21-469 8/30/21

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OH: Wed 10-11:10am,
WEH 8206

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* Go to recitation!
WEH 5202 8:35-9:55am Tuesday

OH: (Zoom) 4-5:30pm Thursday

PARTIAL DIFFERENTIAL EQUATIONS (PDEs)

- Model for Science / Economic processes
- Quantitative Description

Algebraic Equations $x^2 - 2 = 0 \Rightarrow x = \pm\sqrt{2}$

Solution is number (or set of numbers / vectors ...)

Differential Equations (Exception - non-differentiable 'weak' solutions)

Solution is function(s) and the equation relates the function values and its derivatives to other given functions.

(i) $u'(t) = u(t)$. $u : [0,T] \to \mathbb{R}$, $T > 0$ fixed.

(alt. notation: $u'(t) = \frac{d}{dt} u(t) = \dot{u}(t)$) derivative wrt single variable $\Rightarrow$ ordinary diff. eq. $\boxed{ODE}$

- we need an initial (or terminal) condition
e.g. $u(0) = u_0 \in \mathbb{R}$, $u_0 = \frac{1}{2}$

$\frac{1}{u}\frac{du}{dt} = 1$

$\ln u = t + c$

$c = \ln \frac{1}{2}$

$u = e^c \cdot e^t = \frac{1}{2}e^t$

$u(t) = u_0 e^t = \frac{1}{2} e^t$

(ii) Other examples :

- $u'(t) = (u(t))^2$ (non-linear)
- $u''(t) = u(t)$ (second-order)
- $u''(t) + 2u'(t) = -u(t)$
- $u'(t) = \sin(u(t))$ still homogeneous!
- $u'(t) = \sin(t)$ not homogeneous anymore...
- $u'(t) = \sin(v(t))$, $v'(t) = \cos(u(t))$ 1st order system of ODE's

$(u,v) : [0,T] \to \mathbb{R}^2$

Can be rewritten as ~~order~~ system of 1st order ODE's

$(v = u')$ $\begin{cases} u' = v(t) \\ v'(t) = u(t) \end{cases}$

$(v = u')$ $\begin{cases} u' = v(t) \\ v'(t) + 2v(t) = -u(t) \end{cases}$

$\boxed{\text{PDE}}$

- Solutions may depend on several variables
- PDE relates this function to its derivatives in different variables (partial derivative)
- and other given functions

Examples: (1) $u = u(t,x) : \mathbb{R}^2 \to \mathbb{R}$

$$\partial_t u(t,x) + \partial_x u(t,x) = 0$$

(notation) $\partial_t u(t,x) = \frac{\partial}{\partial t} u(t,x) = u_t(t,x) = \frac{\partial u}{\partial t}(t,x) = \lim_{|h| \to 0} \frac{u(t+h,x) - u(t,x)}{h}$

Transport Equation (First-order, Linear PDE)

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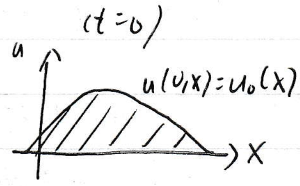
Transport Equation $u(t,x)$

$u : [0,T] \times \mathbb{R} \to \mathbb{R}$

open interval!

$u_t + u_x = 0 \qquad (0,T) \times \mathbb{R}$

$u(0,x) = u_0(x) \qquad$ (initial condition) e.g.

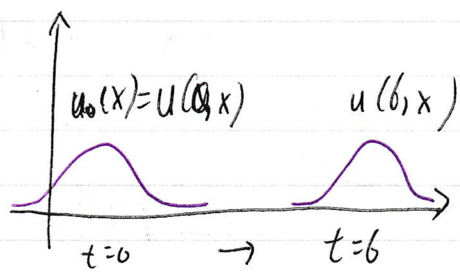


CLAIM: Solution is $u(t,x) = u_0(x-t)$

Verifying. $u_t(t,x) = u_0'(x-t) \times (-1)$ Chain rule

$u_x(t,x) = u_0'(x-t) \times (1)$ sum to 0

* $u_0(x)$ is just a function, x need not have an interpretation.



'wave propagation'

LINEAR EQUATIONS (e.g. Transport Eqn)

If $u_t + u_x = 0$ and $v_t + v_x = 0$, and $w = \alpha u + \beta v$ $(\alpha, \beta \in \mathbb{R})$

$$w_t + w_x = (\alpha u + \beta v)_t + (\alpha u + \beta v)_x$$

'superposition'

$$= \underbrace{\alpha(u_t + u_x) + \beta(v_t + v_x)}_{=0}$$

Exercise. If $u_t + (t+x^2)u_x = 0$ and $v_t + (t+x^2)v_x = 0$, $W = \alpha u + \beta v$ $(\alpha, \beta \in \mathbb{R})$ then $w_t + (t+x^2)w_x = 0 \to$ still linear!

Recall from Linear Algebra...

1) Homogeneous equations $\quad A\underset{\sim}{u} = \underset{\sim}{0}, \; A\underset{\sim}{v} = \underset{\sim}{0} \Rightarrow A(\alpha \underset{\sim}{u} + \beta \underset{\sim}{v}) = \underset{\sim}{0}$

2) Non-homogeneous $\quad A\underset{\sim}{u} = \underset{\sim}{f}, \; A\underset{\sim}{v} = \underset{\sim}{g} \Rightarrow A(\alpha \underset{\sim}{u} + \beta \underset{\sim}{v}) = \alpha \underset{\sim}{f} + \beta \underset{\sim}{g}$

Ex. If $u_t + u_x = f$ and $v_t + v_x = g$, $w = \alpha u + \beta v$,
then $w_t + w_x = \alpha f + \beta g$

NON-LINEAR EQNs $\quad u_t + u \cdot u_x = 0, \; v_t + v v_x = 0$

If $w = \alpha u$ then
$$w_t + w \cdot w_x = (\alpha u)_t + (\alpha u)(\alpha u)_x$$
$$= \alpha u_t + \alpha^2 u u_x$$
$$= \alpha (u_t + \alpha u \cdot u_x) \neq 0 \text{ if } \alpha \neq 1.$$

Origins of PDEs

$\boxed{\text{TRANSPORT EQUATION}}$

$u(t,x)$ = concentration of pollutant
= mass of pollutant / unit length

River

x (direction of flow)

velocity $= v$

$x_1 \quad x_2$

Amount in (x_1, x_2) at time t $= \int_{x_1}^{x_2} u(t,x)\,dx$

Rate of mass entering (x_1, x_2) $= v\, u(t, x_1)$

" exiting (x_1, x_2) $= v u(t, x_2)$

$$\frac{d}{dt} \underbrace{\int_{x_1}^{x_2} u(t,x)\,dx}_{\text{'amount'}} = v\, u(t,x_1) - v\, u(t,x_2)$$

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$$\int_{x_1}^{x_2} u_t(t,x)\,dx = -v\int_{x_1}^{x_2} u_x(t,x)\,dx$$

$$\int_{x_1}^{x_2} (u_t + v u_x)(t,x)\,dx = 0 \quad \text{for } \underline{\text{all}}\ x_1 < x_2$$

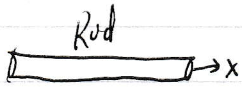
(By conservation of energy) $\boxed{u_t + v u_x = 0}$ (this is also known as the Fundamental Theorem of Variational Calculus)

Second order:

Heat transfer depends on $\underline{\text{time}}$.

Acceleration is 2nd derivative in $\underline{\text{space}}$.

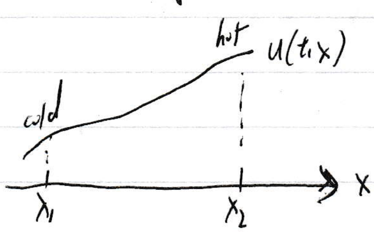
$\boxed{\text{HEAT EQUATION}}$



$u(t,x)$ = temperature at time t, position x.

Thermal Energy in $(x_1, x_2) = \int_{x_1}^{x_2} c\, u(t,x)\,dx$

"specific heat"



Fourier's Law of Heat Conduction states that...

Rate of heat energy flow $= -k\, u_x$

conductivity

heat flows from hot to cold

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heat flowing to x_1 if there is a constant heat source somewhere

$$\frac{d}{dt}\int_{x_1}^{x_2} c\,u(t,x)\,dx = k\,u_x(t,x_2) - k u_x(t,x_1) \quad (+f)$$

heat flowing away from x_1

$$\int_{x_1}^{x_2} c u_t\,dx = k\int_{x_1}^{x_2} u_{xx}(t,x)\,dx$$

$$\int_{x_1}^{x_2} (c u_t - k u_{xx})\,dx = 0 \qquad \forall x_1 < x_2$$

$$\therefore \boxed{c u_t - k u_{xx} = 0.}$$

Exercise: If $\hat{u}(t,x) = u(t/c, x/\sqrt{k})$, then

$$\hat{u}_t - \hat{u}_{xx} = 0. \qquad (*)$$

A solution of $(*)$ is $\hat{u}(t,x) = \frac{1}{\sqrt{t}} e^{-x^2/4t}$, $t>0$.

'General' Conservation Laws.

$$\frac{d}{dt}\int_{x_1}^{x_2} u = q^t(x_1) - q^t(x_2) = -\int_{x_1}^{x_2} q(t,x)\,dx$$

$$u_t + q_x = 0.$$

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Stability

Consider $u'(t) = -u(t)$, $u(0) = 1$

Then $u(t) = e^{-t} u_0$

Suppose $v'(t) = -v(t)$, $v(0) = u(0) + \varepsilon$ ← error term

Then $v(t) = e^{-t}(u_0 + \varepsilon)$

The error $v(t) - u(t) = e^{-t}\varepsilon \to 0$ as $t \to \infty$

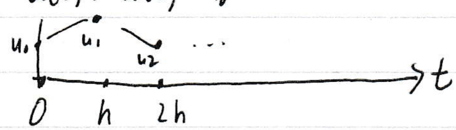
But if $u'(t) = u(t)$, $u(0) = 1$, $u(t) = e^{t} u(0)$,

e.g. "weather forecast" Then $v(t) = e^{t} v_0 = e^{t}(u(0) + \varepsilon)$ under the same assumption of constant initial error.

$v(t) - u(t) = e^{t}\varepsilon \to \infty$ as $t \to \infty$. The error 'blows up'.

Numerical Approximation

e.g. $u'(t) = -u(t)$, $u(0) = u_0$



h: time step, write $t_m = mh$ discrete time

$$\frac{u_{m+1} - u_m}{h} \simeq -u_m \qquad \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e \qquad \left(1+\frac{t}{x}\right)^x = e^t?$$

$$u_{m+1} = (1-h)u_m = (1-h)^2 u_{m-1} \ldots$$

$$u_m = (1-h)^m u_0 \quad \text{but recall} \quad \lim_{h\to 0}(1-h)^{\frac{1}{h}} = e^{-1}$$

For h small, $u_m \approx \left[(1-h)^{1/h}\right]^{mh} u_0 \simeq e^{-1\cdot t_m} u_0 = e^{-t_m} u_0$

If $v_m = (1-h)^m (u_0 + \varepsilon)$, then

$$v_m - u_m = (1-h)^m \varepsilon = [(1-h)^{1/h}]^{mh} \varepsilon$$

$t \to \infty$

$$\approx e^{t_m} \varepsilon \to 0 \text{ as } t_m = mh \, ; h \to 0$$

BUT, if $u'(t) = u(t)$, $\frac{u_{m+1} - u_m}{h} = u_m$

$$u_{m+1} = (1+h) u_m = (1+h)^2 u_{m-1} \dots \quad u_m = (1+h)^m u_0$$

If $v_m = (1+h)^m (u_0 + \varepsilon)$, then

$$v_m - u_m = [(1+h)^{1/h}]^{mh} \varepsilon$$

$$\approx e^{t_m} \varepsilon \to \infty$$

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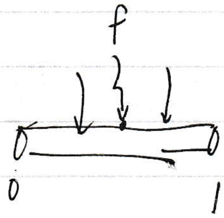
Poisson Equation (in 1D)



Recall Heat Equation: $u_t - u_{xx} = f$, $u(t,x)$ = temperature.

If $f(t,x) = f(x)$ is independent of time, at some large time an equilibrium (steady state) temperature is attained, i.e. $u_t = 0$, $u(t,x) = u(x)$

$$\text{so } \boxed{-u''(x) = f(x), \quad u(0) = u(1) = 0}$$

Lemma (Uniqueness) If $-u'' = f$, $u(0) = u(1) = 0$ and $-v'' = f$, $v(0) = v(1) = 0$, then $u(x) = v(x)$ $\forall x\ [0,1]$.

Proof. $w := v - u$, $w(0) = w(1) = 0$ and $-w'' = -(v-u)'' = -v'' - u'' = f - f = 0$

So $w''(x) = 0$, $0 < x < 1$. Then $w'(x) = a \in \mathbb{R}$ constant, $w(x) = ax + b$.

$w(0) = 0$, $w(1) = 0 \Rightarrow b = 0, a = 0$. Thus $w(x) = 0$

$\Rightarrow v(x) = u(x)$, $0 \le x \le 1$.

Exercise:
1) Solutions of $-u'' = f$ on $(0,1)$, $u(0) = \alpha$, $u(1) = \beta$ are unique.
2) Solutions of $-u'' = f$ on $(0,1)$, $u(0) = \alpha$, $u'(1) = \beta$ are unique.
3) If $-u'' = f$ on $(0,1)$, $u'(0) = \alpha$, $u'(1) = \beta$, and $\underbrace{v(x) = u(x) + C}_{\text{non-uniqueness!}}$, then $-v'' = f$ and $v'(0) = \alpha$, $v'(1) = \beta$.

~ see textbook ~p 50.

Claim: The solution of $-u''=f,\ u(0)=u(1)=0$ is

$$u(x) = x\int_0^1 (1-y)f(y)\,dy - \int_0^x (x-y)f(y)\,d(y)$$

Note: $\frac{d}{dx}\overbrace{\int_0^x F(x,y)\,dy}^{g(x)}$

$= \underbrace{F(x,x)}_{1} + \underbrace{\int_0^x F_x(x,y)\,dy}_{2}$

Proof:
$$= \frac{\int_0^{x+h} F(x+h,y)\,dy - \int_0^x F(x,y)\,dy}{h}$$

$$\int_x^{x+h} F(x+h,y)\,dy + \int_0^x F(x+h,y) - F(x,y)\,dy$$

$$= \frac{\int_0^{x+h} F(x+h,y) + F(x,y) - F(x,y)\,dy - \int_0^x F(x,y)\,dy}{h}$$

$$= \frac{\int_0^{x+h} F(x+h,y) - F(x,y)\,dy + \int_0^{x+h} F(x,y)\,dy - \int_0^x F(x,y)\,dy}{h}$$

$$\xrightarrow{\lim h\to 0} \int_0^x F_x(x,y)\,dy + F(x,x)$$

Proof. $u(0) = 0\cdot\int_0^1 (\quad)\,dy - \int_0^0 (\quad)\,dy = 0.$

$u(1) = \int_0^1 (1-y)f(y)\,dy - \int_0^1 (1-y)f(y)\,dy = 0.$

$u'(x) = \int_0^1 (1-y)f(y)\,dy - \underbrace{0}_{1} - \underbrace{\int_0^x f(y)\,dy}_{2}$

$u''(x) = 0 - 0 - f(x)$

(Motivation: $u'(x) = u'(0) - \int_0^x f(y)\,dy \dots$)

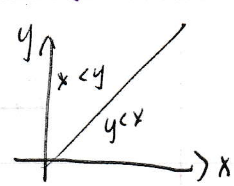
Rewrite the solution as $u(x) = x\int_0^x (1-y)f(y)\,dy + x\int_x^1 (1-y)f(y)\,dy - \int_0^x (x-y)f(y)\,dy$

$$= \int_0^x \left[x(1-y)f(y) - (x-y)f(y)\right]dy + x\int_x^1 (1-y)f(y)\,dy$$

$$= \int_0^x \overbrace{y(1-x)}^{y\le x} f(y)\,dy + \int_x^1 \underbrace{x(1-y)}_{x\le y} f(y)\,dy$$

$$= \int_0^1 G(x,y)\,f(y)\,dy.$$

$$G(x,y) = \begin{cases} y(1-x) & 0\le y\le x\le 1 \\ x(1-y) & 0\le x\le y\le 1 \end{cases}$$



('Green's Function')

Properties of Green's Function:

1) $G(x,y)$ is continuous on $[0,1]^2$

(applicable in higher dimensions) ✓ 2) $G(x,0) = G(x,1) = G(0,y) = G(1,y) = 0.$

✓ 3) $G(x,y) = G(y,x)$

4) $G(x,y)$ is piecewise linear in x, y separately

✓ 5) $G(x,y) \ge 0$ on $[0,1]^2$

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$$\boxed{u(x)=\int_0^1 G(x,y)\,f(y)\,dy}$$

The formula makes sense for $f\in C[0,1]=\{f:[0,1]\to\mathbb{R}\mid f \text{ continuous}\}$.

(also for other integrable functions...) $\hookrightarrow u\in C_0^2((0,1))$,

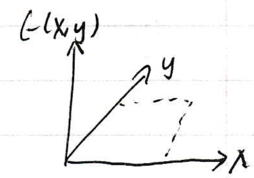
but if f is not, u might still be nice

Example: e.g. $f(x):=-\frac{1}{x}$, then

$$\begin{cases} u(x)=x\log(x) \\ u'(x)=\log(x)+x\cdot\frac{1}{x}=\log(x)+1 \\ u''(x)=1/x \end{cases}$$

satisfies $u\in C[0,1]$, $u(0),u(1)=0$, $-u''(x)=f(x)=-\frac{1}{x}$.

Note that if $f(x)\geq 0$, $0\leq x\leq 1$, then $\underbrace{u(x)\geq 0}_{\text{'positive' map}}$ for $0\leq x\leq 1$



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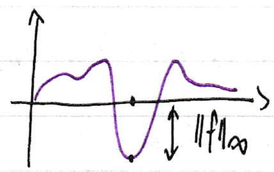
Def. If $f:[0,1]\to\mathbb{R}$ is continuous then $\|f\|_\infty = \max_{0\le x\le 1} |f(x)|$

closed!

e.g. 

No $f(x)=\frac{1}{x}$, continuous on $(0,1)$ only

NOTE

1) $\|f\|_\infty \ge 0$ and $\|f\|_\infty = 0$ iff $f(x)=0$ all x

positive semi-definite   positive definite

2) $\|\alpha f\|_\infty = |\alpha|\,\|f\|_\infty$, $\alpha\in\mathbb{R}$ (absolutely homogeneous)

3) $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$ (Δ-inequality)

Theorem If $f:[0,1]\to\mathbb{R}$ is continuous and $-u''=f$ on $(0,1)$, $u(0)=u(1)=0$, then

$$\|u\|_\infty \le \frac{1}{8}\|f\|_\infty$$

Proof. $u(x)=\int_0^1 G(x,y)f(y)\,dy$

$$\forall x\in[0,1]\ |u(x)| = \left|\int_0^1 G(x,y)f(y)\,dy\right| \le \int_0^1 |G(x,y)f(y)|\,dy$$

$$\le \int_0^1 G(x,y)\,|f(y)|\,dy$$

$$\le \underbrace{\int_0^1 G(x,y)\,dy}_{\le 1/8}\ \|f\|_\infty$$

We have to show

Now $$\int_0^1 G(x,y)\,dy = \int_0^x y(1-x)\,dy + \int_x^1 x(1-y)\,dy$$

$$= \frac{y^2}{2}\Big|_0^x (1-x) + x\,\frac{(1-y)^2}{-2}\Big|_x^1$$

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$$= \frac{x^2}{2}(1-x) + \frac{x}{2}(1-x)^2$$

$$= (1-x)\left[\frac{x^2}{2} + \frac{x}{2}(1-x)\right]$$

$$= (1-x)\left(\frac{x}{2}\right)$$

$$= \frac{1}{2}\underbrace{x(1-x)}_{\leq 1/4 \text{ on } [0,1]} \leq \boxed{1/8} \quad \text{as required.}$$

NUMERICS

Taylor's Theorem

$$u(x+h) = u(x) + h\,u'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{3!}u'''(x) + \frac{h^4}{4!}u^{(4)}(x) + \ldots$$

$$+\quad u(x-h) = u(x) - h\,u'(x) + \frac{h^2}{2}u''(x) - \frac{h^3}{3!}u'''(x) + \frac{h^4}{4!}u^{4}(x) + \ldots$$

$$u(x+h) + u(x-h) = 2u(x) + 0 + h^2 u''(x) + 0 + \frac{2h^4}{4!}u^4(x) + \ldots$$

$$\frac{1}{h^2}\left[-u(x+h) + 2u(x) - u(x-h)\right] = -u''(x) - O(h^2)$$

$$= f(x) - O(h^2) \quad \text{when } -u'' = f$$

$x_0 \quad x_1 \quad x_2 \quad \ldots \quad x_n$

Let $x_i = ih$, $h = 1/n$ then $0 = x_0 < x_1 < \ldots < x_n = 1$ is a uniform partition of $(0,1)$.

If $u_i = u(x_i) = u(ih)$, $0 \leq i \leq n$

$$u_{i+1} = u((i+1)h) = u(x_i + h)$$

$$u_{i-1} = u((i-1)h) = u(x_i - h)$$

So $\frac{1}{h^2}\left[-u_{i+1} + 2u_i - u_{i-1}\right] = f(x_i) + O(h^2)$

So $-u_{i+1} + 2u_i - u_{i-1} \simeq h^2 f(x)$, $1 \le i \le n-1$

$u_0 = u(0) = 0$, $u_n = 1 = 0$.

Replacing with equality, we have

$$[-1, 2, -1] \begin{pmatrix} u_{i-1} \\ u_i \\ u_{i+1} \end{pmatrix} = h^2 f_i$$

matrix vector

Doing this for $1 \le i \le n-1$,

$$\boxed{\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-1} \end{bmatrix} h^2 \\ A u = h^2 f}$$

$\Rightarrow$ $u = (h^2 A^{-1}) f$

w.T.s. not singular (in fact, pos def) see p16.

w.T.s ≥ 0.

Linear Equations have solutions that can be linearly combined

↳ The solutions form a vector space.

$$\int_{a(x+h)}^{b(x+h)} f(x+h,t)\,dt - \int_{a(x)}^{b(x)} f(x,t)\,dt$$

Use semicolon at the end of a line to suppress output

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(x,t)\,dt = f(x,b(x)) \cdot b'(x) - f(x,a(x)) \cdot a'(x) + \int_{a(x)}^{b(x)} \frac{d}{dx} f(x,t)\,dt$$

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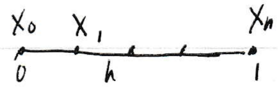
Recitation 9/14

$$u(x_i) = \sum_{j=1}^{n} h\, G(x_i, y_j) f(y_j)$$

$$G_{ij} = h\, G(x_i, y_j),\ f_j = f(y_j) \rightarrow u = Gf \qquad \begin{array}{c|c} & j \\ \hline i \rightarrow & \begin{bmatrix} & & \end{bmatrix} \begin{bmatrix} \ \end{bmatrix} j \end{array}$$



$$-u_{i-1} + 2u_i - u_{i+1} = h^2 f_i,\ f_i \approx f(x_i)$$

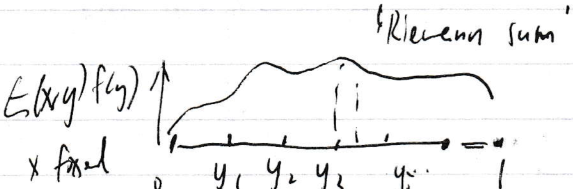
$$[-1, 2, -1] \begin{pmatrix} u_{i-1} \\ u_i \\ u_{i+1} \end{pmatrix} = f_i$$

$$A u = h^2 f \rightarrow u = h^2 A^{-1} f$$

↑ constant

Consider another approach:

$$u(x) = \int_0^1 G(x,y) f(y)\, dy \qquad G(x,0) = G(x,1) = 0.$$



Expect $[G] \approx h A^{-1}$.

$$u(x) \approx \sum_{j=1}^{n-1} G(x, y_j) f(y_j) h$$

Now $u(x_i) \approx u_i = \sum_{j=1}^{n} G(x_i, y_j) f(y_j) h$

This becomes $u = [G] h f,\ G_{ij} = G(x_i, y_j)$

Positive Definiteness of A.

Motivation Suppose $-u'' = f$, $u(0)=u(1)=0$

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & & \ddots & \\ & & & -1 & 2 \end{bmatrix}$$

Multiply by u and integrate...

$$\int_0^1 -u''u = -u'u\Big|_0^1 - \int_0^1 -u'u'$$

$$\boxed{\approx A\underset{\sim}{u}\,!!} = 0 + \int_0^1 (u')^2 > 0\,!$$

Now, $\underset{\sim}{u}^T A \underset{\sim}{u} = \sum_{i=1}^{n-1} u_i (A\underset{\sim}{u})_i = \sum_{i=1}^{n-1} u_i \sum_{j=1}^{n-1} A_{ij} u_j$

$= \sum_{i=1}^{n-1} u_i(-u_{i-1} + 2u_i - u_{i+1})$ (with $u_0 = u_n = 0$)

$= \sum_{i=1}^{n-1} u_i(u_i - u_{i-1}) + u_i(u_i - u_{i+1})$

summation by parts (1st occurrence!) $= \sum_{i=1}^{n-1} u_i(u_i - u_{i-1}) + \sum_{i=2}^{n} u_{i-1}(u_{i-1} - u_i)$

$= u_1(u_1 - u_0) + \sum_{i=2}^{n-1} (u_i - u_{i-1})^2 + u_{n-1}(u_{n-1} - u_n)$

$\therefore\ \underset{\sim}{u}^T A \underset{\sim}{u} = u_1^2 + \sum_{i=2}^{n-1} (u_i - u_{i-1})^2 + u_{n-1}^2 \geq 0$

'discrete derivative'

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Proposition $\underset{\sim}{u}^T A \underset{\sim}{u} \geq 0$, and $\underbrace{\underset{\sim}{u}^T A \underset{\sim}{u} = 0 \text{ iff } \underset{\sim}{u} = \underset{\sim}{0}}$.

Proof. ($\Leftarrow$) Clearly $\underset{\sim}{0}^T A \underset{\sim}{0} = 0$.

($\Rightarrow$) If $\underset{\sim}{u}^T A \underset{\sim}{u} = 0$ then $0 = \underset{\sim}{u}^T A \underset{\sim}{u} = u_1^2 + \sum_{i=2}^{n-1} (u_i - u_{i-1})^2 + u_n^2$

$\underline{0} = u_0 = u_1 = u_2 = u_2 = \cdots u_n = \underline{0}$

Analogously in calculus, Functions have derivatives that vanish iff the function is constant.

(Positive Definite $\Rightarrow$ Non-singular)

Corollary. A is non-singular.

Proof. We want to show $A\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}$.

But $A\underset{\sim}{u} = \underset{\sim}{0} \Rightarrow \underset{\sim}{u}^T A \underset{\sim}{u} = \underset{\sim}{u}^T \cdot \underset{\sim}{0} = 0 \Rightarrow \underset{\sim}{u} = 0$.

Def 1) A matrix $A \in \mathbb{R}^{n \times n}$ is diagonally dominant if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$$

2) A is strictly diagonally dominant if

for at least ONE index i, $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$

* Strictly diagonally dominant $\Rightarrow$ Not singular!

Examples

1) $A = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \\ & & -1 & 2 \end{bmatrix}$ is strictly diagonally dominant

Since the first (and last) rows have $a_{ii} = 2 > 1 = \sum_{j \neq i} |a_{ij}|$

$u_0 - u_2$

2) $B = \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & & \ddots & \\ & & -1 & 1 \end{bmatrix}$ is diagonally dominant but not strictly.

Note: 1) If $\underset{\sim}{x} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ then $B\underset{\sim}{x} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ so B is singular.

Thus the strict DD property of A was important.

2) However, the block matrix

$$\left[\begin{array}{cccc|cccc} 1 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & & \ddots & & & & & \\ & & 2 & -1 & & & & \\ & & -1 & 1 & & & & \\ \hline & & & & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & \ddots & \\ & & & & & & -1 & 2 \end{array}\right] \left[\begin{array}{c} 1 \\ \vdots \\ 1 \\ \hline 0 \\ 0 \\ \vdots \\ 0 \end{array}\right] = \underset{\sim}{0}$$

is ~~strictly~~ DD but singular.

3) The solution of $-u'' = f$, $u'(0) = u'(1) = 0$ does not have a unique solution!

If $u(x)$ is a solution, so too is $u(x) + c$.

$u'(0) \approx \frac{u_1 - u_0}{h}$, $u'(1) \approx 0 \approx \frac{u_n - u_{n-1}}{h}$

We can change top left and bottom right entries, as in B.

delete the first row and column (we don't know that $u_0 = 0$ anymore!)

However, with $u'(0) = 0$, $u(0) = 0$, we can deduce $u(1) = 0$, $u(2) = u(1) = 0$ ... and so on.

Conclusion: We need at least 1 boundary condition in our two conditions for the solution to be unique.

From p13 (basic setup of Poisson Equation)

Consider $u''(x)=-1$, $u(0)=u(1)=0$. We saw previously the approximation

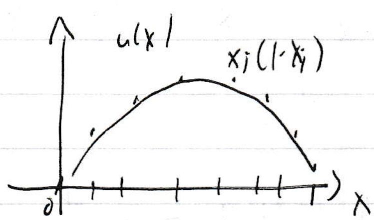
$$-u''(x) = \frac{1}{h^2}\left(-u(x-h)+2u(x)-u(x+h)\right) + \frac{h^2}{4!}\left(u^{(4)}(\xi_1)+u^{(4)}(\xi_2)\right)$$

NOTE. If $\boxed{u(x)=\frac{x(1-x)}{2}}$ then $u(0)=u(1)=0$.
$u''(x)=-1$, $-u''(x)=1$, $u^4(\xi)=0$.] this is the analytic solution

Thus $-u''(x)=1=\frac{1}{h^2}\left(-u(x+h)+2u(x)-u(x-h)\right)$ $\forall x$.

So if $u_i=\frac{x_i(1-x_i)}{2}$ then $\frac{1}{h^2}\left(-u_{i+1}+2u_i-u_{i-1}\right)=1$.

So $A\underset{\sim}{u}=h^2\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$, and $\underset{\sim}{u}=h^2A^{-1}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}x_1(1-x_1)\\ x_2(1-x_2)\\ \vdots\\ x_{n-1}(1-x_{n-1})\end{pmatrix}$ → discrete solution

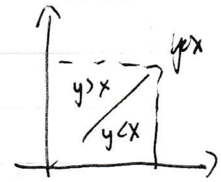


← In this case the ~~piecewise~~ discrete solution exactly coincides with the analytic solution.

Exercise. show $-x_{i-1}(1-x_{i-1})+2x_i(1-x_i)-x_{i+1}(1-x_{i+1})=h^2$ without using the fact that the 4th derivative vanishes
(write $x_{i-1}=x_i-h$ and $x_{i+1}=x_i+h$)

Recall $u(x) = \int_0^1 G(x,y)\, f(y)\, dy$

where $G(x,y) = \begin{cases} y(1-x) & 0 \le y \le x \le 1 \\ x(1-y) & 0 \le x \le y \le 1 \end{cases}$



Define the matrix $\boxed{G^h_{ij} = G(\underset{x_i}{ih}, \underset{x_j}{jh})}$

CLAIM. $\underline{AG^h = hI}$. (We want to show $A \simeq G^{-1}$)

Proof. (a) Along the diagonal, $(AG^h)_{ii} = \sum_k A_{ik} G^h_{ki}$

$$= \underbrace{A_{i,i-1}}_{-1} \underbrace{G^h_{i-1,i}}_{(x<y)} + \underbrace{A_{i,i}}_{2} G^h_{ii} + \underbrace{A_{i,i+1}}_{-1} \underbrace{G^h_{i+1,i}}_{(x>y)}$$

$$= -\underbrace{x_{i-1}}_{=x_i-h}(1-x_i) + 2x_i(1-x_i) - x_i(1-\underbrace{x_{i+1}}_{x_i+h})$$

$$= h(1-x_i) + 0 + x_i h = h.$$

(b) If $j>i$ then $\left[\begin{matrix} x_i<y_j \\ G_{ij} = x_i(1-x_j) \end{matrix}\right]$

$$(AG^h)_{ij} = \sum_k A_{ik} G^h_{kj} = \underbrace{A_{i,i-1}}_{-1} \underbrace{G_{i-1,j}}_{x<y} + \underbrace{A_{ij}}_{2} \underbrace{G_{i,j}}_{x<y} + \underbrace{A_{i,i+1}}_{(-1)} \underbrace{G_{i,j}}_{x\le y}$$

$$= (1-x_j)\left[-1(x_i-1) + 2x_i + (-1)x_{i+1}\right]$$

$$= (1-x_j)\left[-(x_i-h) + 2x_i - (x_i+h)\right] = 0 \checkmark$$

Also we can write $\underline{A^{-1} = \frac{1}{h} G^h}$.

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'infinity norm'

Notation: $\|\underset{\sim}{u}\|_\infty = \max_{1\le i\le n} |u_i|$ $\left(\|\underset{\sim}{u}\|_2 = \sqrt{u \cdot u} = \sqrt{\sum_i u_i^2} \right)$

Corollary: $(A^{-1})_{ij} \ge 0$, thus if $f_i \ge 0$, and $A\underset{\sim}{u} = h^2 f$, then $u_i \ge 0$.

Proof. $A^{-1} = \frac{1}{h} G^h$, $G^h_{ij} = G(ih, jh) \ge 0$, so

$u_i = \sum_j \underbrace{(A^{-1})_{ij}}_{\ge 0} \underbrace{f_j}_{\ge 0} \ge 0$

<u>Theorem.</u> If $A\underset{\sim}{u} = h^2 \underset{\sim}{f}$ then $\|u\|_\infty \le \frac{1}{8} \|f\|_\infty$ (discrete version of result p12)

Proof. $u_i = \sum_j (A^{-1})_{ij} h^2 f_j = h \sum_j \underbrace{G^h_{ij}}_{\ge 0} f_j$

$\therefore |u_i| \le h \sum_j G^h_{ij} |f_j| \le \|f_\infty\| \, h \sum_j G^h_{ij}$

$\le \|f_\infty\| \left(h G^h \underset{\sim}{1} \right)_i$

$\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

$\left[\begin{array}{l} A v = h^2 \begin{pmatrix} 1 \\ \vdots \end{pmatrix} \text{ when } v_i = x_i(1-x_i)/2 \\ \therefore v = A^{-1} h^2 \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = h G^h \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \end{array} \right]$

$\le \|f\|_\infty \frac{1}{2} x_i (1-x_i)$

$\le \|f\|_\infty \cdot \frac{1}{8}$

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F. weber returns!

Matlab Simulation of Poisson Equation

vec = ones(n,1)

A = spdiags([-vec, 2*vec, -vec], -1:1, n,n)

"sparse"

F = h^2 * f(x')

u = A \F

uhat = utrue(x')

error_val = abs [uhat - u]

log-log plot of errors against mesh size is linear. $\alpha = 2$.

$$\log(\text{error}) = C + \alpha \log(h)$$

$$\text{error} = C \cdot h^{\alpha}$$

Maximum Principle (BVP) $\begin{cases} -u''(x) = f(x), & x \in (0,1) \\ u(0) = u(1) = 0 \end{cases}$

$$\boxed{\|u\|_\infty \le \tfrac{1}{8} \|f\|_\infty}$$

(FDM) $\begin{cases} \underbrace{-\left(\dfrac{v_{j+1} - 2v_j + v_{j-1}}{h^2}\right)}_{L_h(v_j)} = f(x_j) & j = 1 \dots n \\ v_0 = v_{n+1} = 0 & \|v\|_{h,\infty} \le \tfrac{1}{8} \|f\|_\infty. \end{cases}$

def. $\|g\|_\infty = \sup_{x \in (0,1)} |g(x)|$, $\|g\|_{h,\infty} = \max_{j=0\dots n+1} |g(x_j)|$

$\|v\|_{h,\infty} = \max_{j=0\dots n+1} |v_j|$

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Def. $f \in C([0,1])$, $u \in C_0^2(0,1)$ the solution to (BVP).

twice differentiable $C([0,1]) \cap C^2((0,1))$

0 on the boundary

Then the <u>truncation error</u> is the discrete vector τ_h defined by

$$\tau_h(x_j) = -\left(\frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{h^2}\right) - f(x_j), \quad j = 1, 2, \ldots n$$

exact solution

not exactly u''

We say (FDM) is <u>consistent</u> with (BVP) if $\lim_{h \to 0} \|\tau_h\|_{h,\infty} = 0$.

<u>Lemma</u> If $f \in C^2([0,1])$, then $\|\tau_h\|_{h,\infty} \leq \frac{\|f''\|_\infty}{12} h^2$

Green's

(1D) $G(x,y) = \begin{cases} y(1-x) & 0 \leq y \leq x \\ x(1-y) & x \leq y \leq 1 \end{cases}$

(2D) $\simeq C \log|x-y|$

(3D) $\simeq C \frac{1}{|x-y|}$

Proof. $$\left| -\frac{(u(x_{j+1}) - 2u(x_j) + u(x_{j-1}))}{h^2} - \underbrace{f(x_j)}_{-u''(x_j)} \right|$$

$$= \left| -\frac{(u(x_{j+1}) - 2u(x_j) + u(x_{j-1}))}{h^2} + u''(x_j) \right|$$

PB1.19...

$$= \frac{h^2}{24} \left| u^{(4)}(x+\xi_1) + u^{(4)}(x-\xi_2) \right|, \quad \xi_1, \xi_2 \in [0,h]$$

$$\leq \frac{h^2}{12} \|\underbrace{u^{(4)}}_{=-f^{(2)}}\|_\infty = \frac{h^2}{12} \|f''\|$$

(convergence)

Theorem: Assume $f \in C^2([0,1])$, u solves (BVP) and V solves (FDM)

Then $\|u-v\|_{h,\infty} \leq \frac{\|f''\|_\infty}{96} h^2$

Proof. def. $e(x_j) := u(x_j) - v_j$, $j = 1, 2, \dots n$

$$-\frac{(e(x_{j+1}) - 2e(x_j) + e(x_{j-1}))}{h^2} = \frac{-(u(x_{j+1}) - 2u(x_j) + u(x_{j-1}))}{h^2} + \frac{v_{j+1} - 2v_j + v_{j-1}}{h^2}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\; \parallel \qquad\qquad\qquad\qquad\qquad\qquad\qquad -f(x_j)$$

$$= \qquad\qquad$$

$$\overset{def}{=} \tau_h(x_j)$$

e solves scheme (FDM) with (RHS) τ_h.

$\Rightarrow$ By the maximum principle,

$$\|e\|_{h,\infty} \leq \frac{1}{8} \|\tau_h\|_{h,\infty} \underset{\text{lemma}}{\leq} \frac{1}{8} \cdot \frac{h^2}{12} \|f''\|_\infty = \frac{\|f''\|_\infty}{96} h^2$$

Implication: As $h \to 0$, $\|u-v\|_{h,\infty} \to 0$.

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(Recitation)

$J : [0,1] \to [a,b]$, $J(x) = bx + (1-x)a$

Function and its inverse are all differentiable

Let $v(x) := u(J(x))$

$-v''(x) = -(b-a)^2 u''(J(x)) = (b-a)^2 f(J(x))$

$t = J(s)$ $\quad \frac{ds}{dt} = (J^{-1})'(t)$

$s = J^{-1}(t)$

$\vdots$

$G(x,y) = (b-a)^2 \tilde{G}(J^{-1}(x), J^{-1}(y))(J^{-1})'(y) = (b-a)\tilde{G}\left(\frac{x-a}{b-a}, \frac{y-a}{b-a}\right),$

$(x,y) \in [a,b] \times [a,b]$

$$\begin{matrix} \alpha & 0 & 0 & \cdots & & \\ -\alpha & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & 2 & & & \\ & & & \ddots & & \\ & & & -1 & 2 & -\beta \\ & & & & & \beta \end{matrix}$$

$f'(x_0) = f(x_0) + \frac{v_0}{h^2}$

$f'(x_{n+1}) = f(x_n) + \frac{v_{n+1}}{h^2}$

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Eigenvalues / Eigenvectors

(Review) Given $A \in \mathbb{R}^{n \times n}$, then an eigenvalue μ satisfies

$$\underbrace{Av = \mu v}_{\text{'stretched'}} \text{ for some } v \neq 0 \in \mathbb{R}^n$$

eigenvector corresponding to μ.

Given (BVP) $\begin{cases} -u''(x) = f(x) \text{ on } [0,1] \\ u(0) = u(1) = 0 \end{cases}$, we define $-u''(x) =: Lu(x)$

An eigenvalue (λ) of L (or $-u''(x)$) satisfies

$$\boxed{(-v'') = Lv = \lambda v} \text{ for some } \boxed{\text{eigenfunction } v \in C^2([0,1])}$$

(L symmetric $\rightarrow$ λ real (as in 'normal' matrices))

$$\boxed{\Rightarrow v''(x) + \lambda v(x) = 0.}$$

We also have the discrete problem: $-\left(\dfrac{v_{j+1} - 2v_j + v_{j-1}}{h^2}\right) = f(x_j)$, $j = 1 \dots n$

↳ An eigenvalue / eigenvector pair (μ, v) satisfies

$$-\left[\frac{v_{j+1} - 2v_j + v_{j-1}}{h^2}\right] = \mu v_j, \quad j = 1 \dots n$$

↳ $Av = \mu v$

$$v = (v_1, v_2 \dots v_n)^T, \quad A = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}$$

Inner Products

For vectors $(u,v) = \sum_{j=1}^{n} v_j u_j$

For functions $u,v : [0,1] \to \mathbb{R}$, $u,v \in C([0,1])$

$$\langle u,v \rangle := \int_0^1 u(x)\, v(x)\, dx$$

Symmetry
non-negative
0 only if one of them 0

$$\boxed{\text{Claim: We have } \langle Lu,u \rangle = \langle -u'',u \rangle \underset{\text{'Dirichlet'}}{>0} \quad (u \neq 0,\ u(0)=u(1)=0)}$$

Proof. $$\int_0^1 \underbrace{-u''(x)}_{dv} \underbrace{u(x)}_{u}\, dx = -\int_0^1 -u'(x)\,u'(x)\,dx + \left[-u'(x)\cancel{u(x)}\right]_{x=0}^{x=1}$$

$$= \int_0^1 (u'(x))^2\, dx \geq 0$$

$$\int_0^1 (u'(x))^2\, dx = 0 \iff u'(x) = 0,\ x \in [0,1]$$

$\Rightarrow u(x) = c$, c constant $\in \mathbb{R}$
$c = 0$ by boundary conditions.

If u is an eigenfunction, then

$$0 < \langle Lu,u \rangle = \langle \lambda u, u \rangle = \lambda \underbrace{\langle u,u \rangle}_{\int (u(x))^2 dx > 0}$$

$\therefore\ \lambda > 0$ (all eigenvalues are positive!)

$\therefore$ We can write $\beta = \sqrt{\lambda}$, $\beta \in \mathbb{R}^+$ and solve

$$u''(x) + \beta^2 u(x) = 0.$$

$u''(x) + \beta^2 u(x) = 0$, $\beta > 0$ has the solution

$u(x) = c_1 \cos(\beta x) + c_2 \sin(\beta x)$

To find c_1 and c_2, we plug in the boundary conditions:

$$0 = u(0) = c_1 \cos(0) + c_2 \sin(0) = c_1 \cos(0) = c_1 \Rightarrow c_1 = 0.$$

To find β, $0 = c_2 \sin\beta \Rightarrow \beta = k\pi,\ k \neq 0$

$\Rightarrow \underline{\underline{\lambda_k}} = (k\pi)^2,\ k \in \mathbb{N}$ and eigenfunctions

$\underline{\underline{u_k}}(x) = \sin(k\pi x), k \in \mathbb{N}$ for $-u''(x) = \lambda u(x)$, $u(0) = u(1) = 0$.

<u>Lemma</u>: The eigenfunctions u_k are orthogonal with respect to $\langle\cdot,\cdot\rangle$:

$$\underbrace{\langle \sin(k\pi x)}_{u_k(x)}, \underbrace{\sin(m\pi x)}_{u_m(x)}\rangle = \begin{cases} 0 & k \neq m \\ 1/2 & k = m \end{cases}$$

Proof:
(similar to proof of Fourier series)

$$\sin\alpha \sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$$

$$\int_0^1 \sin(k\pi x)\sin(m\pi x)\,dx = \frac{1}{2}\int_0^1 \left[\cos((k-m)\pi x) - \cos((k+m)\pi x)\right] dx$$

$$= \frac{1}{2}\left[\frac{1}{(k-m)\pi}\sin((k-m)\pi x) - \frac{1}{(k+m)\pi}\sin((k+m)\pi x)\right]_{x=0}^{x=1}$$

($k \neq m$, $k,m \geq 1$)

$k = m \ldots$

$$= \frac{1}{2}\int_0^1 (1 - \cos(2k\pi x))\,dx = \frac{1}{2} - \frac{1}{2}\left[\frac{1}{2k\pi}\sin(2k\pi x)\right]_0^1$$

$$= \frac{1}{2}.$$

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Recall $\underset{:= Lu(x)}{-u''(x) = \lambda u(x)}$ $\rightarrow U_k(x) = \sin(k\pi x), \lambda_k = (k\pi)^2$ for the continuous case.

Finite Difference Scheme

$$\underbrace{-\left[\frac{v_{j+1} - 2v_j + v_{j-1}}{h^2}\right]}_{:= L_h} = \mu v_j \ , j = 1 \ldots n$$

We rewrite this as

$$Av = \mu v, \text{ where } A = A\frac{1}{h^2}\begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & & \\ & & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}$$

and we want to find v and μ.

Guess: Eigenvectors are $v_k = (\underbrace{\sin(\pi k x_1)}_{= v_k^1}, \underbrace{\sin(\pi k x_2)}_{= v_k^2} \ldots, \underbrace{\sin(\pi k x_j)}_{= v_k^j}, \ldots \underbrace{\sin(\pi k x_n)}_{= v_k^n})$

for $k \in \mathbb{N}$, $v_k^0 = v_k^{n+1} = 0$, $h = (n+1)^{-1}$

For $j = 2, 3, \ldots n-1$,

$$-\left(\frac{\sin(\pi x_{j+1} k) - 2\sin(\pi k x_j) + \sin(\pi k x_{j-1})}{h^2}\right)$$

$$= -\left(\frac{\sin(\pi kjh + \pi kh) - 2\sin(\pi kjh) + \sin(\pi kjh - \pi kh)}{h^2}\right)$$

(using $\sin(\alpha+\beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta$)

$$= -\frac{2\sin(\pi kjh)\cos(\pi kh) - 2\sin(\pi kh)}{h^2}$$

$$= \frac{2}{h^2}(1 - \cos(\pi kh))\sin(\pi khj) = \underbrace{\frac{2}{h^2}(1 - \cos(\pi kh))}_{=: \mu_k} v_j^k$$

$\sin(\pi x_{n+1} k) = \sin(\pi k(n+1)h) = \sin(\pi k) = 0, k \in \mathbb{N}$
$\sin(\pi x_0 k) = \sin(\pi \cdot 0 \cdot k) = 0$, so calculation works for $j=1$ and $j=n$ as well.

$\therefore$ The eigenvalues and eigenvectors for $Av = \mu v$ are given by $\mu_k = \frac{2}{h^2}(1-\cos(\pi k h))$ and $v^k = (\sin(\pi x_1 k), \dots, \sin(\pi x_n k))^T$

We can show there is only a finite number of v's and μ's, even though $k \in \mathbb{N}$

Note ① Periodicity : $\mu_{n+1+k} = \mu_{n+1-k}$, $k = 1 \dots n$

$$\mu_{n+1+k} = \frac{2}{h^2}(1-\cos(\pi(k+n+1)h)) = \frac{2}{h^2}(1-\cos(\pi(n+1)h+\pi k h))$$

$$= \frac{2}{h^2}(1-\cos(\pi+\pi k h))$$

$1, 2, \dots, n+1, n+2, \dots, 2n+2$

(symmetry of cosine about $x=\pi$)
$$= \frac{2}{h^2}(1-\cos(\pi-\pi k h))$$

$$= \frac{2}{h^2}(1-\cos(\pi h(n+1)-\pi k h))$$

$$= \mu_{n+1-k}.$$

② $\mu_{2(n+1)+k} = \mu_k$

$$\mu_{2(n+1)+k} = \frac{2}{h^2}(1-\cos(\pi(2(n+1)+k)h)) = \frac{2}{h^2}(1-\cos(\pi\, 2\underbrace{(n+1)h}_{=1} + \pi k h))$$

$$= \frac{2}{h^2}(1-\cos(2\pi+\pi k h))$$

③ $\mu_{n+1} \Rightarrow$ case

$$= \frac{2}{h^2}(1-\cos(\pi k h)) = \mu_k$$

①+②+③ $\Rightarrow$ only n of them are distinct!

Eigenfunctions: $\cdot\ v^{(n+1)}(x_j) = \sin((n+1)\pi x_j) = \sin((n+1)h\pi j) = \sin(\pi j) = 0$ for all j

$\cdot\ v^{k+n+1} = v^k$: $v^{k+n+1}(x_j) = \sin((k+n+1)\pi h j)$
$= \sin((n+1)h\pi j + k h \pi j)$
$= \pm\sin(k h \pi j) = \pm v^k(x_j) \Rightarrow$ multiple of v^k

$\Rightarrow$ Again, only n are distinct.

Example of Operator (Heat Equation)

$$u_t - u_{xx} = f$$

$$\underbrace{\partial_t u(t,x) - \partial_{xx} u(t,x)}_{Lu(t,x)} = f(t,x)$$

$Lu(t,x)$, where $L := \partial t - \partial_{xx}$

Another Example: $-\partial_x \underbrace{(\alpha(x)\, \partial_x u(x))}_{} = f(x)$ for $\alpha : [0,1] \to \mathbb{R}^+$

$\frac{d^2}{dx^2}$ can be rewritten as $-(\alpha(x) u'(x))'$

In this case $L_\alpha = -\partial_x (\alpha(x) \partial x)$

A possible finite scheme is

$$D_- v_j = \frac{1}{h}(v_j - v_{j-1})$$

$$D_+ v_j = \frac{1}{h}(v_{j+1} - v_j)$$

Then $D_- D_+ v_j = D_- \left(\frac{1}{h}(v_{j+1} - v_j)\right) = \frac{1}{h}\left(\frac{1}{h}(v_{j+1} - v_j) - \frac{1}{h}(v_j - v_{j-1})\right)$

$= \frac{1}{h^2}(v_{j+1} - v_j - v_j + v_{j-1})$

* converges as $O(h)$?

$= \frac{1}{h^2}(v_{j+1} - 2v_j + v_{j-1})$, same as p29.

[whereas $D_- D_- v_j = D_- \left(\frac{1}{h}(v_j - v_{j-1})\right)$

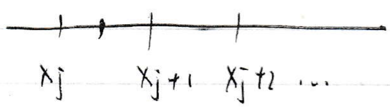
$= \frac{1}{h}\left(\frac{1}{h}(v_j - v_{j-1}) - \frac{1}{h}(v_{j-1} - v_{j-2})\right)$

$= \frac{1}{h^2}(v_j - 2v_{j-1} + v_{j-2})$]

but we would need to handle boundary conditions separately 'midpoint'

Let's do $\quad -D_-\left(k_{j+1/2}\, D_+ v_j\right) = f(x_j)$

where $\quad k_{j+1/2} = k(x_j + h/2)$



Notice if $k=1$, we get the usual scheme for $-u''(x) = f(x)$

By expanding the LHS, $\quad -D_-\left(k_{j+1/2}\frac{1}{h}(v_{j+1} - v_j)\right) = f(x_j)$

$$= -\frac{1}{h}\left[k_{j+1/2}\frac{1}{h}(v_{j+1} - v_j) - k_{j-1/2}\frac{1}{h}(v_j - v_{j-1})\right]$$

$$= -\frac{1}{h^2}\left[k_{j+1/2} v_{j+1} - k_{j+1/2} v_j - k_{j-1/2} v_j + k_{j-1/2} v_{j-1}\right]$$

$$= -\frac{1}{h^2}\left[k_{j+1/2} v_{j+1} - (k_{j+1/2} + k_{j-1/2}) v_j + k_{j-1/2} v_{j-1}\right]$$

Going back to the discrete problem $\boxed{Av = \mu v}$

$$A = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}$$

$v^k = (v_1^k, \ldots, v_n^k)^T \qquad v_j^k = \sin(k\pi x_j)$

$k = 1, \ldots, n \qquad \mu_k = \frac{2}{h^2}(1 - \cos(k\pi h))$

$$\overset{\text{trigonometry}}{=} \frac{4}{h^2}\sin^2\left(\frac{k\pi h}{2}\right)$$

We can show that the vectors v_j^k ($\cancel{j \text{ fixed}}$) are linearly independent with respect to inner product.

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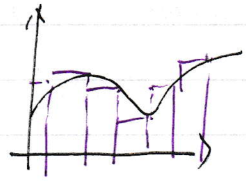
We need an approximate integral.



Define $\underline{\langle v,w\rangle_h} = h\left(\frac{v_0w_0 + v_{n+1}w_{n+1}}{2} + \sum_{i=1}^{n} v_j w_j\right)$

$$\simeq \int_0^1 v(x)w(x)\,dx$$

__Lemma__ $\langle v^k, v^m\rangle_h = \begin{cases} 0 & \text{if } k\neq m \\ 1/2 & \text{if } k=m \end{cases}$

Proof of Case 1.
$$\mu_k \langle v^k, v^m\rangle_h = \langle \mu_k v^k, v^m\rangle_h$$
$$= \langle Av^k, v^m\rangle_h$$
$$= \langle v^k, A^Tv^m\rangle_h$$
$$(A=A^T) \quad = \langle v^k, Av^m\rangle_h$$
$$= \langle v^k, \mu_m v^m\rangle_h$$
$$= \mu_m \langle v^k, v^m\rangle_h$$

$$h\left(\sum_{i=1}^{n} (Av^k)_i\, v^m_i\right)$$
$$= h\left(\sum_{i=1}^{n}\left[\sum_{j=1}^{n} A_{ij} v^k_j\right] v^m_i\right)$$
$$= h\left(\sum_{i=1}^{n}\left[\sum_{j=1}^{n} A^T_{ji} v^k_j\right] v^m_i\right)$$
$$= h\left(\sum_{j=1}^{n}\left[\sum_{i=1}^{n} A^T_{ji} v^m_i\right] v^k_j\right)$$

$k\neq m$

$\therefore \mu_k \neq \mu_m \Rightarrow \langle v^k, v^m\rangle_h = 0.$

Alternate proof

$$h\,(Av)^Tw = h\,v^TA^Tw$$
$$= h\,v^T(A^Tw)$$

Proof of Case 2: Trigonometry.

Go to pg 37

Midterm Review Recitation 9/28

A symmetric $\Rightarrow$ Definite if A has positive eigenvalues

Symmetric + Definite $\Rightarrow$ Full set of eigenvectors

Q3. For an operator L, L is positive definite if $\langle Lu, u\rangle \geq 0$ $\forall u \in C_0^2((a,b))$

with equality iff $u \equiv 0$

(i.e. $\langle Lu, u\rangle > 0$ $\forall u \in C_0^2((a,b)) \setminus \{0\}$)

Q2 $$\langle Lg_1, g_2\rangle = \int_a^b Lg_1(x)\, g_2(x)\, dx$$

$$= \int_a^b (-g_1''(x) + \alpha^2 g_1(x)) \cdot g_2(x)\, dx$$

$$= \int_a^b -g_1''(x) g_2(x) + \alpha^2 g_1(x) g_2(x)\, dx$$

Using integration by parts

$$\int_a^b -g_1''(x) g_2(x) dx = \cancel{-g_1'(x) g_2(x)\Big|_a^b}^{=0} + \int_a^b g_1'(x) g_2'(x)\, dx$$

$$= (\text{apply again}) \quad g_2'(x) g_1(x)\Big|_a^b - \int_a^b g_1(x) g_2''(x)\, dx$$

$\vdots$

$$= \int_a^b g_1(x)(-g_2''(x) + \alpha^2 g_2(x))\, dx$$

$$= \langle g_1, Lg_2\rangle$$

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Def. truncation error (see p64 of textbook), (p23 notes)

$$\tau_h(x_j) = (\mathcal{L}_h u)(x_j) - f(x_j) = o(h^\alpha).$$

$$-\frac{v_{j+1} - 2v_j + v_{j-1}}{h^2} + \alpha^2 v_j$$

$$v_{j+1} = v(x_{j+1}) = v(x_j + h) = v(x_j) + hv'(x_j) + \frac{h^2}{2!}v''(x_j) + \frac{h^3}{3!}v'''(x_j) + \underbrace{R_3(x)}_{o(h^4)}$$

v_j

v_{j-1}

$$\max_{1\le j\le n} |\tau_h(x_j)| \to 0 \text{ as } h\to 0^+$$

(5) $Av = F$, $A = \text{tridiag}_n(-1, 2+\alpha^2 h^2, -1)$

$$v = (v_j)_{j=1}^n = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$$F = h^2 \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix}$$

(6) Non-homogeneous Dirichlet $\longrightarrow$ $F' = h^2 \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix} + \begin{pmatrix} \sigma \\ 0 \\ \vdots \\ 0 \\ \gamma \end{pmatrix}$

(7) $u'(t) = -\alpha u(t)$, $v_0 = u_0 + \varepsilon$

$v(t) = (u_0 + \varepsilon)e^{-\alpha t}$

at $t = -T$, $|u(t) - v(t)| = |\varepsilon| e^{T\alpha} \to 0$, $\varepsilon \to 0$

(Pg 40-41 of textbook)

Dirichlet $u(0)=0$, Neumann $u'(1)=0$

HW Exercise 2.5

FTOC. $u(x)-u(0)=\int_0^x u'(y)\,dy = \int_0^x \left[u'(0)+\int_0^y u''(z)\,dz\right] dy$

$u'(y)-u'(0)=\int_0^y u''(z)\,dz$ $= u(0)+u'(0)x+\int_0^x F(y)\,dy$,

where $F(y)=\int_0^y u''(z)\,dz$

$= u(0)+u'(0)-\int_0^x (x-y)f(y)\,dy$

(by parts) $\int_0^x F(y)\,dy = \int_0^x \overset{dv}{(1)}\, \overset{u}{F(y)}\,dy$

$= yF(y)\Big|_{y=0}^{y=x} - \int_0^x y\,F'(y)\,dy$

$= xF(x) - \int_0^x y\,(-f(y))\,dy$

$= -x\int_0^x f(y)\,dy + \int_0^x y f(y)\,dy$

$= -\int_0^x (x-y)f(y)\,dy$

Recall: (Leibniz) $\dfrac{d}{dx}\int_{a(x)}^{b(x)} f(x,y)\,dy = f(x,b(x))\,b'(x) - f(x,a(x))\,a'(x) + \int_{a(x)}^{b(x)} \dfrac{\partial}{\partial x} f(x,y)\,dy$

$= \int_0^x y f(y)\,dy + \int_x^1 x f(y)\,dy$

$u'(x)=u(0) - \dfrac{d}{dx}\int_0^x (x-y)f(y)\,dy$

$= u'(0) - \int_0^x f(y)\,dy$

$0=u'(1)=u'(0)-\int_0^1 f(y)\,dy$, so $u(x)=u(0)+u'(0)-\int_0^x (x-y)f(y)\,dy$

$= \int_0^1 x f(y)\,dy - \int_0^x (x-y)f(y)\,dy$

from p33

Recall...

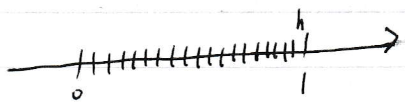
$$\begin{cases} -u''(x) = f(x) \\ u(0) = u(1) = 0 \end{cases} \qquad (BVP)$$

$$-u''(x) = \lambda u(x) \longrightarrow \begin{array}{l} v^k(x) = \sin(k\pi x) \\ \lambda_k = (k\pi)^2 \end{array} \quad k = 1,2,3\ldots$$

Discrete versions...

$$(FDM) \begin{cases} -\dfrac{v_{j+1} - 2v_j + v_{j-1}}{h^2} = f(x_j) \quad j = 1,2,\ldots n \\ v_0 = v_{n+1} = 0 \end{cases}$$

$$\begin{cases} -\dfrac{v_{j+1} - 2v_j + v_{j-1}}{h^2} = \mu v_j . \quad j = 1,2,\ldots n \\ v_0 = v_{n+1} = 0 \end{cases} \longrightarrow \begin{array}{l} v_j^k = \sin(\pi k x_j) \quad j = 1\ldots n, \ k = 1\ldots n \\ \mu_k = \dfrac{4}{h^2}\sin^2\left(\dfrac{\pi h k}{2}\right) \quad k = 1\ldots n \\ \quad = \dfrac{2}{h^2}\left(1 - \cos(\pi h k)\right) \end{array}$$



$$\langle v^k, v^m \rangle_h = \begin{cases} 0 & \text{if } k \neq m \\ 1/2 & k = m \end{cases} \quad (*)$$

WTS $\mu_k \to \lambda_k$ as $h \to 0$

since $\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} \ldots$

$$\therefore \mu^k = \frac{2}{h^2}\left(1 - \cos(k\pi h)\right) = \frac{2}{h^2}\left(1 - 1 + \frac{(k\pi h)^2}{2} - \frac{(k\pi h)^4}{4!} \ldots\right)$$

$$= (k\pi)^2 - \frac{(k\pi)^4}{12}h^2 + o(h^4)$$

$$\to (k\pi)^2 = \lambda^k.$$

Remarks

- Tacoma Bridge resonance example
- $\{v^k\}_{k=1}^n$ are a basis of $\mathbb{R}^n$

$\Rightarrow$ Any vector in $\mathbb{R}^n$ can be written as a linear combination of v^k's.

$$v = \sum_{k=1}^n c_k v_k$$

let $v = \sum_{k=1}^{n} c_k v_k$

$\langle v, v_m \rangle_h = \sum_{k=1}^{n} c_k \langle v_k, v_m \rangle_h \quad = c_m \cdot \frac{1}{2}$

by (*)

$= \langle \sum_{k=1}^{n} c_k v_k, v_m \rangle_h \qquad \Downarrow$

$c_m = 2 \langle v, v_m \rangle_h$

$\Rightarrow v = 2 \sum_{k=1}^{n} \langle v, v_k \rangle_h v_k \qquad (*A)$

(Finite Fourier series)

If v solves (FDM), v is the unknown and we don't know $\langle v, v_k \rangle_h$ a priori

$F = (f(x_1) \dots f(x_n))^T \Rightarrow Av = F$ want to solve

So we must have $\langle Av, v^m \rangle_h = \langle F, v^m \rangle_h$

Rewrite this as $\langle A \cdot 2 \sum_{k=1}^{n} \langle v, v^k \rangle_h v^k, v^m \rangle = \langle F, v^m \rangle_h$

$\downarrow$ A is linear

$= \langle \sum_{k=1}^{n} A(2 \langle v, v^k \rangle_h v^k), v^m \rangle_h$

$\downarrow$ linearity

$= \langle \sum_{k=1}^{n} 2 \langle v, v^k \rangle_h \underbrace{Av^k}_{\mu_k v_k}, v^m \rangle_h$

$\mu_k v_k$ because v^k is an eigenvector

$= \langle \sum_{k=1}^{n} 2 \langle v, v^k \rangle_h \mu_k v^k, v^m \rangle_h$

$$= \sum_{k=1}^{n} 2\langle v, v^k\rangle_h \mu_k \langle v^k, v^m\rangle_h$$

$$= \cancel{2\langle v, v^m\rangle_h \mu_m \cdot \tfrac{1}{2}}$$

$$= \mu_m \langle v, v^m\rangle_h = \langle F, v^m\rangle_h \quad \text{(from RHS)}$$

$$\langle v, v^m\rangle_h = \frac{1}{\mu^m}\langle F, v^m\rangle_h$$

Substituting into $(**)$, we get

$$v = \sum_{m=1}^{n} \underbrace{\frac{2}{\mu^m}}_{\sim O(\frac{1}{h^2})} \langle F, v^m\rangle_h \underbrace{v^m}_{\text{between -1 and 1 (sin's)}}$$

$h = \frac{1}{n+1} \to n = \frac{1}{h} - 1$

$h \to 0 \Rightarrow n \to \infty$

Only low-order terms will matter.

Formally, any $u \in C^1([0,1])$ with $u(0) = u(1) = 0$ can be written as

$$u(x) = \sum_{k=1}^{\infty} 2\langle u, u_k\rangle u_k(x), \text{ where}$$

$$\langle u, u_k\rangle = \int_0^1 u(x) u_k(x)\,dx \text{ and } u_k(x) = \sin(\pi k x)$$

'real' Fourier series!!

10/4/21

Heat Equation / Diffusion Equation

In 1D , $u = u(t,x) : [0,\infty) \times \mathbb{R} \to \mathbb{R}$

$$(H) \begin{cases} \partial_t u(t,x) - \partial_{xx} u(t,x) = 0 & t > 0, x \in \mathbb{R} \\ u(0,x) = u_0(x) & x \in \mathbb{R} \end{cases}$$

Multi-Dimensional $u = u(t,x) : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$, $x = (x_1, x_2, \dots x_d)$

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) = 0 & t > 0, x \in \mathbb{R}^d \\ u(0,x) = u_0(x) & x \in \mathbb{R}^d \end{cases}$$

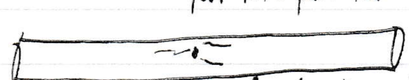
$$\Delta u(t,x) = \partial_{x_1 x_1} u(t,x) + \partial_{x_2 x_2} u(t,x) + \dots + \partial_{x_d x_d} u(t,x)$$

The heat equation arose out of the study of Brownian Motion (Robert Brown 1826, pollen particles under microscope)

Observations
- particle paths are very irregular
- motion of any 2 particles appears independent

1905 (Albert Einstein)

Pour ink particles at time $t=0$



long thin tube filled with wate

- How does the ink particle density change over time?

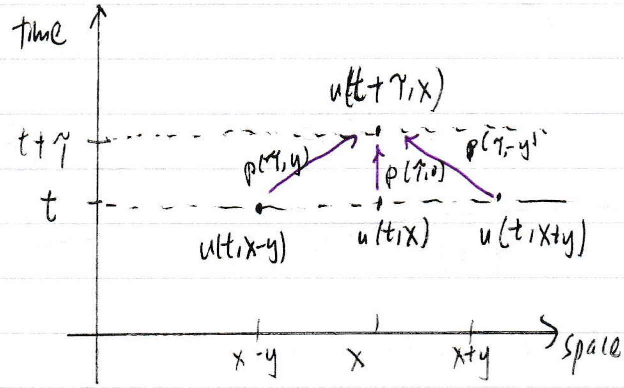
$= u(t,x)$, time t and location x

- Let the probability density that a particle moves by y in time span τ $= p(\tau, y)$

*This is independent of local time t and location x

(the particle doesn't know what time / where it is . . .)

What is $u(t+\tau,x)$?



$N(0,\tau)$ Brownian motion!

Assumption. $\cdot\, p(\tau,y) = p(\tau,-y)$

$\cdot\, p(\tau,y)$ is small for small τ and large y (a nice Gaussian would work)

$$u(t+\tau,x) = \int_{\mathbb{R}} u(t,x+y)\, p(\tau,y)\, dy$$

$$\overset{\text{Taylor}}{=} \int_{\mathbb{R}} \left(u(t,x) - y\,\partial_x u(t,x) + \frac{y^2}{2}\partial_{xx} u(t,x) + O(y^3)\right) p(\tau,y)\, dy$$

$$= u(t,x)\underbrace{\int_{\mathbb{R}} p(\tau,y)\,dy}_{=1} - \partial_x u(t,x) \underbrace{\int_{\mathbb{R}} y\,p(\tau,y)\,dy}_{(*)} + \frac{\partial_{xx}u(t,x)}{2}\underbrace{\int y^2 p(\tau,y)\,dy}_{\substack{\text{second moment} \\ =\text{variance in this case} \\ = D\tau,\ D>0 \text{ constant}}} + \underbrace{\int O(y^3)\, p(\tau,y)\,dy}_{\text{"small"}}$$

$$(*) = \underbrace{\int_{-\infty}^{0} y\,p(\tau,y)\,dy}_{} + \int_0^{\infty} y\,p(\tau,y)\,dy = 0$$

$$\left(= \int_{\infty}^{0} -y\,p(\tau,-y)(-dy) = -\int_0^{\infty} y\,p(\tau,-y)\,dy = -\int_0^{\infty} y\,p(\tau,y)\,dy \right)$$

$$\therefore\ u(t+\tau,x) = u(t,x) + \frac{D\tau}{2}\partial_{xx}u(t,x) + \text{"small error"}$$

$O(\tau^2)$ assumptions

$$\frac{u(t+\tau,x)-u(t,x)}{\tau} = \frac{D}{2}\partial_{xx} u(t,x) + \text{"small error"}\ O(\tau)$$

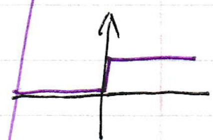
just need $\alpha > 1$

send $\tau \to 0$, $\boxed{\partial_t u(t,x) = \frac{D}{2}\partial_{xx} u(t,x)}$

Heat Equation, encore...

Heaviside initial conditions

$$\boxed{\begin{array}{ll} u_t - u_{xx} = 0 & , t > 0, x \in \mathbb{R} \\ u(0,x) = H(x) & x \in \mathbb{R} \quad , \text{ where } H(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases} \end{array}}$$



let $c \in \mathbb{R}^+ \setminus \{0\}$, and let $v(t,x) = u(c^2 t, cx)$

$$\partial_t v(t,x) = \partial_t u(c^2 t, cx) \cdot c^2$$

$$\partial_x v(t,x) = \partial_x u(c^2 t, cx) \cdot c$$

$$\partial_x (\partial_x v(t,x)) = \partial_{xx} u(c^2 t, cx) \cdot c^2$$

$$\partial_t v - \partial_{xx} v = c^2 (\partial_t u - \partial_{xx} u) = 0$$

$$v(0,x) = u(0,cx) = H(cx) = H(x) \text{ if } c > 0.$$

$\therefore$ If u is a solution, there are many 'other solutions' v.

This tells us that the solution should be constant along lines $(c^2 t, cx)$, $c > 0$

$\Rightarrow$ Solution is a function of $y = \frac{x}{\sqrt{t}} = \frac{cx}{\sqrt{c^2 t}}$

So we write $u(t,x) := w(y)$, $y = \frac{x}{\sqrt{t}}$

$$u_t - u_{xx} = 0, \quad \partial_t \left(w\left(\tfrac{x}{\sqrt{t}}\right)\right) = w'\left(\tfrac{x}{\sqrt{t}}\right)\left(-\tfrac{1}{2}\tfrac{x}{t^{3/2}}\right)$$

$$\partial_x \left(w\left(\tfrac{x}{\sqrt{t}}\right)\right) = w'\left(\tfrac{x}{\sqrt{t}}\right)\tfrac{1}{\sqrt{t}}$$

$$\partial_{xx} \left(w\left(\tfrac{x}{\sqrt{t}}\right)\right) = w''\left(\tfrac{x}{\sqrt{t}}\right)\tfrac{1}{t}$$

Substituting back to $u_t - u_{xx} = 0$

$$w'\left(\frac{x}{\sqrt{t}}\right)\left(-\frac{1}{2}\frac{x}{t^{3/2}}\right) - w''\left(\frac{x}{\sqrt{t}}\right)\frac{1}{t} = 0.$$

$\times -t$

$$\frac{1}{2}w'(y)\frac{x}{\sqrt{t}} + w''(y) = 0$$

$$\underline{\frac{1}{2}yw' + w''} = 0$$

We have reduced the PDE to an ODE

10/8/2021

How to solve this PDE? Note that $\left(e^{y^2/4}\right)' = \frac{y}{2}\left(e^{y^2/4}\right)$ - integrating factor

So we multiply by $e^{y^2/4}$ to get

$$\frac{1}{2}ye^{y^2/4}w'(y) + e^{y^2/4}w''(y) = 0$$

$\Rightarrow$ product rule

$$\left(e^{y^2/4}w'(y)\right)' = 0$$

$$\int_{z_0}^{z}\left(e^{y^2/4}w'(y)\right)'dy = \alpha \quad \text{for some } \alpha \in \mathbb{R} \text{ [FTOC 1]}$$

should be 0 already...

$$e^{z^2/4}w'(z) - e^{z_0^2/4}w'(z_0) = \alpha$$

$$w'(z) = e^{-z^2/4}\underbrace{\left(\alpha + e^{z_0^2/4}w'(z_0)\right)}_{\text{call this } \alpha \text{ henceforth}}$$

$$= e^{-z^2/4}\alpha$$

$$w(y) = \int_{y_0}^{y} e^{-z^2/4}\alpha\, dz + w(y_0)$$

$$u(t,x) = w\left(\frac{x}{\sqrt{t}}\right), \; y = \frac{x}{\sqrt{t}}$$

Continuing from $w(y) = \int_{y_0}^{y} e^{-z^2/4} \alpha \, dz + w(y_0)$,

If $y_0 \to -\infty$, then $w(y_0) = 0$ by our initial condition. $y = \frac{x}{\sqrt{t}} \to > 0$ always $\nearrow < 0$

$$w(y) = \alpha \int_{-\infty}^{y} e^{-z^2/4} \, dz$$

$$\theta = \frac{z}{2} \qquad \overset{\frac{dz}{d\theta}}{=} 2\alpha \int_{-\infty}^{y/2} e^{-\theta^2} \, d\theta$$

let $y \to \infty$. we get

$y \to \infty \Rightarrow x > 0$

$$1 = \lim_{y \to \infty} w(y) = 2\alpha \int_{-\infty}^{\infty} e^{-\theta^2} \, d\theta = 2\alpha \sqrt{\pi}$$

recall $\frac{1}{\sqrt{2\pi} \cdot \sqrt{2}} \int_{-\infty}^{\infty} e^{-\theta^2/2} \, d\theta = 1$.

So $\alpha = \frac{1}{2\sqrt{\pi}}$.

Finally, $w(y) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{y/2} e^{-\theta^2} \, d\theta$.

$$\therefore \boxed{u(t,x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x/2\sqrt{t}} e^{-\theta^2} \, d\theta}$$

Using the fact that $u(t,x)$ solves $u_t - u_{xx} = 0 \Rightarrow V(t,x) = \int_{-\infty}^{\infty} u(x-y, t) g(y) \, dy$ is another solution for any function $g(y)$, as does $u_x(x,t)$, we eventually deduce the explicit form

$$\boxed{u(t,x) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y) \, dy = \int_{-\infty}^{\infty} S(t, x-y, \cancel{t}) \phi(y) \, dy}$$ to $\begin{cases} u_t - k u_{xx} = 0 & (x \in \mathbb{R}, t > 0) \\ u(x,0) = \phi(x) \end{cases}$

where $\boxed{S(t,x) = \frac{1}{\sqrt{4\pi k t}} e^{-x^2/4kt}}$ is known as the heat kernel.

not homogeneous $\Rightarrow$ harder to solve

open (to avoid contradicting boundary conditions)

Going back to the Heat Equation, now with a different initial condition:

(bounded!)

$$(H)\begin{cases} u_t - u_{xx} = 0, \quad t>0,\ x\in(0,1) & -(1) \\ u(t,0)=u(t,1)=0,\ t>0 & -(2) \\ u(0,x)=u_0(x) \leftarrow \text{defined later} \end{cases}$$

(Separation of variables) Assume $u(t,x)=T(t)X(x)$, then by substitution into (1), we get

$$T'(t)X(x) - T(t)X''(x) = 0$$

Assuming $T(t)\neq 0$, $X(x)\neq 0$, $\quad \dfrac{T'(t)}{T(t)} = \dfrac{X''(x)}{X(x)}$

Notice that if we vary t on LHS, RHS should stay the same as it is independent of t

$$\Rightarrow \frac{T'(t)}{T(t)} \text{ is also independent of } t$$

Similarly, $\dfrac{X''(x)}{X(x)}$ is independent of x.

This means $\dfrac{T'(t)}{T(t)} = \dfrac{X''(x)}{X(x)} = -\lambda$ for some $\lambda\in\mathbb{R}$.

Two ODEs to solve:

(1) $\begin{cases} X''(x) = -\lambda X(x) \\ X(1)=X(0)=0 \end{cases}$ $\leftarrow$ eigenvalue problem!

let $X_k(x) = \sin(k\pi x)$, then $X_k''(x) = -(k\pi)^2\sin(k\pi x)$

So let $\lambda = (k\pi)^2 \Rightarrow \sin(k\pi x)$ is a solution.

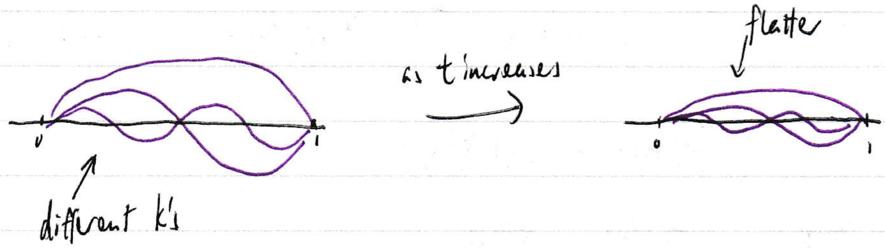
(2) $T'(t) = -\lambda T(t)$, $T(t) = C_k e^{-\lambda t} = C_k e^{-(k\pi)^2 t}$

So $\underbrace{u(t,x) = C_k \sin(k\pi x)e^{-(k\pi)^2 t}}_{T(t)X(x)}$ are special solutions of the heat equation.

$$u(t,x) = C_k \sin(k\pi x)\, e^{-(k\pi)^2 t}$$

We can check this satisfies $u(t,0) = u(t,1) = 0$.

So $u(t,x) = C_k \sin(k\pi x)\, e^{-(k\pi)^2 t}$ are special solutions of the heat equation with $u(t,0) = u(t,1) = 0$.



Superposition

→ If v, w are solutions to (H), then $\forall c_1, c_2 \in \mathbb{R}$, $c_1 v + c_2 w$ is also a solution to (H).

If $u_0(x) = \sum_{k=1}^{N} C_k \sin(k\pi x)$, then by superposition and the special solutions formula,

$$u(t,x) = \sum_{k=1}^{N} C_k e^{-(k\pi)^2 t} \sin(k\pi x)$$

Use Fourier series to approximate!

$$(H)\begin{cases} u_t - u_{xx} = 0, & x\in(0,1),\ t>0 \\ u(0,x) = u_0(x), & x\in[0,1] \\ u(t,0) = u(t,1) = 0, & t>0 \end{cases}$$

Energy Methods (different way of arguing)

$$E(t) := \int_0^1 (u(t,x))^2\,dx \quad \text{'energy'}$$

(same sort of norm: Pg 12 ≥ 0, abs. homogeneous, Δ-inequality)

Assumptions
- u satisfies (H)
- $u, u_t, u_x, u_{xx} \in C([0,\infty)\times[0,1])$

Let's differentiate $E(t)$:

$$\frac{d}{dt}E(t) = \frac{d}{dt}\int_0^1 (u(t,x))^2\,dx$$

$$\boxed{\underset{(u(t,x)^2 \text{ cont. diff.})}{\overset{\text{Leibniz rule}}{=}} \int_0^1 \partial_t (u(t,x))^2\,dx}\quad \text{proof p 54.}$$

$$\overset{\text{chain rule}}{=} 2\int_0^1 u(t,x)\,\partial_t u(t,x)\,dx$$

$$\overset{(H)}{=} 2\int_0^1 u(t,x)\,\partial_{xx} u(t,x)\,dx$$

$$\overset{\text{by parts}}{=} \underbrace{\cancel{2u(t,x)\,\partial_x u(t,x)\Big|_{x=0}^{x=1}}}_{\substack{=0 \text{ by boundary conditions} \\ \text{(could be } \neq 0 \text{ in other cases)}}} - 2\int_0^1 \underbrace{(\partial_x u(t,x))^2}_{\geq 0}\,dx \leq 0$$

$\frac{d}{dt}E(t) \leq 0 \leftrightarrow$ Energy is non-increasing

$\Rightarrow E(t) \leq E(0) = \int_0^1 (u_0(x))^2\,dx = \int_0^1 f(x)^2\,dx$ (sometimes)

- The energy stays finite.

Energy/stability Estimate

Recall $\|u\|_\infty \leq \frac{1}{8}\|f\|_\infty$ (pg 12)

Now define $\|f\|_{L^2} := \left(\int_0^1 (f(x))^2 dx \right)^{1/2}$

"L^2/energy-norm"

(take sqrt of previous page's inequality)

So we have $\|u(t,x)\|_{L^2} \leq \|u_0\|_{L^2}$

→ This is a bound on solution with respect to input data.

Uniqueness of solutions?

Let u, v be 2 solutions of (H) with initial data u_0, v_0

u solves $\begin{cases} u_t - u_{xx} = 0 \\ u(0,x) = u_0(x) \\ u(t,1) = u(t,0) = 0 \end{cases}$, and v solves $\begin{cases} v_t - v_{xx} = 0 \\ v(0,x) = v_0(x) \\ v(t,1) = v(t,0) = 0 \end{cases}$

Assume $u, v, u_x, v_x, u_{xx}, v_{xx}, u_t, v_t$ are continuous.

Theorem: $\boxed{\int_0^1 \left(u(t,x) - v(t,x)\right)^2 dx \leq \int_0^1 \left(u_0(x) - v_0(x)\right)^2 dx}$

$\|u(t,\cdot) - v(t,\cdot)\|_{L^2} \leq \|u_0 - v_0\|_{L^2}$

Proof. Define $w(t,x) = u(t,x) - v(t,x)$

Observe that $\partial_t w = \partial_t u - \partial_t v = \partial_{xx} u - \partial_{xx} v = \partial_{xx}(u-v) = \partial_{xx} w$ (Boundary conditions also hold)
Initial conditions

Let $w(0,x) = u_0(x) - v_0(x) := w_0(x)$

Do energy estimate for w:

$$E(t) = \int_0^1 (w(t,x))^2 dx \le E(0) = \int_0^1 (w_0(x))^2 dx$$

$$\Rightarrow \int_0^1 (u(t,x) - v(t,x))^2 dx \le \int_0^1 (u_0(X) - v_0(X))^2 dx \quad ✓$$

Corollary: If u is a solution of (H) with $u, u_x, u_{xx}, u_t \in C([0,\infty) \times [0,1])$ then it is unique.

Reason: AFSOC there are 2 solutions u_1 and u_2. $w(t,x) := u_1(t,x) - u_2(t,x)$

$w_t = (u_1 - u_2)_t = (u_1)_t - (u_2)_t$
$= (u_1)_{xx} - (u_2)_{xx}$
$= (u_1 - u_2)_{xx} = w_{xx}$
$w(t,0) = u_1(t,0) - u_2(t,0) = 0 - 0 = 0$
$w(t,1) = u_1(t,1) - u_2(t,1) = 0 - 0 = 0$
$w(0,x) = u_1(0,x) - u_2(0,x)$
$= u_0(x) - u_0(x) = 0$
w solves (H) with $w_0 \equiv 0$

Plug in, noting that $u_{1,0}(x) = u_{2,0}(x)$

$$\int_0^1 \overbrace{(u_1(t,x) - u_2(t,x))^2}^{\ge 0} dx \le 0$$

$\therefore u_1(t,x) = u_2(t,x)$, $x \in [0,1]$, $t \ge 0$

$e(x) = u_1(x) - u_2(x)$

$\Rightarrow e''(x) = 0 \quad , \quad \int_0^1 e(x)\,dx = 0$

$$\int_0^1 e''(x)\, e(x)\, dx = 0$$

$\downarrow$ integration by parts

$$= -\int (e'(x))^2\, dx + e'e \Big|_{x=0}^{x=1}$$

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Rec 7.

1. $u_t = u_{xx}$

$u(0,t)=0,\ u(L,t)=0$ for all $t \geq 0$

$u(x,0) = f(x)$

compare to 'direct proof' p 47.

W.T.S $\int_0^L (u(x,t))^2 dx \leq \int_0^L (f(x))^2 dx \quad \forall t \geq 0.$

Hint: $u \cdot u_t = \frac{1}{2} \partial_t u^2$ (chain rule)

Then $u \cdot u_t = \underline{u \cdot u_{xx} = \frac{1}{2} \partial_t u^2}$

$\int \frac{1}{2} \partial_t u^2 \, dt = \int u \cdot u_{xx} \, dt$

(integrating w.r.t t)

LHS: $\int_0^t \frac{1}{2} \underline{\partial_t (u(x,s))^2} \, ds = \frac{1}{2} (u(x,t)^2 - u(x,0)^2)$

$= \frac{1}{2}(u^2 - f^2).$

RHS

Integrating w.r.t x, $\frac{1}{2} \int_0^L u^2 - f^2 \, dx = \int_0^L \int_0^t u \cdot u_{xx} \, ds \, dx$

$= \int_0^t \int_0^L u \cdot u_{xx} \, dx \, ds$

(Note $\int_0^L u \cdot u_{xx} \, dx$

$= u \cdot u_x \Big|_{x=0}^{x=L} - \int_0^L (u_x)^2 dx \leq 0$)

$= \int_0^t \underbrace{F(L)}_{\leq 0} \, ds$

≤ 0

$\therefore \int_0^L (u(x,t))^2 dx \leq \int_0^L (f(x))^2 dx.$

L2 norm: $\|g\|_{L^2} = \int g^2$ energy method controls L^2 $\|u\|_{L^2} \le \|f\|_{L^2}$ ← initial data

Sup norm/ L^∞ norm $\|g\|_{L^\infty} = \sup |g(x)|$ max principle controls L^∞ $\|u\|_\infty \le C \cdot \|\ldots\|_\infty$

Let u_1 and u_2 be two solutions to the heat equation given.

Then $v(x,t) := u_1(x,t) - u_2(x,t)$ solves the same equation

with $v(x,0) = 0 = g(x)$

$$\frac{d}{dt}\int_0^L (v(x,t))^2\,dx \le \int_0^L (\underbrace{g(x)}_{0})^2\,dx = 0.$$

We conclude that $(v(x,t))^2 := 0$,

i.e. $u_1(x,t) = u_2(x,t)$

So the solution ~~would~~ unique.

Another variant of the Heat Equation... Non-homogeneous!!

$$E(t) = \int_0^1 (u(t,x))^2\,dx$$

$$\begin{cases} u_t - u_{xx} = -u^3 & x \in (0,1),\ t>0 \\ u(0,x) = u_0(x) & x \in (0,1) \\ u(t,0) = u(t,1) = 0 & t \geq 0 \end{cases}$$

$u, u_t, u_x, u_{xx} \in C([0,1] \times [0,\infty))$

We see that $\frac{d}{dt} E(t) = \int_0^1 \partial_t (u(t,x))^2\,dx$

$$= 2\int_0^1 u(t,x)\,\partial_t u(t,x)\,dx$$

plug in equations $= 2\int_0^1 u(t,x)\left(\partial_{xx} u(t,x) - (u(t,x))^3\right)dx$

$$= 2\int_0^1 \overset{u}{u(t,x)}\,\overset{dv}{\partial_{xx} u(t,x)}\,dx - 2\int_0^1 (u(t,x))^4\,dx$$

$$= \underbrace{-2\int_0^1 (\partial_x u(t,x))^2\,dx}_{\text{integration by parts and boundary conditions}} - 2\int_0^1 (u(t,x))^4\,dx \leq 0$$

Note ① doesn't work for $u_t - u_{xx} = u^3$

② Cannot control stability of solutions

$-u^3 - (-v^3) \neq 0$! Non-linear RHS

A detour to real analysis. When can we exchange integration and differentiation?

$f: [a,b] \times [c,d] \to \mathbb{R}$, $f = f(x,y)$

$F(y) := \int_a^b f(x,y)\, dx$

$F'(y) \overset{?}{=} \int_a^b \partial_y f(x,y)\, dx$

Proposition: Assume f and $\partial_y f$ are continuous on $[a,b] \times [c,d]$.
Then $F'(y)$ exists for all $y \in [c,d]$ and $F'(y) = \int_a^b \partial_y f(x,y)\, dx$.

Proof: $F'(y) = \lim_{h \to 0} \dfrac{F(y+h) - F(y)}{h}$

w.T.S For any $\tilde{\varepsilon}$, there exists δ such that for all $|h| < \delta$

$$\left| \frac{F(y+h)-F(y)}{h} - \int_a^b \partial_y f(x,y)\, dx \right| < \tilde{\varepsilon}.$$ (show that limit exists)

For $\tilde{\varepsilon} > 0$, by continuity on $[a,b] \times [c,d]$ we can find $\underset{= \delta(x)}{\delta} > 0$ s.t.

$|f_y(x,y_1) - f_y(x,y_2)| < \varepsilon$ whenever $|y_1 - y_2| < \delta$.

In fact,

$\sup_{x \in [a,b]} |f_y(x,y_1) - f_y(x,y_2)| < \varepsilon$ for some δ, $|y_1 - y_2| < \delta$. $(*)$

~~closed~~, compact.. see 21-355 Chp 6

since $\partial_y f$ continuous on $[a,b] \times [c,d]$ $\Rightarrow$ uniformly continuous.

21-469 $\quad LHS = \int_a^b \frac{f(x,y+h)-f(x,y)}{h}\,dx$  $\quad$ 10/13/2021

mean value theorem.

$$\frac{F(y+h)-F(y)}{h} = \int_a^b \partial_y f(x,y+\tilde{h})\,dx, \quad \tilde{h} \in [\min(0,h), \max(0,h)]$$

$$\Rightarrow \left| \frac{F(y+h)-F(y)}{h} - \int_a^b \partial_y f(x,y)\,dx \right| = \left| \int_a^b \partial_y f(x,y+\tilde{h})\,dx - \int_a^b \partial_y f(x,y)\,dx \right|$$

$$= \left| \int_a^b \left(\partial_y f(x,y+\tilde{h}) - \partial_y f(x,y)\right) dx \right|$$

$$\leq \int_a^b \left| \partial_y f(x,y+\tilde{h}) - \partial_y f(x,y) \right| dx \quad (\triangle)$$

In $(*)$ pick $\varepsilon = \frac{\tilde{\varepsilon}}{|b-a|}$, then for this ε there is $\delta > 0$ s.t.

for $|h| < \delta$, we have $\quad \sup_x |\partial_y f(x,y) - \partial_y f(x,y+h)| < \varepsilon$

$$\therefore \quad (\triangle) \leq \int_a^b \underbrace{\sup_x |\partial_y f(x,y) - \partial_y f(x,y+\tilde{h})|}_{<\varepsilon}\,dx \leq \int_a^b \varepsilon\,dx = (b-a)\varepsilon = \tilde{\varepsilon} \checkmark$$

JJ

21-469

Finite Differences

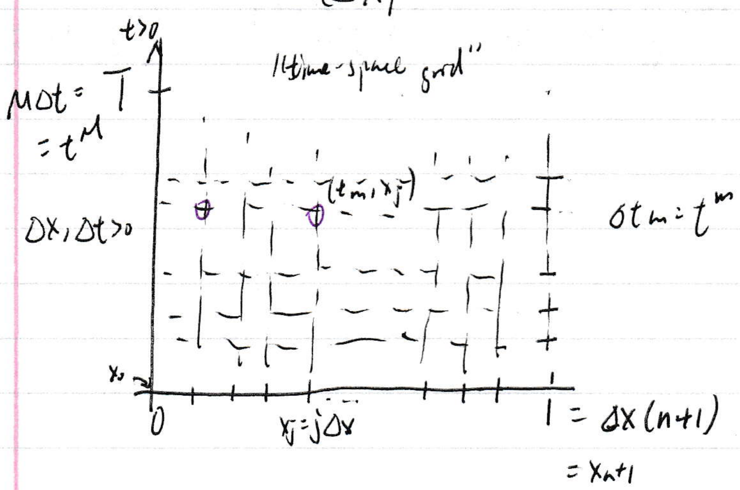
Useful for 'strange' equations like non-homogeneous heat eqn (or $u_{xx} - \sum_{k=0}^{\infty}(\cdot) = 0 \ldots$)

$$(H) \begin{cases} u_t - u_{xx} = 0 & x \in (0,1), t > 0 \\ u(0,x) = u_0(x) & x \in (0,1) \\ u(t,0) = u(t,1) = 0 & t \geq 0 \end{cases}$$

By Taylor,

$$\frac{u(t+\Delta t, x) - u(t,x)}{\Delta t} = u_t(t,x) + \underbrace{\frac{\Delta t}{2} u_{tt}(t,x)}_{1^{st}\text{ order}} + O((\Delta t)^2), \ \Delta t > 0.$$

$$\frac{u(t, x+\Delta x) - 2u(t,x) + u(t, x-\Delta x)}{(\Delta x)^2} = u_{xx}(t,x) + \underbrace{O((\Delta x)^2)}_{2^{nd}\text{ order}}$$



where $M, n \in \mathbb{N}$

We try to find $v_j^m \simeq u(t^m, x_j)$ (m ← time, j ← space)

21-469 'Forward Euler' / 'Explicit Heat' 10/15/2021

$$(FH)\begin{cases} \dfrac{v_j^{m+1} - v_j^m}{\Delta t} - \kappa \dfrac{v_{j+1}^m - 2v_j^m + v_{j-1}^m}{\Delta x^2} = 0 & j = 1, 2, \ldots n, \; m \geq 0 \\ v_j^0 = u_0(x_j) & j = 1, \ldots n \quad \text{initial condition} \\ v_0^m = v_{n+1}^m = 0 & \text{boundary condition} \end{cases}$$

$$\boxed{v_j^{m+1} = v_j^m + \overset{\text{optional}}{\kappa} \frac{\Delta t}{\Delta x^2}\left(v_{j+1}^m - 2v_j^m + v_{j-1}^m\right)}$$

↑ march in time

$\Rightarrow$ Compute v_j^1 based on v_j^0

$\Rightarrow$ Then compute v_j^2 based on v_j^1

$\vdots$

$\Rightarrow$ Compute v_j^{m+1} based on v_j^m

let $\kappa \frac{\Delta t}{\Delta x^2} =: r$.

(explicitheat.m)

Then $v_j^{m+1} = (1-2r)v_j^m + r v_{j+1}^m + r v_{j-1}^m$

Simulation:

$u_0(x) = \sin(2\pi x) \rightarrow$ exact solution is $u(t, x) = e^{-(2\pi)^2 t} \sin(2\pi x)$

** Need time step (Δt) to be much smaller than space (Δx)! why? $\approx c\,\Delta x^2$ (0.1, $\frac{0.01}{2}$)

$$|v_j^{m+1}| = |(1-2r)v_j^m + r v_{j-1}^m + r v_{j+1}^m|$$

$$\leq |(1-2r)v_j^m| + |r v_{j-1}^m| + |r v_{j+1}^m|$$

$$|v_j^{m+1}| = |(1-2r)v_j^m + rv_{j-1}^m + rv_{j+1}^m|$$

$$\leq |(1-2r)v_j^m| + |rv_{j-1}^m| + |rv_{j+1}^m|$$

$$= |(1-2r)v_j^m| + r|v_{j-1}^m| + r|v_{j+1}^m|$$

(if $1-2r \geq 0$)

$$= (1-2r)|v_j^m| + r|v_{j-1}^m| + r|v_{j+1}^m|$$

$$=: \square$$

$$(\square) \leq (1-2r)\max(|v_j^m|, |v_{j-1}^m|, |v_{j+1}^m|) + r\max(|v_j^m|, |v_{j+1}^m|, |v_{j-1}^m|) + r\max(|v_j^m|, |v_{j-1}^m|, |v_{j+1}^m|)$$

$$= \underbrace{(1-2r+r+r)}_{=1}\max(|v_j^m|, |v_{j-1}^m|, |v_{j+1}^m|)$$

$$\therefore |v_j^{m+1}| \leq \max(|v_j^m|, |v_{j-1}^m|, |v_{j+1}^m|) \leq \max_{j=1\ldots n}|v_j^m| = \|v^m\|_{n,\infty},$$

where $v^m = (v_1^m \ldots v_n^m)$

$$\|v^{m+1}\|_{n,\infty} = \max_{j=1\ldots n}|v_j^{m+1}| \leq \max_{k=1\ldots n}|v_k^m| = \|v^m\|_{n,\infty} \leq \|v^{m-1}\|_{n,\infty}$$

($\Delta x, \infty$)

$$\vdots$$

$$\leq \|v_0\|_{n,\infty} \leq \|u_0\|_{n,\infty} < \infty$$

We really needed $1-2r \geq 0 \Leftrightarrow r \leq \frac{1}{2}$

$$\Leftrightarrow \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$$

$$\Leftrightarrow \Delta t \leq \frac{\Delta x^2}{2}\,!$$

The problem is that this scheme is <u>slow</u>, Δt must be small, very small...

$\Delta t \simeq \Delta x$

$\Delta t > \Delta x$ catastrophe – solution unbounded, but energy equation insists that it should be