21-355 Midterm I Cheatsheet

3/13/2021

- Distribute ε into $\varepsilon/2$ / $\varepsilon/3$ buckets and add (e.g. $(a_n - b_m)$ vs $(a_m - a)$)
- Topology → Use openness / closeness, squeeze into open balls $\mathbb{Q}$ vs $\mathbb{R}$.
- Concoct special subsequences, midpoints… go infinitely small or infinitely large
- Add and subtract

1. Totally ordered Field $(F, +, \cdot, 1, 0)$ + "$>$"

 - Abelian group w.r.t $(F, +, 0)$ $\qquad x<y \Rightarrow x+z<y+z$
 - Abelian group w.r.t $(F\setminus\{0\}, \cdot, 1)$ $\qquad x>0, y>0 \Rightarrow x\cdot y>0$
 - Distributivity

$\mathbb{Q}$ & $\mathbb{R}$ are both totally ordered. Furthermore $\mathbb{R}$ is complete (every C-S in $\mathbb{R}$ converges in $\mathbb{R}$)

2. Construction of Reals

 - Convergence $\Leftrightarrow \exists\, a$ s.t. $\forall \varepsilon>0, \exists N=N(\varepsilon)$ s.t. $\forall n\geq N, |a_n-a|<\varepsilon$.
 - Cauchy Sequence $\Leftrightarrow \forall \varepsilon>0, \exists m=m(\varepsilon)$ s.t. $\forall k,n\geq m, |a_n-a_k|<\varepsilon$.

- Every Convergent sequence is Cauchy. Convergent $\Rightarrow$ C-S.
- Every C-S is bounded. $|a_n|\leq M$.
- $\mathcal{C} = \{(a_n) \mid \forall n: a_n \in \mathbb{Q}, (a_n) \text{ is Cauchy}\}$, $\mathbb{R} := (\mathcal{C}/\sim)$, where $(a_n)\sim(b_n) \Leftrightarrow (a_n-b_n)\to 0$.

set of Cauchy sequences $\quad q\in\mathbb{Q} \xrightarrow{\text{embedding}} (q,q,q\ldots)\in\mathbb{R}$.

- $a\in\mathcal{C} > 0 \Leftrightarrow \begin{cases} a\not\sim 0 \\ \exists N \text{ s.t. } \forall n\geq N: a_n>0 \end{cases} \Leftrightarrow \exists\, \delta>0 \in \mathbb{Q}^+, \exists N$ s.t. $\forall n\geq N, a_n>\delta$.
- $a\sim\alpha, b\sim\beta \in \mathcal{C}$ $\Rightarrow (\alpha+\beta)\sim a+b$
 $\Rightarrow \alpha\beta \sim a\cdot b$
- $a>0, a\sim b \Rightarrow b>0$

$\mathbb{R}$ is complete (there are other ways to show this, construction-independently)

Given this construction of $\mathbb{R}$, we can show Every C-S in $\mathbb{R}$ has limit point in $\mathbb{R}$. (HW1-7) Furthermore this limit point is unique (HW1-1)

- If $S\subseteq\mathbb{R}$, S is bounded $\Rightarrow \exists\, \alpha\in\mathbb{R}, \alpha:=\sup(S)$

Archimedean Property

- $x,y\in\mathbb{R}, x>0 \Rightarrow \exists\, n\in\mathbb{N}^+, nx>y$ $\qquad$ $n_0=\sup\{n \mid nx<y\}$, $n:=n_0+1$ $\quad$ ($\forall x,y\in\mathbb{R}, \exists\, n_1,n_2\in\mathbb{N}$ s.t. $n_1>x, n_2<y$)
- $x,y\in\mathbb{R}, x<y \Rightarrow \exists\, q\in\mathbb{Q}, x<q<y$.

$k_0 = \sup\{k \mid \frac{k}{n}\leq x\}$ $\qquad$ w.t.s $x<\frac{k_0+1}{n}<y$

Topology

Special sets: · $\{\frac{1}{n} \mid n \in \mathbb{N}^+\}$ · no interior points, only limit point is 0.

· $\{F_n\}$, $F_n = (0, \frac{1}{n}]$ · every finite subcollection has non-empty intersection but $\cap \{F_n\} = \emptyset$

· $\{F_n\}$, $F_n = (n, \infty)$

· $(0,1)$ not compact! I give you $\{(0, \frac{1}{n}) \mid n \in \mathbb{N}\}$

· $\mathbb{Q}, \mathbb{R}$. $\mathbb{Q}^o \subsetneq (\overline{\mathbb{Q}})^o$

· $\cup F_\alpha' \subseteq (\cup F_\alpha)'$

Metric Space. $d: X \times X \to \mathbb{R}_0^+$ $\begin{cases} a = b \iff d(a,b) = 0 \\ d(a,b) = d(b,a) \\ \triangle\text{-inequality}, \ d(a,b) \le d(a,c) + d(b,c) \end{cases}$

$X \neq \emptyset$, $d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$ is the discrete metric

E' $(\forall r>0\ (B_r(x) \cap E \neq \emptyset))$ Limit Point $\not\Leftarrow$ Isolated point

$\overset{o}{E}$ $(\exists r>0\ (B_r(x) \subseteq E))$ Interior Point!

An interior point need not be a limit point. e.g. $\{0,1\}$ discrete metric, $E = \{0\}$, 0 is interior point but not limit point

– (limit points that are not interior points are everywhere

– Convergent (monotonically increasing) sequence $(a_n) \to \sup(a_n)$.

– x is limit point of $A \subseteq X \iff \exists$ sequence in A, $(a_n) \to x$.

Open: all points are interior

Closed: all limit points are inside

– $\emptyset \neq A \subseteq \mathbb{R}$ bdd $\Rightarrow \sup A \in A$ or $\sup A$ is limit point of A.

· Sequence converges to $x \in X \Rightarrow x \in \overline{A}$.

– E is open $\iff E^c$ is closed.

– $(A_\alpha)_{\alpha \in I}$ all open $\Rightarrow \bigcup_\alpha A_\alpha$ open $\quad (A_n)_{n<\infty}$ closed $\Rightarrow \bigcup_{k=1..n} A_k$ closed

$(A_n)_{n<\infty}$ open $\Rightarrow \bigcap_{k=1..n} A_k$ open $\quad (A_\alpha)_{\alpha \in I}$ closed $\Rightarrow \bigcap_\alpha A_\alpha$ closed

Counter-example: $A_n = (-\frac{1}{n}, \frac{1}{n})$, $\bigcap_\infty A_n = \{0\}$ not open!

Let (Z, d'), $X \subseteq Z$, (X, d).

E open in $X \iff \exists G$ open in Z, $E = G \cap X$.

Also $\sup A \in (\overline{A})^c$

– $(E^o)^c = \overline{E^c}$, $\overline{E}$ always closed, $(\overline{E})^c$ always open.

Compactness (p23-p30) Every open cover contains a finite subcover.

*intrinsic property · $K \subseteq Y \subseteq X$. K is compact in $Y \iff K$ is compact in X.

· Compact $\Rightarrow$ Closed (complement open) min dist... 1/3

· F closed in X is subset of K, compact in $X \Rightarrow F$ is compact. (Pf: F^c open, add that to open cover of F)

· $(K_\alpha)_{\alpha \in I}$ compact, every finite subcollection has non-empty intersection $\Rightarrow \bigcap_{\alpha \in I} K_\alpha \neq \emptyset$.

– take out one set, intersect ~~union~~ with the rest

$K_{\alpha_0} \cap (\bigcap_I K_\alpha) = K_{\alpha_0} \cap (\cup K_\alpha^c)^c = \emptyset$

$\to K_n \downarrow$ compact, $K_n \neq \emptyset\ \forall n \Rightarrow \bigcap_n K_n \neq \emptyset$, $\bigcap_n K_n$ is compact.

$K_{\alpha_0} \cup (\bigcup_n K_n^c)^c = \emptyset$

$\to$ If $I_n = [a_n, b_n] \neq \emptyset$, $n \ge 1$, $I_n \downarrow \Rightarrow \bigcap_n I_n \neq \emptyset$

$K_{\alpha_0} \cap (\bigcap_n K_n^c) = \emptyset$

compact!

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· $E \subseteq X$, $|E| = \infty$, $E \subseteq K$ compact $\Rightarrow E' \neq \emptyset$

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$X = \mathbb{R}^d, K \subseteq X$

TFAE: p25.
1) K closed + bounded
2) K compact
3) K sequentially compact

General Metric Spaces ($A \subseteq (X,d)$)

TFAE
1) A complete + totally bounded
2) K compact (p35.)
3) K sequentially compact.

Hw4, Hw5

let (X,d). $A \subseteq X$ sequentially compact $\iff$ Every sequence in A has a convergent subsequence with limit in A.

E connected $\iff$ not disconnected
- $E = E_1 \cup E_2$, where $E_1 \cap \bar{E_2} = \bar{E_1} \cap E_2 = \emptyset$, $E_1 \neq \emptyset$, $E_2 \neq \emptyset$
- $E = \underbrace{(A \cap E)}_{\neq \emptyset} \sqcup \underbrace{(B \cap E)}_{\neq \emptyset}$, where A and B are disjoint and open.

$E \subseteq \mathbb{R}$ connected $\iff$ $x, y \in E$, $x < z < y$ $\Rightarrow z \in E$

($E = (1,5) / [1,5) / (1,5] / [1,5]$)

- $\text{diam}(E) := \sup\{d(x,y) \mid x, y \in E\}$

$\rightarrow \text{diam}(\bar{E}) = \text{diam}(E)$

$\rightarrow$ $K_n \downarrow$ compact, $\text{diam}(K_n) \to 0$ $\Rightarrow$ $K := \bigcap_{n \geq 1} K_n$ contains exactly one point. (proof: not empty, but cannot contain >1 point either)

Def. $A \subseteq X$ complete $\iff \forall$ C-S in A, it is convergent in A.

- Compact $\Rightarrow$ complete* $K_n := \{a_k \mid k \geq n\}$ is closed hence compact (subset of compact set), intersection non-empty. $\text{diam}(\cdot)$ also decreases mon. to 0.
- $\mathbb{R}^k$ is complete. (C-S is bounded, put interval, we prev)

(X,d) complete, $A \subseteq X$ closed $\Rightarrow A$ is complete.

p40 Rudin.

$\nearrow$ Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

$B_\alpha \subseteq Y,\ A_\alpha \subseteq X$

- $(f^{-1}(B))^c = f^{-1}(B^c)$
- $f^{-1}(\bigcup_{\alpha\in I} B_\alpha) = \bigcup_{\alpha\in I} f^{-1}(B_\alpha)$
- $f^{-1}(\bigcap_{\alpha\in I} B_\alpha) = \bigcap_{\alpha\in I} f^{-1}(B_\alpha)$
- $f(\bigcup_{\alpha\in I} A_\alpha) = \bigcup_{\alpha\in I} f(A_\alpha)$
- * $f(\bigcap_{\alpha\in I} A_\alpha) \subset \bigcap_{\alpha\in I} f(A_\alpha)$ $\quad f(A^c)=f(A)$ for constant map.
- $f(A^c)=(f(A))^c$ for bijective maps.

$\boxed{\text{Continuity}}$ $f: (X,d) \to (Y,\rho)$

$\to$ Def. $\boxed{\forall \varepsilon>0, \exists \delta>0 \text{ s.t. } f(B_\delta(x)) \subseteq B_\varepsilon(f(x))}$ $\iff$ "The inverse image of any open set containing $f(x)$ contains x as an interior point."

$\to$ Sequence Criterion

$\boxed{f \text{ continuous at } x \iff (x_n \to x \Rightarrow f(x_n) \to f(x))}$

$\to f \in \mathcal{C} \iff (G \subseteq Y \text{ open} \Rightarrow f^{-1}(G) \text{ open}) \iff (F \subseteq Y \text{ closed} \Rightarrow f^{-1}(F) \text{ closed})$

$K \subseteq X$ compact $\Rightarrow f(K)$ compact $-$ $\sup f(K) = \max(f(K))$ since f_K is closed, and $f(K)$ bounded

$\hookrightarrow$ Intermediate Value Theorem $\to$ $f \in \mathcal{C}(\mathbb{R}), f(a) \le \alpha \le f(b)\ [a \le b]$ $\Rightarrow \exists x \in [a,b]$ s.t. $f(x)=\alpha$.

$A \subseteq X$ connected $\Rightarrow f(A)$ connected

$\to$ $f: (K,d) \to (Y,\rho)$ continuous, K compact

f bijective $\Rightarrow f^{-1}$ continuous.

$\to$ f, g continuous $\to g\circ f$ continuous (use sequence criterion)

note also that $(g\circ f)^{-1} = f^{-1} \circ g^{-1}$... (use open $\to$ open criterion)

* All complex polynomials are continuous

* Transcendental functions are those that are defined as the limit of power series, so they are continuous

$-$ $\boxed{\text{Uniform Convergence}}$ $f_n \to f$ uniformly iff $\forall \varepsilon, \exists N=N(\varepsilon)$ s.t. $\forall n \ge N$, $\forall x \in X: \rho(f_n(x), f(x)) \le \varepsilon$.

e.g. $X=[0,1)$ $f_n(x)=x^n$ does not converge uniformly to $f(x)=0$

$\Rightarrow$ * If (f_n) is a sequence of continuous functions converging to f uniformly, then f is continuous itself. (use sequence criterion)

$\Rightarrow$ * $f: X \to Y$, $f \in \mathcal{C}$, X compact $\Rightarrow f$ is $\boxed{\text{uniformly continuous}}$ $(\forall \varepsilon>0, \exists \delta=\delta(\varepsilon) \text{ s.t. } d(x,y)<\delta \Rightarrow \rho(f(x),f(y))<\varepsilon)$.

$\boxed{\text{SEQUENCES \& SERIES}}$

$\to$ If $(x_n) \nearrow$, then $\lim_n x_n = \begin{cases} \infty, & \text{if } (x_n) \text{ unbounded} \\ \sup_n x_n, & \text{if } (x_n) \text{ bounded} \end{cases}$

Soln #1: use compactness of X

Soln #2: use compactness of $f(X)$, maximizer x_n $\quad f_n(x_n)\ldots$

(Dini's Theorem, HW6) Q1 (X,d) compact, $f_n: X \to \mathbb{R} \in \mathcal{C}$
$\forall n \ge 1, \forall x: f_n(x) \ge f_{n+1}(x)$; $\lim_{n\to\infty} f_n = f$ continuous
$\Rightarrow f_n \to f$ uniformly

$\Rightarrow a_n \to a, b_n \to b \Rightarrow \frac{1}{a_n} \to \frac{1}{a}$

$\Rightarrow a_n \cdot b_n \to a\cdot b$ ... (δ, ε proofs)

$\Rightarrow |a_n| \Rightarrow |a|$, use $||a|-|b|| \le |a-b|$

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Use the comparison tests ("squeeze theorem"): we have

1) $0 \le x_n \le c_n,\ c_n \to 0 \Rightarrow x_n \to 0$

2) $b_n \le x_n \le a_n,\ a_n \to a,\ b_n \to a \Rightarrow x_n \to a$

Examples of limits of sequences. $(x_n) =$

ⓐ $p > 0, p \in \mathbb{Q}$: $\frac{1}{n^p} \to 0$

ⓑ $\sqrt[n]{a} \to 1\ \forall a > 0$ (show $\sqrt[n]{a} - 1 \to 0$) know $\sqrt[n]{a} \ge 1$ when $a > 1$, $a \ge (1 + (\sqrt[n]{a} - 1))^n \ge 1 + n(\sqrt[n]{a} - 1)$

ⓒ $\sqrt[n]{n} \to 1$

ⓓ let $a > 0, \alpha \in \mathbb{R}$ (distinct.): $\frac{n^\alpha}{(1+a)^n} \to 0$ lower-bound $(1+a)^n$

$\frac{a^n}{n!} \to 0$ likewise (choose $n > a$) $\frac{a^a}{a!} \cdot \underbrace{\frac{a^{n-a}}{(a+1)(a+2)\cdots n}}_{\searrow 0}$

ⓔ $|a| < 1 \Rightarrow a^n = \frac{1}{(\frac{1}{a})^n} \to 0$

Liminf, Limsup.

$\overline{\lim_n}\, a_n = \inf_n \sup_{k \ge n} a_k$ $\quad l = \overline{\lim_n}\, a_n \iff \begin{cases} \forall \varepsilon > 0 \text{ only } \underline{\text{finitely many}}\ a_n\text{'s are} > l + \varepsilon \\ \infty \text{ many } a_n\text{'s are } > l - \varepsilon \end{cases}$

$\underline{\lim_n}\, a_n = \sup_n \inf_{k \ge n} a_k$ $\quad \to$ largest subsequential limit point (limit of some convergent subsequence)

$\underline{\lim}\, a_n = \overline{\lim}\, a_n =: a \iff a_n \to a$

'4 limits'

$\underline{\lim}_{h \downarrow 0} f(x+h)$ $\quad \overline{\lim}_{h \downarrow 0} f(x-h)$

$\underline{\lim}_{h \downarrow 0} f(x+h)$ $\quad \underline{\lim}_{h \downarrow 0} f(x-h)$

* we can use $x \le \underline{\lim}(y_n) \le \overline{\lim}(y_n) \le x$ to show $\lim(y_n) = x$

eg. p54, p50, p75

(series) $\Rightarrow$ Define for $(a_n) \subseteq \mathbb{R}$ or $\mathbb{C}$ the sequence of partial sums

$$S_n := \sum_{k=1}^{n} a_k. \quad \lim_{n \to \infty} S_n =: s \in \mathbb{R} \cup \{+\infty, -\infty\} \text{ (or } \mathbb{C}) \text{ if } \underline{\text{the limit exists}}.$$

$s \in \mathbb{R} \iff$ series is <u>convergent</u>

Theorems. • $S_n = \sum_{k=1}^{n} a_k$ is convergent $\iff S_n$ is Cauchy. ($|S_n - S_m| < \varepsilon$ etc.) (corollary: $\sum a_k$ is convergent $\Rightarrow a_n \to 0$)

• If $|a_n| < c_n\ \forall n \ge N_0$, $\sum_n c_n < \infty$, then $\sum a_n$ is convergent.

Furthermore $\sum |a_n| < \infty$, and we call $\sum_n a_n$ an <u>absolutely convergent series</u>.

can be complex

• Assume $(a_n) \subseteq \mathbb{R}^+$, $a_n \downarrow 0$. Then $\sum_{k \ge 1} a_k$ converges $\iff \sum_{k \ge 0} 2^k a_{2^k} < \infty$

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Examples of Converging and Non-converging Sequences

(1) (Geometric series) $z \in \mathbb{C}$ (in particular, $x \in \mathbb{R}$)

$$0 \leq |z| < 1 \Rightarrow \sum z^k = \frac{1}{1-z}$$

$$|z| \geq 1 \Rightarrow \text{series diverges}$$

(2) (Harmonic series) $\sum_{n\geq 1} \frac{1}{n^p}$ converges for $p>1$ and diverges for $p\leq 1$

(3) $$\sum_{n\geq 2} \frac{1}{n(\log n)^p} \begin{cases} < \infty & \text{if } p>1 \\ = \infty & \text{if } p\leq 1 \end{cases}$$

(4) (Euler's constant) $e := \sum_{n\geq 0} \frac{1}{n!} < \infty$, furthermore $e = \lim_{n\to\infty} (1+\frac{1}{n})^n$

(5) (Power series) $z \in \mathbb{C}$, $\varphi(z) := \sum_{n\geq 0} c_n z^n$ is the power series associated with (c_n).

see p7.

Other Convergence Tests:

(1) (Root Test) Given $\sum a_n$, $a_n \in \mathbb{C}$, define $r := \overline{\lim} \sqrt[n]{|a_n|}$.

- $r<1 \Rightarrow$ series convergent (in particular, absolutely convergent)
- $r>1 \Rightarrow$ not convergent
- $r=1 \Rightarrow$ inconclusive

(2) (Ratio Test) For given series $\sum a_n$,

- $\overline{\lim} \left|\frac{a_{n+1}}{a_n}\right| < 1 \Rightarrow$ convergent (in particular, absolutely convergent)
- $\exists n_0$ s.t. $\forall n \geq n_0$, $\left|\frac{a_{n+1}}{a_n}\right| \geq 1 \wedge a_n \neq 0 \Rightarrow$ not convergent.

(3) Let $A := \sum_n a_n$, $B := \sum_n b_n$. Then (1) $\sum (a_n+b_n) = A+B$

(2) $\sum \gamma \cdot a_n = \gamma A$

(4) Assume $A := \sum_n a_n$ is absolutely convergent, $B := \sum_n b_n$ is convergent.

Then $A \cdot B = \sum c_n$ is convergent with $c_n := \sum_{k=0\ldots n} a_k b_{n-k}$

(5) (Rearrangements) Let $\varphi: \mathbb{N} \to \mathbb{N}$ bijective, $(a_n) \subseteq \mathbb{C}$. Define $a_n' := a_{\varphi(n)}$, $S_n' = \sum_{k=0\ldots n} a_k'$.

If $\sum a_n$ is absolutely convergent $\Rightarrow$ $\sum a_n'$ is also absolutely convergent and $\sum a_n' = \sum_k a_k =: s$.

$f_n(x) := x^n$ for $x \in [0,1]$
no uniform convergence to $f=0$, not equicontinuous

$f_n = \frac{1}{\sqrt{n}} \sin(nx)$ $f_n \to f$ uniformly but $f'_n \not\to f$

$f(x) := x \sin(1/x)$, $x \in [0,1]$
$\Rightarrow$ continuous but not differentiable at 0
$f'(x) = \sin(1/x) + (-\frac{1}{x^2}) x \cos(\frac{1}{x})$
$= \sin(1/x) - \frac{1}{x}\cos(\frac{1}{x})$
or consider the difference quotients

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$\boxed{\text{POWER SERIES}}$ $\varphi(z) := \sum_{n \geq 0} c_n z^n$.

Examples 1) $c_n := \frac{1}{n!} \Rightarrow \varphi(z) = \exp(z) := \sum \frac{1}{n!} z^n$

2) $c_n = \begin{cases} (-1)^{n+1}/n! & n = 2k+1 \quad k \geq 0 \\ 0 & n = 2k \quad k \geq 0 \end{cases} \Rightarrow \sum c_n z^n =: \sin(z)$

set $R := (\overline{\lim} \sqrt[n]{|c_n|})^{-1} \in [0, \infty]$. Then

(root test) — R is the radius of absolute convergence i.e. $R = \sup \{r \mid \sum c_n z^n$ is absolutely convergent in $|z| < r\}$.

Analytic functions $\cup$ Transcendental functions

Ex. $R = \infty$ for $\varphi(z) = \exp(z)$. By the ratio test: $\left|\frac{c_{n+1} z^{n+1}}{c_n z^n}\right| = \frac{1}{n+1}|z| \to 0, \forall z$.

* $f_n(z) := \sum_{k \geq 0}^{n} c_k z^k \xrightarrow{\text{uniformly}} f(z)$ on $\overline{B_r(0)}$, for every $r < R$.

In particular, $f(z)$ is continuous on $B_R(0)$. What about $r = R$?

① $\sum_k |c_k| R^k < \infty \Rightarrow$ ok
② $\sum_k |c_k| R^k = \infty \Rightarrow$ If $R = 1$, $\sum_n c_n < \infty$, then $f(1) = \sum_n c_n$ exists and $\lim_{x \to 1} f(x) = f(1)$

misc facts:

continuous $\Leftrightarrow$ lcont + rcont

limit $= x \Leftrightarrow$ right limit $=$ left limit $= x$

right continuous $\Leftrightarrow$ right limit exists, $= f(x)$
$\lim_{h \downarrow 0} f(x+h) = \lim_{h \to 0} f(x+h)$

for monotone functions, $(x \leq y \Rightarrow f(x) \leq f(y))$
- $\forall x$, $f(x^+)$ and $f(x^-)$ exist, $f(x^+) = f(x) = f(x^-)$.
- If $D = \{x \mid f(x^-) \lneq f(x^+)\}$ is the set of jump discontinuities, then D is countable.

DIFFERENTIATION For $f: I \to \mathbb{R}$, $f'(x) := \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$ if RHS exists and $\in \mathbb{R}$.

- Diff at $x \Rightarrow$ Cont at x
- Product rule, quotient rule...
- Chain Rule. $I \xrightarrow{f} J \xrightarrow{g} \mathbb{R}$
  f continuous everywhere, diff at $x \in I$, $f(I) \subseteq J$, g diff. at $f(x)$ $\Rightarrow (g \circ f)'(x) = g'(f(x)) f'(x)$

'sandwich' with $f'(x^-)$ and $f'(x^+)$
- At a local maximum x ($\exists \delta$ s.t. $\forall y \in B_\delta(x), f(y) \leq f(x)$), if $f'(x)$ exists then $f'(x) = 0$.
- (Mean Value Theorem) $f \in \mathcal{C}([a,b], \mathbb{R})$, diff in (a,b)
  $\Rightarrow \exists x \in (a,b)$ s.t. $f(b) - f(a) = (b-a) f'(x)$

($f_n \to f \Rightarrow f'_n \to f'$ under certain extra conditions)

- (Helper) Let $f_n, f: E \to \mathbb{R}$ not necessarily continuous, $f_n \to f$ uniformly on $E \subseteq X$ for any $x \in E'$ s.t. $\lim_{t \to x} f_n(t) =: A_n \in \mathbb{R}$, then $\lim_n [\lim_{t \to x} f_n(t)] = \lim_{t \to x} [\lim_n f_n(t)]$

* (Main result) If f_n diff on $[a,b]$, exists 'pivot' $x_0 \in [a,b]$ s.t. $f_n(x_0) \longrightarrow := f(x_0)$, f'_n converges uniformly on $[a,b]$ $\Rightarrow$ ① $f_n \to f$ uniformly on $[a,b]$, ② $f'(x) = \lim_{n \to \infty} f'_n(x)$ $\forall x \in [a,b]$]*

Using the 'main result', and the fact that for power series $g(x) := \lim_{n\to\infty} \sum_{k=0}^{n} c_k x^k$, $h(x) = \sum_{k\geq 1} k c_k x^{k-1}$ has the same radius of convergence as $g(x)$,

(termwise derivative) $h_n(x) \xrightarrow{\text{uniformly}} h(x) \Rightarrow g'(x) = \lim_{n\to\infty} g_n'(x) = \lim_{n\to\infty} h_n(x) = h(x)$

This shows that every power series is infinitely differentiable.

Taylor's Theorem (Remainder) $f \in \mathcal{C}([a,b],\mathbb{R})$, $n \in \mathbb{N}^+$, $f^{(n-1)} \in \mathcal{C}([a,b])$ and differentiable on (a,b).
let $a \leq \alpha \neq \beta \leq b$. $P(t) := \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k$ is the "expansion" around a.
$\Rightarrow \exists\, \xi \in (\alpha,\beta)$ or (β,α) s.t. $f(\beta) = P(\beta) + \frac{f^{(n)}(\xi)}{n!}(\beta-\alpha)^n$.

L'Hôpital's Rule let $-\infty < a,b \leq \infty$, g,f differentiable on (a,b), $g'(x) \neq 0$ for $x \in (a,b)$.
Assume $\frac{f'(x)}{g'(x)} \xrightarrow[x\to a]{} A \in [-\infty,\infty]$ and either $\begin{cases} f(x), g(x) \to 0 \\ f(x), g(x) \to +\infty \end{cases}$ as $x \to a$
Then $\frac{f(x)}{g(x)} \to A$ (as $x \to a$).

(Riemann) Integration. let $f: [a,b] \to \mathbb{R}$ be bounded, P a partition $\{(a=)p_0 < p_1 < p_2 \dots < p_n (=b)\}$.

$U(P,f) = \sum_{k=0\dots n-1} \Delta_k p \cdot \sup_{x\in[p_k,p_{k+1}]} f(x)$, $\overline{\int} f = \inf_P U(P,f)$

, $L(P,f) = \sum_k \Delta_k p \cdot \inf_{x\in[p_k,p_{k+1}]} f(x)$, $\underline{\int} f = \sup_P L(P,f)$
$= -U(P,-f)$

$f: [a,b] \to \mathbb{R}$ R-integ $\Leftrightarrow$ ① f bounded
② $\underline{\int} f\,dx = \overline{\int} f\,dx$

$\to$ ε-criterion: if f bounded, f is R-integ $\Leftrightarrow \exists P = P(\varepsilon)$ s.t. $U(P_\varepsilon) - L(P_\varepsilon) \leq \varepsilon$.

Monotone (increasing) ✓

f continuous except at finitely many points ✓

f_n R-integ, $f_n \to f$ uniformly on $[a,b] \to f$ R-integ ✓, $\int f_n \xrightarrow{n\to\infty} \int f$ ✓

21-355 Final Cheatsheet May 2021

$\boxed{\text{Fundamental Theorems of Calculus}}$ Part I. Let 1) $f: [a,b] \to \mathbb{R}$ be R-intg.
2) $\exists F$ diff on $[a,b]$ with $F'(x) = f(x)\ \forall x$
Then $\int_a^b = F(b) - F(a)$.

II. Let $g: [a,b] \to \mathbb{R}$ be R-intg.
Then 1) $x \mapsto G(x) := \int_a^x g$ is uniformly continuous
2) If g is continuous at $c \in [a,b]$, then G is diff. at c, and $G'(c) = g(c)$

$\boxed{\text{Theorem (Lebesgue)}}$ For $f: [a,b] \to \mathbb{R}$ bounded,

f R-intg $\iff$ set of discontinuities of f has measure zero

Def: $A \subseteq \mathbb{R}$, $\lambda(A) = 0 \iff \exists I_1, I_2 \dots$ open $((a,b))$ s.t.
1) $\bigcup_k I_k \supseteq A$, and
2) $\sum_k \operatorname{diam}(I_k) < \varepsilon$

Lemmas. $\lambda(I_k) = 0\ \forall k = 1,2,\dots$ (countably many) $\Rightarrow \lambda(\bigcup_k I_k) = 0$
(HW10)
$f: [a,b] \to \mathbb{R}^+$ R-intg, $\int_a^b f = 0 \Rightarrow \lambda(\{x \in [a,b] \mid f(x) \neq 0\}) = 0$
i.e. $f = 0$ "almost everywhere"

The Lebesgue Integral for 'simple functions' is $\int f_n\, d\lambda = \sum_{k=1\dots n} \mu(\underbrace{f_n^{-1}(\{a_k\})}_{\text{preimage}})$

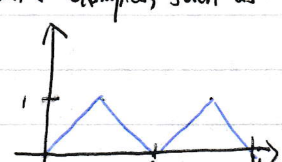
If $f_n \to f$, $f \geq 0$, then $\int f_n\, d\lambda \to \int f\, d\lambda$.

If we don't have $f \geq 0$, then we write $f = f^+ - f^- = \max(f,0) - \max(-f,0)$

$\boxed{\text{Theorem (Weierstrass)}}$ There exists a function f that is everywhere continuous but nowhere differentiable.

Several functions can serve as counter-examples, such as $f(x) \sim \sum_n \frac{1}{n^4} \sin(nx)$, or the following:

$$\varphi(x) = \begin{cases} \varphi(x+2) & x < 0 \\ |1-x| & 0 \leq x \leq 2 \\ \varphi(x-2) & x > 2 \end{cases}$$



Families of Functions

Theorem (HW5) (X,d) totally bounded $\Rightarrow \exists D \subseteq X$, D is dense, i.e. $\bar{D} = X$ and D is countable.
(we also say that X is separable.)

Define a family $\mathcal{F} = \bigcup_{\alpha \in I} f_\alpha$, where each $f: X \to \mathbb{R}$ (or $\mathbb{C}$)

e.g. $X=[0,1]$ $f_n(y)=y^n$ is not equicontinuous.

$\mathcal{F}$ can be 1) Pointwise Bounded $\Leftrightarrow \forall x \in X, M(x) := \sup_{f \in \mathcal{F}} |f(x)| < \infty$.

2) Uniformly Bounded $\Leftrightarrow \exists M < \infty: \forall x \in X, f \in \mathcal{F}: |f(x)| < M \Leftrightarrow \sup_{x \in X} \sup_{f \in \mathcal{F}} |f(x)| =: M' < \infty$

'family of uniformly continuous functions'? 3) Equicontinuous $\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0$ s.t. $d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon, \forall f \in \mathcal{F}$

$\Leftrightarrow \sup_{f \in \mathcal{F}} W_f(\delta) =: W(\delta) \xrightarrow[\delta \downarrow 0]{} 0$, where $W_f(\delta) := \sup |f(x) - f(y)|, x,y \in X, d(x,y) < \delta$

We measure the distance between functions f and g by the 'sup-norm': $(\mathcal{C}(X,\mathbb{R}), d(f,g) := \sup_x |f(x) - g(x)|)$ is a metric space.
(X compact)

* $f_n \to f$ w.r.t d $\Leftrightarrow f_n \to f$ uniformly
* X compact: $(\mathcal{C}(X), d)$ is complete

Lemmas: 1) $\forall n: f_n \in \mathcal{C}(X)$, f_n uniformly continuous, $f_n \to f$ uniformly $\to \{f_n\}$ is equicontinuous.

2) Let $K \subseteq (X,d)$ be compact, $\mathcal{F} \subseteq \mathcal{C}(K)$. If $\mathcal{F}$ is pointwise bounded and equicontinuous, then $\mathcal{F}$ is uniformly bounded.

*3) $f_n: A \to \mathbb{R}, n \geq 1$, A countable. If $(f_n)_{n \geq 1}$ is pointwise bounded
$\Rightarrow \exists$ subsequence $(f_{n_k})_k$ s.t. $(f_{n_k})(x)$ converges $\forall x \in A$.

Arzela-Ascoli Let (X,d) compact, $\mathcal{F} \subseteq \mathcal{C}(X)$.

① If $\mathcal{F}$ is pointwise bounded and equicontinuous,

(①⇒②) $\forall (f_n) \subseteq \mathcal{F} \Rightarrow \exists (f_{n_k})$ subsequence s.t. $f_{n_k} \to f$ w.r.t. d, and $f \in \bar{\mathcal{F}}$ (if $\mathcal{F}$ closed, then $f \in \mathcal{F}$).

② If $\mathcal{F}$ is uniformly bounded and equicontinuous $\Leftrightarrow \bar{\mathcal{F}}$ is compact ($\mathcal{F}$ is 'quasi-compact')

Stone-Weierstrass $\mathcal{P}[0,1]$ (polynomial ring on $\mathbb{R}/\mathbb{C}$) is dense in $(\mathcal{C}[0,1], d)$.

i.e. $f \in \mathcal{C}[0,1] \Rightarrow \exists (p_n) \in \mathcal{P}[0,1]$ s.t. $d(f,p_n) = \sup_x |f(x) - p_n(x)| \xrightarrow[n \to \infty]{} 0$

Pf. Considering $B_{f,n}(x) = \sum_{k=0 \ldots n} f(\frac{k}{n}) \underbrace{\binom{n}{k} x^k (1-x)^{n-k}}_{b_{k,n}}$, Chebyshev's Inequality
'family of Binomial Coeffs'
$(\forall c > 0: \sum_{|d_k - \mu| > c} p_k \leq \frac{\sigma^2}{c^2}) \; \frac{1}{n} x(1-x)$ for binomial distribution
We can show that $\lim_{n \to \infty} B_{f,n}(x) = f$.