x_j)) + \alpha(u(t^m,x_{j+1}))}{2}\left(\frac{\frac{u(t^m,x_{j+1}) + u(t^{m+1},x_{j+1})}{2} - \frac{u(t^m,x_j) + u(t^{m+1},x_j)}{2}}{\Delta x}\right) + \frac{\alpha(u(t^m,x_j)) + \alpha(u(t^m,x_{j-1}))}{2}\left(\frac{\frac{u(t^m,x_j) + u(t^{m+1},x_j)}{2} - \frac{u(t^m,x_{j-1}) + u(t^{m+1},x_{j-1})}{2}}{\Delta x}\right)\right]}$$

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$$\underbrace{\qquad\qquad\qquad}$$

$$\tfrac{1}{2} u_x(t^m, x_j) - \frac{\Delta x}{4} u_{xx}(t^m, x_j) + \frac{\Delta x^2}{12} u_{xxx}(t^m, x_j) + O(\Delta x^3)$$

$$+ \tfrac{1}{2} u_x(t^{m+1}, x_j) - \frac{\Delta x}{4} u_{xx}(t^{m+1}, x_j) + \frac{\Delta x^2}{12} u_{xxx}(t^{m+1}, x_j) + O(\Delta x^3)$$

option 1

$$= \frac{1}{\Delta x}\left[\frac{\alpha(u(t^m, x_j)) + \alpha(u(t^m, x_{j+1}))}{2}\left(\tfrac{1}{2} u_x(t^m, x_j) + \tfrac{1}{2} u_x(t^{m+1}, x_j)\right)\right.$$

...

$$-\left(\alpha(u(t^m, x_j)) + \alpha(u(t^m, x_{j-1}))\right)\left(\tfrac{1}{2} u_x(t^m, x_j) + \tfrac{1}{2} u_x(t^{m+1}, x_j) + \frac{\Delta x}{4} u_{xx}(t^m, x_j)\right.$$

$$\left. - \frac{\Delta x}{4} u_{xx}(t^{m+1}, x_j) + \frac{\Delta x^2}{12} u_{xxx}(t^m, x_j) + \frac{\Delta x^2}{12} u_{xxx}(t^{m+1}, x_j)\right]$$

Product rule:

$(\alpha(u) u_x)_x = \alpha(u)_x u_x + \alpha(u) u_{xx}$

$$\simeq \frac{1}{\Delta x}\underbrace{\left(\alpha(u(t^m, x_{j+1})) + \alpha(u(t^m, x_{j-1}))\right)}_{\partial_x[\alpha(u)]}\underbrace{\left(\tfrac{1}{2} u_x(t^m, x_j) + \tfrac{1}{2} u_x(t^{m+1}, x_j)\right)}_{u_x}$$

$$+ \left(\underbrace{\frac{2\alpha(u(t^m, x_j)) + \alpha(u(t^m, x_{j-1})) + \alpha(u(t^m, x_{j+1}))}{2}}_{\simeq 2\alpha(u(t^m, x_j))}\ \underbrace{\frac{u_{xx}(t^m, x_j) + u_{xx}(t^{m+1}, x_j)}{4}}_{\frac{1}{2} u_{xx}(t^{m+1/2}, x_j)}\right)$$

$$\simeq \boxed{\partial_x \alpha(u(t^m, x_j)) u_x(t^m, x_j) + \alpha(u(t^m, x_j)) u_{xx}(t^m, x_j) + O(\Delta t + \Delta x^2)}$$

note: $u_x(t^{m+1}, x_j) = u_x(t^m, x_j) + \Delta t\, u_{xt} + \dots$

will ~~knock~~ out earlier $u_t(t^m, x_j)$ term in C-N derivation

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Non-linear equations : Need 'compactness' to prove convergence...

But we can prove stability using an Energy estimate.

Define $E^{m} := \Delta x \sum_{j=1}^{n} (v_j^m)^2$ (time)

We want to show $E^{m+1} \le E^m$, $m = 0, 1, \ldots$

"Compute a discrete version of $\frac{d}{dt}E(t)$, show that it is non-positive"

Focus on the key ideas:

$$\frac{E^{m+1} - E^m}{\Delta t} = \frac{\Delta x}{\Delta t} \sum_{j=1}^{n} \left[(v_j^{m+1})^2 - (v_j^m)^2 \right]$$

$a^2 - b^2 = (a+b)(a-b)$

$$= \Delta x \sum_{j=1}^{n} (v_j^{m+1} + v_j^m) \left(\frac{v_j^{m+1} - v_j^m}{\Delta t} \right)$$

def $v_j^{m+1/2}$

$$= 2 \Delta x \sum_{j=1} v_j^{m+1/2} \left(\frac{v_j^{m+1} - v_j^m}{\Delta t} \right)$$

Plug in scheme (semi-implicit)

plug in scheme

$$= 2 \sum_{j=1}^{n} v_j^{m+1/2} \alpha_{j+1/2} \left(\frac{v_{j+1}^{m+1/2} - v_j^{m+1/2}}{\Delta x} \right) - 2 \sum_{j=1}^{n} v_j^{m+1/2} \alpha^m_{j-1/2} \left(\frac{v_j^{m+1/2} - v_{j-1}^{m+1/2}}{\Delta x} \right)$$

shift indices

$$= 2 \sum_{j=1}^{n} v_j^{m+1/2} \alpha^m_{j+1/2} \cdot \frac{v_{j+1}^{m+1/2} - v_j^{m+1/2}}{\Delta x} - 2 \sum_{j=0}^{n-1} v_{j+1}^{m+1/2} \alpha^m_{j+1/2} \cdot \frac{v_{j+1}^{m+1/2} - v_j^{m+1/2}}{\Delta x}$$

$$= 2 \sum_{j=1}^{n-1} (v_j^{m+1/2} - v_{j+1}^{m+1/2}) \alpha^m_{j+1/2} \cdot \left(\frac{v_{j+1}^{m+1/2} - v_j^{m+1/2}}{\Delta x} \right) \underbrace{+ 2 v_n^{m+1/2} \alpha^m_{n+1/2} \frac{v_{n+1}^{m+1/2} - v_n^{m+1/2}}{\Delta x} - 2 v_n^{m+1} \alpha^m_{1/2} \left(\frac{v_1^{m+1/2} - v_0^{m+1/2}}{\Delta x} \right)}_{\Delta}$$

$$= \underbrace{-\frac{2}{\Delta x} \sum_{j=1}^{n-1} \alpha^m_{j+1/2} \left(v_{j+1}^{m+1/2} - v_j^{m+1/2} \right)^2}_{\le 0} \underbrace{- \frac{2}{\Delta x} \alpha^m_{n+1/2} (v_n^{m+1/2})^2}_{\le 0} \underbrace{- \frac{2}{\Delta x} \alpha_{1/2} (v_1^{m+1/2})^2}_{\le 0} \le 0$$

Works for periodic boundary conditions ... $v_0^m = v_n^m$, $v_{n+1}^m = v_1^m$

$\alpha^m_{n+1/2} = \frac{1}{2}(\alpha(v_n^m) + \alpha(v_{n+1}^m)) = \frac{1}{2}(\alpha(v_n^m) + \alpha(v_1^m))$

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$$\alpha^m_{1/2} = \tfrac{1}{2}(\alpha(v_1^m) + \alpha(v_0^m)) = \tfrac{1}{2}(\alpha(v_1^m) + \alpha(v_n^m))$$

$$\therefore\ \alpha^m_{n+1/2} = \alpha^m_{1/2}.$$

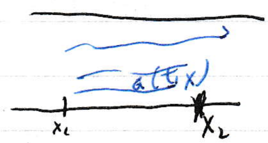
So $\triangle$ becomes $2V_n^{m+1/2}\alpha^m_{1/2}\dfrac{V_1^{m+1/2} - V_n^{m+1/2}}{\Delta x} - 2V_n^{m+1/2}\alpha^m_{1/2}\dfrac{V_1^{m+1/2} - V_n^{m+1/2}}{\Delta x}$

$$= -\frac{2}{\Delta x}\alpha^m_{1/2}\left(V_1^{m+1/2} - V_n^{m+1/2}\right)^2 \leq 0$$

Transport Equation

$$\begin{cases} u_t + a u_x = 0, & x \in \mathbb{R},\ t > 0 \\ u(0,x) = u_0(x) & x \in \mathbb{R} \end{cases}$$

$a = a(t,x)$

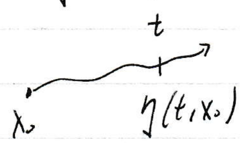


$$\int_{x_1}^{x_2} \partial_t u(t,x)\,dx = a u(t,x_1) - a u(t,x_2) = -\int_{x_1}^{x_2} a u_x(t,x)\,dx$$

Method of Characteristics

Consider the trajectory of a particle $y(t,x_0)$

$y(0,x_0) = x_0$



$$\frac{d}{dt}y(t,x_0) =: \overset{\text{'rate of flow'}}{a(t, y(t,x_0))}$$

If u is the particle concentration, how does the ~~trajectory~~ concentration change along a trajectory y?

$$\frac{d}{dt}u(t,y(t,x_0)) = \partial_t u(t,y(t,x_0)) + \partial_x u(t,y(t,x_0))\underbrace{\frac{d}{dt}y(t,x_0)}_{a(t,y(t,x_0))} \overset{\text{eqn of } u}{=} 0$$

$\therefore$ (Characteristic Equations)

$$\begin{cases} \dfrac{d}{dt}y(t,x_0) = a(t,y(t,x_0)), & y(0,x_0) = x_0 \\ \dfrac{d}{dt}u(t,y(t,x_0)) = 0, & u(0,y(0,x_0)) = u(0,x_0) = u_0(x_0) \end{cases}$$

Let us suppose $a(t,x_0) = a \in \mathbb{R}$ constant

By integration: 2nd Equation is how $u(t,\eta(t,x_0)) = u(0,\eta(0,x_0)) = u_0(x_0)$ $(*)$

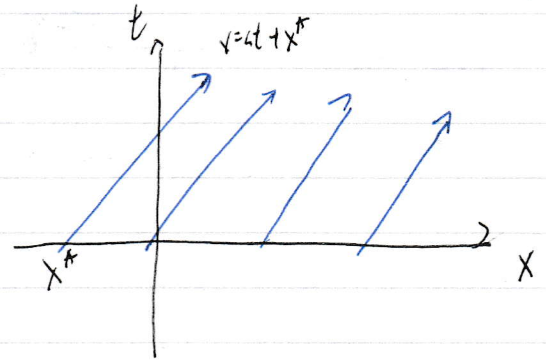
1st condition gives $\frac{d}{dt}\eta(t,x_0) = a$, $\eta(0,x_0) = x_0$

$$\eta(t,x_0) = a\cdot t + \eta(0,x_0) = a\cdot t + x_0$$

Plug into $(*)$: $u(t, x_0 + at) = u_0(x_0)$

$$\Downarrow$$

$$u(t,x) = u_0(x-at)$$



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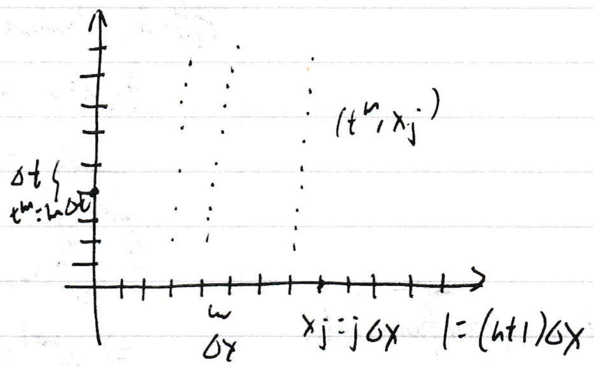
Finite Difference schemes for the Transport Equation

Assume a is constant, and we have periodic boundary conditions on $[0,1]$

Space-time grid:



Hope that $v_j^m \approx u(t^m, x_j)$ $\forall m \geq 0,\ j = 1 \ldots n$

$$\frac{u(t+\Delta t, x) - u(t,x)}{\Delta t} = u_t(t,x) + \underbrace{O(\Delta t)}_{\frac{\Delta t}{2} u_{tt}(\tilde{t}, x),\ \tilde{t} \in [t, t+\Delta t]}$$

$$\frac{u(t, x+\Delta x) - u(t, x-\Delta x)}{2\Delta x} = u_x(t,x) + \underbrace{O(\Delta x^2)}_{\frac{\Delta x^2}{6} u_{xxx}(t,\tilde{x})}$$

$$\Rightarrow \frac{v_j^{m+1} - v_j^m}{\Delta t} + \frac{a(v_{j+1}^m - v_{j-1}^m)}{2\Delta x} = 0$$

(Explicit) $$v_j^{m+1} = v_j^m - \frac{a\Delta t}{2\Delta x} v_{j+1}^m + \frac{a\Delta t}{2\Delta x} v_{j-1}^m, \quad m \geq 0$$

$$v_j^0 = u_0(x_j),\ j = 1, \ldots n$$

$v_j^0 = u(x_j) \qquad j = 1, \dots, n$

for $m = 0 \dots M-1$

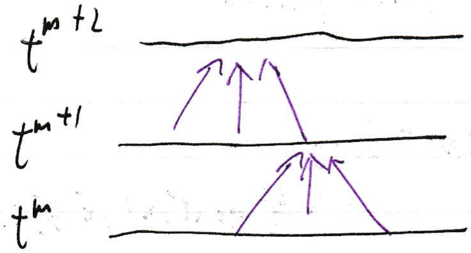
$$v_j^{m+1} = v_j^m - \frac{a\Delta t}{2\Delta x} v_{j+1}^m + \frac{a\Delta t}{2\Delta x} v_{j-1}^m$$

$$v_0^{m+1} = v_n^{m+1}$$

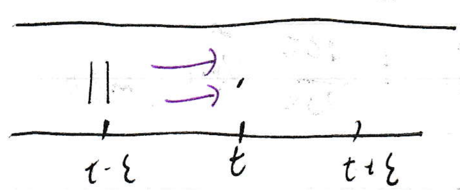
$$v_{n+1}^{m+1} = v_1^{m+1}$$

end for

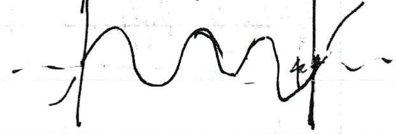
However, this ~~sch~~ scheme doesn't really perform well.
Heuristic explanation: what we're doing is



However, we shouldn't use 'downstream' information to figure out what is at an earlier location at the next time step (information flows left to right).



Interpretation (Periodic B.C.)



'[illegible]' or 'circular pool'

Alt. B.C.: Homogeneous Neumann conditions

$\partial_x u = 0$ at $x=0$ & $x=1$

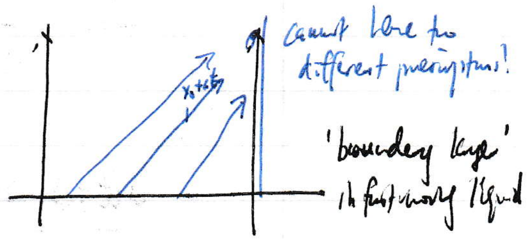
'specify flux across boundary'

$$\frac{v_1^{m+1} - v_0^{m+1}}{\Delta x} = 0, \quad \frac{v_{n+1}^{m+1} - v_n^{m+1}}{\Delta x} = 0$$

$$\Rightarrow v_0^{m+1} = v_1^{m+1}$$

$$v_{n+1}^{m+1} = v_n^{m+1}$$

Dirichlet B.C. does not lead to stable solutions



$$y(t; x_0) = x_0 + ct$$

$$u(t, y(t, x_0)) = u(0, x_0) = u_0(x_0)$$

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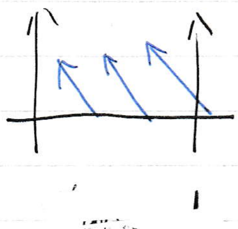
'upwind scheme' a_{jm}

So we should use $\boxed{\dfrac{v_j^{m+1}-v_j^m}{\Delta t} + a\,\dfrac{v_j^m - v_{j-1}^m}{\Delta x} = 0}$. (* for $a>0$)

By trial-and-error: $\Rightarrow v_j^{m+1} = \left(1 - \dfrac{a\Delta t}{\Delta x}\right) v_j^m + \dfrac{a\Delta t}{\Delta x} v_{j-1}^m$

$$dt \approx \frac{dx}{a}$$

For $a<0$: we should use $\boxed{\dfrac{v_j^{m+1}-v_j^m}{\Delta t} + a\,\dfrac{v_{j+1}^m - v_j^m}{\Delta x} = 0}$.



$$v_j^{m+1} = \left(1 + \frac{a\Delta t}{\Delta x}\right) v_j^m - \frac{a\Delta t}{\Delta x} v_{j+1}^m$$

What if a is not constant? e.g. $a(x) = -1 + 0.5 * \cos(2\pi x)$

we should take the minimum of a ~~e.g.~~ i.e. $\|a(x)\|_\infty$

We define now $a^+(x) := \max(0, a(x))$, $a^-(x) = \min(0, a(x))$

$$a_j^+ = a^+(x_j) \qquad a_j^- = a^-(x_j)$$

$$\boxed{\frac{v_j^{m+1}-v_j^m}{\Delta t} + a_j^+ \frac{v_j^m - v_{j-1}^m}{\Delta x} + a_j^- \frac{v_{j+1}^m - v_j^m}{\Delta x} = 0.}$$

$$\frac{v_j^{m+1}-v_j^m}{\Delta t} + a\,\frac{(v_{j+1}^m - v_{j-1}^m)}{2\Delta x} = \frac{|a|}{2\Delta x}\underbrace{(v_{j+1}^m - 2v_j^m + v_{j-1}^m)}$$

Go to page 100 for proof of stability $\approx \dfrac{\Delta x |a|}{2} u_{xx}$ 'numerical viscosity/diffusion'

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$$(T) \begin{cases} u_t + a u_x = 0 & x \in (0,1), t>0 \\ u(0,x) = u_0(x) & x \in (0,1) \\ u(t,0) = u(t,1) \\ u_x(t,0) = u_x(t,1) \end{cases}$$

Assume $a = a(t,x) \in \mathbb{R}$ in this case.

$$E(t) = \int_0^1 (u(t,x))^2\, dx.$$

Lemma: Assume the solution of (T) is continuously differentiable. Then $(\in C^1)$

$$E(t) = E(0) \text{ for all } t \geq 0.$$

Proof. Multiply (T) by u:

$$u u_t + a u \cdot u_x = 0$$

By the chain rule, $\frac{1}{2}\partial_t(u^2) + \frac{1}{2} a \partial_x(u^2) = 0$

$$\Leftrightarrow \frac{1}{2}\partial_t(u^2) + \frac{1}{2}\partial_x(au^2) = 0$$

Integrate in space over $[0,1]$:

$$\frac{1}{2}\int_0^1 \partial_t (u(t,x)^2)\,dx + \frac{1}{2}\int_0^1 \partial_x(a(u(t,x))^2)\,dx = 0$$

$\Downarrow$ because $u \in C^1$

$$\frac{1}{2}\frac{d}{dt}\underbrace{\int_0^1 (u(t,x))^2 dx}_{E(t)} + \frac{1}{2}\int_0^1 \partial_x(a(u(t,x))^2)\,dx = 0$$

$$\int_0^1 \partial_x(a(u(t,x))^2)\,dx = \frac{1}{2}a(u(t,1))^2 - \frac{1}{2}a(u(t,0))^2$$

(by FTC: $f(1) - f(0) = \int_0^1 \partial_x f(x)\,dx$)

$$\frac{d}{dt}E(t) + \underbrace{\frac{a}{2}(u(t,1))^2 - \frac{a}{2}(u(t,0))^2}_{=0 \text{ by periodic boundary conditions}} = 0$$

$$\frac{d}{dt}E(t) = 0 \Rightarrow E(t) = E(0)$$

Central scheme : (CS) $\dfrac{v_j^{m+1}-v_j^m}{\Delta t} + a\dfrac{v_{j+1}^m - v_{j-1}^m}{2\Delta x} = 0$

(doesn't seem to work: let's find out why)

Discrete Energy: $E^m := \Delta x \sum_{j=1}^{n} (v_j^m)^2$

Lemma: Approximations computed by (CS) satisfy

$$E^{m+1} = E^m + \underbrace{\Delta x \sum_{j=1}^{n} (v_j^{m+1} - v_j^m)^2}_{\geq 0 \text{ : bad!}}$$

Proof. Multiply the scheme (CS) by v_j^m to get

$$\frac{v_j^m(v_j^{m+1}-v_j^m)}{\Delta t} + \frac{a}{2\Delta x} v_j^m (v_{j+1}^m - v_{j-1}^m) = 0$$

$c(b-c) = \frac{1}{2}b^2 - \frac{1}{2}c^2 - \frac{1}{2}(b-c)^2 \quad c, b \in \mathbb{R}$

Use with $c = v_j^m$, $b = v_j^{m+1}$

$$\frac{1}{2\Delta t}\left((v_j^{m+1})^2 - (v_j^m)^2 - (v_j^{m+1} - v_j^m)^2\right) + \frac{a}{2\Delta x} v_j^m (v_{j+1}^m - v_{j-1}^m) = 0$$

Multiplying by $2\Delta t \Delta x$,

$$\Delta x\left((v_j^{m+1})^2 - (v_j^m)^2 - (v_j^{m+1}-v_j^m)^2\right) + a\Delta t\, v_j^m (v_{j+1}^m - v_{j-1}^m) = 0$$

$$\underbrace{\Delta x \sum_{j=1}^{n} (v_j^{m+1})^2}_{E^{m+1}} - \underbrace{\Delta x \sum_{j=1}^{n} (v_j^m)^2}_{E^m} - \Delta x \sum_{j=1}^{n} (v_j^{m+1} - v_j^m)^2 + a\Delta t \sum_{j=1}^{n} v_j^m (v_{j+1}^m - v_{j-1}^m) = 0.$$

$$E^{m+1} - E^m - \Delta x \sum_{j=1}^{n} (V_j^{m+1} - V_j^m)^2 + a\Delta t \Big(\underbrace{\sum_{j=1}^{n} V_j^m V_{j+1}^m - \sum_{j=1}^{n} V_j^m V_{j-1}^m}_{-V_1^m V_0^m + V_n^m V_{n+1}^m} \Big) = 0$$

$= 0$ by periodic B.C.

$$\therefore E^{m+1} = E^m + \Delta x \sum_{j=1}^{n} (V_j^{m+1} - V_j^m)^2$$

Upwind scheme (US) $a > 0$, $\dfrac{V_j^{m+1} - V_j^m}{\Delta t} + a \dfrac{V_j^m - V_{j-1}^m}{\Delta x} = 0$

$\left(a < 0 : \dfrac{V_j^{m+1} - V_j^m}{\Delta t} + a \dfrac{V_{j+1}^m - V_j^m}{\Delta x} = 0 \right)$

'CFL condition'

lemma: Assume $|a| \frac{\Delta t}{\Delta x} \le 1$. Then (US) satisfies $E^{m+1} \le E^m$ $\forall m$.
'L2-stability'

Proof. Assume $a > 0$. Rewrite scheme as

$$\frac{V_j^{m+1} - V_j^m}{\Delta t} + a \frac{V_{j+1}^m - V_{j-1}^m}{2\Delta x} = \frac{a}{2\Delta x} (V_{j+1}^m - 2V_j^m + V_{j-1}^m)$$

$b - b = \frac{a-c}{2} - \left(\frac{a-2b+c}{2}\right)$

Multiply scheme by V_j^m and use $a(b-a) = \frac{1}{2}b^2 - \frac{1}{2}a^2 - \frac{1}{2}(b-a)^2$

$$\frac{1}{2\Delta t} \left((V_j^{m+1})^2 - (V_j^m)^2 - (V_j^{m+1} - V_j^m)^2 \right) + \frac{a}{2\Delta x} V_j^m (V_{j+1}^m - V_{j-1}^m) = \frac{a}{2\Delta x} \begin{bmatrix} V_j^m (V_{j+1}^m - V_j^m) \\ - V_j^m (V_j^m - V_{j-1}^m) \end{bmatrix}$$

Multiply by $2\Delta t \Delta x$

$$\underbrace{\Delta x \sum_{j=1}^{n} (V_j^{m+1})^2}_{E^{m+1}} - \underbrace{\Delta x \sum_{j=1}^{n} (V_j^m)^2}_{E^m} - \Delta x \sum_{j=1}^{n} (V_j^{m+1} - V_j^m)^2 + \Delta t\, a \sum_{j=1}^{n} V_j^m (V_{j+1}^m - V_{j-1}^m)$$

$$= \Delta t\, a \left(\sum_{j=1}^{n} V_j^m (V_{j+1}^m - V_j^m) - \sum_{j=1}^{n} V_j^m (V_j^m - V_{j-1}^m) \right)$$

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#4

$$E^{n+1} - E^n - \Delta x \sum_{j=1}^{n} (v_j^{n+1} - v_j^n)^2 = \Delta t a \left(\underbrace{\sum_{j=1}^{n} v_j(v_{j+1}^n - v_j^n) - \sum_{j=1}^{n} v_j^n (v_j^n - v_{j-1}^n)}_{} \right)$$

$$+ \cancel{\Delta t a \sum_{j=1}^{n} v_j^n (v_{j+1}^n - v_{j-1}^n)} = 0$$

$$= \sum_{j=0}^{n-1} v_{j+1}^n (v_{j+1}^n - v_j^n)$$ 11/15/2021

2D:

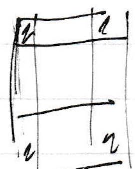
$$\frac{v_{ij}^{n+1} - v_{ij}^n}{\Delta t} + k_1 \frac{v_{i,j}^n - v_{i-1,j}^n}{\Delta x} + k_2 \frac{v_{i,j}^n - v_{i,j-1}^n}{\Delta x} = 0$$

$$v_{i,j}^{n+1} = v_{ij}^n - k_1 \frac{\Delta t}{\Delta x} (v_{i,j}^n - v_{i-1,j}^n) - k_2 \frac{\Delta t}{\Delta x} (v_{i,j}^n - v_{i,j-1}^n)$$

Summation by parts

$$= \sum_{j=1}^{n-1} (v_j^n - v_{j+1}^n)(v_{j+1}^n - v_j^n) - v_1^n (v_1^n - v_0^n) + v_n^n (v_{n+1}^n - v_n^n)$$

$$= \quad '' \quad - v_1^n (v_1^n - v_0^n) + v_0^n (v_1^n - v_0^n) + \cancel{v_n^n (v_{n+1}^n - v_n^n) - v_{n+1}^n (v_{n+1}^n - v_n^n)}$$



HW7Q1 see photo

$$= \Delta t a \left(-\sum_{j=2}^{n} (v_j^n - v_{j-1}^n)^2 - (v_1^n - v_0^n)^2 \right)$$

How to show uniqueness of solution(?)
- Energy method (same equations as problem) $u_1 - u_2 = 0$
- Lipschitz → show uniqueness (see ODEs)

$$E^{n+1} - E^n - \Delta x \sum_{j=1}^{n} (v_j^{n+1} - v_j^n)^2 = -a \Delta t \sum_{j=1}^{n} (v_j^n - v_{j-1}^n)^2$$

$$\Leftrightarrow E^{n+1} = E^n + \Delta x \underbrace{\sum_{j=1}^{n} (v_j^{n+1} - v_j^n)^2}_{= -\frac{a \Delta t}{\Delta x}(v_j^n - v_{j-1}^n)} - a \Delta t \sum_{j=1}^{n} (v_j^n - v_{j-1}^n)^2$$

by plugging in the scheme

$$= E^n + \underbrace{\frac{a^2 \Delta t^2}{\Delta x} \sum_{j=1}^{n} (v_j^n - v_{j-1}^n)^2 - a \Delta t \sum_{j=1}^{n} (v_j^n - v_{j-1}^n)^2}_{} \overset{\text{want}}{\leq} 0.$$

So we needed $\frac{a^2 \Delta t^2}{\Delta x} \leq a \Delta t \Leftrightarrow \frac{a \Delta t}{\Delta x} \leq 1$

~~Next~~ Future directions

$$\|v^{m+1}\|_{\Delta x,\infty} \le \|v^m\|_{\Delta x,\infty} \iff \max_{j=1\dots n}|v_j^{m+1}| \le \max_{j=1\dots n}|v_j^m|$$

Transport Equation = 1-way $\boxed{\text{Wave Equation}}$

(Assume 1D on $\mathbb{R}$ for now)

$$u_{tt} = u_{xx} \qquad x\in\mathbb{R}, t>0$$

$$u(0,x) = f(x)$$
$$u_t(0,x) = g(x) \qquad x\in\mathbb{R}$$

$f, g \in C(\mathbb{R})$ compactly supported non-zero on compact set...

(sometimes $u_{tt} = c^2 u_{xx}$, where c is the wave speed)

Explicit solutions

We can rewrite $u_{tt} - u_{xx} = 0$

$$\iff (\partial_t - \partial_x)(\partial_t + \partial_x)u = 0$$

$$(\partial_t\partial_t u - \cancel{\partial_x\partial_t u} + \cancel{\partial_t\partial_x u} - \partial_x\partial_x u)$$

assume u is smooth (C^2 is enough)

2 transport equations!

Define $w = \partial_t u + \partial_x u$. Then if u solves the wave equation,

$\Rightarrow$ w solves $\partial_t w - \partial_x w = 0$ (transport equation with $a = -1$)

can sub. in to check that $\partial_t w - \partial_x w = \partial_t(\partial_t u + \partial_x u) - \partial_x(\partial_t u + \partial_x u) = 0$

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with initial condition $w(0,x) = \partial_t u(0,x) + \partial_x u(0,x)$
$= g(x) + f'(x)$ [initial conditions for u]

Idea: find w, then find u

We can find w using the method of characteristics, since it solves the transport equation with $c=1$. In particular,

$$\begin{cases} w_t - w_x = 0 \\ w(0,x) = g(x) + f'(x) \end{cases}$$

$$\Rightarrow \quad w(t,x) = \underbrace{w_0(x-ct)}_{\text{here } c=-1} = w_0(x+t) = g(x+t) + f'(x+t)$$

How to find u from this?

$u_t(t,x) + u_x(t,x) = w(t,x) = g(t+x) + f'(x+t)$ (nonhomogeneous transport equation)

Method of characteristics: particle trajectories η satisfy

in general: $a(\eta(t,x_0),t)$ for $a = a(x,t)$

① $\left\{ \frac{d}{dt}\eta(t,x_0) = a = 1, \quad \eta(0,x_0) = x_0 \right.$

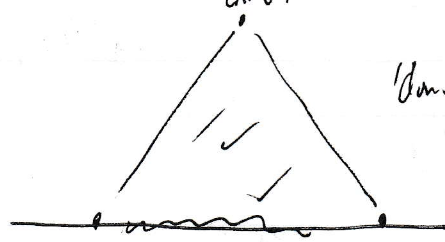
② $\left\{ \begin{aligned} \frac{d}{dt} u(t,\eta(t,x_0)) &= u_t(t,\eta(t,x_0)) + u_x(t,\eta(t,x_0)) \underbrace{\frac{d}{dt}\eta(t,x_0)}_{=1} \\ &= u_t(t,\eta(t,x_0)) + u_x(t,\eta(t,x_0)) \\ &= w(t,\eta(t,x_0)) \\ &= f'(t+\eta(t,x_0)) + g(t+\eta(t,x_0)) \\ u(0,\eta(0,x_0)) &= f(x_0) \end{aligned} \right.$

(Go to page 101)

Recitation 11/16
(And sketch work for HW)
8Q2

Wave Equation
$(\partial_{tt} u = \partial_{xx} u)$

(x,t)



'Domain of influence'/
'history'

Heat Equation: $(\partial_t u = \partial_{xx} u)$ Propagation speed is infinite

(fundamental solution of $\begin{cases} u_t = \Delta u, \\ u(x,0) = g(x) \end{cases}$ is $u(x,t) = \int_{\mathbb{R}} \Phi(x-y,t) g(y)\,dy$

$$= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} g(y)\,dy$$

$$\frac{V_{i,j}^{m+1} - V_{i,j}^{m}}{\Delta t} + k_1^{+} \frac{V_{i,j}^{m} - V_{i-1,j}^{m}}{\Delta x} + k_1^{-} \frac{V_{i+1,j}^{m} - V_{i,j}^{m}}{\Delta x} + k_2^{+} \frac{V_{i,j}^{m} - V_{i,j-1}^{m}}{\Delta x} + k_2^{-} \frac{V_{i,j+1}^{m} - V_{i,j}^{m}}{\Delta x} = 0$$

from PS7

$$E^m = \Delta x_1 \Delta x_2 \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} (v_{i,j}^{m})^2$$

$$E^{m+1} - E^m = \Delta x_1 \Delta x_2 \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} [(v_{i,j}^{m+1})^2 - (v_{i,j}^{m})^2]$$

$$\frac{V_{i,j}^{m+1} - V_{i,j}^{m}}{\Delta t} + k_1 \frac{V_{i+1,j}^{m} - V_{i-1,j}^{m}}{2\Delta x_1} + k_2 \frac{V_{i,j+1} - V_{i,j-1}}{2\Delta x_2} = \frac{a}{\Delta x_1}(V_{i+1,j}^{m} - 2V_{i,j}^{m} + V_{i-1,j}^{m})$$

Multiply by $V_{i,j}^{m}$,

$$\frac{1}{2\Delta t}\left((V_{i,j}^{m+1})^2 - (V_{ij}^{m})^2 - (V_{ij}^{m+1} - V_{ij}^{m})^2\right) + \frac{k_1}{2\Delta x} \ldots$$

$$+ \frac{a}{2\Delta x_2}(V_{i,j+1}^{m} - 2V_{i,j}^{m} + V_{i,j-1}^{m}) v_j^m$$

Use $a(b-a) = \frac{1}{2}b^2 - \frac{1}{2}a^2 - \frac{1}{2}(b-a)^2$

Multiply by $2\Delta t \Delta x_1 \Delta x_2$

$$\Delta x_1 \Delta x_2 (\qquad) - \Delta x_1 \Delta x_2 \sum_{i=1}^{N} \sum_{j=1}^{N} (v_{ij}^{m+1} - v_{ij}^{m})^2 + \Delta t \Delta x_2 k_1 \sum_{i=1}^{N} \sum_{j=1}^{N} V_{ij}^{m} (v_{i,j+1}^{m} - v_{i,j-1}^{m})$$

$+ \Delta t \Delta x_1 k_2 \ldots$

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(from page 99) We were trying to solve for u which satisfies the following:

$$\begin{cases} \frac{d}{dt}\eta(t,x_0)=1,\ \eta(0,x_0)=x_0 & -① \\ \frac{d}{dt}u(t,\eta(t,x_0))=g(t+\eta(t,x_0))+f'(t+\eta(t,x_0)) & -② \\ u(0,\eta(0,x_0))=f(x_0) \end{cases}$$

$$\overset{①}{\Rightarrow} \eta(t,x_0)=x_0+t$$

$$\overset{\text{Subst into (2)}}{\Rightarrow} \frac{d}{dt}u(t,x_0+t)=g(2t+x_0)+f'(2t+x_0)$$

By integration, $u(t,x_0+t)-u(0,x_0)=\int_0^t (g(2s+x_0)+f'(2s+x_0))\,ds$

$$u(t,x_0+t)=f(x_0)+\int_0^t (g(2s+x_0)+f'(2s+x_0))\,ds$$

Define $x:=x_0+t$, $x_0=x-t$

$$u(t,x)=f(x-t)+\int_0^t\big(g(\underbrace{2s}_{=:z}+x-t)+f'(2s+x-t)\big)\,ds$$

$$=f(x-t)+\frac{1}{2}\int_0^{2t}\big(g(z+x-t)+f'(z+x-t)\big)\,dz$$

$$\overset{FTC}{=} f(x-t)+\frac{1}{2}\int_0^{2t}g(z+x-t)\,dz+\frac{1}{2}f(x+t)-\frac{1}{2}f(x-t)$$

$$=\frac{1}{2}(f(x+t)-f(x-t))+\frac{1}{2}\int_0^{2t}g(\underbrace{z+x-t}_{=:s})\,dz$$

d'Alembert's formula

$$\therefore \boxed{u(t,x)=\frac{1}{2}(f(x+t)+f(x-t))+\frac{1}{2}\int_{x-t}^{x+t}g(s)\,ds}$$

$\left(\begin{array}{l} s=z+x-t \Rightarrow \frac{dz}{ds}=1 \\ 2t=z \Rightarrow s=2t+x-t=x+t \\ z=0 \Rightarrow s=x-t \end{array}\right)$

If $G(x)=\int_0^x g(s)\,ds$ is primitive, then d'Alembert's formula can be written as

$$u(t,x)=\frac{1}{2}(f(x+t)+f(x-t))+\frac{1}{2}(G(x+t)-G(x-t))$$

'two-sided' / finite speed of propagation

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(0,x) = f(x) \\ u_t(0,x) = g(x) \end{cases}$$

Another perspective...

Define $v := u_t$, $w := u_x$

What equations do v & w satisfy?. $v_t = (u_t)_t = u_{tt} = u_{xx} = (u_x)_x = w_x$

and similarly $w_t = (u_x)_t = u_{tx} = (u_t)_x = v_x$

$$\begin{cases} v_t - w_x = 0 \\ w_t - v_x = 0 \end{cases}$$

Define $U = \begin{pmatrix} v \\ w \end{pmatrix}$, $A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \Rightarrow \boxed{U_t + AU_x = 0}$

"vector-transport" equation

Wave Equation on a bounded domain

$$(W) \begin{cases} u_{tt} - u_{xx} = 0 \\ u(0,x) = f(x),\ u_t(0,x) = g(x) \\ u(t,0) = u(t,1) = 0 \end{cases}$$

We try to find a "special" solution using separation of variables

$u(t,x) = T(t)X(x)$ "Ansatz" (assumption)

Plug this ansatz into the wave equation:

$$\begin{cases} T''(t)X(x) - T(t)X''(x) = 0 \\ X(0) = X(1) = 0 \leftarrow \text{boundary conditions} \end{cases}$$

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Rewrite this as $\underbrace{\dfrac{T''(t)}{T(t)}}_{\text{constant in } x} = \underbrace{\dfrac{X''(x)}{X(x)}}_{\text{constant in time}} = -\lambda$, $\lambda \in \mathbb{R}$

↑ this should be constant

The LHS is constant in the variable x and the RHS is constant in t
$\Rightarrow$ both expressions should be constant in x & t.

1) we need to solve an ODE and 2-point BVP for X.

$$\begin{cases} -X''(x) = \lambda X(x) \ , \ x \in (0,1) \\ X(0) = X(1) = 0 \end{cases}$$

know $\lambda > 0$ because of B.C.

let $\lambda_k = (k\pi)^2$ $\Rightarrow$ $X_k(x) = \sin(k\pi x)$ is a solution.

$k = 1, 2, 3 \ldots$

(k<0 would lead to complementary c_k and d_k's below, so nothing has been omitted by having $k>0$)

2) T_k now satisfies $T_k''(t) = -\lambda_k T_k(t)$

$$T_k(t) = c_k e^{ik\pi t} \ , \ T_k(t) = d_k e^{-ik\pi t}$$

$u_{-k}(t,x) = (a_{-k}\cos(-k\pi t) + b_{-k}\sin(-k\pi t))\sin(-k\pi x)$
$= (-a_{-k}\cos(k\pi t) + b_{-k}\sin(k\pi t))\sin(k\pi x)$
$u_{-k} = u_k$ if we choose $a_k = -a_{-k}$, $b_k = b_{-k}$
so $u_{-k} \in \Delta$ if we choose a_k and b_k suitably

We are interested in the real solution: (add and eliminate...)

$$T_k(t) = a_k \cos(k\pi t) + b_k \sin(k\pi t) \ , \ a_k, b_k \in \mathbb{R}$$

$(1)+(2) \Rightarrow u_k(t,x) = T_k(t) X_k(x) = (a_k \cos(k\pi t) + b_k \sin(k\pi t)) \sin(k\pi x) \quad (\Delta)$

Take $t=0$, fit $f(x) = \sum_{k=1,2,\ldots} (a_k \cos(0) + b_k \sin(0)) \sin(k\pi x) = \sum_k a_k \sin(k\pi x)$

$g(x) = \frac{d}{dt}[\ \ldots\ ] = \sum_k \pi k b_k \sin(k\pi x)$

Example: $f(x) = 2\sin(\pi x)$
$g(x) = -\sin(2\pi x)$

$\to$ $a_1 = 2$, $a_k = 0$ for $k \neq 1$

$b_2 = -\frac{1}{2\pi}$, $b_k = 0$ for $k \neq 2$

$\Rightarrow 2\cos(\pi t)\sin(\pi x) - \frac{1}{2\pi}\sin(2\pi t)\sin(2\pi x)$ solves the wave equation with this initial data.

For more complicated initial data, we can combine the u_k:

Since u_k all solve (W) also $\sum_{k=1}^{N} u_k(t,x)$ solves (W) with initial data

$$f(x) = \sum_{k=1}^{N} u_k(0,x), \quad g(x) = \sum_{k=1}^{N} \partial_t u_k(0,x)$$

Let's check this: $\partial_{tt}\left(\sum_{k=1}^{N} u_k(t,x)\right) - \partial_{xx}\left(\sum_{k=1}^{N} u_k(t,x)\right)$

$$= \sum_{k=1}^{N} \partial_{tt} u_k(t,x) - \sum_{k=1}^{N} \partial_{xx} u_k(t,x)$$

$$= \sum_{k=1}^{N} \underbrace{(\partial_{tt} u_k(t,x) - \partial_{xx} u_k(t,x))}_{0 \text{ by construction}} = 0$$

Boundary condition: (left) $\left(\sum_{k=1}^{N} u_k(t,x)\right)\Big|_{x=0} = \sum_{k=1}^{N} u_k(t,0) = 0$ because u_k solves (W)

(right) Identical.

$\Rightarrow$ Superposition Principle

To be able to represent an even larger class of initial data, we can take infinite sums ($N \to \infty$)
$u(t,x) = \sum_{k=1}^{\infty} u_k(t,x)$. This requires some conditions on a_k, b_k $k \in \mathbb{N}$ so that the sum is finite
(a_k, b_k need to be "small" for large k, $\sum_k |a_k| < \infty$ etc.)

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Energy Methods

- This is a way to gain "qualitative" information about the solution to (W) without knowing its explicit form.

Assume $u \in C^2([0,\infty) \times [0,1])$

Define $E(t) := \int_0^1 [(u_t(t,x))^2 + (u_x(t,x))^2]\,dx$ to be the "energy" of the wave equation for $t \geq 0$.

At $t=0$: $E(0) = \int_0^1 [(g(x))^2 + (f'(x))^2]\,dx$

How does the energy change over time? To figure this out, we take the time derivative ("rate of change")

$$E'(t) = \frac{d}{dt}\int_0^1 [(u_t(t,x))^2 + (u_x(t,x))^2]\,dx$$

since u_{tt}, u_{xx} continuous

$$= \int_0^1 [\partial_t((u_t(t,x))^2) + \partial_t((u_x(t,x))^2)]\,dx$$

$$= 2\int_0^1 (u_t(t,x)\,u_{tt}(t,x) + \underbrace{u_x(t,x)\,u_{xt}(t,x)}\,)\,dx$$ (note also that $u_{tx} = u_{xt}$ because u is continuously differentiable)

$$= 2\int_0^1 u_t u_{tt}\,dx - 2\int_0^1 u_{xx} u_t\,dx + [2u_t(t,x)\,u_x(t,x)]_{x=0}^{x=1}$$

By the boundary condition, $u(t,0)=0$ and $u(t,1)=0$.
Therefore $u_t(t,0)=0$ and $u_t(t,1)=0$.

So the boundary term vanishes, and $E'(t) = 2\int_0^1 \underbrace{(u_{tt}-u_{xx})}_{=0 \text{ by (W)}} u_t\,dx = 0$

$\therefore$ The energy stays constant; $\boxed{E(t) = E(0) = \int_0^1 ((g(x))^2 + (f'(x))^2)\,dx.}$

p 102 'Another perspective'

$$v = u_t,\ w = u_x \Rightarrow \begin{matrix} v_t = w_x \\ w_t = v_x \end{matrix}$$

For $U = \binom{v}{w}$, $U_t + AU_x = 0$, $A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$$\rightsquigarrow E(t) = \int_0^1 \left((v(t,x))^2 + (w(t,x))^2 \right) dx = \int_0^1 |U(t,x)|^2\, dx$$

Stability

$(W)\begin{cases} u_{tt} - u_{xx} = 0 \\ u(0,x) = f(x),\ u_t(0,x) = g(x) \\ u(t,0) = u(t,1) = 0 \end{cases}$

Consider (W) with initial conditions (f_1, g_1) and denote by u_1 the corresponding solution, and for initial conditions (f_2, g_2) let the solution be u_2. How does a perturbation affect the solution?

$$\tilde{u} := u_1 - u_2$$
$$\tilde{f} := f_1 - f_2$$
$$\tilde{g} := g_1 - g_2$$

By superposition, $\tilde{u}$ satisfies (W) with initial conditions $\tilde{f}$ and $\tilde{g}$.

$\tilde{u}$ satisfies the energy principle:

$$E(t) = \int_0^1 \left[(\tilde{u}_t(t,x))^2 + (\tilde{u}_x(t,x))^2 \right] dx$$

$$= E(0) = \int_0^1 \left[(\tilde{g}(x))^2 + (\tilde{f}'(x))^2 \right] dx$$

By plugging in $\tilde{u}$, $\tilde{g}$, and $\tilde{f}$,

$$\int_0^1 \left[\left((u_1)_t - (u_2)_t \right)^2 + \left((u_1)_x - (u_2)_x \right)^2 \right] dx = \int_0^1 \left[(g_1 - g_2)^2 + (f_1' - f_2')^2 \right] dx \qquad (\square)$$

Note also that $\int_0^1 (u(t,x))^2 dx \le \int_0^1 (u_x(t,x))^2 dx$ so we get a bound on LHS

(Proof sketch. $u(t,x) = \underbrace{u(t,0)}_{0 \text{ by B.C.}} + \int_0^x \partial_y u(t,y)\, dy$

Hölder?

$\therefore$ $\cancel{\int_0^1 (u(t,x))^2 dx = \int_0^1 \left| \int_0^x \partial_y u(t,y) dy \right|^2 dx} \le \int_0^1 |\partial_x u(t,x)|^2 dx$) ~~see~~

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Prop (Uniqueness follows from stability)

Assume u_1, u_2 are two different solutions for the same initial data. Then $\tilde{u} = u_1 - u_2$ satisfies (W) with $f(x) = g(x) = 0$.

Then ($\square$) becomes $\int_0^1 ((u_1)_t - (u_2)_t)^2 + ((u_1)_x - (u_2)_x)^2 \, dx = 0$.

But the integrand is non-negative, so it must be exactly 0.

$\Rightarrow (u_1)_t = (u_2)_t$ and $(u_1)_x = (u_2)_x$.

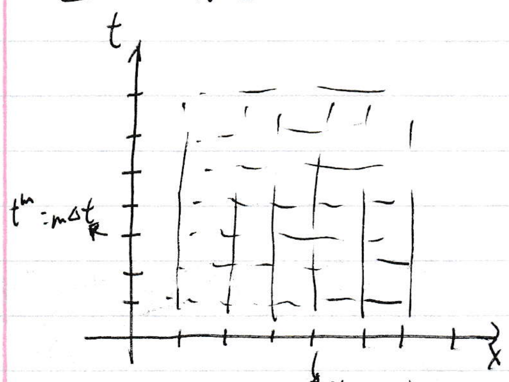
$\Rightarrow u_1(t,x) = u_2(t,x) + c$ for some constant $c \in \mathbb{R}$.

Plug in boundary conditions: $0 = u_1(t,0) = u_2(t,0) + c = c$

$\Rightarrow c = 0$

$\therefore u_1 = u_2$ and the solution is unique.

Finite Differences



$x_j = j\Delta x \quad j \in \{0, 1, \dots, n+1\}$

$\Delta x = \frac{1}{n+1}$ is the mesh size

$\Delta t > 0$ time step size, $t^m = m\Delta t$ time levels, $m \in \mathbb{N}_0$

We use differences to approximate the derivatives.

$$\frac{u(t, x+\Delta x) - 2u(t,x) + u(t, x-\Delta x)}{\Delta x^2} = u_{xx}(t,x) + O(\Delta x^2)$$

(by Taylor expansion)

$$\frac{u(t+\Delta t, x) - 2u(t,x) + u(t-\Delta t, x)}{\Delta t^2} = u_{tt}(t,x) + O(\Delta t^2)$$

$$\Rightarrow v_j^m \approx u(t^m, x_j) \Rightarrow \frac{v_j^{m+1} - 2v_j^m + v_j^{m-1}}{\Delta t^2} = \frac{v_{j+1}^m - 2v_j^m + v_{j-1}^m}{\Delta x^2}$$

$v_0^m = v_{n+1}^m = 0$ $\quad m = 1, 2, 3, \dots$

$j = 1, \dots, n$

$v_j^0 = f(x_j), j = 1 \dots n$

How to compute v_j^1?

$$u(\Delta t, x) = u(0,x) + \Delta t\, u_t(0,x) + \frac{\Delta t^2}{2} u_{tt}(0,x) + O(\Delta t^3)$$

$$= f(x) + \Delta t\, g(x) + \frac{\Delta t^2}{2} f''(x) + O(\Delta t^3)$$

$$\therefore v_j^1 = f(x_j) + \Delta t\, g(x_j) + \frac{\Delta t^2}{2\Delta x^2}\left(f(x_{j+1}) - 2f(x_j) + f(x_{j-1})\right) + O(\Delta t^3, \Delta t^2 \Delta x^2)$$

$$= v_j^0 + \Delta t\, g(x_j) + \frac{\Delta t^2}{2\Delta x^2}\left(v_{j+1}^0 - 2v_j^0 + v_{j-1}^0\right) \quad [07]$$

(*) $= \int_0^1 \left| \int_0^1 1_{[0,x]}(y)\, u_y(t,y)^2 \, dy \right|^2 dx$

see p. 109

$$\frac{v_j^{m+1} - 2v_j^m + v_j^{m-1}}{\Delta t^2} = \frac{v_{j+1}^m - 2v_j^m + v_{j-1}^m}{\Delta x^2}, \quad \begin{matrix} m = 1,2,3\ldots \\ j = 1,2,\ldots n \end{matrix}$$

$v_0^m = v_{n+1}^m = 0$

$$v_j^0 = f(x_j), \quad v_j^1 = v_j^0 + \Delta t\, g(x_j) + \frac{\Delta t^2}{2\Delta x^2}(v_{j+1}^0 - 2v_j^0 + v_{j-1}^0)$$

(in matrix form)

Define $A = \frac{\Delta t^2}{\Delta x^2}\begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}$, $v^m = (v_1^m, \ldots v_n^m)^T$

$$\begin{pmatrix} 2 - 2\frac{\Delta t^2}{\Delta x^2} & \frac{\Delta t^2}{\Delta x^2} & & \\ & 2 & & \\ & & 2 & \\ & & & \end{pmatrix}$$

$\Rightarrow v^{m+1} = (2I_n - A)v^m - v^{m-1}, \; m = 1,2,\ldots$

$v^1 = v^0 + \Delta t\,\tilde{g} - \frac{1}{2}Av^0$

$v_j^0 = f(x_j)$

$\tilde{g} = (g(x_1), \ldots g(x_n))^T$

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$$v_j^{m+1} = \frac{\Delta t^2}{\Delta x^2}(v_{j+1}^m - 2v_j^m + v_{j-1}^m) + 2v_j^m - v_j^{m-1}$$

'dispersion'

$u_{tt} + \gamma u_t = u_{xx}$ 'damped'

$u_{tt} = u_{xx} + \gamma u_{txx}$

wave_explicit1D.m

v^{m-1} : v_veryold

v^m : v_old

v^{m+1} : v_new

$2I_n - A$: 'A_update'

'A'

2nd-order Accurate in space

$\Delta t = \Delta x$ works better than $\Delta t = 0.8\Delta x$

Truncation Error

Plug in exact solution into scheme:

$$\tau_\Delta(t^m, x_j) = \underbrace{\frac{u(t^{m+1}, x_j) - 2u(t^m, x_j) + u(t^{m-1}, x_j)}{\Delta t^2}} - \frac{u(t^m, x_{j+1}) - 2u(t^m, x_j) + u(t^m, x_{j-1})}{\Delta x^2}$$

numerator $\frac{1}{\Delta t^2}\left(\begin{matrix} u_j^m + \Delta t\,\partial_t u_j^m + \frac{\Delta t^2}{2}\partial_{tt}u_j^m + \frac{\Delta t^3}{6}\partial_{ttt}u_j^m + \frac{\Delta t^4}{24}\partial_t^{(4)}u_j^m + O(\Delta t^5) \\ -2u_j^m \\ + u_j^m - \quad + \quad - \quad + \quad \underbrace{- O(\Delta t^5)}_{O(\Delta t^4)} \end{matrix}\right)$

$$= \frac{O(\Delta t^4)}{\Delta t^2} + \frac{O(\Delta x^4)}{\Delta x^2} + \underbrace{\partial_{tt}u(t^m, x_j) - \partial_{xx}u(t^m, x_j)}_{=0}$$

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Recitation, etc. 11/23/2021

Hölder's Inequality - If p, p' satisfy $\frac{1}{p}+\frac{1}{p'}=1$, $f \in L^p(\Omega)$, $g \in L^{p'}(\Omega)$,

then $fg \in L^1(\Omega)$ and $\|fg\|_1 \leq \|f\|_p \|g\|_{p'}$

Want to show $\int_0^1 (u(t,x))^2 dx \leq \int_0^1 (u_x(t,x))^2 dx$, given $u(t,x) = \cancel{u(t,0)} + \int_0^x \partial_y u(t,y)dy$

$= \int_0^x \partial_y u(t,y) dy$ by D.C.

$$\cancel{\int_0^1 (u(t,x))^2 dx = \int_0^1 \left[\int_0^1 1_{[0,x]}(y)\, u_y(t,y)^2 dy\right] dx}$$

$$\text{~~Hölder?~~} \cancel{\leq \int_0^1 \sqrt{\int_0^1 1_{[0,x]}(y)^2 dy} \cdot \sqrt{\int_0^1 (u_y(t,y))^2 dy}\ dx}$$

??

$$\leq \int_0^1 \cancel{1_{[0,x]}(y)\ (u_x(t,x))^2 dx}$$

First note that $|u(x)|^2 = \left| \int_0^x u'(y)dy \right|^2 = \left| \int_0^1 1_{[0,x]}(y) u'(y)\, dy \right|^2$

$(a_1b_1+a_2b_2)^2 \leq (a_1^2+a_2^2)(b_1^2+b_2^2)$

$|\langle u,v\rangle|^2 \leq \langle u,u\rangle \cdot \langle v,v\rangle$

$$\leq \left| \int_0^1 1_{[0,x]}(y)\, |u'(y)|\, dy \right|^2$$

$\int_0^1 f g\, dy$ is inner product space

Hölder $\leq \int_0^1 (1_{[0,x]}(y))^2 dy \int_0^1 |u'(y)|^2 dy$

$\overset{1}{=} x \int_0^1 |u'(y)|^2 dy \ \underset{x\in[0,1]}{\overset{2}{\leq}} \int_0^1 |u'(y)|^2 dy$

From 2 we now have

$$\int_0^1 |u(t,x)|^2 dx \leq \int_0^1 \underbrace{\int_0^1 |u'(y)|^2 dy}_{\text{const}} dx = \int_0^1 |u'(y)|^2 dy \text{ as required.}$$

1 we have a better estimate:

$$\int_0^1 |u(t,x)|^2 dx \leq \int_0^1 x \int_0^1 |u'(y)|^2 dy\, dx = \frac{1}{2}\int_0^1 |u'(y)|^2 dy$$

'Generalized' d'Alembert

$d_{tt}u - c^2 d_{xx}u = f(t,x)$

$u(0,x) = g(x)$ $\Rightarrow u(t,x) = \frac{g(x+ct)+g(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)ds + \frac{1}{2c}\int_0^t \int_{x-c(t-\gamma)}^{x+c(t-\gamma)} f(\gamma,s)\,ds\,d\gamma$

$u_t(0,x) = h(x)$

Let $g(x) = h(x) = 0$, $c=1$. Then the goal is to show that $u(t,x) = \frac{1}{2}\int_0^t \int_{x-t+\gamma}^{x+t-\gamma} f(\gamma,s)\,ds\,d\gamma$.

Own attempt:

~~Let $w = u_t - u_x$. Then we solve $w_t + w_x = f(t,x)$ with $w(0,x) = w_t(0,x) = 0$.~~

~~$\begin{cases} \eta_t(t,x_0) = 1 \\ \eta(0,x_0) = x_0 \end{cases} \Rightarrow \eta(t,x_0) = t + x_0$~~

~~$\begin{cases} w_t(t,\eta(t,x_0)) = f(t,\eta(t,x_0)) \\ w(0,\eta(0,x_0)) = 0 \end{cases} \Rightarrow w(t,\eta(t,x_0)) = \int_0^t f(s,\eta(s,x_0))\,ds$~~

~~$w(t, t+x_0) = \int_0^t f(s, s+x_0)\,ds$~~

~~$w(t,x) = \int_0^t f(s, s+x_0-t)\,ds$~~

$u_{tt} + u_t = u_{xx}$

$u(t,0) = u(t,1) = 0$. Assume $u(t,x) = X(x)T(t)$

$u(0,x) = f(x)$, $u_t(0,x) = g(x)$ $X(x)T''(t) + X(x)T'(t) = X''(x)T(t)$

$$\frac{T''(t) + T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

$X''(x) = -\lambda X(x) \Rightarrow X(x) = a\sin(k\pi x)$ for $\lambda_k = (k\pi)^2$

$T''(t) + T'(t) + \lambda T(t) = 0$

The equation $x^2 + x + \lambda = 0$ has roots $y_1, y_2 = \frac{-1 \pm \sqrt{1-4\lambda}}{2} \notin \mathbb{R}$ $\quad e^{iy_1 t}$? $-\frac{1}{2} \pm \frac{\sqrt{4\lambda-1}}{2} i$

so $T(t) = \left[a\sin\left(\frac{\sqrt{4\lambda-1}}{2}t\right) + b\cos\left(\frac{\sqrt{4\lambda-1}}{2}t\right)\right]e^{-\frac{1}{2}t}$

$u(t,x) = \left[a_k \sin\left(\frac{\sqrt{4\pi^2k^2-1}}{2}t\right) + b_k\cos\left(\frac{\sqrt{4\pi^2k^2-1}}{2}t\right)\right]\sin(k\pi x)\,e^{-\frac{1}{2}t}$.

$u(0,x) = f(x) = \sum b_k \sin(k\pi x)$

$u_t(0,x) = g(x) = \sum a_k\left(\frac{\sqrt{4\pi^2k^2-1}}{2}\right)\sin(k\pi x)\cancel{e^{-\frac{1}{2}t}} - \frac{1}{2}b_k\cos\left(\frac{\sqrt{4\pi^2k^2-1}}{2}t\right)\cancel{e^{-\frac{1}{2}t}}\sin(k\pi x)$

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To be precise,

$$T_\Delta(t^m, x_j) = \partial_{tt} u(t^m,x_j) - \partial_{xx} u(t^m,x_j) + \frac{\Delta t^2}{12}\partial_{tttt} u(t^m,x_j) - \frac{\Delta x^2}{12}\partial_{xxxx} u(t^m,x_j)$$
$$+ O(\Delta t^4 + \Delta x^4)$$

$u_{tt} = u_{xx} \Rightarrow u_{tttt} = u_{xxtt}$
$= u_{ttxx}$
$= u_{xxxx}$

So if $\Delta t = \Delta x$, we have that these two terms cancel out. This might explain why $\Delta t = \Delta x$ outperforms $\Delta t = 0.8\Delta x$

Let's look at this particular scheme from another angle:

$$(W) \begin{cases} u_{tt} = u_{xx}, & x \in (0,1), t>0 \\ u(0,x) = f(x), u_t(0,x) = g(x) & x \in (0,1) \\ u(t,0) = u(t,1) = 0 & t>0 \end{cases}$$

$$\frac{v_j^{m+1} - 2v_j^m + v_j^{m-1}}{\Delta t^2} = \frac{v_{j+1}^m - 2v_j^m + v_{j-1}^m}{\Delta x^2} \quad (FD)$$

Exact explicit solution is
$$\sum_{k=1}^{N} \sin(k\pi x)\left(a_k \cos(k\pi t) + b_k \sin(k\pi t)\right)$$

(Special case: $\Delta t = \Delta x$) Multiplying by $\Delta t^2 = \Delta x^2$ to get

$$v_j^{m+1} - \cancel{2v_j^m} + v_j^{m-1} = v_{j+1}^m - \cancel{2v_j^m} + v_{j-1}^m$$
$$v_j^{m+1} = v_{j+1}^m + v_{j-1}^m - v_j^{m-1} \quad (*)$$

Consider $u_k(t,x) = \sin(k\pi x)\cos(k\pi t)$

$$u_k(t^m, x_j) = \sin(k\pi\Delta x_j)\cos(k\pi m\Delta t)$$
$$= \sin(k\pi\Delta x_j)\cos(k\pi m\Delta x)$$

Plug $u_k(t^m, x_j)$ into $(*)$

If $v_j^m = u_k(t^m, x_j)$, then $v_j^{m+1} = \sin(k\pi\Delta x(j+1))\cos(k\pi m\Delta x) + \sin(k\pi\Delta x(j-1))\cos(k\pi m\Delta x)$
$- \sin(k\pi\Delta x_j)\cos(k\pi(m-1)\Delta x)$

$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$

$$= 2\sin(k\pi\Delta x_j)\cos(k\pi\Delta x)\cos(k\pi m\Delta x) - \sin(k\pi\Delta x_j)\cos(k\pi\Delta x(m-1))$$

By trigonometry, $\cos(\alpha\pi(m+1)) = 2\cos(\alpha\pi m)\cos(\alpha\pi) - \cos(\alpha\pi(m-1))$

(The only error would be estimating v^1)

Then $v_j^{m+1} = \cos(k\pi\Delta x(m+1))\sin(k\pi\Delta x_j) = u_k(t^{m+1}, x_j)$ ✓

///

(Energy stability)

Continuous case: $E(t) = \int_0^1 (|u_t(t,x)|^2 + |u_x(t,x)|^2)\,dx$

$u_{tt} = u_{xx} \Rightarrow u_t u_{tt} = u_t u_{xx}$

Integrating in space, $\int_0^1 \underbrace{u_t u_{tt}}_{=\partial_t \frac{|u_t|^2}{2}}\,dx = \int_0^1 u_t u_{xx}\,dx$

Assume $u \in C^2$, then we have

$$\frac{1}{2}\frac{d}{dt}\int_0^1 |u_t(t,x)|^2 dx = \int_0^1 u_t(t,x)\,u_{xx}(t,x)\,dx$$

$$\overset{\text{by parts}}{=} -\int_0^1 u_{tx}(t,x)u_x(t,x)\,dx + \underbrace{u_t(t,x)u_x(t,x)\Big|_{x=0}^{x=1}}_{=0 \text{ at } x=0,1}$$

$$= -\int_0^1 u_{tx}u_x\,dx$$

check exact solution!

$$= -\frac{1}{2}\frac{d}{dt}\int |u_x|^2 dx$$

$$\boxed{\therefore\ \frac{1}{2}\frac{d}{dt}\int_0^1 |u_t|^2 + |u_x|^2\,dx = 0 \Rightarrow \frac{1}{2}\frac{d}{dt}E(t) = 0}$$

Discrete case: Try to replicate? u_t corresponds to $\frac{v_j^{m+1} - v_j^m}{\Delta t}$ or $\frac{v_j^m - v_j^{m-1}}{\Delta t}$ or the average of the two

Let's take the average $\frac{v_j^{m+1}-v_j^m}{2\Delta t} + \frac{v_j^m - v_j^{m-1}}{2\Delta t}$ $(\approx u_t)$ and multiply both sides of $\frac{v_j^{m+1} - 2v_j^m + v_j^{m-1}}{\Delta t^2} = \frac{v_{j+1}^m - 2v_j^m + v_{j-1}^m}{\Delta x^2}$ with this.

After multiplication, LHS $= \left(\frac{v_j^{m+1} - v_j^m}{2\Delta t} + \frac{v_j^m - v_j^{m-1}}{2\Delta t}\right)\left(\frac{v_j^{m+1} - v_j^m}{\Delta t^2} - \frac{v_j^m - v_j^{m-1}}{\Delta t^2}\right)$

$$= \frac{1}{2\Delta t}\left|\frac{v_j^{m+1} - v_j^m}{\Delta t}\right|^2 - \frac{1}{2\Delta t}\left|\frac{v_j^m - v_{j-1}^{m-1}}{\Delta t}\right|^2$$ (after summing over j)

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Multiply by Δx and sum over j:

"summation by parts" $\sum_{j=1}^{n} G_j(H_j - H_{j-1}) = -\sum_{j=1}^{n-1}(G_{j+1}-G_j)H_j + G_1H_0 + G_{n+1}H_n$

$$\frac{\Delta x}{2\Delta t}\sum_j \left(\left|\frac{v_j^{m+1}-v_j^m}{\Delta t}\right|^2 - \left|\frac{v_j^m - v_j^{m-1}}{\Delta t}\right|^2\right) = \Delta x \sum_{j=1}^{n} \underbrace{\left(\frac{v_j^{m+1}-v_j^m}{2\Delta t} + \frac{v_j^m - v_j^{m-1}}{2\Delta t}\right)}_{G_j}\left(\underbrace{\frac{v_{j+1}^m - v_j^m}{\Delta x^2}}_{H_j} - \underbrace{\frac{v_j^m - v_{j-1}^m}{\Delta x^2}}_{H_{j-1}}\right)$$

Neumann $\partial_x u = 0$ at $x=0, x=1$

Dirichlet $u(t,0) = u(t,1) = 0$

Periodic $u(t,0) = u(t,1)$, $\partial_x u(t,0) = \partial_x u(t,1)$

know from exact solution that $u(0)=0$

$v_0^m = v_{-1}^m = 0$

$= v_{n+1}^m = v_{n+2}^m$

$G_{n+1} = H_0 = 0$

$\Rightarrow G_1 H_0 = G_{n+1}H_n$

$$= -\Delta x \sum_{j=1}^{n}\left(\frac{v_{j+1}^{m+1} - v_{j+1}^m - (v_j^{m+1} - v_j^m)}{2\Delta t\Delta x} + \frac{v_{j+1}^m - v_{j+1}^{m-1} - (v_j^m - v_j^{m-1})}{2\Delta t \Delta x}\right)\frac{v_{j+1}^m - v_j^m}{\Delta x}$$

$\simeq u_{tx}$ $\qquad u_x$

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$$= -\Delta x \sum_{j=1}^{n}\left(\frac{v_{j+1}^{m+1} - v_{j+1}^m}{2\Delta t\Delta x} - \frac{v_j^{m+1} - v_j^m}{2\Delta t\Delta x} + \frac{v_{j+1}^m - v_{j+1}^{m-1}}{2\Delta t\Delta x} - \frac{v_j^m - v_j^{m-1}}{2\Delta t\Delta x}\right)\frac{v_{j+1}^m - v_j^m}{\Delta x}$$

$a(b-a) = \frac{b^2}{2} - \frac{a^2}{2} - \frac{1}{2}(b-a)^2 \Rightarrow f^m(f^{m+1} - f^m) = \frac{1}{2}(f^{m+1})^2 - \frac{(f^m)^2}{2} - \frac{(f^{m+1} - f^m)^2}{2}$

$b(b-a) = \frac{b^2}{2} - \frac{a^2}{2} + \frac{1}{2}(b-a)^2 \Rightarrow f^m(f^m - f^{m-1}) = \frac{1}{2}(f^m)^2 - (f^{m-1})^2 + \frac{1}{2}(f^m - f^{m-1})^2$

$$= -\frac{\Delta x}{2\Delta t}\sum_{j=1}^{n}\left(\underbrace{\frac{v_{j+1}^{m+1} - v_j^{m+1}}{\Delta x}}_{f^{m+1}} - \underbrace{\frac{v_{j+1}^m - v_j^m}{\Delta x}}_{f^m} + \underbrace{\frac{v_{j+1}^m - v_j^m}{\Delta x}}_{f^m} - \underbrace{\frac{v_{j+1}^{m-1} - v_j^{m-1}}{\Delta x}}_{f^{m-1}}\right)\left(\underbrace{\frac{v_{j+1}^m - v_j^m}{\Delta x}}_{f^m}\right)$$

$$= -\frac{\Delta x}{4\Delta t}\sum_{j=1}^{n}\left[\left|\frac{v_{j+1}^{m+1} - v_j^{m+1}}{\Delta x}\right|^2 - \left|\frac{v_{j+1}^m - v_j^m}{\Delta x}\right|^2 + \left|\frac{v_{j+1}^m - v_j^m}{\Delta x}\right|^2 - \left|\frac{v_{j+1}^{m-1} - v_j^{m-1}}{\Delta x}\right|^2\right]$$

$$+ \frac{\Delta x}{4\Delta t}\sum_{j=1}^{n}\left|\frac{v_{j+1}^{m+1} - v_j^{m+1} - (v_{j+1}^m - v_j^m)}{\Delta x}\right|^2 - \left|\frac{v_{j+1}^m - v_j^m - (v_{j+1}^{m-1} - v_j^{m-1})}{\Delta x}\right|^2$$

Multiply by $2\Delta t$ and sum over m (time):

telescoping

$$\underbrace{\Delta x \sum_{j=1}^{n}\left|\frac{v_j^{m+1} - v_j^m}{\Delta t}\right|^2}_{E^m} - \underbrace{\Delta x \sum_{j=1}^{n}\left|\frac{v_j^1 - v_j^0}{\Delta t}\right|^2}_{E^0} = -\frac{\Delta x}{2}\sum_{j=1}^{n}\left(\left|\frac{v_{j+1}^{m+1} - v_j^{m+1}}{\Delta x}\right|^2 + \left|\frac{v_{j+1}^m - v_j^m}{\Delta x}\right|^2\right) + \frac{\Delta x}{2}\sum\left|\frac{v_{j+1}^1 - v_j^1}{\Delta x}\right|^2 + \left|\frac{v_{j+1}^0 - v_j^0}{\Delta x}\right|^2$$

$\simeq |u_t|^2$ $\quad \simeq |u_x|^2$ $\quad |u_x|^2$

$$+ \underbrace{\frac{\Delta x}{2}\sum_{j=1}^{n}\left|\frac{v_{j+1}^{m+1} - v_j^{m+1} - v_{j+1}^m + v_j^m}{\Delta x}\right|^2}_{\text{bad}} - \underbrace{\frac{\Delta x}{2}\sum_{j=1}^{n}\left|\frac{v_{j+1}^1 - v_j^1 - v_{j+1}^0 + v_j^0}{\Delta x}\right|^2}_{\text{good}}$$

$(a-b)^2 \le 2(a^2+b^2)$

$\frac{\Delta x}{2}\sum_{j=1}^{n}\left|\frac{v_{j+1}^{m+1}-v_j^{m+1}-v_{j+1}^{m}+v_j^{m}}{\Delta x}\right|^2$

$[\text{Bad}] \le \frac{1}{2}\Delta x \sum_{j=1}^{n}\left|\frac{v_{j+1}^{m+1}-v_j^{m+1}}{\Delta x}\right|^2+\left|\frac{v_{j+1}^{m}-v_j^{m}}{\Delta x}\right|^2 \quad (A)$

(since $(A) \le (B)$)

$+\frac{1}{2}\Delta x \sum_{j=1}^{n}\left|\frac{v_{j+1}^{m+1}-v_{j+1}^{m}}{\Delta t}\right|^2+\left|\frac{v_j^{m+1}-v_j^{m}}{\Delta t}\right|^2 \quad (B)$

$\left(\text{Wanted } \frac{\Delta t}{\Delta x}<1 \iff \frac{1}{\Delta t} > \frac{1}{\Delta x}\right)$

$\le E^m + \frac{1}{2}\sum_{j=1}^{n}\left|\frac{v_{j+1}^{m+1}-v_{j+1}^{m}}{\Delta t}\right|^2 - \frac{1}{2}\sum_{j=1}^{n}\left|\frac{v_j^{m+1}-v_j^{m}}{\Delta t}\right|^2 = E^m + \frac{1}{2}\underbrace{\left|\frac{v_{n+1}^{m+1}-v_{n+1}^{m}}{\Delta t}\right|^2}_{=0} - \frac{1}{2}\underbrace{\left|\frac{v_1^{m+1}-v_1^{m}}{\Delta t}\right|^2}_{=0} \le E^m$

$\underbrace{-\frac{\Delta x}{2}\sum_{j=1}^{n}\left|\cancel{\frac{v_{j+1}^{1}-v_j^{1}-v_{j+1}^{0}+v_j^{0}}{\Delta x}}\right|^2}_{\text{good}}$ $\overset{?\text{ (wrong direction)}}{\cancel{\ge}}$ $\cancel{E^0 - \frac{1}{2}\sum_{j=1}^{n}\left|\frac{v_{j+1}^{1}-v_{j+1}^{0}}{\Delta t}\right|^2 + \frac{1}{2}\sum}$

This part was left unproved... see (p168-169 of Tveito/Winther...)

We just have to (?) show that $\sum_{j=1}^{n}\left|\frac{v_{j+1}^{1}-v_j^{1}-v_{j+1}^{0}+v_j^{0}}{\Delta x}\right|^2 \ge \sum_{j=1}^{n}\left|\frac{v_{j+1}^{1}-v_j^{1}}{\Delta x}\right|^2+\left|\frac{v_{j+1}^{0}-v_j^{0}}{\Delta x}\right|^2$

Then $E^m - E^0 = [\text{Bad}] - [\text{Good}]$

$\le E^m - \text{Good}$

≤ 0 (hopefully)