SU-413 11/4/2021

- Propositions-as-types
- "proofs as programs"

$\boxed{\text{Naturality (Ch 7)}}$

Observation: $\underline{\text{Cat}}$ (of small categories) is Cartesian closed.

$$\begin{array}{cccl}
\mathbb{A}, \mathbb{B} & & \mathbb{B}^{\mathbb{A}} & \\
1 & \to & \mathbb{B}^{\mathbb{A}} & \text{'exponential'} \\
1 \times \mathbb{A} & \to & \mathbb{B} & \text{'product'} \\
\mathbb{A} & \to & \mathbb{B} &
\end{array}$$

categories: $\mathbb{A}, \mathbb{B}$

Morphisms between functors are natural transformations

first appeared in 1944: "A general theory of Natural Equivalences" Eilenberg & Mac Lane

Def Given functors $F, G : \mathbb{A} \to \mathbb{B}$, a natural transformation

$$\eta : F \to G$$

is given by morphisms

$$\eta_A : FA \to GA$$

for all A in $\mathbb{A}$, such that for all arrows $f: A \to B$ in $\mathbb{A}$, the square

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "\eta_A"'] & F(B) \arrow[d, "\eta_B"] \\
G(A) \arrow[r, "G(f)"'] & G(B)
\end{tikzcd}$$

commutes. "naturality square"

Examples: (1) $A \times B \xrightarrow{p_1} A \qquad A \times B \xrightarrow{p_2} B$

- let $\mathbb{C}$ be a category with binary products
- let $\mathbb{P}, Q : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ (product) (projection)

$$\mathbb{P}(A,B) = A \times B \quad , \mathbb{P}(f,g) = f \times g$$

$$Q(A,B) = A \qquad Q(f,g) = f$$

Then the product projections $A \times B \to A$ $\eta_{(A,B)} = p_1$: form a natural transformation.

$$\eta : \mathbb{P} \to Q$$

Check naturality square

$$\begin{array}{c} \mathbb{P} \\ \eta \downarrow \\ Q \end{array} \qquad (A,B) \xrightarrow{(f,g)} (C,D) \text{ in } \mathbb{C} \times \mathbb{C}$$

$$\begin{tikzcd}
\mathbb{P}(A,B) \arrow[r, "\mathbb{P}(f,g)"] \arrow[d, "\eta(A,B)"] & \mathbb{P}(C,D) \arrow[d, "\eta(C,D)"] \\
Q(A,B) \arrow[r, "Q(f,g)"] & Q(C,D)
\end{tikzcd}
\iff
\begin{tikzcd}
A \times B \arrow[r, "f \times g"] \arrow[d, "p_1"] & C \times D \arrow[d, "p_1"] \\
A \arrow[r, "f"] & C
\end{tikzcd}$$

$$p_1 \circ (f \times g) \overset{HW3}{=} p_1 \circ \langle f \circ p_1, g \circ p_2 \rangle = f \circ p_1$$

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(2)

$$\langle p_2, p_1\rangle : A \times B \xrightarrow{\cong} B \times A$$

$\sigma : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, $P \to P'$, where $P(A,B) = A \times B$

$P'(A,B) = B \times A$

$\sigma_{(A,B)} = \langle p_2, p_1 \rangle$

$$(A \times B) \xrightarrow{(f,g)} (C, D)$$

$$\begin{array}{ccc} A \times B & \xrightarrow{f \times g} & C \times D \\ \downarrow{\scriptstyle \langle p_2, p_1\rangle} & & \downarrow{\scriptstyle \langle p_2, p_1\rangle} \\ B \times A & \xrightarrow[g \times f]{} & D \times C \end{array}$$

$$\langle p_2, p_1 \rangle \circ (f \times g) \overset{\text{Hw 3}}{=} \langle p_2, p_1 \rangle \circ \langle f \circ p_1, g \circ p_2 \rangle$$

$$= \langle p_2 \circ \langle f \circ p_1, g \circ p_2 \rangle, p_1 \circ \langle f \circ p_1, g \circ p_2 \rangle \rangle$$

$$= \langle g \circ p_2, f \circ p_1 \rangle$$

$$= (g \times f) \circ \langle p_2, p_1 \rangle$$

$F : (A, B, C) \to A \times (B \times C)$

$F(A) \xrightarrow{f} F(A')$

$E(A) \xrightarrow{f} E(A')$

$$\begin{array}{ccc} A \times (B \times C) & \xrightarrow{\eta} & (A \times B) \times C \\ \downarrow & & \downarrow \\ A' \times (B \times C) & \xrightarrow[\eta]{} & (A' \times B) \times C \end{array}$$

(3) $T \to T' : \mathbb{C}^3 \to \mathbb{C}$, since $(A \times B) \times C \cong A \times (B \times C)$

e.g. F, G

(4)? Evaluations in CCC also form nat. trans!

Def: For small categories $\mathbb{A}, \mathbb{B}$, $\underline{\text{Fun}(\mathbb{A}}, \mathbb{B})$ is the category of functors and natural transformations

Composition: Given functors $F, G, H : \mathbb{A} \to \mathbb{B}$ and NT's $F \xrightarrow{\eta} G \xrightarrow{\theta} H$,

$\theta \circ \eta : F \to H$ is defined by $(\theta \circ \eta)_A = \theta_A \circ \eta_A$

$$A \xrightarrow{f} B$$

Composition

$\Rightarrow$

$$\begin{tikzcd} F \arrow[d, "\eta"'] \\ G \arrow[d, "\theta"'] \\ H \end{tikzcd} \qquad \begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\eta_A"'] & FB \arrow[d, "\eta_B"] \\ GA \arrow[r, "Gf"] \arrow[d, "\theta_A"'] & GB \arrow[d, "\theta_B"] \\ HA \arrow[r, "Hf"'] & HB \end{tikzcd}$$

Identity

$\Rightarrow$ $1_F : F \to F : \mathbb{A} \to \mathbb{B}$ $\qquad (1_F)_A = 1_{F(A)}$

Given monoid homomorphisms $f, g : (M, \cdot, e) \to (N, \cdot, e)$ these can be viewed as a $\overline{NT}$ $\eta : \overline{f \to g}$. This has one component

$\eta_* : * \to *$ s.t. $\forall m \in M$,

$$\begin{tikzcd} * \arrow[r] \arrow[d, "fm"'] & * \arrow[d, "gm"] \\ * \arrow[r, "\eta"'] & * \end{tikzcd}$$

$\forall m,\ g(m) \circ \eta = \eta \circ f(m)$

(write $90°$)

Groups ... $\eta^{-1} \circ g(m) \circ \eta = f(m)$

Recall: Given functors $F, G : \mathbb{C} \to \mathbb{D}$, a natural transformation $\eta : F \to G$ consists of arrows $\eta_C : FC \to GC$ for $C \in \mathbb{C}_0$, such that for all $C \to C'$ in $\mathbb{C}$, the square

$$\begin{array}{cccc} & C & \xrightarrow{f} & C' \\ F & FC & \xrightarrow{Ff} & FC' \\ \eta\downarrow & \downarrow \eta_C & & \downarrow \eta_{C'} \\ G & GC & \xrightarrow[Gf]{} & GC' \end{array}$$

Sizes: (see p 97.

* [
- If $\mathbb{C}$ & $\mathbb{D}$ are ~~locally~~ small, then $\mathrm{Fun}(\mathbb{C}, \mathbb{D})$ small. η's form a set
- If $\mathbb{C}$ small and $\mathbb{D}$ is locally small, then $\mathrm{Fun}(\mathbb{C}, \mathbb{D})$ is locally small.
- If $\mathbb{C}$ is locally small and $\mathbb{D}$ is locally small, then $\mathrm{Fun}(\mathbb{C}, \mathbb{D})$ is not necessarily l.-s.

Def: An isomorphism in a functor category $\mathrm{Fun}(\mathbb{C}, \mathbb{D})$ is called a natural isomorphism.

Lemma: A natural transformation $\{\eta : F \to G\} : \mathbb{C} \to \mathbb{D}$ is a natural isomorphism iff $\eta_C : FC \to GC$ is also an iso $\forall\, C \in \mathbb{C}_0$.

Proof: ($\Rightarrow$) If $\eta : F \to G$ is ~~an inverse~~ a natural iso, then the inverse to η_C is given by $(\eta^{-1})_C$.

($\Leftarrow$) Conversely, assume that all η_C are invertible. Then define $\eta^{-1} : G \to F$ by $(\eta^{-1})_C = (\eta_C)^{-1}$. It remains to show that η^{-1} is a natural transformation.

$$C \xrightarrow{f} C'$$

$$\begin{tikzcd}
G \arrow[d] & GC \arrow[r, "Gf"] \arrow[d, "(\eta_C)^{-1}"] & GC' \arrow[d, "(\eta_{C'})^{-1}"] \\
F & FC \arrow[r, "Ff"'] & FC'
\end{tikzcd}$$

We have to show that this square commutes.

By naturality of η:

$$\begin{tikzcd}
FC \arrow[r, "Ff"] \arrow[d, "\eta_C"'] & FC' \arrow[d, "\eta_{C'}"] \\
GC \arrow[r, "Gf"'] & GC'
\end{tikzcd}$$

commutes.

$\Rightarrow \eta_{C'} \circ Ff = Gf \circ \eta_C$

$\Rightarrow \eta_{C'} \circ Ff \circ (\eta_C)^{-1} = Gf$

$\Rightarrow Ff \circ (\eta_C)^{-1} = (\eta_{C'})^{-1} \circ Gf$

$\Rightarrow$ upper square commutes.

* Cartesian closedness does not depend on local smallness (unless we characterize them as hom-sets).

* Goal is to show that $\underline{Cat}$ is cartesian closed (with $Fun(\mathbb{C}, \mathbb{D})$ as the exponential)

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First, we need the

Bifunctor Lemma : $\mathbb{A} \times \mathbb{B} \to \mathbb{C}$

Let $\mathbb{A}, \mathbb{B}, \mathbb{C}$ be categories and assume that we have

- a function $F_0 : \mathbb{A}_0 \times \mathbb{B}_0 \to \mathbb{C}_0$
- for each $A \in \mathbb{A}_0$, a functor

  $F(A, -) : \mathbb{B} \to \mathbb{C}$ extending the (object) mapping $B \mapsto F_0(A, B)$

$(-)\times(-)\times 3 :$

$\mathbb{C}^3 \to \mathbb{C}$

- for each $B \in \mathbb{B}_0$, a functor $F(-, B) : \mathbb{A} \to \mathbb{C}$, extending the mapping $A \mapsto F_0(A, B)$
- such that the square

$$\begin{tikzcd}
F_0(A, B) \arrow[r, "F(A, g)"] \arrow[d, "F(f, B)"] & F_0(A, B') \arrow[d, "F(f, B')"] \\
F_0(A', B) \arrow[r, "F(A', g)"'] & F_0(A', B')
\end{tikzcd}$$

commutes for all $f : A \to A'$ in $\mathbb{A}$ and all $g : B \to B'$ in $\mathbb{B}$.

Then there exists a unique functor $F : \mathbb{A} \times \mathbb{B} \to \mathbb{C}$ such that

- $F(A, B) = F_0(A, B) \quad \forall A, B$
- $F(1_A, g) = F(A, g)$ for all $A \in \mathbb{A}_0$ & $g : B \to B'$ in $\mathbb{B}$
- $F(f, 1_B) = F(f, B)$ for all $f : A \to A'$ & $B \in \mathbb{B}_0$.

Proof: Given any morphism $(f,g):(A,B)\to(A',B')$ in $\mathbb{A}\times\mathbb{B}$, i.e. $f:A\to A'$, $g:B\to B'$, we have a commutative diagram in $\mathbb{A}\times\mathbb{B}$

$$\begin{tikzcd}
(A,B) \arrow[r, "(1_A, g)"] \arrow[d, "(f,1_B)"'] \arrow[dr, "(f,g)"] & (A,B') \arrow[d, "(f,1_{B'})"] \\
(A',B) \arrow[r, "(1_{A'},g)"'] & (A',B')
\end{tikzcd}$$

Note that $(1_{A'},g)\circ(f,1_B)=(1_{A'}\circ f,\ g\circ 1_B)=(f,g)$

similarly $\quad (f,1_{B'})\circ(1_A,g)=(f,g)$

$\Rightarrow (1,g)\circ(f,1)=(f,1)\circ(1,g)$ so we needed the square on $F(\cdot)$

If F satisfying conditions exists

$$F(f,g)=F((f,1_{B'})\circ(1_A,g))=F(f,1_{B'})\circ F(1_A,g)$$

$$:=F(f,B')\circ F(A,g)$$

It remains to show that the functor with this morphism part is well-defined and satisfies the conditions (This is straightforward...?)

Applications: ① Given locally small $\mathbb{C}$, the covariant hom functors $\hom(A,-):\mathbb{C}\to set$ can be combined with the contravariant hom functor $\hom(-,B):\mathbb{C}^{op}\to set$ into a "mixed covariance hom-functor" $\hom(-,-):\mathbb{C}^{op}\times\mathbb{C}\to set$

② Similarly, given a CCC $\mathbb{C}$, the (co-, contra) contravariant exponentiation functors

$E_A=(-)^A:\mathbb{C}\to\mathbb{C}$

$B^{(-)}:\mathbb{C}^{op}\to\mathbb{C}$

can be combined into a functor $(-)^{(-)}:\mathbb{C}^{op}\times\mathbb{C}\to\mathbb{C}$

$(A,B)\mapsto B^A$

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Theorem $\mathbf{Cat}$ is cartesian-closed.

Proof. Given small categories $\mathbb{A}, \mathbb{B}$, define $\mathbb{B}^{\mathbb{A}} = \mathrm{Fun}(\mathbb{A}, \mathbb{B})$

Define (functor) $\varepsilon^{\mathbb{A}}_{\mathbb{B}} : \underbrace{\mathrm{Fun}(\mathbb{A},\mathbb{B})}_{\text{category}} \times \underbrace{\mathbb{A}}_{\text{category}} \to \mathbb{B}$ using the bifunctor lemma:

Define $\varepsilon(F, A) = F(A)$

$\varepsilon(F, -) : \mathbb{A} \to \mathbb{B}$

$\varepsilon(F, A) = F(A)$

the regularities (plus) $\varepsilon(F, f) = F(f)$

'evaluation functor' $\varepsilon(-, A) : \mathrm{Fun}(\mathbb{A}, \mathbb{B}) \to \mathbb{B}$

object $\varepsilon(F, A) = F(A)$

morphism $\varepsilon(\eta, A) = \eta_A \quad (FA \to GA \in \mathbb{B})$

$$A \xrightarrow{f} A'$$

need

$$\begin{tikzcd} F \arrow[d, "\eta"'] & F(A) \arrow[r, "F(f)"] \arrow[d, "\eta_A"'] & F(A') \arrow[d, "\eta_{A'}"] \\ G & G(A) \arrow[r, "G(f)"'] & G(A') \end{tikzcd}$$

to commute,

but this follows from naturality of Fun()

In this case, $\varepsilon(\eta, f) = \varepsilon(\eta, A') \circ \varepsilon(F, f)$
$= \eta_{A'} \circ F(f)$

naturality of η_A

Currying: Given $F : \mathbb{A} \times \mathbb{B} \to \mathbb{C}$, define $\tilde{F} : \mathbb{A} \to \mathrm{Fun}(\mathbb{B}, \mathbb{C})$,
$\tilde{F}(A) = F(A, -)$ functor

to be the functor given by $\tilde{F}(A)(B) = F(A, B)$

$\varepsilon \circ (\tilde{F} \times 1_{\mathbb{B}}) = F$

$\tilde{F}(A)(\overset{B \to B'}{g}) = F(1_A, g)$

Given $f : A \to A'$, $\overset{\in \mathbb{A}}{\tilde{F}(f)} : \tilde{F}(A) \to \tilde{F}(A')$ ('η') is given by

$\tilde{F}(f)_B = F(f, 1_B) : \overset{\in \mathbb{C}}{F(A, B)} \to F(A', B) \in \mathbb{C}$

$$\begin{tikzcd} F(A,B) \arrow[r, "g"] \arrow[d, "F(f)_B"'] & F(A,B') \\ F(A',B) & \end{tikzcd}$$

The proof that $\tilde{F}$ is well-defined and unique such that $\varepsilon \circ (\tilde{F} \times 1) = F$ is left as an exercise.

We need $\varepsilon \circ (\tilde{F} \times 1_{\mathbb{B}})(f, 1_B)$
$= \varepsilon \circ (\tilde{F}(f) \times 1_{\mathbb{B}})$
$= \tilde{F}(f)_B \circ F(1_A, 1_B) \equiv F(f, 1_B)$

Example: for small $\mathbb{C}$, $\hom : \mathbb{C}^{op} \times \mathbb{C} \to set$ gives transposes

$$\left\{\begin{array}{lcl} \mathbb{C}^{op} & \to & Fun(\mathbb{C}, set) \\ \text{object } A & \mapsto & \left[\hom(A,-) = \underset{\mathbb{C}\to set}{\text{functor}} \text{ (covariant functor p85)}\right] \end{array}\right.$$

$$\left\{\begin{array}{lcl} \mathbb{C} & \to & Fun(\mathbb{C}^{op}, set) \\ B & \mapsto & \underset{\mathbb{C}^{op}\to set}{\hom(-,B)} \end{array}\right\} \text{ the Yoneda embedding}$$

see p111

$\underset{\sim}{2} = \{0, 1\}$

$set/\underset{\sim}{2}$

$$\begin{array}{ccc} X & \to & Y \\ \downarrow & \swarrow g & \\ \underset{\sim}{2} & & \end{array}$$

$set^2 = set \times set$

$(A \times B)$

$S : \quad set^2 \to set/\underset{\sim}{2}$

$\quad (A, B) \to \left(A + B \xrightarrow{[0,1]} \underset{\sim}{2}\right)$

$T : \quad set/\underset{\sim}{2} \to set^2 = set \times set$

$\quad (X \xrightarrow{f} \underset{\sim}{2}) \mapsto (X_0, X_1)$

where $X_0 = \{x \mid fx = 0\}$, $X_1 = \{x \mid fx = 1\}$

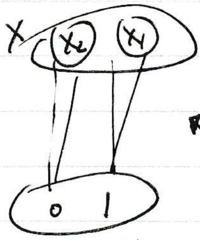
$$S(T(f : X \to 2)) = S(X_0, X_1)$$
$$= (X_0 + X_1 \to 2) \cong (X \to 2)$$

$$\therefore S \circ T \cong 1_{set/2}$$

Equivalences 'indexed presentation'

'fiber presentation'

$$S: Set^2 = Set \times Set \to Set/\underset{\sim}{2}$$



$$(A,B) \mapsto (A+B \xrightarrow{[0,1]} \underset{\sim}{2})$$

tagged $0 \to 0$, $1 \to 1$

$$T: Set/\underset{\sim}{2} \to Set^2$$

$$(f: X \to \underset{\sim}{2}) \mapsto (X_0, X_1)$$

where $X_0 = \{x \in X \mid f(x) = 0\}$

$X_1 = \{x \in X \mid f(x) = 1\}$

$$S(T(f: X \to \underset{\sim}{2})) = (X_0 + X_1 \to \underset{\sim}{2}) \cong (X \to \underset{\sim}{2})$$

$$\begin{tikzcd} X \arrow[dr] & X_0 + X_1 \arrow[l, "p_2"'] \arrow[d] \\ & \underset{\sim}{2} \end{tikzcd}$$

$= \{0\} \times X_0 \cup \{1\} \times X_1$ 'tag'

$$T(S(A,B)) = T(A+B \to \underset{\sim}{2})$$

$$= (\{0\} \times A, \{1\} \times B)$$

$$\cong (A,B)$$

We have 'natural isos' $S \circ T \cong 1_{Set/\underset{\sim}{2}}$, $T \circ S \cong 1_{Set^2}$

Definition: An equivalence of categories between $\mathbb{C}$ and $\mathbb{D}$ consists of

- functors $F: \mathbb{C} \to \mathbb{D}$, $E: \mathbb{D} \to \mathbb{C}$
- natural isomorphisms (p99) $\alpha: 1_{\mathbb{C}} \to E \circ F$, $\beta: 1_{\mathbb{D}} \to F \circ E$

Terminology:
- Say "$\mathbb{C}$ and $\mathbb{D}$ are equivalent"
- write $\mathbb{C} \simeq \mathbb{D}$

Remark: 'Equality up to isomorphism' $\mathbb{C} \cong \mathbb{D}$ $F: \mathbb{C} \to \mathbb{D}$, $E: \mathbb{D} \to \mathbb{C}$. $E \circ F = 1$, $F \circ E = 1$, $\forall C \in \mathbb{C}_0$, $E(F(C)) = C$

→ we would require objects to be equal → against spirit of cat theory

→ With equivalence, objects are not necessarily the same

Example:

Finite Dimension Vector space over $\mathbb{R}$ dimension → basis → n elements ...

$$\mathrm{Mat}_*(\mathbb{R}) \underset{F}{\overset{F}{\rightleftarrows}} \mathrm{FinVect}(\mathbb{R})$$

natural numbers ≥ 0
& $n \times k$-matrices

$$n \overset{F}{\longmapsto} \mathbb{R}^n \qquad \mathbb{R}^n \to \mathbb{R}^k$$

$E \circ F = 1$, $F \circ E \cong 1$ (not all finite vector spaces are $\mathbb{R}^n$)

<u>Theorem</u> A functor $F: \mathbb{C} \to \mathbb{D}$ is (part of) an equivalence of categories

$\Leftrightarrow$ F is
1. faithful
2. full $F_{A,B}$ surjective
3. "essentially surjective" i.e. $\forall D \in \mathbb{D}_0$, $\exists C \in \mathbb{C}_0$, $F(C) \cong D$

Proof. ($\Rightarrow$) Assume $E: \mathbb{D} \to \mathbb{C}$,

$\alpha: 1_{\mathbb{C}} \xrightarrow{\cong} E \circ F$

$\beta: 1_{\mathbb{D}} \xrightarrow{\cong} F \circ E$

Show 1. F faithful. Let $f, g: C \to C'$ such that $Ff = Fg: FC \to FC'$

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\alpha_C"', "\cong"] & C' \arrow[d, "\alpha_{C'}", "\cong"'] \\ EFC \arrow[r, "EFf"'] & EFC' \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "g"] \arrow[d, "\alpha_C"'] & C' \arrow[d, "\alpha_{C'}"] \\ EFC \arrow[r, "EFg"'] & EFC' \end{tikzcd}$$

reverse the arrow!!

, all components are isomorphic in a natural isomorphism (p58.)

$$f = (\alpha_{C'})^{-1} \circ GFf \circ \alpha_C$$
$$= (\alpha_{C'})^{-1} \circ EFg \circ \alpha_C$$
$$= g$$

$$\begin{tikzcd} D \arrow[r, "f"] \arrow[d] & D' \arrow[d] \\ FED \arrow[r, "FEf"'] & D' \end{tikzcd}$$

$Ef = Eg \Rightarrow FEf = FEg \Rightarrow \dots f = g$

$\therefore$ E is also faithful.

② F is full.

Let $h: FC \to FC'$

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\alpha_C"'] & C' \arrow[d, "\alpha_{C'}"] \\ EF(C) \arrow[r] & EF(C') \end{tikzcd}$$

$FC \xrightarrow{h} FC'$

$EF(f) = Eh$ ← since $h: FC \to FC'$

Proof in class 11/16

Consider $h: FC \to FC'$

$$\begin{tikzcd} C \arrow[r, "k"] \arrow[d, "\alpha_C \cong"'] & C' \arrow[d, "\cong \alpha_{C'}"] \\ EFC \arrow[r, "Eh"'] & EFC' \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "k"] \arrow[d, "\alpha_C"'] & C' \arrow[d, "\alpha_{C'}"] \\ EFC \arrow[r, "EFk"'] & EFC' \end{tikzcd}$$

Enough to show $EFk = Eh$

'reverse arrows'

Define $f := (\alpha_C)^{-1} \circ (Eh) \circ \alpha_C$

w.t.s $Ff = F((\alpha_C)^{-1}) \circ FEh \circ F(\alpha_C) = h$

$\cancel{\Leftrightarrow FEh \circ F(\alpha_C) = F(\alpha_{C'}) \circ h}$

But since E is faithful, $E(F(f)) = E(h) \Rightarrow F(f) = h$

③ F is essentially surjective. $D \in \mathbb{D}_0$, we have $D \xrightarrow{\cong} F(ED)$

Choose C in essentially surjective statement to be ED

($\Leftarrow$) Assume $F: \mathbb{C} \to \mathbb{D}$ satisfies ①, ②, ③.

Define E as follows: Using Axiom of Choice and essential surjectivity,

$\mathbb{D} \to \mathbb{C}$ Choose an object $ED \in \mathbb{C}_0$ and an iso

we know that at least one exists

$\beta_D : D \xrightarrow{\cong} FED$ for each $D \in \mathbb{D}$

This fixes action of E on objects.

Given $g: D \to D'$, want to construct $Eg: ED \to ED'$ ($ED \in \mathcal{C}$)

$$\begin{array}{ccc} D & \xrightarrow{g} & D' \\ B_D \downarrow & ED \xrightarrow{k} ED' & \downarrow B_{D'} \\ FED & \xrightarrow[FEg]{} & FED' \end{array}$$

EFC ($\mathcal{C}$)

We know that $FEg = (B_{D'}) \circ g \circ B_D^{-1}$

Since F is fully faithful, $\exists! \, k: ED \to ED'$ with $F(k) = B_{D'} \circ g \circ (B_D)^{-1}$

Define $E(g) = k$.

(faithfulness precedes fullness)

Faithful + Full

$\Downarrow$

$F_{C,C'}: \hom_{\mathcal{C}}(C,C') \to \hom_{\mathcal{D}}(FC, FC')$ is a bijection $\forall C, C'$

Given $g: D \to D'$, define $E(g) = (F_{ED,ED'})^{-1}(B_{D'} \circ g \circ B_D^{-1})$

It remains to show that E is a functor (identity + morphism)

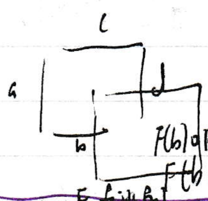
[$1_D \mapsto FE(1_D) = B_D \circ 1_D \circ B_D^{-1} = 1_D$
$\Rightarrow E 1_D = 1_C$ etc.]

and that we can find $\beta: 1_{\mathcal{D}} \to FE$ ✓ and $\alpha: 1_{\mathcal{C}} \to E \circ F$.

$$\begin{array}{ccc} C & \xrightarrow{\alpha_C} & EF(C) \\ \downarrow & \beta_{FC} & \downarrow \\ F(C) & \longrightarrow & FEF(C) \end{array}$$

(define $\alpha_C = (F_{C,EFC})^{-1}(\beta_{FC})$)

α_C is isomorphism



w.T.S

$$\begin{array}{ccc} C & \to & C' \\ \downarrow & & \downarrow \alpha_{C'} \\ EFC & \to & EFC' \end{array}$$ commutes

$F(b) \circ F(a) = F(d) \circ F(c)$

$F(b \circ a) = F(d \circ c)$

F faithful

$b \circ a = d \circ c$

More examples

(1) Stone Duality

Boolean Algebra ✓

$(\text{Fin})\text{Set}^{op} \to (\text{Fin})\,BA$

all finite meets and join

$(B, \wedge, \top, \vee, \bot, \neg: B^{op} \to B)$

$$\begin{array}{ccc} A & \longmapsto & P(A) \\ f \downarrow & & \uparrow f^{-1} \\ B & \longmapsto & P(B) \end{array}$$

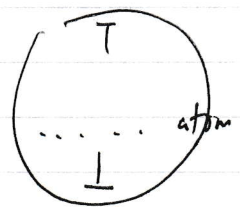
all powersets are posets

$b \vee \neg b = \top$

$b \wedge \neg b = \bot$

Poset!

$f^{-1}: PB \to PA$, $f^{-1}(V) = \{a \mid fa \in V\}$



$$\text{FinSet}^{op} \xleftrightarrow[\simeq]{} \text{FinBA}$$

$\downarrow$ filtered colimit completion

'finitely presented' Boolean Algebra corresponds to finite BA's

'profinite'

$$\text{ProFinSet}^{op} \xleftrightarrow{\simeq} BA$$

'clopen' subsets

$\simeq$

Stone space^{op} · totally disconnected
· compact Hausdorff space

Cantor space is an example of Stone spaces...

(2) $(-)^* : \text{FinVect}^{op} \xrightarrow{\simeq} \text{FinVect}$

$$V \longmapsto V^* = \text{Lin}(V, \mathbb{R})$$

$$V^{**} \simeq 1$$

(3) Gelfand Duality.

Let X be a compact Hausdorff topological space

$$(\underbrace{C(X,\mathbb{C}), \|\cdot\|_\infty}_{\text{Banach Space}}, \cdot, \underset{\substack{\text{involutive}\\ \text{conjugation}}}{(\ )^*}) \text{ is called a } C^*\text{-Algebra.}$$

$$\underset{\text{(geometry!!)}}{\text{CHaus}} \xleftrightarrow{op} \text{Comm-}C^*\text{-Alg} \subseteq (C^*\text{-Alg})$$

Given a preorder $(A, \le)$, we can take its 'poset reflection'

$$\begin{tikzcd} (A,\le) \arrow[r, "p"] & (A/\equiv, \le) \arrow[l, bend left, dashed, "q"] \end{tikzcd}$$

$a \equiv b \iff a \le b,\ a \ge b$
i.e. $a \cong b$

category of preordered sets

Viewing preordered sets as category $\underline{(A,\le)} \xrightarrow{\bar{p}} \underline{(A/\equiv,\le)}$ 'p-bar'

$\circlearrowright$ set $\quad p \circ q = 1, \quad (q \circ p)\, a \equiv a,$

$1 \cong q \circ p$

Observation: Cat's cannot be "quotiented by iso".

e.g. $\overset{\text{preorder}}{m \le n} \iff m \mid n$, $\quad 1 \mid -1, \quad -1 \mid 1$ but $-1 \neq 1$

$[m]=[n] \iff |m| = |n|$

$\bar{p}$ is an essentially surjective functor...

w.T.s $g \circ f = 1_A$

$\langle f, 1\rangle$ is split mono $\therefore A$ is initial

by Jc

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Yoneda Lemma

For $\mathbb{C}$ locally small, $\hom(-,-) : \mathbb{C}^{op} \times \mathbb{C} \to Set$, we have derived the following functors

'dual Yoneda embedding' $Z : \mathbb{C}^{op} \to Set^{\mathbb{C}}$

(Yoneda Embedding) $Y : \mathbb{C} \to Set^{\mathbb{C}^{op}}$

$*$ $Y = \mathbb{C} \to Fun(\mathbb{C}^{op}, Set) = \mathbb{C} \to [\mathbb{C}^{op}, Set]$ are alternate notations.

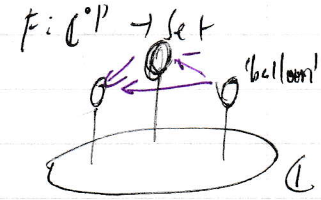
$\left(= \hat{\mathbb{C}} \text{ category of } \underline{\text{presheaves}} \text{ on } \mathbb{C} \right)$

A presheaf is a contravariant functor $F : \mathbb{C}^{op} \to Set$ but we think of functors as objects in $Fun(\cdot,\cdot)$

e.g. $G : \mathbb{C} \to Set$ is a presheaf on $\mathbb{C}^{op}$

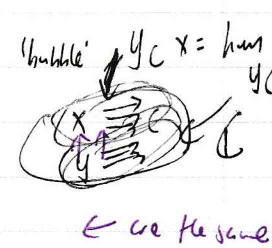
$Y : \mathbb{C} \to [\mathbb{C}^{op}, Set]$

$$\begin{array}{ccc} C & \mapsto & YC = \hom(-,C) \\ f \big\downarrow & & \big\downarrow Yf = \hom(-,f) \\ D & \mapsto & YD = \hom(-,D) \end{array}$$



$\hom(X,f) = (Yf)_X : \hom(X,C) \to \hom(X,D)$

$\hom(X,f)(h) = f \circ h$



Statement of <u>Yoneda Lemma</u>. It is about natural transformations

$$y_C \to F \quad \text{for } F : \mathbb{C}^{op} \to Set,\ C \in \mathbb{C}_0.$$

View as 'generalized elements'

fruitful for representable functors (on RHS)

Given $\underset{\text{eta}}{\overset{\text{N.T.}}{\eta}} : \overset{\text{functor}}{y_C} \to F$

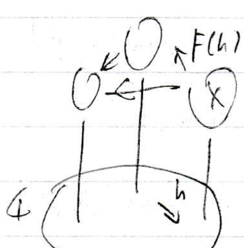
$$\eta_C : y_C\, C \to FC$$

$$\text{i.e. } hom(C, C) \to FC$$

'cheat' $\hat{\eta} \overset{\text{always exists!}}{=} \eta_C(1_C) \in FC$ i.e. an object in some set

> **Lemma:** Let $\mathbb{C}$ be locally small, $C \in \mathbb{C}_0$, $F : \mathbb{C}^{op} \to Set$. Then the function
>
> $$hom_{\hat{\mathbb{C}}}(yC, F) \to FC \quad \text{is a bijection.}$$
>
> homset in functor category   same set
>
> $$\eta \longmapsto \hat{\eta}$$

Proof. We want to construct an inverse $FC \to hom_{\hat{\mathbb{C}}}(yC, F)$



Define $x \mapsto \hat{x} : yC \xrightarrow{\text{natural transformation}} F$, where

natural transformation $\hat{x}_D : yC\,D = hom(D, C) \to FD$

$$\hat{x}_D(h : D \to C) = F(h)(x)$$

First, note that $\hat{x} : y_C \to F$ is natural: we have the following square

$$\begin{tikzcd}
 & h \circ f \arrow[r, mapsto] & F(h\circ f)(x) & \\
D \arrow[d, "f"'] & \hom(D,C) \arrow[r] & F(D) & \\
E & \hom(E,C) \arrow[u, "\hom(f,C)"'] \arrow[r] & F(E) \arrow[u, "Ff"'] & (F(f)\circ F(h))(x) \\
 & h \arrow[r, squiggly, mapsto] & F(h)(x) &
\end{tikzcd}$$

Next we show that the two mappings are inverses

- First, Given $x \in FC$, $\check{(\hat{x})} = (\hat{x}_C)(1_C) = \underbrace{F(1_C)}_{=1_{FC}} x = x$

- On the other hand, given $\eta : y_C \to F$, want to show $\eta = \hat{(\check{\eta})}$

$$\Leftrightarrow \forall D,\ \eta_D = (\hat{\check{\eta}}) : \hom(D,C) \to FC$$

$$\Leftrightarrow \forall D,\ (h : D \to C),\quad \eta_D(h) = (\hat{\check{\eta}})_D(h)$$

$$RHS = (\hat{\check{\eta}})_D(h) = F(h)(\check{\eta}) = F(h)\,(\eta_C(1_C)) \overset{?}{=} \eta_D(h)$$

where for the last step (?) we used the square

$$\begin{tikzcd}
y_C \arrow[d, "\eta"'] & \hom(D,C) \arrow[d] & \hom(C,C) \arrow[l, "\hom(h,D)"] \arrow[d] \\
F & FD & FC \arrow[l]
\end{tikzcd}$$

$$1_C \overset{D \xrightarrow{h} C}{\longmapsto} h \qquad \eta_C(1_C) \longmapsto \eta_D(h) = F(h)(\eta_C(1_C))$$

□

$\mathrm{Fun}(\mathbb{C}^{op}, \mathrm{Set})$

$$\boxed{\text{Corollary: } \overset{(Y)}{y} : \mathbb{C} \to \hat{\mathbb{C}} \text{ is full and faithful}}$$

Proof: Have to show $\hom_{\mathbb{C}}(C,D) \to \hom_{\hat{\mathbb{C}}}(y_C, y_D)$ is a bijection $\forall\, C,D \in \mathbb{C}_0$

Instantiate Yoneda lemma with $F = y_D$

$$\Rightarrow FC \to \hom_{\hat{\mathbb{C}}}(y_C, F) \quad \text{is a bijection}$$

i.e. $\hom(C,D) = y_D C \to \hom_{\hat{\mathbb{C}}}(y_C, y_D)$

$$h \longmapsto \hat{h}$$

Enough to show $(h \mapsto \hat{h})$ & $(h \mapsto Yh)$ coincide, i.e. $\forall\, h : C \to D$,

(NT) $\underline{\hat{h} \overset{\text{w.t.s}}{=} \hom(-,h)} : \hom(-,C) \to \hom(-,D)$

Let $X \in \mathbb{C}_0,\ k : X \to C$ $\qquad$ $* \, y_D = \hom(-,D)$

On the LHS, $(\hat{h}_X)(k) = y_D\, k\, h$

$= \hom(k, D)(h) = h \circ k$ $\quad$ added this!

On the RHS, $\hom(X,h)(k) = h \circ k$.

added this!

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Recall: Given hom: $\mathbb{C}^{op}\times\mathbb{C}\to Set$, and its 'left curry' $y:\mathbb{C}\to Set^{\mathbb{C}^{op}}$

For $F:\mathbb{C}^{op}\to Set$ and $A\in\mathbb{C}_0$,

(Yoneda lemma) $FA\cong hom(yA, F)$

$\eta_A(1_A) \leftarrow\!\shortmid\ (\eta: y_A\to F)$

(Corollary) $y:\mathbb{C}\to\hat{\mathbb{C}}$ is full and faithful.

When $(A,\le)$ is a pre-order,

$(\underline{A,\le})^{op}\times(\underline{A,\le})\to Set \supseteq \{\overset{\text{empty}}{0}<\overset{\text{singleton}}{1}\}$ poset of truth values $[p<q \iff p\to q]$ here, 'false' implies 'true'

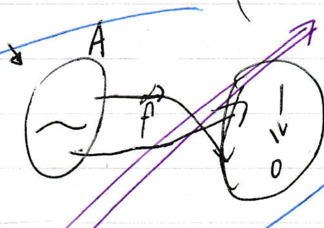
$(a,b)\longmapsto\begin{cases}\{*\} & \text{if } a\le b\\ \emptyset & \text{else}\end{cases}$

Remark: Preord $\xhookrightarrow{\text{inclusion}}$ Cat preserves exponentials

Henceforth we omit the underlines.

hom: $(A,\le)^{op}\times(A,\le)\to\{0<1\}=\mathbb{2}$

By transposition, we have $(A,\le)\to\mathbb{2}^{(A,\le)^{op}}$ $\quad (f: A^{op}\to\mathbb{2}) \mapsto (f^{-1}(\{1\}))$



pointwise ordering $f\le g \iff \forall x\in X, f(x)\le g(x)$

$\mathbb{2}^{A^{op}}\cong(\{\text{lower sets in } A\},\subseteq)$

Intuition: preimage of $\{1\}$ must be the lower set to satisfying monotonicity requirements

$\hookrightarrow U\subseteq A$ such that $\forall a\in U, b\in A$, if $b\le a$ then $b\in U$

$\boxed{y:(A,\le)\to low(A)}$

$a\longmapsto\ \downarrow\{a\}=\{b\in A\mid b\le a\}$

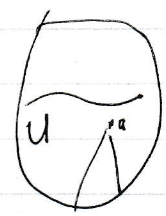
('down closure') smallest lower set containing 'a' as an element

The 'regular' Yoneda lemma tells us

lower set

$$\hom(\mathcal{Y}A, F) \cong F(A) \qquad U \subseteq_{low} A$$

(actually not used!) $\boxed{\downarrow\{a\} \subseteq U \iff a \in U}$



By corollary, $\exists$ bijection between A and $low(A)$ ordered by inclusion

$\to \quad a \le b \iff \downarrow\{a\} \subseteq \downarrow\{b\}$

$\Leftarrow$

Representability (roughly as expressive as 'limits')

For $\mathbb{C}$ locally small, $A \in \mathbb{C}_0$, we call $\hom(-,A) = \mathcal{Y}A$ the contravariant hom functor.

Def: $F: \mathbb{C}^{op} \to Set$ is called representable if it is isomorphic to one of the form $\mathcal{Y}A$, i.e. if there exists an object A and an iso $f: \mathcal{Y}A \xrightarrow{\cong} F$.

Equivalently, F is representable iff $\exists A \in \mathbb{C}_0, x \in FA$ s.t.

$\hat{x}: \mathcal{Y}A \to F$ is an iso (recall $\hat{x}$ is N.T. 'generated' by x. If there is an iso, by Yoneda it must be generated by some x).

In this case x is called ~~an~~ a universal element of F

(How to 'Terminal Object': (D, y) universal element)

Many category-theoretic concepts can be reformulated in terms of representability.

'A locally small cat $\mathbb{C}$ has a terminal object (1) iff $F: \mathbb{C}^{op} \to Set$, $A \mapsto \{*\}$ is representable ($\cong \hom(-,1)$)'

This translates to $|\hom(X,1)| = |\hom(X,\{*\})| = 1$.

- Given $A, B \in \mathcal{C}_0$ there exists a product of A & B if and only if

$$G : \mathcal{C}^{op} \to Set$$
$$X \longmapsto hom(X,A) \times hom(X,B)$$

is representable. Here we want it to be isomorphic to $hom(-,P)$, i.e. $\forall X \in \mathcal{C}$

$$hom(X,P) \overset{\Phi}{\cong} hom(X,A) \times hom(X,B)$$

* - Universal element in $G(P) = hom(P,A) \times hom(P,B)$ is precisely the pair of projections. P here is the product.

- If $\mathcal{C}$ has binary products, then an exponential of $A, B \in \mathcal{C}_0$ exists if and only if $\begin{cases} H : \mathcal{C}^{op} \to Set \\ H(X) = hom(X \times A, B) \end{cases}$ is representable.

$$\underset{\tilde{f}}{hom(X, E)} \cong \underset{f}{hom(X \times A, B)}$$

Identify E with B^A

$\varepsilon : B^A \times A \to B \in hom(B^A \times A, B) \cong hom(B^A, B^A)$

define properly...

- A free monoid on S exists iff

$$K : Mon \longrightarrow Set$$
$$M \longmapsto hom(S, U(M)) \text{ is representable.}$$

$$\begin{tikzcd} & S \arrow[d, "\text{def free monoid}"'] \arrow[dr] & \\ & UF(S) \arrow[r] & U(M) \\ & F(S) \arrow[r] & M \end{tikzcd}$$

$hom(S, U(M)) \cong hom(F(S), M)$

'functions' $\cup$ 'homomorphisms'

information is contained... review??

see p 56.

Applications of Yoneda Lemma

We know that $y : \mathcal{C} \to Set^{\mathcal{C}^{op}}$ full and faithful.

$$\Rightarrow \forall A, B \in \mathcal{C}_0, \quad \begin{array}{ccc} hom(A,B) & \xrightarrow{\cong} & hom(yA, yB) \\ \cup\!\! \mid & & \cup\!\! \mid \\ Iso(A,B) & \xrightarrow{\cong} & Iso(yA, yB) \end{array}$$

full + faithful reflects isomorphisms ("conservative", see there)

Yoneda Principle: $\boxed{A \cong B \text{ iff } yA \cong yB}$

i.e. (Iso(A,B) is not empty)

Examples

- Assume $\mathbb{C}$ Cartesian closed, $A, B, C \in \mathbb{C}_0$. We can give a new proof of

$$C^{A \times B} \cong (C^B)^A$$

Enough to show that $y(C^{A\times B}) \cong y((C^B)^A)$

We do this pointwise: $y(C^{A\times B})(X) = \hom(X, C^{A\times B})$

$$\cong \hom(X \times (A \times B), C)$$

functors preserve isos

$$\boxed{\begin{aligned} &\cong \hom((X\times A)\times B, C) \\ &\cong \hom(X \times A, C^B) \end{aligned}}$$

$$\cong \hom(X, (C^B)^A) = y((C^B)^A)(X)$$

Remark: These bijections are "natural in X". e.g.

e.g. $X \xleftarrow{f} Y$

$$\begin{tikzcd} h \in \hom(X\times A, C^B) \arrow[r] \arrow[d, "\sim"] & \hom(Y\times A, C^B) \ni h\circ(f\times 1_A) \arrow[d, "\sim"] \\ \tilde{h} \in \hom(X, (C^B)^A) \arrow[r, "{\hom(f,\, (C^B)^A)}"'] & \hom(Y, (C^B)^A) \ni \tilde{h}\circ f = \widetilde{h\circ(f\times 1_A)} \end{tikzcd}$$

Another Example

For A, B, C objects of cartesian-closed $\mathbb{C}$ with binary coproducts, we have $(A+B)\times C \cong A\times C + B\times C$

'distributivity'

Proof. We show that $\hom((A+B)\times C, -) \cong \hom(A\times B + B\times C, -)$

$$\hom((A+B)\times C, Y) \cong \hom(A+B, Y^C)$$

'non-constructive' → uses definition but no proof statement

$$\cong \hom(A, Y^C) \times \hom(B, Y^C) \quad \text{product category}$$

$\hom(A+B, X) \cong \hom(A, X) \times \hom(B, X)$

$$\cong \hom(A\times C, Y) \times \hom(B\times C, Y)$$

$$\cong \hom(A\times C + B\times C, Y)$$

$$\begin{tikzcd} A \arrow[d, "\sigma_1"] \arrow[dr, "f"] & \\ A+B \arrow[r, "{[f,g]}"] & X \\ B \arrow[u, "\sigma_2"'] \arrow[ur, "g"'] & \end{tikzcd}$$

$[f,g] \leftarrow\!\shortmid (f,g)$

$g \mapsto (g\circ\sigma_1, g\circ\sigma_2)$

Proposition. Let $\mathbb{C}$ and $\mathbb{D}$ be categories and assume that $\mathbb{D}$ has limits of type $\mathbb{J}$ ('index category'). Then $\mathrm{Fun}(\mathbb{C}, \mathbb{D})$ has limits of type $\mathbb{J}$, and they are computed "pointwise" (see below).

p68-69

Proof Assume that $\mathbb{D}$ is equipped with a choice $\lim(D)$ of limit for every diagram $D : \mathbb{J} \to \mathbb{D}$ 'functor'

① Def Then we get a limit functor — assigns each diagram its limit 'functor'

$$\lim : \mathbb{D}^{\mathbb{J}} \longrightarrow \mathbb{D} \qquad \mathrm{Fun}(\mathbb{J}, \mathbb{D})$$

$$D \longmapsto \lim(D)$$

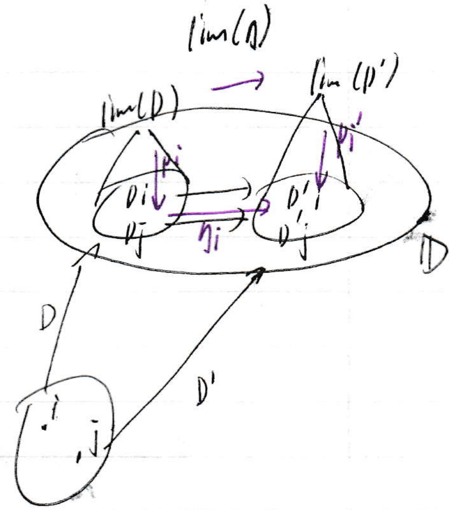
$$\eta \downarrow \qquad\qquad \downarrow \lim(\eta) \quad \text{(see definition on next page)}$$

$$D' \longmapsto \lim(D')$$

$\lim : \mathbb{D}^{\mathbb{J}} \to \mathbb{D}$

$\overset{\mathbb{J}\to\mathbb{D}}{D} \longmapsto \lim(D)$

$$\begin{array}{ccccc} D_i & & D & \longmapsto & \lim(D) \\ \downarrow \eta_i & & \eta \downarrow & & \downarrow \lim(\eta) \quad \mathbb{J} \\ D'_i & & D' & \longmapsto & \lim(D') \end{array}$$



$\lim(\eta)$ is the unique arrow such that

$\forall i \in \mathbb{J}, \quad p_i' \circ \lim(\eta) = \eta_i \circ p_i$

($\eta_i \circ p_i$ creates a new cone to D', but $\lim(D')$ is terminal)

(2) Actual construction

the limit of which we are after

$$\boxed{\mathbb{J} \xrightarrow{D} \mathbb{D}^{\mathbb{C}}}$$

$\cong$

$\mathbb{J} \times \mathbb{C} \xrightarrow{\bar{D}} \mathbb{D}$

CCCs...

$\cong$

$\mathbb{C} \xrightarrow{D^*} \mathbb{D}^{\mathbb{J}} \xrightarrow{\lim} \mathbb{D}$

Switch argument

$\boxed{\lim(D) = \lim \circ D^*} \in (\mathbb{D}^{\mathbb{C}})$.

'Functor preserves limits'

Pg 194-195 of Awodey $\underset{j\in J}{\lim}\,(D_j)(C) \cong \underset{j\in J}{\lim}\,(\underbrace{D_j C}_{\text{set}})$

$\lim (D_j) \in \mathrm{Set}^{\mathbb{C}^{op}}$, $\quad \mathbb{C}^{op} \to \mathrm{set}$

$\mathbb{J} \to \mathrm{Set}^{\mathbb{C}^{op}}$

$\mathbb{J} \times \mathbb{C}^{op} \to \mathrm{Set}$

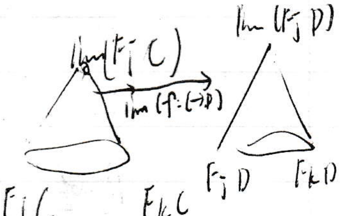
$\mathbb{C}^{op} \to \mathrm{Set}^{\mathbb{J}} \to \mathrm{Set}$

Limit in $\mathbb{D}^{\mathbb{C}}$ is a functor $\mathbb{C} \to \mathbb{D}$!

Then $f \in \mathbb{C}_1$ (from last topic) $D_j f$ is an arrow in $\mathbb{D}$

$\lim(D_j)(f) \cong \lim \circ \underbrace{D^*(f)}$

(from above) use the limit functor to get $\lim(\eta)$



need some arrow

$\mathrm{Hom}(X, \lim D_j^C) \cong$

$\mathbb{D} \cong$

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Recall, a presheaf category is ~~a~~ $\mathrm{Fun}(\mathbb{C}^{op}, \mathrm{set}) = \hat{\mathbb{C}} = [\mathbb{C}^{op}, \mathrm{set}] = \mathrm{set}^{\mathbb{C}^{op}}$

If $\mathbb{C}$ and $\mathbb{D}$ are arbitrary categories and $\mathbb{D}$ has limits of type $\mathbb{J}$, then $\mathbb{D}^{\mathbb{C}}$ has limits of type $\mathbb{J}$, and they are computed pointwise.

$\mathbb{J} \xrightarrow{D} \mathbb{D}^{\mathbb{C}}$ $\left[\begin{array}{l} \text{we have } \lim : \mathbb{D}^{\mathbb{J}} \to \mathbb{D} \\ \text{where } (D : \mathbb{J} \to \mathbb{D}) \mapsto \lim(D) \end{array}\right]$

We now want $\lim : \underline{(\mathbb{D}^{\mathbb{C}})^{\mathbb{J}}} \to \mathbb{D}^{\mathbb{C}}$ which we construct by:

~~We did this in p 119-120, basically~~

$$(\mathbb{D}^{\mathbb{J}})^{\mathbb{C}} \times \mathbb{C} \xrightarrow{\varepsilon} \mathbb{D}^{\mathbb{J}} \xrightarrow{\lim} \mathbb{D}$$

apply, take limit

$$\frac{(\mathbb{D}^{\mathbb{C}})^{\mathbb{J}} \times \mathbb{C} \to \mathbb{D}}{(\mathbb{D}^{\mathbb{C}})^{\mathbb{J}} \to \mathbb{D}^{\mathbb{C}}}$$

currying

$$\mathbb{D}^{\mathbb{C} \times \mathbb{J}} \cong (\mathbb{D}^{\mathbb{C}})^{\mathbb{J}} \cong (\mathbb{D}^{\mathbb{J}})^{\mathbb{C}}$$

$\lim(D) = \lim_{j \in \mathbb{J}} D_j : \mathbb{C} \to \mathbb{D}$, given by

$\mathbb{J} \underset{D'}{\overset{D}{\Downarrow \eta}} \mathbb{D}^{\mathbb{C}}$ where

$$\boxed{\left(\lim_{j \in \mathbb{J}} D_j\right)(C) = \lim_{j \in \mathbb{J}} \underbrace{(D_j(C))}_{\in \mathbb{D}}}$$

$$\left(\lim_{j \in \mathbb{J}} D_j\right)(C) \xrightarrow{\lim(\eta)_C} \left(\lim_{j \in \mathbb{J}} D'_j\right)(C)$$

def

$$\begin{array}{ccc} \lim_j (D_j C) & \xrightarrow{\lim(\eta)_C} & \lim(D'_j C) \\ \downarrow p_j & & \downarrow p'_j \\ D_j C & \xrightarrow{\eta_j C} & D'_j C \end{array}$$

defined to be unique arrow that makes sq. ever commute

Claim: $\boxed{[\mathbb{C}^{op}, \mathrm{set}] \text{ is cartesian closed whenever } \mathbb{C} \text{ is small.}}$

Let $G, H : \mathbb{C}^{op} \to \mathrm{set}$. What is H^G?
functors

By Yoneda lemma, $(H^G)(C) \cong \hom(yC, H^G) \overset{ccc}{\cong} \hom(yC \times G, H)$
if the latter exists

This allows us to "guess" the exponentials

Fun$(\mathcal{C}^{op}, set)$ is locally small if $\mathcal{C}$ is small (p99?)
$\Leftrightarrow$ between functors F and $G: \mathcal{C}^{op} \to set$, $\hom(F, G)$ is a set

Based on our intuition,

Define $H^G = \hom(y(-) \times G, H): \mathcal{C}^{op} \to set$

$$\begin{tikzcd} C \arrow[d, "f"'] & \longmapsto & \hom(yC \times G, H) \\ D & \longmapsto & \hom(yD \times G, H) \arrow[u, "\hom(yf \times 1_G, H)"'] \end{tikzcd}$$

(Eval)

Define NT $\varepsilon: H^G \times G \to H$

$$\varepsilon_C: (H^G)(C) \times G(C) \to H(C) \quad \text{a set}$$

$$\parallel$$

$$\hom_{\hat{\mathcal{C}}}(yC \times G, H) \times G(C) \to H(C)$$

where $(\underset{\sim}{\eta}: yC \times G \to H, \underset{\sim}{x} \in GC) \mapsto \eta_C(\underbrace{1_C}, x) \in H(C)$

N.T. see def 'Czech' operator

$$\begin{tikzcd} C & \hom(yC \times G, H) \times GC \arrow[r, "\varepsilon_C"] \arrow[d, "\hom(yf \times 1_G, H) \times Gf"] & HC \arrow[d] \\ D \arrow[u, "f"] & \hom(yD \times G, H) \times GD \arrow[r, "\varepsilon_D"] & HD \end{tikzcd}$$

is a set

$$(\eta, x) \longmapsto \eta_C(1_C, x)$$

$$(\eta \circ (yf \times 1), Gf\,x) \longmapsto (\eta \circ (yf \times 1))_D(1_D, Gf\,x)$$

$$Hf(\eta_C(1_C, x)) \overset{?}{=} \underbrace{(\eta \circ (yf \times 1))_D}(1_D, Gf\,x)$$

$$\eta_D(f, Gf\,x)$$

equal by naturality of η

$$\hom(-, C) \times G-$$

$$\begin{tikzcd} \underset{C \to C}{1_C}, \underset{\in GC}{x} \arrow[r] \arrow[d, "\eta_C"] & (\underset{D \to C}{f}, \underset{\in GD}{Gf\,x}) \arrow[d, "\eta_D"] \\ H(C) \arrow[r, "Hf"] & H(\;) \end{tikzcd}$$

(Currying)

Given $\overset{(N.T.)}{\phi}: F \times G \to H: \mathcal{C}^{op} \to set$, we need to define

N.T. $\tilde{\phi}: \overset{\text{functor}}{F} \to H^G$.

'specialization to C in F' Let $C \in \mathcal{C}$, $\tilde{\phi}_C: FC \to \hom(yC \times G, H)$

$\hookrightarrow$ Let $x \in FC$ $\quad \tilde{\phi}_C(x): yC \times G \to H$ $\quad$ LHS is a NT.

$\hookrightarrow$ Let $D \in \mathcal{C}$ $\quad (\tilde{\phi}_C(x))_D: \hom(D, C) \times GD \to HD$ $\quad$ this is another specialization in G now we get a regular morphism between objects

$h: D \to C$

$\hookrightarrow$ let $h: D \to C$, $y \in D$, $(\tilde{\phi}_C(x))_D(h, y) = \phi_D(\tilde{F}h x, y)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \in D \qquad \in FD \quad \in D$

$x \in FC$, so need h to 'link things together'

Applications: Simplicial sets $[\Delta^{op}, \text{set}] = \text{Sset}$

Category Homotopy Theory

~~Clifford Logic~~? (not sure what was said)

Higher CT?

(cs) Separation logic (presheaves on monoidal categories)

Take $x \in FD$, $y \in ED$. Can show

$\varepsilon \circ (\tilde{\phi} \times 1_C)$

$= \phi_D(x, y)$

$\eta_D(1_D, y) = (\tilde{\phi}_D(x))_D(1_D, y)$

ADJUNCTIONS

Given forgetful functor $U: \underline{\text{Mon}} \to \underline{\text{set}}$

A free monoid on $S \in \underline{\text{set}}$ is (up to iso) $\longrightarrow$ Given by $FS \in \text{Mon}$, $\eta_S: S \to UFS$ such that $\forall M \in \text{Mon}$, $\hom(FS, M) \to \hom(S, UM)$, $g \mapsto Ug \circ \eta_S$ is a bijection.

set:

$$\begin{tikzcd} S \arrow[d, "\eta_S"'] \arrow[dr, "f"] & \\ UFS \arrow[r, dashed, "U\hat{f}"] & UM \end{tikzcd}$$

'list' $FS \in \underline{\text{Mon}} \dashrightarrow M$, $\hat{f}$

Unique $\hat{f}$ such that $U\hat{f}$ completes the triangle

We get a functor $F: \text{set} \to \text{Mon}$

$$\begin{tikzcd} S \arrow[r, mapsto] \arrow[d, "f"'] & FS \arrow[d, "\widehat{\eta_T \circ f}"] \\ T \arrow[r, mapsto] & FT \end{tikzcd}$$

($\wedge$ removes U on RHS, adds F on LHS)

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\eta_S"'] \arrow[dr, dashed] & T \arrow[d, "\eta_T"] \\ UFS & UFT \\ FS \arrow[r, "\widehat{\eta_T \circ f}"] & FT \end{tikzcd}$$

The maps η_T give a NT

$$\eta : 1_{Set} \to U \circ F$$

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\eta_S"'] & T \arrow[d, "\eta_T"] \\ UFS \arrow[r, "U(\widehat{\eta_T \circ f})"'] & UFT \end{tikzcd}$$

<u>Preliminary definition of adjunctions</u>

An adjunction between categories $\mathbb{A}$, $\mathbb{B}$ consists of

- functors $F: \mathbb{A} \to \mathbb{B}$, $U: \mathbb{B} \to \mathbb{A}$
- a NT $\eta: 1_{\mathbb{A}} \longrightarrow U \circ F$ such that for all $A \in \mathbb{A}$, $B \in \mathbb{B}$, the function

$$\varphi_{A,B}: \begin{array}{ccc} \hom_{\mathbb{B}}(FA, B) & \to & \hom_{\mathbb{A}}(A, UB) \\ \underset{: B \to B}{g} & \mapsto & U_g \circ \eta_A \end{array}$$ is a bijection.

<u>Terminology</u>

- F is left adjoint to U.
- U is right adjoint to F.
- η is the unit of the adjunction
- $F \underset{\text{"is left adjoint to"}}{\dashv} U$

i.e. $(- \times B)$ ← can put anything here in place of A

It turns out that φ is natural in $A \times B$:

The φ_{AB} constitute a NT of type

$$\varphi : \hom_{\mathbb{B}}(F-,-) \longrightarrow \hom_{\mathbb{A}}(-,U-) : \mathbb{A}^{op} \times \mathbb{B} \to \mathrm{Set}$$

$h : FA \to B \longmapsto Uh \circ \eta_A : A \to UB$

$Uh : UFA \to UFB$ (object)

$$\begin{array}{ccc} \hom_{\mathbb{B}}(FA,B) & \longrightarrow & \hom(A,UB) \\ \downarrow & & \downarrow \\ \hom(FA',B') & \longrightarrow & \hom(A',UB') \end{array} \qquad \begin{array}{cc} A & B \\ \uparrow f & \downarrow g \\ A' & B' \end{array}$$

pre- and post-composed

$Ug \circ Uh \circ \eta_A \circ f$

$\overset{?}{=}$

$g \circ h \circ Ff \longmapsto Ug \circ Uh \circ UFf \circ \eta_{A'}$

ok: $\begin{array}{ccc} \cdot & \xrightarrow{f} & \cdot \\ \eta_{A'} \downarrow & & \downarrow \eta_A \\ \cdot & \xrightarrow[UFf]{} & \cdot \end{array}$

Def: (Actual) An adjunction between $\mathbb{A}$ & $\mathbb{B}$ is given by

- functors $F : \mathbb{A} \to \mathbb{B}$, $U : \mathbb{B} \to \mathbb{A}$
- natural iso $\varphi : \hom(F-,-) \cong \hom(-,U-)$ ('left' / 'right'; $\uparrow_A$ $\uparrow_B$)

(we don't need η explicitly) $1_A \to U \circ F$

① Example: let $\mathbb{C}$ be a Cartesian closed category.

For $A, B, C \in \mathbb{C}$, $\hom(\underbrace{A \times B}_{F(A)}, C) \cong \hom(A, \underbrace{C^B}_{U(C)})$ is natural in A and C.

$$\begin{array}{ccc} \hom(A \times B, C) & \xrightarrow{(\sim)} & \hom(A, C^B) \\ \downarrow \hom(f \times 1, g) & & \downarrow \hom(f, g^B) \\ \hom(A' \times B, C') & \to & \hom(A', C'^B) \end{array}$$

This means that $F = (- \times B) : \mathbb{C} \to \mathbb{C}$ is left adjoint to $U = (-)^B : \mathbb{C} \to \mathbb{C}$

unit ~ counit

equivalent categories

② Again in CCC $\mathbb{C}$, $\hom_{\mathbb{C}}(A, R^B)$ ← $R^{(-)} : \mathbb{C}^{op} \to \mathbb{C}$ is right adjoint to $R^{(-)} : \mathbb{C} \to \mathbb{C}^{op}$

$\cong \hom_{\mathbb{C}}(A \times B, R)$

product comm. up to isomorphism

$\cong \hom_{\mathbb{C}}(B \times A, R)$

$\cong \hom_{\mathbb{C}}(B, R^A)$

$\cong \hom_{\mathbb{C}^{op}}(R^A, B)$ ←

$\times \quad 1 \quad (-)^{(-)}$

$\wedge \quad \top \quad \Rightarrow$ Propositional Intuitionistic Logic

$R^{(A)} \equiv A \Rightarrow R$ (generalized negation) $A \Rightarrow \bot$

→ continuation monads

R : response type

CPS

(3) Assume $\mathbb{C}$ has binary products.

$$\begin{tikzcd} & & A \\ X \arrow[urr] \arrow[r, dashed] \arrow[drr] & & A\times B \arrow[u] \arrow[d] \\ & & B \end{tikzcd}$$

Then $\hom_{\mathbb{C}}(X, A\times B) \cong \hom_{\mathbb{C}}(X,A) \times \hom_{\mathbb{C}}(X,B)$

right $\quad U(\,)$

$\varphi : \hom(F-,-) \cong \hom(-,U-)$

$$\underbrace{\hspace{4cm}}_{\text{switching to product categories}}$$

$$\hom_{\mathbb{C}\times\mathbb{C}}((X,X),(A,B)) = \hom_{\mathbb{C}\times\mathbb{C}}(\underset{\text{left}}{\Delta X}, (A,B))$$

where $\underset{\text{'F'}}{\Delta} : \mathbb{C} \to \mathbb{C}\times\mathbb{C}$

$A \mapsto (A,A)$

Then $(-\times-) : \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ is right adjoint to $\Delta : \mathbb{C} \to \mathbb{C}\times\mathbb{C}$

More generally, a category $\mathbb{C}$ has all limits of type $\mathbb{J}$ iff the diagonal functor $\Delta_{\mathbb{J}} : \mathbb{C} \to \mathbb{C}^{\mathbb{J}}$ (transpose of $\mathbb{C}\times\mathbb{J} \xrightarrow{\pi_1} \mathbb{C}$) has a right adjoint.

$\mathbb{J} = (\cdot \quad \cdot)$

$\Delta_{\mathbb{J}}(C) \in \mathrm{Fun}(\mathbb{J}, \mathbb{C})$

For $j \in \mathbb{J}$, map to same C

Dually, $\mathbb{C}$ has all colimits of type $\mathbb{J}$ iff $\Delta_{\mathbb{J}}$ has a left adjoint.

Note that ~~Recall~~: $\mathbb{D} : \mathbb{J} \to \mathbb{C}$ has a limit iff $\hom_{\mathbb{C}^{\mathbb{J}}}(\Delta -, D)$ is representable.

In particular, product of A and B exists iff $\hom(-,A) \times \hom(-,B)$ is representable.

('more granular': existence of individual products)

Assume $F: \mathbb{A} \to \mathbb{B}$, $U, U': \mathbb{B} \to \mathbb{A}$ such that $F \overset{\text{left adjoint}}{\dashv} U$ & $F \dashv U'$

i.e. $\varphi_{AB}: \hom_{\mathbb{B}}(FA, B) \xrightarrow{\cong} \hom_{\mathbb{A}}(A, UB)$

$\varphi'_{AB}: \hom_{\mathbb{B}}(FA, B) \xrightarrow{\cong} \hom_{\mathbb{A}}(A, U'B)$

are natural in A and B

Claim: Then $U \cong U'$. (exists natural isomorphism...)

Proof: $\hom(A, UB) \cong \hom(FA, B) \cong \hom(A, U'B)$

$\hom(A, UB) = y(UB)(A)$, $\hom(A, U'B) = y(U'B)(A)$

$\overset{\text{naturality in } A}{\Longrightarrow}$ $y(UB) \cong y(U'B)$

iff componentwise isomorphism

$\overset{\text{naturality in } B}{\Longrightarrow}$ $y \circ U \cong y \circ U'$

$\overset{\text{fully faithfulness of } y}{\Longrightarrow}$ $U \cong U'$

Remark (self duality): $F \dashv U: \mathbb{B} \to \mathbb{A}$ $\quad \hom_{\mathbb{B}}(FA, B) \cong \hom_{\mathbb{A}}(A, UB)$

$U^{op} \dashv F^{op}: \mathbb{A}^{op} \to \mathbb{B}^{op}$ $\quad \hom_{\mathbb{B}^{op}}(B, FA) \cong \hom_{\mathbb{A}^{op}}(UB, A)$

Prop TFAE for $U: \mathbb{B} \to \mathbb{A}$:

① U has a left adjoint 'free object'

② $\forall A \in \mathbb{A}$, $\exists FA \in \mathbb{B}$, $\eta_A: A \to UFA$ such that $\forall B \in \mathbb{B}$, $f: A \to UB$, $\exists! \tilde{f}: FA \to B$

$$\in \mathbb{A}: \begin{tikzcd} A \arrow[d, "\eta_A"'] \arrow[dr, "f"] & \\ UFA \arrow[r, "U\tilde{f}"'] & UB \end{tikzcd} \qquad U\tilde{f} \circ \eta_A = f$$

$$FA \xrightarrow{\tilde{f}} B$$

③ $\forall A \in \mathbb{A}$, the co-presheaf $\hom(A, U-): \mathbb{B} \to \mathbf{Set}$ is representable. 'local'

④ p.t.o!

④ For all A, $A \downarrow U$ has an initial object. Element of hom$(A, U-)$

objects of $A \downarrow U$: $(B \in \mathbb{B}, f: A \to UB)$

$\downarrow (h: B \to C)$

$(C \in \mathbb{B}, g: A \to UC)$

Morphisms:

$$\begin{tikzcd} & UB \arrow[d] & B \arrow[d] \\ A \arrow[ur, "f"] \arrow[r, "g"'] & UC & C \end{tikzcd}$$

such that the diagram commutes.

Example: $J: Pos \hookrightarrow Preord$ has a left adjoint.

By formulation ③, we have for all preorders A, $\exists$ poset FA

$\eta A: A \to FA \ldots$

$$\begin{tikzcd} A \arrow[d] \arrow[dr] & \\ FA \arrow[r, dashed] & B \end{tikzcd}$$

$A \in Preord$, $FA \in Pos$, $B \in Pos$ $(\subseteq Preord)$

F Quotients out by symmetric part (see HW5 Q2a)

inclusion

JFA $\qquad$ J is 'inclusion'

Def: A full subcategory $\mathbb{B} \hookrightarrow \mathbb{A}$ ('inclusion') is call (co)reflective if the inclusion has a left (right) adjoint.

More generally, a fully faithful functor $J: \mathbb{B} \to \mathbb{A}$ is called (co-)reflective inclusion if it has a left (right) adjoint.

Examples: T_0-spaces are reflective in topological spaces.

(any 2 points can be separated by open set)

$$\begin{tikzcd} X \arrow[d] \arrow[dr] & \\ X/\sim & Y \end{tikzcd}$$

$\mathbb{B}$

- Banach spaces are reflective in normed spaces

$$\begin{tikzcd} (V, \|\cdot\|) \arrow[d, "uc"'] \arrow[dr, "f"] & \\ (\bar{V}, \|\cdot\|) \arrow[r] & B \end{tikzcd}$$

Banach space $(\bar{V}, \|\cdot\|)$

* Cauchy sequences bring completeness

- Complete metric spaces are reflective in metric spaces of unif. continuous maps.

- The full inclusion $J: \mathrm{Set} \to \mathrm{Cat}$

$$I \mapsto \Delta(I) \quad (\text{discrete category, only identities})$$

is reflective AND co-reflective.

$$\begin{tikzcd} \mathbb{C} \arrow[d] \arrow[dr] & \\ \Delta(L\mathbb{C}) \arrow[r, "\in \mathrm{Cat}"] & \Delta S \end{tikzcd}$$

'left'

information is lost...

$$L\mathbb{C} \xrightarrow{\in \mathrm{Set}} S$$

$= \pi_0(\mathbb{C})$ connected components i.e. equivalence classes for equivalence relations generated by arrows

$$\begin{tikzcd} S \arrow[r, dashed, "\tilde{f}"] & R\mathbb{C} \arrow[d] \\ & \mathbb{C} \end{tikzcd}$$

'right' (set of objects)

$\Delta S \xrightarrow{f} \mathbb{C}$, $\Delta(\mathbb{C}_0) \to \mathbb{C}$

$$(- \times B) \dashv (-)^B$$

$$A \to C^B$$

$$\begin{tikzcd} A \times B \arrow[r, "(-\times B)"] \arrow[dr] & C^B \times B \arrow[d] \\ & C \end{tikzcd}$$

$$\underbrace{\pi_0 \dashv \Delta}_{\text{reflection}} \overbrace{\dashv (-)_0}^{\text{co-reflection}} : \mathrm{Cat} \to \mathrm{Set}$$

- Discrete spaces are reflective in locally connected spaces.

$\overset{\text{left}}{F} \dashv \overset{\text{right}}{U} : \text{Mon} \to \text{Set}$

$F \underbrace{U F}_{\text{list of lists} \ldots} S$ is not idempotent.

On the other hand, co-reflective adjunctions are always idempotent. ?

Lemma: Given any adjunction $F \dashv U : B \to A$
we have U fully faithful iff $\varepsilon : FU \xrightarrow{B \to B} 1_B$, (counit of the adjunction)
$\varepsilon_B = \varphi^{-1}_{UB, B}(1_{UB})$ is a natural isomorphism.

$\overset{\text{unit}}{\eta} : 1_A \to U \circ F$

$\eta_A : A \to UFA$

$\eta_A = \varphi_{A, FA}(1_{FA})$

$\hookrightarrow$ intuition
Things in subcategory are not changed by reflection.

(F, U, φ)

$\Updownarrow$

$(F, U, \eta, \varepsilon)$

$$\begin{tikzcd} FA \arrow[r, "F\eta_A"] \arrow[dr, "1"'] & FUFA \arrow[d, "\varepsilon_{FA}"] \\ & FA \end{tikzcd}$$

$$\begin{tikzcd} UB \arrow[r, "\eta_{UB}"] \arrow[dr, "1"'] & UFUB \arrow[d, "U\varepsilon_B"] \\ & UB \end{tikzcd}$$

$F : A \to B$

$U : B \to A$

$\eta : 1 \to UF$

$\varepsilon : FU \to 1$

$\Rightarrow F \dashv U$

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Consider $F: \mathbb{A} \to \mathbb{B}$

$U: \mathbb{B} \to \mathbb{A}$

'Pre order Categories' / "thin"

$$\varphi_{AB}: \hom_{\mathbb{B}}(FA, B) \xrightarrow{\cong} \hom_{\mathbb{A}}(A, UB)$$

one side non-empty iff other side non-empty

Consider monotone functions $f: A \to B$, $U: B \to A$ (posets)

They are adjoint iff $\forall a, b.\ fa \leq b \Leftrightarrow a \leq Ub$

E.g. $R \subseteq \mathbb{C}^n \times \mathbb{C}[x_1 \ldots x_n]$ (complex numbers)

'Galois connection' in algebraic geometry $R = \{(z_1 \ldots z_n), p[x_1 \ldots x_n] \mid p[z_1 \ldots z_n] = 0\}$

$$f: P(\mathbb{C}^n) \to P(\mathbb{C}[\vec{x}])$$

$$g: P(\mathbb{C}[\vec{x}]) \to P(\mathbb{C}^n)$$

'radical ideals' $f(U) := \{p[\vec{x}] \mid \forall \vec{z} \in U, (\vec{z}, p) \in R\}$

'joint zeros of polynomials' $g(V) = \{\vec{z} \mid \forall p \in V, (\vec{z}, p) \in R\}$

$$U \subseteq f(V) \Leftrightarrow \forall \vec{z} \in U, \forall p \in V, (\vec{z}, p) \in R \Leftrightarrow V \subseteq g(U)$$

$\boxed{\text{Theorem: Right adjoints preserve limits.}}$ (Dually, left adjoints preserve colimits)

$F \dashv U : \mathbb{B} \to \mathbb{A}$

Proof: let $D : \mathbb{J} \to \mathbb{B}$ be a diagram with a limit in $\mathbb{A}$.

Claim: $U(\lim(D))$ is a limit of $U \circ D : \mathbb{J} \to \mathbb{A}$

$$\hom(X, U(\lim_j D_j)) \overset{\text{adjoint}}{\cong} \hom(FX, \lim_j D_j)$$

Yoneda principle (Pg 118)

hom functors preserve limits? (p74) $\cong \lim_j \hom(FX, D_j)$

$$\cong \lim_j \hom(X, UD_j) \; [= \mathrm{cone}(X, U \circ D)]$$

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ UD_i & & UD_j \end{array}$$

set of cones with apex X

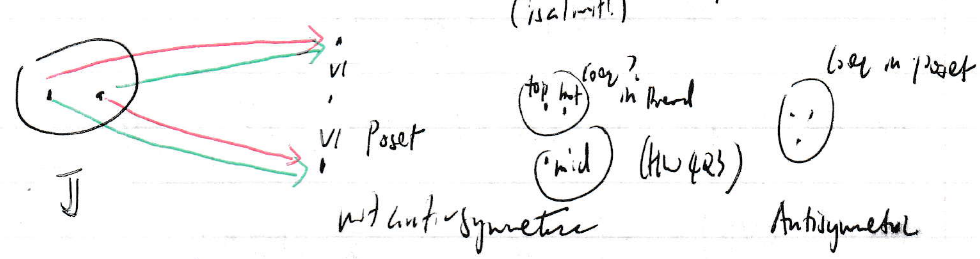
$$\cong \hom(X, \underbrace{\lim(UD_j)}_{\text{assume this exists}}) = \hom(X, \lim(U \circ D))$$

$U(\lim_j D_j)$ represents the presheaf of cones.

$Y(\cdot) = \hom(-, \cdot)$ $\quad \mathrm{Cone}(-, UD) : \mathbb{A}^{op} \to Set$

Corollary: Pos $\hookrightarrow$ Preord does not have a right adjoint.

Proof: WTS limits are not preserved: Coequalizers are not preserved. (is a limit!)



The coequalizer in Preord lumps top and bot together.

- Reflective subcategories are closed under limits. (up to iso)
- Coreflective subcategories are closed under colimits. (up to iso)

However, if $\mathbb{J} \to \mathbb{B} \underset{\perp}{\overset{F}{\hookrightarrow}} \mathbb{A}$ "left" is reflection and $\mathbb{D}: \mathbb{J} \to \mathbb{B}$ diagram which has a colimit A,

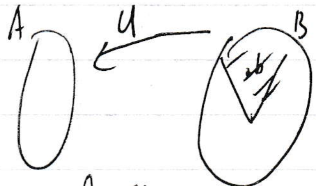
then it has a colimit in $\mathbb{A}$ given by $F(\operatorname{colim} D)$

? Are limit-preserving functors right adjoints?

$\boxed{\text{Prop: Let } B \text{ be a complete lattice (poset in which all infima exist) and let } U: B \to A \text{ be monotone. Then } U \text{ has a left adjoint iff it preserves all infima.}}$

Proof: Define $f: A \to B$, $f(a) = \inf\{b \mid a \le Ub\}$

Check $fa \le b \iff a \le UB$



$fa \le b$

Theorem (Freyd, General Adjoint Functor Theorem)

Assume $\mathbb{B}$ locally small with all small limits. Then $U: \mathbb{B} \to \mathbb{A}$ has a left adjoint iff U preserves small limits and satisfies the "solution set condition": $\forall A \in \mathbb{A}_0$, $\underbrace{A \downarrow U}_{?\text{ comma category}}$ has a weakly initial family.
(small)

Application: Haus $\to$ Top

(Def A family $(A_i)_{i \in I}$ of objects in $\mathbb{C}$ is called <u>weakly initial</u> if $\forall B \in \mathbb{C}_0$, $\exists i \in I$, $f: A_i \to B$ $\quad A_1 \to B$, A_2, $A_3 \ldots$)

MONAD:

A monad is a triple (T, η, μ)

$T : \mathcal{C} \to \mathcal{C}$
(endofunctor)

η (NT) $: 1_{\mathcal{C}} \to T$ (unit)

$\mu : T \circ T \to T$ (multiplication) such that

$$\begin{tikzcd} TA \arrow[r, "T\eta_A"] \arrow[dr, "1"'] & T^2A \arrow[d, "\mu_A"] & TA \arrow[l, "\eta_{TA}"'] \arrow[dl, "1"] \\ & TA & \end{tikzcd} \qquad \begin{tikzcd} T^3A \arrow[r, "\mu_{TA}"] \arrow[d, "T\mu_A"'] & T^2A \arrow[d, "\mu_A"] \\ T^2A \arrow[r, "\mu_A"'] & TA \end{tikzcd}$$

commutes.

Example (List Monad) $T : \mathrm{Set} \to \mathrm{Set}$

$T(A) = A^*$

$$A \xrightarrow{\eta_A} TA \xleftarrow{\mu_A} T^2A$$

$a \longmapsto [a]$

$\longleftarrow [l_1, \ldots l_n]$

concatenation

Any monad is a monoid
in the category of endofunctors

μ = composition

η = identity

T = object! ('set')

see Mac Lane...

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Given an adjunction $F \dashv U : \mathbb{B} \to \mathbb{A}$, we get a monad on $\mathbb{A}$

$T = U \circ F$

η is the unit of the monad $(\varepsilon_B : FUB \to B)$

$\mu_A = U(\varepsilon_{FA} : FUFA \to FA)$

$UFUFA \to UFA$

$T^2A \to TA$

Decomposing monads into adjunctions $(T : \mathbb{C} \to \mathbb{C})$: $T = (\mathbb{C} \xrightarrow{F_T} \mathbb{C}_T \xrightarrow{U_T} \mathbb{C})$ initial decomposition

$\mathbb{C}_T$ is Kleisli category

$T = (\mathbb{C} \xrightarrow{F^T} \mathbb{C}^T \xrightarrow{U^T} \mathbb{C})$ terminal decomposition

$\mathbb{C}^T$ = category of Eilenberg-Moore algebras

How is $\mathbb{C}_T$ defined?

free objects

$(\mathbb{C}_T)_0 = \mathbb{C}_0$

$\hom_{\mathbb{C}_T}(A, B) = \hom_{\mathbb{C}}(A, TB)$

$$\begin{tikzcd}
A \arrow[dr, "f"'] & B \arrow[dr, "g"] & C \\
& TB \arrow[r, "g^\dagger"'] \arrow[dr, "Tg"'] & TC \\
& & T^2C \arrow[u, "\mu_C"']
\end{tikzcd}$$

how to compose? $g \circ_T f = g^\dagger \circ f$ ('dagger')

$= \mu_C \circ Tg \circ f$

Identities in $\mathbb{C}_T$ given by η $(\eta_A : A \to TA)$

$F_T(A) = A$

$F_T(A \xrightarrow{f} B) = (\eta_B \circ f)$

$U_T(A) = TA$

$U_T(g) = g^\dagger$

Monads model "side effects" in CS.

printing / non-termination

$\mathcal{C}$ category of "types"

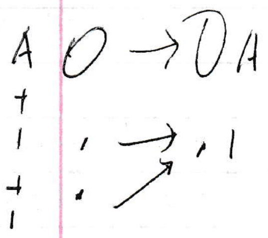
$$1 \xrightarrow[[[t]]]{} TA$$

t closed program of type A involving some side-effects

Examples • Maybe monad

$A \to B+1$

$B \to C+1$



$T: \text{Set} \to \text{Set}$

$T(A) = A + 1$ ← error

$\eta_A(a) = (0, a)$ (inclusion)

$\mu_A : A + 1 + 1 \to A + 1$ identifies the first two '1's '1'. (sends to 1)

$f \in \hom_{\text{Set}_T}(A, B)$

$f : A \to B + 1$

• $\underbrace{\mathbb{R}^{(-)} \dashv \mathbb{R}^{(-)}}_{\text{Some random Monad}}$

• State monad (S fixed set of "states") $\quad T(A) = \mathbb{R}^{(\mathbb{R}^A)}$

$T: \text{Set} \to \text{Set}$

$T(A) = (A \times S)^S$

$f : A \to_T B$

$f : A \to (B \times S)^S$

$\bar{f} : A \times S \to B \times S$

In general, $T(\cdot)$ adds more information in a 'pseudo' space.

Another example: $X \mapsto [[X, S], S]$

↑ object in category, S fixed (continuation)