NEGATIVE LONGITUDINAL CORRELATION FOR ISOTROPIC FLOWS

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Abstract. Examples of physically reasonable homogeneous, isotropic, three-dimensional divergence-free vector fields with longitudinal correlation negative on some interval are presented. The negativity of the longitudinal correlation persists in the Galerkin approximation of the hydrodynamic equations at least for some time. Both the outline of the mathematical arguments and the numerical implementation are included.

1. Introduction

The longitudinal correlation function of homogeneous and isotropic flows is the correlation of the velocity components parallel to the separation vector as a function of the separation distance. For incompressible flows this function determines all components of the correlation, see Batchelor (1953), p. 46. Kolmogorov (1941) examines the longitudinal correlation and its derivatives at zero separation before formulating the famous hypotheses on the statistics of locally homogeneous and isotropic turbulence.

It is often argued that this longitudinal correlation function is positive for all separation lengths in physically significant cases, e.g. Batchelor (1953), p. 48, Davidson (2004), p. 328, Gustafsson & George (2008), § 3.1. This note points out that in some physically significant cases the longitudinal correlation function can be negative on whole intervals of separation, and that this persists in the Galerkin approximation of the equations of fluid motion.

In particular, examples are given of Gaussian, homogeneous and isotropic fields with the property that velocity fields in any proximity of smooth fields with compact support occur with non-zero probability. Their longitudinal correlation function is negative on intervals. These are the contents of § 3. With such fields as initial conditions, the Galerkin approximation of homogeneous and isotropic flows continues to have negative longitudinal correlation, at least on some time interval, see § 4. A numerical implementation of a periodic analogue and the numerical simulation of its evolution are presented in § 5.

2. Random fields

Random flows are described here in terms of a space $\mathcal{H}$ of 3-dimensional solenoidal vector fields equipped with a probability law (probability measure) μ. (This description is, of course, equivalent to the more common description of $\mathbf{u}$ as $\mathbf{u}(x,\omega)$ for ω some event in $(\Omega,\mathbb{P})$, by taking $(\Omega,\mathbb{P}) = (\mathcal{H},\mu)$ and $\mathbf{u}(x,\omega) = \omega(x)$.)

One starts with the space H of vector fields $\mathbf{u}(\mathbf{x}) = (u_1(\mathbf{x}), u_2(\mathbf{x}), u_3(\mathbf{x}))$, $\mathbf{x} = (x_1, x_2, x_3)$ on 3-space, with the property that

$$\int_{\mathbb{R}^3} \left(1 + |\mathbf{x}|^2\right)^r |\mathbf{u}(\mathbf{x})|^2 \, d\mathbf{x}, \quad r < -\tfrac{3}{2}, \tag{1}$$

is finite. Then distance and scalar product on H are given by

$$\|u - v\| = \left\{ \int_{\mathbb{R}^3} \left(1 + |\mathbf{x}|^2\right)^r |\mathbf{u}(\mathbf{x}) - \mathbf{v}(\mathbf{x})|^2 \, d\mathbf{x} \right\}^{1/2}, \tag{2}$$

$$< \mathbf{u}, \mathbf{v} > = \left(1 + |\mathbf{x}|^2\right)^r \mathbf{u}(\mathbf{x})\mathbf{v}(\mathbf{x}) \, d\mathbf{x}, \tag{3}$$

respectively.

Here r can be any real number strictly less than $-3/2$ so that the "weight" $\left(1 + |\mathbf{x}|^2\right)^r$ has finite integral over $\mathbb{R}^3$. Observe that the restriction on r implies that constant and periodic vector fields are in H.

$\mathcal{H}$ will denote the subspace of solenoidal vector fields in H.

For $\mathbf{u}$ in H let $T_{\mathbf{h}}$ be the translation operation defined by

$$T_{\mathbf{h}}\mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{x} + \mathbf{h}). \tag{4}$$

A measure μ defined on H is **homogeneous** if it is translation invariant:

$$\int_H f(T_{\mathbf{h}}\mathbf{u}) \, \mu(d\mathbf{u}) = \int_H f(\mathbf{u}) \, \mu(d\mathbf{u}), \tag{5}$$

for any integrable f on H and for all $\mathbf{h}$ in $\mathbb{R}^3$. Or, using the over-line from now on to denote averaging with respect to the given measure μ,

$$\overline{f(T_{\mathbf{h}}\mathbf{u})} = \overline{f(\mathbf{u})}. \tag{6}$$

The significance of the weight in the definition of H is to ensure that non-trivial probability measures invariant under translations exist, see Vishik & Fursikov (1988), p. 208.

In addition, for W an orthogonal matrix, define its action on vector fields by

$$(R_W\mathbf{u})(\mathbf{x}) = W\mathbf{u}(W^{-1}\mathbf{x}). \tag{7}$$

and call a measure μ on H **isotropic** if it is invariant under this action, i.e. for all W and any (integrable) f on H

$$\overline{f(R_W\mathbf{u})} = \overline{f(\mathbf{u})}. \tag{8}$$

2.0.1. *Averages.* All averages here are ensemble averages, i.e. they are taken with respect to probability measures on spaces of vector fields. Nevertheless, the measures constructed here have decaying (in space) correlations. For homogeneous measures, this decay, regardless of rate, guarantees that ensemble averages equal (for μ-almost all fields) the space averages used in measurements or numerical simulations, see Androulakis & Dostoglou (2004), p. 8, Theorem 4.6, Tempelman (1992), p. 66, Corollary 2.7.

3. Gaussian measures

3.1. **The probability measures.** This section constructs measures of negative longitudinal correlation. The construction starts with choosing a reasonable candidate for the correlation $\overline{u_i(\mathbf{x})u_j(\mathbf{x}+\mathbf{h})}$ for the i and j components of the random $\mathbf{u}$, cf. Yaglom (1957), § 4.

For each $\mathbf{h}$ in 3-space, and setting $h = |\mathbf{h}|$, define

$$B_{ij}(\mathbf{h}) = \int_0^\infty \left\{ \left[\frac{3\sin(kh)}{(kh)^3} - \frac{3\cos(kh)}{(kh)^2} - \frac{\sin(kh)}{kh} \right] \frac{h_i h_j}{h^2} \right. \tag{9}$$

$$\left. + \left[\frac{\cos(kh)}{(kh)^2} - \frac{\sin(kh)}{(kh)^3} + \frac{\sin(kh)}{kh} \right] \delta_{ij} \right\} \Phi'(k)\ dk, \tag{10}$$

$i, j = 1, 2, 3$, for any $\Phi : [0, +\infty) \to [0, +\infty)$ non decreasing, differentiable on $(0, \infty)$, with $\Phi(0) = 0$, and of finite variation:

$$\int_0^\infty \Phi'(k) dk < \infty. \tag{11}$$

(For isotropic fields that will be constructed from B_{ij}, it can be seen that Φ' is the energy spectrum function E.) Then formulas (4.20) and (4.25) in Yaglom (1957) yield

$$B_{ij}(\mathbf{h}) = \int_{\mathbb{R}^3} e^{i\mathbf{k}\cdot\mathbf{h}} \left(\delta_{ij} - \frac{k_i k_j}{|\mathbf{k}|^2} \right) \Phi'(|\mathbf{k}|)\ d\mathbf{k}. \tag{12}$$

Define now $\mathcal{B} : H \to H$ by

$$(\mathcal{B}\mathbf{v}(\mathbf{y}))_i = \sum_{j=1}^{3} \int_{\mathbb{R}^3} B_{ij}(\mathbf{x}-\mathbf{y}) v_j(\mathbf{x})(1+|\mathbf{x}|^2)^r\ d\mathbf{x}. \tag{13}$$

As an operator on H, it can be shown that $\mathcal{B}$ is continuous, symmetric, non-negative and of trace class, see Dostoglou & Gastler (2009). These are exactly the properties guaranteeing the existence of a Gaussian measure μ on H with zero mean and correlation $\mathcal{B}$, see Gikhman & Skorohod (1980), p. 350. That is, there exists a Gaussian probability measure μ on H such that

$$< \mathcal{B}\mathbf{v}_1, \mathbf{v}_2 > = \overline{< \mathbf{u}, \mathbf{v}_1 >< \mathbf{u}, \mathbf{v}_2 >}. \tag{14}$$

In particular, the characteristic functional $\chi_\mu(\mathbf{v}) = \overline{e^{i\mathbf{u}\cdot\mathbf{v}}}$ of μ is given by

$$\chi_\mu(\mathbf{v}) = e^{-\frac{1}{2}\mathcal{B}\mathbf{v}\cdot\mathbf{v}}, \tag{15}$$

for $\mathbf{v}$ any test vector field and for $\mathbf{u}\cdot\mathbf{v} = \int_{\mathbb{R}^3} \mathbf{u}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})\ dx$. It follows by straight forward calculations that χ_μ is invariant under shifts and rotations:

$$\chi_\mu(T_\mathbf{h}\mathbf{v}) = \chi_\mu(\mathbf{v}), \quad \chi_\mu(R_\omega \mathbf{v}) = \chi_\mu(\mathbf{v}), \tag{16}$$

see Dostoglou & Gastler (2009). Repeating standard arguments that establish how the characteristic functional determines uniquely the measure, the Gaussian measure μ can be shown to be homogeneous and isotropic, Dostoglou & Gastler (2009).

Note that up to this point the measure is constructed on H. However B_{ij}, by construction, satisfies

$$\sum_{j=1}^{3} \partial_j B_{ij} = 0 \tag{17}$$

for all i. This implies that $\mathcal{B}(\nabla\phi) = 0$ for any test function ϕ, and μ is thus supported on vector fields satisfying $\mathbf{u} \cdot \nabla\phi = 0$, i.e. on divergence-free vector fields. Therefore, the measure μ, although initially defined on H, assigns zero probability on anything outside $\mathcal{H}$ and can therefore be considered as a measure on $\mathcal{H}$.

3.2. **The longitudinal correlation.** For $\mathbf{e}_L$ unit vector, h a positive number, and $\mathbf{u}_L$ the projection of $\mathbf{u}$ on $\mathbf{e}_L$, the longitudinal correlation function

$$B_{LL}(h) = \overline{u_L(x + h\mathbf{e}_L)u_L(x)} \tag{18}$$

corresponding to (9) is

$$B_{LL}(h) = \int_0^\infty \left[\frac{2\sin(hk)}{(hk)^3} - \frac{2\cos(hk)}{(hk)^2}\right] \Phi'(k)dk, \tag{19}$$

see Yaglom (1957), p. 305. Choosing

$$\Phi(k) = \begin{cases} 0 & \text{if } 0 \le k < 3\pi/2 \\ A(1+\cos(2k)) & \text{if } 3\pi/2 \le k \le 2\pi \\ 2A & \text{if } 2\pi < k, \end{cases} \tag{20}$$

where $A > 0$ is an arbitrary constant, all conditions for (9) are satisfied and (19) becomes

$$B_{ll}(h) = -4A \int_{3\pi/2}^{2\pi} \left[\frac{\sin(hk)}{(hk)^3} - \frac{\cos(hk)}{(hk)^2}\right] \sin(2k)\ dk. \tag{21}$$

In addition, as

$$B_{LL}(1) = -4A \int_{3\pi/2}^{2\pi} \left[\frac{\sin k}{k^3} - \frac{\cos k}{k^2}\right] \sin(2k)\ dk < 0, \tag{22}$$

and B_{LL} is continuous, B_{LL} is negative on an interval. (In fact, a slight modification exhibits a whole family $\Phi_\epsilon, \epsilon \le \epsilon_0$, such that $\Phi_\epsilon > 0$ for all k and the longitudinal correlations B_{LL}^ϵ are arbitrarily negative on intervals.)

The measure constructed in the previous section corresponding to this choice of Φ is therefore Gaussian, homogeneous, isotropic, sees only divergence-free fields, and has negative longitudinal correlation on intervals.

3.3. **The support of the measure.** An important feature of the measure μ is that its support contains reasonable solenoidal vector fields. In particular, all smooth, divergence free vector fields vanishing outside a compact subset of $\mathbb{R}^3$ are included in the support of μ. In other words, for each such vector field $\mathbf{u}$ any ball of vector fields in $\mathcal{H}$ around $\mathbf{u}$ has non-zero probability with respect to μ.

To see this, recall from Itô (1970) that the support of any Gaussian measure is the subspace orthogonal to the kernel of the correlation $\mathcal{B}$. If $\mathbf{v}$ is in the kernel then for the current setting $< \mathcal{B}\mathbf{v}, \mathbf{v} >= 0$, or, by (12),

$$\int_{\mathbb{R}^3} \left(|\widetilde{\phi}|^2(\mathbf{k}) - \frac{|\mathbf{k}\cdot\widetilde{\phi}(\mathbf{k})|^2}{|\mathbf{k}|^2} \right) \Phi'(|\mathbf{k}|)d\mathbf{k} = 0, \tag{23}$$

where $\phi(\mathbf{x}) = (1+|\mathbf{x}|^2)^r\mathbf{v}(\mathbf{x})$ and $\tilde{\phi}$ is its Fourier transform. For $\Phi' > 0$ everywhere, it follows that the kernel consists of $\mathbf{v}$'s such that $\widetilde{\phi}(\mathbf{k}) = M(\mathbf{k})\mathbf{k}$ for all $\mathbf{k}$, and M can be shown square integrable. One then obtains an m such that $\nabla m = \phi$. In this manner, elements in $\mathcal{H}$ perpendicular to the kernel of $\mathcal{B}$ are perpendicular to gradients. In particular, all solenoidal square integrable vector fields are in the support of μ.

4. Galerkin approximation and time evolution

Dostoglou, Fursikov & Kahl (2006) shows how to obtain homogeneous and isotropic statistical solutions P of the Navier-Stokes equations for any finite time interval as the (weak) limit of homogeneous and isotropic statistical solutions P_l of a Galerkin approximation of the Navier-Stokes equations. This isotropic Galerkin approximation takes place on the space the finite-dimensional space of divergence-free, vector valued trigonometric polynomials of degree l and period $2l$

$$\mathcal{M}_l = \left\{ \sum_{|k|\leq l} a_k e^{ik\cdot x} : a_k \cdot k = 0,\ a_k = \overline{a}_{-k},\ k \in \frac{\pi}{l}\mathbb{Z}^3 \right\}, \tag{24}$$

augmented with all its rotations to a space $\widehat{\mathcal{M}_l}$. The Navier-Stokes equations are at each step approximated by

$$\partial_t u - \nu\Delta u + \widehat{P}_l[(u,\nabla)u] = 0, \tag{25}$$

where $\widehat{P}_l$ projects onto $\widehat{\mathcal{M}_l}$.

The correlation of the l-Galerkin P_l at time t

$$B^l_{ij}(t,\mathbf{h}) = \mathbb{E}_l\ [u_i(t,\mathbf{x})u_j(t,\mathbf{x}+\mathbf{h})] \tag{26}$$

is continuous in t, see Dostoglou & Gastler (2009). (This relies on the fact that the support of the measure P_l consists of continuous in t solutions, and does not use the evolution equation for the correlations that suffer from the well-known closure problem. This argument breaks down for the correlation of the statistical solution itself as its support does not consist of continuous in t functions. In addition, there is no rigorous proof at this point that the Galerkin correlations at time $t > 0$ converge to the correlation of the statistical solution.)

Now since the correlation of any isotropic measure satisfies

$$B_{ij}(\mathbf{h}) = [B_{LL}(h) - B_{KK}(h)]\,\frac{h_i h_j}{h^2} + B_{KK}(h)\delta ij \tag{27}$$

for some appropriate function B_{KK}, (for example, see Yaglom (1957), formula (4.36), p. 305), then

$$B_{11}(h,0,0) = B_{LL}(h). \tag{28}$$

On the other hand, as shown in Dostoglou & Kahl (2009) there is sequence of l's such that $B^l_{ij}(\mathbf{h}) \to B_{ij}(\mathbf{h}), l \to \infty$ uniformly in h on compact subsets of $\mathbb{R}^3$. Therefore, for large enough l's B^l_{LL} is negative on intervals. At the same time, the continuity of B^l_{ij} and (28) give that B^l_{LL} is continuous in time. Therefore, for the example of § 3.2 there exists sequence $\{l_n\}_{n\geq 1}$ and an N such that for $n \geq N$, $B^{l_n}_{LL}(t,h) < 0$ on h-intervals for $t \in [0, t_n)$.

5. Numerical implementation

For $\mathbf{v}$ is a (possibly random) vector field that is $L \times L \times L$ periodic

$$\mathbf{v}(\mathbf{x}) = \sum_{\mathbf{k} \in 2\pi L^{-1}\mathbb{Z}^3} \hat{\mathbf{v}}_{\mathbf{k}} e^{i\mathbf{x}\cdot\mathbf{k}}, \tag{29}$$

define a random vector field by

$$\mathbf{u}(\mathbf{x}) = W\mathbf{v}(W^{-1}(\mathbf{x} - \xi)), \tag{30}$$

where W is a randomly chosen orthogonal matrix (using the standard probability measure on the group of orthogonal matrices), and ξ is chosen uniformly from $[0, L]^3$. Thus $\mathbf{u}$ is distributed according to a homogeneous, isotropic measure, and

$$B_{ij}(\mathbf{h}) = \overline{u_i(\cdot)u_j(\cdot - \mathbf{h})} \tag{31}$$

and Φ in equation (9) is now replaced by the (not differentiable any more)

$$\Phi(k) = \sum_{\mathbf{k} \in 2\pi L^{-1}\mathbb{Z}^3 : |\mathbf{k}| < k} \tfrac{1}{2}\overline{|\hat{\mathbf{v}}_{\mathbf{k}}|^2}. \tag{32}$$

Furthermore

$$B_{LL}(h) = \sum_{\mathbf{k} \in 2\pi L^{-1}\mathbb{Z}^3} \tfrac{1}{2}\overline{|\hat{\mathbf{v}}_{\mathbf{k}}|^2} \left(\frac{2\sin(hk)}{(hk)^3} - \frac{2\cos(hk)}{(hk)^2} \right). \tag{33}$$

For example, suppose that $\mathbf{v}$ is chosen randomly using

$$\hat{\mathbf{v}}_{\mathbf{k}} = \sigma_{|\mathbf{k}|} P_{\mathbf{k}}(\gamma_{\mathbf{k},1}, \gamma_{\mathbf{k},2}, \gamma_{\mathbf{k},3}). \tag{34}$$

Here $P_{\mathbf{k}}\mathbf{w} = \mathbf{w} - |\mathbf{k}|^{-2}\mathbf{w} \cdot \mathbf{k}\,\mathbf{k}$ and $\gamma_{\mathbf{k},1}$, $\gamma_{\mathbf{k},2}$, $\gamma_{\mathbf{k},3}$ are independent mean zero, variance one, complex Gaussian random variables. The complete joint distribution between all the $\gamma_{\mathbf{k},i}$'s is not specified, although in all numerical simulations they are independent, except, of course, that $\hat{\mathbf{v}}_{\mathbf{k}}$ is the complex conjugate of $\hat{\mathbf{v}}_{-\mathbf{k}}$. Then

$$\tfrac{1}{2}\overline{|\hat{\mathbf{v}}_{\mathbf{k}}|^2} = |\sigma_{|\mathbf{k}|}|^2. \tag{35}$$

Now for the case $L = 2\pi$, set

$$\Psi(k) = \begin{cases} 0 & \text{if } k < 8 \\ \frac{1}{2}(1 - \cos(2\pi k/8)) & \text{if } 8 \le k \le 12 \\ 1 & \text{if } k > 12 \end{cases} \tag{36}$$

and set

$$\sigma_k = \left(\frac{\Psi'(k)}{4\pi k^2} \right)^{1/2}. \tag{37}$$

Then $\Phi(k)$ and $\Psi(k)$ are related as in figure 1.

A sample $\mathbf{v}$ was chosen using the distribution equation 34, and it was assumed that $\mathbf{u}$ was built as described in equation 30. Putting this into a numerical simulation of the Navier-Stokes equation on a $64 \times 64 \times 64$ grid with viscosity equal to 10^{-4}, and noting that solving the Navier-Stokes for $\mathbf{u}$ is equivalent to solving the Navier-Stokes equation for $\mathbf{v}$ and then applying equation 30 to the solution obtained, the plots of $f(h) = B_{LL}(h)/B_{LL}(0)$ for various values of t are as in figure 2.

The complete software and data can be downloaded at:
`http://www.math.missouri.edu/~stephen/navier3d/navier3d-1.1-d.tar.gz`

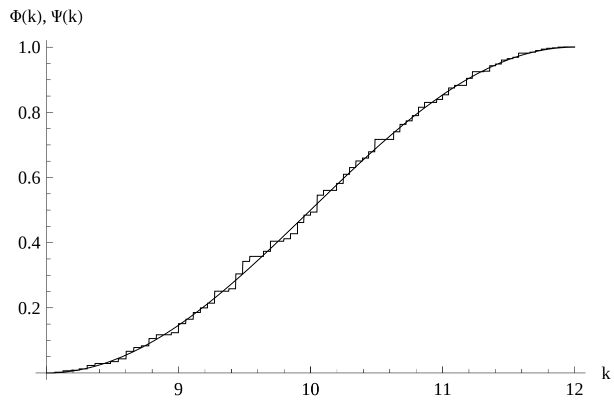

FIGURE 1. $\Phi(k)$ vs. $\Psi(k)$.

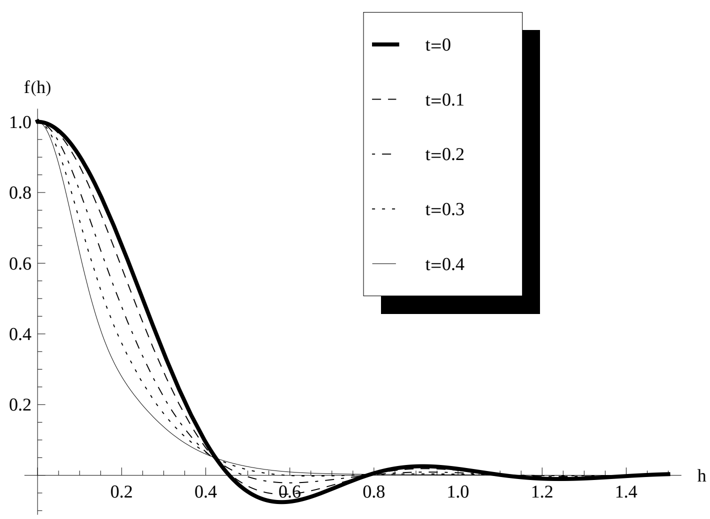

FIGURE 2. Time evolution of $f(h) = B_{LL}(h)/B_{LL}(0)$:
▬▬, t=0; - - - - , t=0.1; · − ·−, t=0.2; · · · · · ·, $t = 0.3$; ——, t=0.4.

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References

ANDROULAKIS G.; DOSTOGLOU S. 2004 Space averages and homogeneous fluid flows, *Math. Phys. Electron. J.* **10**, No. 4.

BATCHELOR G.K. 1953 *The theory of homogeneous turbulence*. Cambridge University Press

DAVIDSON, P. A. 2004 *Turbulence. An introduction for scientists and engineers.* Oxford University Press.

Dostoglou, S.; Fursikov, A. V.; Kahl, J. D. 2006 Homogeneous and isotropic statistical solutions of the Navier-Stokes equations. *Math. Phys. Electron. J.* **12**, No 2.

Dostoglou S., Gastler, R. R. 2009 On the longitudinal correlation function of isotropic flows. *Preprint.*

Dostoglou S., Kahl J.D. 2009 Approximation of homogeneous measures in the 2-Wasserstein metric Math. Phys. Electron. J. **15**, No. 1.

Gustafsson J., George W.K. 2008 Reconsidering integral invariants in homogeneous incompressible turbulence *Preprint.*

Gikhman, I. I.; Skorohod, A. V. 1980 *The theory of stochastic processes. I, Grundlehren der Mathematischen Wissenschaften 210.* Springer-Verlag.

Itô, K. 1970 The topological support of Gauss measure on Hilbert space. *Nagoya Math. J.* **38**, 181–183.

Kolmogorov, A.N. 1941 The local structure of turbulence in incompressible viscous fluid for very large Reynold's numbers. *C. R. (Doklady) Acad. Sci. URSS (N.S.)* **30**, 301–305.

Tempelman, A. 1992 *Ergodic Theorems for Group Actions.* Kluwer.

Vishik, M.J; Fursikov A.V. 1988 *Mathematical Problems of Statistical Hydromechanics* Kluwer.

Yaglom, A. M. 1957 Certain types of random fields in n-dimensional space similar to stationary stochastic processes. *Teor. Veroyatnost. i Primenen* **2**, 292–338.

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