REARRANGEMENT INVARIANT NORMS OF SYMMETRIC SEQUENCE NORMS OF INDEPENDENT SEQUENCES OF RANDOM VARIABLES

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Abstract. Let $X_1, X_2, \ldots, X_n$ be a sequence of independent random variables, let M be a rearrangement invariant space on the underlying probability space, and let N be a symmetric sequence space. This paper gives an approximate formula for the quantity $\| \|(X_i)\|_N \|_M$ whenever L_q embeds into M for some $1 \le q < \infty$. This extends recent work of Gordon, Litvak, Schütt and Werner who obtained similar results for Orlicz spaces.

1. Introduction

Let M be a rearrangement invariant space on $[0,1]$, or equivalently on any probability space, and let N be a symmetric sequence space. Let $X_1, X_2, \ldots, X_n$ be positive independent random variables. Define a function $Y : [0,n] \to [0,\infty]$ to be a non-increasing function such that

$$\text{measure}\{Y > t\} = \sum_{i=1}^{n} \Pr(X_i > t).$$

Notice that Y has the same law as the function $t \mapsto X_{[t]+1}(t - [t])$ where $[t]$ denotes the integer part of t. The purpose of this paper is to investigate conditions under which the following approximation holds:

$$\| \|(X_i)\|_N \|_M \approx \|Y|_{[0,1]}\|_M + \|(Y(i))\|_N. \tag{1}$$

Here $A \approx B$ means that the ratio of A/B is bounded below and above by constants. In the case of equation (1), the constants of approximation may be allowed to depend upon M.

There is much support to make such a conjecture. First, Rosenthal's inequality [9] can be interpreted (see for example [1]) as the truth of this in the case that $N = \ell_1$ or $N = \ell_2$, and $M = L_p$ for $1 \le p < \infty$.

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This was extended by Carothers and Dilworth [1] to the case when M is a Lorentz space $L_{p,q}$, $1 \le p < \infty$, $1 \le q \le \infty$, and then by Johnson and Schechtman [5] to the case when M is any rearrangement invariant space, even including the case when M is a quasi-Banach space, as long as M satisfies certain restrictions on the Boyd indices that disallow the case $M = L_\infty$. It is not hard to extend this last result to also allow $N = \ell_p$ for any $1 \le p < \infty$.

The next step was taken in a paper by Gordon, Litvak, Schütt and Werner [3]. They found a formula for $\| \| (X_i) \|_N \|_M$ in the case that $M = L_1$ and that N is an Orlicz space, and that $X_i = a_i \xi_i$, where a_1, $a_2, \dots, a_n$ are real numbers, and $\xi_1, \xi_2, \dots, \xi_n$ are identically distributed random variables. Their paper was a major inspiration for this work, and there is quite some overlap with the techniques.

In this paper we will assume that all the vector spaces in question satisfy the triangle inequality. It seems quite likely that many of the formulae will extend to at least some quasi-Banach situations, but we do not explore this possibility here. We will normalize the spaces so that $\|1\|_M = \|(1,0,\dots,0)\|_N = 1$.

Theorem 1. *Equation (1) is true if there exists* $1 \le q < \infty$ *such that* L_q *embeds continuously via the natural embedding into* M. *In that case, the constants of approximation in equation (1) depend only upon* q *and the constant of embedding.*

It is clear that at least some restriction must be placed upon M. For example, if $M = L_\infty$ then $\| \| (X_i) \|_N \|_M = \| (\|X_i\|_\infty) \|_N$, and so equation (1) does not necessarily hold.

The author would like to express his sincere appreciation to Mark Rudelson for bringing this problem and reference [3] to his attention, and also to Joel Zinn for pointing out the reference [7].

2. Proof of Main Theorem

If (x_i) is a sequence we will denote its non-increasing rearrangement by (x_i^*). If f is a function or random variable, we will denote its non-increasing rearrangement by $f^\#$.

Lemma 2. *Equation (1) is true if* $M = L_1$ *and* $N = \ell_\infty$.

Proof. This follows because

$$(2) \quad \tfrac{1}{2} \operatorname{measure}\{ Y|_{[0,1]} > t \} \le \Pr(\max_i X_i > t) \le \operatorname{measure}\{ Y|_{[0,1]} > t \}.$$

This has an elementary proof — see for example [4, Proposition 2.1] or [2]. Thus

$$\| \|(X_i)\|_\infty \|_1 \approx \int_0^1 Y(t)\, dt. \tag{3}$$

□

For each integer $1 \le m \le n$, let k_m denote the sequence space $\|(x_i)\|_{k_m} = \sum_{i=1}^m x_i^*$.

Lemma 3. *For each positive integer m, equation (1) is true if $M = L_1$ and $N = k_m$, with constants of approximation independent of m.*

Proof. Let $I_1, I_2, \dots, I_n$ be $\{0,1\}$-valued independent random variables that are also independent of (X_i), where $\Pr(I_i = 1) = 1/m$. Applying equation (3) to the sequence $(I_i X_i)$ we obtain

$$\| \|(I_i X_i)\|_\infty \|_1 \approx \int_0^1 Y(mt)\, dt \approx \frac{1}{m} \left(\int_0^1 Y(t)\, dt + \|(Y(i)\|_{k_m} \right). \tag{4}$$

Next, let $\mathcal{M}$ denote the σ-field generated by (I_i). Then applying equation (4), we see that

$$\mathbb{E}(\|(I_i X_i)\|_\infty | \mathcal{M}) \approx \frac{1}{m} \|(X_i)\|_{k_m}.$$

Thus we also obtain that

$$\| \|(I_i X_i)\|_\infty \|_1 = \|\mathbb{E}(\|(I_i X_i)\|_\infty | \mathcal{M})\|_1 \approx \frac{1}{m} \| \|(X_i)\|_{k_m} \|_1.$$

The result follows. □

Let P denote the space of functions f on $[0,n]$ for which its quasi-norm

$$\|f\|_P = \|f^\# |_{[0,1]}\|_M + \|(f^\#(i))\|_N$$

is finite. In fact this quasi-norm is equivalent to a norm, *viz*, $\|f\|_{P'} = \|f^\#|_{[0,1]}\|_M + \left\| \left(\int_{i-1}^i f^\#(t)\, dt \right) \right\|_N$ (see for example [8, Section 7]). However we will content ourselves with proving the following statement.

Lemma 4. *For any function f on $[0,n]$ we have $\|f(\cdot/100)\|_P \le 200 \|f\|_P$.*

Proof. First, since M satisfies the triangle inequality, it follows that $\|f(\cdot/100)^\#|_{[0,1]}\|_M \le 100 \|f^\#|_{[0,1/100]}\|_M$. Next, since $f^\#(i/100) \le f^\#([i/100])$, where $[t]$ denotes the integer part of t, we see that

$$\begin{aligned}
\|(f^{\#}(i/100))\|_N &\leq 100\|(f^{\#}(i))\|_N + \sum_{i=1}^{99} f^{\#}(i/100) \\
&\leq 100\|(f^{\#}(i))\|_N + 100\int_0^1 f^{\#}(t)\,dt \\
&\leq 100\|f\|_P.
\end{aligned}$$

□

Finally we need to cite a couple of results [4, Theorem 6.1 and Theorem 7.1]. These concern maximal sums of vector valued random variables $U = \max_k \left\|\sum_{i=1}^k Z_i\right\|$, where $Z_1,\ Z_2, \dots,\ Z_n$ are Banach-valued independent random variables. Let $V : [0,1] \to [0,\infty]$ be defined so that

$$\text{measure}\{V > t\} = \min\left\{1, \sum_{i=1}^n \Pr(\|Z_i\| > t)\right\}.$$

Theorem 5. *If $p \geq 1$, then $\|U\|_p \approx U^{\#}(e^{-p}/4) + \|V\|_p$.*

Theorem 6. *Suppose that L_q embeds continuously into M via the natural embedding, where $1 \leq q < \infty$. Then $\|U\|_M \approx \|U\|_1 + \|V\|_M$, where the constant of approximation depends only upon q and the embedding constant.*

Proof of Theorem 1. Let us first show the lower bound. Here the proof is very similar to the proof of [8, Theorem 27]. We know that

$$\begin{aligned}
\|(x_i)\|_N &= \sup_{\|y\|_{N^*}\leq 1} \sum_{i=1}^n x_i^* y_i^* \\
&= \sup_{\|y\|_{N^*}\leq 1} \sum_{m=1}^n (y_m^* - y_{m+1}^*)\|(x_i)\|_{k_m},
\end{aligned}$$

where by convention $y_{n+1}^* = 0$, and N^* denotes the dual space to N. From this, we immediately see that

$$\begin{aligned}
\mathbb{E}\|(X_i)\|_N &\geq \sup_{\|y\|_{N^*}\leq 1} \sum_{m=1}^n (y_m^* - y_{m+1}^*)\mathbb{E}\|(X_i)\|_{k_m} \\
&\approx \sup_{\|y\|_{N^*}\leq 1} \sum_{m=1}^n (y_m^* - y_{m+1}^*)\left(\int_0^1 Y(t)\,dt + \|(Y(i))\|_{k_m}\right) \\
&\approx \int_0^1 Y(t)\,dt + \|(Y(i))\|_N
\end{aligned}$$

since $y_1^* \le 1$ whenever $\|y\|_{N^*} \le 1$. To finish the lower bound, we see that

$$2\|\|(X_i)\|_N\|_M \ge \|\|(X_i)\|_\infty\|_M + \mathbb{E}\|(X_i)\|_N,$$

and the result follows by equation (2).

Now let us focus on the upper bound. Really the first part of this proof follows by an inequality obtained independently by van Zuijlen [10], [11], [12], and Marcus and Pisier [6]. But we shall provide a self contained proof that is essentially a copy of the proof of this same result that may be found in [7, Theorem 5.1]. From Lemma 4, it follows that $\|Y(\cdot/100)\|_P \le 200\|Y\|_P$. We have that

$$\begin{aligned}
\Pr(\|(X_i)\|_N > 200\|Y\|_P) &\le \Pr(\|(X_i)\|_N > \|Y(\cdot/100)\|_P) \\
&\le \Pr(\|(X_i)\|_N > \|(Y(i/100))\|_N) \\
&\le \Pr(\exists i : X_i^* > Y(i/100)) \\
&\le \sum_{i=1}^n \Pr(X_i^* > Y(i/100)) \\
&\le \sum_{i=1}^n \sum_{j_1<j_2<\cdots<j_i} \prod_{k=1}^i \Pr(X_{j_k} > Y(i/100)) \\
&\le \sum_{i=1}^n \frac{1}{i!}\left(\sum_{j=1}^n \Pr(X_j > Y(i/100))\right)^i \\
&\le \sum_{i=1}^n \frac{i^i}{100^i i!} \\
&\le \frac{1}{4e}.
\end{aligned}$$

Now we may apply Theorems 5 and 6 to $Z_i = X_i e_i \in N$, where e_i denotes the ith unit vector. In that case we see that $U = \|(X_i)\|_N$, and $V = Y|_{[0,1]}$, and the result follows. □

3. Application to Orlicz Spaces

In this section we will recover some of the results of Gordon, Litvak, Schütt and Werner [3].

Lemma 7. *Suppose that M and N are Orlicz spaces constructed from Orlicz functions Φ and Ψ respectively. Define a function*

$$\Theta(x) = \begin{cases} \Psi(x) & \text{if } 0 \le x \le 1 \\ \Phi(x) & \text{if } x \ge 1. \end{cases}$$

Then P is equivalent to the Orlicz space L_Θ.

Proof. Note that because of the normalization on M and N that $\Phi(1) = \Psi(1) = 1$. Also while Θ need not be an Orlicz function, it does satisfy the property that $\Theta(x)/x$ is an increasing function, and hence it is easily seen to be equivalent to the Orlicz function: $\tilde\Theta(x) = \int_0^x \frac{\Theta(t)}{t}\,dt$.

Suppose that $\|f\|_{L_\Theta} \le 1$, that is $\int_0^n \Theta(f^\#(t))\,dt \le 1$. Then in particular $f^\#(1) \le \int_0^1 \Theta(f^\#(t))\,dt \le 1$. Thus

$$\sum_{i=1}^n \Psi(f^\#(i)) \le \Theta(f(1)) + \int_1^n \Theta(f^\#(t))\,dt \le 2,$$

and so $\sum_{i=1}^n \Psi(f^\#(i)/2) \le 1$, that is $\|(f^\#(i))\|_N \le 2$. Also, if $a = \text{measure}\{f > 1\}$ (so $a \le 1$), then

$$\int_0^1 \Phi(f^\#(t))\,dt \le \int_0^a \Theta(f^\#(t))\,dt + (1-a) \le 2,$$

that is, $\|f^\#|_{[0,1]}\|_M \le 2$. Therefore $\|f\|_P \le 4$.

Now suppose that $\|f\|_P \le 1$. Again we see that $f^\#(1) \le 1$, and $a = \text{measure}\{f > 1\} \le 1$. Hence

$$\int_0^n \Theta(f^\#(t))\,dt \le \int_0^a \Phi(f^\#(t))\,dt + (1-a) + \sum_{i=1}^n \Psi(f^\#(i)) \le 3.$$

Since $\Theta(x/3) \le \Theta(x)/3$, it follows that $\|f\|_{L_\Theta} \le 3$. □

Now we will give a formulation of one of the results of [3].

Theorem 8. *Suppose that $\xi_1, \xi_2, \dots, \xi_n$ are identically distributed random variables, and that M and N are Orlicz spaces where M is constructed from an Orlicz function Φ satisfying* $\limsup_{x\to\infty} \log\Phi(x) < \infty$. *Then there exists an Orlicz function Λ such that for all real numbers $a_1, a_2, \dots, a_n$ we have*

$$\| \|(a_i \xi_i)\|_N \|_M \approx \|(a_i)\|_{L_\Lambda}.$$

Proof. Construct Θ as in Lemma 7. The condition on Φ tells us that L_q embeds into M for some $1 \le q < \infty$. The result follows from the definitions and Theorem 1 by setting $\Lambda(x) = \mathbb{E}(\Theta(x\xi_1))$. (Again Λ is not actually an Orlicz function, but it is equivalent to the Orlicz function $\tilde\Lambda(x) = \int_0^x \frac{\Lambda(t)}{t}\,dt$.) □

Now we will give another proof of the following result that appears in [3].

Corollary 9. *Suppose that $\gamma_1, \gamma_2, \dots, \gamma_n$ are identically distributed normalized Gaussian random variables, and let $1 \le m \le n$ be an integer. Let Λ be an Orlicz function equivalent to $xe^{-1/(mx)^2}$. Then for all* *real numbers $a_1, a_2, \dots, a_n$ we have*

$$\left\| \|(a_i\gamma_i)\|_{k_m} \right\|_1 \approx \|(a_i)\|_{L_\Lambda} \approx \sum_{i=1}^m a_i^* + m \sup_{m\le i\le n} a_i^* \sqrt{\log(1+i/m)}.$$

Proof. An easy argument shows that the Orlicz function Θ constructed in Lemma 7 is equivalent to the Orlicz function $x \mapsto (x-1/m)^+$. Then the function $\Lambda(x)$ constructed in the proof of Theorem 8 equals

$$\sqrt{\frac{2}{\pi}} \int_0^\infty (xt-1/m)^+ e^{-t^2/2}\,dt = x\sqrt{\frac{2}{\pi}} \int_0^\infty (t-1/mx)^+ e^{-t^2/2}\,dt,$$

and the rest of the result follows by simple calculations. □

Actually it looks like if $E|\xi_1| = 1$, then

$$\left\| \|(a_i\xi_i)\|_{k_m} \right\|_1 \approx \sum_{i=1}^m a_i^* + m \left\| \|(a_{mi}^*\xi_i)\|_\infty \right\|_1$$

Proof when I get back.

Finally let us finish with a remark. In [3], the authors showed in the case that $M = L_1$ that their upper bound held even if the random variables were not independent. This can also hold in our more general case. In [8] was introduced the concept of what it means for a rearrangement invariant space to be D^*-convex. This property is held, for example, by all Orlicz spaces. Following the proof of [8, Theorem 27], it can be shown that equation (1) holds even if the sequence (X_i) is not necessarily independent, as long as $M = L_1$ and P is D^*-convex. It is easy to see from the definition that the condition that P be D^*-convex cannot be dropped. We leave the details to the interested reader.

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