21-378 Theorems 2/5/2021

## Replicating a Forward Loan

$R^{for}_{\tau, \eta, T}$ ← loan end
↑ agreement ↑ loan start

- Buy a ZCB, sell ZCB of different maturity

- Receive \$1 at time $\eta$
- Repay $1 \cdot (1+R^{for}_{0,\eta,T})^{T-\eta} \stackrel{def}{=} A_T$ at $T$

replicate the cash flow!

Buy ZCB with $F=1$, maturity $\eta$.

Sell ZCB with $F = A_T$, maturity $T$.

$\Rightarrow d(\eta) - A_T\, d(T) = 0.$

$A_T = \dfrac{d(\eta)}{d(T)}$ (can expression in terms of spot rate

$\therefore \quad (1+R^{for}_{0,\eta,T})^{T-\eta} = \dfrac{d(\eta)}{d(T)}$

(2.13)
Forward rate > compo rate:
longer-duration bond better

$$\left(\frac{1+\frac{\hat r(t)}{2}}{1+\frac{\hat r(t-0.5)}{2}}\right)^{2t} = \frac{(1+\frac{f(t)}{2})}{(1+\frac{\hat r(t-0.5)}{2})}$$

→ Upward-Sloping Spot Rate Curve $\Leftrightarrow$ Forward rate $f(t) > \hat r(t)$

Let $\lambda = \dfrac{1+\frac{\hat r(t)}{2}}{1+\frac{\hat r(t-0.5)}{2}}$ we have (1) $\left(1+\frac{\hat r(t)}{2}\right)^{2t} = \left(1+\frac{\hat r(t-0.5)}{2}\right)^{2t-1}\left(1+\frac{f(t)}{2}\right)$

$\Updownarrow$

$>1$ iff $\hat r(t) > \hat r(t-0.5)$

$$\left(\frac{1+\frac{\hat r(t)}{2}}{1+\frac{\hat r(t-0.5)}{2}}\right)^{2t-1} = \frac{(1+\frac{f(t)}{2})}{(1+\frac{\hat r(t)}{2})}$$

→ Positivity of forward rates is equivalent to discount factors decreasing as maturity increases.

Yield to Maturity (let $g$ be the annual coupon rate)

$\lambda = \frac{1}{1+\frac{y}{2}}$, want to solve for $y$.

$$\boxed{P = F\lambda^{2T} + F\frac{g}{2}\sum_{i=1}^{2T}\lambda^i}$$

$$= F\left(\lambda^{2T} + \frac{g}{2}\frac{\lambda(1-\lambda^{2T})}{1-\lambda}\right)$$

* Effective YTM $Y$ satisfies

$$P = \sum_{i=1}^{N}\frac{F_i}{(1+Y)^{T_i}}.$$

If $F_i$ can be both $>0$ and $<0$, there can be no or $>1$ YTM. In such situations the term "internal rate of return" is used.

$$\boxed{\begin{array}{l} P > F \iff g > y \\ P = F \iff g = y \\ P < F \iff g < y. \end{array}}$$

Note that $\frac{\lambda}{1-\lambda} = \frac{\frac{1}{1+\frac{y}{2}}}{1-\frac{1}{1+\frac{y}{2}}} = \frac{1}{\frac{y}{2}} = \frac{2}{y}$.

$$\therefore P = F\left(\lambda^{2T} + \frac{g}{y}(1-\lambda^{2T})\right).$$

Perpetuity: as $T \to \infty$, $A = (F\cdot\frac{g}{2})\ldots$

$$P \to A\frac{\lambda}{1-\lambda} = \frac{2A}{y}, \quad y = \frac{2A}{P}.$$

An Important Observation:

$$f\left(\frac{N}{2}\right) \simeq \frac{1}{N}\sum_{i=1}^{N} f\left(\frac{i}{2}\right)$$

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## Existence and Uniqueness of YTM

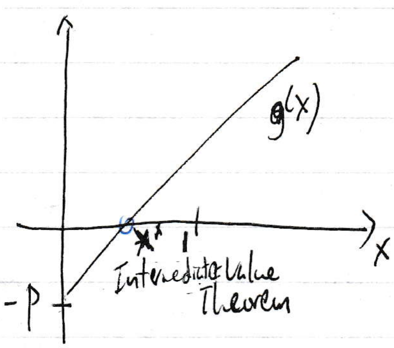
- Deterministic payments $F_i \ge 0$ at times $\frac{i}{2}$, $i = 1, 2, 3 \ldots 2T$

  $F_i > 0$ for at least 1 value of $i$.
- Assume the price is $P > 0$
- A yield to maturity $y^{(2)}$ satisfies $P = \sum_{i=1}^{2T} \frac{F_i}{(1+\frac{y}{2})^i}$.

Proposition: $y$ exists and is unique.

Moreover, $y > 0 \iff P < \sum_{i=1}^{2T} F_i$.

Define $x = \frac{1}{1+\frac{y}{2}}$, $g(x) = -P + \sum_{i=1}^{2T} F_i x^i$




$0 < x^* < 1 \Rightarrow y > 0$.

$x^* > 1 \Rightarrow y < 0$

If the spot rate curve is flat at some level $\alpha > 0$ ($\hat{r}(t) = \alpha$ for $t > 0$), all securities with deterministic payments have yield to maturity $y = \alpha$.

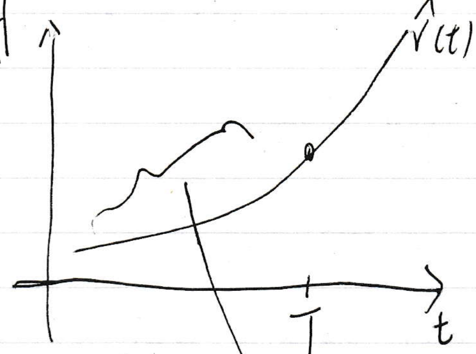
T3

- Yield to Maturity is a Blend of Spot Rates. $\min\{\hat{r}(t) : t \in \mathcal{T}\} \le y \le \max\{\hat{r}(t) : t \in \mathcal{T}\}$
- Spot Rate Curve := Zero Coupon Yield Curve

Annuity: $P = A \sum_{i=1}^{2T} d(\frac{i}{2}) = A \sum_{i=1}^{2T} \lambda^i$.

Coupon Bond: $P = F d(T) + F \frac{q}{2} \sum_{i=1}^{2T} d(\frac{i}{2}) = F \lambda^{2T} + F \frac{q}{2} \sum_{i=1}^{2T} \lambda^i$.

**Coupon Effect**




"discount things cheaply at the start"

· Bigger coupons: more weight on front part

Let $q^{(2)} \ge q^{(1)} \ge 0$. Let $y^{(1)}$ and $y^{(2)}$ be their yields to maturity.

(i) If $\hat{r}(t) \ge \hat{r}(t-0.5)$ $\forall t \in \{1, 1.5, \dots T\}$ then $y^{(1)} \ge y^{(2)}$

(ii) $\le$ $\le$

Combining Securities

Consider $S^{(1)}, S^{(2)}$ with to prices $P^{(1)}, P^{(2)}$ and payments $F_i^{(1)}, F_i^{(2)}$.
Let $S$ have $P = P^{(1)} + P^{(2)}$ and $F_i = F_i^{(1)} + F_i^{(2)}$.
Then YTM $y$ for $S$ satisfies $\min\{y^{(1)}, y^{(2)}\} \le y \le \max\{y^{(1)}, y^{(2)}\}$.

Par Bond. $P = F,\ y = q$. $F = F d(T) + F \frac{q}{2} \sum_{i=1}^{2T} d(\frac{i}{2})$ variable payments with float notes

$$y = q = \frac{2[1 - d(T)]}{\sum_{i=1}^{2T} d(\frac{i}{2})}$$

This is the swap rate!

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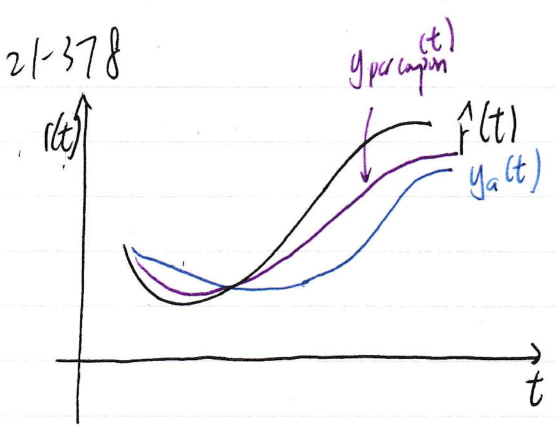


$y_a(t)$ (annuity yield) is a 'moving average'.
$\hat{r}(t)$ is the ZCB yield curve.

A par-coupon bond is "in between". If coupons dominate, $y_{pc}(t)$ will tend towards $y_a(t)$.

If $\hat{r}(t+0.5) \geq \hat{r}(t)$ $\forall t = 0.5, 1, 1.5, \dots T-1.5$

then $y_a(t+0.5) \geq y_a(t)$ and $y_{pc}(t+0.5) \geq y_{pc}(t)$ for all $t = 0.5, 1, 1.5 \dots$

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$$\begin{cases} P_a(t) = \sum_{i=1}^{2t} d\left(\frac{i}{2}\right) \text{ pays \$1 every 6 months} \\ P_a(t+0.5) = P_a(t) + \frac{1}{\left(1+\frac{\hat{r}(t+0.5)}{2}\right)^{2t+1}} \end{cases}$$

Using annuity yields,

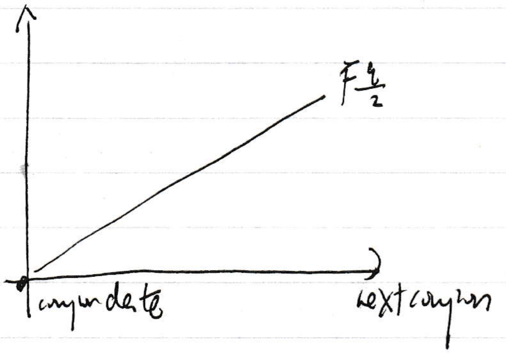
$$P_a(s) = \sum_{i=1}^{2s} \frac{1}{\left(1+\frac{y_a(s)}{2}\right)^i}.$$

$$\sum_{i=1}^{2t+1} \frac{1}{\left(1+\frac{y_a(t+0.5)}{2}\right)^i} = \sum_{i=1}^{2t} \frac{1}{\left(1+\frac{y_a(t)}{2}\right)^i} + \frac{1}{\left(1+\frac{\hat{r}(t+0.5)}{2}\right)^{2t+1}}$$

Since the spot-rate is upward-sloping, $\frac{1}{1+\frac{y_a(t)}{2}} > \frac{1}{1+\frac{y_a(t+0.5)}{2}}$

$y_a(t+0.5) \leq \hat{r}(t+0.5)$. $y_a(t+0.5) > y_a(t)$

$\Rightarrow$ LHS > RHS




$P_{full} = P_{flat} + \underbrace{P_{AI}}_{\text{totally arbitrary!}}$ (accrued interest)

$\gamma = 1 - \eta$ = Amount of time until next coupon

- $P^{full} = F \lambda^{k+\gamma} + F \frac{q}{2} \sum_{i=0}^{k} \lambda^{\gamma+i}$ YTM between Coupon Payments

- Current yield $:= \frac{qF}{P^{flat}}$

- Discount yield: $P = 100\left[1 - \frac{n y_d}{360}\right]$

$\iff y_d = \frac{(100-P)360}{100n}$

- $y_{be} = \frac{(100-P)365}{P_n}$ (for $n < 82$)

$n > 182$: $P\left(1+\frac{y_{be}}{2}\right) + \frac{y_{be}}{365}\left(n - \frac{365}{2}\right)\left(1+\frac{y_{be}}{2}\right)P = 100$

$\iff P\left(1+\frac{y_{be}}{2}\right)\left[1 + \text{rate} \cdot \text{time}\right] = 100$

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## One Factor Models

$P = f(y)$

1st order: $\Delta P \approx f'(y)\Delta y$

2nd order: $\Delta P \approx f'(y)\Delta y + \frac{1}{2} f''(y)(\Delta y)^2$

e.g. ZCB, $P = f(y) = F \cdot (1+\frac{y}{2})^{-2T}$, where $y = \hat{r}(T)$.

Then $f'(y) = (-2T)(\frac{1}{2}) F (1+\frac{y}{2})^{-2T-1}$

$$= -\left(\frac{T}{1+\frac{y}{2}}\right) f(y)$$

$$\therefore \Delta P = -\left(\frac{T}{1+\frac{y}{2}}\right) f(y)\,\Delta y$$

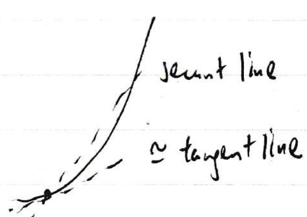
$$= -\left(\frac{T}{1+\frac{y}{2}}\right) P \Delta y$$

$\Delta y > 0 \Rightarrow \Delta P < 0.$

- 1 Basis point = 0.01% = 0.0001

- $DV01 = \dfrac{-\Delta P}{10{,}000\,\Delta y}$, change in dollar value in price resulting from a change in the interest rate factor of 1 b.p.

(dollar value of an '01)




So $\frac{\Delta P}{\Delta y} \approx f'(y)$,

and $DV01 = -\frac{f'(y)}{10{,}000}$

If the DV01s of securities A and B have the same sign, then hedging a short position in security A requires a long position in security B.

$$DV01 = -\frac{\Delta P}{10000\,\Delta y}$$

$$\boxed{\text{Duration} = -\frac{1}{P}\frac{\Delta P}{\Delta y}} = \frac{10{,}000}{P} DV01$$

('relative change', 'elasticity')

$$\boxed{\text{Convexity: } C = \frac{f''(y)}{P}} = \frac{1}{P}\frac{d^2 P}{d^2 y} \qquad \text{(no negative sign!)}$$

So we can write $\quad \frac{\Delta P}{P} = -D\,\Delta y \quad$ (first-order)

and $\quad \frac{\Delta P}{P} = -D\Delta y + \frac{1}{2} C (\Delta y)^2 \quad$ (second-order)

$\uparrow$ convexity

$C > 0$: $(\Delta y)^2$ is good $\Rightarrow$ long on volatility

<u>Portfolios</u>

Suppose a portfolio is constructed with $S^1, S^2, \dots S^N$ each with prices $P^1, P^2, \dots P^N$ (in absolute $$ amounts)

The total initial capital of buying $\alpha^i$ shares of $S^i$ is

$$X = \alpha^1 P^1 + \alpha^2 P^2 + \dots + \alpha^N P^N$$

$$DV01 = \sum_{i=1}^{N} \alpha^i \, DV01^i$$

$$\left.\begin{aligned} D &= \sum_{i=1}^{N} \left(\frac{\alpha^i P^i}{X}\right) D^i \\ C &= \sum_{i=1}^{N} \left(\frac{\alpha^i P^i}{X}\right) C^i \end{aligned}\right\} \text{weighted averages}$$

- Given $P, \delta, \gamma$ with $P > 0$ and any 3 maturities $T_3 > T_2 > T_1 > 0$, we can construct a portfolio of ZCBs of maturities $T_1, T_2, T_3$, such that the portfolio has price $P$, duration $\delta$, and convexity $\gamma$.

  "Van der Monde matrix"

- Yield-Based DV01 for Coupon Bonds

$$P = f(y) = F\left[(1+\tfrac{y}{2})^{-2T} + \frac{q}{y}\left(1-(1+\tfrac{y}{2})^{-2T}\right)\right]$$

$$DV01 = \frac{F}{10{,}000}\left[\frac{q}{y^2}\left(1-\frac{1}{(1+\frac{y}{2})^{2T}}\right) + \left(1-\frac{q}{y}\right)\frac{T}{(1+\frac{y}{2})^{2T+1}}\right]$$

$\downarrow$ if $(q=y)$ (par bonds)

$$= \frac{F}{10{,}000}\left[\frac{1}{y}\left(1-\frac{1}{(1+\frac{y}{2})^{2T}}\right)\right]$$

$\downarrow$ $T \to \infty$

$$= \frac{F}{10{,}000}\left[\frac{1}{y}\right]$$

e.g. 2/19/2021

10-yr $y = 1.346$

$DV01 = \frac{1}{0.01346}\left[1 - \left(\frac{1}{1+\frac{0.01346}{2}}\right)^{20}\right]$ (for $F = 10{,}000$) $\approx 9.3269$

30-yr $DV01 \approx 22.06$ $(y = 2.138)$ $\quad \frac{1}{0.02138}\left[1-\frac{1}{(1+\frac{0.02138}{2})^{60}}\right]$

2.356 F of 10's to hedge each for 30's.

$\Rightarrow$ deterministic cash flows!

$\boxed{\text{Macaulay Duration}}$

If $P = \sum_{i=1}^{2T} \frac{F_i}{(1+\frac{y}{2})^i}$, then $D = -\frac{f'(y)}{P} = \frac{1}{P(1+\frac{y}{2})}\sum_{i=1}^{2T}\frac{\frac{i}{2}F_i}{(1+\frac{y}{2})^i}$

def $D_{Mac} = \frac{1}{P}\sum_{i=1}^{2T}\frac{\frac{i}{2}F_i}{(1+\frac{y}{2})^i} = \frac{1}{P}\sum_{i=1}^{2T}\frac{T_iF_i}{(1+\frac{y}{2})^i}$ weighted average by present values

$$\Leftrightarrow D_{Mac} = (1+\tfrac{y}{2})D$$

Ex. ZCB. $P = F(1+\frac{y}{2})^{-2T}$, $\frac{dP}{dy} = -2TF(1+\frac{y}{2})^{-2T-1}(\frac{1}{2}) = \frac{-TP}{1+\frac{y}{2}} \Rightarrow D = \frac{T}{1+\frac{y}{2}}$

$\Rightarrow D_{Mac} = T$ (!)

**Duration for Coupon Bonds:** $D = \frac{F}{P}\left[\frac{q}{y^2}\left(1-\frac{1}{(1+\frac{y}{2})^{2T}}\right) + \left(1-\frac{q}{y}\right)\frac{T}{(1+\frac{y}{2})^{2T+1}}\right]$ $\simeq 9$ for 10-yr bond (higher at lower yields: $\propto \frac{1}{y}$)

so $D_{Mac} = \frac{F}{P}\left(1+\frac{y}{2}\right)\left[\quad\quad\right]$

$\xrightarrow{\text{Par Coupon Bonds}}$ $\frac{F}{P}=1$, $\frac{q}{y}=1$, $1-\frac{q}{y}=0$

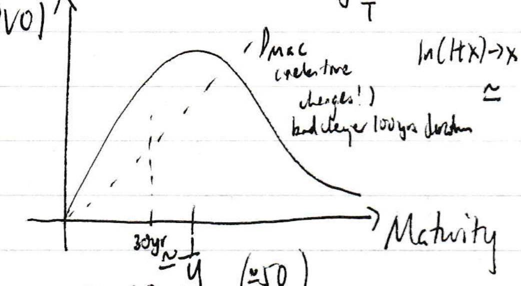
$D = \frac{1}{y}\left[1-\frac{1}{(1+\frac{y}{2})^{2T}}\right]$,

$D_{Mac} = \left(\frac{1}{y}+\frac{1}{2}\right)\left[1-\frac{1}{(1+\frac{y}{2})^{2T}}\right]$

$DV01 = \frac{F}{10{,}000\,y}\left[1-\frac{1}{(1+\frac{y}{2})^{2T}}\right]$

**For a Zero-Coupon Bond:** $DV01 = \frac{FT}{10{,}000}\left(1+y/2\right)^{-2T-1} \propto Ae^{-xT}\cdot T$

$\arg\max_T DV01 = \frac{1}{2\ln(1+\frac{y}{2})}$ $\quad \ln(1+x)\to x$ $\quad \simeq \frac{1}{y}$.

→ 


*Recall that Macaulay Duration is a weighted average - use that to compute $D_{Mac}$ and then DV01 of a portfolio.

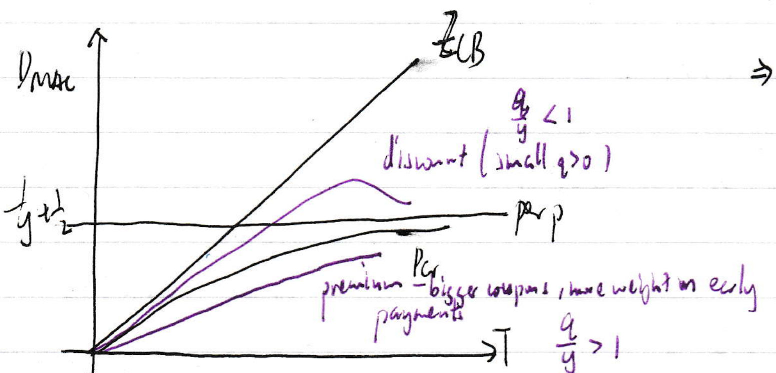
→ $C = \frac{T(T+\frac{1}{2})}{(1+\frac{y}{2})^2} \simeq T^2$.

$\cdot\, P_{perp} = A\left(\frac{1}{1-\frac{1}{1+\frac{y}{2}}}\right)\underbrace{\left(\frac{1}{1+y/2}\right)}_{\text{first payment}} = A\left(\frac{1+\frac{y}{2}}{1+\frac{y}{2}-1}\right)\left(\frac{1}{1+y/2}\right)$

$= 2A/y$.

$\Rightarrow \frac{dP}{dy} = -\frac{2A}{y^2} \Rightarrow D = -\frac{1}{P}\frac{dP}{dy}$

$= \frac{1}{y}$, $D_{Mac} = (1+\frac{y}{2})\frac{1}{y} = \frac{1}{y}+\frac{1}{2}$.




→ $\boxed{D = \frac{T}{1+\frac{y}{2}}}$

*DV01's add.

*Duration of a portfolio is a weighted average of the durations of the components.

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$$D^{Ann}_{Mac} < D^{ZCB}_{Mac}$$

$$D^{Ann}_{Mac} < D^{Coup}_{Mac} < D^{ZCB}_{Mac}$$

Big Coupon Bond = Small Coupon Bond + Annuity

$\therefore\ c \nearrow$ : "Bigger annuity part"

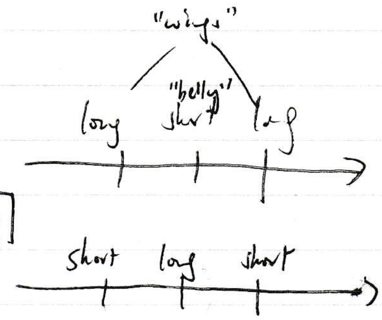
Bullets, Barbells...

Rules of Thumb.

$T\uparrow \to D\uparrow^*$

$c\uparrow \to D\downarrow$

$y\uparrow \to D\downarrow$ $\boxed{\text{Butterfly}}$

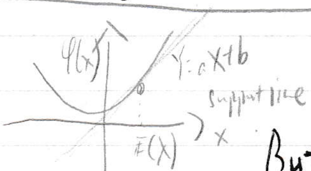



$\boxed{\text{Barbells}}$: Portfolio of short-term and long-term bonds rather than intermediate term bonds

Recap: Jensen's Inequality

$\boxed{\text{Bullets}}$: A single intermediate-term ZCB $D_{Mac,ZCB} = T$

$$\varphi(\mathbb{E}(X)) \le \mathbb{E}(\varphi(X))$$




Use $C_{ZCB} = \dfrac{T(T+\frac{1}{2})}{(1+\frac{y}{2})^2}$ (strictly) Convex!

$$\mathbb{E}(X) = \alpha\,(\underbrace{\mathbb{E}(X \mid X<\beta)}_{X_1}) + (1-\alpha)(\underbrace{\mathbb{E}(X \mid X>\beta)}_{X_2})$$

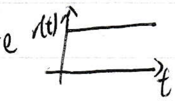
Butterflies...

$\mathbb{E}(\varphi(X)) \ge \mathbb{E}(cX+b)$

$= c\,\mathbb{E}(X) + b$

$= \varphi(\mathbb{E}(X))$

Assuming flat spot-rate curve




and parallel shifts

$$\varphi(\alpha x_1 + (1-\alpha)x_2) \le \alpha\,\varphi(x_1) + (1-\alpha)\,\varphi(x_2)$$

this proof is wrong

$\varphi(x_1) \ne \mathbb{E}_{X_\beta}(\varphi(X))$

when the yield curve changes the price of the barbell portfolio will be above that of the bullet portfolio.

9 year ZCB (Bullet)

$0.06 \to 0.064$

$\frac{9}{(1+0.03)}$ $0.004$ $80.3155$

$$\frac{\Delta P}{P} = -D\,\Delta y + \frac{1}{2}C(\Delta y)^2 \approx -0.034307$$

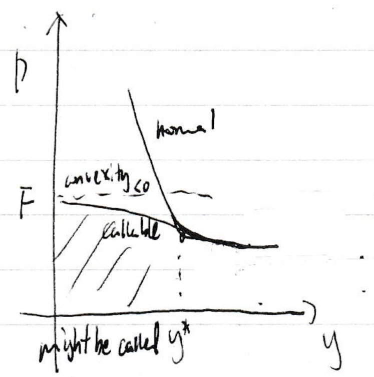
$\approx 96.564$ T11

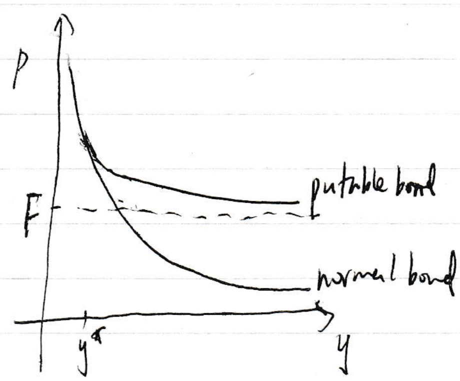
Price of Callable = Plain Bond − "European" Call on Bond (advantage on issuer) can buy back

Putable = Plain Bond + Put on Bond (advantage on holder) demand full payment of principal

$$D^{callable} = \frac{P^{Bond} \cdot D^{Bond}}{P^{Bond} + \underbrace{P^{option}}_{<0 \text{ since we are short}}} + \frac{P^{option} \cdot D^{option}}{P^{Bond} + P^{option}}$$

Callable bonds have >0 convexity when market rates are high




Floating Rates + Forward Rates

time-0 prices to receive a sum of $\frac{F}{2} r_{t-.5,t}$ at time t is $F(d(t-0.5) - d(t)) = \frac{F}{2} f(t) d(t)$
① ②

(Replicating forward loan)

$r_{t-.5,t}$: floating rate

① = ②: $d(t-0.5) = d(t)\left(1+\frac{f(t)}{2}\right)$

$\therefore d(t-0.5) - d(t) = d(t)\left(1+\frac{f(t)}{2} - 1\right)$

$= \frac{d(t) f(t)}{2}$.

remember the factor 1/2!

T12

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## Inverse Floaters

- Pay coupons $\frac{F}{2}(q^* - r_{t-0.5,t})$

(Capped and floored?)

"Double whammy" if rates increase → lower present value of future payments
→ lower the future payment amounts

Suppose an inv. floater pays coupons $\frac{F}{2}(2q - r_{t-0.5,t})$ at times 0.5, 1, 1.5 ... T,
$q$ = 10-year p.c. yield. Then we can replicate with long 2F p.c. bond, short floater face F.

$$D^{\text{inv float}} = \frac{2D - D^{\text{float}}}{2-1} \simeq 2D. \quad (D^{\text{float}} \simeq 0)$$

* Duration of floating rate bond is between 0 and 0.5. In time $0 < \alpha < 0.5$ you will get a brand new floater with value F, plus the (known) dividends for this period

$$D_{\text{float}} = \frac{\alpha}{1 + \frac{r(\alpha)}{2}}.$$

Duration $= -\frac{\Delta P/P}{\Delta y}$, in this case duration of ZCB, maturity $\alpha$

## Volatility-Weighted Hedging

E.g. change in 1 b.p. in yield A => change in 1.1 bp in yield B

→ match up Face · relative volatility · DV01.

Continuous time:

$f_c(t) = \hat{r}_c(t) + t\hat{r}_c'(t)$

$f_c'(t) = 2\hat{r}_c'(t) + t\hat{r}_c''(t)$

**Misc Formulas:**

HW 1#2 $f(t+0.5) \geq f(t) \Rightarrow \hat{r}(t+0.5) \geq \hat{r}(t)$

But $\hat{r}(t+0.5) \geq \hat{r}(t) \not\Rightarrow f(t+0.5) \geq f(t)$

# Multiple Interest Rate Factors

$$p = f(\underset{\text{2-year}}{y_1}, \underset{\text{5-year}}{y_2}, \underset{\text{10-year}}{y_3}, \underset{\text{30-year}}{y_4})$$

First order approximation:

$$\Delta p = \frac{\partial f}{\partial y_1}\Delta y_1 + \frac{\partial f}{\partial y_2}\Delta y_2 + \frac{\partial f}{\partial y_3}\Delta y_3 + \frac{\partial f}{\partial y_4}\Delta y_4$$

$\dfrac{-\frac{\partial f}{\partial y_i}}{10{,}000} \rightarrow$ DV01 of $i^{th}$ key rate , $\dfrac{-\frac{\partial f}{\partial y_i}}{p}$ ≃ durations $[\ \ ]$

Second order Approx:

$$\Delta p = \sum_{i=1}^{4} \frac{\partial f}{\partial y_i}\Delta y_i + \frac{1}{2}\sum_{i,j=1}^{4} \frac{\partial^2 f}{\partial y_i \partial y_j}\Delta y_i \Delta y_j .$$

Convexity Matrix $[\ \ ]$

- Assuming Key Rate shifts 'update linearly', $Y_1(t) + Y_2(t) + Y_3(t) + Y_4(t) = 1$

$\Delta y$ for 1 b.p. change in 1-year rate on $\hat{r}(0)$ to $\hat{r}(30)$

- Assume we started with $y_1^* = y(2)$, $y_2^* = y(5)$, $y_3^* = y(10)$, $y_4^* = y(30)$, $y(t)$ given.

Then after perturbation our par-coupon yield curve becomes

$$y(t) + \underbrace{(y_1 - y_1^*)\,Y_1(t) + (y_2 - y_2^*)\,Y_2(t) + (y_3 - y_3^*)\,Y_3(t) + (y_4 - y_4^*)\,Y_4(t)}_{\Delta p}$$

Hedging: Use $F_1$ of 2-year bond (par coupon)

$F_2$ of 5-year bond

$F_3$ of 10-year bond

$F_4$ of 30-year bond

mutually independent (each shift only affects one of the bonds)

→ Match each of the 4 keyrate DV01s of the security with the corresponding keyrate DV01 of the hedging portfolio.

spreadsheet

Recall: Par-coupon yield for coupon bond with maturity $\frac{n+1}{2}$ is

$$y\left(\frac{n+1}{2}\right) = \frac{2\left[1-d\left(\frac{n+1}{2}\right)\right]}{\sum_{i=1}^{n+1} d(i/2)}$$

$$d\left(\tfrac{1}{2}\right) = \frac{1}{1+y(1/2)/2} \;;\; d\left(\tfrac{n+1}{2}\right) = \frac{2 - y\left(\frac{n+1}{2}\right)\sum_{i=1}^{n} d(i/2)}{2+y\left(\frac{n+1}{2}\right)}$$

Reference Yield + Perturbations

↓

New Yield Curve

↓

New Discount Factors

↓

New Price $\qquad P = f(y_1, y_2, y_3, y_4)$

## Alternative Formulation

Original security $P = f(x_1, x_2, x_3, x_4)$ where $x_1 = y_1 - y_1^*$ etc.

2-year spot rate

Hedging portfolio $P^{(h)} = g(x_1, x_2, x_3, x_4)$

Then matching the key rate DV01s is equivalent to having

$$\frac{\partial f}{\partial x_j}(0,0,0,0) = \frac{\partial g}{\partial x_j}(0,0,0,0), \quad j = 1,2,3,4$$

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# PRINCIPAL COMPONENTS ANALYSIS (PCA)

$f(x) = x^T \mathbb{C}(x) x$, $g(x) = \|x\|^2$, for all $x \in V^N$.

$\vdots$

$\nabla f(x) = \lambda \nabla g(x)$

$\underset{\uparrow \text{eigenvector}}{\mathbb{C}(x) x} = \lambda x$

PCA for Yield Curves

Choose maturities $T_1 < T_2 < \dots < T_N$ $(\approx 10)$

0.25 0.5 1 2 3 5 7 10 20 30

Let $y(t,T)$ be the 'spot yield' for investing from $t$ to $T$.

$T_j^{(k)} = y(t_k, t_k + T_j) - y(t_{k-1}, t_{k-1} + T_j)$

where $t_k - t_{k-1} = \Delta t$ constant, $1 \leq k \leq M$ $T^{(1)}, T^{(2)} \dots T^{(M)}$ are all $N \times 1$ vectors.

M data points!

$V$ is their $N \times N$ covariance matrix.

$$\underset{\substack{\uparrow \\ \text{maturity} (\leq N)}}{y_j} = y_j^* + z_1 \underbrace{e_j^{(1)}}_{j^{\text{th}} \text{ entry, PC1}} + z_2 \underbrace{e_j^{(2)}}_{j^{\text{th}} \text{ entry, PC2}} + z_3 e_j^{(3)}$$

- Number of securities in portfolio = number of principal components (usually 3)

Let $S = f(z_1, z_2, z_3)$, hedge with portfolio with $g(z_1, z_2, z_3)$.

Match DV01's $\Longrightarrow \frac{\partial f}{\partial z_i}(0,0,0) = \frac{\partial g}{\partial z_i}(0,0,0)$

$\left(= \frac{-\Delta p}{10{,}000 \Delta y}\right)$

Example. Relative Maturity $e_j^{(1)}$ $e_j^{(2)}$ $e_j^{(3)}$

⋮

| | | | | |
|---|---|---|---|---|
| 1 year | 0.5775 | -0.1077 | -0.4783 | $\to \hat{r}_c^{new}(1) = \hat{r}_c^{(1)} + 0.5775 Z_1 - 0.1077 Z_2 + 0.4783 Z_3$ |
| 3 year | 0.4528 | 0.1532 | 0.0980 | |
| 4 years | 0.4428 | 0.2013 | 0.2043 | $\to \hat{r}_c^{new}(4) = \hat{r}_c^{(4)} + 0.4428 Z_1 + 0.2013 Z_2 + 0.2043 Z_3$ |

e.g. continuous compounding, hedging 3-year ZCB: then

$$f(z_1, z_2) = 100 \exp[-3(\underset{y_3^*}{0.04} + \underset{e_j^{(1)}}{0.4528} z_1 + \underset{e_j^{(2)}}{0.1532} z_2)]$$

---

## Interest Rate Models

No arbitrage $\Leftrightarrow$ $\tilde{p}, \tilde{q} > 0$, $\tilde{p} + \tilde{q} = 1$

(*) $X_0(1+R_0) = \tilde{p} X_1(H) + \tilde{q} X_1(T)$

$\downarrow$

No strong arbitrage $\Leftrightarrow$ $\tilde{p} \geq 0$, $\tilde{q} \geq 0$, $\tilde{p} + \tilde{q} = 1$, (*) holds

$\downarrow$

Law of One Price $\Leftrightarrow$ $\tilde{p}, \tilde{q} \in \mathbb{R}$, $\tilde{p} + \tilde{q} = 1$, (*) holds.

A probability measure on $\Omega$ is a binomial product measure (BPM) if $\exists\, a, b > 0$, $a + b = 1$ such that $\mathbb{P}(\omega) = a^{\#H(\omega)} b^{\#T(\omega)}$ $\forall \omega \in \Omega$.

- $Y$ on $\Omega$ is Time-$n$ measurable. $\Leftrightarrow$ $Y(\omega) = Y(\omega_1, \omega_2, \dots \omega_n, \underbrace{\mu_{n+1}, \dots \mu_N}_{\forall \text{ possible values} \in \{H,T\}^{N-n}})$

- $(Y_n)_{k \leq n \leq m}$ is adapted process $\Leftrightarrow$ $Y_n$ is time-measurable $\forall n$ s.t. $k \leq n \leq m$.

Ex. $X_j(\omega) = \begin{cases} 1, & \omega_j = H \\ -1, & \omega_j = T \end{cases}$, define $(M_n)_{0 \leq n \leq N}$ by $M_0 = 0$, $M_n = \sum_{j=1}^{n} X_j$, $n = 1, \dots N$ is adapted

$M_n(\omega_1,\omega_2,\dots\omega_n) = \#H_n(\omega_1,\omega_2\dots\omega_n) - \#T_n(\omega_1,\omega_2\dots\omega_n)$

Difference between number of heads and number of tails

$\Rightarrow$ 'Discrete' Brownian Motion!

Let $W$ be time-$m$ measurable R.V.

The price to receive $W(\omega_1\dots\omega_m)$ at time $m$ is

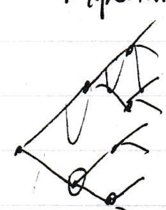
$$\tilde{\mathbb{E}}[D_m W], \quad D_m = \frac{1}{(1+R_0)(1+R_1)\cdots(1+R_{m-1})}$$

$(D_n)_{0\le n\le N}$ is the discount process

Similarly, time-$n$ price to receive $W = W(\omega_1,\dots\omega_n)$ at $m \ge n$ is

$$\tilde{\mathbb{E}}_n\Big[\underbrace{\frac{D_m}{D_n}}_{\text{discounts from } m \text{ back to } n} W\Big] = \frac{1}{D_n}\tilde{\mathbb{E}}_n[D_m W], \quad W_0 = \tilde{\mathbb{E}}[D_m W].$$

Conditional Expectation $\rightarrow$ Best Estimator among all time-$n$ measurable RVs



If $X$ is time-$n+1$ measurable,

$$\tilde{\mathbb{E}}_n[X](\omega_1,\omega_2\dots\omega_n) = \tilde{p}X(\omega_1\dots\omega_n,H) + \tilde{q}X(\omega_1\dots\omega_n,T)$$

If we set $W=1$, we obtain the discount factor for time $n$ as

$$d(n) = \tilde{\mathbb{E}}_n[D_n].$$

$$d(n) = \frac{1}{(1+\hat{R}(n))^n} = \downarrow \quad , D_0 := 1$$

In interest rate models, the pricing measure is usually a Binomial Product Measure:

$$\tilde{\mathbb{P}}(\omega) = \tilde{p}^{\#H(\omega)} \tilde{q}^{\#T(\omega)} \quad \text{for all } \omega \in \Omega.$$

$$X_j(\omega) = \begin{cases} 1 & \text{if } \omega_j = H, \\ -1 & \omega_j = T \end{cases}$$

$M_0 = 0$, $M_n(\omega) = \sum_{j=1}^{n} X_j(\omega)$, $n = 1, 2, 3, \dots N$, $M_n = \#H_n - \#T_n$.

$\Delta R_n = R_{n+1} - R_n$ (drt in continuous time)

Ho-Lee Model

$$\Delta R_n = \underset{\text{drift}}{\lambda_{n+1}} + \sigma X_{n+1} \quad \text{noise / volatility}$$

$$R_n = R_0 + \sum_{i=1}^{n} \lambda_i + \sigma M_n$$

$$R_n(\omega_1, \omega_2, \dots \omega_n) = a_n + b \cdot \#H_n(\omega_1, \dots \omega_n)$$

$b \to 2\sigma$

$$a_n = R_0 - \sigma n + \sum_{i=1}^{n} \lambda_i$$

$M_n = \#H_n - \#T_n$
$= \#H_n - (n - \#H_n)$
$= 2\#H_n - n.$

Vasicek Model

"Mean Reversion"

$$\Delta R_n = k(\theta - R_n) + \sigma X_{n+1}$$

$k \in (0,1)$

$R_n > \theta \Rightarrow \theta - R_n < 0$

$$R_n = (1-k)^n (R_0 - \theta) + \theta + \sigma \sum_{j=1}^{n} (1-k)^{n-j} X_j$$

## Hull-White Model

$K \in (0,1)$,

$$\Delta R_n = k(\theta_{n+1} - R_n) + \sigma_{n+1} X_{n+1}$$

Bill ~

## Black-Derman-Toy Model

$$\Delta(\ln R_n) = -\frac{\Delta\sigma_n}{\sigma_n}(\ln\theta_{n+1} - \ln R_n) + \sigma_{n+1} X_{n+1}$$

$\vdots$

$$R_n(\omega_1 \dots \omega_n) = a_n b_n^{\#H_n(\omega_1, \omega_2 \dots \omega_n)}$$

Let $Y_n = \ln(R_n)$, $A_n = \ln(a_n)$, $B_n = \ln(b_n)$

$\Delta Y_n = A_n + B_n \cdot \#H_n$ "Ho-Lee for log of short rates"

ZCB: $\underline{B_{n,m}}$ is price at time $n$ of a ZCB with maturity $m$ and face value \$1.

discount process

$$B_{n,m} = \tilde{E}_n\left[\frac{D_m}{D_n} \cdot 1\right] = \frac{1}{D_n}\tilde{E}_n[D_m].$$

→ $C^q_{n,m}$, ex-coupon price at time $n$, maturity $m$, for a coupon bond that pays rate $q$

$$C^q_{n,m} = B_{n,m} + q\sum_{i=n+1}^{m} B_{n,i}$$

→ $A_{n,m} = \sum_{i=n+1}^{m} B_{n,i}$

## One-Period Forward Rate

Agree at time $n$, initiate loan at time $m$, settle at time $m+1$

Buy ZCB, maturity $m$, short ZCB $(1+F_{n,m})$, maturity $m+1$

⋮

## Float Notes

$$D_n(1+R_{n-1}) = D_{n-1}$$

$$\Rightarrow D_n + D_n R_{n-1} = D_{n-1}$$

$$\tilde{\mathbb{E}}[D_n R_{n-1}] = B_{0,n-1} - \tilde{\mathbb{E}}[D_n] = B_{0,n-1} - B_{0,n} \quad (*)$$

$$\text{Float}_{0,m} = \tilde{\mathbb{E}}\left[D_m + \sum_{n=1}^{m} D_n R_{n-1}\right] = \tilde{\mathbb{E}}[D_m] + \sum_{n=1}^{m} \tilde{\mathbb{E}}[D_n R_{n-1}]$$

$$\overset{(*)}{=} \tilde{\mathbb{E}}[D_m] + \sum_{n=1}^{m} (B_{0,n-1} - B_{0,n})$$

$$= B_{0,0} = 1.$$

Thus we verify that the price of a float note is equal to its face.

- Model-Independent (like Put-Call Parity)

## M-period Interest Rate Swap

(Receive) contract pays $K - R_{n-1}$
receive fixed

$SR_{0,m} :=$ m-period swap rate.

→ price is model independent (long coupon bond, short floater)

T22

→ M-period Interest Rate Cap pays $(R_{n-1}-K)^+$ at each time $n=1,2,\dots m$ (string of call options)

→ " Floor pays $(K-R_{n-1})^+$

$\underbrace{\qquad\qquad}_{\text{* Model Dependent!!}}$

Recall: $x^+ - (-x)^+ = x$

$x^+ = x + (-x)^+$

$$\text{Swap}^K_{0,m} = \tilde{\mathbb{E}}\left[\sum_{i=1}^{m} D_i\,(K-R_{i-1})\right] = C^K_{0,m} - 1 = \left(K\sum_{n=1}^{m} B_{0,n}\right) + B_{0,m} - 1$$

$SR_{0,m} = \dfrac{1-B_{0,m}}{\sum_{i=1}^{m} B_{0,i}}$

alt expression

$\text{Swap}^K_{0,m} = \sum_{n=1}^{m} B_{0,n}(K - F_{0,n-1})$ (price of the swap)

replacing with forward rates, we $F_{n,m} = \dfrac{B_{n,m}}{B_{n,m+1}} - 1$

$$\text{Cap}^K_{0,m} = \tilde{\mathbb{E}}\left[\sum_{i=1}^{m} D_i (R_{i-1}-K)^+\right]$$

$$\text{Floor}^K_{0,m} = \tilde{\mathbb{E}}\left[\sum_{i=1}^{m} D_i (K-R_{i-1})^+\right]$$

$$\boxed{\Rightarrow \text{Swap}^K_{0,m} + \text{Cap}^K_{0,m} = \text{Floor}^K_{0,m}}$$

$(R_{k-1}-K)^+$ : interest rate caplet → put option on ZCB

$(K-R_{k-1})^+$ : interest rate floorlet → call option on ZCB

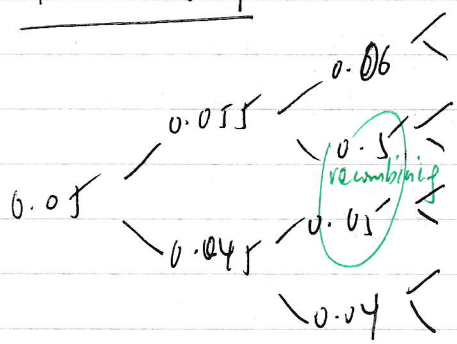
$(P_N - K)^+$ Eur (stock price) : $r>0$, discounted stock price is MRT ⇒ price to receive $P_N$ at time $N$ is $P_K$

$(P_n - K)^+$ Am ⇒ no early exercise

But for caplets, early exercise is possible!

For a contract to receive $R_N$ (an interest rate) at time $N$, price at time $k \neq R_k$.

## Ho-lee Example




$R_n = a_n + b_n \# H_n \quad , \quad P(H_n) = 0.5$

$a_0 = 0.05 \qquad b_1 = 0.01$

$a_1 = 0.045 \qquad b_2 = 0.01$

$a_2 = 0.04$

Discount process depends on the history, and will <u>not</u> be recombining.

$\rightarrow$ Backward Induction.

Example. $V$ European call, exercise date 2, strike 95 on ZCB, $F = 100$, $T = 3$.

$\rightarrow$ Calculate $B_{2,3}(H,H) = \frac{1}{1.06} \approx .94 \qquad B_{2,3}(H,T) = \frac{1}{1.05}$

$B_{2,3}(T,T) = \frac{1}{1.04} \qquad B_{2,3}(T,H) = \frac{1}{1.05} \ldots$

$V_2(H,H) = 0, \; V_2(H,T) = 100[0.95238 - 0.95] = 0.238.$

$$V_1 = \frac{1}{D_1} \tilde{E}_1[D_2 V_2] = \frac{D_2}{D_1} \tilde{E}_1[V_2]$$

e.g. $V_1(H) = \frac{1}{1.055}[0.5(0) + 0.5(0.23810)] \ldots$

$$V_0 = \tilde{E}[D_2 V_2] = \tilde{E}_0[D_2 V_2] = \tilde{E}[\tilde{E}_1[D_2 V_2]]$$

$$= \tilde{E}[D_1 V_1]$$

$$= \frac{1}{1+R_0} \tilde{E}[V_1].$$

Let $v_n(k)$ be the price of the security at time $n$, after the payment has been made.
Assume that each payment $a_n = f_n(\#H_{n-1})$.
Then $$V_n(k) = \frac{a_{n+1}(k)}{1+r_n(k)} + \frac{1}{1+r_n(k)}\left[\tilde{p}\, V_{n+1}(k+1) + \tilde{q}\, V_{n+1}(k)\right].$$

<u>American & Bermudean Options</u>

American Option $(E_n)_{0\le n\le m}$ intrinsic value process,

$E_n \ge 0$ always ...

$\rightarrow V_n = \max\{E_n, \underbrace{\frac{1}{1+R_n}\tilde{\mathbb{E}}_n[V_{n+1}]}_{\text{latent value}}\}$

$(V_m = E_m)$

$$\tilde{p} = \frac{V_0(1+R_0) - V_1(T)}{V_1(H) - V_1(T)}$$

$\rightarrow$ Always optimal to exercise the option at the smallest time <u>$n$</u> s.t. <u>$V_n = E_n$</u>

Bermudan Option (between America and Europe)

Exercise dates $\mathcal{E} \in [N]$

$m := \max \mathcal{E}$, set $V_m = E_m$. $\forall n < m$,

$$V_n = \begin{cases} \frac{1}{1+R_n}\tilde{\mathbb{E}}_n[V_{n+1}] & \text{if } n \notin \mathcal{E}, \\ \max\{E_n, \frac{1}{1+R_n}\tilde{\mathbb{E}}_n[V_{n+1}]\} & n \in \mathcal{E}. \end{cases}$$

Example. $R_0 = 0.06$

$R_1(H) = 0.066$, $R_1(T) = 0.052$

spot rate between $n$ and $n+1$

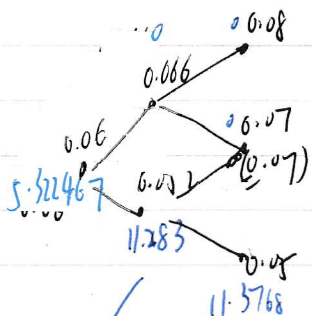
$R_2(H,H) = 0.08$, $R_2(H,T) = R_2(T,H) = 0.07$, $R_2(T,T) = 0.05$

2-periods ($T=2$), $K = 1000$, call option on p.c. bond with $T=3$, $F=1000$

$$\rightarrow B_{0,3} = \frac{1}{1.06}\left[0.25\left(\frac{1}{1.066 \times 1.08} + \frac{1}{1.066 \times 1.07} + \frac{1}{1.052 \times 1.07} + \frac{1}{1.052 \times 1.05}\right)\right]$$

no early exercise of American call

$(1-r_0)\,K$




should exercise here

$\ldots V_0 = 5.322467$

$C_0^E = 2.5506$

$P_0^A = 8.7857$

$x-(x-k)^+ = \min\{x,k\}$,

$x+(k-x)^+ = \max\{x,k\}$

U, V Bermudean, $G_n^{(U)} = (FC_{n,m}^{\xi} - F_n)^+$, $G_n^{(W)} = (F_n - FC_{n,m}^{\xi})^+$

price of caryon bond from n to m, rate $\xi$

If $n \notin \mathcal{E}$, $n \le m-1$, then $V_n^c = \frac{1}{1+R_n}\tilde{E}_n[V_{n+1}^c + F_\xi]$,

$V_n^p = \frac{1}{1+R_n}\tilde{E}_n[V_{n+1}^p + F_\xi]$,

If $n \in \mathcal{E}$, then $V_n^c = \min\{F_n, \frac{1}{1+R_n}\tilde{E}_n[V_{n+1}^c + F_\xi]\}$

$V_n^p = \max\{F_n, \frac{1}{1+R_n}\tilde{E}_n[V_{n+1}^p + F_\xi]\}$

where $V_n^c = FC_{n,m}^{\xi} - U_n$, $V_n^p = FC_{n,m}^{\xi} + W_n$

<u>No early Exercise</u>

1. $G_n^V = (P_n - K)^+$

$D_n P_n = \tilde{E}_n[P_{n+1}D_{n+1}]$, $\left(D_{n+1} = \frac{D_n}{1+R_n}\right)$

$D_n P_n = \frac{D_n}{1+R_n}\tilde{E}_n[P_{n+1}]$

$P_n = \frac{1}{1+R_n}\tilde{E}_n[P_{n+1}]$ if no dividends

$\therefore \frac{1}{1+R_n}\tilde{E}_n[V_{n+1}] \ge \frac{1}{1+R_n}\tilde{E}_n[P_{n+1} - K] \ge P_n - \frac{K}{1+R_n} \ge P_n - K$

2. $\phi \ge 0$, $\phi$ convex, $\phi(0)=0$; $G_n^V = \phi(P_n)$. $\frac{1}{1+R_n}\tilde{E}_n[V_{n+1}] \ge \frac{1}{1+R_n}\tilde{E}_n[G_{n+1}]$

$\ge \frac{1}{1+R_n}\phi(\tilde{E}_n[P_{n+1}])$

$= \phi(P_n)$

4/2/2021

Forwards & Futures

$$\mathrm{For}_{n,m} = \frac{\tilde{\mathbb{E}}_n[D_m P_m]}{\tilde{\mathbb{E}}_n[D_m]} \underset{\text{intermediate payments}}{\overset{\text{if no}}{=}} \frac{D_n P_n}{\tilde{\mathbb{E}}_n[D_m]}$$

Proof. $\tilde{\mathbb{E}}_n\left(D_m(P_m - \mathrm{For}_{n,m})\right) = 0.$

$$= \frac{D_n P_n}{D_n B_{n,m}}$$

$$= \frac{P_n}{B_{n,m}}$$

Futures: made with exchange

→ no default risk (margin account, daily adjustment)

→ can always find counterparty

If interest rates are deterministic, forward = future.

→ All futures expiring on the same date have the same ~~settlement~~ delivery price.

(i) $\mathrm{Fut}_{m,m} = P_m$

(ii) $\tilde{\mathbb{E}}_n\left[\sum_{i=n}^{m-1} D_{i+1}\left(\mathrm{Fut}_{i+1,m} - \mathrm{Fut}_{i,m}\right)\right] = 0$

value at $n$ of all future margin payments is $0$ → pay nothing to enter futures contract

→ $\mathrm{Fut}_{n,m} = \tilde{\mathbb{E}}_n[P_m]$ easy to state, hard to derive.

intuition: price is paid at maturity, no discounting

→ check that it satisfies (i) & (ii)

Remark: $(D_n B_{n,m} \mathrm{For}_{n,m})_{0 \le n \le m}$ $\tilde{\mathbb{E}}_n\left[D_{n+1} B_{n+1,m} \mathrm{For}_{n+1,m}\right] = \tilde{\mathbb{E}}_n\left[D_{n+1} B_{n+1,m} \frac{\tilde{\mathbb{E}}_{n+1}[D_m P_m]}{\tilde{\mathbb{E}}_{n+1}[D_m]}\right]$

$\in$ MRT $\left(= \frac{1}{D_{n+1}} \tilde{\mathbb{E}}_{n+1}[D_m]\right)$

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Notice that $For_{n,m} = \dfrac{\tilde{E}_n[D_m P_m]}{\tilde{E}_n[D_m]}$, $Fut_{n,m} = \tilde{E}_n[P_m]$, so

$$For_{n,m} = Fut_{n,m} \iff \tilde{E}_n[D_m P_m] = \tilde{E}_n[P_m]\,\tilde{E}_n[D_m]$$

$\iff Corr[P_m, D_m] = 0$ under $\tilde{P}$ (e.g. if $D_m$ is known (interest rates constant))

$Corr > 0 \Rightarrow For_{n,m} > Fut_{n,m}$

$Corr < 0 \Rightarrow For_{n,m} < Fut_{n,m}$

Ex. If a security pays 10,000 $R_2$ at time 3, there will be no futures correction at time 3, because the price of the underlying is already known at time 2, ($\tilde{E}_2(P_3) = P_3$, informally)

Ex. (HW6) Delivery at time 2 for ZCB with maturity $T=3$.

→ Since $D_2$ is known at $t=1$, $D_2$ and $P_2$ are conditionally uncorrelated.

$$\tilde{E}_n[D_m B_{m,k}] = \tilde{E}_n\left[D_m \frac{\tilde{E}_m D_k}{D_m}\right] = \tilde{E}_n[D_k] = D_n B_{n,k}$$

$\boxed{\text{Forwards and Futures for 1-Period ZCB's}}$

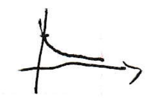
$$For_{n,m} = \underbrace{100}_{\text{face}} \frac{\tilde{E}_n[D_m B_{m,m+1}]}{\tilde{E}_n[D_m]} = 100 \cdot \frac{D_n B_{n,m+1}}{D_n B_{n,m}} = 100 \frac{B_{n,m+1}}{B_{n,m}} = \frac{100}{1+F_{n,m}}$$

forward rate agreed at time $n$ for borrowing between times $m$ and $m+1$

At time, $\dfrac{100}{1+F_{n,m}}$ is paid for the ZCB. Thus profit is made iff

$$\frac{100}{1+F_{n,m}} < \frac{100}{1+R_m} \iff F_{n,m} > R_m$$

$$\mathrm{Fut}_{n,m} = 100\,\tilde{\mathbb{E}}_n[B_{m,m+1}] \overset{\text{definition}}{=} 100\,\tilde{\mathbb{E}}_n\left[\frac{1}{1+R_m}\right]$$



Using Jensen's, $\cancel{\mathrm{Fut}_{n,m} \geq}$ " $\overset{f(x)=\frac{1}{1+x} \text{ convex}}{>} 100\,\dfrac{1}{1+\tilde{\mathbb{E}}_n[R_m]}$

If $\mathrm{For}_{n,m} > \mathrm{Fut}_{n,m}$, this implies $\tilde{\mathbb{E}}_n[R_m] > F_{n,m}$ ← forward rate

price at time $n$ to get the amount $R_m$ at time $m$, no arbitrage concerns

Note:

1) Once the delivery price is known then subsequent adjustments to the margin account become 0.

2) Adding a constant to the delivery price does not change the adjustments.

Thus two futures contracts are <u>financially equivalent</u> if the delivery date of $P_m$ pushed back to $k > m$, or if a constant is added to $P_m$.

Eurodollar Futures: $QP_{m,m} = 100(1 - kR_m)$
(quoted price at maturity)

$$\mathrm{Fut}_{n,m} = 100(1 - k\,\tilde{\mathbb{E}}_n[R_m])$$

$k = 4$ for Eurodollars (see p. 32)

$$R^{\mathrm{fut}}_{n,m} := \tilde{\mathbb{E}}_n[R_m]$$

$P_m$ is a linear function of $R_m$

→ market prediction of $R_m$ is inferred from the futures price

Observation: DV01 of a 6% coupon bond = DV01 of 6% cap float + DV01 of 6% floored float

Recall Ho-Lee:

$$\boxed{R_n = R_0 + \sum_{i=1}^{n} \lambda_i + \sigma M_n} \quad = \#H_n - \#T_n$$

If $P_m = 100(1 - R_m)$, we have

$$Fut_{n,m} = 100\left(1 - R_0 - \sum_{i=1}^{m} \lambda_i - \sigma M_n\right), \quad 0 \le n \le m \le N-1$$

- we can fit $\lambda_i$'s and compute the historical volatility.

Vasicek:

$$\Delta R_n = k(\theta - R_n) + \sigma X_{n+1}$$

Write $\gamma = 1-k$.

$$R_{i+1} - R_i = (1-\gamma)(\theta - R_i) + \sigma X_{i+1}$$

$$R_{i+1} = \gamma R_i + k\theta + \sigma X_{i+1}$$

$$\frac{R_{i+1}}{\gamma^{i+1}} = \frac{R_i}{\gamma^i} + \frac{k\theta + \sigma X_{i+1}}{\gamma^{i+1}}$$

$$\vdots$$

$$\frac{1-\gamma}{\gamma} \frac{1-\gamma^{-n}}{1-\gamma^{-1}} = (\gamma^{-n} - 1)$$

$$R_n = (1-k)^n (R_0 - \theta) + \theta + \sigma \sum_{j=1}^{n} (1-k)^{n-j} X_j$$

$$\tilde{E}_n(P_m) = \tilde{E}_n(100(1-R_m)) = \dots = 100\left[1 - \theta - (R_0 - \theta)(1-k)^m - \sigma \sum_{j=1}^{n} (1-k)^{n-j} X_j\right]$$

Treasury Bills < Notes (2-10) < Bonds

(Hull-White) $\Delta R_n = k(\theta_{n+1} - R_n) + \sigma_{n+1} X_{n+1}$

$R_{n+1} - R_n = k\theta_{n+1} + (\gamma - 1)R_n + \sigma_{n+1} X_{n+1}$ (setting $k = 1-\gamma \Leftrightarrow -k = \gamma - 1$)

$$\frac{R_{n+1}}{\gamma^{n+1}} = \cancel{*}\left(\frac{\theta}{\gamma}\right)^{n+1} + \frac{R_n}{\gamma^n} + \frac{\sigma_{n+1} X_{n+1}}{\gamma^{n+1}}$$

$Z_{n+1}$

$$Z_n = Z_0 + \sum_{i=1}^{n}\left[\left(\frac{\theta}{\gamma}\right)^i + \frac{\sigma_i X_i}{\gamma^i}\right]$$

龍

Options on Futures E.g. European call with strike price $K^C$, maturity $k$ on a futures contract with delivery date $m$, the value of the option at maturity (i.e. $t=k$) is

$$V_k = (Fut_{k,m} - K^C)^+$$

For an American/Bermudan call with strike price .. (same as above), the intrinsic values at time $n$ are

$$G_n^{(V)} = (Fut_{n,m} - K^C)^+,\ \text{for } n = 0,1,2,\dots k \text{ (or a subset of possible exercise dates)}$$

The holder gets a long position upon exercising and \$\$ is deposited into his/her account.

A symmetric floor is worth more than the cap!.

INTEREST RATE FLOORS / CAPS

= series of 'caplets' / 'floorlets'
put options on ZCB

$(K-F)^+ - (F-K)^+ = K - F$ (floating)

Long Floor + Short Cap

= Long bond with coupon $K$,
Short a floater.

When $\sigma\uparrow$, both prices are expected to increase.

DV01 Cap DV01 < 0
Floor DV01 > 0

---

Eurodollar Futures. $P_T = 1{,}000{,}000\left(1 - \frac{L_{T,T+1/4}}{4}\right) = 750{,}000 + 250{,}000(1 - L_{T,T+1/4})$ ('Actual Adjustments' based on; maturity at time T)

$= 750{,}000 + 2{,}500\underbrace{[100(1 - L_{T,T+1/4})]}_{DV01 = 25}$

$QP_T = 100(1 - L_{T,T+1/4})$

$\hookrightarrow L^{fut}(T+0.25) = 1 - \frac{QP_{0,T}}{100}$

We Expect $L^{fut}(T+0.25) > L^{fwr}(T+0.25)$ [both LHS and RHS represent rates agreed at $t=0$ for loans between $T$ and $T+0.25$]

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## Mortgage-Backed Securities

(Monthly compounding) $\lambda = \dfrac{1}{1+\frac{y}{12}}$ ← mortgage rate

$\Leftrightarrow \dfrac{y}{12} = \dfrac{1}{\lambda} - 1$

$$P = A \sum_{i=1}^{12T} \lambda^i = A \frac{\lambda}{1-\lambda}(1-\lambda^{12T})$$

get monthly payment amount

Given we have $A$, we can write

$$A = \underbrace{\left(A - B_n \frac{y}{12}\right)}_{\text{principal payment}} + \underbrace{B_n \left(\frac{y}{12}\right)}_{\text{interest payment}}$$

principal balance

$$B_{n+1} = B_n - \left(A - B_n \frac{y}{12}\right) = B_n\left(1+\frac{y}{12}\right) - A$$

$$B_{n+1}\lambda^{n+1} = B_n \lambda^n - A\lambda^{n+1}$$

$$\lambda^{n+1} B_{n+1} - \lambda^n B^n = -A\lambda^{n+1}$$

$\vdots$ (telescoping)

$$\lambda^k B_k - B_0 = -A \sum_{n=0}^{k-1} \lambda^{n+1} = -A \sum_{j=1}^{k} \lambda^j$$

$$B_k = \frac{\overset{P}{B_0} - A \sum_{j=1}^{k} \lambda^j}{\lambda^k}$$

$$= \frac{A \sum_{i=1}^{12T} \lambda^i - A \sum_{j=1}^{k} \lambda^j}{\lambda^k}$$

$$= A \sum_{i=1}^{12T-k} \lambda^i$$

GNMA, FNMA, FHLMC - agency-backed MBS

FHA: Federal Home Assoc. VA: Veteran Affairs

$pf(t) = \dfrac{B(t)}{P}$

pool factor

$P^{full} = P^{flat} + AI$

$pf \cdot$ original \$ $\cdot \dfrac{\text{no. of days since last payment}}{30} \cdot \dfrac{\text{coupon rate}}{12}$

quoted

accrued interests (computed linearly)

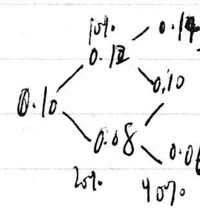
...

### Simple Prepayment Model

$$\gamma_n = \alpha + \beta \cdot (Y - \mu - R_n)^+$$

e.g. $\alpha = 0.10$, $\beta = 10$, $\mu = 0.01$

$Y = 0.10$




then $\gamma_n = 0.10 + 10(0.09 - R_n)^+$

IO strip - hates prepayments

PO strip - likes prepayments

- Pay scheduled interest → reduce outstanding principal
- Repayments processed → "
- Subtract from scheduled payment

Cancelable Swaps, Swaptions

Let $A_{n,m}$ be the time-$n$ price of an annuity with maturity $m$. Then

$$A_{n,m} = \sum_{i=n+1}^{m} B_{n,i}$$

$\uparrow$ price at time $n$ of ZCB that pays \$1 at time $i \geq n$.

$C^{K}_{0,m} = B_{0,m} + K \sum_{i=1}^{m} B_{0,i}$

Let $\underline{\mathrm{Swap}^{K}_{n,m}}$ be the price of a receiver swap ('receive fixed') that receives $K$ at $n+1, n+2, \ldots m$ in exchange for $R_n, R_{n+1} \ldots R_{m-1}$ at the same times. Then

$C^{K}_{n,m} - 1$

$$\mathrm{Swap}^{K}_{n,m} = \frac{1}{D_n} \tilde{E}_n \left[ \sum_{i=n+1}^{m} D_i (K - R_{i-1}) \right]$$

'float', see p22.

$$= K A_{n,m} - (1 - B_{n,m})$$

$D_i R_{i-1} = D_i - D_{i-1} \ldots$

$$SR_{n,m} = \frac{1 - B_{n,m}}{A_{n,m}}$$

A receiver swaption is an option to enter a receiver swap at a future date.

Time 0 price = that of $V_0$ for $V_n = (K - SR_{n,m})^{+} A_{n,m}$

$P^{\text{Receive}}$ (swaption) $- P^{\text{payer}}$ (swaption) $=$ Long forward-start swap ($\mathrm{Swap}^{K}_{n,m}$)

$K > SR$: will exercise /

$K < SR$: will be exercised

Also equivalent to a call option ($K^{C} = 1$) on a bond with coupon rate $K$, maturity $m \ldots$ (since par coupon rate = swap rate)

"Cancelable"

- Callable Swap = Receive swap − Receiver Swaption (price lower than normal)
- Putable Swap = Receive swap + Payer Swaption (price higher than normal)

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"Additional Topics" * note ~~payment~~ interest rates can be annual/"per period"

$$f'(y) \simeq \frac{f(y+h) - f(y-h)}{2h}$$ "centered difference quotient"

- A capped ~~&~~ floored floater has duration very close to 0.

capped floater = floater − $\underbrace{\text{interest rate cap option}}_{(D < 0)}$

floored floater = floater + $\underbrace{\text{interest rate floor option}}_{D > 0}$

Constructing a Zero-Coupon Yield Curve

Total N securities, each one making last payment at time $T_i$

$$P^{(i)} = \sum_{j=1}^{2T_N} F_j^{(i)} d\left(\frac{j}{2}\right), \quad i = 1, 2, \ldots N$$

If $T_1 < T_2 \ldots < T_N$ and $N = 30$, we will have 60 unknowns and far less equations.

"instantaneous forward rate" $f_c(t) = \hat{r}_c(t) + t\hat{r}_c'(t)$

$$= \frac{d}{dt}(t\hat{r}_c(t))$$

$$\Rightarrow d(t) = e^{-t\hat{r}_c(t)} = e^{-\int_0^t f_c(s)ds}$$

"piecewise constant forwards" $f_c(t) = \alpha_i, \quad T_{i-1} < t < T_i$

e.g. when $T_1 = 1$, $T_2 = 2$, we could have $d(t) = \begin{cases} \exp(-\alpha_1 t), & 0 \le t \le 1 \\ \exp(-\alpha_1 - \alpha_2(t-1)), & 1 < t \le 2 \end{cases}$ →

$$d(t) = \begin{cases} \exp(-\alpha_1 t), & 0 \le t \le 1 \\ \exp(-\alpha_1 - \alpha_2(t-1)), & 1 < t \le 2 \end{cases}$$

Given swap rates
$q^{swap}(1) = 0.02$,
$q^{swap}(2) = 0.04$,

$$100 = 2e^{-0.5\alpha_1} + 2e^{-\alpha_1} + 2e^{-\alpha_1 - 0.5\alpha_2} + e^{-\alpha_1 - \alpha_2}(102)$$

$$100 = e^{-0.5\alpha_1} + 101 e^{-\alpha_1}$$

Solving, $100 = 2x + 2x^2 + 2x^2 y + 102 x^2 y^2$ ①

$= x + 101 x^2$ ②

Solve for $x$, subst into ① ...