Autoparametric resonance

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Abstract

We attempt to analyze a pendulum from youtuber Steve Mould.

1 Context and motivation

Steve Mould, an educational youtuber, recently posted a video with an interesting demonstration of a pendulum hanging from a spring [1]. The system starts out bobbing up-and-down, but begins to sway side-to-side and eventually is completely swaying with no bobbing. At that point, the system goes back to purely bobbing up and down and continues to cycle through this state. It turns out that this is a case of auto parametric resonance – the bobbing of the spring occurred at a specific frequency that increased the amplitude of pendulum swings. In the video, it's briefly noted that this only occurs with certain ratios of spring constants, masses, and lengths of string. I want to analyze this system to find out what ratio is needed to produce this resonance.

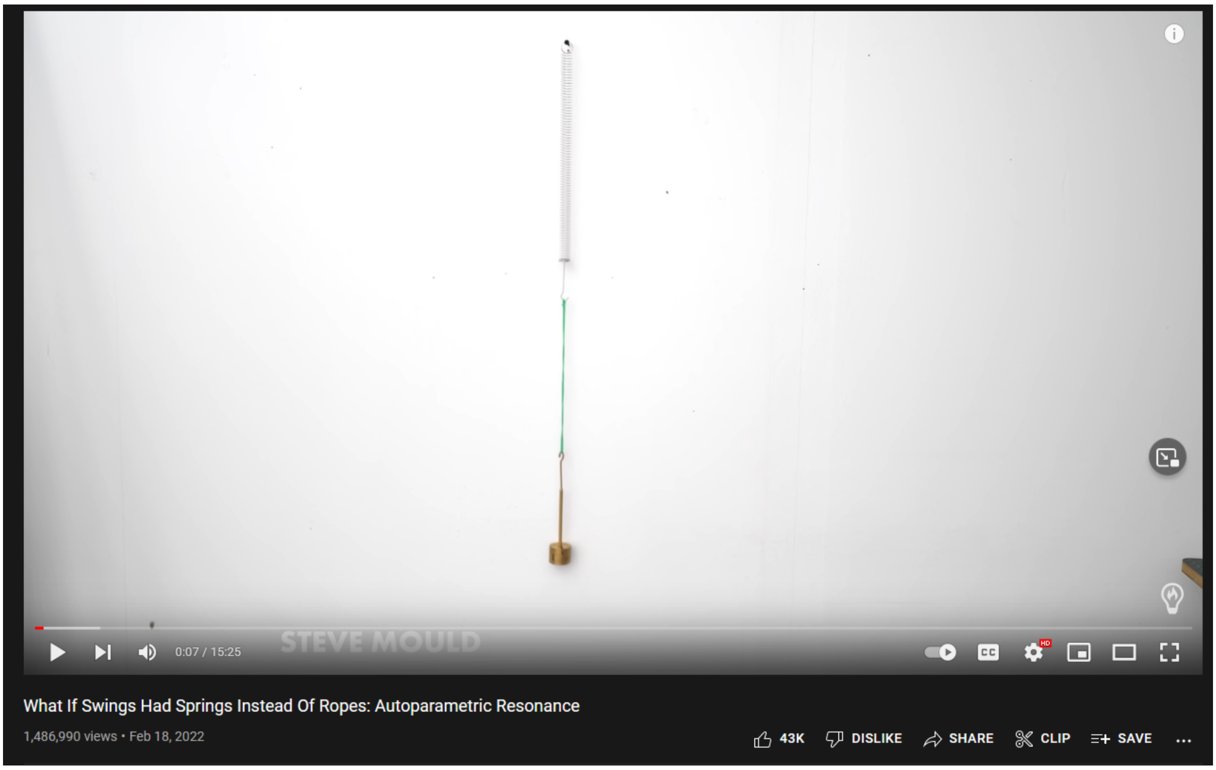

Figure 1: Screenshot from the video which inspired this project.

[1] https://www.youtube.com/watch?v=MUJmKl7QfDU

2 Analysis

2.1 Parametric(External) Resonance

Before we can analyze the spring system, we should first analyze a simpler situation, with external resonance: A pendulum whose string is modulated in length. Let us suppose we have the system as shown in Figure 2. The question is, what range of frequencies ω in $L(t)$ and what initial conditions will produce resonance (ie. increase the amplitude of the pendulum)?

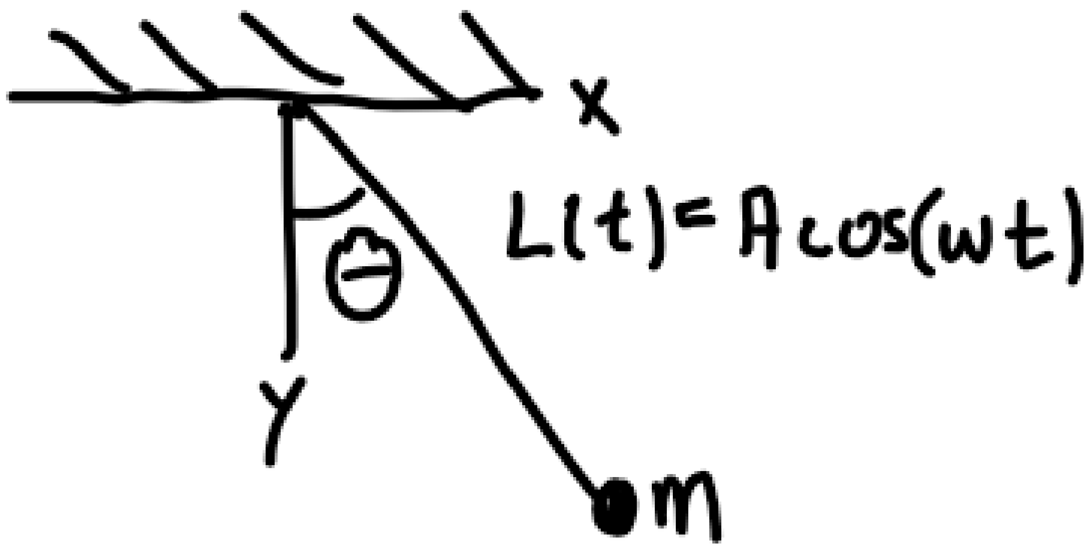

Figure 2: The length of the string is changing in length by a known function $A\cos(\omega t)$.

Set inertial reference frame (iRF) The axes as drawn in the diagram will be our inertial reference frame. It is not attached to the pendulum.

Define degrees of freedom Since $L(t)$ is known, we have one degree of freedom. Our generalized coordinate is θ.

The Lagrangian Since the entire string is changing in length, both x and y coordinates will be affected.

$$x = A\cos(\omega t)\sin(\theta) \tag{1}$$
$$y = A\cos(\omega t)\cos(\theta) \tag{2}$$

$$\dot{x} = -A\omega\sin(\omega t)\sin(\theta) + A\dot{\theta}\cos(\omega t)\cos(\theta) \tag{3}$$
$$\dot{y} = -A\omega\sin(\omega t)\cos(\theta) - A\dot{\theta}\cos(\omega t)\sin(\theta) \tag{4}$$

With some trig substitution, we get our kinetic energy. The cross terms cancel out.

$$\begin{aligned} T =& \frac{m}{2}(\dot{x}^2 + \dot{y}^2) \\ =& \frac{m}{2}(A^2\omega^2\sin^2(\omega t) + A^2\dot{\theta}^2\cos^2(\omega t)) \end{aligned} \tag{5}$$

The first term does not contain θ or $\dot{\theta}$ so it will not contribute to the langrangian. Hence, we have that

$$T = \frac{m}{2} A^2 \dot{\theta}^2 \cos^2(\omega t) \tag{6}$$

The potential energy is easy:

$$U = -mgA\cos(\omega t)\cos(\theta) \tag{7}$$

So our Lagrangian is:

$$\mathcal{L} = T - U = \frac{m}{2} A^2 \dot{\theta}^2 \cos^2(\omega t) + mgA\cos(\omega t)\cos(\theta) \tag{8}$$

We'll need to use approximations with small oscillations to find the normal modes, which will come in handy later when determining the range of ω.

Finding a stable equilibrium This step is pretty simple – take the first derivative of U to find where it's 0, then plug in θ to the second derivative of U to see if it's stable.

$$\frac{dU}{d\theta} = mgA\cos(\omega t)\sin(\theta) = 0 \tag{9}$$

Which only occurs if $\theta = 0, \pi$. Now let's check for stability.

$$\frac{d^2U}{d\theta^2} = mgA\cos(\omega t)\cos(\theta) \tag{10}$$

$$mgA\cos(\omega t)\cos(0) > 0 \tag{11}$$
$$mgA\cos(\omega t)\cos(\pi) < 0 \tag{12}$$

Hence $\theta = 0$ is a stable equilibrium, whereas $\theta = \pi$ is not.

Approximated Lagrangian By taking the Taylor expansion of $\cos(\theta)$ near 0, we get a new Lagrangian (with non-contributing terms removed)

$$\mathcal{L}_{approx} = T - U = \frac{m}{2} A^2 \dot{\theta}^2 \cos^2(\omega t) - mgA\cos(\omega t)\theta^2$$

Euler-Lagrange Equations Let's calculate the Euler Lagrange Equations:

$$\frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{\theta}}\right) = \frac{d\mathcal{L}}{d\theta} \tag{13}$$
$$\frac{d}{dt}(mA^2\dot{\theta}\cos^2(\omega t)) = -2mgA\theta\cos(\omega t) \tag{14}$$
$$mA^2\ddot{\theta}\cos^2(\omega t) - mA^2\dot{\theta}2\cos(\omega t)\omega\sin(\omega t) = -mgA\theta\cos(\omega t) \tag{15}$$
$$\ddot{\theta} - \dot{\theta}2\omega\tan(\omega t) + \theta\frac{g}{A\cos(\omega t)} = 0 \tag{16}$$

Unfortunately, Energy isn't conserved and we don't have any cyclic variables. In this form, this ODE isn't solvable. However, we can apply a transform to get rid of the $\dot{\theta}$ term.[2]

First, let

$$q = l(t)\theta(t) = A\cos(\omega t)\theta(t)$$

Then take first and second derivatives:

$$\dot{q} = -A\theta\omega\sin(\omega t) + A\dot{\theta}\cos(\omega t)$$

$$\ddot{q} = -A\omega^2\cos(\omega t)\theta - A\omega\sin(\omega t)\dot{\theta} + A\ddot{\theta}\cos(\omega t) - A\omega\dot{\theta}\sin(\omega t) \tag{17}$$
$$= -A\theta\omega^2\cos(\omega t) - 2A\dot{\theta}\omega\sin(\omega t) + A\ddot{\theta}\cos(\omega t) \tag{18}$$

[2]Method taken from page 4 of http://www.mat.uniroma3.it/users/gentile/ricerca/lavori/wbg3.pdf

Rearrange equations for θ, $\dot{\theta}$, and $\ddot{\theta}$:

$$\theta = \frac{q}{A\cos(\omega t)}$$

$$\begin{aligned}
\dot{\theta} &= \frac{\dot{q} + A\theta\omega\sin(\omega t)}{A\cos(\omega t)} && (19)\\
&= \frac{\dot{q} + A\frac{q}{A\cos(\omega t)}\omega\sin(\omega t)}{A\cos(\omega t)} && (20)\\
&= \frac{\dot{q} + q\omega\tan(\omega t)}{A\cos(\omega t)} && (21)
\end{aligned}$$

$$\begin{aligned}
\ddot{\theta} &= \frac{\ddot{q} + A\theta\omega^2\cos(\omega t) + 2A\dot{\theta}\omega\sin(\omega t)}{A\cos(\omega t)} && (22)\\
&= \frac{\ddot{q} + A\frac{q}{A\cos(\omega t)}\omega^2\cos(\omega t) + 2A\frac{\dot{q}+q\omega\tan(\omega t)}{A\cos(\omega t)}\omega\sin(\omega t)}{A\cos(\omega t)} && (23)\\
&= \frac{\ddot{q} + q\omega^2 + 2(\dot{q} + q\omega\tan(\omega t))\omega\tan(\omega t)}{A\cos(\omega t)} && (24)\\
&= \frac{\ddot{q} + \dot{q}2\omega\tan(\omega t) + q(\omega^2 + 2\omega^2\tan^2(\omega t)}{A\cos(\omega t)}) && (25)
\end{aligned}$$

Now we can plug $\ddot{\theta}$, $\dot{\theta}$, and θ into our original ODE (Equation 13):

$$\frac{\ddot{q} + \dot{q}2\omega\tan(\omega t) + q(\omega^2 + 2\omega^2\tan^2(\omega t))}{A\cos(\omega t)} - \frac{\dot{q} + q\omega\tan(\omega t)}{A\cos(\omega t)}2\omega\tan(\omega t) + \frac{q}{A\cos(\omega t)}\frac{g}{A\cos(\omega t)} = 0 \quad (26)$$

$$\ddot{q} + \dot{q}2\omega\tan(\omega t) + q(\omega^2 + 2\omega^2\tan^2(\omega t)) - \dot{q}2\omega\tan(\omega t) - q2\omega^2\tan^2(\omega t) + \frac{qg}{A\cos(\omega t)} = 0 \quad (27)$$

$$\ddot{q} + \dot{q}(2\omega\tan(\omega t) - 2\omega\tan(\omega t)) + q(\omega^2 + 2\omega^2\tan^2(\omega t) - 2\omega^2\tan^2(\omega t) + \frac{g}{A\cos(\omega t)}) = 0 \quad (28)$$

$$\ddot{q} + q(\omega^2 + \frac{g}{A\cos(\omega t)}) = 0 \quad (29)$$

This is an ODE with only a second derivative and the initial function, ~~and can be solved with the characteristic equation:~~

Unfortunately, since the q term has a function of t, if we try to write a solution for q like:

$$q = C\cos(st + \phi), \text{where } s = \sqrt{\omega^2 + \frac{g}{A\cos(\omega t)}}$$

Then taking the second derivative of q will introduce other terms. So we're a little bit stuck. We still have an equation that is unsolvable (even after small oscillation approximations!), and we still haven't been able to relate ω with the natural frequency ω_n of a pendulum of fixed length.

There are some guides online describing similar problems (see references), but unfortunately my understanding stops at this point so I cannot continue following those guides in good faith. At this point, I'd like to pivot into numerically solving our ODE.

Numerical Solutions We will be solving Equation 13, which is not transformed and still in terms of θ instead of q. First, let's solve the integral for a simple pendulum of fixed length a with no friction. Note that all code can be found in a Jupyter Notebook at https://github.com/kxing28/Variable-Length-Pendulum.

Simple Pendulum. The equation of a simple pendulum is as follows:

$$\ddot{\theta} + \frac{g}{a}\sin(\theta) = 0 \tag{30}$$

We will use initial conditions

$$\theta(0) = 0, \dot{\theta}(0) = 1\text{rad/s}$$

$$g = 9.8, a = 1$$

Using the odeint function from the SciPy package, we can solve this integral with ease. The results are plotted in Figure 3.

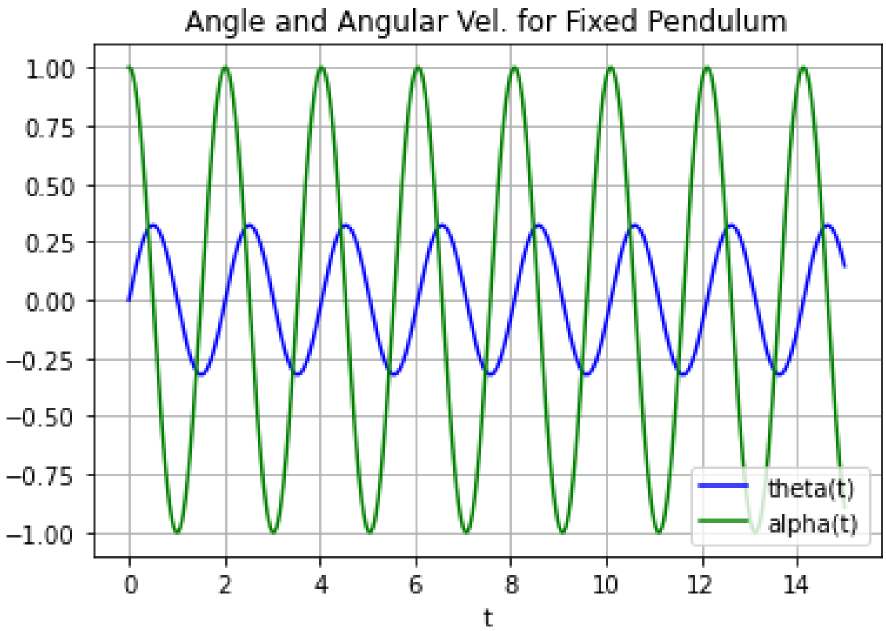

Figure 3: The angle and angular velocity of a pendulum of fixed length.

Note that in this case, both θ and $\dot{\theta}$ have constant period and constant amplitude, as expected.

Variable Length Pendulum. Now let us try to model the variable length pendulum, reproduced here:

$$\ddot{\theta} - \dot{\theta}2\omega\tan(\omega t) + \theta\frac{g}{A\cos(\omega t)} = 0 \tag{31}$$

If we put initial conditions of

$$\theta(0) = 0, \dot{\theta}(0) = 1$$

$$w = 0.1, g = 9.8, a = 1$$

Then we obtain the following graph in Figure 4.

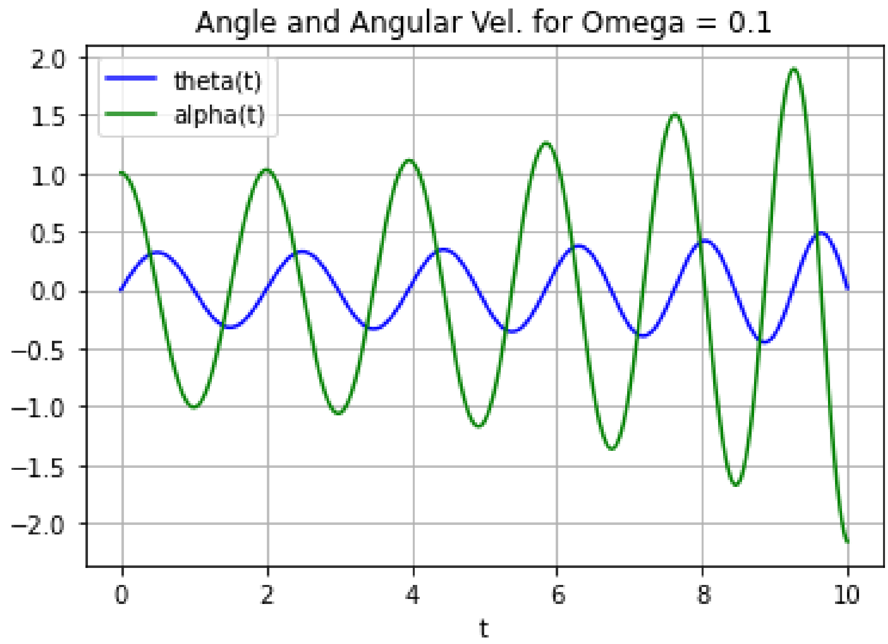

Figure 4: The angle and angular velocity of a pendulum of variable length, with a driver frequency of 0.1.

Note how both θ and $\dot{\theta}$ have steadily increasing amplitudes. Actually, at around $t = 16$, we get a huge spike in our graph, presumably where our small angle approximations fail us (see Figure 5).

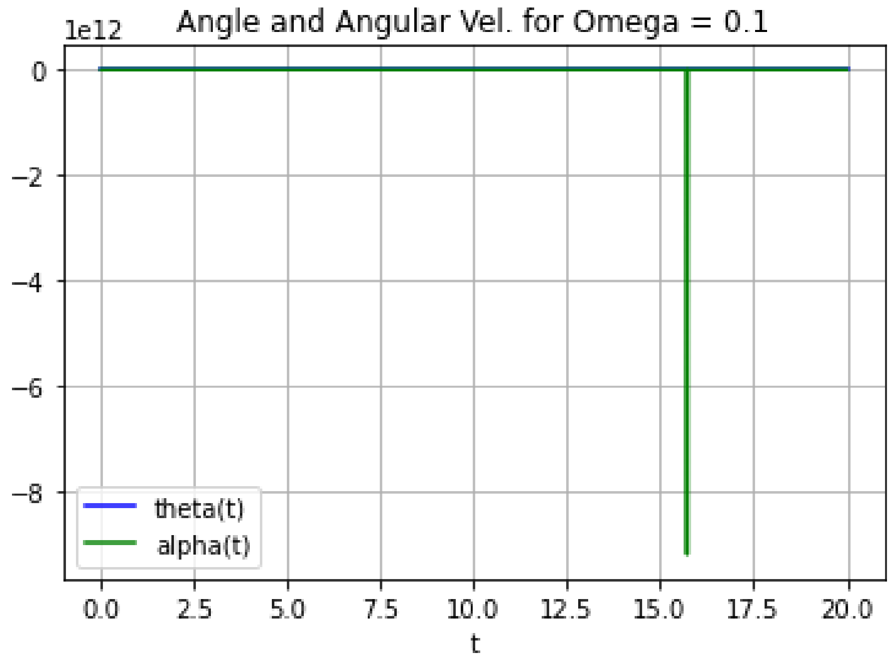

Figure 5: The same system as shown in Figure 4, but with a larger range.

~~Theoretically, our driver ω should be 2π-periodic. However, when we plug in $\omega = 2\pi$, we don't recover the same graph.~~ Okay, ω isn't necessarily 2π-periodic. However, this graph keeps blowing up, when it should only blow up at certain resonant frequencies. If I set $\omega = 0.01$, I get the same blowups, I just need to extend my range:

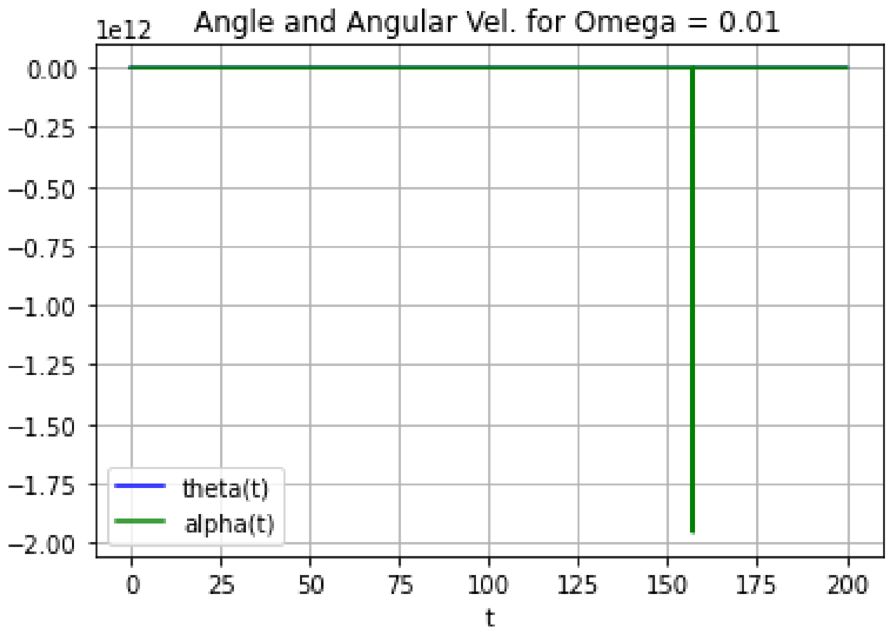

Figure 6: The angle and angular velocity of a pendulum of variable length, with a driver frequency of 0.01. Note that the solution suddenly blows up at around $t = 160$.

I think the reason behind this is that then $\omega t = \frac{\pi}{2}$, 2 terms in our ODE (Equation 13) blow up. $\tan \pi/2 = \infty$ so the second term blows up, and $\frac{1}{\cos \pi/2} = \infty$ so the third term also blows up. Additionally, if we look at Figure 6, the graph blows up at $t \approx 160$, which is very close to $\frac{\pi}{2} * 100 \approx 157$.

Something's gone wrong here...let's try to numerically solve the original Lagrangian, before we approximated for small oscillations.

Original Lagrangian If we obtain Euler-Lagrange equations from Eqn. 8, we get:

$$\ddot{\theta} - 2\dot{\theta}\omega \tan(\omega t) + \frac{g \sin(\theta)}{A \cos(\omega t} = 0$$

But this only differs from the approximated Lagrangian by a factor of $\sin(\theta)$, so this doesn't solve the $\frac{\pi}{2}$ issue.

At this point, I'm just feeling like I really should've gone to office hours earlier. Since this integral still has the $\frac{\pi}{2}$ issue, I know it'll just blow up like the other integrals.

A thought. My function $l(t)$ which describes the length of the pendulum may be incomplete. In its current state, the rope length begins with $l(0) = A\cos(0) = A$, and ends with $l(\pi/2) = A\cos(\pi/2) = 0$. The rope disappears at $\frac{\pi}{2}$, which is physically impossible! By conservation of angular momentum, the speed of the pendulum goes to infinity when the rope is of 0 length. This explains why the graphs blow up – we'll need to modify our $l(t)$ function and rewrite our Lagrangian.

3 Variable Length Pendulum: The correct way

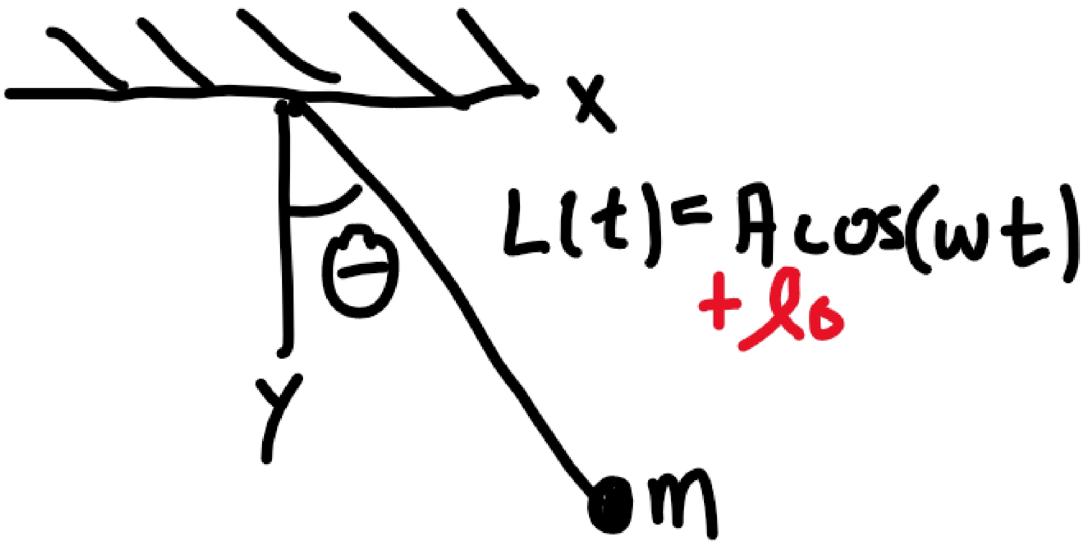

Figure 7: The length of the string is changing in length by a known function $L(t) = l_0 + A\cos(\omega t)$. It is assumed that $|A| < l_0$.

We've learned our lesson. $L(t)$ cannot go all the way to zero, since by conservation of momentum, the tangential velocity ends up going to infinity. So let's go through these steps again for Figure 7.

Set inertial reference frame (iRF) The axes as drawn are fine. It is not attached to the pendulum. We have assumed that $|A| < l_0$.

Define degrees of freedom We have one degree of freedom. Our generalized coordinate is θ.

The Lagrangian We're going to keep L(t) in its labelled form, and expand it at the very end.

$$x = L\sin(\theta) \tag{32}$$
$$y = L\cos(\theta) \tag{33}$$

$$\dot{x} = \dot{L}\sin(\theta) + L\dot{\theta}\cos(\theta) \tag{34}$$
$$\dot{y} = \dot{L}\cos(\theta) - L\dot{\theta}\sin(\theta) \tag{35}$$

The cross terms cancel in kinetic energy:

$$T = (\dot{x}^2 + \dot{y}^2) \tag{36}$$
$$= (\dot{L}^2 + L^2\dot{\theta}^2) \tag{37}$$

The potential energy is, as usual, simple.

$$U = -mgy = -mgL\cos(\theta) \tag{38}$$

Thus, our Lagrangian is (removing unused terms):

$$\mathcal{L} = T - u = \frac{m}{2}(L^2\dot{\theta}^2) + mgL\cos(\theta) \tag{39}$$

Euler-Lagrange Equations Calculation is straightforward:

$$\frac{d}{dt}(\frac{d\mathcal{L}}{d\dot{\theta}}) = \frac{d\mathcal{L}}{d\theta} \tag{40}$$

$$\frac{d}{dt}(mL^2\dot{\theta}) = -mgL\sin(\theta) \tag{41}$$

$$2mL\dot{L}\dot{\theta} + mL^2\ddot{\theta} = -mgL\sin(\theta) \tag{42}$$

$$\ddot{\theta} + \frac{2\dot{L}\dot{\theta}}{L} + \frac{g\sin(\theta)}{L} = 0 \tag{43}$$

Plugging in for $L(t) = l_0 + A\cos(\omega t)$:

$$\ddot{\theta} - \frac{2A\omega\sin(\omega t)\dot{\theta}}{l_0 + A\cos(\omega t)} + \frac{g\sin(\theta)}{l_0 + A\cos(\omega t)} = 0 \tag{44}$$

If we do the transformation from before with $q = L(t)\theta(t)$, and replace $\sin(\theta)$ with θ for small oscillations, we can still get rid of the first derivative:

$$\ddot{q} + \frac{q}{L}(-\ddot{L} + \frac{g}{L}) = 0$$

but L is time-dependent, so this is unsolvable by our simpler methods. Instead, we'll numerically solve the exact solution.

Numerical Solutions We will be solving equation 44 with the "odeint" function from the SciPy package. Our goal in plotting this ODE is to NOT have it blow up at $\omega = \frac{\pi}{2}$. It's okay if it grows at some points (i.e. when we've achieved resonance), but it shouldn't blow up for all ω. Can we do some pre-verification to see if our ODE is correct?

Plugging in $\omega = \pi/2$, the second term goes to 0 and the third term stays finite. Great.

Plugging in $\omega = 0$, we get

$$\ddot{\theta} + \frac{g\sin(\theta)}{l_0 + A} = 0$$

Which is the equation of motion for a simple pendulum with length $l_0 + A$.

It looks like we'd only have a singularity if $l_0 + A\cos(\omega t) = 0$. But $|A| < l_0$, and $\max(A\cos(\omega t)) = A$, so $l_0 + A > 0$. I.e. the numerator is always greater than 0.

This means the terms themselves won't blow up! I'm feeling good about this. We'll plot with the following initial conditions:

$$\theta(0) = 0, \dot{\theta}(0) = 1 \tag{45}$$

$$g = 9.8, A = 0.1, l_0 = 1, \omega = 0.1 \tag{46}$$

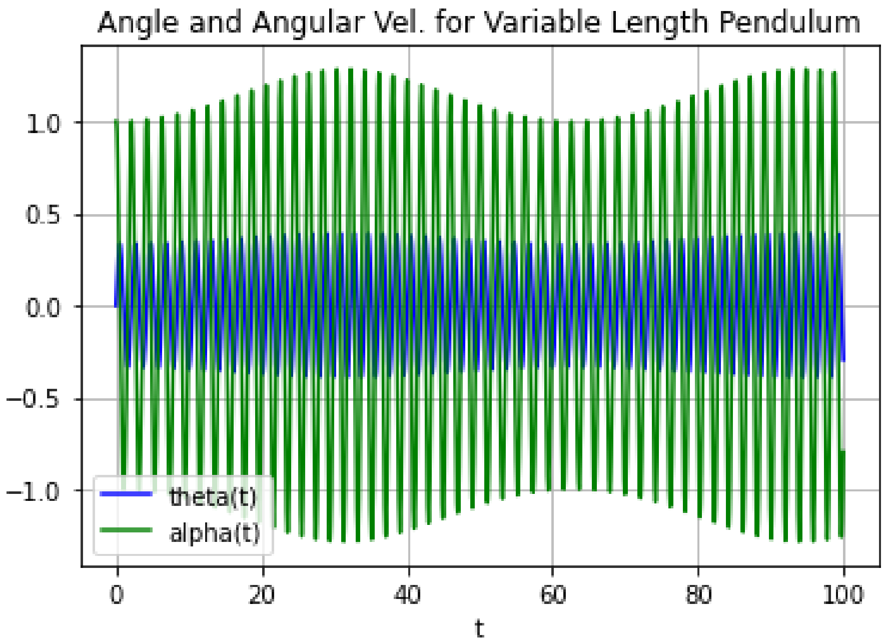

Figure 8: With $\omega = 0.1$, we can see that the angular velocity has varying amplitude and just barely that θ is varying too. We can also see that this is periodic and does not blow up.

In Figure 8, we can see that the solution does not blow up – this is good. We also see that the driving frequency alters $\dot{\theta}$ more than θ. Actually, the amplitude of $\dot{\theta}$ is always larger than θ. I'm not sure why.

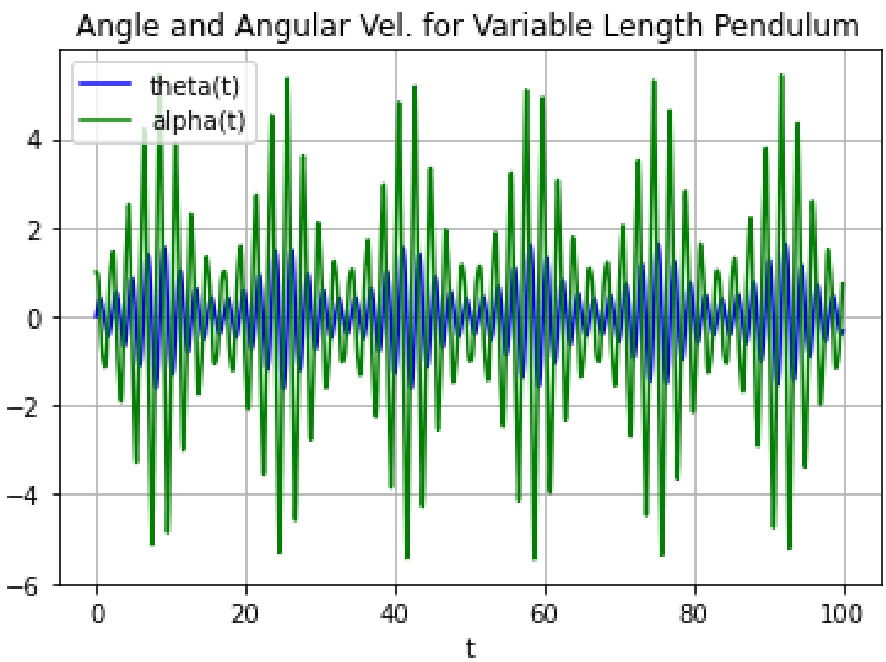

Figure 9: When we apply $\omega = \omega_0 * 2 = \sqrt{\frac{g}{l_0}} * 2$, we get some very strong resonance! Note that this is periodic as well.

Steve Mould states in his video that "..the lengthening and shortening of the string happens at twice the frequency of the pendulum"[3]. We can see if that's the case for our ODE. Given that the natural

[3] https://www.youtube.com/watch?v=MUJmKl7QfDU&t=218s at the 3:38 mark.

frequency ω_0 of a pendulum is $\sqrt{\frac{g}{l_0}}$, then if we apply a frequency of $\omega = 2*\omega_0 = 2\sqrt{\frac{g}{l_0}} = 2\sqrt{9.8} \approx 6.26$ for our constants, we should get resonant behaviour – the amplitudes of θ and $\dot{\theta}$ should increase dramatically. In Figure 9, we see exactly that! It remains to show that resonance is greatest at $\omega = 2 * \omega_0$.

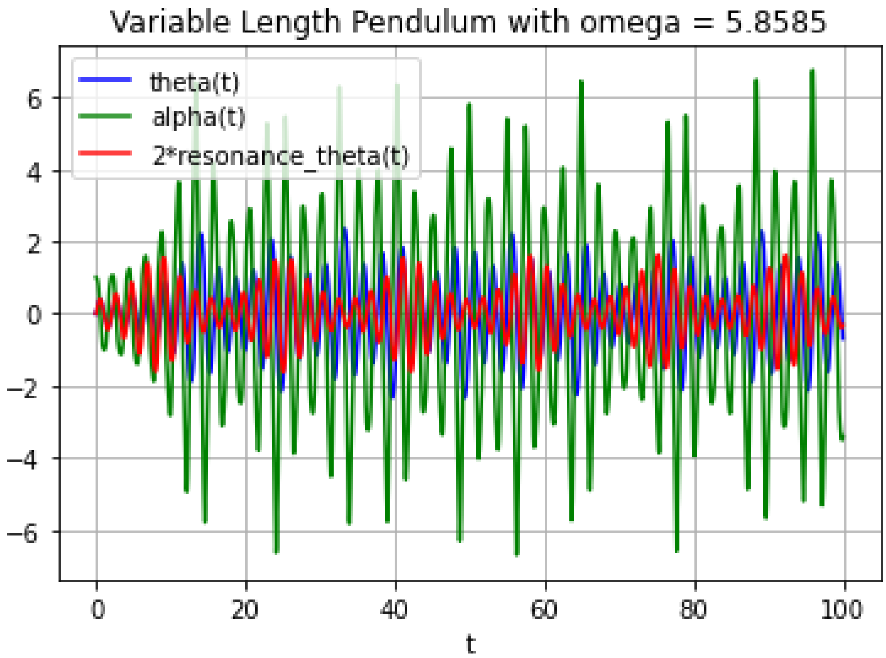

Figure 10: With $A = 0.1$ we calculate the maximum amplitude of θ for a given ω, and plot said ω. We note that the amplitude is indeed larger than the alleged 'max' resonance function, which was reportedly $2\omega_0$.

I wrote a for-loop that calculates the maximum value of the solutions to the ODE with values of ω from 0 to 10. It turns out that the max amplitudes for θ AND $\dot{\theta}$ occur at $\omega = 5.\overline{85}$. I'm not sure why this number is the maximum instead of $2 * \omega_0 \approx 6.26$. It's a little smaller than what's supposed to be the more resonant frequency, and its relation to ω_0 is

$$\frac{5.\overline{85}}{\sqrt{9.8}} \approx 1.87$$

Perhaps Steve Mould used small angle approximations to discover this frequency. If I change the value of l_0 to $l_0 = 2$ then we get a maximum $\omega \approx 1.92\omega_0$, which is a different ratio from all 3 numbers. I'm not sure where these numbers come from, but a good future move would be to try properly deriving the resonant frequency to see which value is 'correct'. There is most likely a window where resonance occurs, and the maximum may vary within that window based on other parameters.

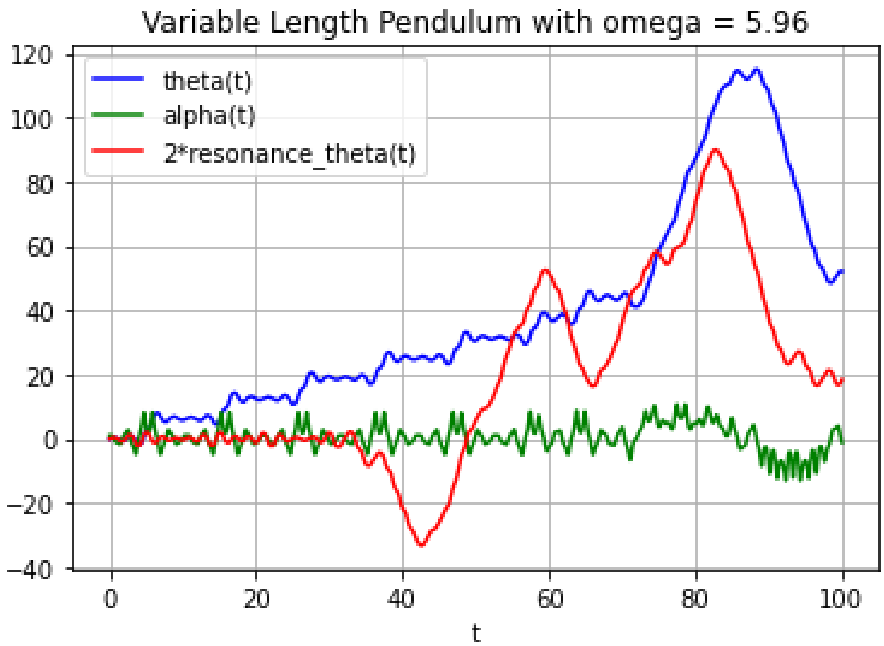

Figure 11: With $A = 0.2$ and $\omega = 5.96$, we don't get a steady solution, and instead θ diverges.

The value of the most resonant ω might also depend on A, the amplitude of the driver oscillations. If we put in $A = 0.2$, θ is maximized at $\omega \approx 5.96$. However, look at Figure 11 to see what happens when I plug in $\omega = 5.96$. I'm not sure why θ diverges. Notice that even $2\omega_0$ produces a divergent θ. If $A = 0.2$, we still have $A < l_0 = 1$, so that shouldn't be the issue. Again, not super sure what's going on here, we probably need a deeper dive into the math. There could be an error in our Lagrangian, or this could be a case of resonance being so powerful that the pendulum eventually spins uncontrollably. The 'step' motion of θ is curious behaviour that deserves to be investigated.

4 The Spring Pendulum

Unfortunately I ran out of time and wasn't able to get to this. However, by looking at the simpler example, I learned a lot about new techniques to find analytic solutions to ODEs, as well as numerical methods to calculate solutions to ODEs in Python. This problem was surprisingly more complex than I thought it would be!

5 Conclusion

We were inspired by a youtube video to analyze a Spring Pendulum. We started with a variable-length pendulum determined by $L(t) = A\cos(\omega t)$. We found the Lagrangian and attempted to solve it analytically but could not progress due to lack of pre-requisite knowledge. We then moved onto numerical methods and discovered an issue with our length function: at $\omega t = \pi/2$, the pendulum's length goes to 0 and the equation blows up.

We restart with a new length function $L(t) = l_0 + A\cos(\omega t)$, where $|A| < l_0$. We find the Lagrangian, move onto numerical methods, and find more promising results. However, the results are inconclusive – we were hoping to find the relation that $\omega = 2\omega_0$ would produce the greatest resonance, but that appears to be untrue – more attention deserves to be paid to this problem.

I owe a lot of my progress on my report to various guides, papers, and demonstrations on similar problems online. I never wrote anything down without understanding it and have cited all websites in the references below.

6 Python Notebook

Note that all code for calculating numerical solutions to the ODEs can be found in a Jupyter Notebook at https://github.com/kxing28/Variable-Length-Pendulum.

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