week12_lec1

Main Topics

- An Experiment with Photons
- Qubits
- No cloning theorem
- Multiple Qubits
- Quantam Entanglement
- Quantum Gates and Circuits
- Quantum Teleportation

An Experiment with Photons

The experiment is as follows — If we have a light source and a screen, we try to observe
what happens in three scenarios:

1. If we have a horizontal polarizer A between the source and the screen, it is observed
that about 50% of the photons pass through to the screen.
2. If we have a horizontal polarizer A followed by a vertical polarizer C between the source
and the screen, it is observed that 0% of the photons pass through to the screen.
3. If we have a horizontal polarizer A followed by a polarizer B at 45° from the plane and
then, the vertical polarizer C, it is observed that about 12.5% of the photons pass through to the screen.

Qubits

If there are two possible outcomes, the superposition of the two is what will be the observed value when one measures. This can be extended to more than two outcome possibilities.

A quantum bit, or qubit, is a unit vector in a two dimensional complex vector space for which a
particular basis has been fixed and is denoted by: { |0> , |1> }

Mathematically, qubits can be in a superposition of $|0\rangle$ and $|1\rangle$ like $a|0\rangle + b|1\rangle$ where a and
b are complex numbers such that $|a|2 + |b|2 = 1$. We can understand this as follows — The
probability that the measured value is $|0\rangle$ is $|a|2$ after which the state collapses to $|0\rangle$ and
the probability that the measured value is $|1\rangle$ is $|b|2$ after which the state collapses to $|1\rangle$.

No cloning theorem

says that there is no unitary transformation that clones, ie, there is no U such that U(|a0>) = |aa>
Proof:

Proof: Assume U is a unary transformation that clones a quantum state as follows —

$U(|a0\rangle) = |aa\rangle$ for all quantum states $|a\rangle$.
Consider

$$|c\rangle = (1/2)(|a\rangle + |b\rangle).$$

Then, $U(|c0\rangle) = 1/2(U(|a0\rangle) + U(|b0\rangle)) = 1/2(|aa\rangle + |bb\rangle).$

This means that we get the states $|aa\rangle$ and $|bb\rangle$ each with 50% probability in the composed

state.
However, if U is a cloning transformation, then

$$U(|c0\rangle) = |cc\rangle = 12(|aa\rangle + |ab\rangle + |ba\rangle + |bb\rangle)$$

because there are four possible states each with equal (25%) probability.
This is a contradiction and thus, the theorem is proved.

Multiple Qubits

- Individual state spaces of *n* particles combine classically through the cartesian product.
- **Quantum states**, however, combine through the tensor product
  for example, the basis set for a three qubit system is

$$\{|000>, |001>, |010>, |011>, |100>, |101>, |110>, |111>$$

- In general, an n qubit system has 2n basis vectors.
  This means that there is an exponential growth of the state space with the number of quantum particles.

Quantam Entanglement

- Suppose we have a 2 qubit system synthesised in such a way that with pobability 1/2 we get |00> and |11> each, and the others are not allowed.
- We can say that there are no *a*1 , *a*2 , *b*1 , *b*2 such that

$$(a1 \mid 0\rangle + b1 \mid 1\rangle) \otimes (a2 \mid 0\rangle + b2 \mid 1\rangle) = \mid 00\rangle + \mid 11\rangle$$

  because the expression would equal

  $a1a2 \mid 00\rangle + a1b2 \mid 01\rangle + b1a2 \mid 10\rangle + b1b2 \mid 11\rangle$ and since $a_1 b_2 = 0,$

  it implies that either *a*1*a*2 = 0 or *b*1*b*2 = 0 and this proves the initial statement.

- Such states that cannot be decomposed are called **entangled states**.

Quantum Gates and Circuits

- Quantum logic gates are nothing but unitary transformations.
- An n q-bit system will be represented by a 2nx2n matrix.

$$
\begin{aligned}
I &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
X &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
Y &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
Z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
H &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
\end{aligned}
$$

The unitary transformations performed by each of these gates are as follows:

$$
\begin{aligned}
I &: |0\rangle \to |0\rangle, |1\rangle \to |1\rangle \\
X &: |0\rangle \to |1\rangle, |1\rangle \to |0\rangle \\
Y &: |0\rangle \to -|1\rangle, |1\rangle \to |0\rangle \\
Z &: |0\rangle \to |0\rangle, |1\rangle \to -|1\rangle \\
H &: |0\rangle \to \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), |1\rangle \to \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)
\end{aligned}
$$

$$
C_{not} : \begin{array}{ccc} |00\rangle & \to & |00\rangle \\ |01\rangle & \to & |01\rangle \\ |10\rangle & \to & |11\rangle \\ |11\rangle & \to & |10\rangle \end{array} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
$$

Quantum Teleportation

The objective is to transmit the quantum state of a particle using classical bits and reconstruct the exact quantum state at the receiver.

We have a quantum system at Alice's end. We need to convert it to information and send this to Bob. Bob should be able to reconstruct the quantum system at his end.

This doesnt violate the no cloning theorem because by the time the state is assembled by bob, it would have changed in Alice's end.

- Initial qubit as Alice's end : φ = a|0> + b|1> which she wants to send Bob.
- Assume alice and bob each possess one qubit of the entangled pair (bell)

  $\varphi 0 = (1/\sqrt{2})(|00> + |11>)$
- The initial state here is as follows:

$$
\begin{aligned}
\varphi \otimes \psi 0 = 1(a \mid 0\rangle \otimes (\mid 00\rangle + \mid 11\rangle) + b \mid 1\rangle \otimes (\mid 00\rangle + \mid 11\rangle)) \\
= 1(a \mid 000\rangle + a \mid 011\rangle + b \mid 100\rangle + b \mid 111\rangle)
\end{aligned}
$$

- In this, Alice controls the first 2 bits and Bob controls the third.
- Alice applies $Cnot \otimes IandH \otimes I \otimes I$to the inital state

$$
\begin{aligned}
&(H \otimes I \otimes I)(C_{not} \otimes I)(\phi \otimes \psi_0) \\
&= (H \otimes I \otimes I)(C_{not} \otimes I)\frac{1}{\sqrt{2}}\big(a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle\big) \\
&= (H \otimes I \otimes I)\frac{1}{\sqrt{2}}\big(a|000\rangle + a|011\rangle + b|110\rangle + b|101\rangle\big) \\
&= \frac{1}{2}\big(a(|000\rangle + |011\rangle + |100\rangle + |111\rangle) + b(|010\rangle + |001\rangle - |110\rangle - |101\rangle)\big) \\
&= \frac{1}{2}\big(|00\rangle(a|0\rangle + b|1\rangle) + |01\rangle(a|1\rangle + b|0\rangle) + |10\rangle(a|0\rangle - b|1\rangle) + |11\rangle(a|1\rangle - b|0\rangle)\big)
\end{aligned}
$$

This means that if the first 2 bits are measured as 00, the 3rd will collapse as a|0> + b|1> and so on Alice measures these first 2 qubits and sends the result of her measurement as classical bits to Bob:

$$
\begin{aligned}
00 &\rightarrow a \mid 0\rangle + b \mid 1\rangle \\
01 &\rightarrow a \mid 1\rangle + b \mid 0\rangle \\
10 &\rightarrow a \mid 0\rangle - b \mid 1\rangle \\
11 &\rightarrow a \mid 1\rangle - b \mid 0\rangle
\end{aligned}
$$

Now, he can reconstruct the original state of Alice's qubit by applying the quantum gates *I*, *X*, *Y* , *Z* respectively based on the bits received.