21-355 4/14/2021

Def :

upper $\nearrow$ $\overline{\int_a^b} f\,dx = \overline{\int} f\,dx := \inf_P U(P,f)$

lower $\searrow$ $\underline{\int_a^b} f\,dx = \underline{\int} f\,dx = \sup_P L(P,f)$

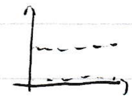
$\forall P: L(P) \le \overline{\int} f\,dx$

$\overset{\text{take sup}}{\Rightarrow} \underline{\int} f\,dx \le \overline{\int} f\,dx$

$$\boxed{\begin{array}{l}\text{Def. } f:[a,b]\to\mathbb{R} \text{ is Riemann-Integrable (R-intg)} \\ \quad \text{iff} \quad 1)\ f \text{ bounded} \\ \qquad\qquad 2)\ \underline{\int} f\,dx = \overline{\int} f\,dx\end{array}}$$

* f does not need to be continuous

* However there certainly exist non Riemann-Integrable functions!

e.g. $1_{\mathbb{Q}\cap[0,1]}$  no matter how fine the partitions are, sup is always 1, inf is always 0 (of a partition)

Theorem

$\boxed{\begin{array}{l}\text{Assume } f \text{ is bounded.} \\ f \text{ is R-intg} \underset{\text{iff!}}{\Longleftrightarrow} \forall \varepsilon\ \exists P = P_\varepsilon \text{ s.t. } U(P_\varepsilon) - L(P_\varepsilon) \le \varepsilon. \\ \qquad\qquad\qquad (\Rightarrow \int f\, dx = U(P_\varepsilon) \pm \varepsilon)\end{array}}$

Proof

$(\Rightarrow)$ $\cdot\ \exists Q$ s.t. $\overset{\text{"sup"}}{\underline{\int} f} - \varepsilon/2 \le L(Q)$

$\cdot\ \exists Q'$ s.t. $U(Q') \le \underset{\text{"inf"}}{\overline{\int}} + \varepsilon/2$

For $P := Q \cup Q'$,

$$\underline{\int} f - \varepsilon/2 \le L(Q) \le L(P) \le U(P) \le U(Q') \le \overline{\int} + \varepsilon/2$$

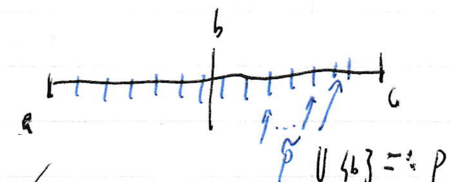
$$\Rightarrow U(P) - L(P) \le \varepsilon$$

$(\Leftarrow)$ $\forall \varepsilon\ \exists P_\varepsilon$ s.t. $0 \le \overline{\int} f - \underline{\int} f \le U(P_\varepsilon) - L(P_\varepsilon) \le \varepsilon.$

$$\Rightarrow \overline{\int} f - \underline{\int} f = 0.$$

Application

$\boxed{\begin{array}{l}\text{Let } a < b < c \quad f : [a,c] \text{ R-intg:} \\ \quad \Rightarrow \int_a^c f = \int_a^b f + \int_b^c f.\end{array}}$



Proof Let $\varepsilon > 0$. Choose $P'_{\varepsilon/2}$ of $[a,b]$ s.t. $U(P') - L(P') < \varepsilon/2$

$P''_{\varepsilon/2}$ of $[b,c]$ $\quad U(P'') - L(P'') < \varepsilon/2$

How to choose P', P''? Let $P' = P\restriction[a,b]$, $P'' = P\restriction[b,c]$ $\quad 0 \le (\cdot) \le \varepsilon/2$

we have $U(P',f) + \underbrace{U(P'',f) - L(P',f)}\overline{ - L(P'',f)} \le \varepsilon/2$

So we have proved $\int_a^b f$ and $\int_b^c f$ both exist.

$\int f\,dx \pm \varepsilon$

Lastly, $U(P,f) = U(P',f) + U(P'',f)$

$= \int_a^b f + \int_b^c f \pm \varepsilon$

Lemma Let $f(x) = g(x)$ $\forall x \in [a,b] \setminus \{r_1, \dots r_n\}$

i.e. f and g differ at most at finitely many points on $[a,b]$.

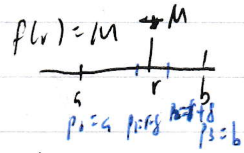
Then f int $\iff$ g int and in this case $\int f = \int g$.

proof Start with $f: [a,b] \to \mathbb{R}$,

$f(x) = 0$ except at points $r_1, r_2, \dots r_n$.

Note f is bounded, $M := \max\{f(r_1) \dots f(r_n)\}$.

Claim $\int f\,dx = 0$. WLOG let $n = 1$ ($\exists r,\ f(r) = M$)



Want $M \cdot 2\delta < \varepsilon$. Then $U(P,f) = 0 + M \cdot 2\delta + 0$

$L(P,f) = 0$

$U(P,f) - L(P,f) = \varepsilon$, $U(P,f) \le \varepsilon$.

So $\int f\,dx = U(P,f) \pm \varepsilon \dots \forall \varepsilon$

$\implies = 0$.

For any finite n, ~~( ) ( ) ( )~~ ('take small radii')

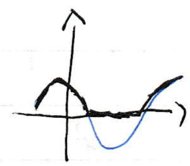
Assuming $\int f \pm \int g = \int f \pm g$ has been proven, for any (f, g), we prove that

$\int f - g = \int 0 = 0$.

Theorems (HW9)

f, g integrable on $[a,b] \Rightarrow$

1) $f+g$ is R-intg & $\int(f+g) = \int f + \int g$
2) $c f(x)$ is R-intg, and $\int cf = c\int f$
3) (monotonicity) $f \le g \Rightarrow \int f \le \int g$
4) $f^+ = \underset{\max}{f \vee 0}$, $f^- = \underset{\min}{f \wedge 0}$ are R-intg.



5) $|f|$ is integrable and $|\int f| \le \int |f|$.
6) $0 \le f \le M \Rightarrow \int_a^b f \le M(b-a)$

Theorem $\boxed{\text{If } f \in \mathcal{C}[a,b] \Rightarrow f \text{ is R-intg.}}$

Proof. Note f is also uniformly continuous, since $[a,b]$ compact (see p. 44).

So $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\left(|x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{b-a}\right)$

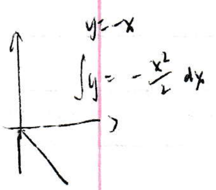
Choose any partition P s.t. $\Delta_k P = p_{k+1} - p_k < \delta$

So $\forall k$, $\sup\limits_{x \in [p_k, p_{k+1}]} f(x) - \inf\limits_{x \in [p_k, p_{k+1}]} f(x) < \frac{\varepsilon}{b-a}$

$\Rightarrow U(P,f) - L(P,f) < \frac{\varepsilon}{b-a} \cdot (b-a) = \varepsilon$.

Theorem $\boxed{\text{If } f \text{ is continuous except at finitely many points, then it is still R-intg.}}$

HW 9



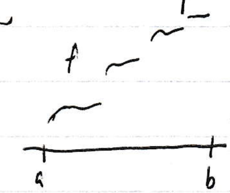
Theorem (Fundamental Theorem of Calculus Part I)
or, how to calculate $\int_a^b f$

let
1) $f : [a,b] \to \mathbb{R}$: R intg

not guaranteed! 2) $\exists F$ diff on $[a,b]$ with $F'(x) = f(x)\ \forall x$

$$\Rightarrow \int_a^b f = F(b) - F(a)$$

counter-example.



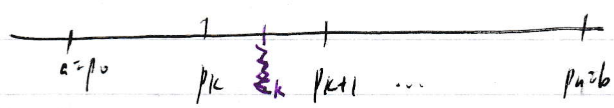
$f \uparrow$, bounded $\Rightarrow$ R-intg

$\nexists F$ s.t. $F' = f$!

The derivative of F cannot equal $f(x)\ \forall x$

Let P be given. By the mean-value theorem,

$$F(p_{k+1}) - F(p_k) = f(\xi_k) \cdot \Delta_k P$$



$$\Delta_k P \cdot \underline{f}_k \le \Delta_k P \cdot f(\xi_k) \le \bar{f}_k \cdot \Delta_k P$$

$$\Rightarrow L(P,f) \le \underbrace{\sum_{k=0,\dots,n-1} \Delta P \cdot f(\xi_k)}_{= F(b) - F(a)} \le U(P,f)$$

$$\therefore \int_a^b f = \underline{\int} f = \sup_P L(P,f) \le F(b) - F(a) \le \bar{\int} f = \int_a^b f$$

so $\int f = F(b) - F(a)$

**

Theorem (Fundamental Theorem of Calculus Part II
or : regularity of indefinite integral)

Let $g: [a,b] \to \mathbb{R}$ R-integrable

$\Rightarrow$ 1) $x \mapsto G(x) := \int_a^x g$ is uniformly continuous (Lipschitz $|f(y)-f(x)| \le C|y-x|$ $\forall y \in S$ sufficiently near x)

2) If g is continuous at $c \in [a,b]$

$\Rightarrow G$ is diff at c, and $G'(c) = g(c)$

1) Exc. use theorem 6. on p80 $\int_a^x g \le M(b-a)$ $\rightarrow$ $|x-y| \le \frac{\varepsilon}{M} \Rightarrow |\int_a^y g - \int_a^x g| \le |\int_x^y g| \le \frac{\varepsilon}{M} \cdot M = \varepsilon.$

2) Want to show $\left| \frac{G(x)-G(c)}{x-c} - \overset{\text{constant}}{g(c)} \right| \xrightarrow{(x \to c)} 0$

$$\frac{1}{x-c} \Big(\underbrace{\int_a^x g - \int_a^c g}_{\int_c^x g(t)dt} \Big) - \frac{1}{x-c} \int_c^x g(c) dt$$

$$\left| \frac{1}{x-c} \int_c^x (g(t) - g(c))\, dt \right| \le \frac{1}{|x-c|} \int_c^x \underbrace{|g(t) - g(c)|}_{<\varepsilon} dt$$

$t \in [c,x]$ if $|t-c| < |x-c| < \delta(\varepsilon) > 0.$

$$\le \frac{1}{|x-c|} \int_c^x \varepsilon \, dt$$

$$= \varepsilon, \text{ q.e.d}$$

use continuity at c.

Theorem (Uniform Convergence)

Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of R-intg functions, $f_n \to f$ uniformly
$\Rightarrow f$ is R-intg and $\int f_n \xrightarrow{n\to\infty} \int f$

Proof sketch $|f - f_n| < \varepsilon \to \int (f - f_n) \leq \varepsilon/(b-a)$ | hw 9 ...

Application Let $g(x) = \sum_{k\geq 0} a_k x^k$ be power series with radius of convergence $R \in [0,\infty]$

$\Rightarrow$ ① $G(x) := \sum_{k\geq 0} \frac{a_k}{k+1} x^{k+1}$ has conv. radius R, and $G'(x) = g(x)$.

Moreover, ② $G(x) = \int_0^x g(t)\,dt \quad \forall x \in (-R,R)$.

Proofs. ① root criterion, differentiability...

② $g_n(x) := \sum_{k=0\ldots n} a_k x^k \xrightarrow[\text{uniformly}]{} g(x)$, then use .

What if the convergence is not uniform?

$$\underset{\text{R-intg}}{f_n(x)} \xrightarrow{n\to\infty} f(x) \,?$$

A: No in general. Example: $f(x) = 1_{\mathbb{Q}\cap[0,1]}(x) \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ - sup always 1 - inf always 0

Let $(q_k)_{k\geq 1}$ be an enumeration of $\mathbb{Q} \cap [0,1]$

Take $f_n(x) = \sum_{k=1\ldots n} \underbrace{1_{\{q_k\}}(x)}_{:=\ 1 \text{ if } x = q_k,\ 0 \text{ otherwise}}$ (every rational x is eventually 'hit' as $n \to \infty$)

Each f_n is continuous except at finitely many points $\to$ R intg.

But $f(x)$ is not R-intg.

Recall
- $1_{\mathbb{Q}}$ is not R-intg (too many discontinuities)
- $f \nearrow \Rightarrow$ countable jumps are ok (every monotonous function is discontinuous at at most countably many points)
- f finitely many discontinuities
  $\Rightarrow$ still R-intg

Theorem (Lebesgue)

let $f : [a,b] \to \mathbb{R}$ bounded.

f R-intg $\not\Leftarrow$ at most countable discontinuities (eg. HW10, Q1) $f : \frac{k}{n} \mapsto \frac{1}{n}$

$\Updownarrow$ $\Downarrow$

set of discontinuities has measure zero.

the direction $\Downarrow$ sets the 'limitations' of Riemann Integrals, can only deal with 'essentially' continuous functions

(def. $A \subseteq \mathbb{R}$ has measure 0
$\Leftrightarrow \forall \varepsilon$ you can cover $A \subseteq \bigcup_{k \geq 1} B(x_k, \varepsilon_k)$
and $\sum_k \varepsilon_k \leq \varepsilon$.)

Note:
Countable set has measure 0

For countable A: $(x_k)_{k \geq 1} \supseteq A$
$= (x_1 \dots x_k \dots)$

$\uparrow$

$B(x_k, \underbrace{\varepsilon \cdot 2^{-k}}_{\leq 2\varepsilon})$

21-355 4/23/2021

"Collections" / "Sets" / "Families" of Functions
→ Equip them with a metric, study them...

Useful! E.g. PDE/ODE/ Maximization. Existence of solution? Use compactness.

Recall def. $\boxed{\text{totally bounded } A \subseteq X \text{ iff } \forall \varepsilon > 0 \ \exists \text{ finitely many points in } X \ x_1, \dots x_n \text{ s.t. } \bigcup_{k=1,\dots n} B_\varepsilon(x_k) \supseteq A}$

Theorem (X,d) totally bounded $\Rightarrow \exists D \subseteq X$, D is dense i.e. $\bar{D} = X$ and D is countable i.e. X is separable.

Proof. For $\varepsilon = \frac{1}{N}$, choose $x_1^N \dots x_{n(N)}^N$ centers by total boundedness

$D := \bigcup_{N=1,2,\dots} \{x_1^N, \dots x_{n(N)}^N\}$ is a countable set, and it is dense since $\forall x \in X, \forall N \ \exists$ center that 'overlaps', $d(x, x_j^N) < \frac{1}{N}$.

Families of Functions

$f: X \to \mathbb{R}$ (or $\mathbb{C}$)

A family $\mathcal{F}$ is 1) Pointwise Bounded: $\forall x \in X$, $\underline{\underline{M(x)}} := \sup_{f \in \mathcal{F}} |f(x)| < \infty$

2) Uniformly Bounded: $\exists M < \infty : \forall x \in X, f \in \mathcal{F} : |f(x)| < M$ (a number)

$\Leftrightarrow \sup_{x \in X} \underbrace{\sup_{f \in \mathcal{F}} |f(x)|}_{M(x)} =: M' < \infty$

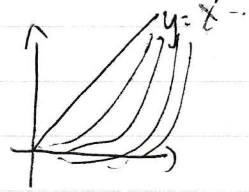
3) $\mathcal{F}$ is ~~equicontinuous~~ uniformly equicontinuous $\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0$ s.t. $d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \ \forall f \in \mathcal{F}$

* choice does not depend on x, and not even f!
⇒ every f is uniformly continuous

$\Leftrightarrow \sup_{f \in \mathcal{F}} \underbrace{w_f(\delta)}_{\sup_{x,y \in X,\, d(x,y) < \delta} |f(x) - f(y)|} =: w(\delta) \xrightarrow[\delta \downarrow 0]{} 0$

* Family of unif continuous fns
$\forall f \in \mathcal{F}, \forall \varepsilon > 0, \exists \delta = \delta(\varepsilon, f)$

Ex ① $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^n$



$\{f_n\}$ is not equicontinuous.

directly, or Arzelà-Ascoli (see pg.

Lemma $\forall n : f_n \in \mathcal{C}(X)$, f_n uniformly continuous, $f_n \to f$ uniformly.

$\Rightarrow \{f_n\}$ is equicontinuous.

Pf. First use uniform convergence:

$$\forall n \geq N, \forall x, \ |f_n(x) - f(x)| < \varepsilon$$

Then $\forall n \leq N$, $\exists \delta = \delta(\varepsilon)$ s.t. $\forall x, y$ with $d(x,y) > \delta$, $\forall n \leq N$, $|f_n(x) - f_n(y)| < \varepsilon$

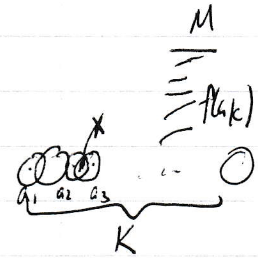
take min of $\delta_1, \delta_2, \ldots \delta_n$

if $n > N$, $d(x,y) < \delta$: $|f_n(x) - f_n(y)| \leq \overbrace{|f_n(x) - f_N(x)|}^{2\varepsilon} + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)| < 5\varepsilon$.

Then take min of these two cases.

Lemma Let $K \subseteq (X,d)$ be compact, $\mathcal{F} \subseteq \mathcal{C}(K)$

If $\mathcal{F}$ is $\begin{cases} \text{pointwise bounded} \\ \text{and} \\ \text{equicontinuous} \end{cases}$ $\Rightarrow \mathcal{F}$ is uniformly bounded.



Equicontinuous $\Rightarrow$ let $\varepsilon = 1$, $\exists \delta$ s.t. $\forall x, y$, $d(x,y) < \delta$

$$\Rightarrow \forall f \in \mathcal{F} : |f(x) - f(y)| < 1$$

Cover K with δ-balls $\Rightarrow \exists a_1, a_2 \ldots a_n \in K$ s.t. $\bigcup_{k=1\ldots n} B_\delta(a_k) \supseteq K$

$$M := \max_{k=1\ldots n} \sup_{f \in \mathcal{F}} |f(a_k)| < \infty$$

let $x \in K \Rightarrow \exists a_k$ s.t. $x \in B_\delta(a_k)$

So $|f(x)| \leq \underbrace{|f(x) - f(a_k)|}_{<1\ \forall f \in \mathcal{F}} + \underbrace{|f(a_k)|}_{\leq M} < M+1$

$\therefore \sup_{f \in \mathcal{F}} \sup_{x \in K} |f(x)| < M+1$

Def. $(\mathcal{C}(X,\mathbb{R}),\ d(f,g) := \sup_x |f(x) - g(x)|$ is a metric space

compact, needed for $d < \infty$ 'sup-norm'

Note: $f_n \to f$ w.r.t d $\Leftrightarrow$ $f_n \to f$ uniformly

What are the compact subsets? ($\Leftrightarrow$ Sequentially compact). Given by Arzela-Ascoli.

Lemma: $f_n : A \to \mathbb{R}$, $n \geq 1$. A countable.

If $(f_n)_{n\geq 1}$ is pointwise bounded $\Rightarrow \exists$ subsequence $(f_{n_k})_k$ s.t. $f_{n_k}(x)$ converges $\forall x \in A$.

"completeness of $\mathbb{R}$" halve, take point, halve again...

compact $\downarrow$ totally bounded $\downarrow$ dense countable set (separable)

Proof. Let $(x_k)_k$ be an enumeration of A.

$(f_n(x_1))_{n \geq 1}$ is bounded

$\Rightarrow \exists$ subsequence $(f_{1,j})_{j\geq 1}$ such that $f_{1,j}(x_1)$ is convergent.

Choose a second subseq. itself a subseq of S_0

$S_0 : f_1, f_2, f_3, f_4$

$S_0 \supseteq S_1 : f_{1,1}\ f_{1,2}$

$S_0 \supseteq S_1 \supseteq S_2 : f_{2,1}\ f_{2,2}\ f_{2,3} \ldots$

$f_{3,3}$

$\vdots$

$S_n \subseteq S_{n-1}$

Taking the diagonal $D = f_{1,1}, f_{2,2}, f_{3,3} \ldots$

The 'tail' converges. $f_{n,n}, f_{n+1,n+1}, \ldots \subseteq S_n$

So $f_{j,j}(x_n)$ is convergent.

Theorem X compact. Then $(\mathcal{C}(X), d)$ is a complete metric space.

Let (f_n) be a C-S $\Rightarrow \forall x, f_n(x)$ is C-S
$\sup \ldots < \varepsilon$
$\Rightarrow$ Convergent to $f(x) \in \mathbb{R}$ ($\mathbb{R}$ complete)

So $f_n \to f$ pointwise, and $\sup_x |f_n(x) - f(x)| \to 0$ (since f_n is C-S)
$\Rightarrow f_n \to f$ uniformly.

Completeness: Cauchy sequence has limit in set

uniform limit of cont functions is continuous

$\Rightarrow f \in \mathcal{C}(X)$ so $\mathcal{C}(X)$ complete.

Theorem (Arzela-Ascoli) (X, d) compact, $\mathcal{F} \subseteq \mathcal{C}(X)$

① If $\mathcal{F}$ is pointwise bounded and equicontinuous,
$\forall (f_n) \subseteq \mathcal{F} \Rightarrow \exists (n_k)$ s.t. $f_{n_k} \rightrightarrows f$ wrt d $(f \in \overline{\mathcal{F}})$
($f \in \mathcal{C}(X)$ and $f \in \mathcal{F}$ if $\mathcal{F}$ is closed)
i.e. uniform

② $\mathcal{F} \subseteq \mathcal{C}(X)$
(i) $\mathcal{F}$ uniformly bounded (pointwise also ok)
(ii) Equicontinuous
$\Longleftrightarrow \overline{\mathcal{F}}$ is compact.
($\mathcal{F}$ is 'quasi-compact')

Application: Riemann disc mapping

① $\Rightarrow$ ② Homework 10.

'skeleton of X'

Proof of ① Since X is compact $\Rightarrow$ totally bounded $\Rightarrow \exists A \subseteq X$, A countable and dense.

(f_n) is pointwise bounded $\Rightarrow \exists f_{n_k}$ s.t. $\forall a \in A$ (countable): $f_{n_k}(a) \to f(a)$, the limiting function. p87.

We will now show that $f_{n_k}(x) \rightrightarrows f(x)$ $\forall x \in X$ (pointwise) convergence by showing that $\forall x$ fixed, $f_{n_k}(x)$ is a Cauchy sequence (then use completeness)

21-355 4/28/2021

Claim: $\forall$ x fixed, $f_{n_k}(x)$ is a C-S.

For simpler notation rewrite the sequence as $f_k(x)$.

Let $\varepsilon>0$. For x fixed, $|f_n(x)-f_m(x)| \le \underbrace{|f_n(x)-f_n(a)|}_{①} + \underbrace{|f_n(a)-f_m(a)|}_{②} + \underbrace{|f_m(a)-f_m(x)|}_{①}$

$\le \underbrace{|f_n(x)-f_n(a_k)|}_{①' \text{ (see below)}} + \underbrace{|f_n(a_k)-f(a_k)|}_{②'} + \underbrace{|f(a_k)-f(x)|}_{③'}$

Go ① → ②.

① : Equicontinuous $\Rightarrow \exists \delta(\varepsilon)$ s.t. $\forall n, x, y : d(x,y)<\delta$
$\Rightarrow |f_n(x)-f_n(y)|<\varepsilon/3$

We can choose $a \in A$ s.t. $d(a,x)<\delta$, so $a=a(x,\delta)=a(x,\varepsilon)$.

② Having chosen a, we know $(f_n(a))$ is a Cauchy Sequence
$\Rightarrow \exists N=N(a,\varepsilon)=N(x,\varepsilon)$ s.t. $\forall n,m \ge N$, $|f_n(a)-f_m(a)|<\varepsilon/3$.

So for $\varepsilon>0$ we found $N=N(\varepsilon,x)$ s.t. $\forall n,m \ge N$, $|f_n(x)-f_m(x)|<\varepsilon$.

Now we try the same argument but independent of the choice of x. We have 'unlocked' f!

①' Now we choose $a_1 \dots a_l$ s.t. $\bigcup_{k=1\dots l} B_\delta(a_k) \supseteq X$ (we total boundedness of X).

Then $\forall x$ we find $a_k=a_k(x)$ s.t. $d(a_k,x)<\delta \Rightarrow \forall x, \forall n: |f_n(x)-f_n(a_k)|<\varepsilon/3$.

②' Since $\forall k=1\dots l$ $f_n(a_k)\to f(a_k) \Rightarrow$
$\exists \underbrace{N=N(\varepsilon)}_{\text{maximum for each } a_k}$ s.t. $\forall k, \forall n \ge N$; $|f_n(a_k)-f(a_k)|<\varepsilon/3$.

③' $|\lim_n f_n(a_k) - \lim_n f_n(x)| = \underbrace{|\lim_n (f_n(a_k)-f_n(x))|}_{\text{absolute convergence}} = \lim_n |f_n(a_k)-f_n(x)| \le \varepsilon/3$

So we have found $N(\varepsilon)$ s.t. $\forall n \geq N$, $|f_n(x) - f(x)| \leq \varepsilon$, and this estimate holds for all $x \in X$ (as $N(\varepsilon)$ does not depend on x).

$\Rightarrow f_n \to f$ uniformly $\boxtimes$

Theorem (Stone-Weierstrass)

$\mathcal{P}[0,1]$ ($:=$ set of polynomials $\in \mathbb{R}$ (or $\mathbb{C}$)) is <u>dense</u> in $(\mathcal{C}[0,1], d)$ ← sup-metric

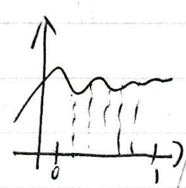
i.e. $f \in \mathcal{C}[0,1] \Rightarrow \exists\, p_n \in \mathcal{P}[0,1]$ s.t. $d(f, p_n) = \sup_X |f(x) - p_n(x)| \xrightarrow[n\to\infty]{} 0$.

– ∃ sequence of polynomials converges uniformly to given continuous function

Application: numerical methods

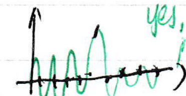
Proof. We will show that the <u>Bernstein polynomials</u> are already dense.

Motivation

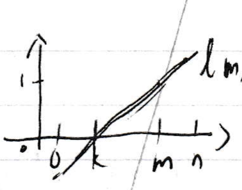


"numerical": divide into intervals, fit polynomial in each

Lagrange polynomial?



yes, it does what it promises, but not much else...



$l_{m,k}(x) \begin{cases} 1 & x = m \\ 0 & x = k \end{cases}$

$$\prod_{\substack{k=0\ldots n,\\ k \neq m}} \left(\frac{x-k}{m-k}\right) = \mathcal{L}_{n,m}(x) \begin{cases} 1 & x = m \\ 0 & x = k \quad \forall k \neq m \end{cases}$$

or $\left(\frac{x - k/n}{\frac{m}{n} - \frac{k}{n}}\right)$ on $[0,1]$

$$f(x) \simeq \underbrace{\sum_{m=0\ldots n} f(m)\, \mathcal{L}_{n,m}(x)}_{= f(m)\ \forall m}$$

Not good enough! Oscillates wildly when $x \neq m$.

B: For $n \geq 1$, $b_0 \ldots b_n \in \mathbb{R}$:

$$B_{\vec{b},n}(x) := \sum_{k=0\ldots n} b_k \binom{n}{k} x^k (1-x)^{n-k}$$

Note that $\sum_{k=0\ldots n} \binom{n}{k} x^k (1-x)^{n-k} = (x + (1-x))^n = 1$ (binomial theorem)

Lemma (Chebyshev)

let $p_k \geq 0$, $\sum_{k=0\ldots n} p_k = 1$, $d_0 \ldots d_n \in \mathbb{R}$ ("probability measure")

$$\mu = \sum_k d_k p_k, \quad \sigma^2 := \sum_k (d_k - \mu)^2 \cdot p_k$$

$$\boxed{\forall c > 0: \quad \sum_{|d_k - \mu| > c} p_k \leq \frac{\sigma^2}{c^2}}$$

$\forall c > 0$

$\left(P[|X| \geq c] \leq \frac{1}{c^p} \mathbb{E}[|X|^p] \right)$

$p = 2$ $\qquad p \geq 1.$

$X \to X - \mu$

Proof.

$$LHS \leq \sum_{k, |d_k - \mu| > c} p_k \cdot \overset{*}{\frac{|d_k - \mu|^2}{c^2}} \quad \left(|d_k - \mu| > c \Rightarrow \frac{|d_k - \mu|^2}{c^2} > 1 \right)$$

$$\leq \frac{1}{c^2} \underbrace{\sum_{k=0\ldots n}}_{\text{all weights / events}} p_k (d_k - \mu)^2$$

$$\overset{\text{def.}}{=} \frac{\sigma^2}{c^2}$$

Special case (binomial distribution)

$$p_k = \binom{n}{k} x^k (1-x)^{n-k}, \; k \geq 0 \Rightarrow \sum_{k=0}^{n} p_k = 1$$

setting $d_k = k/n \Rightarrow \mu = \sum_k p_k d_k = \sum_{k=0}^{n} \frac{k}{n} \binom{n}{k} x^k (1-x)^{n-k}$

$$= \sum_{k=1}^{n} \frac{k}{n} \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k}$$

$$= \sum_{k=1\ldots n} \frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} x^k (1-x)^{n-k}$$

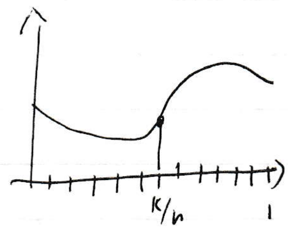
$$= x \sum_{k=1\ldots n} \binom{n-1}{k-1} x^{k-1} (1-x)^{(n-1)-(k-1)}$$

$$= x$$

$$\sigma^2 = \sum_k (d_k - x)^2 p_k = \frac{1}{n} x(1-x)$$

let $f \in \mathcal{C}[0,1]$. For $n \ge 1$ arbitrary,

set $B_{f,n}(x) = \sum_{k=0\dots n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$



let $\varepsilon > 0 \Rightarrow \exists\, \delta > 0$ s.t. $|x-y| < \delta \Rightarrow |f(x)-f(y)| < \varepsilon/2$

WTS. $\forall \varepsilon, \exists N = N(\varepsilon)$ s.t. $\forall n \ge N$, $|B_{f,n}(x) - f(x)| \le \varepsilon$.

$$|B_{f,n}(x) - f(x)| \le \sum_{k=0\dots n} \left|f\left(\tfrac{k}{n}\right) - f(x)\right| \binom{n}{k} x^k (1-x)^{n-k}$$

$$= \underbrace{\sum_{k,\, |k/n - x| < \delta} \underbrace{|f(k/n) - f(x)|}_{<\varepsilon/2} \binom{n}{k} x^k (1-x)^{n-k}}_{<\varepsilon/2 \ \forall n}$$

→ Choose δ

→ Choose n

$$+ \underbrace{\sum_{|k/n - x| \ge \delta} \underbrace{|f(k/n) - f(x)|}_{\le 2M \text{ (compact} \to \text{bounded)}} \underbrace{\binom{n}{k} x^k (1-x)^{n-k}}_{=:\, p_k}}$$

$x(1-x) \le \frac{1}{4},\ 0 \le x \le 1$

$$\underbrace{\le 2M \Big(\sum_{\substack{|k/n - x| > \delta \\ \text{"}d_k\text{"}}} p_k \Big) \le 2M\left(\frac{\sigma^2}{\delta^2}\right) = 2M\left(\frac{x(1-x)}{n\delta^2}\right) \le 2M\left(\frac{1/4}{n\delta^2}\right) = \frac{1}{n} \cdot \frac{M}{2\delta^2}}_{<\varepsilon/2 \text{ when } n \text{ is large enough}}$$

Recall (p 84) the theorem of Lebesgue:

For $f:[a,b] \to \mathbb{R}$ bounded,

f R-intg $\Leftarrow$ at most countable discontinuities

WTS $\Updownarrow$

set of discontinuities has <u>measure</u> zero

Ex (Dirichlet) $f(x) = 1_{\mathbb{Q} \cap [0,1]}(x)$

Points of discontinuity $= [0,1]$, measure $= 1$

Indeed f is not integrable. X

("more complex") $f\left(\frac{k}{n}\right) = \frac{1}{n}$ set of discontinuities is $\mathbb{Q} \cap [0,1]$, measure $= 0$.

This is integrable ✓

Def. $A \subseteq \mathbb{R}$ has measure 0 ($\lambda(A) = 0$) iff $\forall \varepsilon > 0$, $\exists I_1, I_2, \dots$ open

1) $\bigcup_k I_k \supseteq A$, 2) $\sum_k \operatorname{diam}(I_k) < \varepsilon$

Lemmas (HW10)
- $\lambda(I_k) = 0 \ \forall k = 1, 2, \dots$ (countable),

  $\Rightarrow \lambda(\bigcup_k I_k) = 0$ ($\Rightarrow$ any countable A has $\lambda(A) = 0$)
- If $f: [a,b] \to \mathbb{R}^+$ is R-intg with $\int_a^b f = 0$

  $\Rightarrow \lambda(\{x \in [a,b] \mid f(x) \neq 0\}) = 0$ i.e. $f = 0$ "almost everywhere".

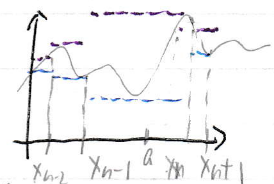
Def. $D = D_f$: set of discontinuities at x

Proof. (Notation)

$P = \{a = x_0 \dots < x_n = b\}$

$U(P,f) = \sum_{k=1\dots n} \bar{f}_k (x_k - x_{k-1})$

Piecewise upper Envelope $u(x) := \bar{f}_1 1_{[x_0,x_1]}(x) + \sum_{k\ge 2} \bar{f}_k 1_{(x_{k-1},x_k]}(x)$



$u \ge f$, u is R-intg, $\int u = U(P,f) \ge \int f$.

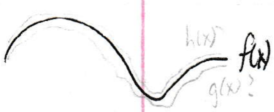
Similarly $\ell(x)$

(R-intg $\Rightarrow \lambda(D)=0$) $\quad$ f is R-intg $\Rightarrow \exists P_k, k \ge 1$ such that

$P_k \subseteq P_{k+1} \dots$ and $U(P_k,f) - L(P_k,f) \xrightarrow[k\to\infty]{} 0$

Then $\ell_k(x) \nearrow$ in k, and $u_k(x) \searrow$,

and $\int \ell_k(x) \nearrow \int f$, $\int u_k(x) \searrow \int f$ $(k \to \infty)$



Def. $g(x) := \lim_k \nearrow \ell_k(x) \le f(x)$ (pointwise)

$h(x) := \lim_k \searrow u_k(x) \ge f(x)$

Then, $\int \ell_k \overset{\ell_k \text{ R-intg}}{=} \underline{\int} \ell_k \le \underline{\int} g \le \overline{\int} g \le \overline{\int} f \overset{f \text{ is R-intg}}{=} \underline{\int} f \le \underline{\int} h \le \overline{\int} h \le \overline{\int} u_k \overset{u_k \text{ R-intg}}{=} \int u_k$

$\lim_k \int \ell_k = \lim_k \int u_k \Rightarrow \underline{\int} g = \overline{\int} g = \int f = \underline{\int} h = \overline{\int} h$

so g, h are R-intg, and $\int g = \int f = \int h$.

$\Rightarrow \int (h-g) = 0 \underset{\text{lemma}}{\Rightarrow} \lambda(\{x \mid h(x) \ne g(x)\}) = 0$.

$\Rightarrow B := \underbrace{(\bigcup_k P_k)}_{\text{countable}} \cup \{x \mid h \ne g\}$, $\lambda(B) = 0$.

It remains to show that f is continuous on B^c.

let $a \in B^C$ and $\varepsilon > 0$ $\Rightarrow g(a) = h(a)$

$$\lim_k \uparrow l_k(a) \qquad \lim_k \downarrow u_k(a)$$

(see picture on previous page)

$\Rightarrow \exists\, k = k(\varepsilon)$ s.t. $u_k(a) - l_k(a) < \varepsilon$.

Furthermore, a is within a segment of P_k i.e. $a \in (x_{n-1}, x_n)$.

$\Rightarrow \exists\, \delta > 0$ s.t. $B_\delta(a) \subseteq (x_{n-1}, x_n]$

Note that $u_k(x)$ and $l_k(x)$ are constant over the segment, (x_{n-1}, x_n) and

$l_k(x) \leq f(x) \leq u_k(x)$

$l_k(a) \leq f(x) \leq u_k(a)$, also $l_k(a) \leq f(a) \leq u_k(a)$

$\Rightarrow \forall x \in B_\delta(a): |f(x) - f(a)| < \varepsilon$

($\lambda(D) = 0 \Rightarrow R$-intg) Proof omitted :(

Comments on the <u>Lebesgue Integral</u>

(1) $\lambda(A) = 0$ ... definition.

Consider μ on $\mathbb{R}$ or I, define the measure.

$\mu : \mathcal{B} \to [0, \infty)$

weight?

$\mathcal{B} \subseteq \mathcal{P}(\mathbb{R})$

$A \in \mathcal{B}$, $A \subseteq \mathbb{R}$

σ-field! (see 21-325)
1) $\emptyset \in \mathcal{B}$
2) $A \in \mathcal{B} \Rightarrow A^C \in \mathcal{B}$
3) $A_1, A_2, \ldots A_k \ldots \in \mathcal{B} \Rightarrow \bigcup_k A \in \mathcal{B}$

$\mathcal{B}$:= smallest σ-field containing all intervals (Borel σ-field)

$\neq (\mathcal{P}(\mathbb{R}))$ too big!

(Simpler Definition)

$\mu : \mathcal{B} \to [0, \infty)$
$A \mapsto \mu(A) \in [0, \infty)$ is a measure iff 1) $\mu(\emptyset) = 0$
2) μ is countably additive.
$A_1, A_2, \ldots A_k \ldots$ disjoint $\Rightarrow \mu(\bigcup_k A_k) = \sum_k \mu(A_k)$

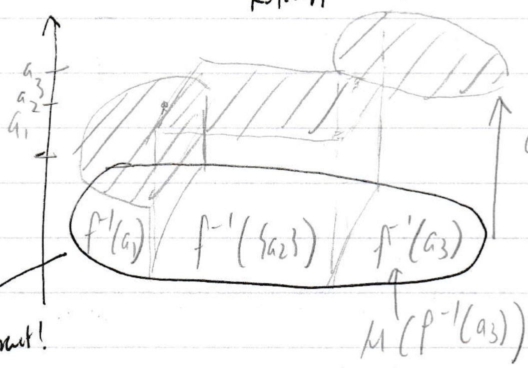
For $(\mathbb{R}, \mathcal{B}, \mu)$ given, $f: \mathbb{R} \to \mathbb{R}$, how do we define the integral?

σ field

21-420!

Imagine 'simple functions' $f: \mathbb{R} \to \{a_1, \dots, a_n\}$ (a.k.a discrete function)

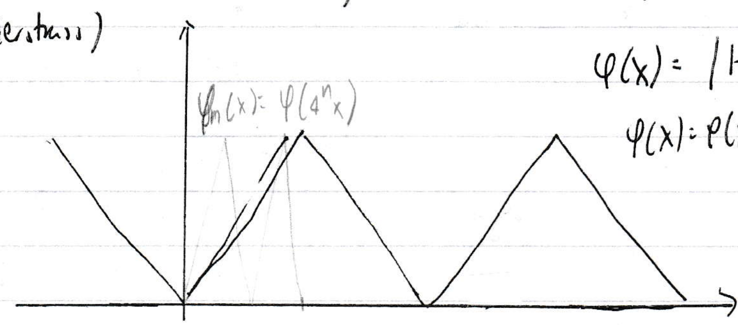
$$\int f \, d\mu = \sum_{k=1 \dots n} a_k \, \mu(f^{-1}\{a_k\})$$



this could be abstract!

When $\mu = \lambda$ (Lebesgue measure), we get the Lebesgue Integral.

Theorem (Weierstrass): $\exists f \in \mathcal{C}(\mathbb{R})$ s.t. f is nowhere differentiable



$\varphi(x) = |1-x|, \ 0 \le x \le 2$

$\varphi(x) = \varphi(x+2)$

$$f(x) := \sum_{n \ge 0} \left(\frac{3}{4}\right)^n \varphi(4^n \cdot x) \text{ uniformly convergent} \Rightarrow f \in \mathcal{C}(\mathbb{R})$$

"damping", to take pointwise

21-355 5/5/2021

Another method (original paper)

$\sum_n \sin(nx) \cdot \frac{1}{n^2} \xrightarrow[n\to\infty]{\text{uniformly}}$ limit, $f(x)$

p=2 harmonic series

but then the derivative is $n\cos(nx)\frac{1}{n^2} = \frac{1}{n}\cos(nx)$ which $\to 0$

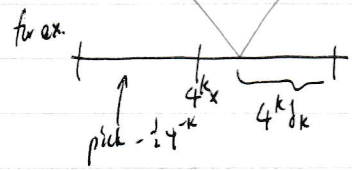
what about $\sum_n \sin(nx) \cdot \frac{1}{n^4} \xrightarrow{?} \frac{d}{dx}(\cdot): \sum_n n^{\cancel{4}\,2}\cos(n^4 x)\frac{1}{n^4} \to \infty$

---

we will show that f is not differentiable at x

Proof. Let x be fixed, $k \in \mathbb{N}$. $\delta_x := \pm \frac{1}{2} \cdot 4^{-k}$, with sign chosen such that the interval $(4^k x, 4^k(x+\delta_k))$ contains no integers.

for ex.



$\Rightarrow$ no 'kink' for $\varphi(4^k x)$ in the interval. and also for all $n \le k$.

Then $\varphi_n' = \pm 4^n$ here $[x, x+\delta_n]$

$$\left| \frac{f(x+\delta_k) - f(x)}{\delta_k} \right| \overset{k \text{ fixed}}{=} \left| \cancel{\sum_{n>k} \left(\tfrac{3}{4}\right)^n \frac{1}{\delta_k}\left(\varphi(4^n(x+\delta_k)) - \varphi(4^n x)\right)} + \sum_{n=0\ldots k} \left(\tfrac{3}{4}\right)^n \frac{1}{\delta_k}\left(\varphi(4^n(x+\delta_k)) - \varphi(4^n x)\right) \right|$$

note $\pm 4^n \cdot \frac{1}{2} 4^{-k} \in 2\mathbb{Z}$

Note φ has period 2! So $\varphi(x+2b) - \varphi(x) = 0$.

$$= \left| \sum_{n=0\ldots k} \left(\tfrac{3}{4}\right)^n \cdot \frac{1}{\delta_k} \underbrace{\left(\varphi_n(x+\delta_k) - \varphi_n(x)\right)}_{= \pm\varphi_n' = \pm 4^n \text{ exactly!}} \right|$$

$$= \left| \sum_{n=0\ldots k} \pm 3^n \right| \underset{\text{assume } 3^k \text{ term} > 0 \text{, all others negative}}{\ge} 3^k - \sum_{j=0}^{k-1} 3^j = 3^k - \frac{3^k - 1}{3-1}$$

$$= \frac{1}{2}(3^k - 1) \nearrow \infty$$

Since $\delta_k \to 0$ ($k \to \infty$) but $|\cdot| \to \infty$, f is not differentiable at x.

Boundary behavior of Power Series (important for complex analysis)

Recall $f(x) := \sum_{n\geq 0} c_n x^n$, $f_n(x) = \sum_{k\geq 0}^{n} c_k x^k$

if exists

1) (f_n) is a C-S in $\mathcal{C}[a,b]$ (sup-metric) $\iff f_n \to f$ uniformly

In particular, if $\sum_k |c_k| r^k < \infty$, then $\Rightarrow f_n \to f$ uniformly in $[-r,r]$

(proof: $\forall x \in [-r,r]$: $|\underset{f(x)-}{f_n}(x)| \leq \sum_{k>n}^{n} |c_k||x|^k \to 0$ see p. 61)

2) Defining $R := \sup\{r \mid \sum |c_k| r^k < \infty\}$ = radius of absolute convergence.

Then ① $\sum_k |c_k| R^k < \infty$
② $\sum_k |c_k| R^k = \infty$.

① For instance $c_k = \frac{1}{k^2} \Rightarrow R = 1$, and $f(R)$ converges.

(However, note that $f_n' \not\to f'$ for $R=1$) the derivative diverges

② $[-R+\varepsilon, R-\varepsilon]$ $\forall \varepsilon > 0$ ok, unclear at R

Theorem (Abel) Assume $R=1$ (for simplicity). If $\sum_n c_n < \infty \Rightarrow \lim_{x \nearrow 1} f(x) = f(1) = \sum_n c_n$. converges nicely

Remark. If $\sum |c_k| < \infty$ ($\Rightarrow$ ①) then ✓

So this statement is interesting only when $\sum |c_k| = \infty$ but $\sum c_n < \infty$

Ex. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n =: f(x)$ Here $R=1$, according to Abel $\lim_{x\to 1} f(x) = f(1) = \sum \frac{(-1)^{n+1}}{n} = \log 2$

However, $f(-1) = \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n} = \sum -\frac{1}{n}$ which is divergent

Proof. Define $s_n = \sum_{k=0}^{n} c_k$, $s_{-1} := 0$.

$s \; (= f(1))$

$s_m x^m - s_{m-1}x^m + s_{m-1}x^{m-1} - s_{m-2}x^{m-1}$

$$\sum_{n=0\ldots m} c_n x^n = \sum_{n=0\ldots m} (s_n - s_{n-1}) x^n = s_m (x^m) + (1-x)\sum_{n=0\ldots m-1} s_n x^n$$

Take $|x|<1$

$$\xrightarrow{m\to\infty} \boxed{f(x) = 0 + (1-x)\sum_{n\geq 0}^{\infty} s_n x^n}$$

$$(1-x)\sum x^n = 1 \;\forall |x|<1$$

let $\varepsilon > 0$. $\exists N(\varepsilon)$ s.t. $|s_n - s| < \varepsilon/2$ $\forall n \geq N$ (convergence)

$$|f(x) - f(1)| = \left| (1-x)\sum_{n\geq 0} s_n x^n - s(1-x)\sum_n x^n \right|$$

$$\leq \left| (1-x)\sum_{n\geq 0}^{N} (s_n - s)x^n \right| + \left| (1-x) \sum_{n>N} \underbrace{(s_n - s)}_{|\cdot|<\varepsilon/2} x^n \right|$$

$$\leq (1-x)\sum_{n\geq 0}^{N} |s_n - s| \cdot |x|^n + \varepsilon/2$$

$$\leq \underbrace{\max_{k\leq N} |s_k - s|}_{M := M(\varepsilon) < \infty} (1-x) \underbrace{\sum_{n\geq 0}^{N} |x|^n}_{\leq 2N(\varepsilon)} + \varepsilon/2$$

$\forall |x|<1$

$$\leq \underbrace{M(\varepsilon)(1-x)2N(\varepsilon)} + \varepsilon/2 < \varepsilon$$

for x large enough s.t. $(1-x) \leq \varepsilon / 4M(\varepsilon)N(\varepsilon)$.

Every complex differentiable function is a power series

In real numbers, (Hw11)
$f(x) = \begin{cases} e^{-1/x^2} & : x \neq 0 \\ 0 & : x = 0 \end{cases}$
cannot be a power series: All coefficients 0.

"Taylor Series"

Let $f(x) = \sum_{n\geq 0} c_n x^n$, with $0 < R \leq \infty$.

$a \in (-R, R)$, $\forall x$ s.t. $|x-a| < R - |a|$. ($=: R'$)

$$\Rightarrow f(x) = \sum_{n\geq 0} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

reverse limits of summation etc. see partial proof

Take the difference, show that it $\to 0$?

For $x \in B_{R'}(a)$:

$$f(x) = \sum_{n \geq 0} c_n((x-a)+a)^n$$

$$= \sum_{n \geq 0} c_n \sum_{k \geq 0}^{n} \binom{n}{k} a^{n-k} (x-a)^k$$

Lemma (absolute convergence, p60).

$$= \sum_{k \geq 0} \underbrace{\left[\sum_{n \geq k} \binom{n}{k} c_n a^{n-k} \right]}_{d_k} (x-a)^k$$

Now we identify d_k as $\frac{f^{(k)}(a)}{k!}$.