# POWER-BOUNDED OPERATORS AND RELATED NORM ESTIMATES

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ABSTRACT. We consider whether $L = \limsup_{n\to\infty} n\|T^{n+1} - T^n\| < \infty$ implies that the operator $T$ is power bounded. We show that this is so if $L < 1/e$, but it does not necessarily hold if $L = 1/e$. As part of our methods, we improve a result of Esterle, showing that if $\sigma(T) = \{1\}$ and $T \neq I$, then $\liminf_{n\to\infty} n\|T^{n+1} - T^n\| \geq 1/e$. The constant $1/e$ is sharp. Finally we describe a way to create many generalizations of Esterle's result, and also give many conditions on an operator which imply that its norm is equal to its spectral radius.

## 1. Introduction

Let $T$ be a bounded linear operator on a complex Banach space $X$. One of the classical problems in operator theory is to determine the relation between the size of the resolvent $(T - \lambda I)^{-1}$ when $\lambda$ is near the spectrum $\sigma(T)$, and the asymptotic properties of orbits $\{T^n x : n \geq 0\}$ for each $x \in X$. The inequality

$$\|(T - \lambda I)^{-1}\| \leq \frac{C}{\operatorname{dist}(\lambda, \sigma(T))}, \quad \lambda \in \mathbb{C} \setminus \sigma(T),$$

has been extensively studied by, for example, Benamara and Nikolski [3] and also, very recently, by El-Fallah and Ransford [9]; see also [15], [16], [21], [24]. Such an inequality is extreme in the sense that the converse inequality (with $C = 1$) is always satisfied. In most cases the

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relationship to such an inequality and the properties of the orbits are very difficult to determine.

Thus it is interesting that one has a very clean equivalence for the resolvent condition introduced by Ritt [22], which says there is a constant $C > 0$ such that

$$\|(T - \lambda I)^{-1}\| \leq \frac{C}{|\lambda - 1|} \qquad (|\lambda| > 1).$$

Nagy and Zemánek [16], and independently Lyubich [14], proved the following result (see also [20, Theorem 4.5.4]).

**Theorem 1.1.** *Let $T$ be an operator on a complex Banach space. Then $T$ satisfies the Ritt resolvent condition if and only if*

(1) *$T$ is power bounded, and*
(2) $\sup_n n\|T^{n+1} - T^n\| < \infty$.

We recall a result of Esterle [10] saying that if $\sigma(T) = \{1\}$ and $T$ is not the identity operator, then $\liminf_{n\to\infty} n\|T^{n+1} - T^n\| \geq 1/12$. (The citation given only has 1/96; this was improved by Berkani [2] to 1/12.) Moreover it was noted in [20, Theorem 4.5.1] that if 1 is a limit point of $\sigma(T)$, then $\limsup n\|T^{n+1} - T^n\| \geq 1/e$. Thus both the Ritt resolvent condition and condition (2) are extremal, and it is natural to ask whether these two conditions are equivalent, at least in the case when $\sigma(T) = 1$. Note it was only recently that Lyubich [15] constructed operators satisfying the Ritt condition and $\sigma(T) = \{1\}$.

Another reason that such a question is interesting is because of the famous Esterle-Katznelson-Tzafriri Theorem [10], [12], which states that if $T$ is power bounded, and its spectrum meets the unit circle only at the point 1, then $\|T^{n+1} - T^n\| \to 0$ as $n \to \infty$. Thus a positive answer to our question would provide a partial converse.

Towards this conjecture, it is known that if $\limsup_{n\to\infty} n\|T^{n+1} - T^n\| < 1/12$, then $T$ is power bounded in a rather trivial manner, that is, it is the direct sum of an identity operator and an operator whose spectral radius is less than 1. This follows directly from the result of Esterle cited above.

In this paper, we improve these results, in effect showing that in all these results that the constant 1/12 may be replaced by $1/e$. We will also provide examples to show that the constant $1/e$ is sharp. In particular, we show that the condition $\sup_n n\|T^{n+1} - T^n\| < \infty$ does not necessarily imply that $T$ is power bounded. We leave open the question as to whether it implies power boundedness in the case that $\sigma(T) = \{1\}$.

Finally we create a general framework which shows how to easily create results in the same vein as Esterle's result. For example, one can give conditions concerning $\|T^n - T^m\|$ that imply that an operator with $\sigma(T) = \{1\}$ is the identity. We also give results similar to the special case of Sinclair's Theorem [23] considered by Bonsall and Crabb [6], giving many different conditions on an operator that imply that its norm is equal to its spectral radius.

Let us finish this introduction by noting that Blunck [4], [5] gives many applications of the condition $\sup_n n\|T^{n+1} - T^n\| < \infty$ to maximal regularity problems.

Throughout this paper, we will take the Fourier transform to be $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi}\,dx$ and the inverse Fourier transform to be $\check{g}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\xi)e^{ix\xi}\,d\xi$. All Banach spaces will be complex in the remainder of the paper.

## 2. Esterle's Result

To illustrate the ideas, let us first give a continuous time version. The methods used are similar to those in a paper by Bonsall and Crabb [6] in their proof of a special case of Sinclair's Theorem [23]. The function $W$ described below is often called the Lambert function (see [8]).

**Theorem 2.1.** *Let $A$ be a bounded operator on a Banach space such that $\sigma(A) = \{0\}$. For each $t > 0$ such that $\|Ae^{tA}\| \le 1/et$, we have that $\|A\| \le 1/t$. In particular, if $\liminf_{t\to\infty} t\|Ae^{tA}\| < 1/e$, then $A = 0$.*

*Proof.* Let $f(z) = ze^z$. There is analytic function $W$ such that $W(f(z)) = z$ in some neighborhood of 0. In particular, by the Riesz-Dunford functional calculus, $W(tAe^{tA}) = tA$. Now

$$W(z) = \sum_{m=1}^{\infty} p_m z^m$$

where, by Lagrange's inversion formula [1, Ch. 5, Ex. 33],

$$p_m = \frac{1}{m!}\frac{d^{m-1}}{dz^{m-1}}\left(\frac{z}{f(z)}\right)^m\Bigg|_{z=0} = \frac{(-m)^{m-1}}{m!}.$$

The radius of convergence of $W$ is $1/e$, and $\sum_{m=1}^{\infty} |p_m|e^{-m} = 1$, since $f(-1) = -1/e$. Therefore $\|W(tAe^{tA})\| \le 1$, and the result follows. $\square$

**Theorem 2.2.** *Let $T$ be a bounded operator on a Banach space such that $\sigma(T) = \{1\}$. For each positive integer $n$ such that $\|T^{n+1} - T^n\| \le n^n/(n+1)^{n+1}$, we have that $\|T - I\| \le 1/(n+1)$. In particular, if $\liminf_{n\to\infty} n\|T^{n+1} - T^n\| < 1/e$, then $T = I$.*

*Proof.* Let $f_n(z) = z(1 + z/n)^n$. There is analytic function $W_n$ such that $W_n(f_n(z)) = z$ in some neighborhood of 0. In particular, by the Riesz-Dunford functional calculus, $W_n(n(T^{n+1} - T^n)) = n(T - I)$. Now

$$W_n(z) = \sum_{m=1}^{\infty} p_{nm} z^m$$

where

$$p_{nm} = \frac{1}{m!} \frac{d^{m-1}}{dz^{m-1}} \left( \frac{z}{f_n(z)} \right)^m \Bigg|_{z=0} = \frac{(-1)^{m-1}}{n^{m-1}(nm+m-1)} \binom{nm+m-1}{m}.$$

The radius of convergence of $W_n$ is $r_n = (n/(n+1))^{n+1}$, and $\sum_{m=1}^{\infty} |p_{nm}| r_n^m = n/(n+1)$, since $f_n(-n/(n+1)) = -r_n$. Therefore $\|W_n(n(T^{n+1} - T^n))\| \leq n/(n+1)$ and the result follows. ☐

In Section 4 below, we will generalize this approach and give many extensions of these results.

Now let us turn out attention to whether the constant $1/e$ in Theorems 2.1 and 2.2 can be improved. By the results of Lyubich [15] combined with Theorem 1.1, we know that there must be some upper bound on the numbers $C > 0$ such that $\sigma(T) = \{1\}$ and $\liminf_{n\to\infty} n\|T^{n+1} - T^n\| < C$ imply that $T = I$. In fact we will be able to modify the examples of Luybich to show that $C = 1/e$ is sharp.

We will consider the fractional Volterra operators, parameterized by $\alpha > 0$, on $L_p([0,1])$ for $1 \leq p \leq \infty$, given by the formula

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-y)^{\alpha-1} f(y)\, dy,$$

and also modified fractional Volterra operators

$$L^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-y)^{\alpha-1} e^{y-x} f(y)\, dy.$$

It is well known (and easy to show) that $(J^\alpha)_{\alpha>0}$ is a $C_0$-semigroup Similarly $(L^\alpha)_{\alpha>0}$ is also a $C_0$-semigroup. Thus it is easily seen that $\|(L^\alpha)^n\| = \|L^{\alpha n}\| \leq 1/\Gamma(\alpha n + 1)$, and hence the spectral radius of $L^\alpha$ is zero.

Let us also consider an extension of this operator $\tilde{L}^\alpha$ on $L_p(\mathbb{R})$ given by the formula

$$\tilde{L}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (x-y)^{\alpha-1} e^{y-x} f(y)\, dy.$$

This is a convolution operator. Therefore, $\widehat{\tilde{L}^\alpha f}(\xi) = m_\alpha(\xi)\hat{f}(\xi)$, where $m_\alpha$ is the Fourier Transform of $x_+^{\alpha-1}e^{-x}/\Gamma(\alpha)$. Direct calculation shows that $m_\alpha(\xi) = (1+i\xi)^{-\alpha}$, where here we are taking the principle branch.

Next, let $M$ denote the operator of multiplication by the indicator function of $[0,1]$, then it is not so hard to see that for any entire function $f$ we have that $f(L^\alpha) = M f(\tilde{L}^\alpha) M$, and so $\|f(L^\alpha)\| \le \|f(\tilde{L}^\alpha)\|$.

Now we see that $\widehat{\tilde{L}^\alpha e^{-t\tilde{L}^\alpha} f}(\xi) = k(\xi)\hat{f}(\xi)$, where $k(\xi) = m_\alpha(\xi)e^{-tm_\alpha(\xi)}$. If $0 < \alpha < 1$, then $\mathrm{Re}(m_\alpha(\xi)) > 0$, and $\lim_{\xi\to\pm\infty} \arg(m_\alpha(\xi)) = \alpha\pi/2$. Hence it is easy to see that

$$\limsup_{t\to\infty} t\|L^\alpha e^{-tL\alpha}\| \le \limsup_{t\to\infty} t\|\tilde{L}^\alpha e^{-t\tilde{L}^\alpha}\| \le 1/e\cos(\alpha\pi/2).$$

This is enough to show that the constant $C = 1/e$ is sharp in Theorem 2.1. However, we can do a little better.

**Theorem 2.3.** (1) *There exists an operator $A \ne 0$ on a Hilbert space, with $\sigma(A) = \{0\}$, and $\limsup_{t\to\infty} t\|Ae^{tA}\| \le 1/e$.*

(2) *There exists an operator $T \ne I$ on a Hilbert space, with $\sigma(T) = \{1\}$, and $\limsup_{n\to\infty} n\|T^{n+1} - T^n\| \le 1/e$.*

*Proof.* Let us consider the operator on $L_2([0,1])$

$$A = -\int_0^{1/2} L^\alpha \, d\alpha.$$

Lyubich [15] showed that the operator $B = \int_0^\infty J^\alpha \, d\alpha$ has spectral radius equal to 0 on $L_p([0,1])$ for all $1 \le p \le \infty$. Now both $-A$ and $B$ are operators with positive kernels, and the kernel of $-A$ is bounded above by the kernel of $B$. It follows that on $L_p([0,1])$ for $p = 1$ or $p = \infty$ that $\|A^n\| \le \|B^n\|$ for all positive integers $n$. Thus $A$ has spectral radius equal to 0 on $L_p([0,1])$ for $p = 1$ and $p = \infty$, and hence, by interpolation, for all $1 \le p \le \infty$.

We also define the operator on $L_2(\mathbb{R})$

$$\tilde{A} = -\int_0^{1/2} \tilde{L}^\alpha \, d\alpha.$$

Following the above argument, we see that $\|Ae^{tA}\| \le \|\tilde{A}e^{t\tilde{A}}\|$, and that $\widehat{\tilde{A}e^{t\tilde{A}} f}(\xi) = k(\xi)\hat{f}(\xi)$, where

$$|k(\xi)| = |h(\xi)| \exp(-t\mathrm{Re}(h(\xi))),$$

and

$$h(\xi) = \int_0^{1/2} m_\alpha(\xi) \, d\alpha.$$

One sees that $\arg(h(\xi)) \to 0$ as $\xi \to \infty$, and hence it is an easy matter to see that $\limsup_{t\to\infty} t\|Ae^{tA}\| \le 1/e$.

The second example is given by $T = e^A$. Note that $T \ne I$, because otherwise $A = \log(T) = 0$. The estimate is easily obtained since $T^{n+1} - T^n = \int_n^{n+1} Ae^{tA}\,dt$. □

## 3. Power Boundedness

**Theorem 3.1.** *Let $T$ be a bounded operator on a Banach space $X$ such that $\limsup_{n\to\infty} n\|T^{n+1} - T^n\| < 1/e$. Then $X$ decomposes as the direct sum of two closed $T$-invariant subspaces such that $T$ is the identity on one of these subspaces, and the spectral radius of $T$ on the other subspace is strictly less than 1. In particular, $T^n$ converges to a projection.*

*Proof.* First note that $\sigma(T)$ must be contained in $\{1\} \cup \{z : |z| < \alpha\}$ for some $\alpha < 1$, otherwise it is easy to see that limit superior of the spectral radius of $T^{n+1} - T^n$ is at least $1/e$ (see, for example [20, Theorem 4.5.1]). Thus there is a projection $P$ that commutes with $T$ such that $\sigma(T|_{\mathrm{image}(P)}) = \{1\}$, and the spectral radius of $T|_{\ker(P)}$ is strictly less than 1. The result now follows by applying Theorem 2.2 to $T|_{\mathrm{image}(P)}$. □

A very similar proof works also for the following continuous time version. However, we were also able to produce a different proof of this same result.

**Theorem 3.2.** *Let $A$ be a bounded operator on a Banach space $X$ such that $L = \limsup_{t\to\infty} t\|Ae^{tA}\| < 1/e$. Then $X$ decomposes as the direct sum of two closed $A$-invariant subspaces such that $A$ is the zero operator on one of these subspaces, and on the other subspace the supremum of the real part of the spectrum is strictly negative. In particular, $e^{tA}$ converges to a projection.*

*Proof.* To illustrate the ideas, let us first prove that $e^{tA}$ converges in the case that $L < 1/4$, that is, there are constants $c < 1/4$ and $t_0 > 0$ such that $\|Ae^{tA}\| \le c/t$ for $t \ge t_0$. It follows that $\|A^2e^{2tA}\| \le c^2/t^2$ for $t \ge t_0$, or $\|A^2e^{tA}\| \le 4c^2/t^2$ for $t \ge 2t_0$. Then for $t \ge 2t_0$ we have

$$\|Ae^{tA}\| = \left\| \lim_{\tau\to\infty} \int_t^\tau A^2e^{sA}\,ds \right\| \le \frac{4c^2}{t},$$

since $Ae^{\tau A} \to 0$ as $\tau \to \infty$. Iterating this process, we get that $\|Ae^{tA}\| \le (4c)^{2^k}/4t$ for $t \ge 2^k t_0$. To put this another way, $\|Ae^{tA}\| \le (4c)^{t/2t_0}/4t$

for $t \geq t_0$. It follows that

$$e^{t_1 A} - e^{t_2 A} = \int_{t_2}^{t_1} A e^{sA} \, ds$$

converges to zero as $t_1, t_2 \to \infty$, that is, $e^{tA}$ is a Cauchy sequence. Hence it converges.

The case when $L < 1/e$ is only marginally more complicated. Again, there are constants $c < 1/e$ and $t_0 > 0$ such that $\|Ae^{tA}\| \leq c/t$ for $t \geq t_0$. For any integer $M \geq 2$ we have that $\|A^M e^{tA}\| \leq (cM)^M / t^M$ for $t \geq M t_0$. Integrating $(M-1)$ times we obtain that

$$\|Ae^{tA}\| \leq \frac{(cM)^M}{t(M-1)!} \quad \text{for } t \geq M t_0.$$

A simple computation shows that

$$\frac{(cM)^M}{(M-1)!} \leq \frac{M}{e} (ce)^M,$$

and hence iterating we obtain that if $t > M^k t_0$ then

$$\|Ae^{tA}\| \leq \left(\frac{M}{e}\right)^{-1/(M-1)} \left( ce \left(\frac{M}{e}\right)^{1/(M-1)} \right)^{M^k} \frac{1}{t}.$$

By choosing $M$ is sufficiently large, we see that there exist constants $c_1, c_2 > 1$ such that $\|Ae^{tA}\| \leq c_1 c_2^{-t}/t$ for $t \geq t_0$, and hence $\|e^{tA}\|$ converges.

Now it is clear that $S = \lim_{t \to \infty} e^{tA}$ is a bounded projection (because $S^2 = S$) such that $Se^{tA} = e^{tA} S = S$. Let $X_1 = \operatorname{Im}(S)$, and $X_2 = \operatorname{Ker}(S)$, so $X = X_1 \oplus X_2$. These spaces are clearly invariant under $e^{tA}$, and hence invariant under $A = \lim_{t \to 0}(e^{tA} - I)/t$. Since $S|_{X_1} = I|_{X_1}$ we see immediately that $e^{tA}|_{X_1} = I|_{X_1}$, and so $A|_{X_1} = \lim_{t \to 0}(e^{tA}|_{X_1} - I|_{X_1})/t = 0$. Furthermore, we have that $e^{tA}|_{X_2} \to 0$. Let $t_0$ be such that $\|e^{t_0 A}|_{X_2}\| \leq 1/2$. Then the spectral radius of $e^{t_0 A}|_{X_2}$ is bounded by $1/2$, and so $\sup \operatorname{Re}(A|_{X_2}) < -\log(2)/t_0$. $\square$

We also point out that that one could prove Theorem 3.1 in a similar manner. But the details can be quite complicated. It is also possible to deduce Theorem 3.1 from Theorem 3.2. Briefly, if $\|T^{n+1} - T^n\| \leq (1+\epsilon)L/(n+1)$ for large enough enough $n$, then by writing out the power series for $(T-I)e^{tT}$ about $t = 0$ one obtains that $\|(T-I)e^{tT}\| \leq (1+2\epsilon)Le^t/t$ for large enough $t$. The result now follows quickly by applying Theorem 3.2 to $A = T - I$, remembering that $\sigma(T) \subset \{1\} \cup \{z : |z| < 1\}$.

Now we give some counterexamples to show that in general the condition $\sup_n n\|T^{n+1} - T^n\| < \infty$ does not necessarily imply power boundedness.

**Theorem 3.3.** *There exists a bounded operator $T$ on $L_1(\mathbb{R})$ such that $\sup_n n\|T^{n+1} - T^n\| < \infty$, and $\|T^n\| \approx \log n$.*

*Proof.* The example is a multiplier on $L_1(\mathbb{R})$ given by $\widehat{Tf}(\xi) = m(\xi)\hat{f}(\xi)$. It is well known that such an operator is bounded if the inverse Fourier transform $\check{m}$ is a measure of bounded variation, and indeed that the norm is equal to the variation of $\check{m}$.

Let us consider the case

$$m(\xi) = \begin{cases} 1 & \text{if } |\xi| \le 1 \\ \exp(1 - |\xi|) & \text{if } |\xi| > 1. \end{cases}$$

An explicit computation shows that the inverse Fourier transform of $m^n$ is

$$\frac{nx\cos(x) + n^2 \sin(x)}{\pi x(x^2 + n^2)}$$

and that the inverse Fourier transform of $m^{n+1} - m^n$ is

$$\frac{(x^2 - n(n+1))\cos(x) + (2nx + x)\sin(x)}{\pi(x^2 + n^2)(x^2 + (n+1)^2)},$$

and it is now easy to verify the claims. □

**Theorem 3.4.** *On any Banach space $X$, there exists a bounded operator $T : X \to X$ such that $\limsup_{n\to\infty} n\|T^{n+1} - T^n\| < \infty$, and $\|T^n\| \to \infty$. Furthermore there is an equivalent norm $|\cdot|$ on $X$ so that $\limsup_{n\to\infty} n|T^{n+1} - T^n| \le 1/e$.*

*Proof.* In any Banach space $X$ we may find a sequence $e_n \in X$ with $\|e_n\| = 1$ and bi-orthogonal functionals $e_n^* \in X^*$ such that $\sup_n \|e_n^*\| = M < \infty$ and such that $(e_n)_{n=1}^\infty$ is not a basic sequence. Indeed, by [19], any subspace of $X$ with a basis has a normalized conditional basis, which may be re-ordered to give the example. (We remark that if $X$ is separable, then one can choose $(e_n)_{n=1}^\infty$ to be fundamental by using [17] or [18]). We refer to [13] for details.

Let $E = [e_n]_{n=1}^\infty$ be the closed linear span of $(e_n)_{n=1}^\infty$. Define $T : X \to X$ by

$$Tx = x + \sum_{k=1}^\infty (\lambda_k - 1)e_k^*(x)e_k$$

where $\lambda_k = \exp(-1/k!)$. Since $|\lambda_k - 1| \le 1/k!$ it follows that $T$ is bounded and $\|T\| \le e + 1$.

Consider

$$(T^n - T^{n+1})x = \sum_{k=1}^{\infty}(\lambda_k^n - \lambda_k^{n+1})e_k^*(x)e_k.$$

Hence

$$n\|T^n - T^{n+1}\| \le M\sum_{k=1}^{\infty}\frac{ne^{-n/k!}}{k!}.$$

To estimate this sum suppose $m! < n \le (m+1)!$. Then

$$\sum_{k=1}^{\infty}\frac{ne^{-n/k!}}{k!} = \left(\sum_{k=1}^{m-1}\frac{n}{k!}e^{-n/k!}\right) + \frac{n}{m!}e^{-n/m!} + \left(\sum_{k=m+1}^{\infty}\frac{n}{k!}e^{-n/k!}\right).$$

Simple estimates show that the two sums converge to 0 as $n \to \infty$, and it is easy to see that the middle term is bounded by $1/e$. Hence $\limsup_n n\|T^n - T^{n+1}\| \le M/e$.

Now we claim that if $\sup\|T^n\| < \infty$ then $(e_n)$ is a basic sequence, giving a contradiction. To do this we estimate $\|P_n\|$ where

$$P_n x = \sum_{k=1}^{n} e_k^*(x)e_k.$$

Then

$$P_n x + T^{n!}x = x + \sum_{k=1}^{n}\lambda_k^{n!}e_k^*(x)e_k + \sum_{k=n+1}^{\infty}(\lambda_k^{n!} - 1)e_k^*(x)e_k.$$

Thus

$$\|P_n + T^{n!} - I\| \le \sum_{k=1}^{n} e^{-n!/k!} + \sum_{k=n+1}^{\infty}\frac{n!}{k!}.$$

As before we can estimate both sums to uniformly bounded in $n$. So if $T$ is power-bounded then $(P_n)$ is uniformly bounded, and hence $(e_n)_{n=1}^{\infty}$ is basic.

Let us remark that the above construction also yields a counter-example if $X$ is reflexive and $(e_n)_{n=1}^{\infty}$ is a basis of an uncomplemented subspace of $X$, since in that case one can show that $P_n$ converges in the weak-operator topology to a projection on $E$.

To obtain the equivalent norm on $X$, set $|x| = \max(\|x\|, \sup_n |e_n^*(x)|)$. Let $X = (X, |\cdot|)$ and note that in this case $M = 1$. $\square$

## 4. A general approach

In this section we will discuss how to extend Theorems 2.1 and 2.2 by a more general approach. We first isolate the argument used.

To do this, let us introduce a class of analytic functions. Let $f$ be an analytic function defined on a disk $\{z : |z| < R\}$ (we allow the case when $f$ is entire and $R = \infty$).

We will say that $f \in \mathcal{P}$ if:

(1) $f(0) = 0$.
(2) $f'(0) \neq 0$.
(3) $f(x) \in \mathbb{R}$ if $-R < x < R$.
(4) The local inverse function $\varphi = f^{-1}$ of $f$ at the origin, which is defined in a neighborhood of 0 with $\varphi(0) = 0$, satisfies the conditions $\varphi^{(n)}(0) \geq 0$ for all $n \geq 1$.

We remark that in [6] the key idea is that $f(z) = \sin z$ is in class $\mathcal{P}$. In §2, we essentially used the fact that the functions $ze^{-z}$ and $z(1-\frac{z}{n})^n$ are in class $\mathcal{P}$. Before proceeding let us include another simple example which illustrates the basic ideas. During the late 1960's a series of papers investigated conditions on the sequence of norms $\|I-T^n\|$ which imply that $T = I$. A typical result is that of Chernoff [7], that says if $\sup_{n\geq 0} \|I - T^{2^n}\| < 1$ then $T = I$. Later Gorin [11] considered similar results for sequences $(q_n)_{n=0}^{\infty}$ replacing $(2^n)$; he showed the result is also true for sequences $q_n = 3^n, 4^n, 5^n$ but not $6^n$. More generally the conclusion is true if $q_0 = 1$ and $q_{n+1}/q_n \leq 5$. Let us prove the following simple result:

**Theorem 4.1.** *Suppose $T$ is a bounded operator on a Banach space $X$. Suppose $\lambda = 1$ is the only complex solution of the system of inequalities*

$$|1 - \lambda^n| \leq \|I - T^n\| \qquad n = 1, 2, \ldots$$

*Then $T = I$.*

*Proof.* It is clear that $\sigma(T) = \{1\}$. Assume $0 < a < 1$. Then there exists $n \in \mathbb{N}$ so that $\|I - T^n\| < 1 - a^n$. Consider the function $f(z) = 1 - (1 - z)^n$. This is in class $\mathcal{P}$ and $\varphi$ is given by $\varphi(z) = 1 - (1 - z)^{\frac{1}{n}}$ for $|z| < 1$. Let $A = I - T$ so that $A, f(A)$ are quasi-nilpotent. By the Riesz-Dunford functional calculus

$$A = \varphi(f(A)) = \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} f(A)^k.$$

In particular $\|A\| \leq \varphi(\|f(A)\|) < 1 - a$. It follows that $A = 0$ and $T = I$. ∎

We now derive a Corollary which is a slightly stronger form of the results of Gorin cited above. Note that if $c < 5$ we have $2\sin(\pi/(c+1)) > 1$.

**Corollary 4.2.** *Suppose $T$ is an operator on a Banach space such that $\liminf_{n\to\infty} \|I - T^n\| < 1$. Suppose for some $c > 1$ there is a sequence $(q_n)_{n=0}^{\infty}$ with $q_0 = 1$ and $q_{n+1} \leq cq_n$ if $n \geq 0$ such that $\|I - T^{q_n}\| < 2\sin(\pi/(c+1))$ for $n \geq 0$. Then $T = I$.*

*Proof.* Both statements follow very simply from the Theorem. Indeed if $|1 - \lambda^n| \leq \|I - T^n\|$ for all $n$ then the fact that $\liminf_{n\to\infty} \|I - T^n\| < 1$ is enough to imply $|\lambda| = 1$. Now if $\lambda = e^{i\theta}$ where $|\theta| \leq \pi$ we have $|\theta| < 2\pi/(c+1)$. If $\theta \neq 0$ let $N$ be the least integer such that $q_{N+1}|\theta| \geq 2\pi/(c+1)$. Then $q_{N+1}|\theta| \leq cq_N|\theta| \leq 2c\pi/(c+1)$ so that $|1 - \lambda^{q_{N+1}}| \geq 2\sin(\pi/(c+1))$. This yields a contradiction and so $\lambda = 1$. □

Our next Lemma gives us a recipe for constructing next examples of functions in class $\mathcal{P}$, when explicit calculation of the inverse function $\varphi$ may be difficult.

**Lemma 4.3.** *Let $f, h$ be analytic functions on the disk $\{z : \ |z| < R\}$. Suppose $f \in \mathcal{P}$ and that $h$ satisfies $h(0) > 0$, $h^{(n)}(0) \geq 0$ for all $n \geq 1$ and $h$ is nonvanishing. Then if $F(z) = f(z)/h(z)$ we have $F \in \mathcal{P}$.*

*Proof.* The first three conditions are obvious. For the last condition, let $\varphi$ be the local inverse of $f$ at the origin defined on some disk centered at the origin. Let $0 < \rho < \frac{1}{2}$ be chosen so that $\rho$ is smaller than the radius of convergence of the power series expansions of $h$ and $\varphi$ around the origin and let $M \geq 1$ be an upper bound for $|h|, |h'|, |\varphi|$ and $|\varphi'|$ on the disk $\{z : \ |z| \leq \rho\}$. For fixed $w$ consider the map $\Phi_w(z) = \varphi(wh(z))$ for $|z| \leq \rho$. Then if $M|w| < \rho$, we have $|\Phi_w(z)| \leq M|w||h(z)| \leq M^2|w|$. Thus if $|w| < M^{-2}\rho$ we have that $\Phi_w$ maps $\{z : \ z \leq \rho\}$ to itself. We also have $|\Phi_w'(z)| \leq M^2|w| < \rho$. We conclude that if $|w| < M^{-2}\rho$ then $\Phi_w$ maps the disk $\{z : \ |z| \leq \rho\}$ to itself and satisfies $|\Phi_w'(z)| \leq \frac{1}{2}$ for $|z| \leq \rho$. By the Banach contraction mapping principle if $|w| < M^{-2}\rho$ we can define $g_n(w)$ by $g_n(0) = 0$ and then $g_n(w) = \Phi_w(g_{n-1}(w))$ and $g_n(w)$ converges to the unique fixed point $\psi(w)$ of $\Phi_w$. The convergence is uniform on the disk $\{w : \ |w| < M^{-2}\rho\}$. By induction each $g_n$ is analytic and has non-negative coefficients in its Taylor series expansion about the origin. It follows that $\psi$ has the same properties, and $\psi$ is clearly the inverse function of $F$. □

Let us say $f \in \mathcal{P}$ is *admissible* if there exists $0 < x < R$ such that $f'(x) = 0$. If $f$ is admissible let $\xi$ be the least positive solution of

$f'(x) = 0$ and suppose $\delta$ is the radius of convergence of the power series expansion of $\varphi$.

**Lemma 4.4.** *If $f$ is admissible then $\delta = f(\xi)$ and*

$$\xi = \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} f(\xi)^k.$$

*Proof.* Clearly we have $\varphi(x) < \xi$ if $0 < x < \delta$. Let $\eta = \lim_{x\to\delta} \varphi(x)$ so that $\eta \le \xi$. If $\eta = \xi$ we are done. Assume $\eta < \xi$. Then it is clear that $\varphi'$ is bounded above by $L = f'(\eta)^{-1}$. Let $U = \{\varphi(z) : \ |z| < \delta\})$. Let $U_n = \{z : \ d(z, U) < \frac{1}{n}\}$. Then $U$ is contained in the disk $\{z : |z| < \eta\}$ and so for large enough $n$, $U_n$ is contained in the domain of $f$. Then $f$ cannot be univalent on any $U_n$, for, if it were, $\varphi$ could be extended to an analytic function on a disk of radius greater than $\delta$. Pick $z_n, w_n \in U_n$ so that $w_n \ne z_n$ and $f(w_n) = f(z_n)$. We can find $w, z \in \overline{U}$ so that $(w, z)$ is an accumulation point of $(w_n, z_n)$. If $w = z$ then $f'(w) = 0$ and this implies $\varphi'$ cannot be bounded above, yielding a contradiction. If $w \ne z$ then we choose $u_n, v_n$ with $|u_n| < r, |v_n| < r$ and $\varphi(u_n) \to w,\ \varphi(v_n) \to z$. Then $u_n, v_n \to f(w) = f(z)$ but

$$|w - z| \le \limsup_{n\to\infty} L|u_n - v_n| = 0.$$

This also yields a contradiction and the proof is complete. □

**Theorem 4.5.** *Let $A$ be a quasi-nilpotent operator on a Banach space $X$. Suppose $f$ is an admissible analytic function defined on a disk $\{z : |z| < R\}$ and suppose $\xi$ is the smallest positive solution of $f'(x) = 0$. Then if $\|f(A)\| < f(\xi)$ we have $\|A\| < \xi$.*

*Proof.* Let $\varphi$ be the local inverse at the origin. Then we have

$$A = \varphi(f(A)) = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} (f(A))^n.$$

Hence by Lemma 4.4

$$\|A\| < \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} f(\xi)^n = \xi.$$

□

Let us note that at this point that we can recapture Theorems 2.1 and 2.2 (without computing derivatives explicitly). Indeed $z$ belongs to $\mathcal{P}$ and hence $f(z) = ze^{-z}$ is admissible with $\xi = 1$ and $f(\xi) = 1/e$. Similarly $f(z) = (1-z)^n - (1-z)^{n+1} = z(1-z)^n$ is admissible with $\xi = 1/(n+1)$ and $f(\xi) = n^n(n+1)^{-n-1}$.

Let us now extend these results slightly. The first theorem below is a trivial application of the same ideas.

**Theorem 4.6.** *Suppose $A$ is a quasi-nilpotent operator and for some positive integer $m$, $\|Ae^{-A^m}\| < (me)^{-1/m}$. Then $\|A\| < m^{-1/m}$. Hence if $\liminf_{t\to\infty} \|tAe^{-t^m A^m}\| < (me)^{-1/m}$ then $A = 0$.*

**Theorem 4.7.** *Suppose $T$ is a bounded operator with $\sigma(T) = \{1\}$ and for some $m > n \in \mathbb{N}$ we have*

$$\|T^m - T^n\| < \left(1 - \frac{n}{m}\right)\left(\frac{n}{m}\right)^{n/(m-n)}.$$

*Then $\|T - I\| < 1 - (\frac{n}{m})^{1/(m-n)}$.*

*Proof.* We show that $f(z) = (1-z)^n - (1-z)^m$ is admissible. This follows from Lemma 4.3 since $f(z) = (1-z)^n(1-(1-z)^{m-n})$ and the function $1-(1-z)^{m-n}$ is in $\mathcal{P}$ since its local inverse at the origin is given by $1-(1-z)^{1/(m-n)}$. Now apply Theorem 4.5 to $I - T$. $\square$

It is possible to derive other formulas of the type of Theorem 2.2 from Theorem 4.7. For example we have the following Corollaries:

**Corollary 4.8.** *Suppose $T$ is a bounded operator with $\sigma(T) = \{1\}$. If*

$$\liminf_{m/n\to\infty} \|T^m - T^n\| < 1$$

*then $T = I$.*

*More precisely if*

$$\limsup_{m/n\to\infty} \frac{m}{n\log(m/n)}(1 - \|T^m - T^n\|) > 1$$

*then $T = I$.*

**Corollary 4.9.** *Suppose $T$ is a bounded operator with $\sigma(T) = \{1\}$. If*

$$\liminf_{p/n\to 0} \frac{n}{p}\|T^{n+p} - T^n\| < \frac{1}{e}$$

*then $T = I$.*

**Corollary 4.10.** *Suppose $T$ is a bounded operator with $\sigma(T) = \{1\}$. Suppose $0 < s < 1$. If*

$$\liminf_{\substack{m/n\to s\\ m,n\to\infty}} \|T^m - T^n\| < (1-s)s^{s/(1-s)}$$

*then $T = I$.*

The next theorem is a generalization of the argument used by Bonsall and Crabb [6] to prove a special case of Sinclair's Theorem [23], namely that the norm of an hermitian element $A$ of a Banach algebra coincides with its spectral radius $r(A)$.

**Theorem 4.11.** *Suppose $f$ is an admissible entire function. Suppose that for every $-\pi < \theta \le \pi$ we have either:*

(1) $\sup_{t>0} |f(te^{i\theta})| > f(\xi)$, *or*

(2) $|f(te^{i\theta})| < f(\xi)$ *for* $0 < t < \xi$.

*Let $A$ be any operator satisfying*

$$\sup_{t>0} \|f(tA)\| \le f(\xi).$$

*Then $r(A) = \|A\|$. In particular, if $A$ is quasi-nilpotent then $A = 0$. Furthermore if*

$$\sup_{t>0} \|f(tA)\| < f(\xi)$$

*then $A = 0$.*

*Proof.* We start by observing that if $\lambda \in \sigma(A)$ then $\sup_{t>0} |f(t\lambda)| \le f(\xi)$. Let $r = r(A)$. If $tr < \xi$ then by (1) and (2) we have $|f(t\lambda)| < f(\xi)$ for every $\lambda \in \sigma(A)$. Thus applying the Riesz-Dunford functional calculus to $tA$ we have $tA = \varphi(f(tA))$ and so

$$t\|A\| < \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!} f(\xi)^n = \xi.$$

Hence $\|A\| < \xi/t$ and it follows that $\|A\| \le r(A)$.

For the last part of the theorem, assume that $\sigma(A) \neq \{0\}$. Then there exists $-\pi < \theta \le \pi$ with $\sup_{t>0} |f(te^{i\theta})| < f(\xi)$. It is easy to see that this implies that $\varphi$ is unbounded on the disk $\{z : \ |z| < f(\xi)\}$ which contradicts Lemma 4.4. Hence $A$ is quasi-nilpotent and the conclusion follows. ∎

In the Bonsall-Crabb argument for Sinclair's theorem one takes $f(z) = \sin z$ and shows that it verifies the hypotheses and hence $\|\sin tT\| \le 1$ for all $t > 0$ implies that the norm and spectral radius of $T$ coincide. Other functions are permissible however, and lead to more general results of this type:

**Theorem 4.12.** *Let $A$ be an operator on a Banach space $X$. Then each of the following conditions implies that $r(A) = \|A\|$.*

(1) $\sup_{t>0} t\|Ae^{-tA}\| \le e^{-1}$.

(2) $\sup_{t>0} t\|Ae^{-tA^m}\| \le (me)^{-1/m}$ *for $m > 1$ an integer.*

(3) $\sup_{t>0} \|e^{-tA} - e^{-stA}\| \le (s-1)s^{-s/(s-1)}$ *for some $s > 1$.*

(4) $\sup_{t>0} \|e^{-(s+i)tA} - e^{-(s-i)tA)}\| \le \dfrac{2e^{-s\arctan(1/s)}}{\sqrt{1+s^2}}$ *for some $s \ge 0$.*

*In each case a strict inequality implies that $A = 0$.*

*Proof.* The first two are immediate deductions from the preceding Theorem 4.11. We then must show for the remaining cases that $e^{-z} - e^{-sz}$ for $s > 1$ and $e^{-sz}\sin z$ for $s > 0$ satisfy the conditions of Theorem 4.11 (the case $s = 0$ is Sinclair's theorem).

Note first that $f(z) = e^{-z}(1 - e^{-(s-1)z})$ is admissible by Lemma 4.3, since $1 - e^{(s-1)z} \in \mathcal{P}$. In this case $\xi = (s-1)^{-1}\log s$ and $f(\xi) < 1$. Let us assume $-\pi < \theta < \pi$ and $\theta \neq 0$. If $|\theta| > \frac{\pi}{2}$ then $f(te^{i\theta})$ is unbounded; if $|\theta| = \frac{\pi}{2}$ then $\sup_{t>0}|f(te^{i\theta})| = 2 > 1$. If $|\theta| < \frac{\pi}{2}$ then we observe that

$$|f(te^{i\theta})| = e^{-t\cos\theta}|1 - e^{-(s-1)te^{i\theta}}|.$$

Assume that $\sup_{t>0}|f(te^{i\theta})| \le f(\xi)$. Pick $t_0$ so that $(s-1)t_0|\sin\theta| = \frac{\pi}{2}$. Then

$$e^{-\xi} > f(\xi) \ge |f(t_0e^{i\theta})| \ge e^{-t_0\cos\theta}.$$

Hence $t_0\cos\theta > \xi$. Choose $t_1 < t_0$ so that $t_1\cos\theta = \xi$. Then $|f(t_1e^{i\theta})| \le f(\xi)$ implies that $(s-1)t_1|\sin\theta|$ is a multiple of $2\pi$. Since $t_1 < t_0$ this is impossible.

Next consider $f(z) = e^{-sz}\sin z$ where $0 < \theta < \frac{\pi}{2}$. In this case $\xi = \arctan s^{-1}$. We can again use Lemma 4.3 to see that $f$ is admissible. Clearly if $|\theta| \ge \frac{\pi}{2}$ then $f(te^{i\theta})$ is unbounded on $\{t > 0\}$. If $0 < |\theta| < \frac{\pi}{2}$ we use the fact that if $z = x + iy$ then

$$|f(z)| \ge e^{-sx}\cosh y|\sin x|.$$

Hence $|f(te^{i\theta})| > |f(t\cos\theta)|$ and so $\sup_{t>0}|f(te^{i\theta})| > f(\xi)$. $\square$

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