Chapter 8.6 Question 25
Evaluate the following sum

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

Answer: use partial fractions

$$n^2 - 1 = (n-1)(n+1)$$

$$\frac{1}{n^2-1} = \frac{A}{n-1} + \frac{B}{n+1}$$

$$= \frac{A(n+1) + B(n-1)}{(n-1)(n+1)}$$

$$1 = An + A + Bn - B$$

Equate coefficients

$$0 = A - B$$

$$1 = A + B$$

add equations

$$1 = 2A$$

$$A = \frac{1}{2}$$

$$B = -\frac{1}{2}$$

$$\frac{1}{n^2-1} = \frac{1}{2(n-1)} - \frac{1}{2(n+1)}$$

$$S_N = \sum_{n=2}^{N} \frac{1}{n^2 - 1}$$

$$= \left(\frac{1}{2} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{8}\right) + \left(\frac{1}{6} - \frac{1}{10}\right) + \left(\frac{1}{8} - \frac{1}{12}\right) + \ldots +$$

$$+ \left(\frac{1}{2\left(N-3\right)} - \frac{1}{2\left(N-1\right)}\right) + \left(\frac{1}{2\left(N-2\right)} - \frac{1}{2N}\right)$$

$$+ \left(\frac{1}{2\left(N-1\right)} - \frac{1}{2\left(N+1\right)}\right)$$

$$= \frac{1}{2} + \frac{1}{4} - \frac{1}{2N} - \frac{1}{2\left(N+1\right)}$$

$$\lim_{N \to \infty} S_N = \frac{3}{4}$$