A CONDITION IMPLYING REGULARITY OF THE THREE DIMENSIONAL NAVIER-STOKES EQUATION

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Abstract. It is shown that if u is the solution to the three dimensional Navier-Stokes equation, then a sufficient condition for regularity is that $\int_0^T \|u(t)\|_q^p/(1+\log^+\|u(t)\|_q)\,dt < \infty$, for all $T > 0$, and some $2 < p < \infty$, $3 < q < \infty$, $\frac{2}{p}+\frac{3}{q} = 1$. This represents a logarithmic improvement over the usual Prodi-Serrin conditions.

1. Introduction

The version of the three dimensional Navier-Stokes equation we will study is the differential equation in $u = u(t) = u(x,t)$, where $t \ge 0$, and $x \in \mathbb{R}^3$:

$$\frac{\partial u}{\partial t} = \Delta u - u \cdot \nabla u + \nabla P, \quad \operatorname{div} u = 0, \quad u(0) = u_0.$$

We will also work with the vorticity form. For the remainder of the paper we denote $w = w(t) = w(x,t) = \operatorname{curl} u$. Then

$$\frac{\partial w}{\partial t} = \Delta w - u \cdot \nabla w + w \cdot \nabla u, \quad w(0) = \operatorname{curl} u_0.$$

A famous open problem is to prove regularity of the Navier-Stokes equation, that is, if the initial data u_0 is in L_2 and is regular (which in this paper we will define to mean that it is in the Sobolev spaces $W^{n,q}$ for some $2 \le q < \infty$ and all positive integers n), then the solution $u(t)$ is regular for all $t \ge 0$. Such regularity would also imply uniqueness of the solution $u(t)$. Currently the existence of weak solutions is known. Also, it is known that for each regular u_0 that there exists $t_0 > 0$ such that $u(t)$ is regular for $0 \le t \le t_0$. We refer the reader to [1], [3], [4], [9], [15].

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In studying this problem various conditions that imply regularity have been obtained. For example, the Prodi-Serrin conditions ([11], [13]) state that for some $2 \le p < \infty$, $3 < q \le \infty$ with $\frac{2}{p} + \frac{3}{q} \le 1$ that

$$\int_0^T \|u(t)\|_q^p \, dt < \infty$$

for all $T > 0$. If u is a weak solution to the Navier-Stokes equation satisfying a Prodi-Serrin condition, with regular initial data u_0, then u is regular (see [14]). (Recently Escauriaza, Seregin and Sverák [5] showed that the condition when $q = 3$ and $p = \infty$ is also sufficient.) This is a long way from what is currently known for weak solutions:

$$\int_0^T \|u(t)\|_q^p \, dt < \infty$$

for $\frac{2}{p} + \frac{3}{q} \ge \frac{3}{2}$, $2 \le q \le 6$. The purpose of this paper is to slightly reduce this rather large gap as follows.

Theorem 1.1. *Let $2 < p < \infty$, $3 < q < \infty$ with $\frac{2}{p} + \frac{3}{q} = 1$. If u is a solution to the Navier-Stokes equation satisfying*

$$\int_0^T \frac{\|u(t)\|_q^p}{1 + \log^+ \|u(t)\|_q} \, dt < \infty$$

for some $T > 0$, then $u(t)$ is regular for $0 < t \le T$.

The hypothesis of Theorem 1.1 imply that, given $\epsilon \in (0, T)$, there exists $t' \in (0, \epsilon)$ with $u(t') \in L_q$. Then by known results (for example Theorem 3.3 below), it follows that there exists $0 < T_0 < \epsilon$ such that $\|w(T_0)\|_r$ is bounded for all $r \in [q, \infty]$.

Let $T^* > T_0$ be the first point of non-regularity for $u(t)$. It is well known that in order to show that $T^* > T$, it is sufficient to show an *a priori* estimate, that is $\sup_{T_0 \le t < \min\{T^*, T\}} \|w(t)\|_q < \infty$. This is because it is then possible to extend the regularity beyond T^* if $T^* \le T$. Without loss of generality, it is sufficient to consider the case $T = T^*$ (so as to obtain a contradiction), and this we do in the remainder of the paper.

Also, from now on, let us fix p and q satisfying the hypothesis of Theorem 1.1, and allow all constants to implicitly depend upon p and q.

2. A Stochastic Description

This is a more rigorous formulation of the description given in [10]. As we have just stated, we are supposing that $u(t)$ is regular for all $T_0 \le t < T$.

If $f\colon \mathbb{R}^3 \to \mathbb{R}$ is regular, and $T_0 \le t_0 \le t_1 < T$, define $A_{t_0,t_1} f(x) = \alpha(x,t_1)$, where α satisfies the transport equation

$$\frac{\partial \alpha}{\partial t} = \Delta\alpha - u\cdot\nabla\alpha, \qquad \alpha(x,t_0) = f(x).$$

Since $\operatorname{div}(u) = 0$, an easy integration by parts argument shows that

$$\frac{\partial}{\partial t}\int \alpha(x,t)\,dx = 0,$$

and hence if f is also in L_1, then

$$\int A_{t_0,t_1} f(x)\,dx = \int f(x)\,dx.$$

Since stochastic differential equations traditionally move forwards in time, it will be convenient to consider a time reversed equation. Let $b(t)$ be three dimensional Brownian motion. For $T_0 \le t_0 \le t_1 < T_1$, define the random function $\varphi_{t_0,t_1}\colon \mathbb{R}^3 \to \mathbb{R}^3$ by $\varphi_{t_0,t_1}(x) = X(-t_0)$, where X satisfies the stochastic differential equation:

$$dX(t) = u(X(t),t)\,dt + \sqrt{2}\,db(t), \qquad X(-t_1) = x.$$

It follows by the Ito Calculus [7] that if $T_0 \le t_0 \le t_1 < T$, then

$$A_{t_0,t_1} f(x) = \mathbb{E} f(\varphi_{t_0,t_1}(x)).$$

(Here as in the rest of the paper, $\mathbb{E}$ denotes expected value.) Note that if f is also in L_1, then

$$\int \mathbb{E} f(\varphi_{t_0,t_1}(x))\,dx = \int f(x)\,dx.$$

Applying the usual dominated and monotone convergence theorems, it quickly follows that the last equality is also true if f is any function in L_1, or if f is any positive function.

Now, we note that w is the unique solution to the integral equation

$$w(t) = A_{T_0,t} w(T_0) + \int_{T_0}^{t} A_{s,t}(w(s)\cdot\nabla u(s))\,ds \quad (T_0 \le t < T).$$

Uniqueness follows quickly by the usual fixed point argument over short intervals, remembering that $u(t)$ is regular for $T_0 \le t < T$.

Consider also the random quantity $\tilde w = \tilde w(x,t)$ as the solution to the integral equation for $T_0 \le t < T$

$$\tilde w(x,t) = w(\varphi_{T_0,t}(x), T_0) + \int_{T_0}^{t} \tilde w(\varphi_{s,t}(x), s)\cdot\nabla u(\varphi_{s,t}(x), s)\,ds.$$

Again, it is very easy to show that a solution exists by using a fixed point argument over short time intervals.

It is seen that $\mathbb{E}\tilde{w}$ satisfies the same equation as w, and hence $\mathbb{E}\tilde{w} = w$. By Gronwall's inequality, if $T_0 \le t < T$

$$|\tilde{w}(x,t)| \le \exp\left(\int_{T_0}^t |\nabla u(\varphi_{s,t}(x),s)|\,ds\right) |w(\varphi_{T_0,t}(x),T_0)|.$$

(This is essentially the Feynman-Kac formula.) The goal, then, is to find uniform estimates on the quantity

$$\exp\left(\int_{T_0}^t |\nabla u(\varphi_{s,t}(x),s)|\,ds\right).$$

This we will proceed to do in the next section.

Remark 2.1. Let us give a little motivation. If instead we defined $\varphi_{t_0,t_1}(x)$ to be $X(-t_0)$, where X satisfies the equation

$$dX(t) = u(X(t),t)\,dt, \qquad X(-t_1) = x,$$

then φ_{t_0,t_1} would be the "back to coordinates map" that takes a point at $t = t_1$ to where it was carried from by the flow of the fluid at time $t = t_0$. See for example [2]. Thus with the addition of Brownian motion, φ_{t_0,t_1} becomes a "randomly perturbed back to coordinates map." In essence, we are considering a kind of Lagrangian approach to the Navier-Stokes equation.

3. Inequalities

Let us introduce some functions defined for $\lambda \ge 0$:

$$\Phi(\lambda) = \left(\frac{e^\lambda - 1}{e-1}\right)^q, \qquad \Theta(\lambda) = \frac{\lambda}{2 + \log^+ \lambda}.$$

Notice that for any $a > 0$, that there exist constants $c_1, c_2 > 0$ such that $c_1\Theta(\lambda) \le \lambda/(1 + \Phi^{-1}(\lambda^a)) \le c_2\Theta(\lambda)$; that Θ is a strictly increasing bijection on $[0,\infty)$; and that Θ and Θ^{-1} obey a moderate growth condition, that is, given $c_1 > 1$ there exists $c_2, c_3 > 1$ such that $c_2\Theta(\lambda) \le \Theta(c_1\lambda) \le c_3\Theta(\lambda)$.

We define the Φ-Orlicz norm on the space of measurable functions by the formula

$$\|f\|_\Phi = \inf\left\{\lambda > 0 : \int \Phi(|f(x)|/\lambda)\,dx \le 1\right\}.$$

The triangle inequality is a consequence of the fact that Φ is convex (see [8]). We extend the definition of the Orlicz norm to random functions F as follows

$$\|F\|_\Phi = \inf\left\{\lambda > 0 : \int \mathbb{E}\Phi(|F(x)|/\lambda)\,dx \le 1\right\}.$$

Using the notation from the previous section, we see for $T_0 \le t_0 \le t_1 < T$ that $\|f \circ \varphi_{t_0,t_1}\|_\Phi = \|f\|_\Phi$.

Let us fix the function

$$M(\lambda) = \int_{\{t\in[0,T]\colon \|u(t)\|_q \ge \lambda\}} \Theta(\|u(t)\|_q^p)\, dt.$$

The hypothesis of Theorem 1.1 tells us that $M(\lambda) \to 0$ as $\lambda \to \infty$.

Lemma 3.1. *There are constants $c_1, c_2, c_3, c_4 > 0$ such that if $\lambda > \max\{c_1, T^{-1}\}$, then*

$$\int_{\{t\in[\lambda^{-1},T]\colon \|\nabla u(t)\|_\infty \ge c_2\lambda\}} \|\nabla u(t)\|_\Phi \, dt \le c_3 M(c_4 \lambda^{1/p}).$$

Let us first show how to use this result.

Proof of Theorem 1.1. By Lemma 3.1, there exists $\lambda > T_0^{-1}$ such that

$$\int_B \|\nabla u(t)\|_\Phi \, dt \le \frac{1}{q}.$$

where $B = \{t \in [T_0, T]\colon \|\nabla u(t)\|_\infty \ge c_2\lambda\}$. Thus for $T_0 \le t < T$, we have that $|\tilde{w}(x,t)|$ is bounded by

$$e^{c_2\lambda(t-T_0)} \exp\left(\int_{B\cap[T_0,t]} |\nabla u(\varphi_{s,t}(x), s)|\, ds\right) |w(\varphi_{T_0,t}(x), T_0)|.$$

Hence by Jensen's and Hölder's inequalities, $\|w(t)\|_q^q \le \int \mathbb{E}|\tilde{w}(t)|^q\, dx \le e^{c_2 q\lambda(t-T_0)}(N_q^q + N_{qq'}^q \tilde{N})$, where $q' = q/(q-1)$,

$$N_r = \left(\int \mathbb{E}|w(\varphi_{T_0,t}(x), T_0)|^r\, dx\right)^{1/r} = \|w(T_0)\|_r \qquad (r \ge 1),$$

and

$$\tilde{N} = \int \mathbb{E}\left(\exp\left(q\int_{B\cap[T_0,t]} |\nabla u(\varphi_{s,t}(x), s)|\, ds\right) - 1\right)^q dx.$$

Since the Orlicz norm satisfies the triangle inequality,

$$\left\|\int_{B\cap[T_0,t]} |\nabla u(\varphi_{s,t}(\cdot), s)|\, ds\right\|_\Phi \le \frac{1}{q},$$

that is, $\tilde{N} \le (e-1)^q$. Since $a^q + b^q \le (a+b)^q$ for $a, b \ge 0$, we conclude that

$$\|w(t)\|_q \le \|w(T_0)\|_q + (e-1)e^{c_2\lambda(t-T_0)}\|w(T_0)\|_{qq'},$$

and the result follows. □

Lemma 3.2. *There is a constant $c > 0$ such that if f is a measurable function, then*

$$\|f\|_\Phi \le c\left(\|f\|_q + \frac{\|f\|_\infty}{1+\Phi^{-1}((\|f\|_\infty/\|f\|_q)^q)}\right).$$

Proof. Let us assume that $\|f\|_\infty = 1$, and set $a = \|f\|_q$, $b = \Phi^{-1}(a^{-q})$ and $n = a + 1/(1+b)$. Let $f^* : [0,\infty] \to [0,\infty]$ be the non-increasing rearrangement of $|f|$, that is,

$$f^*(t) = \sup\{\lambda > 0 : |\{x : |f(x)| > \lambda\}| > t\},$$

so $\int F(|f(x)|)\,dx = \int_0^\infty F(f^*(t))\,dt$ for any Borel measurable function F. Notice that $f^*(t) \le \min\{1, at^{-1/q}\}$.

Let us first consider the case $a \le 1$, so that $b \ge 1$, $2n \ge 1/b$, and $n \ge a$. Then

$$\int \Phi(|f(x)|/2n)\,dx \le \int_0^\infty \Phi(f^*(t)/2n)\,dt.$$

We split this integral up into three pieces. First,

$$\int_0^{a^q} \Phi(f^*(t)/2n)\,dt \le \int_0^{a^q} \Phi(b)\,dt = 1.$$

Next, since $(\Phi(\lambda))^{1/2q}$ is convex for $\lambda \ge 1$,

$$\begin{aligned}\int_{a^q}^{a^qb^q} \Phi(f^*(t)/2n)\,dt &\le \int_{a^q}^{a^qb^q} \Phi(abt^{-1/q})\,dt \\ &\le \int_{a^q}^{a^qb^q} \frac{a^{2q}\Phi(b)}{t^2}\,dt \\ &\le 1.\end{aligned}$$

Next, for $t \ge a^qb^q$, $f^*(t) \le 1/b \le 2n$, and $\Phi(\lambda) \le \lambda^q$ for $0 \le \lambda \le 1$, so

$$\int_{a^qb^q}^\infty \Phi(f^*(t)/2n)\,dt \le \int_{a^qb^q}^\infty (f^*(t)/2n)^q\,dt \le 1.$$

Since $\Phi(\lambda/3) \le \Phi(\lambda)/3$ for $\lambda \ge 0$,

$$\int \Phi(|f(x)|/6n)\,dx \le 1,$$

that is, $\|f\|_\Phi \le 6n$.

The case $a \ge 1$ (so $b \le 1$ and $2n \ge 1 + 2a$) is simpler, as it is easy to estimate

$$\int_0^\infty \Phi(f^*(t)/2n)\,dt \le \int_0^1 \Phi(1)\,dt + \int_1^\infty (f^*(t)/2n)^q\,dt \le 2.$$

□

The following result is due to Grujić and Kukavica [6].

Theorem 3.3. *There exist constants $a, c > 0$ and a function $T : (0,\infty) \to (0,\infty)$, with $T(\lambda) \to \infty$ as $\lambda \to 0$, with the following properties. If $u_0 \in L_q(\mathbb{R}^3)$, then there is a solution $u(t)$ $(0 \le t \le T(\|u_0\|_q))$ to the Navier-Stokes equation, with $u(0) = u_0$, and $u(x,t)$ is the restriction of an analytic function $u(x+iy,t) + iv(x+iy,t)$ in the region $\{x + iy \in \mathbb{C}^3 : |y| \le a\sqrt{t}\}$, and $\|u(\cdot + iy, t) + iv(\cdot + iy, t)\|_q \le c\|u_0\|_q$ for $|y| \le a\sqrt{t}$.*

This next result is related to Scheffer's Theorem [12] that states that the Hausdorff dimension of the set of t for which the solution $u(t)$ is not regular is $1/2$.

Lemma 3.4. *There exists constants $0 < c_5 < 1 < c_6$, such that if $0 < r < 1$, and $u(t)$ is a regular solution to the Navier-Stokes equation with*

$$|\{t \in [t_0 - r^2, t_0] : \|u(t)\|_q \ge c_5 r^{-2/p}\}| < c_5 r^2,$$

then

$$\|\nabla u(t_0)\|_\Phi + \Theta(\|\nabla u(t_0)\|_\infty) < c_6 \Theta(r^{-2}).$$

Proof. Let us first consider the case when $t_0 = 0$ and $r = 1$. By hypothesis, we see that there exists $t \in [-1, -1+c_5]$ with $\|w(t)\|_2 < c_5$. By Theorem 3.3 and the appropriate Cauchy integrals, if c_5 is small enough, then there exists a constant $c_7 > 0$ such that $\|\nabla u(0)\|_q < c_7$ and $\|\nabla u(0)\|_\infty < c_7$

Next, by replacing $u(x,t)$ by $r^{-1}u(r^{-1}x, r^{-2}(t-t_0))$, we can relax the restriction $r = 1$ and $t_0 = 0$, and we see that $\|\nabla u(t_0)\|_q < c_7 r^{-2/p-1}$ and $\|\nabla u(t_0)\|_\infty < c_7 r^{-2}$. Finally, the result follows from Lemma 3.2, and some simple estimates. □

Proof of Lemma 3.1. Let c_5 and c_6 be as in Lemma 3.4. Define the sets

$$A_\mu = \{t \in [1/\Theta^{-1}(\mu), T] : \|\nabla u(t)\|_\Phi + \Theta(\|\nabla u(t)\|_\infty) \ge c_6 \mu\},$$

and

$$B_\mu = \{t \in [0,T] : \Theta(\|u(t)\|_q^p) \ge \mu\}.$$

Notice that

$$\int_{\{t\in[0,T]:\, \|u(t)\|_q \ge \lambda^{1/p}\}} \Theta(\|u(t)\|_q^p)\, dt = \Theta(\lambda)|B_{\Theta(\lambda)}| + \int_{\Theta(\lambda)}^\infty |B_\mu|\, d\mu,$$

and similarly if $c_2 > 0$ is chosen to always exceed $\mu^{-1}\Theta^{-1}(c_6\Theta(\mu))$ for $\mu > 0$, then

$$c_6^{-1} \int_{\{t\in[\lambda^{-1},T]:\, \|\nabla u(t)\|_\infty \ge c_2\lambda\}} \|\nabla u(t)\|_\Phi\, dt \le \Theta(\lambda)|A_{\Theta(\lambda)}| + \int_{\Theta(\lambda)}^\infty |A_\mu|\, d\mu.$$

Thus it is sufficient to show the existence of constants $c_8, c_9 > 0$ such that for $\mu > 1$ that $|A_\mu| < c_8 |B_{c_9\mu}|$. Define r by the relation $\mu = \Theta(r^{-2})$. Note that $0 < r < 1$. Then

$$A_\mu = \{t \in [r^2, T] : \|\nabla u(t)\|_\Phi + \Theta(\|\nabla u(t)\|_\infty) \ge c_6 \Theta(r^{-2})\},$$

and for sufficiently small c_9

$$B_{c_9\mu} \supset \{t \in [0,T] : \|u(t)\|_q \ge c_5 r^{-2/p}\}.$$

It is trivial to find a finite collection $t_1, \dots, t_N$ in the closure of A_r such that the disjoint sets $(t_n - r^2, t_n]$ cover A_r. Furthermore, by Lemma 3.4

$$|\{t \in [t_n - r^2, t_n] : \|u(t)\|_q \ge c_5 r^{-2/p}\}| < c_5 r^2.$$

Hence

$$|A_\mu| \le N r^2 < c_5^{-1} \sum_{n=1}^N |[t_n - r^2, t_n] \cap B_{c_9\mu}| \le c_5^{-1} |B_{c_9\mu}|.$$

$\square$

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References

[1] M. Cannone, Ondelettes, paraproduits et Navier-Stokes, (French) [Wavelets, paraproducts and Navier-Stokes], with a preface by Yves Meyer, Diderot Editeur, Paris, 1995.

[2] P. Constantin, An Eulerian-Lagrangian approach to the Navier-Stokes equations, Comm. Math. Phys. 216 (2001), 663–686.

[3] P. Constantin and C. Foias, Navier-Stokes equations, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1988.

[4] C.R. Doering and J.D. Gibbon, Applied analysis of the Navier-Stokes equations, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 1995.

[5] L. Escauriaza, G. Seregin and V. Sverák, On $L_{3,\infty}$-solutions to the Navier-Stokes equations and backward uniqueness, preprint, `http://www.ima.umn.edu/preprints/dec2002/dec2002.html`.

[6] Z. Grujić and I. Kukavica, Space analyticity for the Navier-Stokes and related equations with initial data in L^p, J. Funct. Anal. 152 (1998), 447–466.

[7] I. Karatzas and S.E. Shreve, Brownian motion and stochastic calculus, second edition. Graduate Texts in Mathematics, 113, Springer-Verlag, New York, 1991.

[8] M.A. Krasnosel'skiĭ and Ja.B. Rutickiĭ, Convex functions and Orlicz spaces, translated from the first Russian edition by Leo F. Boron, P. Noordhoff Ltd., Groningen 1961.

[9] P.G. Lemarié-Rieusset, Recent developments in the Navier-Stokes problem, Chapman and Hall/CRC, 2002.

[10] S.J. Montgomery-Smith and M. Pokorný, A counterexample to the smoothness of the solution to an equation arising in fluid mechanics, Comment. Math. Univ. Carolin. 43 (2002), 61–75.

[11] G. Prodi, Un teorema di unicità per le equazioni di Navier-Stokes, Ann. Mat. Pura Appl. (4) 48 (1959) 173–182.

[12] V. Scheffer, Turbulence and Hausdorff dimension, Turbulence and Navier-Stokes equations (Proc. Conf., Univ. Paris-Sud, Orsay, 1975), 174–183, Lecture Notes in Math., Vol. 565, Springer, Berlin, 1976.

[13] J. Serrin, On the interior regularity of weak solutions of the Navier-Stokes equations, Arch. Rational Mech. Anal. 9 (1962), 187–195.

[14] H. Sohr, Zur Regularitätstheorie der instationären Gleichungen von Navier-Stokes, Math. Z. 184 (1983), no. 3, 359–375.

[15] R. Temam, Infinite-dimensional dynamical systems in mechanics and physics, second edition, Applied Mathematical Sciences, 68, Springer-Verlag, New York, 1997.

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