CONDITIONS IMPLYING REGULARITY OF THE THREE DIMENSIONAL NAVIER-STOKES EQUATION

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Abstract. We obtain logarithmic improvements for conditions for regularity of the Navier-Stokes equation, similar to those of Prodi-Serrin or Beale-Kato-Majda. Some of the proofs will make use of a stochastic approach involving Feynman-Kac like inequalities.

1. Introduction

The version of the three dimensional Navier-Stokes equation we will study is the differential equation in $u = u(t) = u(x,t)$, where $t \ge 0$, and $x \in \mathbb{R}^3$:

$$\frac{\partial u}{\partial t} = \Delta u - L \operatorname{div}(u \otimes u), \quad u(0) = u_0.$$

Here L denotes the Leray projection. We will not usually be working with classical solutions. We will define $u(t)$, $0 \le t \le T$, to be a solution of the Navier-Stokes equation if, whenever $u(t_0)$ is sufficiently regular for a mild solution

$$u(t) = e^{(t-t_0)\Delta} u(t_0) - \int_{t_0}^t e^{(t-s)\Delta} L \operatorname{div}(u(s) \otimes u(s)) \, ds$$

to exist for $t \in [t_0, t_0 + \tau)$ for some $\tau > 0$, then $u(t)$ is equal to that mild solution in $[t_0, t_0 + \tau)$.

For the remainder of the paper we denote the vorticity by $w = w(t) = w(x,t) = \operatorname{curl} u$. If w is sufficiently smooth then

$$\frac{\partial w}{\partial t} = \Delta w - u \cdot \nabla w + w \cdot \nabla u, \quad w(0) = \operatorname{curl} u_0.$$

A famous open problem is to prove regularity of the Navier-Stokes equation, that is, if the initial data u_0 is in L_2 and is regular (which in this paper we will define to mean that it is in the Sobolev spaces $W^{n,q}$

2000 *Mathematics Subject Classification.* Primary 35Q30, 76D05, Secondary 60H30, 46E30.

Key words and phrases. Navier-Stokes equation, vorticity, Prodi-Serrin condition, Orlicz norm, stochastic methods.

The author was partially supported by an NSF grant.

for some $2 \le q < \infty$ and all positive integers n), then the solution $u(t)$ is regular for all $t \ge 0$. Such regularity would also imply uniqueness of the solution $u(t)$. Currently only the existence of weak solutions is known. Also, it is known that for each regular u_0 that there exists $t_0 > 0$ such that $u(t)$ is regular for $0 \le t \le t_0$. We refer the reader to [3], [5], [6], [13], [19].

In studying this problem, various conditions that imply regularity have been obtained. For example, the Prodi-Serrin conditions ([15], [17]) state that for some $2 \le p < \infty$, $3 < q \le \infty$ with $\frac{2}{p} + \frac{3}{q} \le 1$ that

$$\int_0^T \|u(t)\|_q^p \, dt < \infty$$

for all $T > 0$. If u is a weak solution to the Navier-Stokes equation satisfying a Prodi-Serrin condition, with regular initial data u_0, then u is regular (see [18]). (Recently Escauriaza, Seregin and Sverák [7] showed that the condition when $q = 3$ and $p = \infty$ is also sufficient.) This is a long way from what is currently known for the so called Leray-Hopf weak solutions:

$$\int_0^T \|u(t)\|_q^p \, dt < \infty$$

for $\frac{2}{p} + \frac{3}{q} \ge \frac{3}{2}$, $2 \le q \le 6$.

Another condition is that of Beale, Kato and Majda [1]. They show that regularity follows from the condition

$$\int_0^T \|w(t)\|_\infty \, dt < \infty$$

for all $T > 0$. (In fact they proved this for the Euler equation, but the proof works also for the Navier-Stokes equation with only small modifications.) This was strengthened by Kozono and Taniuchi [11] to show that regularity follows from the condition

$$\int_0^T \|\nabla u(t)\|_{\mathrm{BMO}} \, dt \approx \int_0^T \|w(t)\|_{\mathrm{BMO}} \, dt < \infty$$

for all $T > 0$, where here BMO denotes the space of functions with bounded mean oscillation.

The purpose of this paper is threefold. First, we would like to provide some logarithmic improvements to these conditions. Secondly, we would like to present a stochastic approach to the Navier-Stokes equation, obtaining our conditions using Feynman-Kac like inequalities. Thirdly, we would like to present a different process for creating estimates of Foiaş, Guillopé and Temam.

To this end, the first result of this paper will be the logarithmic improvement to the Prodi-Serrin conditions.

Theorem 1.1. *Let $2 < p < \infty$, $3 < q < \infty$ with $\frac{2}{p} + \frac{3}{q} = 1$. If u is a solution to the Navier-Stokes equation satisfying*

$$\int_0^T \frac{\|u(t)\|_q^p}{1 + \log^+ \|u(t)\|_q} \, dt < \infty$$

for some $T > 0$, then $u(t)$ is regular for $0 < t \leq T$.

We will first present a proof of this result (and indeed of a slightly stronger result) that uses a standard approach. Then we will present a stochastic approach to the Navier-Stokes equation. This will be a kind of Lagrangian coordinates approach to the Navier-Stokes equation, but with a probabilistic twist in that we follow the path of each particle with a stochastic perturbation. A similar approach was adopted by Busnello, Flandoli and Romito [2].

From this we will obtain the following Beale-Kato-Majda type condition. For $1 \leq q < \infty$, define the function on $[0, \infty)$

$$\Phi_q(\lambda) = \left(\frac{e^\lambda - 1}{e - 1} \right)^q .$$

Define the Φ_q-Orlicz norm on any space of measurable functions by the formula

$$\|f\|_{\Phi_q} = \inf \left\{ \lambda > 0 : \int \Phi_q(|f(x)|/\lambda) \, dx \leq 1 \right\} .$$

(Thus the triangle inequality is a consequence of the fact that Φ_q is convex, see [12].) It is easily seem that $\|f\|_q \leq (e-1)\|f\|_{\Phi_q}$.

Theorem 1.2. *Let $1 < q < \infty$. If u is a solution to the Navier-Stokes equation satisfying*

$$\int_0^T \|\nabla u(t)\|_{\Phi_q} \, dt < \infty$$

for some $T > 0$, then $u(t)$ is regular for $0 < t \leq T$.

We will then demonstrate how to obtain Theorem 1.1 from Theorem 1.2 using the following result. If u is a solution to the Navier-Stokes equation, we define the sets

$$A^{n,q}_{T_0,T_1}(\lambda) = \{ t \in [T_0, T_1] : \|\nabla^n u(t)\|_q \geq \lambda \}.$$

Theorem 1.3. *Given $3 < q_1 < q_2 \leq \infty$, and a positive integer n, there exists constants $c_1, c_2, c_3 > 0$ such that if $u(t)$, $0 \leq t \leq T_2$ is a* *solution to the Navier-Stokes equation, and if* $0 \le T_1 \le T_2$*, then for all* $r \in (0, \sqrt{T_2 - T_1})$ *we have*

$$|A^{n,q_2}_{T_1+r^2,T_2}(c_1 r^{3/q_2-n-1})| \le c_2 |A^{0,q_1}_{T_1,T_2}(c_3 r^{3/q_1-1})|.$$

A similar result that one can obtain (but we will not prove here) is that for integers $n \ge 2$ we have $|A^{n,2}_{T_1+r^2,T_2}(c_1 r^{1/2-n})| \le c_2 |A^{1,2}_{T_1,T_2}(c_3 r^{-1/2})|$.

Corollary 1.4. *Under the hypotheses of Theorem 1.3, there exists a constant* $c > 0$ *with the following properties. If* $\Theta(\lambda)$ *is a positive increasing function of* $\lambda \ge 0$*, define*

$$\kappa = \int_0^\infty \min\{(c\lambda^{-2} - T_0)^+, T_1\}\, d\Theta(\lambda).$$

Then

$$\int_{T_0}^{T_1} \Theta(\|\nabla^n u(s)\|_{q_2}^{1/(1+n-3/q_2)})\, ds \le c\kappa + c\int_0^{T_1} \Theta(c\|u(s)\|_{q_1}^{1/(1-3/q_1)})\, ds.$$

Similarly,

$$\int_{T_0}^{T_1} \Theta(\|\nabla^n u(s)\|_2^{1/(n-1/2)})\, ds \le c\kappa + c\int_0^{T_1} \Theta(c\|\nabla u(s)\|_2^2)\, ds.$$

Since the Leray-Hopf weak solution to the Navier-Stokes equation satisfies $\int_0^T \|\nabla u(t)\|_2^2\, dt < \infty$, one can quickly recover the results of Foiaş, Guillopé and R. Temam [8] that say that $\int_0^T \|\nabla^n u(t)\|_2^{1/(n-1/2)}\, dt < \infty$.

2. Theorem 1.1

Let $T^* > T_0$ be the first point of non-regularity for $u(t)$. It is well known that in order to show that $T^* > T$, it is sufficient to show an *a priori* estimate, that is $\sup_{T_0 \le t < \min\{T^*, T\}} \|u(t)\|_q < \infty$. This is because it is then possible to extend the regularity beyond T^* if $T^* \le T$. Without loss of generality, it is sufficient to consider the case $T = T^*$ (so as to obtain a contradiction).

Proof of Theorem 1.1. We will allow all constants to implicitly depend upon p and q. Let us define quantities

$$v = u|u|^{q/2-1},$$
$$A = \sum_{i,j=1}^{3} \left(|u|^{q/2-1}\frac{\partial u_i}{\partial x_j}\right)^2,$$
$$B = \sum_{i,j=1}^{3} \left(|u|^{q/2-3}u_i \sum_{k=1}^{3} u_k \frac{\partial u_k}{\partial x_j}\right)^2$$

Note that

$$|\nabla v|^2 := \sum_{i,j=1}^{3}\left(\frac{\partial v_i}{\partial x_j}\right)^2 \approx A + B,$$
$$\sum_{i,j=1}^{3} \frac{\partial}{\partial x_j}\left(|u|^{q-2}u_i\right)\frac{\partial u_i}{\partial x_j} \approx A + B,$$
$$\sum_{i,j=1}^{3}\left(\frac{\partial}{\partial x_j}\left(|u|^{q-2}u_i\right)\right)^2 \le c|u|^{q-2}|\nabla v|^2.$$

We start with the Navier-Stokes equation, take the inner product with $u|u|^{q-2}$, and integrate over $\mathbb{R}^3$ to obtain

$$\|u\|_q^{q-1}\frac{\partial}{\partial t}\|u\|_q = \int |u|^{q-2}u\cdot\Delta u\,dx - \int |u|^{q-2}u\cdot L\,\mathrm{div}(u\otimes u)\,dx.$$

Integrating by parts, we see that

$$\int |u|^{q-2}u\cdot\Delta u\,dx = -\int \sum_{i,j=1}^{3}\frac{\partial}{\partial x_j}\left(|u|^{q-2}u_i\right)\frac{\partial u_i}{\partial x_j}\,dx \approx -\|\nabla v\|_2^2,$$

and

$$\begin{aligned}\int |u|^{q-2}u\cdot L\,\mathrm{div}(u\otimes u)\,dx &= \int \sum_{i,j=1}^{3}\frac{\partial}{\partial x_j}\left(|u|^{q-2}u_i\right)[L(u_j u)]_i\,dx \\ &\le c\||u|^{q/2-1}\|_s\|\nabla v\|_2\|L(u\otimes u)\|_r\end{aligned}$$

where $r = 1 + q/2$ and $s = (2q+4)/(q-2)$. Now the Leray projection is a bounded operator on L_r, and hence $\|L(u\otimes u)\|_r \approx \|u\|_{2+q}^2$. Also $\||u|^{q/2-1}\|_s \approx \|u\|_{2+q}^{q/2-1}$. Hence

$$\int |u|^{q-2}u\cdot L\,\mathrm{div}(u\otimes u)\,dx \le c\|u\|_{2+q}^{1+q/2}\|\nabla v\|_2 = c\|v\|_{2+4/q}^{1+2/q}\|\nabla v\|_2.$$

From the Sobolev and interpolation inequalities

$$\|v\|_{2+4/q} \le c \| |\nabla|^{3/(q+2)} v\|_2 \le c \|v\|_2^{(q-1)/(q+2)} \|\nabla v\|_2^{3/(q+2)},$$

and hence

$$\int |u|^{q-2} u \cdot L \operatorname{div}(u \otimes u)\, dx \le c \|v\|_2^{1-1/q} \|\nabla v\|_2^{1+3/q}.$$

Now apply Young's inequality $ab \le ((q-3)a^{2q/(q-3)} + (q+3)b^{2q/(q+3)})/2q$ for $a, b \ge 0$, to obtain

$$\int |u|^{q-2} u \cdot L \operatorname{div}(u \otimes u)\, dx \le c_1 \|\nabla v\|_2^2 + c_2 \|v\|_2^{2(q-1)/(q-3)},$$

where c_1 may be made as small as required by making c_2 larger. Hence

$$\|u\|_q^{q-1} \frac{\partial}{\partial t} \|u\|_q \le c \|v\|_2^{2(q-1)/(q-3)},$$

that is,

$$\frac{\partial}{\partial t} \|u\|_q \le c \|u\|_q^{p+1},$$

and so

$$\frac{\partial}{\partial t} \log(1 + \log^+ \|u\|_q) \le \frac{c \|u\|_q^p}{1 + \log^+ \|u\|_q}.$$

Integrating, we see that for $T_0 \le t < T$

$$\log(1+\log^+ \|u(t)\|_q) \le \log(1+\log^+ \|u(T_0)\|_q) + c \int_{T_0}^T \frac{\|u(s)\|_q^p}{1 + \log^+ \|u(s)\|_q}\, ds,$$

which provides a uniform bound for $\|u(t)\|_q$. $\square$

Remark 2.1. Note that this proof can easily be adapted to show that a sufficient condition for regularity is that

$$\int_{T_0}^T \frac{\|u(s)\|_q^p}{\Theta(\|u(s)\|_q)}\, ds < \infty,$$

where Θ is any increasing function for which

$$\int_1^\infty \frac{1}{x\Theta(x)}\, dx = \infty.$$

3. A Priori Estimates

This section is devoted to the proof of Theorem 1.3 and Corollary 1.4 The proof is very similar to the proof Scheffer's Theorem [16] that states that the Hausdorff dimension of the set of t for which the solution $u(t)$ is not regular is 1/2. The main tool is the following result is due to Grujić and Kukavica [9].

Theorem 3.1. *There exist constants $a, c > 0$ and a function $T : (0, \infty) \to (0, \infty)$, with $T(\lambda) \to \infty$ as $\lambda \to 0$, with the following properties. If $u_0 \in L_q(\mathbb{R}^3)$, then there is a solution $u(t)$ $(0 \le t \le T(\|u_0\|_q))$ to the Navier-Stokes equation, with $u(0) = u_0$, and $u(x,t)$ is the restriction of an analytic function $u(x+iy,t)+iv(x+iy,t)$ in the region $\{x + iy \in \mathbb{C}^3 : |y| \le a\sqrt{t}\}$, and $\|u(\cdot + iy, t) + iv(\cdot + iy, t)\|_q \le c\|u_0\|_q$ for $|y| \le a\sqrt{t}$.*

Proof of Theorem 1.3. First let us show that there exists a constants $c_1, c_3, c_4 > 0$ such that if $u(t)$, $t_0 - r^2 \le t \le t_0$ is a solution to the Navier-Stokes equation, and $|A^{0,q_1}_{t_0-r^2,t_0}(c_3 r^{3/q_1-1})| < c_4 r^2$, then $\|\nabla^n u(t_0)\|_{q_2} < c_1 r^{3/q_2-n-1}$.

To see this, Let us first consider the case when $t_0 = 0$ and $r = 1$. By hypothesis, we see that there exists $t \in [-1, -1+c_4]$ with $\|u(t)\|_{q_1} < c_3$. By Theorem 3.1 and the appropriate Cauchy integrals, if c_4 is small enough, then there exists a constant $c_7 > 0$ such that $\|\nabla^n u(0)\|_{q_2} < c_1$.

Now, by replacing $u(x,t)$ by $r^{-1}u(r^{-1}x, r^{-2}(t-t_0))$, we can relax the restriction $r = 1$ and $t_0 = 0$, and we obtain the statement we asserted.

Next, given $\epsilon > 0$, it is trivial to find a finite collection $t_1, \dots, t_N$ in $A = A^{n,q_2}_{T_1+r^2,T_2}(c_1 r^{3/q_2-n-1})$ such that the sets $[t_n - r^2, t_n]$ are disjoint, but the sets $[t_n - r^2 - \epsilon, t_n + \epsilon]$ cover A. By the above observation, $|A^{0,q_1}_{t_0-r^2,t_0}(c_3 r^{3/q_1-1})| \ge c_4 r^2$.

Hence

$$\frac{r^2}{r^2+2\epsilon}|A| \le N r^2 < c_4^{-1} \sum_{n=1}^{N} |A^{0,q_1}_{t_n-r^2,t_n}(c_3 r^{3/q_1-1})| \le c_4^{-1} |A^{0,q_1}_{T_1,T_2}(c_3 r^{3/q_1-1})|.$$

Since ϵ is arbitrary, the result follows. □

Proof of Corollary 1.4. We will only prove the first inequality. By Theorem 1.3, there exist constants $c_1, c_2, c_3 > 0$ such that

$$\begin{aligned}
&\int_{T_0}^{T_1} \Theta(\|\nabla^n u(s)\|_{q_2}^{1/(1+n-3/q_2)})\, ds \\
&\qquad = \int_0^\infty |\{s \in [T_0,T_1] : \|\nabla^n u(s)\|_{q_2}^{1/(1+n-3/q_2)} > \lambda\}|\, d\Theta(\lambda) \\
&\qquad \le c_1\kappa + \int_0^\infty |\{s \in [c_2\lambda^{-2},T_1] : \|\nabla^n u(s)\|_{q_2}^{1/(1+n-3/q_2)} > \lambda\}|\, d\Theta(\lambda) \\
&\qquad \le c_1\kappa + c_1 \int_0^\infty |\{s \in [0,T_1] : \|u(s)\|_{q_1}^{1/(1-3/q_1)} > c_3\lambda\}|\, d\Theta(\lambda) \\
&\qquad = c_1\kappa + c_1 \int_0^{T_1} \Theta(c_3^{-1}\|u(s)\|_{q_1}^{1/(1-3/q_1)})\, ds.
\end{aligned}$$

□

4. A Stochastic Description

Let us give a little motivation. Suppose that we defined $\varphi_{t_0,t_1}(x)$ to be $X(t_0)$, where X satisfies the equation

$$dX(t) = u(X(t),t)\, dt, \qquad X(t_1) = x,$$

then φ_{t_0,t_1} would be the "back to coordinates map" that takes a point at $t = t_1$ to where it was carried from by the flow of the fluid at time $t = t_0$. For the Euler equation, this provides a very effective way to describe the solution to the Navier-Stokes equation, for example, the equation for vorticity can be rewritten in a Lagrangian form:

$$w(x,t) = w(\varphi_{0,t}(x),0) + \int_0^t w(\varphi_{s,t}(x),s) \cdot \nabla u(\varphi_{s,t}(x),s)\, ds.$$

For the Navier-Stokes equation this formula is not true, and the Laplacian term can make things complicated. One approach to dealing with this is described in the paper by Constantin [4]. However, we will take a different approach using Brownian motion, using a kind of "randomly perturbed back to coordinates map." Such an approach was already discussed in the paper [14], here we make the discussion more rigorous. The author recently found out that a similar approach was followed by Busnello, Flandoli and Romito in [2].

First, we wish to make sure that all functions involved are sufficiently smooth. The hypothesis of Theorem 1.2 imply that, given $\epsilon \in (0,T)$, there exists $t' \in (0,\epsilon)$ with $u(t') \in L_{\Phi_q} \subset L_q$. Then by known results (for example Theorem 3.1 below), it follows that there exists

$0 < T_0 < \epsilon$ such that $u(T_0) \in W^{n,r}$ for all $r \in [q, \infty]$ and positive integers n. Furthermore, arguing as in Section 2, we only need to prove $\sup_{T_0 \le t < \min\{T^*, T\}} \|w(t)\|_q < \infty$ under the *a priori* assumption that the solution is regular for $t \in [T_0, T]$.

If $f\colon \mathbb{R}^3 \to \mathbb{R}$ is regular, and $T_0 \le t_0 \le t_1 < T$, define $A_{t_0,t_1} f(x) = \alpha(x, t_1)$, where α satisfies the transport equation

$$\frac{\partial \alpha}{\partial t} = \Delta\alpha - u \cdot \nabla\alpha, \qquad \alpha(x, t_0) = f(x).$$

Since $\operatorname{div}(u) = 0$, an easy integration by parts argument shows that

$$\frac{\partial}{\partial t} \int \alpha(x, t)\, dx = 0,$$

and hence if f is also in L_1, then

$$\int A_{t_0,t_1} f(x)\, dx = \int f(x)\, dx.$$

Since stochastic differential equations traditionally move forwards in time, it will be convenient to consider a time reversed equation. Let $b(t)$ be three dimensional Brownian motion. For $T_0 \le t_0 \le t_1 < T_1$, define the random function $\varphi_{t_0,t_1}\colon \mathbb{R}^3 \to \mathbb{R}^3$ by $\varphi_{t_0,t_1}(x) = X(-t_0)$, where X satisfies the stochastic differential equation:

$$dX(t) = -u(X(t), t)\, dt + \sqrt{2}\, db(t), \qquad X(-t_1) = x.$$

It follows by the Ito Calculus [10] that if $T_0 \le t_0 \le t_1 < T$, then

$$A_{t_0,t_1} f(x) = \mathbb{E} f(\varphi_{t_0,t_1}(x)).$$

(Here as in the rest of the paper, $\mathbb{E}$ denotes expected value.) Note that if f is also in L_1, then

$$\int \mathbb{E} f(\varphi_{t_0,t_1}(x))\, dx = \int f(x)\, dx.$$

Applying the usual dominated and monotone convergence theorems, it quickly follows that the last equality is also true if f is any function in L_1, or if f is any positive function.

Now, we note that w is the unique solution to the integral equation

$$w(t) = A_{T_0,t} w(T_0) + \int_{T_0}^{t} A_{s,t}(w(s) \cdot \nabla u(s))\, ds \quad (T_0 \le t < T).$$

Uniqueness follows quickly by the usual fixed point argument over short intervals, remembering that $u(t)$ is regular for $T_0 \le t < T$.

Consider also the random quantity $\tilde{w} = \tilde{w}(x,t)$ as the solution to the integral equation for $T_0 \le t < T$

$$\tilde{w}(x,t) = w(\varphi_{T_0,t}(x), T_0) + \int_{T_0}^t \tilde{w}(\varphi_{s,t}(x), s) \cdot \nabla u(\varphi_{s,t}(x), s)\, ds.$$

Again, it is very easy to show that a solution exists by using a fixed point argument over short time intervals. It is seen that $\mathbb{E}\tilde{w}$ satisfies the same equation as w, and hence $\mathbb{E}\tilde{w} = w$.

Next, $\varphi_{t_0,t_1}(\varphi_{t_1,t_2}(x)) = \varphi_{t_0,t_2}(x)$, since both are $Y(t_0)$ where $Y(t)$ is the solution to the integral equation

$$Y(t) = \varphi_{t_1,t_2}(x) + \int_{t_1}^t u(Y(s),s)\, ds + \sqrt{2}(b_{-t} - b_{-t_1}).$$

Hence

$$\tilde{w}(\varphi_{s_1,t}(x), s_1) - \tilde{w}(\varphi_{s_2,t}(x), s_2) = \int_{s_1}^{s_2} \tilde{w}(\varphi_{s,t}(x), s) \cdot \nabla u(\varphi_{s,t}(x), s)\, ds.$$

Thus, by Gronwall's inequality, if $T_0 \le t < T$

$$|\tilde{w}(x,t)| \le \exp\left(\int_{T_0}^t |\nabla u(\varphi_{s,t}(x), s)|\, ds \right) |w(\varphi_{T_0,t}(x), T_0)|.$$

(This is essentially the Feynman-Kac formula.) The goal, then, is to find uniform estimates on the quantity

$$\exp\left(\int_{T_0}^t |\nabla u(\varphi_{s,t}(x), s)|\, ds \right).$$

This we will proceed to do in the next section.

5. Theorem 1.2

Let us fix p and q satisfying the hypothesis of Theorems 1.1 and 1.2, and allow all constants to implicitly depend upon p and q. We retain the notation from the previous section, in particular the definitions of T_0, T^* and T.

Proof of Theorem 1.2. Since $\|f\|_q \le (e-1)\|f\|_{\Phi_q}$, applying Corollary 1.4, we see that

$$\int_{T_0}^T \|u\|_\infty^{1-3/q}\, dt < \infty.$$

In particular, $\|u\|_\infty < \infty$ for almost every $t \in [T_0, T]$. Hence, there exists $\lambda > T_0^{-1}$ such that

$$\int_B \|\nabla u(t)\|_{\Phi_q}\, dt \le \frac{1}{q},$$

where $B = \{t \in [T_0, T] \colon \|\nabla u(t)\|_\infty \geq c_2\lambda\}$. Thus for $T_0 \leq t < T$, we have that $|\tilde{w}(x,t)|$ is bounded by

$$e^{c_2\lambda(t-T_0)} \exp\left(\int_{B\cap[T_0,t]} |\nabla u(\varphi_{s,t}(x), s)|\, ds\right) |w(\varphi_{T_0,t}(x), T_0)|.$$

Hence by Jensen's and Hölder's inequalities, $\|w(t)\|_q^q \leq \int \mathbb{E}|\tilde{w}(t)|^q\, dx \leq e^{c_2 q\lambda(t-T_0)}(N_q^q + N_{qq'}^q \tilde{N})$, where $q' = q/(q-1)$,

$$N_r = \left(\int \mathbb{E}|w(\varphi_{T_0,t}(x), T_0)|^r\, dx\right)^{1/r} = \|w(T_0)\|_r \qquad (r \geq 1),$$

and

$$\tilde{N} = \int \mathbb{E}\left(\exp\left(q\int_{B\cap[T_0,t]} |\nabla u(\varphi_{s,t}(x), s)|\, ds\right) - 1\right)^q dx.$$

Since the Orlicz norm satisfies the triangle inequality,

$$\left\|\int_{B\cap[T_0,t]} |\nabla u(\varphi_{s,t}(\cdot), s)|\, ds\right\|_{\Phi_q} \leq \frac{1}{q},$$

that is, $\tilde{N} \leq (e-1)^q$. Since $a^q + b^q \leq (a+b)^q$ for $a, b \geq 0$, we conclude that

$$\|w(t)\|_q \leq \|w(T_0)\|_q + (e-1)e^{c_2\lambda(t-T_0)}\|w(T_0)\|_{qq'},$$

and the result follows. □

A second proof of Theorem 1.1 will now follow from this next result.

Lemma 5.1. *There is a constant $c > 0$ such that if f is a measurable function, then*

$$\|f\|_{\Phi_q} \leq c\left(\|f\|_q + \frac{\|f\|_\infty}{1 + \Phi_q^{-1}((\|f\|_\infty/\|f\|_q)^q)}\right).$$

Proof. Let us assume that $\|f\|_\infty = 1$, and set $a = \|f\|_q$, $b = \Phi_q^{-1}(a^{-q})$ and $n = a + 1/(1+b)$. Let $f^* : [0,\infty] \to [0,\infty]$ be the non-increasing rearrangement of $|f|$, that is,

$$f^*(t) = \sup\{\lambda > 0 : |\{x : |f(x)| > \lambda\}| > t\},$$

so $\int F(|f(x)|)\, dx = \int_0^\infty F(f^*(t))\, dt$ for any Borel measurable function F. Notice that $f^*(t) \leq \min\{1, at^{-1/q}\}$.

Let us first consider the case $a \leq 1$, so that $b \geq 1$, $2n \geq 1/b$, and $n \geq a$. Then

$$\int \Phi_q(|f(x)|/2n)\, dx \leq \int_0^\infty \Phi_q(f^*(t)/2n)\, dt.$$

We split this integral up into three pieces. First,

$$\int_0^{a^q} \Phi_q(f^*(t)/2n)\,dt \le \int_0^{a^q} \Phi_q(b)\,dt = 1.$$

Next, since $(\Phi_q(\lambda))^{1/2q}$ is convex for $\lambda \ge 1$,

$$\begin{aligned}\int_{a^q}^{a^q b^q} \Phi_q(f^*(t)/2n)\,dt &\le \int_{a^q}^{a^q b^q} \Phi_q(abt^{-1/q})\,dt \\ &\le \int_{a^q}^{a^q b^q} \frac{a^{2q}\Phi_q(b)}{t^2}\,dt \\ &\le 1.\end{aligned}$$

Next, for $t \ge a^q b^q$, $f^*(t) \le 1/b \le 2n$, and $\Phi_q(\lambda) \le \lambda^q$ for $0 \le \lambda \le 1$, so

$$\int_{a^q b^q}^{\infty} \Phi_q(f^*(t)/2n)\,dt \le \int_{a^q b^q}^{\infty} (f^*(t)/2n)^q\,dt \le 1.$$

Since $\Phi_q(\lambda/3) \le \Phi_q(\lambda)/3$ for $\lambda \ge 0$,

$$\int \Phi_q(|f(x)|/6n)\,dx \le 1,$$

that is, $\|f\|_{\Phi_q} \le 6n$.

The case $a \ge 1$ (so $b \le 1$ and $2n \ge 1 + 2a$) is simpler, as it is easy to estimate

$$\int_0^\infty \Phi_q(f^*(t)/2n)\,dt \le \int_0^1 \Phi_q(1)\,dt + \int_1^\infty (f^*(t)/2n)^q\,dt \le 2.$$

□

Second proof of Theorem 1.1. Using Lemma 5.1 along with Corollary 1.4, we see immediately that the hypothesis of Theorem 1.1 implies the hypothesis of Theorem 1.2. □

Acknowledgments

The author wishes to extend his sincere gratitude to Michael Taksar for help with understanding stochastic processes.

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