AN INTRODUCTION TO KNOT THEORY

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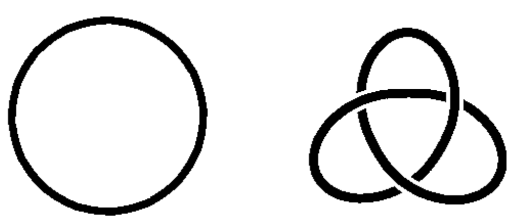
Abstract. We will discuss and delve into the subject of knot theory, a topic that has been researched extensively by mathematicians over the last century, but remains full of mysteries. First, we will cover some fundamental knowledge regarding knots and equivalences, and then move on to the invariants with a primary focus on the crossing number. In the middle, we will discuss the normalized bracket polynomial and Jones polynomial, then use it to prove an interesting theorem attributed to Kauffman. As a forewarning, this paper assumes basic knowledge of algebra and topology. Special thanks to Simon Rubinstein-Salzedo and SUMaC for providing resources and most importantly motivating me to work on this, which I would have never finished otherwise. From the words of a wise student: "Sets is temporary, but math is forever."

1. Introduction

Before going into our major proofs, we need to define some terms for future arguments. We begin with the definition of a knot. Take a rope and tie any sort of knot into it. Then connect both ends of the rope together, and we get a *knot*. More rigorously, we have the following:

Definition 1.1. A *knot* is an injective, continuous map from $\mathbb{S}^1$ to $\mathbb{R}^3$.

For convenience, we will denote the set of all knots to be $\mathbb{K}$. The *trivial knot*, or otherwise known as the *unknot*, is just the identity map $\mathbb{S}^1$. Another famous knot (used for tying shoes and the like) is the *trefoil knot*, which is shown along with the unknot in the figure below. Next, we define a *crossing* to be a point in a knot where one part of the knot goes over another part of the knot. We will denote the number of crossings in a knot k to be $c(k)$. For example, the unknot has $c = 0$ while the trefoil has $c = 3$.



(a) (b)

Figure 1. The unknot and trefoil

Suppose we pick a point in a knot and travel in some direction until we hit the point again. We can mark arrows denoting the *orientation* of the knot. For each crossing in the knot, we

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assign either a $+1$ or -1 depending on the orientation of the crossing. Usually, the crossing is denoted $+1$ if one can use their right hand to curl around the crossing, and -1 otherwise. For a knot k, we denote $\operatorname{sgn}(c)$ to be the sign, either $+1$ or -1, of a crossing c in k.

FIGURE 2. One of two orientations of a knot

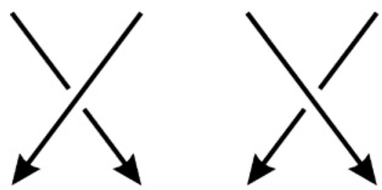


FIGURE 3. $\operatorname{sgn}(c) = +1$ $\operatorname{sgn}(c) = -1$

One way we can create knots is to take the composition of two existing knots. We define the *connected sum* of two knots to be the knot formed when we cut a small piece of each knot and then glue the strands of the knots together so that they do not intersect. We denote the connected sum of $k_1, k_2 \in \mathbb{K}$ as $k_1 \# k_2$.

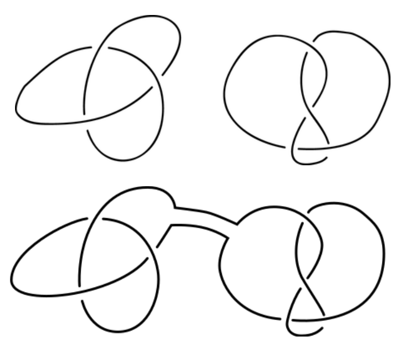


FIGURE 4. The connected sum

Now, if we have a knot $k \in \mathbb{K}$, we can deform k into another knot k' by twisting or moving parts of k. Notice that cutting k or breaking k into pieces does not count as a deformation; we want to preserve k as a simple, closed curve. It turns out that all such deformations can be expressed as a sequence of actions called *Reidemeister moves*, classified as either Type I, II, or III. However, we will not include the proof here due to its complexity.

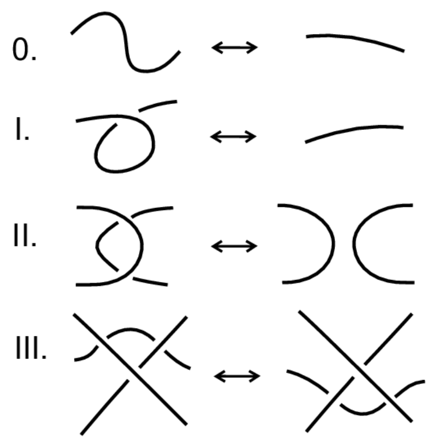


FIGURE 5. The trivial and three Reidemeister moves

The trivial Reidemeister move simply changes the shape of a segment in a knot; no loops or crossings are added or removed. A Type I move consists of forming or removing a loop in the knot, adding or removing one crossing, and a Type II move shifts two parts of the knot to add or remove two crossings. Type III moves work similarly to the trivial move, shifting a part of the knot to the other side of a crossing. We define two knots k, k' to be *equivalent*, denoted as $k \sim k'$, if we can transform one knot to the other using a series of Reidemeister moves. Furthermore, we define the *equivalence class* of k to be $[k] = \{a \in \mathbb{K} \mid a \sim k\}$, or the set of all knots that are equivalent to k.

Theorem 1.2. *There exists an equivalence relation on knots. Specifically, we have the following statements:*

(1) *Reflexivity:* $k \sim k$ *for all* $k \in \mathbb{K}$
(2) *Symmetry: for* $k_1, k_2 \in \mathbb{K}$, $k_2 \sim k_1$ *if* $k_1 \sim k_2$
(3) *Transitivity: for* $k_1, k_2, k_3 \in \mathbb{K}$, $k_1 \sim k_3$ *if* $k_1 \sim k_2$ *and* $k_2 \sim k_3$

Proof. Obviously a knot can be transformed into itself using Reidemeister moves, so we have reflexivity. Now let $k_1, k_2 \in \mathbb{K}$ with $k_1 \sim k_2$. Note that all Reidemeister moves can be reversed, hence $k_2 \sim k_1$ as well and we have symmetry. Now suppose we have $k_1, k_2, k_3 \in \mathbb{K}$ such that $k_1 \sim k_2$ and $k_2 \sim k_3$. Then there exist sequences of Reidemeister moves that transform k_1 into k_2 and k_2 into k_3. By combining these two sequences, we have a sequence of Reidemeister moves that transforms k_1 into k_3, which proves transitivity. ■

Now that we have proved this equivalence relation, we are left wondering about new questions. Given a knot, how can we tell if it is equivalent to another knot? For example, is the trefoil equivalent to the unknot? What about other knots? These are all questions that we will try to answer in the following section, using the concepts of *invariants*.

2. Invariants

Let $k_1 \sim k_2$ be two knots and S be any set. A *knot invariant* is a function $f : \mathbb{K} \to S$ that satisfies $f(k_1) = f(k_2)$ whenever $k_1 \sim k_2$. In other words, if $f(k_1) \neq f(k_2)$ then we must have $k_1 \nsim k_2$. A knot invariant does not tell us which knots are equivalent to each other, but rather which knots are not equivalent. Even still, knot invariants are incredibly useful in discovering new knots.

Previously, we defined a sign of $+1$ or -1 at each crossing. The *writhe* of a knot k, denoted as $\omega(k)$, is the sum of all the signs of its crossings. Is the writhe a knot invariant? That may seem like a difficult question, but all we have to show is that the writhe is invariant under the Reidemeister moves, as applying a series of Reidemeister moves to a knot results in an equivalent knot. It is not hard to see that the Type II and III moves do not change the writhe, though we run into a big problem with the Type I move. This Reidemeister move simply adds or removes a crossing, essentially increasing or decreasing the writhe by one. Therefore, the writhe is not a knot invariant.

Definition 2.1. Let $c(a)$ be the number of crossings in a knot a. The *crossing number* of a knot k is $\min_{j\in[k]} c(j)$, or the minimum number of crossings across all knots equivalent to k.

Theorem 2.2. *The crossing number is a knot invariant.*

Proof. This may seem hard, but it is actually quite easy. Notice that $c(k)$ for some knot k does not actually depend on k, but the equivalence class of k instead. Therefore, every knot equivalent to k also has the same crossing number, and thus it is a knot invariant. ■

This is a good invariant, but it is quite troublesome to find the minimum amount of crossings in the equivalence class of a knot. For the unknot, we know that there cannot be a negative amount of crossings, so the unknot has crossing number zero. But what about the trefoil? How do we know that there isn't an equivalent knot that has fewer crossings?

Proposition 2.3. *There are no nontrivial knots with one or two crossings.*

Proof. We start with knots with one crossing. The crossing can be made in a two ways, one component either going over or under the other component. It is easy to see that the two knots are equivalent to the unknot with the crossing formed by a Type I Reidemeister move.

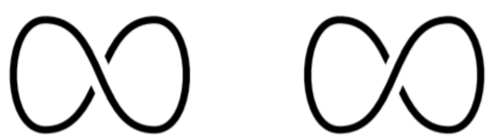


Figure 6. Knots with one crossing

If a knot has two crossings, then there are four possible configurations as shown below:

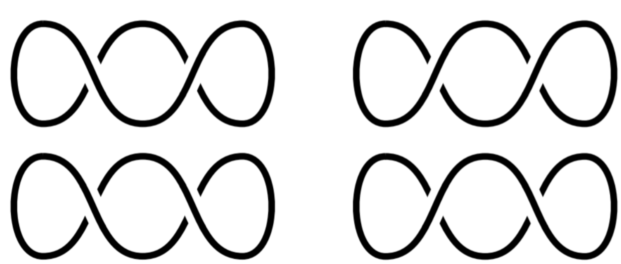


Figure 7. Knots with two crossings

These knots are formed by two Reidemeister I moves on the unknot, and thus are trivial as well. ■

The crossing number is not particularly a good invariant due to the difficulty of proving minimality. Now we turn to polynomial knot invariants, relatively new but powerful ways of distinguishing knots. For the purposes of this paper, we will define a *polynomial* in x to be a *Laurent polynomial*, which can have both positive and negative powers of x. Note that with this definition, $2x + 5$, 0, and $x^{-1} - \sqrt{x}$ are all polynomials.

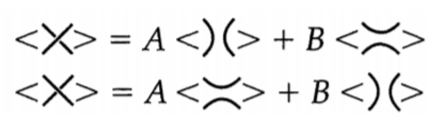
We want to be able to assign a polynomial to any knot such that two polynomials are equal if the corresponding knots are equivalent. We define the *bracket polynomial*, denoted as $< k >$ for a knot k, as one such polynomial. The unknot is the simplest form a knot can take, so we let its bracket polynomial be 1; this is our first rule. Next, we to be able to express crossings in k in some simple way.

Definition 2.4. For a crossing in a knot k, we define the *smoothing* of that crossing to be the process of removing the crossing and gluing together the four leftover strands in a way so they do not go over each other.

FIGURE 8. Vertical smoothing on a crossing

FIGURE 9. Horizontal smoothing on a crossing

We can let a crossing in a knot be the composition of its vertical and horizontal smoothings with parameters A and B as shown.



One consequence of this second rule is that we must define the bracket polynomial for groups of non-intersecting knots, which we will call *links*, in addition to knots. We call the set of all links $\mathbb{L}$.

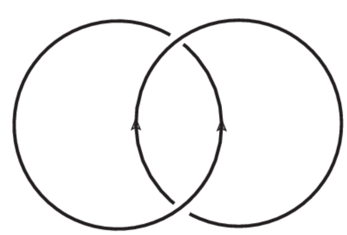
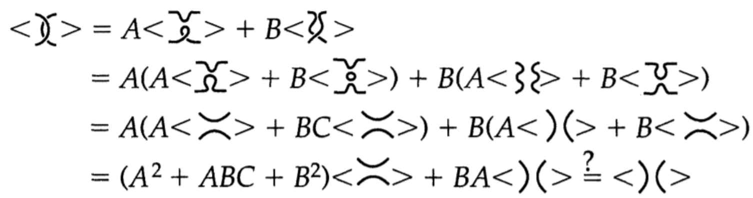


FIGURE 10. The Hopf link, the union of two unknots

Our third and last rule is as follows: for some link L, we define the bracket polynomial corresponding to the union of L and the unknot as $< L \cup \bigcirc >= C < L >$ for some parameter C. These three rules are the axioms of the bracket polynomial, and we will consistently refer to them throughout the rest of the paper.

Theorem 2.5. *The bracket polynomial is invariant under the Type II and III Reidemeister moves.*

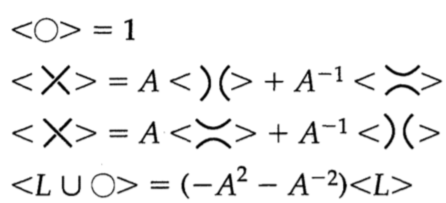
Proof. We begin with the second Reidemeister move, and repeatedly use our predefined three moves:



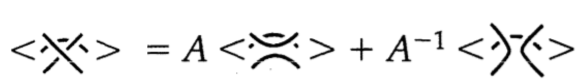
In order to have invariance, we must satisfy the following equations:

$$A^2 + ABC + B^2 = 0$$
$$BA = 1$$

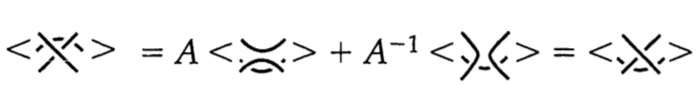
So we immediately get B, C in terms of A; namely, $B = A^{-1}$ and $C = -A^2 - A^{-2}$. Thus our three rules become



which looks much nicer. We can also check that the third Reidemeister move holds under our rules:

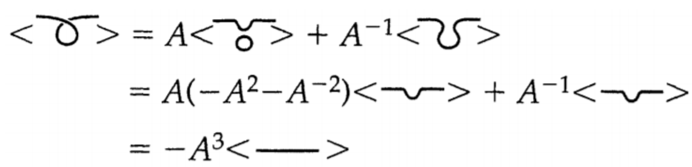


Because we know the bracket polynomial is an invariant under the Type II move, we can shift the top "parentheses" down using second Reidemeister moves in both expressions inside the RHS, resulting in



as desired. ■

Unfortunately, similar to the writhe, we run into a problem when examining the first Reidemeister move.



As shown, we do not get the bracket polynomial of a line segment, but rather it multiplied with the parameter $-A^3$. Similarly, if the sign of the crossing was reversed, we would get the parameter $-A^{-3}$ multiplied by the bracket polynomial of a line segment.

So how do we fix this? Well, recall that the writhe also encounters a problem with the first Reidemeister move, so perhaps we could combine them in some way...

Definition 2.6. We define the *normalized bracket polynomial*, denoted as $X(L)$ for a link L, to be

$$X(L) = (-A)^{-3\omega(L)} < L > .$$

Theorem 2.7. *The normalized bracket polynomial is a link invariant.*

Proof. Because the writhe and bracket polynomial are invariant under Type II and III moves, the normalized bracket polynomial is also invariant, so we only need invariance under Type I moves. For a link L, let L' be the link formed by performing the first Reidemeister move on L. Then we have $< L' >= (-A^{\pm 3}) < L >$ from before and also $\omega(L') = \omega(L) \pm 1$, where the $\pm$ depends on the way the crossing was added/removed. Putting this together,

$$\begin{aligned} X(L') &= (-A)^{-3\omega(L')} < L' > \\ &= (-A)^{-3\omega(L) \mp 3} < L' > \\ &= (-A)^{-3\omega(L) \mp 3}(-A)^{\pm 3} < L > \\ &= (-A)^{-3\omega(L)} < L > \\ &= X(L) \end{aligned}$$

as desired. ■

Definition 2.8. The *Jones polynomial*, denoted as $V(L)$ for a link L, is defined as the normalized bracket polynomial with the substitution $A = t^{-1/4}$.

One might wonder why we have defined this polynomial as a replicate of the normalized bracket polynomial with a substitution, but it turns out that the Jones polynomial satisfies an interesting property.

Theorem 2.9. *Let L_+, L_-, L_0 be three links that are identical (not just equivalent but have the same shape) except for within a region, consisting of at most one crossing, where L_+ has one crossing with sign $+1$, L_- has a crossing with sign -1, and L_0 has no crossing. Then the Jones polynomial satisfies a skein relation, namely*

$$t^{-1}V(L_+) - tV(L_-) = (t^{1/2} - t^{-1/2})V(L_0).$$

Proof. From the definition of the Jones polynomial, we have

$$V(L_+) = (-t^{-1/4})^{-3\omega(L_+)} < L_+ >$$
$$V(L_-) = (-t^{-1/4})^{-3\omega(L_-)} < L_- >$$
$$V(L_0) = (-t^{-1/4})^{-3\omega(L_0)} < L_0 > .$$

However, because L_+, L_-, L_0 are identical except for one small region, we can just let

$$\begin{aligned} V(L_+) &= (-t^{-1/4})^{-3\omega(L_+)} < \times > \\ V(L_-) &= (-t^{-1/4})^{-3\omega(L_-)} < \times > \\ V(L_0) &= (-t^{-1/4})^{-3\omega(L_0)} <)(> . \end{aligned}$$

Note that $\omega(L_+) - 1 = \omega(L_+) + 1 = \omega(L_0)$ because the writhe of L_+, L_- is just L_0 plus the orientation of the added crossing. Now let $A = t^{-1/4}$; we would like to show that $A^4V(L_+) - A^{-4}V(L_-) = (A^{-2} - A^2)V(L_0)$. Substituting in our earlier expressions,

$$A^4V(L_+) - A^{-4}V(L_-) = (A^{-2} - A^2)V(L_0)$$
$$A^4(-A)^{-3\omega(L_0)-3} < \times > -A^{-4}(-A)^{-3\omega(L_0)+3} < \times >= (A^{-2} - A^2)(-A)^{-3\omega(L_0)} <)(> .$$

Cancelling out the exponent of $-3\omega(L_0)$ in each term, we are left with

$$A^4(-A)^{-3} < \times > -A^{-4}(-A)^3 < \times >= (A^{-2} - A^2) <)(>$$
$$A < \times > -A^{-1} < \times >= (A^2 - A^{-2}) <)(> .$$

which is trivial due to how crossings were defined for the bracket polynomial. ■

3. Reduced Alternating Knots

Now that we have proved the existence of the Jones polynomial, we once again turn to knots. We investigate the properties of a special type of knots, ones that are *reduced* and *alternating*.

Definition 3.1. A knot k is *reduced* if it the smoothing of any crossing in k results in a link with one component (a knot).

Definition 3.2. A knot k is called *alternating* if its crossings have alternating signs when one travels along the knot in any direction.

Remark 3.3. The writhe of an alternating knot will either be 1 or -1.

Note that for every crossing in a knot k there is a strand of k that goes over another strand, dividing neighboring space into four regions. Now we rotate the over-strand counterclockwise until it hits the under-strand so that it passes through two opposite regions, which we label as A (not to be confused with the A in the bracket polynomial). To finish, we label the other two regions B.

Figure 11. Labelling the crossings A and B

When we smooth a crossing, either the A's or the B's come together, removing one region. If the A's come together, we call it an A-split, and otherwise a B-split.

Definition 3.4. A *state* S of a knot k is a way of smoothing all crossings in k either in an A-split or a B-split. We denote k_S as the link obtained from applying S on k.

Definition 3.5. Let $a(S)$ and $b(S)$ for a state S be the number of A-splits and B-splits respectively in S. Furthermore, we denote the number of disjoint loops in k_S to be $||S||$.

We have the following proposition:

Proposition 3.6. $< k >= \sum_{\sigma} A^{a(\sigma)-b(\sigma)} C^{||\sigma||-1}$ *where* $C = -A^2 - A^{-2}$.

Proof. Every time we split a crossing in k, we multiply the bracket polynomial of the resulting link by either A if it's an A-split and A^{-1} if it's a B-split. Therefore, for a state σ, $< k_\sigma >$ is multiplied by a factor of $A^{a(\sigma)-b(\sigma)}$. Because k_S consists of the union of $||\sigma||$ unknots, we have $< k_\sigma >= C^{||\sigma||-1}$. Thus, the total contribution to the bracket polynomial of k is $A^{a(\sigma)-b(\sigma)} C^{||\sigma||-1}$, and summing over σ gives the desired result. ∎

Now, we introduce the concept of *checkerboard shading*, a way of coloring a knot k in the colors black and white. We color the regions according to their labels; white for regions labeled A and black for regions labeled B.

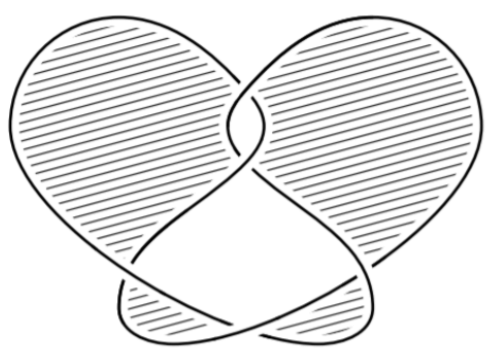


Figure 12. Example of checkerboard shading

Definition 3.7. We define the *span* of a knot, denoted as $Span(< k >)$, to be the difference between the highest and lowest power of A in its bracket polynomial.

Proposition 3.8. *$Span(< k >)$ is a knot invariant.*

Proof. Obviously $Span(< k >)$ is invariant under the Type II and III Reidemeister moves as $< k >$ is also invariant. Note that applying the first Reidemeister move on k multiplies the bracket polynomial by either A^3 or A^{-3}, which does not change $Span(< k >)$. ∎

With all this background knowledge, we are ready to prove the following theorem:

Theorem 3.9. *If a reduced, alternating knot k has n crossings, then $Span(< k >) = 4n$.*

Proof. We find the highest and lowest power of A in $< k >$ individually. Note due to Proposition 3.6 that the highest power is obtained when $a(S)-b(S)+2(||S||-1)$ is maximized. But note that we can just let every smoothing be an A-split, which gives $a(S) = n$ and $b(S) = 0$. As every smoothing connects two A-regions and separates the B-regions, the number of disjoint loops of k_S is just the number of black regions originally in k, which we denote as $||S|| = B$.

Why can't we go higher than $n + 2B - 2$? Suppose we have states S_1, S_2 such that S_2 has exactly one more B-split than S_1. Then the highest power of A for S_1 is $a(S_1) - b(S_1) + 2(||S_1|| - 1)$ while the highest power for S_2 is $a(S_1) - 1 - (b(S_1) + 1) + 2(||S_2|| - 1)$. Note that S_1 and S_2 differ by only one split, meaning that $||S_2|| = ||S_1|| \pm 1$. However, we can clearly see that

$$\begin{aligned} a(S_1) - 1 - b(S_1) + 1) + 2(||S_2|| - 1) &= a(S_1) - b(S_1) + 2||S_1|| \pm 2 - 4 \\ &\leq a(S_1) - b(S_1) + 2(||S_1|| - 1) \end{aligned}$$

as desired.

Similarly, the lowest power of A in $< k >$ will be when $a(S)-b(S)-2(||S||-1)$ is minimized, which happens when every smoothing is a B-split, resulting in a power of $0 - n - 2(W - 1)$ where W is the number of white regions originally in k. Therefore $Span(< k >) = n+2(B-1) - (-n - 2(W - 1)) = 2n + 2(B + W - 2) = 4n$ as the total number regions in k is just $n + 2$ (which can be proved readily using induction). ∎

Corollary 3.10. *Two equivalent reduced alternating knots have the same number of crossings.*

Proof. Let a reduced alternating knot k_1 have n crossings. By Theorem 3.9, we have $Span(< k_1 >) = 4n$. But notice that any reduced alternating knot k_2 equivalent to k_1 also has $Span(< k_2 >) = 4n$ due to invariance, implying that k_2 has n crossings as well. ∎

This is all we can do right now. There is a proof by Kauffman who showed that a reduced alternating knot has crossing number equal to the number of crossings it contains using the above theorem, however it is beyond the extent of this paper. Next, we will discuss *chirality* in knots.

4. Chirality

Definition 4.1. The *mirror image* of a knot k is defined to be the knot obtained from reversing all the signs of the crossings in k.

Note that from how we defined the bracket polynomial, reversing the orientation of all the crossings of a knot k just replaces every A in the bracket polynomial of k with A^{-1} and vice versa. Also, the writhe of the mirror image of k is the negative of the writhe of k.

Definition 4.2. A knot k is *chiral* if it is not equivalent to its mirror image, and *amphichiral* otherwise.

Proposition 4.3. *The trefoil is a chiral knot.*

Proof. We start with finding the bracket polynomial. Using the second move on all three crossings, ∎

Theorem 4.4. *Let an alternating knot k have B black and W white regions in its checkerboard shading. If k is amphichiral, then $B - W = 3\omega(k)$.*

As previously mentioned, this theorem is beyond the scope of the paper and will not be proved here.

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