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Construction of Real Numbers

Naturals as sets:

$\emptyset =$ "0"

$\{\emptyset\} := 1$

$\{0,1\} := 2$

$\{0,1,\dots n-1\} := n$ $\Big]$ $\mathbb{N}$

$S(n) = n+1 = n \cup \{n\}$

$a + 0 = a$ $\quad a \cdot 0 = 0$

$a + S(b) = S(a+b)$ $\quad a \cdot S(b) = a \cdot b + a$

$a+2 = a + S(1) = S(a+1) = S(a+S(0))$
$= S(S(a+0)) = S(S(a))$

Define $(m,n) \sim (p,q) \in \mathbb{N}\times\mathbb{N} \iff m+q = n+p$

, $\mathbb{Z} = \mathbb{N}\times\mathbb{N}/\sim$ $\quad (m-n = p-q)$

or define the +ve integers as $\{\text{"+"}, 0, 1, \dots\}$
-ve $\{\text{"−"}, 1, \dots\}$

$\mathbb{Q} = \{ m/n \mid m \in \mathbb{Z},\ n \in \mathbb{Z} - \{0\}\}$

$= \{(m,n) \mid \dots \}$

define each unique rational number

Equivalence Relations

$\bar{a} := \{x \in S \mid x \sim a\}$ is the equivalence class of $a \in S$.

"System of representatives" help to index eq. classes.

$n-1-\left(\frac{n-1}{2}\right)+1$

$= n-1-\left(\frac{n-1}{2}\right)$

Commutative Diagrams

$$\begin{array}{ccc} a & \in S & \xrightarrow{f} T \\ \downarrow & \downarrow \pi & \nearrow \exists!\,\bar{f} \\ \bar{a} & S/\sim & \end{array}$$

if f is constant on equivalence classes

i.e. $f(a) = \bar{f}\circ\pi(a)$

Ex. $\mathbb{Q}$ Define $\mathcal{Q} := \{(m,n) \mid m \in \mathbb{Z},\ n \in \mathbb{Z} - \{0\}\}$.

Define $\sim$ on $\mathcal{Q}$ by $(m,n) \sim (a,b) \iff \frac{m}{n} = \frac{a}{b}$ $(mb = an)$

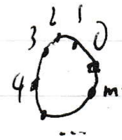
Then $\mathbb{Q} := \mathcal{Q}/\sim$

Ex. $S = \mathbb{Z}$, $m \in \mathbb{N}^+$ fixed. $a \underset{m}{\sim} b \iff \frac{a-b}{m} \in \mathbb{Z}$

$\mathbb{Z}/\sim$ $(= \mathbb{Z}/m\mathbb{Z} = \mathbb{Z}/m$ etc.)

System of Representatives : $\{0, 1, \ldots, m-1\}$

$\{1, 2, \ldots m\}$



$\mathbb{Q}$ is a <u>totally ordered</u> <u>field</u>.

$$(\mathbb{Q}, <) \begin{cases} x \neq y \Rightarrow \text{either } x < y \text{ or } y < x \text{ (antisymmetric)} \\ x < y,\ y < z \Rightarrow x < z \end{cases}$$

Combining order and algebra
$$\begin{cases} x < y \Rightarrow x + z < y + z \\ x > 0,\ y > 0 \Rightarrow x \cdot y > 0 \end{cases}$$

To extend $\mathbb{R}$ from $\mathbb{Q}$, we need to check the same set of properties.

$\sqrt{2}, e, \pi$ are irrational. In general $0.x_1x_2x_3x_4\ldots$ (non-periodic) is irrational

$$= \underbrace{\lim_{n\to\infty} \underbrace{\sum_{k=1..n} x_k \cdot 10^k}_{\in \mathbb{Q}}}_{\text{monotone increasing...}}$$

"limit of a convergent rational sequence?"

What is convergence? (a_n) is convergent $\Rightarrow \exists\, a$ s.t. $\forall \varepsilon > 0, \exists N$ s.t. $\forall n \geq N, |a_n - a| < \varepsilon$.

$\Rightarrow$ What if there exists no rational a?

Cauchy Sequence

(a_n), a rational sequence, is a Cauchy-sequence (C-S)

iff $\forall \varepsilon > 0, \exists m = (M(\varepsilon))$ (m is function of ε)

$\varepsilon \in \mathbb{Q}$ (for now)

s.t. $\forall k, n \geq m, |a_n - a_k| < \varepsilon$

Lemma: $\boxed{\underset{\in \mathbb{Q}}{a_n} \to \underset{\in \mathbb{Q}}{a} \quad (n \to \infty) \Rightarrow (a_n) \text{ is C-S}}$

"Every convergent sequence is Cauchy."

Proof.

look for the "tail of sequence".

Two elements are both close to the common limit, so they must be close to each other.

For $n, m \geq N$ (to be determined), want

$$|(a_n - a) + (a - a_m)| \underset{\triangle\text{-inequality}}{\leq} \underbrace{|a_n - a|}_{\leq \varepsilon/2} + \underbrace{|a - a_m|}_{\leq \varepsilon/2} < \varepsilon$$

motivation

Choose $N = N(\varepsilon/2)$ s.t. $|a_n - a| < \varepsilon/2 \ \forall n \geq N$

Since original sequence was convergent, this N exists.

Now write from bottom-up. Choose m s.t. $\forall n \geq m$, $|a_n - a| < \varepsilon/2 \ldots$

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Lemma: $\boxed{\text{Every C-S is bounded.}}$

$(\mathbb{Z}, \mathbb{R} \text{ etc.})$

Def: (a_n) bounded iff $\exists M \in \mathbb{Q}$ s.t. $\forall k: |a_n| \leq M$.

Proof. Let (a_n) be Cauchy sequence. $\exists m$ s.t. $\forall n \geq m$, $|a_n - a_m| \leq 1 \, (= \varepsilon)$.

$$\Rightarrow |a_n| \leq |a_n - a_m + a_m|$$

$$\leq \underbrace{|a_n - a_m|}_{\leq 1} + |a_m|$$

$$\forall k \geq 1: \quad |a_k| \leq \underbrace{(1 + |a_m|) \underset{\max}{\vee} \big(\max_{j=1\ldots m-1} |a_j|\big)_{<\infty}}_{:= M < \infty}$$

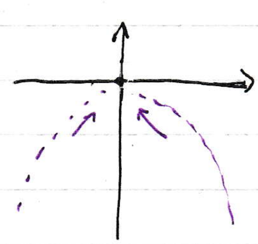
$\boxed{\text{Cauchy's Construction of } \mathbb{R}}$

$$\mathcal{C} = \{(a_n) \mid \forall n: a_n \in \mathbb{Q}, (a_n) \text{ is Cauchy}\}$$

there are many sequences which approach $\sqrt{2}$! Want uniqueness + completeness (no holes)

Define this equivalence relation on $\mathcal{C}$:

$$(a_n) \sim (b_n) \overset{\text{def}}{\Longleftrightarrow} (a_n - b_n) \xrightarrow[n\to\infty]{} 0.$$



1) Reflexive. $(a_n) \sim (a_n)$

2) Symmetric. $(b_n - a_n) \to 0$ iff $(a_n - b_n) \to 0 \Rightarrow$

3) Transitive $a_n \sim b_n$, $b_n \sim c_n \Rightarrow a_n - c_n = \underset{\to 0}{a_n - b_n} + \underset{\to 0}{b_n - c_n} \to 0$.

or $|a_n - c_n| \leq |a_n - b_n| + |b_n - c_n|$

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Algebraic Structure on $\mathcal{C}$

1) Addition. $\underbrace{(a \pm b)_n}_{a,\,b \in \mathcal{C}} := a_n \pm b_n \in \mathcal{C}$

2) Multiplication. $(a \cdot b)_n := a_n \cdot b_n \in \mathcal{C}$

Lemma (Mult. Inv) HW1 $a \in \mathcal{C},\ a \nsim 0 \Rightarrow \exists b \in \mathcal{C}$ s.t. $a \cdot b \sim 1$.

Lemma $+$ and $\cdot$ are invariant i.e. constant on $\sim$ equiv. classes.

WTS 1) $\dfrac{a \sim \alpha,\ b \sim \beta \in \mathcal{C}}{\Rightarrow (\alpha+\beta) \sim (a+b)}$.

$\ulcorner$ $(\alpha+\beta)_n - (a+b)_n = \alpha_n + \beta_n - a_n - b_n$

$= \underbrace{(\alpha_n - a_n)}_{\to 0} + \underbrace{(\beta_n - b_n)}_{\to 0} \to 0$ q.e.d $\lrcorner$

2) W.T.S. $(\alpha\beta)_n \sim (ab)_n$.

('split out') $\alpha_n \beta_n - a_n b_n = \alpha_n \beta_n - a_n \beta_n + a_n \beta_n - a_n b_n$

$= \beta_n (\alpha_n - a_n) + a_n (\beta_n - b_n) \to 0$

$|\beta_n (\alpha_n - a_n)| = |\beta_n|\, |\alpha_n - a_n| \to 0.$

$|\beta_n| \leq M$ (Cauchy Sequences are bounded!)

Corollary. Transfer our operations to $\mathcal{C}/\sim\ (:= \mathbb{R})$

For $\alpha, \beta \in \mathbb{R}$, $\alpha + \beta := \overline{(a+b)} = \overline{(a'+b')}$ using lemma

$\mathcal{C} \ni a \in \alpha$, $b \in \beta$ (representatives!), $(a+b) \in \mathcal{C}$

Comment. $\varepsilon \in \mathbb{Q}$, $\underbrace{\overline{(\varepsilon, \varepsilon, \varepsilon, \ldots \varepsilon)}}_{\in \mathcal{C}} \in \mathbb{R}$. we can pick this 'easy' equivalence class

a rational number!

Ex. $0 \in \mathbb{Q} \longrightarrow \overline{(0,0,\ldots 0)} = 0 \in \mathbb{R}$

$\to R$

Order on $\mathcal{C}$

Def. 1) $a \in \mathcal{C}$ is called (strictly) positive ("$a>0$")

iff 1) $a \not\sim 0$

2) $\exists N$ s.t. $\forall n \geq N$: $a_n > 0$.

2) itself is not sufficient! A mon. dec. sequence can be ~ 0.

2) For $a, b \in \mathcal{C}$, we say $a > b$ if $a-b$ is positive, i.e. $a \not\sim b$, and $\exists N$ s.t. $\forall n \geq N$: $a_n - b_n > 0$.

"<"

a_n "=" b_n since $a_n \sim b_n$

Lemma: $a \in \mathcal{C}$ is positive $\Longleftrightarrow$ $\exists \underset{\in \mathbb{Q}^+}{\delta > 0}$, $\exists N$ s.t. $\forall n \geq N$: $a_n > \delta$.

($\Rightarrow$) WLOG let $a_n > 0$ for all n. (else we can truncate)

If the statement is false, $\forall \delta > 0$ $\not\exists N$ s.t. $\forall n \geq N$, $a_n > 0$

($\Leftarrow$) is less important

$\Leftrightarrow \forall \delta > 0$, $\forall N$ $\exists n \geq N$, $a_n \leq \delta$

1) For $\delta = 1$ choose n_1 s.t. $a_{n_1} \leq \frac{1}{2}$

2) For $\delta = \frac{1}{2}$, choose $n_2 > n_1$ with $a_{n_2} \leq \frac{1}{2}$

$\vdots$

For $\delta = \frac{1}{k}$, choose $n_k > n_{k-1}$ with $a_{n_k} \leq \frac{1}{k}$

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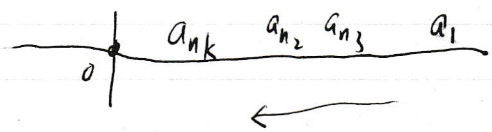

Let $\varepsilon > 0$ given. $\exists N_1$ s.t. $\forall n, m \geq N$, $|a_n - a_m| < \varepsilon/2$.

There exists k such that $\frac{1}{k} < \varepsilon/2$ and $n_k \geq N$ (since $n_k \to \infty$)

$0 \leq a_l \leq |a_l - a_{n_k}| + |a_{n_k}|$

Choose k s.t. $\frac{1}{k} < \varepsilon/2$ and $n_k \geq N(\varepsilon/2)$, then choose $l \geq n_k$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$.

$\therefore a_n \to 0$, $(a_n) \sim 0$.

$\boxed{\text{Corollary.}}$ If $a, b \in \mathcal{C}$, $a > b \Rightarrow \exists \delta > 0$ $\exists N$ s.t. $\forall n \geq N$, $a_n > \delta + b_n$.

$\boxed{\text{Lemma.}}$ Let $a, b, c, d \in \mathcal{C}$. 1) If $a > 0$ and $a \sim b \Rightarrow b > 0$
2) If $a > c$, $a \sim b$, $d \sim c \Rightarrow b > d$.

Proof of 1). • $a > 0 \Rightarrow \exists \delta > 0$, N_1 s.t. $\forall n \geq N$, $a_n > \delta$.
• $a \sim b \Rightarrow \exists N_2 : \forall n \geq N_2$, $|a_n - b_n| < \frac{\delta}{2}$.

$\Rightarrow b_n = (b_n - a_n) + a_n \geq \underbrace{-|b_n - a_n|}_{<\delta/2} + \underbrace{a_n}_{>\delta} \geq \frac{\delta}{2}$. (alternative definition of >0)

2) $a \sim b$, $d \sim c \Rightarrow a - c \sim b - d$

If $a > c$, then $a - c > 0$ then by (1) $b - d > 0$

So $b > d$.

Now we have $+, \cdot, "<", |\cdot|$ on $\mathcal{C}$.

So we define algebraic and order properties on $\mathbb{R} := \mathcal{C}/\sim$.

If $\alpha, \beta \in \mathbb{R} \; (= \mathcal{C}/\sim)$,

- $\alpha + \beta := \overline{(a+b)} \in \mathbb{R}$
- $\alpha \cdot \beta := \overline{(a \cdot b)}$
- $\alpha < \beta \overset{def}{\iff} a < b$ for any representatives $a < b$
- $|\alpha| = \overline{|a|}$

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Def. (Convergence on $\mathbb{R}$)

$(\alpha_k)_{k\geq 1}$, $\alpha \in \mathbb{R}$. We say $\alpha_k \to \alpha \iff \forall \varepsilon \in \mathbb{Q}^+, \exists k_0 = k_0(\varepsilon)$ s.t. $\forall k \geq k_0$,
$|\alpha_k - \alpha| < \bar{\varepsilon}$ $(:= \overline{(\varepsilon, \varepsilon, \dots \varepsilon)})$

(α_k) is a C-S iff $\forall \varepsilon \in \mathbb{Q}^+ \exists K_0$ s.t. $\forall k, l \geq k_0$, $|\alpha_k - \alpha_l| < \bar{\varepsilon}$.

Cauchy seq itself! $a_k \in \mathcal{C}$, $a_k = (a_k)_n = (a_{k,n})_{n\geq 1}$.

how to verify? $|\alpha_k - \alpha| < \bar{\varepsilon} \iff |\underset{\in \alpha_k}{a_k} - \underset{\in \alpha \subseteq \mathcal{C}}{a}| < (\varepsilon, \varepsilon, \dots)$

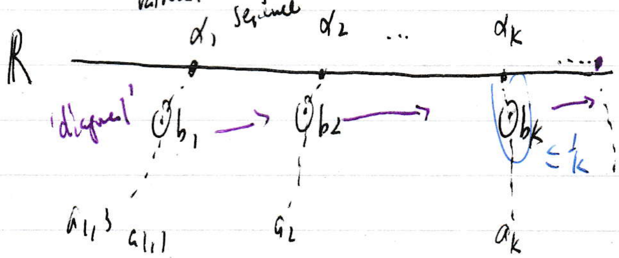
$\iff \exists N_k = N(\varepsilon, k)$ s.t. $|a_{k,n} - a_n| < \varepsilon$ $\forall n \geq N_k$

> **Theorem** $\mathbb{R}$ is complete i.e. every C-S in $\mathbb{R}$ has a unique limit point.
> (in HW 1.1)

Let (α_k) be a C-S in $\mathbb{R}$, with representatives (a_k), $a_k \in \mathcal{C}$.

$a_k = (a_{k,1}, a_{k,2}, \dots a_{k,n}, \dots)$

rational Cauchy sequence



Since $(a_k) \in \mathcal{C} \Rightarrow \exists N_k$ s.t. $\forall n, m \geq N_k$, $|a_{k,n} - a_{k,m}| < \frac{1}{k}$

Choose $b_k := a_{k N_k}$. $\Rightarrow |a_{k,n} - b_k| < \frac{1}{k}$ for all $n \geq N_k$.

Now we show that $\alpha_k \to \underset{(b_1, b_2, \dots)}{\bar{b}}$ (as $k \to \infty$)

Let $\varepsilon \in \mathbb{Q}^+$ be given. Find $k_0 = k_0(\varepsilon)$ s.t. $\forall k \geq k_0$, $\underbrace{|\alpha_k - \bar{b}| < \varepsilon}_{\text{def: } \exists N_k \text{ s.t. } \forall n \geq N_k,\ |a_{k,n} - b_n| < \varepsilon}$.

Then $\forall k \geq k_0$, $\forall n \geq N_k$, $|a_{k,n} - b_n| = |a_{k,n} - a_{k,N_k}| < \frac{1}{k} < \varepsilon$.

$\Rightarrow$ Claim: (b_k) is Cauchy.

Since α_k is C-S $\Rightarrow \exists k_0 = k_0(\varepsilon)$ s.t. $\forall k,l \geq k_0$: $|\alpha_k - \alpha_l| < \varepsilon$.

$\Rightarrow \exists M_{k,l}$ s.t. $\forall n \geq M_{k,l}$, $|a_{k,n} - a_{l,n}| < \varepsilon$.

Then $\forall k,l \geq k_0$,

$$|b_k - b_l| = |a_{k,N_k} - a_{l,N_l}|$$

$$\overset{\triangle\text{-ineq.}}{\leq} \underbrace{|a_{k,N_k} - a_{k,M_{k,l} \vee N_k}|}_{<\varepsilon\ (a_k \text{ is Cauchy})} + |a_{k,M_{k,l} \vee N_k} - a_{l,M_{k,l} \vee N_l}|$$

$$+ \underbrace{|a_{l,M_{k,l} \vee N_l} - a_{l,N_l}|}_{<\varepsilon\ (a_l \text{ is Cauchy})}$$

$$< 3\varepsilon$$

$|a-b| = |a-c+c-d+d-b|$
$\leq |a-c| + |c-d| + |d-b|$

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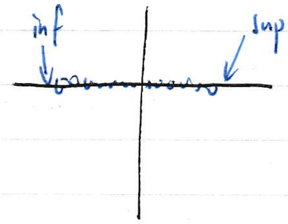
— Least Upper Bound Property

* note: every LUB is unique, called the supremum.

$\alpha := \sup(S)$.

See HW1, Q7.

Greatest lowest bound, $\inf_S := \gamma$



$-\inf S = \sup(-S)$

— $\mathbb{Q}$ is injectively mapped to $\mathbb{R}$ ('embedding')

$q \in \mathbb{Q} \hookrightarrow (q)_n := (q, q, \ldots q) \in \mathcal{C}$

$\left\{ \begin{array}{lll} \mathbb{Q} & \hookrightarrow & \mathcal{C}/\sim \\ q & \longmapsto & \bar{q} \end{array} \right.$ all rational sequences converging to q ... From now on we deal with $\mathbb{R}$ as an ordered field with the LUB property and containing $\mathbb{Q}$.

$\boxed{\text{Archimedean Property}}$ ($\mathbb{Q}$ is dense)

(1) (2)

Theorem (1) $x, y \in \mathbb{R}, x > 0 \Rightarrow \exists n \in \mathbb{N}^+$ s.t. $nx > y$

(2) If $x, y \in \mathbb{R}$, $x < y \Rightarrow \exists q \in \mathbb{Q}$, $x < q < y$.

$n\cancel{(x) \geq (y)}$

$\frac{y}{n} \leq x$ $\exists n$ $\frac{n}{x} > y \ldots$

Proof. (1) $A := \{nx \mid n \in \mathbb{N}\}$, $\alpha := \sup A$. AFSOC $\alpha \geq nx \; \forall n \in \mathbb{N} \Rightarrow A$ is bounded, $A \leq \alpha$

$\alpha - x < \alpha \Rightarrow \alpha - x$ is not an upper bound of A.

$\Rightarrow \exists m$ s.t. $mx > \alpha - x$

$(m+1)x > \alpha$ $\therefore$ α is not an upper bound of A. Contradiction.

(2) WLOG $0<x<y$ (if $[x,y]$ interval includes 0, $x<0<y$ suffices)

Choose $n \in \mathbb{N}$ s.t. $y-x > 1/n$ (we know $\exists$ n s.t. $n(y-x) > 1$)

let $K := \{k \in \mathbb{N} \mid \frac{k}{n} \leq x\}$

K is bounded, so define $k_0 := \sup K$ ($= \max K$)

Then $k_0/n \leq x$ but $k_0+1/n > x$

$$\frac{k_0+1}{n} \leq x + \frac{1}{n} < y.$$

$\boxed{\textbf{Theorem} \quad x \in \mathbb{R}^+ \Rightarrow \exists!\ s \in \mathbb{R}^+ \text{ s.t. } s^n = x}$ (Uniqueness follows from $0<a<b \Rightarrow a^n < b^n$)

Proof of existence

$E := \{t \in \mathbb{R}^+ \mid t^n < x\}$ is non-empty and bounded from above.

$0 < r := \frac{x}{1+x} < 1 \Rightarrow r^n < r < x \Rightarrow E \neq \emptyset.$

Set $b := 1+x \Rightarrow b^n > b > x \Rightarrow b \geq E.$

Def. $s := \sup E$. Prove that $s^n < x$, $s^n > x$ lead to contradiction.

Recall the identity $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \dots + a^{n-1})$

if $0<a<b \Rightarrow b^n - a^n < (b-a)\,n\,b^{n-1}$. (helps prove that n-power function is continuous)

1) Assume $s^n < x$. Choose $0<h<1$ s.t. $h < \frac{x-s^n}{n(s+1)^{n-1}}$

Then let $a := s$, $b := s+h$, $(s+h)^n - s^n < hn\underbrace{(s+h)}_{<1}{}^{n-1} < hn(s+1)^{n-1} < x - s^n$.

$\Rightarrow (s+h)^n < x$, $\Rightarrow s+h \in E$ $\therefore$ s is not $\sup E$.

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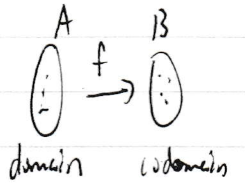
Assume $s^n > x$

2) $h := \dfrac{s^n - x}{n s^{n-1}} \Rightarrow 0 < h < \frac{s}{n} < s$

$t := s-h \Rightarrow s^n - t^n = s^n - (s-h)^n < hns^{n-1} = s^n - x$

$\Rightarrow x < t^n = (s-h)^n \Rightarrow s-h \geq E$

<u>Functions : Terminology</u>



$f : A \to B$

$a \to f(a)$

$f(E) := \{ f(a) \mid a \in E \}$ "Image of E"

$f(A) =$ "Range of A"

Pre-image $f^{-1}(E) = \{ a \in A \mid f(a) \in E \}$.

(not inverse function!)

1) a. $f^{-1}(\bigcup_{\alpha \in I} A_\alpha) = \bigcup_{\alpha \in I} f^{-1}(A_\alpha)$

Pf of 1b: by double containment. "$\subseteq$" let $x \in f^{-1}(A \cap B)$

$\Leftrightarrow f(x) \in A \cap B$

$\Rightarrow f(x) \in A$ and $f(x) \in B$

$x \in f^{-1}(A)$, $x \in f^{-1}(B)$

$x \in f^{-1}(A) \cap f^{-1}(B)$

b. $f^{-1}(\bigcap_{\alpha \in I} A_\alpha) = \bigcap_{\alpha \in I} f^{-1}(A_\alpha)$

2) $f(\bigcup_\alpha A_\alpha) = \bigcup f(A_\alpha)$ ✓

3) $f(\bigcap_\alpha A_\alpha) \subseteq \bigcap_\alpha f(A_\alpha)$

$\neq$ in general!

$f(A) = f(B) \Rightarrow f(A \cap B) \neq f(A)$

$\{a\}$ $\{b\}$ $\emptyset$ $f(A) \cap f(B)$

Cardinalities

Infinite hotel

$1, 2, 3, \ldots n \ldots$

accommodate ~~[illegible]~~
$\mathbb{N}$ extra tourists $\quad \#k \to \#2k$

$$A, B \sim \mathbb{N} \Rightarrow A \cup B \sim \mathbb{N}$$

* Countable union of countable sets is countable.

$$A = \left\{ \begin{array}{cccccc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & \\ a_{31} & a_{32} & a_{33} & a_{34} & & \\ a_{41} & a_{42} & a_{43} & & & \\ a_{51} & a_{52} & & & & \\ a_{61} & & & & & \end{array} \right.$$

Theorem: $\bigotimes_{k \geq 1} A_k$ (countably ∞ product) of $A_k = \{0,1\}$ $\forall k$ is <u>NOT</u> countable.

Proof by contradiction: Assume we have enumerated all sequences... $x_1, x_2 \ldots x_k$

$$y_k = \begin{cases} 1 & \text{if } x_k^k = 0 \\ 0 & \text{if } x_k^k = 1 \end{cases} \quad \forall k \geq 1$$

$0.100 = 0.0999\ldots$

Metric Spaces

Let X be a non-empty set. A metric (or distance) is a function

$$d: X \times X \to \mathbb{R}_0^+ \text{ (not } \overline{\mathbb{R}}!)$$
$$(a,b) \mapsto d(a,b)$$

with

1) $a \neq b \Leftrightarrow d(a,b) > 0$
   $a = b \Leftrightarrow d(a,b) = 0$

2) (Symmetry)
   $d(a,b) = d(b,a)$

3) Δ-inequality:
   $\forall a,b,c \in X : d(a,b) \leq d(a,c) + d(c,b)$

Def. (X,d) is called a <u>metric space</u>.

Ex. ① $\mathbb{R}$, $d(x,y) = |x-y|$

② $\mathbb{R}^n$, $d(x,y) := \|x-y\| = \left(\sum_{k=1,2\ldots} (x-y)_i^2\right)^{1/2}$ (more explicitly, $\|x-y\|_2$)

③ $\mathbb{R}^n$, $d(x,y) := \|x-y\|_1 := \left(\sum_{k=1,2\ldots} |x-y|_i\right)$, the $\mathcal{L}^1$-distance

④ $X = \mathcal{C}[0,1]$, $d(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|$

set of continuous functions $[0,1] \to \mathbb{R}$

$\to$ to understand convergence!

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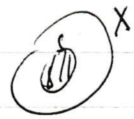
Given (X,d), $S \subseteq d$



(5) $d' : S \times S \to [0,\infty)$

$(x,y) \mapsto d(x,y) := d'(x,y)$. Note $d' = d\restriction_{S\times S}$ (restriction)

"induced metric"

(6) $X \neq \emptyset$, $d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$ for counter-examples

"discrete metric"

Topology of Metric Spaces

Fix (X,d).

1a) $B_r(x) := \{y \in X \mid d(x,y) < r\}$ is the open ball of radius r around x.

$<r$ (boundaries not included)

*Note $\dot{B}_r$ vs B_r

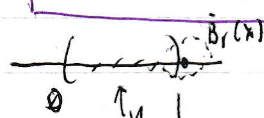
closed $\leq r$

1b) $\dot{B}_r(x) := \{y \in X \mid 0 < d(x,y) < r\} = B_r(x) - \{x\}$

punctured ball

2) Def: $E \subseteq X$. $\boxed{\text{A point } x \in X \text{ is called a limit point of } E \text{ / point of accumulation if } \forall r > 0,\ \dot{B}_r(x) \cap E \neq \emptyset}$

Ex.



$X = \mathbb{R}$.

Claim: $x = 1$ is a limit point of E.

$0 < y < 1$ is also a limit point.

$z > 1$ is not.

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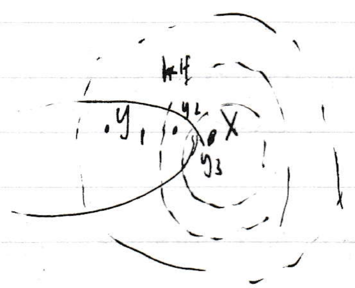
let x be a limit point of E.

For $r>0$: $|\mathring{B}_r(x) \cap E| \neq 0$ ✓

$= \infty$ (!!)



'limiting sequence'

$\Rightarrow \exists\ y_1, y_2, y_3 \ldots \xrightarrow{\text{converges?}} X$ (will prove later)

3) For $X \in E$, Not a limit point $\Leftrightarrow \exists\ r>0,\ \mathring{B}_r(X) \cap E = \emptyset$.

$:=$ "isolated point of E"

let $E = \{\frac{1}{n} \mid n \in \mathbb{N}^+\} \subseteq \mathbb{R} = X$.

- Every point of E is isolated.
- $X=0$ is a limit point of E.

4) $X \in E$ is interior point in E if $\exists r>0$ s.t. $B_r(x) \subseteq E$. The interior points form the interior of E, denoted by $\mathring{E}$.

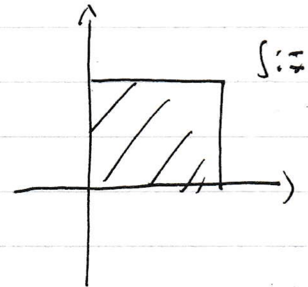
let $E = \{\frac{1}{n} \mid n \in \mathbb{N}^+\}$.

Then E has no interior points.

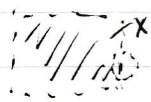
Ex. If $E = [0,1)$, then, $\forall x \in [0,1)$, X is a limit point.

$\forall x \in (0,1)$ X is an interior point.

not interior! $\rightarrow$ 0),



$S := \{(x,y) \mid 0 \le x, y \le 1\}$



We know that $\mathring{S} \supseteq \underbrace{\{(x,y) \mid 0 < x, y < 1\}}_{\tilde{S}}$

$\supseteq$ proof: take $(x,y) \in \tilde{S}$

$r := \frac{1}{2}\left[\left((1-x) \wedge (x-0)\right) \wedge \left((y-0) \wedge (1-y)\right)\right]$ (min)

{interior points of E}

After further work, $\mathring{S} = \tilde{S}$

5) $\boxed{\text{If } E = \mathring{E} \text{ (all points have a neighborhood in } E) \text{ then } E \text{ is called open.}}$

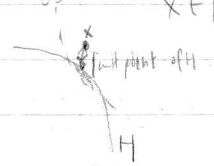
Ex. (a,b) is open. If $X = [0,\infty)$, $E = [0,1)$, then E is open!

6) Let $E' :=$ set of limit points of E.

Define $E' \cup E = \bar{E}$ (closure of E).

$\boxed{E \text{ is closed} \iff E = \bar{E} \iff E' \subseteq E.}$

$U := B_r$ $\quad x \in \bar{H} \Rightarrow U \cap H \neq \emptyset$



($\bar{E}$ is always closed.)

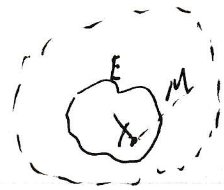
3 cases (1) $E' \setminus E \neq \emptyset$. Take $E = \{\frac{1}{n} \mid n \ge 1\} \subseteq \mathbb{R}$

$E' = \{0\} \cap E = \emptyset$

(2) $E' \subseteq E$. $\quad E := \{0\} \cup \{\frac{1}{n} \mid n \ge 1\}$, $E' = \{0\} \subseteq E$.

(3) $E' = E$ $\quad$ ~~$E := \{0\}$~~

or $E := [0,1] = E'$ $\quad$ "Perfect"



7) E is bounded ("bdd") if $\exists\, x \in X$ s.t. $d(x,e) \overset{\forall e}{\leq} M$ for some $M \in \mathbb{R}$.

$\iff E \subseteq B_M(x)$

no '0', unlike in a vector space $\iff \sup_{e \in E} d(x,e) < \infty$

8) E is <u>dense</u> in X iff $\bar{E} = X$, $X = E \cup E'$

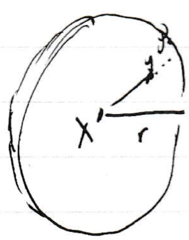
either an element or on limit

$\mathbb{Q}$ is dense in $\mathbb{R}$

Claim: $B_r(x)$ is open

$\iff$ Every $y \in B_r(x)$ is interior.

$\exists\, \rho > 0$ s.t. $B_\rho(y) \subseteq B_r(x)$.



Pf. Let $y \in B_r(x)$.

$\rho := \frac{1}{2}(r - d(x,y))$

Claim. $B_\rho(y) \subseteq B_r(x)$

Let $p \in B_\rho(y)$. W.T.S $d(p,x) < r$.

$d(p,x) \leq \underbrace{d(p,y)}_{< \rho = \frac{1}{2}r - \frac{1}{2}d(y,x)} + d(y,x) = \frac{1}{2}r + \frac{1}{2}d(y,x) < \frac{1}{2}r + \frac{1}{2}r = r$

Theorem $\boxed{E \text{ is open} \iff E^c \text{ is closed.}}$

$\ulcorner$ "$\Rightarrow$" W.T.S $\underbrace{(E^c)'}_{\text{limit points}} \subseteq E^c$

E^c Take $x \in (E^c)'$. To show $x \in E^c$.

$\forall r > 0,\ \dot{B}_r(x) \cap E^c \neq \emptyset$

$\Rightarrow x \notin \mathring{E}\ (= E) \Rightarrow x \in E^c$.

"$\Leftarrow$" Take $x \in E$. $\overset{\text{know } (E^c)' \subseteq E^c}{\Rightarrow} \exists r > 0,\ B_r(x) \cap E^c = \emptyset$.

otherwise $x \in (E^c)' \subseteq E^c$. $\lrcorner$

$\therefore B_r(x) \subseteq E \Rightarrow x$ interior point.

(Every point is interior point.)

Recalling De Morgan's laws, $\begin{bmatrix} (\bigcup_\alpha A_\alpha)^c = \bigcap_\alpha A_\alpha^c \\ (\bigcap_\alpha A_\alpha)^c = \bigcup_\alpha A_\alpha^c \end{bmatrix}$

1) Let $(A_\alpha)_{\alpha \in I}$ be all open $\Rightarrow$ $\boxed{\bigcup_\alpha A_\alpha \text{ is also open.}}$

*finite

2) $A_1, A_2 \dots A_n$ open $\Rightarrow$ $\boxed{\bigcap_{k=1}^n A_k \text{ is still open.}}$ Counter-example, $F_k := (-\frac{1}{k}, \frac{1}{k})$

Pf 2). Pick $x \in \bigcap_{k=1,\dots,n} A_k \Rightarrow \exists r_1, r_2, \dots r_n > 0$ s.t. $B_{r_k}(x) \subseteq A_k\ \forall k = 1, 2, \dots, n$.

Let $r := \min_{k=1\dots n} (r_1, r_2, \dots r_n) > 0$.

$\Rightarrow B_r(x) \subseteq B_{r_k}(x) \subseteq A_k\ \forall k \Rightarrow B_r(x) \subseteq \bigcap_k A_k$, q.e.d.

Pf 1). $x \in \bigcup_\alpha A_\alpha \Rightarrow x \in A_k \Rightarrow x \in \mathring{A}_k\ (B_r(x) \subseteq A_k)$

$\Rightarrow B_r(x) \subseteq \bigcup_\alpha A_\alpha$.

Taking complements, we have

1) $B_1 \dots B_n$ closed $\Rightarrow \bigcup B_n$ is also closed.

2) $B_1 \dots B_n$ closed $\Rightarrow \bigcap B_n$ is closed
arbitrary!!

($\bar{B_1}, \bar{B_2} \dots \bar{B_n}$ open $\Rightarrow \bigcap \bar{B_k}$ open $\Rightarrow \bigcup B_k$ closed)

($\bar{B_1}, \bar{B_2} \dots \bar{B_n}$ open $\Rightarrow \bigcup_\alpha \bar{B_\alpha}$ open $\Rightarrow \bigcap_\alpha B_\alpha$ closed)

Theorem

For (X,d) metric space, $E \subseteq X$:

* 1) $\bar{E}$ is closed. ($\bar{E} = E \cup E'$)

the closure can be seen as an 'operator'

($\Leftarrow\Rightarrow$) $\bar{E}^c$ is open. Take $x \in \bar{E}^c = E^c \cap (E')^c$

Proof.

- So $x \notin E$ and x is not a limit point of E
  $\Rightarrow \exists r > 0$, $\dot{B}_r(x) \cap E \overset{x \notin E}{=} \emptyset$
  Since $x \notin E$, $B_r(x) \cap E = \emptyset$
- Claim: $B_r(x) \cap E' = \emptyset$.

  [Pf by contradiction. Suppose $\exists p \in E'$, $p \in B_r(x)$.
  Then exists $r' > 0$ s.t. $B_{r'}(p) \subseteq B_r(x)$.

  If $p \in E' \Rightarrow \exists e \in E \cap \underbrace{B_{r'}(p)}_{\subseteq B_r(x)}$
  $\Rightarrow e \in E \cap B_r(x)$]

$\boxed{\emptyset = B_r(x) \cap (E \cup E')} \Rightarrow B_r(x) \subseteq \bar{E}^c$.

$\therefore$ So x is an interior point of $\bar{E}^c$.

Every point x is interior $\Rightarrow \bar{E}^c$ is open.

* 2) $\bar{E}$ is the smallest closed set containing E.
Note that X is closed. ($X' \subseteq X$)

let $\bigcap_{F \text{ closed}, F \supseteq E} F = S$, S is closed and $E \subseteq S$. S is the smallest closed set by construction

Claim: $\bar{E} = S$.

1) $\bar{E}$ is closed $\Rightarrow S \subseteq \bar{E}$

2) WTS $\bar{E} \subseteq S$. Note that F closed $\supseteq E \Rightarrow F' \supseteq E'$, furthermore $F \supseteq F'$
$\therefore F \supseteq E' \cup E = \bar{E}$

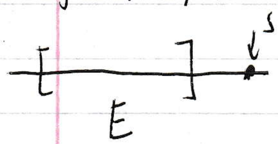
*3) E is closed $\Leftrightarrow E = \bar{E}$.

$(\Rightarrow)$ $E' \subseteq E \Rightarrow E' \cup E \subseteq E$, But $\bar{E} \supseteq E$. $\Big\} \Rightarrow \bar{E} = E$.

$(\Leftarrow)$ If $\bar{E} = E \cup E' = E$, $E' \subseteq E \Rightarrow E$ is closed.

<u>Ex</u> let $E \subseteq \mathbb{R}$ be bounded, $E \leq b \in \mathbb{R}$.

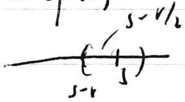
* In general $s \notin E'$!



$\Rightarrow s := \sup E \in \bar{E}$.

AFSOC ~~take~~ suppose that $s \in (\bar{E})^c$. Since $(\bar{E})^c$ is open,

$\exists\, r > 0$ s.t. $B_r(s) \cap \bar{E} = \emptyset$



$B_r(s) \subseteq (\bar{E})^c$ $\quad s - \frac{r}{2} \geq \bar{E} \Rightarrow s - \frac{r}{2} \geq E$

<u>Theorem</u> (Z, d), $X \subseteq Z$, (X, d)

$$\boxed{\begin{aligned} &E \subseteq X \text{ is open in } X \\ &\Leftrightarrow \exists\, G \text{ open in } Z \text{ s.t. } E = G \cap X. \end{aligned}}$$

(open)

$(\Leftarrow)$ intersection of open sets...

$(\Rightarrow)$ Take $G := \bigcup_{x \in E} B_{d(x, \text{edge of } E)}(x)$ $\qquad E = \left[\bigcup_{x \in E} B(x)\right] \cap X$

$\subseteq$

$\supseteq$

$X^c \supseteq X$

Remark. For (X, d) given, 1) X is open 2) $\emptyset$ is open.
(relative properties) Taking complements, 1)' $\emptyset$ is closed 2)' X closed.

Counter-intuitive!

(Prove 1) and 2)'.) $X' \subseteq X$

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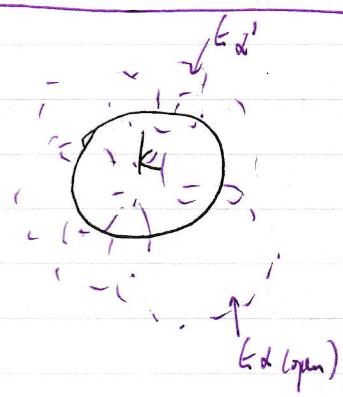
Compactness

"Every open cover contains a finite subcover."



Open cover: If $K \subseteq \bigcup_{\alpha \in I} G_\alpha$

$\therefore$ Then we call $\bigcup_\alpha E_\alpha$ an open cover.

If we can find $\alpha_1, \dots \alpha_n \in I$ such that

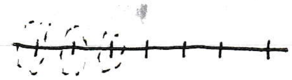
$\bigcup_{k=1,\dots n} E_{\alpha_k} \supseteq K$, then $\bigcup_{k=1\dots n} E_{\alpha_k}$ is a finite subcover.

Ex $K \subseteq \mathbb{R}$, $K = \{a\}$ compact! Select any one E_α from an open cover.

$K = \{a_1, \dots a_n\}$ ✓
(any finite set)

For every point a_k, pick any $E_{\alpha_k} \ni a_k$.

$K = \mathbb{N}$ X



$K = \{ 1/n \mid n \geq 1 \}$ X discrete points! Choose disjoint balls to cover K.

* $K = \{0\} \cup \{1/n \mid n \geq 1\}$ ✓ is compact

Once you cover 0, you cover all but finitely many points to the 'right' of the ball!

$K = \emptyset$ ✓

Theorem

$\boxed{\text{Let } K \subseteq Y \subseteq X. \text{ Then } K \text{ is compact in } Y \Leftrightarrow K \text{ is compact in } X.}$

* "internal" property, unlike closed/open.

Proof. ($\Rightarrow$) Let $G_\alpha \subseteq X$ be open, $K = \bigcup_\alpha G_\alpha$. We find a subcover.

$H_\alpha := Y \cap G_\alpha$ open in Y and

$\bigcup_\alpha H_\alpha = \bigcup_\alpha Y \cap G_\alpha \supseteq Y \cap K = K$

K compact in $Y \Rightarrow \exists \alpha_1, \dots \alpha_n$ s.t.

$$K \subseteq \bigcup_{j=1\dots n} H_{\alpha_j} \subseteq \bigcup_{j=1\dots n} G_{\alpha_j}$$

($\Leftarrow$) "extend" Given $\{G_\alpha\}$ in Y, $\bigcup_\alpha G_\alpha \supseteq K$

$\exists H_\alpha$ open in X s.t. $G_\alpha = Y \cap H_\alpha$.

$K \subseteq \bigcup_\alpha H_\alpha \Rightarrow K \subseteq \bigcup_{k=1\dots n} H_{\alpha_k}$

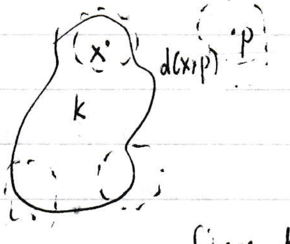
$\Rightarrow K \cap Y \subseteq \bigcup_{j=1\dots n} H_{\alpha_j} \cap Y = \bigcup_{j=1\dots n} G_{\alpha_j}$.

Theorem

$\boxed{K \text{ compact} \Rightarrow K \text{ closed.}}$

We show that K^c is open (every point of K^c is interior).

Let $p \in K^c$. Want to show $\exists r > 0$ s.t. $B_r(p) \subseteq K^c \Leftrightarrow B_r(p) \cap K = \emptyset$



For $x \in K$: $r_x := \frac{1}{3} d(x,p) \Rightarrow B_{r_x}(x) \cap B_{r_x}(p) = \emptyset$

$G_x = B_{r_x}(x)$ open, and $\bigcup_{x \in K} G_x \supseteq K$, so $\bigcup_{x \in K} G_x$ is an open cover.

Since K is compact, $\exists x_1 \dots x_n$ s.t. $K \subseteq \bigcup_{k=1\dots n} G_{x_k}$.

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Want to show that for some r, $B_r(p) \cap \left(\bigcup_{k=1\dots n} E_{x,k}\right) = \emptyset$

Pick $r = \min_k \{ r_{x_k} := \frac{1}{3} d(x_k, p) \}$, so $B_r(p) \cap B_{r_{x_k}}(x_k) = \emptyset$

Then since $\bigcup_k B_{r_{x_k}}(x_k) \supseteq K \Rightarrow B_r(p) \cap K = \emptyset$

Theorem: $\underset{\text{closed in } X}{F} \subseteq \underset{\text{compact}}{K} \subseteq X \Rightarrow F$ compact.

Given  F^c is open $\Rightarrow$ F^c is open cover of F^c

consider $\{G_\alpha\}$ open cover of F. $\{F^c\} \cup \{G_\alpha\}$ open cover of K.

Theorem: $(K_\alpha)_{\alpha \in I}$ compact. Assume that every finite subcollection has nonempty intersection $\Rightarrow \bigcap_{\alpha \in I} K_\alpha \neq \emptyset$.

Counterexample:

$I_n := (0, \frac{1}{n}) \subseteq \mathbb{R}$ not closed

Infinite subcollection could have empty intersection!

$I_n = [0, \frac{1}{n})$ is also not compact, but here we have $\bigcap I_n = \{0\} \neq \emptyset$.

$\rightarrow$ Pick any $K_{\alpha_0} := K$ and $I' := I \setminus \{\alpha_0\}$.

AFSOC $K \cap \left(\bigcap_{I'} K_\alpha\right) = \emptyset$

$K \subseteq \left(\bigcap K_\alpha\right)^c$, deMorgan

$\Leftrightarrow K \subseteq \underbrace{\left(\bigcup_{I'} K_\alpha^c\right)}_{\text{open cover}\dots}$ $\uparrow$ open

$\Leftrightarrow K \subseteq \left(\bigcup_{i=1\dots n} K_{\alpha_i}^c\right)$

$\Leftrightarrow K \cap \underbrace{\left(\bigcap_{i=1\dots n} K_{\alpha_i}\right)}_{\text{finite collection}} = \emptyset$

contradicting the assumption that every finite subcollection has non-empty intersection.

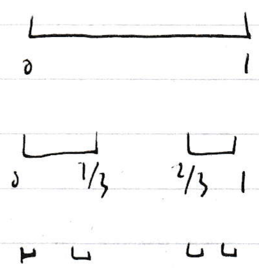
Ex.

If $K_n \Downarrow$ compact, $K_n \neq \phi \; \forall n$,

$\Rightarrow \bigcap_n K_n \neq \phi$, and $\bigcap_n K_n$ is compact!

Ex.

(Cantor set)



$= K_1$ K_2 K_3

* what is the total (volume) length of K_n?

$vol(K_1) = 1$, $vol(K_2) = 2/3$, $vol(K_3) = (\frac{2}{3})^2$

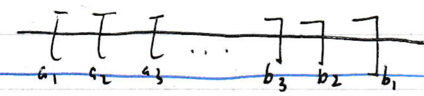
$\downarrow$

$vol(K_\infty) = 0$. (have > 0 not possible)

So the cantor set has measure 0.

But $K_\infty = \bigcap_n K_n \neq \phi$ and is compact (show that each K_n is compact, see below theorem)

not countable!



$\boxed{\text{Lemma: if } I_n = [a_n, b_n] \neq \phi,\ n \geq 1 \text{ and } I_n \Downarrow \Rightarrow \bigcap_n I_n \neq \phi}$

~~Proof~~: We know $a_n \nearrow$ and $b_n \searrow$

$s := \sup_n a_n \; (= \sup\{a_k \mid k \geq 1\})$

We know $a_k \leq b_n \; \forall n \geq 1$.

$\Rightarrow \forall n: b_n \geq \{a_k \mid k \geq 1\}$ so we define

$s := \sup_k a_k \underset{\forall n}{\leq} b_n$

$\Rightarrow \forall k \geq 1: a_k \leq s \leq b_k \Rightarrow s \in I_k$

$\Rightarrow s \in \bigcap_k I_k$

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*$(0,1)$ not compact: take $G_\alpha = \{(1/n, 1)\}$

Theorem:

$\boxed{1)\ X = \mathbb{R},\ a \le b,\ \Rightarrow I = [a,b] \text{ is compact.}}$

Pf. wlog let $I = [0,1]$.

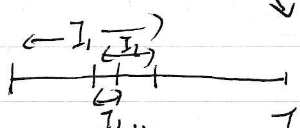
AFSOC $I \subseteq \bigcup_\alpha G_\alpha$ (open) but $\nexists$ finite subcover.

Then $[0, 1/2]$ or $[\frac{1}{2}, 1]$ has no finite subcover. I_1

✓wlog

$[0, \frac{1}{4}], [\frac{1}{4}, 1/2]$ I_2

↘ ...

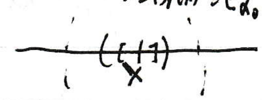


$I \supseteq I_1 \supseteq I_2 \supseteq \ldots$

By the lemma, $\exists x \in \bigcap_k I_k \neq \emptyset$.

$\boxed{\exists \alpha_0 \text{ s.t. } x \in G_{\alpha_0}} \Rightarrow \exists r > 0,\ B_r(x) \subseteq G_{\alpha_0}$

$x \in B_r(x) \subseteq G_{\alpha_0}$



Choose k_0 sufficiently large $|I_{k_0}| = 2^{-k_0} < r$

$\therefore I_{k_0} \subseteq B_r(x)$

So we have covered I_{k_0} with a single element G_{α_0}!

2) Higher dimensions:

$X = \mathbb{R}^d$, $a_i \le b_i$, $i = 1 \ldots d$

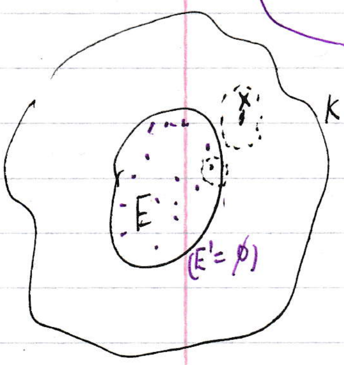
$\Rightarrow I := [a_1, b_1] \times [a_2, b_2] \ldots \times [a_d, b_d]$

$= \{x \in \mathbb{R}^d \mid x_i \in [a_i, b_i]\ i = 1 \ldots d\}$

is compact.

Theorem: If $E \subseteq X$, $|E| = \infty$, $E \subseteq K$ compact.

$\Rightarrow E' \neq \emptyset$ (since K closed $\Rightarrow E \subseteq K \Rightarrow E' \subseteq K$)



AFSOC $E' = \emptyset$. $\forall x \in K: \exists r = r(x) > 0$ s.t. $\dot{B}_r(x) \cap E = \emptyset$

$$\Rightarrow B_r(x) \cap E = \begin{cases} \{x\} & \text{if } x \in E \\ \emptyset & \text{if } x \notin E \end{cases}$$

$(G_x)_{x \in K}$ is a cover for K, and only G_e covers each point in E.

Contradiction; we cannot have finite collection covering K since we need infinitely many G_x's to cover E alone!

(X, d), $A \subseteq X$

Sequentially compact $\Leftrightarrow$ Every sequence in A has a convergent subsequence with limit in A.

Theorem: If K is compact, K is sequentially compact. Pf. Let $(e_k)_{k \geq 1}$ be a sequence in K.

Case 1. $E := \{e_k \mid k \geq 1\}$, $|E| < \infty$ (the sequence repeats)

$\Rightarrow$ Choose indices $k_1 < k_2 < k_3 \dots$ $\quad e_{k_j} \overset{\forall j \geq 1}{=} e_{k_1} \in K$

Case 2. $|E| = \infty$. Then $\exists x \in K \cap E' (\neq \emptyset)$. In other words x is a limit point of E.

As ~~proved~~ in HW3, there exists a sequence of distinct elements

$e_{k_1}, e_{k_2}, \dots e_{k_j} \xrightarrow[j \to \infty]{} x$.

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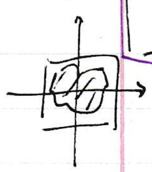
$$\boxed{\begin{array}{l} \text{Theorem: } X = \mathbb{R}^d,\ K \subseteq X \\ \text{TFAE} \\ 1)\ K \text{ is closed and bounded} \\ 2)\ K \text{ compact} \\ 3)\ K \text{ sequentially compact.} \end{array}}$$

Heine-Borel (1), 2))

technical definition

General metric spaces:
totally bounded + complete $\Updownarrow$ compact

"minimum", "maximum" – existence of solutions



closed subset of compact set is compact

Proof.

$1) \Rightarrow 2)$ $K \subseteq [-N, N]^d$ compact! $\Rightarrow$ K itself is compact

$2) \Rightarrow 3)$ Prev. Theorem.

$3) \Rightarrow 1)$

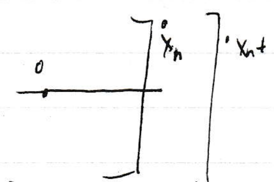
i) AFSOC K is not closed $\Rightarrow \exists x \in K' \setminus K$

Then $\exists$ sequence in K, $x_1, x_2, \cdots \subseteq K$, $x_n \xrightarrow{n \to \infty} x$.

Then no subsequence can converge to an element in K.

let $x \in K'$.
$\exists$ sequence
subsequence
$\vdots$
$x \in K$, $K' \subseteq K$,
K closed

ii) Assume K is not bounded



$\Rightarrow \forall n \geq 1, \exists x_n \in K$ s.t. $d(x_n, 0) > n$.

Consider the subsequence $x_{n_1}, x_{n_2}, \ldots \subseteq (x_n)$

· Want to show $\forall p \in K$, $(x_{n_k}) \not\to p$.

$$d(0,p) + d(p, x_{n_j}) \geq d(0, x_{n_j}).$$

△ inequality in reverse

$$\Rightarrow d(p, x_{n_j}) \geq \underbrace{d(0, x_{n_j})}_{\geq n_j} - \underbrace{d(0,p)}_{\text{constant (in } j)}$$

$n_j \nearrow \ldots\ d(p, x_{n_j}) \underset{j \to \infty}{\not\to} 0$.

· So $(x_n)_{n \geq 1}$ does not have a convergent subsequence with limit in K.

· So K will not be sequentially compact.

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Connectedness

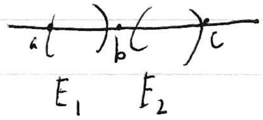
(geometry)

Def $A, B \subseteq X$ are separated iff $A \cap \overline{B} = \overline{A} \cap B = \phi$. (more than disjoint!)

closure

$\boxed{\cdot\ E \subseteq X \text{ is disconnected iff } E = E_1 \cup E_2, \text{ where } E_1, E_2 \text{ are non-empty and separated.}}$

otherwise E is connected. ("negative property")

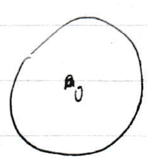
(1) 

$E := (a,c) \setminus \{b\}$

$(a,b) \cap \overline{(b,c)} = (a,b) \cap [b,c] = \phi$

$\therefore$ Disconnected.

(2) $E \subseteq \mathbb{R}^2$, $E = \{x \in \mathbb{R} \mid 0 < |x| \leq 1\}$ is connected



Proof sketch: "pathwise connected", any two points can be connected by a path

$\Downarrow$

connected

$\Leftrightarrow \forall x, y \in E, \exists f: [a,b] \to E$ s.t. $f(a) = x$, $f(b) = y$

$\to$ take a line connecting x and y

(3) $\mathbb{Q} \subseteq \mathbb{R}$ is "super disconnected"!

Pick any irrational number to split $(-\infty, z)$, (z, ∞)

21-355 "only intervals are connected" 3/1/2021

Theorem $E \subseteq \mathbb{R}$ is connected ! not true in $\mathbb{R}^2$

$\Leftrightarrow$ if $x, y \in E$ and $x < z < y \Rightarrow z \in E$

($s := \sup E$, $i := \inf E$
E = (i,s) — $[i,s)$ or $(i,s]$ — $[i,s]$)

The only connected subsets of the real numbers are the intervals

pf. ($\Rightarrow$) AFSOC $\exists x, y$ s.t. $\exists z \in (x,y)$ s.t. $z \notin E$.

Let $x \in A = (-\infty, z) \cap E$, $y \in B := (z, \infty) \cap E$

A is separate from B $\Rightarrow$ $E = A \cup B$ is disconnected

($\Leftarrow$) AFSOC E disconnected $\Rightarrow$ $\exists$ disjoint sets A, B $A \cup B = E$.

Let $x \in A$, $y \in B$, wlog let $x < y$.

Def $z := \sup(A \cap [x,y])$

$z \in \bar{A}$, $\bar{A} \cap B = \emptyset$ (def disconnected) $\Rightarrow z \notin B$ ($z \neq y$). ($\Rightarrow x \leq z < y$) $\Rightarrow$ $z \in A$ ($z \in E$)

$\Rightarrow z \notin \bar{B}$ (def. disconnected)

$B_r(z) \subseteq \bar{B}^c$ (which is open)

So we can find $z_1 \in \bar{B}^c$, $z_1 > z$

If $z_1 \in E \Rightarrow$ In that case z_1 would be the supremum. ($z_1 \notin B \Rightarrow z_1 \in A$)

Theorem $E \subseteq X$ is disconnected iff

$\exists A, B$ open disjoint sets s.t. $\underbrace{A \cap E}_{\neq \emptyset} \cup \underbrace{B \cap E}_{\neq \emptyset} = E$

see HW 4

Sequences

Def. Convergence: $(x_n)_{n\geq 1}$ is called convergent (in X) if $\exists x \in X, \forall \varepsilon > 0$ $\exists n_0 = n_0(\varepsilon)$ s.t. $\forall n \geq n_0$, $d(x_n, x) < \varepsilon$

Theorems

1) $x_n \to x \iff$ every open set containing x also contains all but finitely many (x_n). "tail"

2) If (x_n) is convergent $\to$ bounded.

3) $E \subseteq X$, $x \in E' \Rightarrow \exists (x_n)_{n\geq 1} \subseteq E$.

limit points: $x_n \to x$ and $x_n \neq x_k$, $n \neq k$, $x_n \neq x \ \forall n$.

4) $A \subseteq X$, $(x_n) \subseteq A$, $x_n \to x$ in $X \Rightarrow x \in \bar{A}$.

Furthermore

1) $K \subseteq X$ compact, $(x_n) \subseteq K \Rightarrow \exists x \in K$ and $n_1 < n_2 \ldots < n_k < \ldots$ s.t. $(x_{n_k}) \xrightarrow{k \to \infty} x$

Compact $\to$ seq cpct.

2) In $X = \mathbb{R}^n$, every bounded sequence has a convergent subsequence.

$K = [-R, R]^n$ compact $\Rightarrow$ sequentially compact

(X,d) given, $(x_n)_{n\geq 1}$ is a Cauchy Sequence (no limits are involved)
iff $\forall\, \varepsilon>0,\ \exists N$ s.t. $\forall n,m \geq N,\ d(x_n,x_m)\leq \varepsilon$.

Def. $\boxed{E \subseteq X,\ \operatorname{diam}(E) := \sup\{d(x,y) \mid x,y\in E\}}$

(x_n) is C-S (Cauchy Sequence) $\Leftrightarrow \lim\limits_{N} \operatorname{diam}\{a_k \mid k\geq N\} = 0.$

Theorem 1) $\boxed{\operatorname{diam}(\bar{E}) = \operatorname{diam}(E)}$

Pf. $E \subseteq \bar{E} \Rightarrow \operatorname{diam} E \leq \operatorname{diam} \bar{E}$.

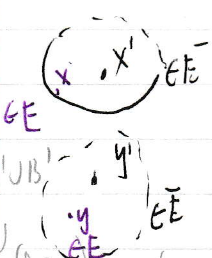
$(A\cup B)' = A'\cup B'$

To show $\operatorname{diam} E \geq \operatorname{diam} \bar{E}$ choose $\varepsilon>0$.

$\Leftarrow$ $a\in A'$ or $a \in B'$ ✓

$\Rightarrow a\in (A\cup B)'$ $\forall B_r(a) \cap (A\cup B)\neq\emptyset$

we will show $\operatorname{diam} E + \varepsilon \geq \operatorname{diam} \bar{E}$, ε arbitrary ✓



Choose $x' \in \bar{E},\ y'\in\bar{E}$

$\Rightarrow \exists\ x\in E$ s.t. $d(x,x')<\varepsilon/2$, similarly $y\in E$

$(x\in E$ or $x\in E'\setminus E)$

If $a\notin A'\cup B'$

$\Rightarrow d(x',y') \leq d(x',x) + d(x,y) + d(y,y')$

$(A\cap B_{r_1}) \cup (B\cap B_{r_2}) = \emptyset$

$\leq \varepsilon + d(x,y)$

$(A\cup B) \cap (\min(B_{r_1}, B_{r_2})) = \emptyset$

$\leq \varepsilon + \sup\limits_{x,y\in E} d(x,y) = \varepsilon + \operatorname{diam}(E)$

~~Take sup on LHS.~~

2) If $K_n \downarrow$ compact with $\operatorname{diam}(K_n)\to 0$
$\Rightarrow K := \bigcap\limits_{n\geq 1} K_n$ contains exactly one point.

We know it's nonempty

Proof by contradiction that it cannot have more than 1 element
(in which case $\operatorname{diam}(K_n) \not\to 0$)

Def. $\boxed{(X,d),\ A \subseteq X \text{ is called complete iff } \forall \text{ Cauchy sequences in } A, \text{ it is convergent in } A.}$

⚠ complete $\not\equiv$ closed ← depends on space

$X = (0,1) \rightarrow X$ is closed but not complete

'inherent property'

$\{\frac{1}{n} \mid n \geq 1\}$ does not converge in X.

Theorem

1) $\boxed{(x_n) \text{ convergent} \Rightarrow (x_n) \text{ is } C\text{-}S}$

2) $\boxed{C \subseteq X \text{ compact}, (x_n) \subseteq C, (x_n)\ C\text{-}S \Rightarrow x_n \to x \in C.}$ (Compact → complete)

"Tail" $K_n := \overline{\underbrace{\{x_k \mid k \geq n\}}_{:=E_n}} = \overline{E_n}$ closed $\Rightarrow$ compact

(decreasing)

$(x_n)\ C\text{-}S \Rightarrow \underset{\to 0}{\operatorname{diam}(E_n)} = \operatorname{diam}(K_n) \to 0$

identify the intersection → Prev. Theorem: $\exists!\ x \in K := \bigcap_n K_n$ ($\Rightarrow x \in C$)

Claim: $x_n \to x$. $\varepsilon > 0$: $\exists n_0$ s.t. $\forall n \geq n_0$,

then $\operatorname{diam} K_n \leq \varepsilon$. Since $x, x_n \in K_n$, $d(x_n, x) \leq \operatorname{diam} K_n \leq \varepsilon$.

3) In $\mathbb{R}^k$ every Cauchy-Sequence is convergent, i.e. $\boxed{\mathbb{R}^k \text{ is complete.}}$

Pf. Let (x_n) be a Cauchy Sequence. $\Rightarrow \exists\ n_0$ s.t. $m,n \geq n_0 \Rightarrow d(x_m, x_n) \leq 1$

$\Rightarrow d(0, x_k) \underset{\forall k \geq n_0}{\leq} d(0, x_{n_0}) + 1$

prove bounded $R := \max_{k \leq n_0} d(0, x_k) \vee d(0, x_{n_0}) + 1 < \infty$

$\Rightarrow$ Put a big ball $\overline{B_R(0)}$ (← compact) $\supseteq \{x_n \mid n \geq 1\} \overset{2)}{\Rightarrow} x_n \to x\ (\in \overline{B_R(0)})$

21-355 — Compact $\Downarrow$ Totally Bounded $\Downarrow$ Bounded — 3/3/2021

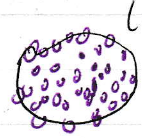
Corollary (X,d) complete, $A \subseteq X$ closed $\Rightarrow A$ is complete.

Pf. Take C-S $(x_n) \subseteq A$, $x_n \to x$ in X, but $x \in \bar{A} = A$.

Given a complete metric space X
- $A \subseteq X$ is complete $\to$ A is closed. Prove that $A' \subseteq A$. There's a sequence converging to l.p. $x \to$ Cauchy $\to x \in A$.
- A is closed $\to$ A is complete (see above)

Theorem
$$\boxed{(X,d)\quad A \text{ is totally bounded} + \text{complete} \iff A \text{ is compact.}}$$
closed in $\mathbb{R}^d$

"$\Leftarrow$" compact $\Rightarrow$ complete (item 3) on previous page), $\Rightarrow$ totally bounded easily verified



$(E_x) = B_r(x),\ x \in A$

HW3

"$\Rightarrow$"

AFSOC A not compact $\Rightarrow \exists (E_x)$ open cover with no finite subcover.

Similar to proof pg 27 of compactness of $[a,b]$

Since A is totally bounded, $\{B_r(x_k)\}$ covers A.
$\exists\, k \le n$ s.t. $A_1 := B_r(x_k) \cap A$ is not finitely covered.

Some $A_1 \in \{A_n\}$ is not finitely covered. Find $a_1 \in A_1$
But A_1 is totally bounded

Some $A_2 \subseteq A_1$ is not finitely covered. Find $a_2 \in A_2$.

$d(a_1, a_2) \le 2 \cdot \frac{1}{2}$.

$d(a_1, \underline{a_k}) \le 2 \cdot \frac{1}{2}$

$d(a_2, \underline{a_k}) \le 2 \cdot \frac{1}{4}$ etc. $\Rightarrow d(a_n, a_k) \le 2 \cdot 2^{-n}$ $\forall k \ge n$

$(a_k) \overset{\in A}{\text{is}}$ C-S, and since A is complete, we have $a_k \to a \in A$.

the limit

Pick α_0 s.t. $a \in E_{\alpha_0} \Rightarrow \exists\, r > 0$ s.t. $B_r(a) \subseteq E_{\alpha_0}$.

But then $A_k \subseteq B_r(a)$, $\forall\, k$ s.t. $2^{-k} < 2r$.

This means that A_k does have a finite subcover among $\{E_\alpha\}$, contradicting the earlier assumption.

From HW:
Sequentially compact $\Rightarrow$ Totally Bounded + Complete

THEOREM. For $A \subseteq (X,d)$, TFAE:

1) A complete + Totally Bounded
2) A compact
3) Sequentially compact