ƒU-413 Cheatsheet

1. Definitions
2. Theorems

Application: Product in $\mathbb{C}/X$ is the pullback in $\mathbb{C}$

$$\begin{tikzcd} & & C \arrow[ddl, bend left, "h_2"] \\ & P \arrow[dl, "f"'] \arrow[d, dashed] \arrow[dr, "g"] & \\ B \arrow[r] & X & C \arrow[l] \end{tikzcd}$$

h_1

Terminal category $\circlearrowleft *$

(𝟚) Arrow category (finite)

$$\begin{tikzcd} A \arrow[loop right] \arrow[d, "f"] \\ B \arrow[loop right] \end{tikzcd}$$

2 Discrete category

$$\begin{tikzcd} A \arrow[loop right] \\ B \arrow[loop right] \end{tikzcd}$$

≠ Preord

"Thin"/"Posetal" category: $|\hom(A,B)| = 1$

Slice category $\mathbb{C}/A$

$$\begin{tikzcd} B \arrow[r, "h"] \arrow[dr, "f"'] & C \arrow[r, "k"] \arrow[d, "g"] & D \arrow[dl, "l"] \\ & A & \end{tikzcd}$$

$A/\mathbb{C}$

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[d, "g"] \arrow[dr, "h"] & \\ B \arrow[r, "i"'] & C \arrow[r, "j"'] & D \end{tikzcd}$$

Subcategories $\leftrightarrows$ 'Inclusion functor'

Opposite category ($\mathbb{C}^{op}$): $A \underset{f}{\overset{f}{\rightleftarrows}} B \underset{g}{\overset{g}{\rightleftarrows}} C$

$f \circ_{\mathbb{C}^{op}} g = g \circ_{\mathbb{C}^{op}} f$

	Initial Objects	Terminal obj
Set	$\emptyset$	$\{*\}$ (terminal cat)
Pos	$\emptyset$	$(1, \leq)$
Mon	$(\{1\}, \cdot, 1)$	$(\{1\}, \cdot, 1)$

→ $\mathbb{N}$ is a free monoid on 1 generator

→ Not cartesian-closed

Arrow category : Objects are morphisms $a: a_0 \to a_1$

Morphisms are the commutative squares

$$\begin{tikzcd} a_0 \arrow[r, "f_0"] \arrow[d, "a"] & b_0 \arrow[r, "g_0"] \arrow[d, "b"] & c_0 \arrow[d, "c"] \\ a_1 \arrow[r, "f_1"'] & b_1 \arrow[r, "g_1"'] & c_1 \end{tikzcd}$$

$f = g = id_{\mathbb{C}}$ (identity functor)

Comma category: Let $F: \mathbb{C} \to \mathbb{E}$ and $G: \mathbb{D} \to \mathbb{E}$ be functors.

f/g is given by

objects $(c_1, d_1, \alpha_1), (c_2, d_2, \alpha_2)$

morphisms (β, γ), $\beta: c_1 \xrightarrow{\in\mathbb{C}} c_2$, $\gamma: d_1 \to d_2$

$$\begin{tikzcd} f(c_1) \arrow[r, "f(\beta)"] \arrow[d, "\alpha_1"] & f(c_2) \arrow[d, "\alpha_2"] \\ g(d_1) \arrow[r, "g(\gamma)"] & g(d_2) \end{tikzcd}$$

$(c_1, d_1, \alpha_1) \xrightarrow{(\beta,\gamma)} (c_2, d_2, \alpha_2)$

(HW5, HW8)

If locally small $\mathbb{C}$ has terminal object 1, any $A \in \mathbb{C}$, the set of global elements is $\hom(1, A)$.

$\mathbb{C}$ is well-pointed if $\hom(1, -): \mathbb{C} \to Set$ is faithful. ($\hom(1,A) \underset{\hom(1,g)}{\overset{\hom(1,f)}{\rightrightarrows}} \hom(1,B)$)

$\Downarrow$

$f = g$

Covariant representable functor $\hom_{\mathbb{C}}(A, -): \mathbb{C} \to Set$

Contravariant " $\hom_{\mathbb{C}}(-, X): \mathbb{C}^{op} \to Set$

'Mixed-variant' $\hom_{\mathbb{C}}(-,-): \mathbb{C}^{op} \times \mathbb{C} \to Set$

$(A, B) \mapsto \hom(A, B)$

$(f, g) \mapsto \hom(f, g): \hom(B, C) \to \hom(A, D)$

$A \to B$ $C \to D$

$h \mapsto g \circ h \circ f$

$$\begin{tikzcd} A \arrow[d, "f"] \\ B \end{tikzcd} \Rightarrow \begin{tikzcd} \hom(A,X) \\ \hom(B,X) \arrow[u] \end{tikzcd} \quad \begin{tikzcd} g \circ f \\ g \arrow[u, mapsto] \end{tikzcd}$$

$= y_X$

see pg 3

Preserves all binary products!

$\forall A, B, \quad \hom(X, A \times B) \cong \hom(X,A) \times \hom(X,B)$

We show that h is a bijection in

$$\begin{tikzcd} A & & \hom(X,A) & \\ A \times B \arrow[u, "p_1"] \arrow[d, "p_2"'] \arrow[r, dashed] & \hom(X, A\times B) \arrow[ur, "\hom(X,p_1)"] \arrow[dr, "\hom(X,p_2)"'] \arrow[rr, "h"] & & \hom(X,A) \times \hom(X,B) \arrow[ul] \arrow[dl] \\ B & & \hom(X,B) & \end{tikzcd}$$

'actual product'

Functor axioms : Maps Identity to Identity
Composition

Monomorphism: $f \circ g = f \circ h \Rightarrow g = h$ (left cancellation)
(w.r.t $f: A \to B$)

split mono: $\exists g$ s.t. $g \circ f = 1_A$ "retraction"

regular $\exists g, h: B \to C$ such that f equalizes g, h

Epimorphism: $g \circ f = h \circ f \Rightarrow g = h$ (right cancellation)

Isomorphism: existence of inverse g (unique!), $g \circ f = 1_A$, $f \circ g = 1_B$

Isomorphism $\underset{\not\Leftarrow}{\Rightarrow}$ Mono + Epi

Generalized Elements

- $C \in \mathbb{C}$ is mapped to $T \to C$ ← test object
- $Y(C): \mathbb{C}^{op} \to Set$ (Yoneda) sends U to $\{x: U \to C\}$ set of generalized elements
- $\{1\}$

'set'

$$\begin{tikzcd} \{1\} \arrow[d] \arrow[dr] & \\ \mathbb{N} \arrow[r] & M \end{tikzcd}$$

represents elems of M

'Mon' $(\mathbb{N}, +, 0) \to (M, +, 0)$

Free Monoids (Congruence Relation) R equivalence relation and $\forall (a,b), (c,d) \in R$, $(a \cdot c, b \cdot d) \in R$

$$\begin{tikzcd} S \arrow[d, "i"'] \arrow[dr, "f"] & \\ S^* \arrow[r, "\bar{f}"'] & M \end{tikzcd}$$

(normal functions)

Presentation by Generators and Relations s.t. 'constraints', 'clusters' $s^2 = t^3 \ldots$

$$\begin{tikzcd} I \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d] & \Sigma^* \arrow[r, "q"] & \Sigma^*/R \\ I^* \arrow[ur, shift left, "s^*"] \arrow[ur, shift right, "t^*"'] & & \end{tikzcd}$$

Coequalizer in Mon

Free Categories (on Graphs)

Intuition: convert category to graph by 'forgetting': $\mathbb{C}_0 \underset{cod}{\overset{dom}{\leftleftarrows}} \mathbb{C}_1$
(vertices) 'edges'

$$\begin{tikzcd} G \arrow[d] \arrow[dr, "f"] & \\ U(\mathbb{C}(G)) \arrow[r, "U(F)"'] & U(\mathbb{D}) \end{tikzcd}$$

Similarly, the category generated by graph G subject to relations $(f_i, g_i)_{i \in I}$ (parallel arrows in $\mathbb{C}(G)$) is $\mathbb{C}(G)/R$, where R is the least congruence relation containing all pairs (f_i, g_i).

$R \subseteq \mathbb{C}_1 \times \mathbb{C}_1$ eq. rel, and $\forall (f,g) \in R \Rightarrow$ parallel, $(h \circ f, k \circ g) \in R$ if $cod(f) = dom(h)$

LIMITS
&
COLIMITS

- A diagram of type $\mathbb{J}$ in $\mathbb{C}$ is a functor $D: \mathbb{J} \to \mathbb{C}$ 'picking out' objects and arrows in $\mathbb{C}$.

index category

- A cone to/over a diagram D consists of object (vertex/apex) C in $\mathbb{C}_0$ and "projection arrows" $(p_j : C \to D_j) \in \mathbb{C}_1$ such that

$$\begin{tikzcd} C \arrow[d, "p_i"'] \arrow[dr, "p_j"] & \\ D_i \arrow[r, "D_\alpha"'] & D_j \end{tikzcd}$$

commutes for all $\alpha : i \to j$ in $\mathbb{J}$.

Cone morphism from $(C,(p_j)_{j\in\mathbb{J}})$ to $(E,(q_j)_{j\in\mathbb{J}})$

$$\begin{tikzcd} C \arrow[rr, "\theta"] \arrow[dr, "p_j"'] & & E \arrow[dl, "q_j"] \\ & D_j & \end{tikzcd}$$

$\Rightarrow$ Cones (to D) + Cone morphisms form a category $\mathrm{Cone}(D)$.

- A limit of the diagram D is a terminal object on $\mathrm{Cone}(D)$.

$\lim(D)$, $\lim_{j\in\mathbb{J}} D_j$, $\varprojlim D$

- A category $\mathbb{C}$ has all (small/finite) limits $\Leftrightarrow$ it has all (small/finite) products and equalizers.

- If $\mathbb{C}$ is locally small and $X \in \mathbb{C}_0$, then $\hom(X,-) : \mathbb{C} \to \mathbf{Set}$ (covariant hom functor) preserves all limits that exist in $\mathbb{C}$

'product category' $\mathbb{C}\times\mathbb{D}$: $(\mathbb{C}\times\mathbb{D})_0 = \mathbb{C}_0 \times \mathbb{D}_0$

$1_{(A,B)} = (1_A, 1_B)$

Products $\mathbb{J} = \cdot \quad \cdot$

n=0 terminal
n=1 obj itself
n=2 ✓

$\to$ F preserves products $\Leftrightarrow$ $F(A\times B) \cong F(A)\times F(B)$

$\to (M,\cdot,e_m)\times(N,\cdot,e_n) = (M\times N, \cdot, (e_m,e_n))$

Equalizers $\mathbb{J} = \cdot \rightrightarrows \cdot$

In $\underline{\mathbf{Set}}$, $E = \{c \in C \mid fc = gc\}$, e is the inclusion function $e(c) = c$

$$\begin{tikzcd} E \arrow[r, "e"] & C \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & D \\ Z \arrow[u, shift left, "x"] \arrow[u, shift right, "y"'] \arrow[ur, "z"'] & & \end{tikzcd}$$

e is always mono!

Pullbacks $\mathbb{J} = \cdot \to \cdot \leftarrow \cdot$

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & C \arrow[d, "g"] \\ B \arrow[r, "p"'] & A \end{tikzcd}$$

Take product then equalizer

$$\begin{tikzcd} P \arrow[d, "eq"] \arrow[drr, bend left, "p_2"] \arrow[dd, bend right, "p_1"'] & & \\ B\times C \arrow[rr] \arrow[d] & & C \arrow[d, "g"] \\ B \arrow[rr, "p"'] & & A \end{tikzcd}$$

* p_1, p_2 are jointly monic! * 'single arrow'

HW7Q1

$$\begin{tikzcd} X \arrow[d, "k"'] \arrow[r, bend left, "f\circ k"] & E \arrow[r, "h"] \arrow[d, "e"'] & B \arrow[d, "\langle 1_B, 1_B\rangle"] \\ & A \arrow[ur, shift left, "f"] \arrow[ur, shift right, "g"'] \arrow[r, "\langle f,g\rangle"'] & B\times B \end{tikzcd}$$

WTS e equalizes $A \overset{f}{\underset{g}{\rightrightarrows}} B$

① WTS $f\circ e = g\circ e$ $\langle f\circ e, g\circ e\rangle = \langle h,h\rangle$

② $\forall X \to A$ s.t. $f\circ k = g\circ k$,

$\exists! X \to E$ s.t. $e \circ l = k$

Show every eq arrow is pb arrow.

$e \circ l_2 = k$, $h \circ l_2 = f\circ e\circ l_2 = f\circ k$

- Explicit construction:

$$\begin{tikzcd} \boxed{E} \arrow[r, "e"] \arrow[drr, bend right] & \prod_{j\in\mathbb{J}_0} D_j \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[dr, "p_j"'] & \prod_{\alpha: i\to j \in \mathbb{J}} D_i \arrow[d, "p_\alpha"] \\ & & D_j \end{tikzcd}$$

p_i D_i D_α

In $\underline{\mathbf{Set}}$, $E = \{(x_i)_{i\in\mathbb{J}} \in \prod_{j\in\mathbb{J}_0} D_j \mid D_\alpha(x_i) = x_j \ \forall D_\alpha : D_i \to D_j\}$

Equalize f and g, same on all α

- A cocone from/under D consists of an object $C \in \mathbb{C}$ and a family $(\sigma_j)_{j\in\mathbb{J}}$ of injection (= projection) maps such that

$$\begin{tikzcd} D_i \arrow[rr, "D_\alpha"] \arrow[dr, "\sigma_i"'] & & D_j \arrow[dl, "\sigma_j"] \\ & C & \end{tikzcd}$$

commutes for all $\alpha : i \to j$ in $\mathbb{J}$.

Cocone morphisms:

$$\begin{tikzcd} & D_i \arrow[dl, "\sigma_i"'] \arrow[dr, "\tau_i"] & \\ C \arrow[rr, "f"'] & & E \end{tikzcd}$$

$\Rightarrow$ Categories of cocones under D form $\mathrm{Cocone}(D)$.

- A colimit of D is an initial cocone.

$\mathrm{colim}(D)$, $\mathrm{colim}_{j\in\mathbb{J}}$, $\varinjlim D$

- A category $\mathbb{C}$ has all (small/finite) colimits $\Leftrightarrow$ it has all small/finite coproducts and coequalizers.

- Dually, $\hom(-,X) : \mathbb{C}^{op} \to \mathbf{Set}$ maps arbitrary colimits in $\mathbb{C}$ to limits in $\underline{\mathbf{Set}}$.

Coproducts In $\underline{\mathbf{Set}}$, $A+B = A\times\{0\} \cup B\times\{1\}$, $in(a) = (a,n)$, $n\in\{1,2\}$ 'tagged union'

$$\begin{tikzcd} A \arrow[d, "i_1"'] \arrow[dr, "f"] & \\ A+B \arrow[r, "h"] & X \\ B \arrow[u, "i_2"] \arrow[ur, "g"'] & \end{tikzcd}$$

$h(x,n) = \begin{cases} f(x), & n=1 \\ g(x), & n=2 \end{cases}$

Coequalizers $A \overset{f}{\underset{g}{\rightrightarrows}} B \hookrightarrow B/S \leftarrow$ least eq rel containing $\{(f(a),g(a)) \mid a\in A\}$

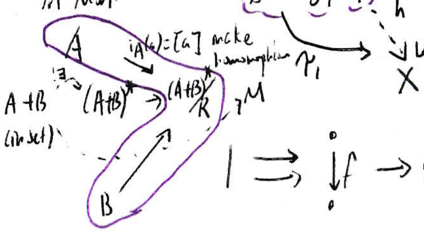
Take congruence relation for monoids (see prev page)

Pushouts

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "\sigma_2"] \arrow[ddr, bend left, "\tau_2"] & \\ B \arrow[r, "\sigma_1"'] \arrow[drr, bend right, "\tau_1"'] & P \arrow[dr, dashed, "h"] & \\ & & X \end{tikzcd}$$

Take coproduct then coequalize

In Mon:



$1 \rightrightarrows \overset{\cdot}{\underset{\cdot}{\downarrow}} f \to \partial f$

$= (\mathbb{N}, +, 0)$

single arrow f being the number 1

$$\begin{tikzcd} \overset{[1,1]}{A} \arrow[r] \arrow[d] & \overset{[0,1]?}{C} \arrow[d] \arrow[ddr, bend left, "\sigma_2"] & \\ \underset{[1],?}{B} \arrow[r] \arrow[drr, bend right, "\sigma_1"'] & B+C \arrow[dr, "\mathrm{coeq}"] & {[0,1]+1} \\ & & P \end{tikzcd}$$

unit circle

↘

Pullback Lemma

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "l"'] & B \arrow[r, "g"] \arrow[d, "m"] & C \arrow[d, "n"] \\ D \arrow[r, "h"'] & E \arrow[r, "k"'] & F \end{tikzcd}$$

$\in \mathbb{C}$. If $\begin{tikzcd} \cdot \arrow[r, "g"] \arrow[d, "m"'] & \cdot \arrow[d, "n"] \\ \cdot \arrow[r, "k"'] & \cdot \end{tikzcd}$ is a pullback, then Left rect is pullback $\Leftrightarrow$ Entire rect is pullback

Pullback Functors : We extend $m: A \to B$ in $\mathbb{C}$ to $m^*: \mathbb{C}/B \to \mathbb{C}/A$.

$m^*(f: X \to B) = (p_1: A \times_B X \to A)$

$m^*(u: X \to Y) = \langle p_1, u \circ p_2 \rangle$

'shift left'

m is used to define the pullback.

$$\begin{tikzcd} A \times_B X \arrow[rr, "p_2, X"] \arrow[dr] \arrow[ddr, bend right, "p_1, X"'] & & X \arrow[dr, "u"] \arrow[ddr, "f"] & \\ & A \times_B Y \arrow[rr, "p_2, Y"] \arrow[d, "p_1, Y"] & & Y \arrow[d, "g"] \\ & A \arrow[rr, "m"'] & & B \end{tikzcd}$$

$\leftarrow$

We have the following square

$$\begin{tikzcd} \mathbb{C}/A & \mathrm{Sub}_{\mathbb{C}}(A) \arrow[l, hook', "(\text{inclusion})"'] \\ \mathbb{C}/B \arrow[u, "m^*"] & \mathrm{Sub}_{\mathbb{C}}(B) \arrow[l, hook', "(\text{inclusion})"] \arrow[u, "m^{-1}"'] \end{tikzcd}$$

$\mathrm{Sub}_{\mathbb{C}}(A) \subseteq \mathbb{C}/A$

$\begin{tikzcd} \cdot \arrow[dr, "(\text{mono})"'] & & \cdot \arrow[dl] \\ & A & \end{tikzcd}$ (thin/preorder)

Since m^* preserves monomorphisms:

If m is mono in a pullback, so is n.

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "n"'] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[d, "m"] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

In $\underline{Set}$, $(P(A), \subseteq) \xrightarrow[\text{inclusion}]{(\text{injective})} (\mathrm{Sub}_{Set}(A), \leq) \xrightarrow[\text{quotient}]{(\text{bijective})} (\mathrm{Sub}_{Set}(A)/\equiv, \leq)$

HW 7 Q5a (Limit-colimit coincidence?)

$$\begin{tikzcd} & & C \text{ colimit} & & \\ A_0 \arrow[urr, "\sigma_0"] \arrow[r, shift right] & A_1 \arrow[l, "e_0"'] \arrow[ur, "\sigma_1"] \arrow[r, shift right] & A_2 \arrow[l, "e_1"'] \arrow[u, "\sigma_2"] \arrow[r, shift right] & A_3 \arrow[l, "e_2"'] \arrow[ul, "\sigma_3"'] & \cdots \\ & & L \text{ imit} \arrow[ull, "p_0"] \arrow[ul, "p_1"] \arrow[u, "p_2"'] \arrow[ur, "p_3"'] & & \end{tikzcd}$$

m_0, m_1, m_2; f_0

$\to e_i \circ m_i = 1_{A_i}$. $(L, (p_i: L \to A_i))$ limit cone over e's

$(C, (\sigma_i: A_i \to C))$ colimit cocone over m's

Want to find morphisms $f: C \to L$ satisfying $p_i \circ f \circ \sigma_i = 1 \ \forall i \in \mathbb{N}$.

① Make all A_i's into cones over e-maps (t_{ij}'s)

② Get universal arrow $f_i: A_i \to L \ \forall i$,

③ $(f_i: A_i \to L \mid i \in \mathbb{N})$ constitutes a cocone over ~~[illegible]~~ m-maps

④ so we get a universal arrow $f: C \to L$ with $f \circ \sigma_i = f_i$ for all i.

⑤ $p_j \circ f \circ \sigma_i = p_j \circ f_i = t_{ij}$, when $i = j$, $t_{ij} = 1_{A_i}$

Representability

For $\mathbb{C}$ locally small, $A \in \mathbb{C}_0$, $\hom(-, A) = y_A$ is the contravariant hom functor.

Def: A functor $F: \mathbb{C}^{op} \to Set$ is representable if it is isomorphic to a functor of the form y_A, i.e. $\exists A$ and an iso $y_A \to F$. (natural)

- Free monoid on S exists iff $K: Mon \to Set$, $M \mapsto \hom(S, U(M))$ is representable.

  (covariant case!)

  $$\hom(S, U(-)) \cong \hom(\underline{F}^{(S)}, (-))$$

- A locally small category $\mathbb{C}$ has a terminal object 1 iff

  $F: \mathbb{C}^{op} \to Set$, $A \mapsto \{*\}$ is representable ($\cong \hom(-, 1)$)

- If $\mathbb{C}$ has binary products, then an exponential of $A, B \in \mathbb{C}_0$ exists iff $\begin{cases} H: \mathbb{C}^{op} \to Set \\ H(X) = \hom(X \times A, B) \end{cases}$ is representable i.e. $\hom(X, E) \cong \hom(X \times A, B)$

  $\overset{\uparrow}{\text{identify with } B^A}$

Since each $x \in FA$ generates a N.T., we just need to find $A \in \mathbb{C}_0$, $x \in FA$ such that $\hat{x}: y_A \to F$ is an isomorphism. This $x \in FA$ is called a universal element of F and forms the terminal object (D, y) in the category of elements $\int F$. ($\exists! f: C \to D$ from (C, x) to (D, y) such that $F(f)(y) = x$. HW10 says that $\eta_C(x) = f_{C,x}$ defines NT from F to y_D) Alt: $F(h)(x): y_D \to F$

As an example, given $A, B \in \mathbb{C}_0$, the product $A \times B$ exists if and only if

$$G: \mathbb{C}^{op} \to Set, \quad X \mapsto \hom(X, A) \times \hom(X, B) \text{ is representable.}$$

$$\cong \hom(X, P) \leftarrow \text{product}$$

Furthermore we choose (p_1, p_2) in $GP = \hom(P, A) \times \hom(P, B)$ and we show that $\widehat{(p_1, p_2)}$ is the required isomorphism, i.e. $y_P(C) \xrightarrow{\widehat{(p_1,p_2)}_C} FC$ is a bijection. For $h: C \to P$ in $y_P(C) = \hom(C, P)$,

$$\begin{tikzcd} A & P \arrow[l, "p_1"'] \arrow[r, "p_2"] & B \\ & C \arrow[ul, "f"] \arrow[u, "h"] \arrow[ur, "g"'] & \end{tikzcd}$$

$$\widehat{(p_1, p_2)}_C(h) = G(h)(\widehat{p_1, p_2}) = (\hom(h, A) \times \hom(h, B))(p_1, p_2)$$

$$= (p_1 \circ h, p_2 \circ h)$$

$$\exists! h \ = \ (f, g) \in \hom(C, A) \times \hom(C, B)$$

which proves that $(P, (p_1, p_2))$ is terminal.

Another example : Choose (B^A, ε_B^A) in $\int H$. Then in $\hom(B^A \times A, B)$

$\exists! \tilde{f}: C \to B^A$ such that

$$H(f)(\varepsilon_B^A) = \hom(\tilde{f} \times A, B)(\varepsilon_B^A)$$

$$= \varepsilon_B^A \circ (\tilde{f} \times 1_A)$$

$$= f \text{ for all } f \text{ in } \hom(C \times A, B)$$

So y in (D, y) is the 'defining arrow' for concepts otherwise expressed as limits.

HW3 'Rules'

$$\begin{tikzcd}
A & A\times B \arrow[l, "p_1"'] \arrow[r, "p_2"] & B \\
& X \arrow[ul, "f"] \arrow[u, "\langle f,g\rangle"] \arrow[ur, "g"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
{} & {} & {} \\
C \arrow[u, "h"] & C\times D \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[u, "h\times k"] & D \arrow[u, "k"] \\
A \arrow[u, "f"] & A\times B \arrow[l, "p_1"] \arrow[r, "p_2"'] \arrow[u, "f\times g"] & B \arrow[u, "g"]
\end{tikzcd}$$

$f\times g := \langle f\circ p_1,\ g\circ p_2\rangle$

$\langle g,h\rangle\circ f = \langle g\circ f, h\circ f\rangle$

$(h\times k)\circ\langle f,g\rangle = \langle h\circ f, k\circ g\rangle$

$(h\times k)\circ(f\times g) = (h\circ f)\times(k\circ g)$

use this notation!

let $f: A\to B$, $g: B\to C$

w.T.S $E'_Y(g\circ f) = E'_Y(f)\circ E'_Y(g)$

$\Leftrightarrow \widetilde{\mathcal{E}^{C}_{Y}\circ(1_{Y^C}\times(g\circ f))} = \widetilde{\mathcal{E}^{B}_{Y}\circ(1_{Y^B}\times f)}\circ\widetilde{\mathcal{E}^{C}_{Y}\circ(1_{Y^C}\times g)}$

$Y^C\times A\to Y$

$\Leftrightarrow \mathcal{E}^{C}_{Y}\circ(1_{Y^C}\times(g\circ f)) = \overline{\widetilde{\mathcal{E}^{B}_{Y}\circ(1_{Y^B}\times f)}\circ\widetilde{\mathcal{E}^{C}_{Y}\circ(1_{Y^C}\times g)}}$

$\boxed{\text{Hw 8 Q4}}$

$= \mathcal{E}^{A}_{Y}\circ((\widetilde{\mathcal{E}^{B}_{Y}\circ(1_{Y^B}\times f)}\circ\widetilde{\mathcal{E}^{C}_{Y}\circ(1_{Y^C}\times g)})\times 1_A)$

$= \underbrace{\mathcal{E}^{A}_{Y}\circ((\widetilde{\mathcal{E}^{B}_{Y}\circ(1_{Y^B}\times f)}\times 1_A)}\circ(\widetilde{\mathcal{E}^{C}_{Y}\circ(1_{Y^C}\times g)}\times 1_A))$

$= \mathcal{E}^{B}_{Y}\circ(1_{Y^B}\times f)\circ(\widetilde{\mathcal{E}^{C}_{Y}\circ(1_{Y^C}\times g)}\times 1_A)$

$= \mathcal{E}^{B}_{Y}\circ(\widetilde{\mathcal{E}^{C}_{Y}\circ(1_{Y^C}\times g)}\times f)$

$= \mathcal{E}^{B}_{Y}\circ(\widetilde{\mathcal{E}^{C}_{Y}\circ(1_{Y^C}\times g)}\times 1_B)\circ(1_{Y^C}\times f)$

$= (\mathcal{E}^{C}_{Y}\circ(1_{Y^C}\times g))\circ(1_{Y^C}\times f)$

$= \mathcal{E}^{C}_{Y}\circ(1_{Y^C}\times(g\circ f)) = LHS$

Cartesian Closed Categories

- $\mathbb{C}$ is cartesian closed if there are finite products and exponentials $\forall B,C\in\mathbb{C}_0$.

Exponential: $C^B\in\mathbb{C}_0$, and $\mathcal{E}^B_C : C^B\times B\to C$ (eval)

such that $\forall A\in\mathbb{C}_0$, $f: A\times B\to C$, $\exists!\ \tilde{f}: A\to C^B$ s.t. $\mathcal{E}\circ(\tilde{f}\times 1_B) = f$.

The inverse of $\tilde{(\cdot)}$ is $\overline{(\cdot)}$: $\underset{A\to C^B}{g}\longmapsto\bar{g} = \mathcal{E}^B_C\circ(g\times 1_B): A\times B\to C$

$\Updownarrow$

$\hom_{\mathbb{C}}(A\times B, C)\xrightarrow{\cong}\hom_{\mathbb{C}}(A, C^B)$

$$\begin{tikzcd}
A \arrow[r, dashed, "\tilde{f}"] & C^B \\
A\times B \arrow[r, dashed, "\tilde{f}\times 1_B"] \arrow[dr, "f"'] & C^B\times B \arrow[d, "\mathcal{E}"] \\
& C
\end{tikzcd}$$

Given X in cartesian-closed $\mathbb{C}$, we have internal hom functor

$E_A: \mathbb{C}\to\mathbb{C}$, $A\mapsto A^X$

$\underset{(A\to B)}{f}\longmapsto\widetilde{f\circ\mathcal{E}^X_A}: A^X\to B^X$, since $A^X\times X\xrightarrow{\mathcal{E}^X_A}A\xrightarrow{f}B$

and $\hom(A^X, B^X)\cong\hom(A^X\times X, B)$

$E'_Y: \mathbb{C}^{op}\to\mathbb{C}$, $A\mapsto Y^A$

$\underset{(A\to B)}{f}\longmapsto\widetilde{\mathcal{E}^B_Y\circ(1_{Y^B}\times f)}: Y^B\to Y^A$

$\hom(Y^B, Y^A)\cong\hom(Y^B\times A, Y)$

$Y^B\times A\xrightarrow{1_{Y^B}\times f}Y^B\times B\xrightarrow{\mathcal{E}^B_Y}Y$

$\hat{\mathbb{C}} = [\mathbb{C}^{op}, \text{set}]$ is cartesian closed whenever $\mathbb{C}$ is small ($\Leftarrow$ $\hat{\mathbb{C}}$ is locally small $\Leftrightarrow$ For any functors F, E, $\hom(F,E)$ is a set)

let functors $E, H: \mathbb{C}^{op}\to\text{set}$. Note that $(H^E)(C)\overset{\text{Yoneda}}{\cong}\hom(Y_C, H^E)$

$\overset{\text{ccc's}}{\cong}\hom(Y_C\times E, H)$

So indeed we define $H^E = \hom(Y_{(-)}\times E, H): \mathbb{C}^{op}\to\text{set}$

by

$$\begin{tikzcd}
C \arrow[d, "f"'] \arrow[r, mapsto] & \hom(Y_C\times E, H) \\
D \arrow[r, mapsto] & \hom(Y_D\times E, H) \arrow[u, "\hom(Y_f\times 1_E, 1_H)"']
\end{tikzcd}$$

$\xrightarrow{H^E}$

Define (Eval) $\underset{(NT)}{\mathcal{E}}: H^E\times E\to H$

$\forall C\in\mathbb{C}$, $\mathcal{E}_C: \underset{\hom_{\hat{\mathbb{C}}}(Y_C\times E, H)}{\overset{\shortparallel}{(H^E)(C)}}\times E(C)\to H(C)$

$(\eta: Y_C\times E\to H, x)\mapsto\eta_C(1_C, x)$

(Currying) Given $\phi: F\times E\to H: \mathbb{C}^{op}\to\text{set}$, we need to define NT $\tilde{\phi}: F\to H^E$.

$\forall C\in\mathbb{C}$, $\tilde{\phi}_C: FC\to\hom(Y_C\times E, H)$

$x\mapsto\tilde{\phi}_C(x): Y_C\times E\to H$ (NT!)

where we have $(\tilde{\phi}_C(x))_D: \hom(D,C)\times ED\to HD$,

For $h: D\to C$, $y\in ED$,

$(\tilde{\phi}_C(x))_D(h,y) = \phi_D(Fhx, y)$

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NATURALITY

- Given functors $F, G : \mathbb{A} \to \mathbb{B}$, a <u>natural transformation</u> $\eta : F \to G$ is given by morphisms $\eta_A : FA \to GA$ for all A in $\mathbb{A}$, such that for all arrows $f : A \to \mathbb{B}$ in A, the "naturality square"

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\eta_A"] & F(B) \arrow[d, "\eta_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

commutes.

$\mathbb{C}$ small, $\mathbb{D}$ locally small $\Rightarrow$ Fun$(\mathbb{C}, \mathbb{D})$ locally small

- For small categories $\mathbb{A}, \mathbb{B}$, Fun$(\mathbb{A}, \mathbb{B})$ is the (small) category of functors and <u>natural transformations</u>. Given functors $F, G, H : \mathbb{A} \to \mathbb{B}$ and NT's $F \xrightarrow{\eta} G \xrightarrow{\theta} H$, $\theta \circ \eta : F \to H$ is given by $(\theta \circ \eta)_A = \theta_A \circ \eta_A$

- A natural transformation $\{\eta : F \to G\} : \mathbb{C} \to \mathbb{D}$ is an isomorphism in Fun$(\mathbb{C}, \mathbb{D})$ (<u>natural isomorphism</u>) iff $\eta_C : FC \to GC$ is also iso $\forall C \in \mathbb{C}_0$.

- Bifunctor Lemma: let $\mathbb{A}, \mathbb{B}, \mathbb{C}$ be categories and assume we have

  function $F_0 : \mathbb{A}_0 \times \mathbb{B}_0 \to \mathbb{C}_0$

  for each $A \in \mathbb{A}_0$, a functor $F(A, -) : \mathbb{B} \to \mathbb{C}$ with $B \mapsto F_0(A, B)$

  for each $B \in \mathbb{B}_0$, a functor $F(-, B) : \mathbb{A} \to \mathbb{C}$ with $A \mapsto F_0(A, B)$

  such that the square

$$\begin{tikzcd} F_0(A,B) \arrow[r, "F(A,g)"] \arrow[d, "F(f,B)"] & F_0(A,B') \arrow[d, "F(f,B')"] \\ F_0(A',B) \arrow[r, "F(A',g)"] & F_0(A',B') \end{tikzcd}$$

(in $\mathbb{C}$) commutes for all $f : A \to A'$ in $\mathbb{A}$ and all $g : B \to B'$ in $\mathbb{B}$.

Then there exists a unique functor $F : \mathbb{A} \times \mathbb{B} \to \mathbb{C}$ such that

- $F(A, B) = F_0(A, B)$ $\forall A, B$, $F(1_A, g) = F(A, g)$ $\forall A \in \mathbb{A}_0$, $g : B \to B'$ in $\mathbb{B}$
  $F(f, 1_B) = F(f, B)$ $\forall f : A \to A' \in \mathbb{A}$, $B \in \mathbb{B}_0$.

We define $F(f, g) = F((f, 1_{B'}) \circ (1_A, g)) = F(f, 1_{B'}) \circ F(1_A, g) := F(f, B') \circ F(A, g)$

<u>Theorem</u> <u>Cat is Cartesian Closed</u>. Given small categories $\mathbb{A}, \mathbb{B}$, $\mathbb{B}^{\mathbb{A}} := \text{Fun}(\mathbb{A}, \mathbb{B})$

we define the 'eval functor' $\varepsilon^{\mathbb{A}}_{\mathbb{B}} : \text{Fun}(\mathbb{A}, \mathbb{B}) \times \mathbb{A} \to \mathbb{B}$ using the bifunctor lemma.

The 'function' is $\varepsilon(F, A) = F(A)$, the functors are $\varepsilon(F, -) : \mathbb{A} \to \mathbb{B}$, $\varepsilon(-, A) : \text{Fun}(\mathbb{A}, \mathbb{B}) \to \mathbb{B}$.

where $\varepsilon(F, A) = F(A)$ $\qquad$ $\varepsilon(F, A) = F(A)$

$\varepsilon(F, f) = F(f)$ $\qquad$ $\varepsilon(\eta, A) = \eta_A$.

$$\begin{tikzcd} & A \arrow[r, "f"] & A' \\ F \arrow[d, "\eta"] & F(A) \arrow[r, "F(f)"] \arrow[d, "\eta_A"] & F(A') \arrow[d, "\eta_{A'}"] \\ G & G(A) \arrow[r, "G(f)"'] & G(A') \end{tikzcd}$$

commutes, so we define $\varepsilon(\eta, f) = \varepsilon(\eta, A') \circ \varepsilon(F, f) = \eta_{A'} \circ F(f)$.

Given $F : \mathbb{A} \times \mathbb{B} \to \mathbb{C}$, define 'curried' functor $\tilde{F} : \mathbb{A} \to \text{Fun}(\mathbb{B}, \mathbb{C})$ as follows:

$\tilde{F}(A) = F(A, -)$ is the functor given by $\tilde{F}(A)(B) = F(A, B)$
$\tilde{F}(A)(g) = F(1_A, g)$ for $g : B \to B' \in \mathbb{B}$

$\tilde{F}(f) : \tilde{F}(A) \to \tilde{F}(A')$ is the natural transformation given by $\tilde{F}(f)_B = F(f, 1_B) : F(A, B) \to F(A', B)$

HW 10 Q1: $\text{Fun}(\mathbb{B}, \mathbb{C}) \times \text{Fun}(\mathbb{A}, \mathbb{B}) \to \text{Fun}(\mathbb{A}, \mathbb{C})$ as transpose of $\mathbb{C}^{\mathbb{B}} \times \mathbb{B}^{\mathbb{A}} \times \mathbb{A} \xrightarrow{\mathbb{C}^{\mathbb{B}} \times \varepsilon^{\mathbb{A}}_{\mathbb{B}}} \mathbb{C}^{\mathbb{B}} \times \mathbb{B} \xrightarrow{\varepsilon^{\mathbb{B}}_{\mathbb{C}}} \mathbb{C}$
$=: \mathbb{F}$

Let $\underbrace{F_1, F_2 : \mathbb{A} \to \mathbb{B}, \eta : F_1}_{\in \text{Fun}(\mathbb{A}, \mathbb{B})} \to F_2$, $G_1, \underbrace{G_2 : \mathbb{B} \to \mathbb{C}, \theta : G_1 \to G_2}_{\in \text{Fun}(\mathbb{B}, \mathbb{C})}$

We want to define $\mathbb{F}$: , where $\underline{\tilde{\mathbb{F}}_1(\theta, \eta) : G_1 F_1 \to G_2 F_2}$, $\tilde{\mathbb{F}}_0(G, F) = G \circ F$

$\forall \alpha : A \to A'$ in $\mathbb{A}$,

$$\begin{tikzcd} G_1F_1A \arrow[r, "G_1F_1\alpha"] \arrow[d, "G_1(\eta_A)"] & G_1F_1A' \arrow[d, "G_1(\eta_{A'})"] \\ G_1F_2A \arrow[r, "G_1F_2\alpha"] \arrow[d, "\theta_{F_2A}"] & G_1F_2A' \arrow[d, "\theta_{F_2A'}"] \\ G_2F_2A \arrow[r, "G_2F_2\alpha"] & G_2F_2A' \end{tikzcd}$$

use η and the fact that the functor G_1 preserves domains and codomains

θ is a N.T. for G_1, G_2

So $\tilde{\mathbb{F}}_1(\theta, \eta)$ is the natural transformation φ such that $\forall A \in \mathbb{A}_0$, $\varphi_A = \underline{\theta_{F_2, A} \circ G_1(\eta_A)}$

Alternatively, to define the action of φ on A, plug θ, η, A into for $\tilde{\mathbb{F}}(\theta, \eta)_A = \mathbb{F}(\theta, \eta, 1_A)$.

First eval gives $\eta_A : F_1(A) \to F_2(A)$, for second eval we with η_A as f, θ as η.

An equivalence of categories between $\mathbb{C}$ and $\mathbb{D}$ consists of

- functors $F: \mathbb{C} \to \mathbb{D}$, $G: \mathbb{D} \to \mathbb{C}$
- natural isomorphisms $\alpha: 1_{\mathbb{C}} \underset{\cong}{\to} G \circ F$, $\beta: 1_{\mathbb{D}} \underset{\cong}{\to} F \circ G$

A functor $F: \mathbb{C} \to \mathbb{D}$ is part of an equivalence of categories (corresponding G exists)

$\iff$ F is ① faithful ② full ③ essentially surjective
$\forall D \in \mathbb{D}_0 \; \exists C \in \mathbb{C}_0, F(C) \cong D$

Example: Given a preorder $(A, \leq)$, we take its 'poset reflection'

$$\underline{(A, \leq)} \xrightarrow{p} \underline{(A/\equiv, \leq)} \text{ where } a \equiv b \Leftrightarrow \begin{smallmatrix} a \leq b, \\ b \leq a \end{smallmatrix}$$

The functor p ("p-bar") is equivalent to some q such that $p \circ q = 1$, $(q \circ p) \cong 1$. If $A = \mathbb{Z}$, $a \leq b \Leftrightarrow a | b$, then $A/\equiv \; = \mathbb{N}$, $[a] = [b] \Leftrightarrow |a| = |b|$.

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② Let $\mathbb{C}$ and $\mathbb{D}$ be categories and assume that $\mathbb{D}$ has limits of type $\mathbb{J}$. Then $\text{Fun}(\mathbb{C}, \mathbb{D})$ has limits of type $\mathbb{J}$ and they are computed "pointwise".

- For diagram $D: \mathbb{J} \to \mathbb{D}^{\mathbb{C}}$, $\overset{\text{functor!}}{\lim(D)} = \lim \circ D^* \in \mathbb{D}^{\mathbb{C}}$,

where $\mathbb{J} \xrightarrow{D} \mathbb{D}^{\mathbb{C}}$
$\mathbb{J} \times \mathbb{C} \xrightarrow{\overline{D}} \mathbb{D}$
$\mathbb{C} \xrightarrow{\boxed{D^*}} \mathbb{D}^{\mathbb{J}} \xrightarrow{\lim} \mathbb{D}$

$$\begin{tikzcd} D \arrow[r, mapsto] \arrow[d, "\eta"] & \lim(D) \arrow[d, "\lim \eta"] \\ D' \arrow[r, mapsto] & \lim(D') \end{tikzcd}$$

- Alternatively, $\lim(D) = \lim\limits_{j \in \mathbb{J}} D_j : \mathbb{C}, \mathbb{D}$, where $(\lim\limits_{j \in \mathbb{J}} D_j)(C) := \lim\limits_{j \in \mathbb{J}} \underbrace{(D_j(C))}_{\in \mathbb{D}}$

In both cases, $\lim \eta : \lim(D) \to \lim(D')$ is the unique NT such that $\forall j \in \mathbb{J}$, C in $\mathbb{C}$,

$$\begin{tikzcd} (\lim\limits_{j \in \mathbb{J}} D_j)(C) \arrow[r, "(\lim \eta)_C"] \arrow[d, "(p_j)_C"'] & (\lim\limits_{j \in \mathbb{J}} D_j')(C) \arrow[d, "(p_j')_C"] \\ D_j C \arrow[r, "\eta_j C"'] & D_j' C \end{tikzcd}$$ commutes.

This works in particular if $\mathbb{D} = \underline{\text{Set}}$, so $\text{Fun}(\mathbb{C}^{op}, \underline{\text{Set}}) = \hat{\mathbb{C}}$ has all limits that $\underline{\text{Set}}$ has.
Theorem 8.14 shows that Y (Yoneda) preserves all products and exponentials as well.

Yoneda Lemma

The Yoneda Embedding sends $C \in \mathbb{C}$ to the corresponding (object) contravariant hom functor in the presheaf category $\hat{\mathbb{C}}$ $(= [\mathbb{C}^{op}, \text{set}] = \text{fun}(\mathbb{C}^{op}, \text{set}))$, locally small. Formally,

$Y: \mathbb{C} \to [\mathbb{C}^{op}, \text{set}]$

$$\begin{tikzcd} C \arrow[r, mapsto] \arrow[d, "f"] & Y_C = \hom(-, C) \arrow[d, "Y_f = \hom(-, f)"] \\ D \arrow[r, mapsto] & Y_D = \hom(-, D) \end{tikzcd}$$

✓functor
✓N.T.
✓functor

$\hom(X, f) = (Y_f)_X : \hom(X, C) \to \hom(X, D)$
$\hom(X, f)(h) = f \circ h$

The Yoneda lemma views natural transformations $Y_C \to F$, where $F: \mathbb{C}^{op} \to \text{Set}$, as generalized elements of the set FC. Every NT corresponds to a unique element in FC:

The function $\boxed{\begin{matrix} \hom_{\hat{\mathbb{C}}}(Y_C, F) \to FC \\ \eta \longmapsto \check{\eta} = \eta_C(1_C) \in FC \end{matrix}}$ is a bijection. → this is also the terminal object if $Y_C \cong F$ (see p 3)

Proof sketch: We construct the inverse $FC \to \hom_{\hat{\mathbb{C}}}(Y_C, F)$ by generating a natural transformation for every $x \in FC$.

$x \mapsto \hat{x} : Y_C \to F$, where $\hat{x}_D : \underset{(= \hom(D, C))}{Y_C D} \to FD$

$\hat{x}_D (h: D \to C) = F(h)(x)$

$\left(\begin{matrix} \text{check naturality of } \hat{x}, \text{ and} \\ \check{\hat{x}} = \hat{\check{x}} = 1. \end{matrix} \right)$

Furthermore, $Y: \mathbb{C} \to \hat{\mathbb{C}}$ is full and faithful $\iff$ $\forall C, D \in \mathbb{C}_0$, (bijection)
$\hom_{\mathbb{C}}(C, D) \cong \hom_{\hat{\mathbb{C}}}(Y_C, Y_D)$

Applications ① Since Y is f.f. (reflecting isomorphisms), $\overset{\text{contains normal iso's}}{\hom(A, B)} \cong \hom(Y_A, Y_B)$ ← contains N.T.'s

$$\begin{matrix} \hom(A, B) & \cong & \hom(Y_A, Y_B) \\ \cup\!| & & \cup\!| \\ \boxed{\text{Iso}(A, B)} & \cong & \boxed{\text{Iso}(Y_A, Y_B)} \end{matrix}$$

a. we can show $C^{A \times B} \cong (C^B)^A \Leftrightarrow Y(C^{A \times B})(X) = \hom(X, C^{A \times B})$ (for $A, B, C \in \mathbb{C}_0$)

natural in $X \Leftarrow$ can show pointwise iso (bijection)

$\cong \hom(X \times (A \times B), C)$
$\cong \hom((X \times A) \times B, C)$
$\cong \hom(X \times A, C^B)$
$= \hom(X, (C^B)^A)$ $\forall X$ in $\mathbb{C}$.

b. $(A+B) \times C \cong A \times C + B \times C \Leftrightarrow \hom((A+B) \times C, -)$
$\cong \hom(A \times B + B \times C, -)$

Again, pointwise computation gives

$\hom((A+B) \times C, Y) \cong \hom(A+B, Y^C)$
$\cong \hom(A, Y^C) \times \hom(B, Y^C)$
$\cong \hom(A \times C, Y) \times \hom(B \times C, Y) \cong \hom(A \times C + B \times C, Y)$