80-413 8/31/21

What is Category Theory?

- objects and arrows
- Higher-level patterns in mathematics, "background"/"meta-theory"
- Higher level of abstractions (mathematics is about rising level of abstractions)

Antiquity
↓
Geometric Shapes
↓
Numbers $\qquad \mathcal{P}(N) \not\cong N \ldots$
↓
Functions → Set theory $\quad \mathcal{P}(N) : \{U \subseteq N\} \cong \{f : N \to \{0,1\}\}$

↓ Continuum hypothesis ... $(2^{\aleph_0} = \aleph_1)$

Mathematical, axiomatically defined structures

Example: $\boxed{\text{Partially ordered set}}$ (POSET) e.g. $(\mathcal{P}(S), \subseteq)$

$(S, \leq)$.

(Reflexivity) $\forall x \in S,\ x \leq x$

(Transitivity) $\forall x, y, z \in S,\ x \leq y,\ y \leq z \Rightarrow x \leq z$

(Anti-symmetry) $\forall x, y \in S.\ x \leq y,\ y \leq x \Rightarrow x = y$.

$$\begin{tikzcd} & \{0,1\} & \\ \{0\} \arrow[ur, phantom, "\subseteq"] & & \{1\} \arrow[ul, phantom, "\supseteq"] \\ & \emptyset \arrow[ul, phantom, "\supseteq"] \arrow[ur, phantom, "\subseteq"] & \end{tikzcd}$$

$\boxed{\text{Preorder}}$: (REFL) + (TRANS) only

POSET $\subseteq$ PREORDER $\subseteq$ TOTAL ORDER

$\boxed{\text{Group}}$: $(G, \cdot, e)$ $(\cdot): G \times G \to G$

- $e \cdot x = x \cdot e = x$
- (inverse)
- (associativity)

Abelian vs Non-abelian Groups

S_3 . . . (permutation)

D_3 . . . (dihedral)

Matrix multiplication ($GL_{\mathbb{R}}$ group)

$\boxed{\text{Monoids}}$: unit, associativity, no need for inverses

e.g. $(\mathbb{N}, \times, 1)$ e.g. string concatenation

Example of non-commutative monoid: $\boxed{\text{Function Composition}}$

(eg. general functions $S \to S$)

* We say 'commutative monoid', not 'abelian ~~monoid~~'

Others: graphs, vector spaces, topological space

('closeness', but distance not numerically measured)

Observation: Most notions of axiomatically defined structure come with an associated notion of structure-preserving mapping.

e.g. POSETs, monotone functions

Monoids, ~~groups~~ homomorphism

80-413 9/2/2021

Def: A CATEGORY contains the following data

- Objects $A, B, C, D \ldots$
- Morphisms (Arrows) $f, g, h \ldots$
- For each arrow there are given objects dom(f), cod(f)
  (domain) (codomain)

  we say $f: A \to B$ to say that f has domain A and codomain B.

- Composition of morphisms $f: A \to B$, $G: B \to C$ gives $H = G \circ F: A \to C$
- For each object A, there is an identity arrow $1_A: A \to A$ ($id_A: A \to A$)
- Axioms : Associativity $h \circ (g \circ f) = (h \circ g) \circ f$

  Identity $1_B \circ f = f = f \circ 1_A$ for $f: A \to B$

Examples. POS (monotone maps) AB (abelian groups)
PREORD (monotone maps) R-VECT
MON (monoids) TOP (topological space, continuous function)
GRP (groups) SET (sets and functions)

Categories of structured sets and structure-preserving maps are called concrete categories.
(opposed to 'abstract')

Note: $id_{\mathbb{Q}}: \mathbb{Q} \to \mathbb{Q}$

$j: \mathbb{Q} \hookrightarrow \mathbb{R}$ "canonical inclusion"

$j(q) = q$

we have to specify triples $(A, B, f: A \to B)$ for morphisms in SET

Examples of abstract (non-categorical) categories:

MAT objects are natural numbers $n \in \mathbb{N}$

morphisms from m to n are $(n \times m)$-matrices

Composition is given by matrix multiplication

$$\begin{pmatrix} 2 & 0 & 2 \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$$

if we allow m, n to equal 0...

$2 \times 0 : \mathbb{R}^0 \to \mathbb{R}^2$

e.g. $\mathbb{R}^3 \to \underset{=\{0\}}{\mathbb{R}^0} \to \mathbb{R}^2$ composition gives $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Remark: $Mat(\mathbb{R})$ is equivalent to the category of finite dimensional real vector spaces & linear maps.

REL Objects are sets

Morphisms are binary relations, i.e. a morphism from A to B is a subset $R \subseteq A \times B$

Composition $A \xrightarrow{R \subseteq A \times B} B \xrightarrow{S \subseteq B \times C} C$

$$(S \circ R) = \{ (a,c) \mid \exists\, b,\ (a,b) \in R \wedge (b,c) \in S \}$$

Identities $id_A = \{ (a,a) \mid a \in A \}$

observations:

$A \xrightarrow{\text{rel arrow}} B$

$P(A) \xrightarrow{\text{function}} P(B)$

Relations between finite sets can be written as matrices $\{1,2,3\} \nrightarrow \{1,2\}$

e.g.

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{array}$$

'almost matrix multiplication' but must define '1+1=1'??

80-413 9/7/2021

Finite Categories

$1 = \ast$ the *terminal* category

$$\mathbf{2} = \begin{tikzcd} A \arrow[d, "f"] \\ B \end{tikzcd}$$ *Arrow* category

$$2 = \begin{tikzcd} A \\ B \end{tikzcd}$$ *discrete* category

$$\begin{tikzcd} A \arrow[d, "f"] \arrow[dd, bend right, "h"'] \\ B \arrow[d, "g"] \\ C \end{tikzcd}$$ $h = g \circ f$ $= 3 = \Delta^2$

$$\begin{tikzcd} A \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \\ B \end{tikzcd}$$ not posetal!

$$\begin{tikzcd} A \arrow[d, shift left, "g"] \\ B \arrow[u, shift left, "f"] \end{tikzcd}$$ (unique isomorphism...) Codiscrete/Indiscrete Posetal

$\boxed{\text{Posetal Categories}}$

Given a preorder $(A, \leq)$ we can define a category $(A, \leq)$ whose objects are the elements of A, and where there is a unique arrow from a to b whenever $a \leq b$ in A.

If $a \leq b$ and $b \leq c$, then the composition of the corresponding arrows is the unique arrow that exists between a, c because $a \leq c$ by transitivity.

id arrows exist because of reflexity.

Given a monoid $(M, \cdot, e)$, we can define a category $(M, \cdot, e) = \Sigma(M, \cdot, e)$ with one object $*$ and arrows being the elements of M

$\to m \in M$, $\mathrm{dom}(m) = \mathrm{cod}(m) = *$

$\to m \circ n = m \cdot n$

$\to 1_* = e$

So we say that monoids are "the same thing" as small categories.

Def: Given a category $\mathcal{C}$ and an arrow $f: A \to B$ in $\mathcal{C}$, an arrow $g: B \to A$ is called an inverse of f, if $g \circ f = 1_A$ and $f \circ g = 1_B$.

* If ~~inverses~~ an inverse exists, it is unique.

$g, g': B \to A$, $g = g \circ 1_B = g \circ (f \circ g') = (g \circ f) \circ g' = 1_A \circ g' = g'$.

Def: f is called an $\boxed{\text{isomorphism}}$ if it has an inverse, say g.

$f: A \to A$, isomorphism $\Rightarrow$ automorphism

$f: A \to A$, arrow $\Rightarrow$ endomorphism, auto = iso + endo

* In sets, $A = \{0, 1\}$, $f: A \to A$, $f \circ f = id$ is an automorphism
$f(0) = 1$
$f(1) = 0$

Observation: A function $f: A \to B$ is isomorphic in Set iff it is bijective.

$(\Leftarrow)$
$\forall B$, exists unique A that maps to B . . . . . this is the inverse

$(\Rightarrow)$ Definition.

80-413 9/7/2021

Prop: A monoid homomorphism $f: (M, \cdot, e) \to (N, \cdot, e)$ is an isomorphism iff it is bijective.

Proof. ($\Leftarrow$) Assuming f is bijective WTS f^{-1} is also a homomorphism.

(1)
$$e = f(f^{-1}(e)) = f(e)$$

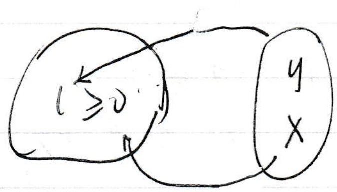
So $f^{-1}(e) = e$.

(2)
$$f(f^{-1}(a \cdot b)) = f(f^{-1}(a) \cdot f^{-1}(b)) = f(f^{-1}(a))\, f(f^{-1}(b)) = a \cdot b.$$
$$= a \cdot b$$

$$\therefore \quad f^{-1}(a \cdot b) = f^{-1}(a) \cdot f^{-1}(b)$$

We conclude that isomorphisms in category Mon are those bijective maps.

In PREORDER



not every bijective monotone map is an isomorphism.

Suppose neither $x \leq y$ nor $y \geq x$ (possible in PRE-ORDER!)

Then the map ($\leftrightarrows$) is monotone ($\neg P \vee Q$!), but the inverse is not:

$1 \geq 0 \not\Rightarrow \underbrace{f^{-1}(1)}_{y} \geq \underbrace{f^{-1}(0)}_{x}$. However it is still a valid bijection.

Definition: Given categories $\mathbb{C}, \mathbb{D}$, a $\boxed{\text{functor}}$ $F : \mathbb{C} \to \mathbb{D}$ is a mapping sending

- objects A in $\mathbb{C}$ to objects $F(A)$ in $\mathbb{D}$
- arrows f in $\mathbb{C}$ to arrows $F(f)$ in $\mathbb{D}$

such that

- $\mathrm{dom}(F(f)) = F(\mathrm{dom}(f))$
- $\mathrm{cod}(F(f)) = F(\mathrm{cod}(f))$
- $F(g \circ f) = F(g) \circ F(f)$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "f \circ g"'] & B \arrow[d, "g"] & \text{in } \mathbb{C} \arrow[dd, "F"] \\ & C & \\ F(A) \arrow[r, "F(f)"] \arrow[dr, "F(g \circ f) = F(g) \circ F(f)"'] & F(B) \arrow[d] & \mathbb{D} \\ & F(C) & \end{tikzcd}$$

Example Forgetful Functors

$U : \mathbf{Preord} \to \mathbf{Set}$

$(A, \leq) \mapsto A$ "underlying"

$$\begin{tikzcd} (A, \leq) \arrow[d, "\cancel{f}"'] & A \arrow[d, "f"] \\ A \arrow[r] & B \end{tikzcd}$$

$U : \mathbf{Mon} \to \mathbf{Set}$

$(M, \cdot, e) \mapsto M$, $f \mapsto f$ "do nothing" "simplify"

Prop. Functors preserve isomorphism. If $F : \mathbb{C} \to \mathbb{D}$, $f : A \to B$ in $\mathbb{C}$ is isomorphism then $F(f)$ is isomorphism in $\mathbb{D}$.

$$F(f) \circ F(f^{-1}) = F(f \circ f^{-1}) = F(1_B) = 1_{FB}, \quad F(f^{-1}) \circ F(f) = F(1_A) = 1_{FA}.$$

→ On the other hand, if $F(f)$ is an isomorphism, then f is not necessarily an isomorphism.

Proof by counterexample ... (Preorder $\xrightarrow{\text{simplifying}}$ Set)

→ If f isomorphic ~~if~~ when $F(f)$ is isomorphic, then the functor is called conservative.

U: Monoid → Set is conservative.

Rings
Groups ...

Top → Set is not conservative

SUBCATEGORIES "special case"

Ab ⊆ Groups , Pos ⊆ Preord $\boxed{\text{see p12.}}$

Formally, a subcategory of C specifies a subclass of the objects and a subclass of the morphisms.

$\boxed{\text{Is Cat} \in \text{Cat?}}$ • 'Free Group'? We cannot have a set of groups. (Grps not a set

• Recall Russell's Paradox, $R = \{M \mid M \notin M\}$. Does $R \in R$?? Sets ⊆ Pos, not set)

↑ trivial ordering

$\{M \mid \cancel{M \notin M}\}$ $\{M \mid \cancel{\text{true}}\}$ 'set of all sets'

→ comprehension (unbounded)

So we only allow bounded comprehension

$\{M \in A \mid \phi(M)\}$

intentional ambiguity

[→ So we think of objects and arrows as 'collections' or 'classes']

Def A category ($\mathbb{C}$) is called '<u>small</u>' if $\mathbb{C}_0$ (collection of objects) and $\mathbb{C}_1$ (collection of arrows) are <u>sets</u>.

SMALL	LARGE	
$1, \mathbb{Z}, \mathbb{B}$	<u>Grp</u>	<u>Pos</u>
$Mat(\mathbb{R})$	<u>Set</u>	<u>Preord</u>

Def A category $\mathbb{C}$ is called <u>locally small</u> if for all objects A, B in $\mathbb{C}$, the collection $Hom_{\mathbb{C}}(A,B) = \{f : A \to B\} = \{f \in \mathbb{C}_1 \mid dom(f) = A, cod(f) = B\}$ is a set.

<u>Ex.</u> All these are locally small!

(*note SMALL $\subseteq$ LOCALLY SMALL)

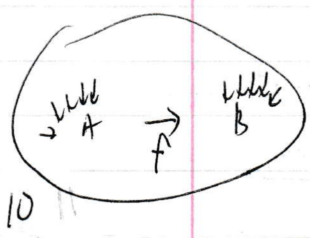
Def. $\mathbb{C}at$ is the (large) categories of small categories.

Theorem Every small category $\mathbb{C}$ is <u>isomorphic</u> to a category of sets of functions.

Proof sketch. For $A \in \mathbb{C}_0$, define $\underline{A} = \{f \in \mathbb{C}_1 \mid cod(f) = A\}$

For $f : A \to B$ in $\mathbb{C}_1$ define $\underline{f} : \underline{A} \to \underline{B}$, $\underbrace{h \mapsto f \circ h}_{\text{"post-composition function"}}$ ← composition of morphisms

Define $\underline{\mathbb{C}}$ by $\underline{\mathbb{C}}_0 = \{\underline{A} \mid A \in \mathbb{C}_0\}$, $\underline{\mathbb{C}}_1 = \{\underline{f} \mid f \in \mathbb{C}_1\}$



80-413 9/9/2021

'Size issues'

→ (locally) small/large distinction comes to its limits if we want to consider

- functor categories

- the category of large categories should be a "very large category"
??!

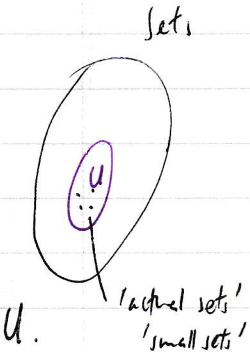
[cannot have class containing other classes]

'conglomerates'??
Grothendieck

To make "very large categories" precise, we can use universes.

Def A Grothendieck universe is a set U such that

- $N \in U$
- 'transitive' $M \in U, x \in M \Rightarrow x \in U$
- $x, y \in U \Rightarrow \{x, y\} \in U$
- $A \in U \Rightarrow P(A) \in U$
- $(A_j)_{j \in J}$ with $J \in U$, all $A_j \in U$ then $\bigcup_{j \in J} A_j \in U$.



With a Grothendieck Universe U, (ZFC + U can be used as a set of axioms)

- a set is small if it is in U
- a category $\mathbb{C}$ is small if $\mathbb{C}_0$ & $\mathbb{C}_1$ are in U
- a category $\mathbb{C}$ is locally small if all $\hom(A,B) \in U$, and $\mathbb{C}_0 \subseteq U$

We could also postulate a 2nd universe U_1 with $U_0 \in U_1 \in U_2 \in U_3$, and call $\mathbb{C}$ locally small if all $\hom_{\mathbb{C}}(A,B) \in U_0$ and $\mathbb{C}_0 \in U_1$.

A $\boxed{\text{subcategory}}$ of a category $\mathbb{D}$ is a category $\mathbb{C}$ s.t.

- $\mathbb{C}_0 \subseteq \mathbb{D}_0$
- $\mathbb{C}_1 \subseteq \mathbb{D}_1$
- Dom, Cod, Identities, Composition in $\mathbb{C}$ are obtained by restriction from $\mathbb{D}$

more ex. $\underline{\text{Pos}} \subseteq \underline{\text{Preord}}$, $\underline{\text{Ab}} \subseteq \underline{\text{Grp}}$,
$\underline{\text{CMon}} \subseteq \underline{\text{Mon}}$, $\underline{\text{Set}}_{inj} \subseteq \underline{\text{Set}}$ ✓

↑ only morphisms allowed are the injective functions.

$\underline{\text{Set}}_{surj}$ ✓

- "X"
- Every preimage / fiber has at most 2 elements "semi-injective"

→ this is not a category!

To define a subcat of $\mathbb{C}$, we can choose an arbitrary subclass $B_0 \subseteq \mathbb{C}_0$ but when choosing B_1 we have to make sure it contains ids, compositions...

Def. A subcategory $\mathbb{C} \subseteq \mathbb{D}$ is called $\boxed{\text{full}}$ if $\hom_{\mathbb{C}}(A,B) = \hom_{\mathbb{D}}(A,B)$ for all $A, B \in \mathbb{C}_0$.

~~Def.~~ Given a cat $\mathbb{C}$ and an object $A \in \mathbb{C}_0$, the $\boxed{\text{slice cat}}$ $\mathbb{C}/A$ is given as follows:

- Objects of $\mathbb{C}/A$ are morphisms f of $\mathbb{C}$ with $\text{cod}(f) = A$
- Morphisms from $(f: B \to A)$ to $(g: C \to A)$ are morphisms $h: B \to C$ s.t. $g \circ h = f$, i.e.

$$\begin{tikzcd} B \arrow[rr, "h"] \arrow[dr, "f"'] & & C \arrow[dl, "g"] \\ & A & \end{tikzcd}$$

80-413 9/9/2021

$$\begin{tikzcd} B \arrow[r, "h"] \arrow[dr, "f"'] & C \arrow[r, "k"] \arrow[d, "g"] & D \arrow[dl, "l"] \\ & A & \end{tikzcd}$$

"Over categories"

- Composition & Identities are inherited from $\mathbb{C}$.

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[d, "g"] \arrow[dr, "h"] & \\ B \arrow[r, "i"'] & C \arrow[r, "j"'] & D \end{tikzcd}$$

"Under categories"

- Objects in $A/\mathbb{C}$ are arrows in $\mathbb{C}$ with domain A
- Morphisms are commutative triangles e.g. $i : B \to C$ such that $i \circ f = g$.

Analogous to quotient groups?

$$\begin{tikzcd} H \arrow[r, hook] & G \arrow[r] \arrow[dr] & G/H \arrow[d] \\ L \arrow[ur, hook] & & G/K \end{tikzcd}$$

G/Grp

coset calculus...

Hw 1 Q4 (sketch) A single category is automatically a pre-order.

But we can think of Cat (each object in Cat is a category)

and the functor $\downarrow F$

Pre (each object in Pre is a pre-ordered set)

$Ca_1 \xrightarrow{\varphi} Ca_2 \quad a_1 \le a_2$

$\Downarrow$

$db_1 \to db_2 \quad b_1 \le b_2$

$Cat_1 \qquad Pre_1$

$F : F_{cat} \longmapsto f$ (monotone function)

$\uparrow$ arrows in Cat are themselves functors

* We can simply take $C_1 \in F_{cat} = (C_0, C_1)$ and prove that it defines a monotone map

let $f : A \to B$

let $a_1 \le a_2$

HW1 Q4 (cont)

Inclusion functor /

arrows / e.g. $f: A \to B$ ($A \subsetneq B$)

inclusion functors preserve structure

$f(a) = a$ is an inclusion but not an identity

(because it cannot be left composed by another arrow $g: A \to C$)

To be pedantic : $(A, f, B) \neq (A, f, A)$

'same function definition' but different object!

"evident inclusion functor of preorders into categories"

preorders
(between any two objects there is at most 1 arrow)

categories

$\xrightarrow{\text{inclusion functor}}$

$\xleftarrow[P]{}$ ✓ $\qquad P \cdot (\text{Incl}) = 1$

So P is the left inverse of Incl.

Abstract "Structures"

Recall: A function $f: A \to B$ is called

injective if $\forall a, a' \in A$, $f(a) = f(a') \Rightarrow a = a'$

surjective if $\forall b \in B$, $\exists a \in A$ s.t. $f(a) = b$

Def. A morphism $f: A \to B$ in a category $\mathbb{C}$ is called

$\boxed{\text{monomorphic / mono / monic}}$, if

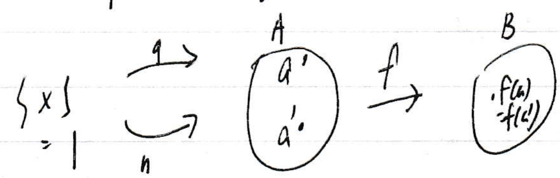
for all objects X of $\mathbb{C}$ and parallel pairs

$g, h: X \to A$

of arrows, we have $f \circ g = f \circ h \Rightarrow g = h$.

Prop. A function $f: A \to B$ is a mono in the category **Set** iff it is injective.

Proof. $(\Rightarrow)$ Assume f is monic, and let $a', a' \in A$ with $f(a) = f(a')$.



$g(*) = a$

$g'(*) = a'$

$f \circ g = f \circ g'$

$\Leftrightarrow \forall x \in 1, f(g(x)) = f(g'(x))$

$\Leftrightarrow f(g(*)) = f(g'(*))$

$f(a) = f(a')$.

we have $f \circ g = f \circ g' \Rightarrow g' = g$, $g'(*) = g(*) \Rightarrow a = a'$

$(\Leftarrow)$ let $g, h: X \to A$ with $f \circ g = f \circ h$. Claim: $g = h$.

we know $f \circ g = f \circ h \Leftrightarrow \forall x,\ f(g(x)) = f(h(x)) \xRightarrow[f \text{ injective}]{} \forall x\ g(x) = h(x))$

Remark. By the same argument, we see that arrow in

$\underline{Pos}$, $\underline{Preord}$, $\underline{Mon}$ (?)

are monic iff they are injective.

Def. $f: A \to B$ in $\mathcal{C}$ is called

epimorphism / epi / epimorphic / epic (?) if

for all parallel pairs $g, h : B \to Y$ we have

$$g \circ f = h \circ f \Rightarrow g = h.$$

(right cancellation)

$X \rightrightarrows A \xrightarrow{f} B \rightrightarrows Y$ (also terminal)

Def. The opposite category $\mathcal{C}^{op}$ of a category $\mathcal{C}$ is as follows:

- Objects and morphisms of $\mathcal{C}^{op}$ are the same as in $\mathcal{C}$

  i.e. $\mathcal{C}^{op}_0 = \mathcal{C}_0$, $\mathcal{C}^{op}_1 = \mathcal{C}_1$

- $\mathrm{dom}_{\mathcal{C}^{op}}(f) = \mathrm{cod}_{\mathcal{C}}(f)$

  $\mathrm{cod}_{\mathcal{C}^{op}}(f) = \mathrm{dom}_{\mathcal{C}}(f)$

$$A \underset{f}{\overset{f}{\rightleftarrows}} B \underset{g}{\overset{g}{\rightleftarrows}} C$$

$A \to B \xrightarrow{f} C$

- $f \circ_{\mathcal{C}^{op}} g = g \circ_{\mathcal{C}} f$

- 1_A in $\mathcal{C}^{op}$ same as 1_A in $\mathcal{C}$

9/14/2021

Observations

- $f: A \to B$ is epi in $\mathbb{C}$ iff it is monic in $\mathbb{C}^{op}$ (duality)

- Slice categories: $(A/\mathbb{C})^{op} = \mathbb{C}^{op}/A$

$$\underbrace{\begin{tikzcd} & A \arrow[dl] \arrow[d] \\ B \arrow[r] & C \end{tikzcd}}_{A/\mathbb{C}} \qquad \underbrace{\begin{tikzcd} B \arrow[r] \arrow[d] & C \arrow[r] \arrow[dl] & D \arrow[dll] \\ A & & \end{tikzcd}}_{\mathbb{C}/A}$$

$$\Rightarrow \left[(A/\mathbb{C})^{op}\right]^{op} = A/\mathbb{C} = (\mathbb{C}^{op}/A)^{op}$$

Prop. A function $f: A \to B$ is epi in Set iff it is surjective.

Proof. Assume first that f is surjective. To show that f is epi, let $g, h: B \rightrightarrows Y$ with $g \circ f = h \circ f$.
($\Leftarrow$)

Claim: $g = h \iff \forall b,\ g(b) = h(b)$

Let $b \in B$. Then by surjectivity of f,

$\exists a$ with $fa = b$, $gb = g(f(a)) = (g \circ f)(a)$
$= (h \circ f)(a) = h(f(a)) = h(b)$.

$\therefore g = h$.

*

($\Rightarrow$) Now assume f epi, and let $b \in B$.

Def $A \xrightarrow{f} B \underset{h}{\overset{g}{\rightrightarrows}} \overset{Bool}{\{0,1\}}$ $\quad g(b) = 1$, $h(b) = \begin{cases} 1 & \text{if } \exists a, f(a) = b \\ 0 & \text{otherwise} \end{cases}$

$\forall a \in A.\ h(f(a)) = 1 = g(f(a)) \Rightarrow h \circ f = g \circ f \Rightarrow h = g$
$\Rightarrow \forall b, \exists a, f(a) = b$.

$0 \le 1$
$1 \le 0$ ... can show preorder.
to discrete $\forall a, b\ a \le b$
(discrete $a = b \iff a \le b$)

Prop. The monoid map (Epimorphisms need not be surjective)

$f : (\mathbb{N}, +, 0) \to (\mathbb{Z}, +, 0)$ $n \mapsto n$

is an epimorphism in CMon. / not in Set because it is not surjective

(commutative monoid)

Proof. let $(\mathbb{Z}, +, 0) \underset{h}{\overset{g}{\rightrightarrows}} (M, +, 0)$

with $g \circ f = h \circ f$. For $z \in \mathbb{Z}$ we have to show $g(z) = h(z)$.

If $z \geq 0$, then $g(z) = g(f(z)) = h(f(z)) = h(z)$

$z < 0$, $g(z) = -g(-z) = -g(f(-z)) = -h(f(-z))$

$= -h(-z) = h(z)$.

[However, this epimorphism is NOT surjective!]

Furthermore, this map, being 'mono' + 'epi', is not an 'isomorphism'

Norm-Vector space → Banach space (a complete normed vector space)

add Cauchy sequences

not surjective unless space is complete

always epimorphic

Ring Theory $\mathbb{Z} \hookrightarrow \mathbb{Q}$ (ring inclusion) is an epimorphism but not surjective.

Lemma: Given $f: A \to B$, $g: B \to C$ in $\mathbb{C}$ with $g \circ f = 1$, we have f monic & g epic.

$$\begin{tikzcd} X \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r, shift left, "m"] \arrow[r, shift right, "n"'] & Y \end{tikzcd}$$

Proof. $f \circ h = f \circ k$

$\Rightarrow g \circ (f \circ h) = g \circ (f \circ k)$

$\Rightarrow (g \circ f) \circ h = (g \circ f) \circ k$

$\Rightarrow h = k$

$\Rightarrow f$ monic, and g epic by duality

$m \circ g = n \circ g$

$\Rightarrow m \circ (g \circ f) = n \circ (g \circ f)$

$\Rightarrow m = n$

$\therefore g$ epic.

Remark. In particular, isomorphisms are monic and epic.

Terminology. We call $f: A \to B$ a split mono ("the mono f is split") if $\exists\, g: B \to A$ s.t. $g \circ f = 1$ (split mono $\Rightarrow$ mono)

- Call g a split epi if $\exists\, f$ with $g \circ f = 1$.
- If $g \circ f = 1$ we also call f a section and g a retraction
- With the same situation, call A a retract of B.

Observation: Retractions $\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \end{tikzcd}$ are preserved by arbitrary functors.

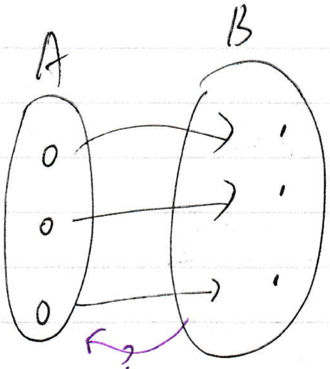
$F: \mathbb{C} \to \mathbb{D}$ $\quad \begin{tikzcd} FA \arrow[r, bend left, "Ff"] & FB \arrow[l, bend left, "Fg"] \end{tikzcd}$, since if $g \circ f = 1$, then $F(g) \circ F(f) = F(g \circ f)$
$= F(1)$
$= 1.$

→ Given a mono $f: A \to B$ in $\underline{\underline{Set}}$, it splits whenever A non-empty.

(Pf: we can find anything for b to map to)

→ Now, which epis in $\underline{Set}$ split?

[Recall $f: A \to B$ is epi iff it is surjective $\Rightarrow \forall b \in B, \exists a \in A, f(a)=b$.]



Find $g: B \to A$ s.t. $f \circ g = 1_B$.

This depends on the Axiom of Choice

Surjection → right inverse
$\exists f$ s.t. $gf = 1$.

· Given a family $(A_i)_{i \in I}$ of non-empty sets, we can pick one of each.

$\Updownarrow$

· Every surjection has a right inverse section.

→ (Need law of Excluded Middle to show that every injection f with non-empty domain has a retraction / left inverse section.)

→ So all epimorphisms in set split.

80-413 9/16/2021

Initial and Terminal Objects

Def. Given category C,

- an object 0 is called <u>initial</u> if for every object A of C there exists exactly one arrow from 0 to A.
- an object 1 is called <u>terminal</u> if for every A there exists exactly one arrow from A to 1.

Comments.
- → Definitions are dual to each other in the sense of opposite cats.
- → These definitions are examples of the Universal Mapping Properties.
- → UMP's are often used in definitions: they characterise defined objects up to <u>unique isomorphism</u>.

Lemma: If 1 and $1'$ are terminal in C, then there exists a unique iso $1 \to 1'$.
"they are the same"

Proof. By terminality of $1'$, we know that there is a unique arrow

$$1 \xrightarrow{f} 1'.$$

By terminality of 1, $\exists$ unique $g: 1' \to 1$ s.t.

$$g \circ f = id_1 \quad , \quad f \circ g = id_{1'}$$

$$\begin{tikzcd} 1 \arrow[r, bend left, "g \circ f"] \arrow[r, bend right, "id_1"'] & 1 \end{tikzcd}$$

By terminality of 1, $\exists$ exactly one arrow $1 \to 1$.

Therefore $id_1 = g \circ f$ & $f \circ g = id_{1'}$.

$\#\mathrm{Fun}(A,B) = (\#B)^{\#A}$

* object can have multiple arrows to itself in general.
{0,1} $f(x) = \neg x$, $f(x) = x$, $f(x) = 0$, $f(x) = 1$ ...

Examples

- Terminal object in $\underline{\underline{set}}$ $\quad A \longrightarrow \underbrace{1 = \{x\}}$ (e.g. $\{0\}, \{1\}$) ✓

- Initial object in $\underline{set}$: Empty set $\emptyset \to A$ - (injective but surjective)

$$\# \mathrm{Fun}(\emptyset, A) = \# A^{\#\emptyset} = \# A^0 = 1.$$

(By convention) $\# \mathrm{Fun}(\emptyset, \emptyset) = 1.$

Category of sets is not isomorphic to its opposite...

In $\underline{Pos}$,

- the initial poset is empty
- posets with exactly one element are terminal.

$$(A, \leq) \to (1, \leq)$$
$$a \leq b \qquad \underbrace{fa}_{=*} \leq \underbrace{fb}_{=*}$$

In $\underline{Mon}$, the initial monoid cannot be empty.

For Grp, Ab this phenomenon holds also (we call $(\{1\}, \cdot, e)$ the 'zero object')

- The initial monoid is $(\{1\}, \cdot, 1)$. $\qquad (\{1\}, \cdot, 1) \xrightarrow{f} (M, \cdot, e)$
$\to f(x) = f(1) = e.$
- The terminal monoid is $(\{1\}, \cdot, 1)$ $\qquad \to f(1 \cdot 1) = f(1) \cdot f(1)$
$\qquad\qquad f(1) \qquad e \cdot e = e$

$(M, \cdot, e) \to (\{1\}, \cdot, 1)$

the unique function is a monoid homomorphism, because in 1 all eqns are true.

Other examples: "pointed set"

"vector spaces" (over any field)

HW 2.

faithful + mono $\to$ $F(\cdot)$ not mono

$$\begin{array}{ccccc} & & A\{1,2\} & \xrightarrow{f} & B\{1\} \\ & & \downarrow & \downarrow F(f) & \downarrow \\ \text{E.g. (mono)} & X \overset{g}{\underset{h}{\rightrightarrows}} & F(A) & \longrightarrow & F(B) \\ & \{a,b\} & \{1,2\} & & \{1\} \end{array} \quad \begin{array}{c} \mathbb{C} \\ \mathbb{D} \end{array}$$

80-413   9/16/2021

(J onesf) $\mathbb{C}$: $0 \xrightarrow{f} 1$ $F(f) = g$, $g(x) = 1$.

$\mathbb{D}$ set

Generalized Elements

Slogan of Category Theory: "Forget about elements, focus on arrows"

In set we can recover elements: elements of a set A correspond to arrows

$$\begin{array}{c} 1 \xrightarrow{a} A \\ * \longmapsto a \end{array} \quad \forall\, a \in A.$$

(In appropriate categories, we call arrows $1 \to A$ 'points' of A)

In Mon, elements of $(M, \cdot, e)$ correspond not to arrows

$1 \to (M, \cdot, e)$ in Mon (since initial = terminal in Mon)

but $(\mathbb{N}, +, 0) \to (M, \cdot, e)$ works!   [$(\mathbb{Z}, +, 0) \to (G, \cdot, e)$ similarly for Groups]

Given $m \in M$, $n \longmapsto \underbrace{m \cdots m}_{n\text{-times}} = m^n$

(Integers form the free monoid on 1 generator) in other words $\hom_{Mon}((\mathbb{N}, +, 0), (M, \cdot, e)) \overset{\text{bijection}}{\cong} M$.

Def. Given an object C of a cat $\mathbb{C}$, a generalized element of C is just any arrow $T \to C$ into C from any object T "test object"

Dually, a generalized co-element of C is an arrow $C \to T$

Recall: (p10) $\underset{\text{small categories}}{\mathbb{C}} \overset{\text{isomorphic}}{\cong} \underset{\text{sets of functions}}{\mathbb{C}}$

$$A \leftrightarrow \underline{A} = \{ f \in \mathbb{C}_1 \mid \mathrm{cod}(f) = A \} = \{\text{generalized elements of } A\}$$

$$
\begin{array}{ccl}
A & \mapsto & \underline{A} = \{ g \in \mathbb{C}_1 \mid \mathrm{cod}(g) = A \} \\
 & & \phantom{\underline{A}} = \{ \text{generalized elements of } A \} \\
f \downarrow & & \underline{f} \downarrow \\
B & \text{—} & \underline{B} \qquad \underline{f}(g) = f \circ g .
\end{array}
$$

__Observation__ An arrow $f: A \to B$ in $\mathbb{C}$ is mono iff $\underline{f}$ is injective

i.e. $\underline{f}$ acts injectively on generalized elements by post-composition.

$$C \rightrightarrows A \to B$$

$$
\begin{array}{l}
C \searrow \\
\qquad A \to B \\
D \nearrow
\end{array}
$$

__Binary Products__

- In set theory, the cartesian product $A \times B = \{ (a,b) \mid a \in A, b \in B \}$

Definition: A $\boxed{\underline{\text{product}}}$ (binary) of objects A and B of a category $\mathbb{C}$ consists of an object P together with two projection arrows

$p_1 : P \to A$, $p_2 : P \to B$ such that

for every object T and pair $f: T \to A$, $g: T \to B$ of arrows, there exists a __unique__ $h: A \to P$ with $p_1 \circ h = f$, $p_2 \circ h = g$.

$\exists!$

Ju-4/3

9/16/2021

(9/21/2021)

$$\begin{tikzcd} & T \arrow[dl, "f"'] \arrow[d, "h"] \arrow[dr, "g"] & \\ A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B \end{tikzcd}$$

object of span category

terminal object of span category

(UMP equivalent $\forall \ldots \exists! \, h$ s.t. ....)

→ We show that the cartesian product is a binary product in $\underline{\text{Set}}$.

i.e. Given sets S, T, the binary product is

$$\boxed{\begin{array}{ll} S & \\ \uparrow p_1 & p_1(s,t)=s \\ S \times T = & \{(s,t) \mid s \in S, t \in T\} \\ \downarrow p_2 & p_2(s,t)=t \\ T & \end{array}}$$

ordered pairs

Proof.

$$\begin{tikzcd} & S \\ X \arrow[ur, "f"] \arrow[r, dashed, "h"] \arrow[dr, "g"'] & S \times T \arrow[u, "p_1"'] \arrow[d, "p_2"] \\ & T \end{tikzcd}$$

$x \in X$

$h(x) = (fx, gx)$

$p_1(h(x)) = f(x)$, $p_2(h(x)) = g(x)$

existence ✓

uniqueness?

Given

$$\begin{tikzcd} & A & \\ P \arrow[ur] \arrow[dr] \arrow[rr, shift left] & & Q \arrow[ul] \arrow[dl] \arrow[ll, shift left] \\ & B & \end{tikzcd}$$

, we can show the UMP more explicitly.

unique isomorphism between products P and Q

Subfields.

$$\begin{array}{cc} K & \mathbb{Q} \\ \downarrow & \downarrow \\ \overline{K} & \overline{\mathbb{Q}} \\ & \cap| \\ & \mathbb{C} \end{array}$$

$\overline{\mathbb{Q}}$ algebraic closure

(unique up to isomorphism, but not unique up to unique isomorphism)

<u>Definition.</u> Given a <u>category</u> $\mathbb{C}$ and objects A, B of $\mathcal{C}$, (span category) define a cat $\boxed{\mathrm{Span}(A,B)}$ as follows.

→ Objects are diagrams ("spans") $A \xleftarrow{x_1} X \xrightarrow{x_2} B$

→ Morphisms between $(A \xleftarrow{x_1} X \xrightarrow{x_2} B)$ and $(A \xleftarrow{y_1} Y \xrightarrow{y_2} B)$

are <u>arrows $f : X \to Y$</u> such that

$$\begin{tikzcd} & X \arrow[dl, "x_1"'] \arrow[dr, "x_2"] \arrow[dd, "f"] & \\ A & & B \\ & Y \arrow[ul, "y_1"] \arrow[ur, "y_2"'] & \end{tikzcd}$$

commutes.

(i.e. we have ↻ and ↺)

→ Composition and identities are inherited from $\mathcal{C}$.

(i.e. $id_{(A \leftarrow X \to B)} = id_X$

e.g. A binary product is an object in a span category. (<u>"product span"</u>)

26

80-413 9/21/2021

Note Moreover, A binary product of A & B is precisely a terminal object of Span(A,B). (unique arrow...)

Corollary. Binary products are unique up to unique isomorphism in Span(A,B).

Remark Arrows arising from UMPs are often called canonical arrows. In particular, the isomorphism between two product spans is called canonical iso.

Lemma Given product span

$$\begin{tikzcd} & X \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & \\ A & P \arrow[l, "p_1"'] \arrow[r, "p_2"] & B \end{tikzcd}$$

and $f,g : X \to P$, we have $f = g$ whenever $p_1 f = p_1 g$ and $p_2 f = p_2 g$.

Proof. only 1 arrow makes them commute...

(Recall the definition of monomorphism $\rightrightarrows A \to B$)

Terminology: we also say that p_1 and p_2 are jointly monic.

→ We say that a category $\mathbb{C}$ has/admits a binary product, if for all pairs A, B of objects there exists a product span

$$\begin{tikzcd} A & A \times B \arrow[l, "p_1"'] \arrow[r, "p_2"] & B \\ & X \arrow[ul, "f"] \arrow[u, "\langle f,g \rangle"] \arrow[ur, "g"'] & \end{tikzcd}$$

→ (more notation)

→ product denoted by $A \times B$

→ canonical arrow by $\langle f,g \rangle$

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \times B \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[dl, "f \circ p_1"] \arrow[d, "{\langle f \circ p_1, g \circ p_2 \rangle = f \times g}"] \arrow[dr, "g \circ p_2"] & B \arrow[d, "g"] \\
C & C \times D \arrow[l, "p_1"] \arrow[r, "p_2"'] & D
\end{tikzcd}$$

Examples Binary products in Preord:

$(A, \leq) \times (B, \leq) = (A \times B, \leq)$ where $(a,b) \leq (c,d)$ iff $a \leq c$ and $b \leq d$

$$\begin{tikzcd}
 & (A, \leq) \\
(X, \leq) \arrow[ur, "f"] \arrow[r, "h\,?"] \arrow[dr, "g"'] & (A \times B, \leq) \arrow[u, "p_1"'] \arrow[d, "p_2"] \\
 & (B, \leq)
\end{tikzcd}$$

(need to state that it is monotone)

→ We know that h has to (fx, gx), following uniqueness from Sets and functions

→ Remains to show that h is monotone, given f, g, monotone.

pf. Assume $x \leq y$ in X, we claim that

$hx \leq hy$ in $(A \times B, \leq)$

$\iff (fx, gx) \leq (fy, gy)$

$\iff fx \leq fy$ & $gx \leq gy$.

(Example of Binary Products) Monoids (and monoid isomorphism)

$$(M,\cdot,e_m) \times (N,\cdot,e_n) = (M \times N, \cdot, (e_m, e_n))$$

$(m \cdot e)$

$$(m,n)\cdot(m',n') = (m\cdot m', n \cdot n')$$

Products in $\underline{Cat}$

"product category"

$\mathbb{C} \times \mathbb{D}$ is given by $(\mathbb{C}\times\mathbb{D})_0 = \mathbb{C}_0 \times \mathbb{D}_0$

$(\mathbb{C}\times\mathbb{D})_1 = \mathbb{C}_1 \times \mathbb{D}_1$

everything is componentwise!

$$1_{(A,B)} = (1_A, 1_B)$$

$$\mathrm{dom}(f,g) = (\mathrm{dom}(f), \mathrm{dom}(g))$$

$$\mathrm{cod}(f,g) = (\mathrm{cod}(f), \mathrm{cod}(g))$$

$$(f,g) \circ (h,k) = (f \circ h, g \circ k)$$

"~~Inclusion functors from ... to ... preserve binary products~~"

Example $\mathbf{2} = (\overset{1}{0} \xrightarrow{f} \overset{1}{1})$

$\mathbf{2}_0 = \{0,1\}$

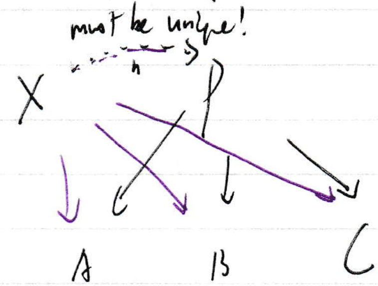
$$\begin{tikzcd} & (0,1) \arrow[dr, "{(f,1)}"] & \\ (0,0) \arrow[ur, "{(1,f)}"] \arrow[rr, "{(f,f)}"] \arrow[dr, "{(f,1)}"'] & & (1,1) \\ & (1,0) \arrow[ur, "{(1,f)}"'] & \end{tikzcd}$$

$\mathbf{2} \times \mathbf{2}$

In the same way as binary products, we can define ternary products:

Given A, B, C in $\mathbb{C}$, a ternary product is given by

P and $p_1 : P \to A$, $p_2 : P \to B$, $p_3 : P \to C$



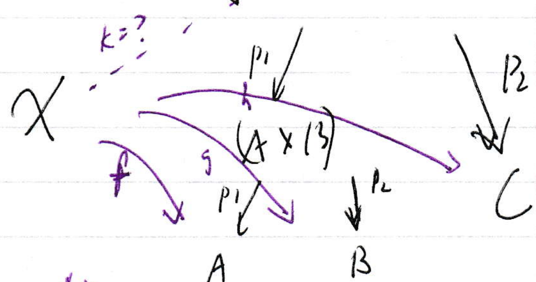
such that for $X \xrightarrow{f} A$, $X \xrightarrow{g} B$, $X \xrightarrow{h} C$ $\exists! \, k : X \to P$

s.t. $p_1 k = f$, $p_2 k = g$, $p_3 k = h$.

Lemma If $\mathbb{C}$ has binary products, then it also has ternary products

Proof A ternary product of A, B, C is given by $(A \times B) \times C$

p_1: take 1st element of tuple
p_2: take 2nd ...



Show that $k = \langle\langle f, g\rangle, h\rangle$ works, and that it is unique.
(see p. 27, lemma. same composition result → same arrow)

Corollary: Given A, B, C in $\mathbb{C}$ with binary products, there is a canonical iso such that

$$(A \times B) \times C \; (\cong) \; A \times (B \times C)$$

Proof Both sides are ternary products (use unique isomorphism property on the ternary span terminal object)

Generally, $(n \geq 2)$-ary products exist.
1-ary product is the object itself. 0-ary product is the terminal object

80-413 $\qquad$ 9/21/21

9/23/21

Lemma $\mathcal{C}$ has finite products ($\forall n \in \mathbb{N}$ (n-ary products)) iff it has bin products and a terminal object.

Proof. unary product always exists...
$2 \to 2^+ \ldots$

Span(A,B): $A \leftarrow X \rightarrow B$

ex binary product

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, "\langle f,g \rangle"] \arrow[dr, "g"] & \\ A & A \times B \arrow[l, "p_1"] \arrow[r, "p_2"'] & B \end{tikzcd}$$

- Ternary ($n=3$) and generally n-ary products for $n>2$ can be constructed as iterated binary products.

$$(A \times B) \times C \cong A \times (B \times C)$$

($n=0$)
- Terminal object

$A \times (B \times (C \times D))$ many variations...

- unary object ($n=1$) $\quad X \xrightarrow{f} A$ $\quad$ let $P = A$, $p_1 = 1_A$.

Infinite Products

$A \times A_2 \times A_3 \times A_4 \ldots$ countable!

Def. A product of a family $(A_i)_{i \in I}$ (set) of objects in a category $\mathcal{C}$ consists of

- An object P
- For each $i \in I$, a projection arrow $p_i : P \to A_i$ such that for all objects X together with arrows $f_i : X \to A_i$ ($i \in I$)
- there exists a unique $X \xrightarrow{h} P$ such that $\forall i \in I$ $p_i \circ h = f_i$

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d] \arrow[dr] \arrow[drr] & P \arrow[dl] \arrow[d] \arrow[dr] & \\ A_{i_1} & A_{i_2} \cdots & A_{i_n} \end{tikzcd}$$

Terminology: We say that a cat $\mathbb{C}$ has (small) products if there exists products for all set-indexed families.

Recall small products in set:

Given a family $(S_i)_{i \in I}$ of sets, its product is given by

$$\prod_{i \in I} S_i = \{ f : I \to \bigcup_{i \in I} S_i \mid \forall i,\ f(i) \in S_i \}$$

Alternatively: consider the 'selector' $p_i : \prod_{i \in I} S_i \to S_i$, $p_i(f) = f(i)$

$(a_i)_{i \in I} \in \prod_{i \in I} S_i$, $p_j((a_i)_{i \in I}) = a_j$.

$$\begin{tikzcd} X \arrow[r, "h"', "\langle f_i \mid i \in I \rangle"] \arrow[dr, "f_j"'] & \prod_{i \in I} S_i \arrow[d, "p_j"] \\ & S_j \end{tikzcd}$$

$h(x) = (f_i(x))_{i \in I}$

another preordered set!

E.g. Small products in Preord. $\prod_{i \in I} (A_i, \le) = \left(\prod_{i \in I} A_i, \le \right)$

$(a_i)_{i \in I} \le (b_i)_{i \in I}$ in

iff $\forall i \in I$, $a_i \le b_i$

→ component-wise comparison against

→ (e.g. function comparison $\prod_{i \in \mathbb{R}} (\mathbb{R}, \le)$)

Finite sets do not admit infinite products.

(Infinite products of sets could be infinite ...)

$$\begin{tikzcd} \mathbb{C} \arrow[r, "F"] & \mathbb{D} \end{tikzcd}$$

$$\begin{tikzcd} A & F(A) \\ A \times B \arrow[u, "p_1"] \arrow[d, "p_2"'] & F(A \times B) \arrow[u, "F(p_1)"'] \arrow[d, "F(p_2)"] \\ B & F(B) \end{tikzcd}$$

Def. If $\mathbb{C}$ has finite/small products, we say that functor F <u>preserves</u> (finite/small) products if product diagrams in $\mathbb{C}$ are mapped to product diagrams in $\mathbb{D}$.

In particular, F is said to preserve binary products if

$F(A) \leftarrow F(A \times B) \rightarrow F(B)$ is a bin product diagram in $\mathbb{D}$ for bin. product diagrams

$A \leftarrow A \times B \rightarrow B$ in $\mathbb{C}$.

If $\mathbb{D}$ also has binary products, then we must have

$$\begin{tikzcd} A & & F(A) & \\ A \times B \arrow[u] \arrow[d] & & F(A \times B) \arrow[u, "F(p_1)"] \arrow[d, "F(p_2)"'] \arrow[r, "\langle F(p_1), F(p_2) \rangle", "\text{unique!}"'] & F(A) \times F(B) \arrow[ul, "p_1"'] \arrow[dl, "p_2"] \\ B & & F(B) & \\ (\mathbb{C}) & & & (\mathbb{D}) \end{tikzcd}$$

$\exists!\, h$

* $\mathbb{D}$ can have fewer products than $\mathbb{C}$.

(e.g. functor F sends everything to the terminal object ...)

h is then the identity.

- If F preserves binary products then h is an isomorphism. (show inverse $F(p_1) \circ h = p_1 h = F(p_1)$.)
- If h is an isomorphism for any A, B then F preserves binary products.
- $F(A \times B) \cong F(A) \times F(B)$

Recall: Given A, B in locally small ~~set~~ cat $\mathbb{C}$

$$\hom_{\mathbb{C}}(A,B) = \{ f \in \mathbb{C}_1 \mid \mathrm{dom}(f) = A,\ \mathrm{cod}(f) = B \} \text{ is a } \underline{\text{set}}.$$

Given $f : B \to C$, we can define

$$\boxed{\hom_{\mathbb{C}}(A,f)} : \hom_{\mathbb{C}}(A,B) \longrightarrow \hom(A,C)$$
$$h \longmapsto f \circ h$$

(illustration of 'preserving binary products')

For fixed A, we get a functor $\underline{\hom_{\mathbb{C}}(A,-) : \mathbb{C} \to \mathbf{set}}$

Currying
sets of functions $A \to B$

$$\begin{tikzcd}
\text{(objects)} & B \arrow[r, mapsto] \arrow[d, "f"'] & \hom_{\mathbb{C}}(A,B) \arrow[d, "\hom_{\mathbb{C}}(A,f)"] \\
\text{(arrows)} & C \arrow[r, mapsto] & \hom_{\mathbb{C}}(A,C)
\end{tikzcd}$$

set of ~~functions~~ morphisms from A to C

We verify that $\hom_{\mathbb{C}}(A,-)$ is indeed a functor:

$$\hom(A, 1_B) : \hom(A,B) \to \hom(A,B)$$
$$h \to 1_B \circ h = h$$

$\hom(A,g) \circ \hom(A,f)$

$$\begin{tikzcd}
B \arrow[dd, "g \circ f"'] \arrow[dr, "f"] & \\
& C \arrow[dl, "g"] \\
D &
\end{tikzcd}
\qquad
\begin{tikzcd}
h \in \hom(A,B) \arrow[dr, "\hom_{\mathbb{C}}(A,f)"] \arrow[dd, "\hom_{\mathbb{C}}(A,g \circ f)"'] & \\
& \hom(A,C) \arrow[dl, "\hom_{\mathbb{C}}(A,g)"] \\
\hom(A,D) &
\end{tikzcd}$$

$$(g \circ f) \circ h = g \circ (f \circ h)$$

Associativity of functions

Prop: Given A in $\mathbb{C}$, $\hom(A, -)$ preserves all binary products that exist in $\mathbb{C}$.

Proof (of binary case)

$$\begin{tikzcd}
A & & \hom(X,A) & \\
A \times B \arrow[u] \arrow[d] \arrow[r, "\text{“generalized element”}"] & \hom(X, A\times B) \arrow[ur, "\hom(X,p_1)"] \arrow[dr, "\hom(X,p_2)"'] \arrow[rr, "h"] & & \hom(X,A) \times \hom(X,B) \arrow[ul] \arrow[dl] \\
B & & \hom(X,B) & \\
\mathbb{C} & & \mathbf{Set} &
\end{tikzcd}$$

binary product in set is a tuple

$h(k : X \to A\times B) = (p_1 \circ k, p_2 \circ k)$

$h = \langle \hom(X,p_1), \hom(X,p_2) \rangle$

Have to check that h is a bijection.

i.e. $\forall (u,v) \in \hom(X,A) \times \hom(X,B), \exists!\, k \in \hom(X, A\times B), (p_1 \circ k, p_2 \circ k) = (u,v)$

$$\begin{tikzcd}
& A \\
X \arrow[ur, "u"] \arrow[r] \arrow[dr, "v"'] & A \times B \arrow[u] \arrow[d] \\
& B
\end{tikzcd}$$

But this is precisely the Universal Mapping Property as applied to $\mathbb{C}$.

Duality (Ch3)

(allowing for opposite categories)

Observation: Definition of category is symmetric! We can exchange domain and codomain, $f \circ g \leftrightarrow f \circ g$

$$1_A \leftrightarrow 1_A$$

Dualizing definitions

initial $\leftrightarrow$ terminal

monomorphisms $\leftrightarrow$ epimorphisms

product $\leftrightarrow$ coproduct

Dualizing propositions,

Terminal objects are unique up to isomorphism

$\leftrightarrow$ initial objects are unique ".

Def. A coproduct of objects A, B in $\mathbb{C}$ is a diagram

$$\begin{tikzcd} A \arrow[d, "i_1 \text{ (injective)}"] \\ A+B \\ B \arrow[u, "i_2"'] \end{tikzcd}$$

such that for every

$$\begin{tikzcd} A \arrow[d, "f"] \\ X \\ B \arrow[u, "g"'] \end{tikzcd}, \quad \exists!\, h : A+B \to X$$

such that

$$\begin{tikzcd} A \arrow[d, "i_1"'] \arrow[dr, "f"] & \\ A+B \arrow[r, "h"', "{[f,g]}"] & X \\ B \arrow[u, "i_2"] \arrow[ur, "g"'] & \end{tikzcd}$$

commutes.

Coproduct of finite sets

$\to$ no. of elements is added.

So-413 9/23/21

Example Coproducts in set

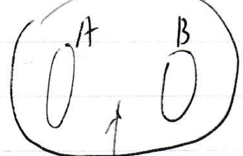
Given sets A, B, $A+B = A \times \{0\} \cup B \times \{1\} = A \amalg B$

'upside down', ~~[illegible]~~ inverse) of product

$\vdots$

$= A \uplus B$

$\vdots$



no overlaps! use tags '0' and '1'

$i_1: A \to A+B$, $a \mapsto (a,0)$

$i_2: B \to A+B$, $b \mapsto (b,1)$

If we take $i_1, i_2 : A \to A+A$

$i_n(a) = (a,n)$ $n \in \{1,2\}$

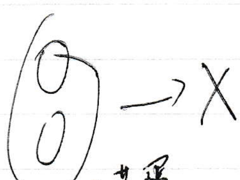
W.T.S $\exists!\, h \ldots$

$$\begin{tikzcd} A \arrow[d, "i_1"'] \arrow[dr, "f"] & \\ A+B \arrow[r, "h"] & X \\ B \arrow[u, "i_2"] \arrow[ur, "g"'] & \end{tikzcd}$$

__Existence__

$h(x,n) := \begin{cases} f(x) & \text{if } n=1 \\ g(x) & \text{if } n=2 \end{cases}$

$h(i_1(a)) = h(a,1) = f(a)$



双黄蛋...

__Uniqueness__ Assume

$h \circ i_1 = f = k \circ i_1$

$h \circ i_2 = g = k \circ i_2$

To show $h = k$

$\Leftrightarrow \forall x \in A+B, h(x) = k(x)$

Let $x \in A+B$. Then either $x = (a,1)$ or $x = (b,2)$, $b \in B$

First case. $h(x) = h(a,1) = h(i_1(a)) = k(i_1(a)) = k(a,1) \overset{\in A}{=} k(x)$

Second case similar.

We proved that

$\hom(X,-): \mathbb{C} \to \mathrm{Set}$ preserves products.

how to formalize the dualization?

If $\mathbb{C}$ has binary coprods, then $\mathbb{C}^{op}$ has products.

We say $\hom_{\mathbb{C}^{op}}(X,-): \mathbb{C}^{op} \to \mathrm{Set}$ preserves products.

$\Updownarrow$ alternative notation

$\hom_{\mathbb{C}}(-,X): \mathbb{C}^{op} \to \mathrm{Set}$ $\qquad B \dashrightarrow A \rightrightarrows X$

$$\begin{tikzcd} A \arrow[d, "f"', "\text{(in } \mathbb{C}^{op}\text{)}"] & \mapsto & \hom(A,X) \ni a \\ B & \mapsto & \hom(B,X) \ni b \arrow[u, dashed, "\hom_{\mathbb{C}}(f,X)\ :\ b \mapsto b \circ f"'] \end{tikzcd}$$

$B \rightrightarrows X$

This is the "representable functor" (also contravariant, reverses arrows)

Coproducts in Set

$$A + B = A \times \{1\} \cup B \times \{2\}$$

- Coproducts are unique up to unique isomorphism
- Coproduct of objects A, B in $\mathcal{C}$

$$\begin{tikzcd}
A \arrow[r, "i_1"] \arrow[dr, "f"'] & A+B \arrow[d, dashed, "{[f,g]}"] & B \arrow[l, "i_2"'] \arrow[dl, "g"] \\
& X &
\end{tikzcd}$$

Coproducts in Preord

$$(A, \leq) + (B, \leq) = (A+B, \leq)$$

↳ defined by cases

$(a,1) \leq (a',1)$ iff $a \leq a'$ in A

$(b,2) \leq (b',2)$ iff $b \leq b'$

$(a,1), (b,2)$ cannot be compared, neither can $(b,2), (a,1)$

* Similar in Top...

* Monoids: deferred

$$\begin{tikzcd}
(A, \leq) \arrow[r, "a \mapsto (a,1)"] \arrow[dr, "f"'] & (A+B, \leq) \arrow[d, dashed, "{[f,g]}"] & (B, \leq) \arrow[l, "(b,2) \mapsfrom b"'] \arrow[dl, "g"] \\
& (X, \leq) &
\end{tikzcd}$$

Def: An equalizer of a parallel pair

$$f, g : A \rightrightarrows B$$

in a cat $\mathcal{C}$ consists of an object E and an arrow $e : E \to A$ such that

- $f \circ e = g \circ e$
- for all $(h : X \to A)$ with $f \circ h = g \circ h$, there exists unique $m : X \to E$ with $e \circ m = h$.

$$\begin{tikzcd} X \arrow[d, dashed, "m"'] \arrow[dr, "h"] & & \\ E \arrow[r, "e"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

(Definition)

Observation: Given $A \overset{f}{\underset{g}{\rightrightarrows}} B$ in $\mathcal{C}$, define the category $E(f,g)$ as follows –

- objects of $E(f,g)$ are arrows $X \xrightarrow{x} A$ in $\mathcal{C}$ such that $f \circ x = g \circ x$
- morphisms in $E(f,g)$ from $(X \xrightarrow{x} A)$ to $(Y \xrightarrow{y} A)$ are arrows $m : X \to Y$ such that $y \circ m = x$.

Given $A \overset{f}{\underset{g}{\rightrightarrows}} B$ we have $f = g$ iff $\forall a, f(a) = g(a)$ (extensional equivalence)

→ Furthermore, an equalizer of f, g is the same thing as a terminal object in $E(f,g)$.

extra condition

→ Note also that $E(f,g) \subseteq \mathcal{C}/A$

80-413 9/28/2021

'u.m.p'

$$\begin{tikzcd} E \arrow[dr] \arrow[d, shift right] & & \\ E' \arrow[u, shift right] \arrow[r] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

Equalizers in set

$e(a) = a$

$$\begin{tikzcd} E = \{a \in A \mid fa = ga\} \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ X \arrow[u, dashed, "m"] \arrow[ur, "h"'] & & \end{tikzcd}$$

If $f \circ h = g \circ h$,

then $\forall x \in X, f(h(x)) = g(h(x))$

$\Rightarrow \forall x, h(x) \in E$

$\Rightarrow m = h$.

Another example:

$$\begin{tikzcd} & E = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\} \arrow[d, "\text{inclusion map}"] \\ X \arrow[ur, dashed] \arrow[r, dashed, "h"] & \mathbb{R}^2 \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \\ & \mathbb{R} \end{tikzcd}$$

$f(x,y) = x^2+y^2$
$g(x,y) = 1$

Equalizers in Preord

$$(\underbrace{\{a \mid fa = ga\}}_{E}, \leq) \hookrightarrow (A, \leq) \underset{g}{\overset{f}{\rightrightarrows}} (B, \leq)$$

$a \leq a'$ in E iff $a \leq a'$ in A

'same set'

Equalizers in Monoids

$$(\underbrace{\{a \mid fa = ga\}}, \cdot, e) \hookrightarrow (M, \cdot, e) \underset{g}{\overset{f}{\rightrightarrows}} (N, \cdot, e)$$

To see that E is a submonoid, have to check

(1) $e \in E$ $(f(e) = e = g(e))$

(2) $\forall a, a' \in E, a \cdot a' \in E$ $\left(f(x \cdot y) = f(x) \cdot f(y) = g(x) \cdot g(y) = g(x \cdot y) \text{ for } x, y \in E \Rightarrow x \cdot y \in E \right.$

$$\begin{tikzcd} (X, \cdot, e) \arrow[d] \arrow[dr, "h"] & & \\ (E, \cdot, e) \arrow[r, "e"] & (M, \cdot, e) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (N, \cdot, e) \end{tikzcd}$$

Lemma: If $E \xrightarrow{e} A \overset{f}{\underset{g}{\rightrightarrows}} B$ is an equalizer in $\mathcal{C}$, then e is monic in $\mathcal{C}$.

\begin{tikzcd}
X \arrow[d, shift right, "h"'] \arrow[d, shift left, "k"] \arrow[dr, "e \circ h = e \circ k"] & & \\
E \arrow[r, "e"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B
\end{tikzcd}

$f \circ e = g \circ e$

$f \circ (e \circ h) = g \circ (e \circ k)$

$\Rightarrow h = k$ ... (by left cancellation)

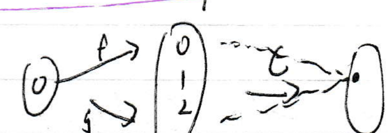
Def. A coequalizer of maps $A \overset{f}{\underset{g}{\rightrightarrows}} B$

is given by an object C and an arrow $c: B \to C$ such that

- $c \circ f = c \circ g$
- $\forall (B \xrightarrow{y} Y)$ with $y \circ f = y \circ g$,
  $\exists!\ m: C \to Y$ with $m \circ c = y$

\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "c"] \arrow[dr, "y"'] & C \arrow[d, "m"] \\
& & Y
\end{tikzcd}

<u>Coequalizer in <u>Set</u></u>

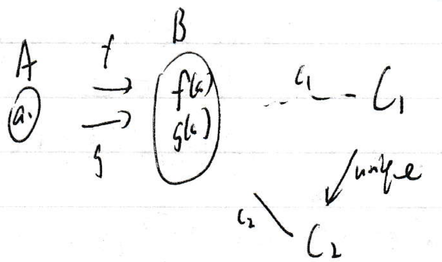


'force elements to become equal'

$f(0) = 0,\ g(0) = 2$ We want $c \circ f = c \circ g$

$\Rightarrow c(f(0)) = c(g(0))$

$\Rightarrow c(0) = c(2)$

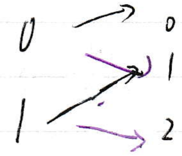


C_1, C_2 should be 'minimal' $\to$ otherwise uniqueness is lost

80-413 9/28/2021

$A = \{0,1\}$, $B = \{0,1,2\}$

$f(a) = a$, $g(a) = a+1$



$A \underset{g}{\overset{f}{\rightrightarrows}} B \longrightarrow B/R$

- Def: A binary relation $R \subseteq B \times B$ on a set B is called an equivalence relation if it is reflexive, symmetric, and transitive.

For R an equivalence relation and $b \in B$ write

$$[b]_R = [b] = \{ b' \in B \mid (b,b') \in R \}$$ for the equivalence class of b w.r.t. R.

Given equivalence relation R on B, the quotient of B by R is the set

$$B/R = \{ [b]_R \mid b \in B \}$$

There is a projection function $B \xrightarrow{c} B/R$, $c(b) = [b]$ $R^o = \{(b',b) \mid (b,b') \in R\}$

Prop: for every binary relation R on B, there exists a least equivalence relation $S \subseteq B \times B$ containing R. Proof. First way: note that $R' = R \cup R^o \cup \Delta$ $\Delta = \{(b,b) \mid b \in B\}$

$S = \overset{dist=1}{R'} \cup \overset{dist=2}{(R' \circ R')} \cup \overset{dist=3}{(R' \circ R' \circ R')} \cup \dots$

So what are coequalizers in Set? second way. $S = \bigcap \{ T \subseteq B \times B \mid R \subseteq T \;\&\; T \text{ equivalence relation} \}$

Prop: The coequalizer of parallel maps $A \underset{g}{\overset{f}{\rightrightarrows}} B$ in Set is given by

$B \xrightarrow{c} B/S$ where S is the least relation containing

$R = \{ (f(a), g(a)) \mid a \in A \}$, and $c(b) = [b]_S$.

Proof

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "c"] \arrow[dr, "h"'] & B/S \arrow[d, dashed, "m"] \\ & & X \end{tikzcd}$$

'collapsing map'

where $h \circ f = h \circ g$.

we are tempted to write $m([b]) := h(b)$ (representative), but we first want to make sure it's 'well-defined'

i.e. $\forall b, b' \in B, (b,b') \in S \Rightarrow h(b) = h(b')$

$$\Leftrightarrow S \subseteq \underbrace{\{(b,b') \mid h(b) = h(b')\}}_{T}$$

Observe that T is an equivalence relation. So it suffices for $R \subseteq T$. (then $R^{\equiv} = S \subseteq T$) 'least'

$$R \subseteq T \Leftrightarrow \{(fa, ga) \mid a \in A\} \subseteq \{(b,b') \mid h(b) = h(b')\}$$

$$\Leftrightarrow \forall a, h(fa) = h(ga)$$

$$\Leftrightarrow h \circ f = h \circ g \quad ✓$$

so we can say

$$h(b) = m(c(b)) = m([b])$$

use the underlying function on any representative

m is correct: " (same proof)

Prove that m is unique: different $m \Rightarrow$ different map for some b $\Rightarrow$ wrong result

use the fact that c is surjective

Remark: c is epic in $\mathbb{C}$ (as with all coequalizers) (obvious?)

A