THE BEST CONSTANTS FOR OPERATOR LIPSCHITZ FUNCTIONS ON SCHATTEN CLASSES

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ABSTRACT. Suppose that f is a Lipschitz function on $\mathbb{R}$ with $\|f\|_{\text{Lip}} \leq 1$. Let A be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$. Let $p \in (1, \infty)$ and suppose that $x \in B(\mathcal{H})$ is an operator such that the commutator $[A, x]$ is contained in the Schatten class $\mathcal{S}_p$. It is proved by the last two authors, that then also $[f(A), x] \in \mathcal{S}_p$ and there exists a constant C_p independent of x and f such that

$$\|[f(A), x]\|_p \leq C_p \|[A, x]\|_p.$$

The main result of this paper is to give a sharp estimate for C_p in terms of p. Namely, we show that $C_p \sim \frac{p^2}{p-1}$. In particular, this gives the best estimates for operator Lipschitz inequalities.

We treat this result in a more general setting. This involves commutators of n self-adjoint operators $A_1, \ldots, A_n$, for which we prove the analogous result. The case described here in the abstract follows as a special case.

1. INTRODUCTION

Recently, the last two authors proved that Lipschitz functions on $\mathbb{R}$ act as operator Lipschitz functions on the Schatten classes $\mathcal{S}_p$ for all $p \in (1, \infty)$, see [18], [21]. That is, suppose that $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function and

$$\|f\|_{\text{Lip}} = \sup_{\xi, \tilde{\xi} \in \mathbb{R}} \frac{|f(\xi) - f(\tilde{\xi})|}{\|\xi - \tilde{\xi}\|_1} \leq 1.$$

Let $p \in (1, \infty)$. Suppose that A, B are bounded, self-adjoint operators such that $A - B \in \mathcal{S}_p$. Then, it was proved in [18] that also $f(A) - f(B) \in \mathcal{S}_p$ and there is a constant $C_p < \infty$ independent of A, B and f such that

$$(1.1) \qquad \|f(A) - f(B)\|_p \leq C_p \|A - B\|_p.$$

We denote C_p for the minimal constant for which the inequality (1.1) holds.

For the case $p = 1$, the analogous result fails. That is, there is no constant C_1 such that the inequality (1.1) holds as was proved in [5]. For the case $p = \infty$ the analogous statement also fails as was proved in [13].

This raises the question of what the growth order of C_p is as p approaches either 1 or ∞. In [14] it was proved that $C_p \preceq p^8$ as $p \to \infty$ and $C_p \preceq (p-1)^{-8}$ as $p \downarrow 1$. In fact, in [14] a more

general result is covered involving an n-tuple of commuting self-adjoint (bounded) operators. We refer to [14, Theorem 5.3] for the precise statement.

In [18] an estimate for the asymptotic behavior of C_p was not mentioned explicitly. However, it is in principle possible to find an upper estimate for C_p from the proof presented in [18]. These proofs involve the Marcinkiewicz multiplier theorem as well as diagonal truncation and do not lead to a sharp upper estimate of C_p.

The main result of this paper is a sharp estimate for C_p. Namely, we prove that $C_p \sim p$ as $p \to \infty$ and we prove that $C_p \sim (p - 1)^{-1}$ as $p \downarrow 1$. Our result is stated in terms of commuator estimates in Schatten classes. In particular, it sharpens the estimates found in [14] for n-tuples of commuting self-adjoint operators.

The novelty of our proof is that we apply the main result of [6]. In [6] sharp estimates were found for the action of a smooth, even multiplier that acts on vector valued L^p-spaces. The norm of such a multiplier can be expressed in terms of the UMD-constant of a Banach space (we recall the definition below). This result together with the so-called transference method forms the key argument that allows us to improve the known estimates for C_p.

This paper relates to the general interest of finding the best constants in non-commutative probability inequalities. In particular, major achievements have been made considering the best constants of Burkholder/Gundy inequalities [12], [20], Doob and Stein inequalities [12] and Khintchine inequalities [9], [10].

The structure of this paper is as follows. Section 2 recalls the necessary theory on Fourier multipliers. In Section 3 we construct a special multiplier that forms a key step for our main result. In Section 4 we recall the theory of double operator integrals and prove the necessary lemmas on discrete approximations. Section 5 contains our main result and the core of our proof.

General conventions. For $p \in [1, \infty)$ we write $\mathcal{S}_p$ for the Schatten-von Neumann classes. These are the non-commutative L^p-spaces associated with the bounded operators on a Hilbert space $\mathcal{H}$ with respect to the standard trace τ. For $p \in (1, \infty)$ we use $p' \in (1, \infty)$ to denote the conjugate exponent, which is defined by $\frac{1}{p} + \frac{1}{p'} = 1$. We use χ to denote an indicator function.

Let $C_p, D_p \in \mathbb{R}^+$ be constants depending on $p \in (1, \infty)$. We write

$$C_p \sim D_p,$$

if there are constants a, b such that $a \leq C_p/D_p \leq b$ for all $p \in (1, \infty)$. In particular, in this case C_p and D_p have the same asymptotic behavior as $p \to \infty$ or $p \downarrow 1$.

2. Construction of Fourier multipliers

We recall the theory of L^p-Fourier multipliers and their vector valued counterparts. In this section $\xi = (\xi_1, \ldots, \xi_n)$ is always a vector in $\mathbb{R}^n$. μ is always a number in $\mathbb{R}$.

2.1. **Multipliers.** A function $m \in L^\infty(\mathbb{R}^n)$ defines a bounded linear map:

$$T_m : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) : f \mapsto (\mathcal{F}_2^{-1} \circ m \circ \mathcal{F}_2)(f).$$

Here, $\mathcal{F}_2 : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is the Fourier transform that is defined for $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ as:

$$(\mathcal{F}_2 f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^n} f(s) e^{-is\cdot\xi} ds.$$

Let $p \in [1, \infty)$. Suppose that for every $f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ we have $T_m(f) \in L^p(\mathbb{R}^n)$ and

$$(2.1) \qquad\qquad T_m : L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$$

extends to a bounded map on $L^p(\mathbb{R}^n)$. Then, we call m an L^p-multiplier and we keep denoting the extension of (2.1) by T_m. m is called *homogeneous* (of degree 0) if for every $\lambda \in \mathbb{R}^+$ we have $m(\lambda\xi) = m(\xi)$. We call m *even* if $m(-\xi) = m(\xi)$. m is called *odd* if $m(-\xi) = -m(\xi)$. A homogeneous L^p-multiplier is called smooth if it is smooth on the unit sphere in $\mathbb{R}^n$ (or equivalently, if it is smooth on $\mathbb{R}^n \backslash \{0\}$).

Remark 2.1. Let $p \in [1, \infty)$. By definition, the set of L^p-multipliers forms an algebra.

2.2. **Vector valued multipliers.** We are mostly concerned with vector valued counterparts of L^p-multipliers. Let E be a Banach space. Let (X, μ) be a σ-finite measure space and $p \in [1, \infty)$. Let $L_E^p(X) = L_E^p(X, \mu)$ denote the space of strongly measurable functions $f : X \to E$ for which there is a separable subspace $E_0 \subseteq E$ such that $f(X) \subseteq E_0$ and

$$\|f\|_{L_E^p(X)} := \left(\int_X \|f(x)\|^p d\mu(x) \right)^{\frac{1}{p}} < \infty.$$

Let $T : L^p(X) \to L^p(X)$ be a bounded operator. Then, T defines a linear map on the simple functions of $L_E^p(X)$ which we denote for the moment by

$$T(E) : \sum_{k=1}^n x_k \chi_{A_k} \mapsto \sum_{k=1}^n x_k T(\chi_{A_k}), \qquad x_k \in E,\ A_k \subseteq X \text{ measurable.}$$

In case,

$$\sup \left\{ \|T(E)(f)\| \mid f \text{ simple and } \|f\|_{L_E^p(X)} \leq 1 \right\}$$

is finite, the map $T(E)$ extends to a bounded map $T(E) : L_E^p(X) \to L_E^p(X)$.

Notation 2.2. We write T for $T(E) : L_E^p(X) \to L_E^p(X)$. It will always be clear from the context if T acts on $L^p(X)$ or $L_E^p(X)$.

In particular, we apply the previous construction to the special case where (X, μ) is $\mathbb{R}^n$ equipped with the Lebesgue measure, E is a non-commutative L^p-space and T is T_m for some L^p-multiplier m.

Remark 2.3. Suppose that m is a L^p-multiplier such that $T_m : L_E^p(\mathbb{R}^n) \to L_E^p(\mathbb{R}^n)$ is bounded. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear invertible transformation. Then, also $T_{m\circ A} : L_E^p(\mathbb{R}^n) \to L_E^p(\mathbb{R}^n)$ is bounded with the same norm. This follows from the observation (see also [6, p. 557]):

$$(T_{m\circ A} f)(\xi) = (T_m(f \circ A^T))((A^T)^{-1}\xi).$$

Here, A^T is the transpose of A.

Remark 2.4. Let $GL_n(\mathbb{R})$ denote the invertible $n \times n$-matrices. Let $A \in GL_n(\mathbb{R})$. For $f \in L^p_E(\mathbb{R}^n)$, let $A^*(f) = f \circ A$. So A^* defines a bounded map on $L^p_E(\mathbb{R}^n)$, (with bound given by the determinant of A^{-1}, as follows from a substitution of variables). Let

$$[0, 1] \to GL_n(\mathbb{R}) : t \mapsto A_t$$

be a continuous path. Then, $t \mapsto A_t^*$ is a strongly continous path with values in the bounded operators on $L^p_E(\mathbb{R}^n)$. Indeed, one can check that for simple functions $f \in L^p_E(\mathbb{R}^n)$ the path $t \mapsto A_t^*(f)$ is continous and then use the fact that $t \mapsto \|A_t^*\|$ is bounded. Let m be a L^p-multiplier and let E be a Banach space such that $T_m : L^p_E(\mathbb{R}^n) \to L^p_E(\mathbb{R}^n)$ is bounded. Then, the strong integral $\int_0^1 T_{m \circ A_t} dt$ exists and defines a bounded map on $L^p_E(\mathbb{R}^n)$.

2.3. **Discrete multipliers.** Let $m \in L^\infty(\mathbb{R}^n)$ be an odd L^p-multiplier. Put $\bar{m} \in L^\infty(\mathbb{Z}^n)$ by setting $\bar{m}(k) = m(k), k \in \mathbb{Z}^n \backslash \{0\}$ and $\bar{m}(0) = 0$. For f a finite trigonometric polynomial, we set

$$(T_{\bar{m}} f)(\theta) = \sum_{k \in \mathbb{Z}^n} \bar{m}(k) \hat{f}(k) e^{ik\theta}.$$

Here, $\hat{f}(k) = \int_{\mathbb{T}^n} f(\theta) e^{-ik \cdot \theta} d\theta$, where the n-torus $\mathbb{T}^n$ is considered with the normalised Lebesgue measure. The following theorem gives a sufficient condition on m in order to extend $T_{\bar{m}}$ to a bounded map on $L^p(\mathbb{T}^n)$.

Theorem 2.5 (Theorem 3.6.7 of [7]). *Let $m \in L^\infty(\mathbb{R}^n)$ be an odd function that is smooth on $\mathbb{R}^n \backslash \{0\}$. Let $p \in [1, \infty)$ and suppose that m is a L^p-multiplier. Let E be a Banach space and suppose that $T_m : L^p_E(\mathbb{R}^n) \to L^p_E(\mathbb{R}^n)$ is bounded. Then,*

$$(2.2) \qquad \|T_{\bar{m}} : L^p_E(\mathbb{T}^n) \to L^p_E(\mathbb{T}^n)\| \leq \|T_m : L^p_E(\mathbb{R}^n) \to L^p_E(\mathbb{R}^n)\|.$$

Remark 2.6. Theorem 2.5 was proved for $E = \mathbb{C}$ in [7, Theorem 3.6.7]. For a general Banach space E, the statement follows from a mutatis mutandis copy of its proof. The exact statement of Theorem 2.5 can also be found as [6, Lemma 2.2].

2.4. **The UMD-property.** Let E be a Banach space. E is said to have the *UMD-property* (Unconditional Martingale Differences) if there exists a constant $C_p(E)$ with $p \in (1, \infty)$ such that for every probability measure space (Ω, Σ, μ) and every sequence of σ-subalgebras $B_1 \subseteq B_2 \subseteq \ldots \subseteq \Sigma$ and every martingale difference sequence $\{d_n\}_{n=1}^\infty$ with respect to $\{B_n\}_{n=1}^\infty$ in $L^p_E(\Omega)$, the sequence $\{d_n\}_{n=1}^\infty$ satisfies:

$$(2.3) \qquad \|\sum_{k=1}^n \epsilon_k \alpha_k d_k\|_{L^p_E(\Omega)} \leq C_p(E) \|\sum_{k=1}^n \alpha_k d_k\|_{L^p_E(\Omega)}$$

for every $\epsilon_k = \pm 1$ and scalars $\{\alpha_k\}_{k=1}^\infty$ and all $n = 1, 2, \ldots$. We will denote the minimal constant $C_p(E)$ for which (2.3) holds by $\mathrm{UMD}_p(E)$. This constant is also called the *UMD-constant* of E.

Theorem 2.7 (Theorem 4.3 and Remark 4.4 of [19]). *The Schatten class $\mathcal{S}_p$ is a UMD-space for every $p \in (1, \infty)$. Moreover,*

$$\mathrm{UMD}_p(\mathcal{S}_p) \sim \frac{p^2}{p - 1}.$$

Remark 2.8. Theorem 2.7 is also valid if $\mathcal{S}_p$ is replaced by a non-commutative L^p-space assocated with an arbitrary von Neumann algebra M. This particularly applies to Haagerup L^p-spaces associated with a non-semi-finite von Neumann algebra M, see [19]. For Haagerup L^p-spaces we refer to [8], [22].

2.5. **The Hilbert transform.** Consider the function $h : \mathbb{R}^n \to \mathbb{C} : \xi \mapsto i \operatorname{sign}(\xi_1)$ where we use the convention $\operatorname{sign}(0) = 0$. Let E be a UMD-space. Then, for every $p \in (1, \infty)$,

$$(2.4) \qquad\qquad T_h : L^p_E(\mathbb{R}^n) \to L^p_E(\mathbb{R}^n)$$

is bounded. In fact, E is a UMD-space if and only if (2.4) is bounded for every $p \in (1, \infty)$ [1], [2]. T_h is also called the *Hilbert transform*, see also [7, Chapter 4].

2.6. **The Riesz transform.** Consider the function $r_j : \mathbb{R}^n \to \mathbb{C} : \xi \mapsto i \frac{\xi_j}{\|\xi\|_2}$ where we use the assumption $r_j(0) = 0$. Let E be a UMD-space. Then, for every $p \in (1, \infty)$,

$$(2.5) \qquad\qquad T_{r_j} : L^p_E(\mathbb{R}^n) \to L^p_E(\mathbb{R}^n)$$

is bounded.

T_{r_j} is also called the *Riesz transform*, see also [7, Chapter 4].

3. Construction of a special multiplier

The goal of this section is to construct a specific smooth homogeneous even multiplier m_j. This multiplier plays an essential role in Section 5. In this section $\xi = (\xi_1, \ldots, \xi_n)$ is always a vector in $\mathbb{R}^n$. μ is always a number in $\mathbb{R}$.

Lemma 3.1. *For $1 \le j \le n$, there exists a function $m_{1,j}$ on $\mathbb{R}^{n+1}$ such that*

$$(3.1) \qquad\qquad m_{1,j}(\xi, \mu) = \begin{cases} \frac{\mu}{\|\xi\|_2} \frac{\xi_j}{\|\xi\|_2} & \text{if } \frac{|\mu|}{\|\xi\|_1} < 1, \\ 0 & \text{if } \frac{|\mu|}{\|\xi\|_1} > 1, \end{cases}$$

and moreover, such that for every $p \in (1, \infty)$ and UMD-space E, the map $T_{m_{1,j}} : L^p_E(\mathbb{R}^{n+1}) \to L^p_E(\mathbb{R}^{n+1})$ is bounded.

Proof. Recall that we use the convention $\operatorname{sign}(0) = 0$. Let $h(\xi) = i \operatorname{sign}(\xi_1)$, see also Section 2.5. We start with observing that

$$(3.2) \qquad\qquad \frac{1}{2}(-ih(\xi) + 1) = \begin{cases} 1 & \text{if } \xi_1 > 0, \\ \frac{1}{2} & \text{if } \xi_1 = 0, \\ 0 & \text{if } \xi_1 < 0. \end{cases}$$

Hence, $\frac{1}{2}(-ih(\xi) + 1)$ is equal to the indicator function $\chi_{[0,\infty)}(\xi_j)$ except for the point 0. Let

$$h_{a,b}(\xi, \mu) = i \operatorname{sign}(b\xi_1 + \ldots + b\xi_n + a\mu)$$

be the multiplier associated with the Hilbert transform (2.4) precomposed with the linear map

$$A_{a,b} : (\xi_1, \ldots, \xi_n, \mu) \mapsto (b\xi_1 + \ldots + b\xi_n + a\mu, \xi_2, \ldots, \xi_n, \mu),$$

see Remark 2.3. Put $k = \frac{1}{2} \int_{-1}^{1} h_{1,t} dt$. So,

$$(3.3) \qquad\qquad k(\xi, \mu) = \frac{1}{2} i \int_{-1}^{1} \operatorname{sign}(t\xi_1 + \ldots + t\xi_n + \mu) dt.$$

If $|\mu| \geq |\xi_1 + \ldots + \xi_n|$, then $\text{sign}(t\xi_1 + \ldots + t\xi_n + \mu) = \text{sign}(\mu)$ for every $t \in [-1, 1]$. If $|\mu| \leq |\xi_1 + \ldots + \xi_n|$, then (3.3) is equal to

$$
\begin{aligned}
k(\xi, \mu) &= \frac{1}{2}i \int_{-1}^{-\frac{\mu}{\xi_1+\ldots+\xi_n}} \text{sign}(t\xi_1 + \ldots + t\xi_n + \mu)dt + \frac{1}{2}i \int_{-\frac{\mu}{\xi_1+\ldots+\xi_n}}^{1} \text{sign}(t\xi_1 + \ldots + t\xi_n + \mu)dt \\
&= \frac{1}{2}i \int_{-1}^{-\frac{\mu}{\xi_1+\ldots+\xi_n}} \text{sign}(-\xi_1 - \ldots - \xi_n)dt + \frac{1}{2}i \int_{-\frac{\mu}{\xi_1+\ldots+\xi_n}}^{1} \text{sign}(\xi_1 + \ldots + \xi_n)dt \\
&= i \frac{\mu}{|\xi_1 + \ldots + \xi_n|}.
\end{aligned}
$$

So we conclude,

$$
k(\xi, \mu) = \begin{cases} i \frac{\mu}{|\xi_1+\ldots+\xi_n|} & \text{if } |\mu| \leq |\xi_1 + \ldots + \xi_n|, \\ i\,\text{sign}(\mu) & \text{if } |\mu| \geq |\xi_1 + \ldots + \xi_n|. \end{cases}
$$

For $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{-1, 1\}^n$, put

$$
k_\epsilon(\xi, \mu) = k(\epsilon_1 \xi_1, \ldots, \epsilon_1 \xi_1, \mu) \cdot \Pi_{j=1}^{n} \frac{1}{2}(-ih(\epsilon_j \xi_j) + 1).
$$

We can explicitly describe k_ϵ. The next formula can be determined by first considering the value of $k_\epsilon(\xi, \mu)$ for the case that for every $1 \leq j \leq n$ we have $\xi_j \geq 0$. In that case, keeping in mind (3.2), one arrives at equation (3.4) below. Similarly, we can compute $k_\epsilon(\xi, \mu)$ for other signs of ξ_j. This results in the following expression:

$$
(3.4) \qquad k_\epsilon(\xi, \mu) = \begin{cases} i \frac{1}{2^{l(\xi)}} \frac{\mu}{|\xi_1|+\ldots+|\xi_n|} & \text{if } |\mu| \leq |\xi_1| + \ldots + |\xi_n| \text{ and } \forall j: \epsilon_j \xi_j \geq 0, \\ i \frac{1}{2^{l(\xi)}} \text{sign}(\mu) & \text{if } |\mu| \geq |\xi_1| + \ldots + |\xi_n| \text{ and } \forall j: \epsilon_j \xi_j \geq 0, \\ 0 & \text{else,} \end{cases}
$$

where $l(\xi)$ is the number of coordinates j for which $\xi_j = 0$. Put $K = \sum_{\epsilon \in \{-1,1\}^n} k_\epsilon$. Then, treating again the different possibilities for the signs of ξ_j separately, one computes

$$
(3.5) \qquad\qquad K(\xi, \mu) = \begin{cases} i \frac{\mu}{\|\xi\|_1} & \text{if } |\mu| \leq \|\xi\|_1, \\ i\,\text{sign}(\mu) & \text{if } |\mu| > \|\xi\|_1. \end{cases}
$$

Next, for $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{-1, 1\}^n$. Consider the multiplier:

$$
\begin{aligned}
r_\epsilon(\xi, \mu) =& i \frac{\epsilon \cdot \xi}{\|\xi\|_2} \cdot \Pi_{j=1}^{n} \frac{1}{2}(-ih(\epsilon_j \xi_j) + 1) \cdot \\
& \left(\frac{1}{2}(-ih(\mu) + 1)\frac{1}{2}(-ih(\epsilon \cdot \xi - \mu) + 1) + \frac{1}{2}(-ih(-\mu) + 1)\frac{1}{2}(-ih(\epsilon \cdot \xi + \mu) + 1) \right).
\end{aligned}
$$

Note that for (ξ, μ) to be in the support of r_ϵ, we must have for all $1 \leq j \leq n$ that $\text{sign}(\xi_j) = \epsilon_j$ or $\xi_j = 0$. Moreover, every (ξ, μ) in the support of r_ϵ must satisfy $\|\xi\|_1 \geq \mu$ in case $\mu \geq 0$ and $\|\xi\|_1 \leq \mu$ in case $\mu \leq 0$. Then, with $l(\xi)$ as before, and taking into account that $\epsilon \cdot \xi = \|\xi\|_1$,

$$
r_\epsilon(\xi, \mu) = \begin{cases} i \frac{1}{2^{l(\xi)}} \frac{\|\xi\|_1}{\|\xi\|_2} & \text{if } |\mu| < \|\xi\|_1 \text{ and } \forall j: \epsilon_j \xi_j \geq 0, \\ 0 & \text{if } |\mu| > \|\xi\|_1 \text{ or } \exists j: \epsilon_j \xi_j < 0. \end{cases}
$$

Put $R = \sum_{\epsilon \in \{-1,1\}^n} r_\epsilon$. Then,

(3.6)
$$R(\xi, \mu) = \begin{cases} i\frac{\|\xi\|_1}{\|\xi\|_2} & \text{if } |\mu| < \|\xi\|_1, \\ 0 & \text{if } |\mu| > \|\xi\|_1. \end{cases}$$

Recall the Riesz transform r_j from Section 2.6. We define the multiplier,

$$m_{1,j}(\xi, \mu) = iK(\xi, \mu)R(\xi, \mu)r_j(\xi).$$

Then, it follows from (3.5), (3.6) and (2.5) that $m_{1,j}$ satisfies (3.1).

Let E be a UMD-space and let $p \in (1, \infty)$. By construction of $m_{1,j}$ the map

$$T_{m_{1,j}} : L_E^p(\mathbb{R}^{n+1}) \to L_E^p(\mathbb{R}^{n+1})$$

is bounded as follows from Remarks 2.1 and 2.4 and the fact that the Hilbert transform and Riesz transform are bounded operators on $L_E^p(\mathbb{R}^n)$.

$\square$

Lemma 3.2. *There exists a homogeneous even function m_j on $\mathbb{R}^{n+1}$ that is smooth on $\mathbb{R}^{n+1} \backslash \{0\}$ such that*

(3.7)
$$m_j(\xi, \mu) = \frac{\mu}{\|\xi\|_2} \frac{\xi_j}{\|\xi\|_2} \quad \text{if } \frac{|\mu|}{\|\xi\|_1} \leq 1,$$

and moreover, for every $p \in (1, \infty)$ and UMD-space E, the map $T_{m_j} : L_E^p(\mathbb{R}^{n+1}) \to L_E^p(\mathbb{R}^{n+1})$ is bounded.

Proof. Let $m_{1,j}$ be the L^p-multiplier of Lemma 3.1. For every $\lambda \in \mathbb{R}^+$ the function $m_{\lambda,j} := \frac{1}{\lambda} m_{1,j}(\xi, \lambda\mu)$ is also a L^p-multiplier, see Remark 2.3. Note that,

(3.8)
$$m_{\lambda,j}(\xi, \mu) = \begin{cases} \frac{\mu}{\|\xi\|_2} \frac{\xi_j}{\|\xi\|_2} & \text{if } \lambda |\mu| < \|\xi\|_1, \\ 0 & \text{if } \lambda |\mu| > \|\xi\|_1. \end{cases}$$

Let $s : [0,1] \to [0, \infty)$ be a smooth function with support contained in $[\frac{1}{2}, \frac{3}{4}]$ and $\int_0^1 s(\theta) d\theta = 1$. Set,

(3.9)
$$m_j = \int_0^1 s(\lambda) m_{\lambda,j} \, d\lambda.$$

Consider the areas

$$A_1 = \left\{ (\xi, \mu) \in \mathbb{R}^{n+1} \mid \|\xi\|_1 < \frac{1}{2}|\mu| \right\},$$

$$A_2 = \left\{ (\xi, \mu) \in \mathbb{R}^{n+1} \mid \|\xi\|_1 > \frac{3}{4}|\mu| \right\},$$

$$A_3 = \left\{ (\xi, \mu) \in \mathbb{R}^{n+1} \mid \frac{1}{3}|\mu| < \|\xi\|_1 < |\mu| \right\}.$$

We have $A_1 \cup A_2 \cup A_3 = \mathbb{R}^{n+1} \backslash \{0\}$. Now, we check that m_j is smooth on each of these areas. For $(\xi, \mu) \in A_1$, we find that $m_j(\xi, \mu) = 0$ as follows from (3.8) together with the fact that the support of s is contained in $[\frac{1}{2}, \frac{3}{4}]$. So m_j is smooth in A_1. For $(\xi, \mu) \in A_2$, we find that

$$m_j(\xi, \mu) = \frac{\mu}{\|\xi\|_2} \frac{\xi_j}{\|\xi\|_2}.$$

Indeed, this follows again from (3.8) together with the fact that the support of s is contained in $[\frac{1}{2}, \frac{3}{4}]$. So m_j is smooth on A_2. Since every $(\xi, \mu) \in A_2$ satisfies $\frac{|\mu|}{\|\xi\|_1} \leq 1$, this also proves that m_j satisfies (3.7). Define $S(t) = \int_0^t s(\lambda)d\lambda$, which is a smooth function on the open interval $(0, 1)$. For $(\xi, \mu) \in A_3$ we find that

$$
\begin{aligned}
m_j(\xi, \mu) &= \int_0^{\frac{\|\xi\|_1}{|\mu|}} s(\lambda)m_{\lambda,j}(\xi, \mu)d\lambda + \int_{\frac{\|\xi\|_1}{|\mu|}}^1 s(\lambda)m_{\lambda,j}(\xi, \mu)d\lambda \\
&= \int_0^{\frac{\|\xi\|_1}{|\mu|}} s(\lambda)d\lambda \cdot \frac{\mu}{\|\xi\|_2} \frac{\xi_j}{\|\xi\|_2} + 0 \\
&= S\left(\frac{\|\xi\|_1}{|\mu|}\right) \frac{\mu}{\|\xi\|_2} \frac{\xi_j}{\|\xi\|_2}.
\end{aligned}
$$

Here, the second equality follows from (3.8). The other equalities follow from the definitions. Hence, we see that m_j is smooth on A_3. We conclude that m_j is smooth on $\mathbb{R}^{n+1}\backslash\{0\}$.

Let E be a UMD-space and let $p \in (1, \infty)$. In Lemma 3.1 we proved that $T_{m_{1,j}} : L_E^p(\mathbb{R}^n) \to L_E^p(\mathbb{R}^n)$ is bounded. The definition of T_{m_j} together with Remark 2.4 implies that also $T_{m_j} : L_E^p(\mathbb{R}^n) \to L_E^p(\mathbb{R}^n)$ is bounded. Since $m_{1,j}$ is even, also m_j is even. $\qquad\square$

The following theorem forms the key step in finding the best constants for commutator estimates in Schatten classes.

Theorem 3.3 (see Theorem 3.1 of [6]). *Let $p \in (1, \infty)$ and let E be a UMD-space. There exists a constant C that is independent of p, j and E such that:*

$$\|T_{m_j} : L_E^p(\mathbb{R}^{n+1}) \to L_E^p(\mathbb{R}^{n+1})\| \leq C \cdot \mathrm{UMD}_p(E).$$

Remark 3.4. Let E be a UMD-space and let $p \in (1, \infty)$. Using [6, Proposition 3.8] it follows directly that for any homogenenous even function $m \in L^\infty(\mathbb{R}^{n+1})$ that is smooth on $\mathbb{R}^{n+1}\backslash\{0\}$, the transform $T_m : L_E^p(\mathbb{R}^{n+1}) \to L_E^p(\mathbb{R}^{n+1})$ is bounded. This observation could be used to supply an alternative proof of Lemma 3.2. Here, we have chosen to give a self-contained proof.

4. DOUBLE OPERATOR INTEGRALS

The goal of this section is to recall the basic notions of double operator integrals [16]. We prove the necessary results in order to see that certain double operator integrals may be approximated by discrete versions of double operator integrals. Troughout this section, $\xi = (\xi_1, \ldots, \xi_n), \tilde\xi = (\tilde\xi_1, \ldots, \tilde\xi_n)$ are vectors in $\mathbb{R}^n$. Recall that τ denotes the standard semi-finite trace on the bounded operators on a Hilbert space $\mathcal{H}$.

Let E be a spectral measure on $\mathbb{R}^n$ having finite support taking values in the orthogonal projections on a Hilbert space $\mathcal{H}$. E generates an n-tuple of commuting self-adjoint bounded operators

$$(4.1) \qquad \mathcal{A} = (A_1, \ldots, A_n), \qquad \text{with } A_k = \int_{\mathbb{R}^n} \xi_k dE(\xi).$$

We also set

$$f(\mathcal{A}) = \int_{\mathbb{R}^n} f(\xi) dE(\xi).$$

Let $A, B \subseteq \mathbb{R}^n$ be measurable subsets. The mapping $(E \otimes E)(A \times B)(x) = E(A)xE(B), x \in \mathcal{S}_2$ defines an orthogonal projection on $\mathcal{S}_2$. The mapping naturally extends to a spectral measure on the Borel sets of $\mathbb{R}^n \times \mathbb{R}^n$. We denote this measure by F.

Let $\phi : \mathbb{R}^n \times \mathbb{R}^n$ be a bounded Borel function. The mapping

$$\mathcal{I}_\phi = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi(\xi, \tilde{\xi}) dF(\xi, \tilde{\xi}),$$

defines a bounded operator on $\mathcal{S}_2$. $\mathcal{I}_\phi$ is called the *double operator integral* of ϕ with respect to the measure E. If $\mathcal{I}_\phi : \mathcal{S}_2 \cap \mathcal{S}_p \to \mathcal{S}_p$ admits a bounded extension to $\mathcal{S}_p$, then we keep denoting this map with $\mathcal{I}_\phi$. Suppose that $\phi(\xi, \tilde{\xi}) = f(\xi) - f(\tilde{\xi})$ for a Borel function f on $\mathbb{R}^n$. Then,

$$(4.2) \qquad \mathcal{I}_\phi(x) = f(\mathcal{A})x - xf(\mathcal{A}), \qquad x \in \mathcal{S}_2.$$

Define $\delta(\xi, \tilde{\xi}) = 1$ if $\xi = \tilde{\xi}$ and $\delta(\xi, \tilde{\xi}) = 0$ if $\xi \neq \tilde{\xi}$. An element $x \in \mathcal{S}_2$ is called *off-diagonal* (with respect to E) if

$$\mathcal{I}_\delta(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \delta(\xi, \tilde{\xi}) F(\xi, \xi)(x) = 0.$$

For $x, y \in \mathcal{S}_2$, we define a finite measure on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$\nu_{y,x}(\Omega) := \tau(y \mathcal{I}_{\chi_\Omega} x), \qquad \Omega \subseteq \mathbb{R}^n \times \mathbb{R}^n \text{ a Borel set.}$$

Remark 4.1. In case the operators $A_1, \ldots, A_n$ are unbounded, the analysis below becomes much more intricate. One has to treat the domains of the various operators and commutators very carefully, see for example [17].

Lemma 4.2. *Let B_j $1 \leq j \leq n$ be a tuple of bounded commuting operators. Similarly, let C_j $1 \leq j \leq n$ be a tuple of bounded commuting operators. Put $\mathcal{B} = (B_1, \ldots, B_n)$ and $\mathcal{C} = (C_1, \ldots, C_n)$. Then, for all $s \in \mathbb{R}^n$,*

$$\|e^{is \cdot \mathcal{B}} - e^{is \cdot \mathcal{C}}\| \leq \sum_{j=1}^n |s_j| \|B_j - C_j\|,$$

where we use the notation $s \cdot \mathcal{B} = s_1 B_1 + \ldots + s_n B_n$.

Proof. Using Duhamel's formula [23], see also [18, Lemma 8] with $r = 1$, one finds that for any two self-adjoint operators B and C we have

$$(4.3) \qquad \|e^{iB} - e^{iC}\| \leq \|B - C\|.$$

Therefore, using first the triangle inequality and then (4.3),

$$\|e^{is \cdot \mathcal{B}} - e^{is \cdot \mathcal{C}}\| \leq \sum_{j=1}^n \|e^{i(s_1 C_1 + \ldots + s_{j-1} C_{j-1} + s_j B_j + \ldots + s_n B_n)} - e^{i(s_1 C_1 + \ldots + s_j C_j + s_{j+1} B_{j+1} + \ldots + s_n B_n)}\|$$

$$\leq \sum_{j=1}^n |s_j| \|B_j - C_j\|.$$

$\square$

For $l \in \mathbb{N}^*$, let

$$U_l = \{(\xi, \tilde{\xi}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \|\xi - \tilde{\xi}\|_2 < \frac{1}{l}\}.$$

Then, U_l is an open neighbourhood of the diagonal of $\mathbb{R}^n \times \mathbb{R}^n$.

Proposition 4.3. *Let $p \in (1, \infty)$. Let $y \in \mathcal{S}_p \cap \mathcal{S}_2$ be such that there exists a $U_l, l \in \mathbb{N}^*$ such that we have $\int_{U_l} dF(\xi, \tilde{\xi})(y) = 0$. Let $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be such that there exists a Schwartz function $\phi_0 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ for which $\phi(\xi, \tilde{\xi}) = \phi_0(\xi, \tilde{\xi})$ for every $(\xi, \tilde{\xi}) \in \mathbb{R}^n \times \mathbb{R}^n \backslash U_{l+1}$. For $m \in \mathbb{Z}$ define a discrete spectral measure E_m*

$$E_m(\Omega) = \sum_{k \in \mathbb{Z}^n, \text{s.t.} \frac{k}{m} \in \Omega} E\left(\left[\frac{k_1}{m}, \frac{k_1 + 1}{m}\right) \times \ldots \times \left[\frac{k_n}{m}, \frac{k_n + 1}{m}\right)\right), \qquad \Omega \subseteq \mathbb{R}^n \ a \ Borel \ set.$$

Consider the double operator integral $\mathcal{I}_\phi$ of ϕ with respect to E. And similarly, let $\mathcal{I}_\phi^m$ be the double operator integral of ϕ with respect to E_m. Then, for any $z \in \mathcal{S}_{p'} \cap \mathcal{S}_2$,

$$\tau\left(z\mathcal{I}_\phi^m y\right) \to \tau(z\mathcal{I}_\phi y), \qquad as \quad m \to \infty.$$

Proof. Let $\mathcal{I}_{\phi_0}$ and $\mathcal{I}_{\phi_0}^m$ be the double operator integrals of ϕ_0 with respect to E and respectively E_m. Our assumption on y and ϕ_0 implies that

$$(4.4) \qquad \begin{aligned} \mathcal{I}_\phi(y) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi(\xi, \tilde{\xi}) dF(\xi, \tilde{\xi})(y) = \int_{U_l} \phi(\xi, \tilde{\xi}) dF(\xi, \tilde{\xi})(y) \\ &= \int_{U_l} \phi_0(\xi, \tilde{\xi}) dF(\xi, \tilde{\xi})(y) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi_0(\xi, \tilde{\xi}) dF(\xi, \tilde{\xi})(y) = \mathcal{I}_{\phi_0}(y). \end{aligned}$$

For $k \in \mathbb{Z}^n$ and $m \in \mathbb{N}$, we set $p_{k,m} = E_m(\frac{k}{m})$. Let m be large (in fact $m \geq \sqrt{n}l(l+1)$ suffices) and let $k, \tilde{k} \in \mathbb{Z}^n$ be such that $(\frac{k}{m}, \frac{\tilde{k}}{m}) \in U_{l+1}$. Then, since $\int_{U_l} dF(\xi, \tilde{\xi})(y) = 0$ we have that $p_{k,m} y p_{\tilde{k},m} = 0$. Hence, we compute

$$(4.5)$$

$$\begin{aligned} \mathcal{I}_\phi^m(y) &= \sum_{k,\tilde{k} \in \mathbb{Z}^n} \phi\left(\frac{k}{m}, \frac{\tilde{k}}{m}\right) p_{k,m} y p_{\tilde{k},m} = \sum_{k,\tilde{k} \in \mathbb{Z}^n, (\frac{k}{m}, \frac{\tilde{k}}{m}) \notin U_{l+1}} \phi\left(\frac{k}{m}, \frac{\tilde{k}}{m}\right) p_{k,m} y p_{\tilde{k},m} \\ &= \sum_{k,\tilde{k} \in \mathbb{Z}^n, (\frac{k}{m}, \frac{\tilde{k}}{m}) \notin U_{l+1}} \phi_0\left(\frac{k}{m}, \frac{\tilde{k}}{m}\right) p_{k,m} y p_{\tilde{k},m} = \sum_{k,\tilde{k} \in \mathbb{Z}^n} \phi_0\left(\frac{k}{m}, \frac{\tilde{k}}{m}\right) p_{k,m} y p_{\tilde{k},m} = \mathcal{I}_{\phi_0}^m(y). \end{aligned}$$

Let $\hat{\phi}_0$ be the Fourier transform of ϕ_0. Then,

$$\phi_0(\xi, \tilde{\xi}) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\phi}_0(s, \tilde{s}) e^{is \cdot \xi} e^{i\tilde{s} \cdot \tilde{\xi}} ds d\tilde{s}.$$

This implies that

$$\mathcal{I}_{\phi_0}(y) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\phi}_0(s, \tilde{s}) e^{is \cdot \mathcal{A}} y e^{i\tilde{s} \cdot \mathcal{A}} ds d\tilde{s}.$$

Let $A_j^m = \int_{\mathbb{R}^n} \xi_j dE_m(\xi)$ and set $\mathcal{A}_m = (A_1^m, \ldots, A_n^m)$. Then,

$$
\begin{aligned}
(4.6) \quad \mathcal{I}_{\phi_0}(y) - \mathcal{I}_{\phi_0}^m(y) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\phi}_0(s, \tilde{s})(e^{is\cdot\mathcal{A}} y e^{i\tilde{s}\cdot\mathcal{A}} - e^{is\cdot\mathcal{A}_m} y e^{i\tilde{s}\cdot\mathcal{A}_m}) ds d\tilde{s} \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\phi}_0(s, \tilde{s})(e^{is\cdot\mathcal{A}} y (e^{i\tilde{s}\cdot\mathcal{A}} - e^{is\cdot\mathcal{A}_m}) + (e^{i\tilde{s}\cdot\mathcal{A}} - e^{is\cdot\mathcal{A}_m}) y e^{i\tilde{s}\cdot\mathcal{A}_m}) ds d\tilde{s}
\end{aligned}
$$

We find the following estimates as $m \to \infty$, by respectively (4.4) and (4.5), then applying (4.6) and Lemma 4.2 and finally using that $\|A_j^m - A_j\| \leq \frac{1}{m}$,

$$
\begin{aligned}
\|\mathcal{I}_\phi(y) - \mathcal{I}_\phi^m(y)\|_2 &= \|\mathcal{I}_{\phi_0}(y) - \mathcal{I}_{\phi_0}^m(y)\|_2 \\
&\leq 2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\hat{\phi}_0(s, \tilde{s})| \sum_{j=1}^n |s_j| \|A_j^m - A_j\| \|y\|_2 ds d\tilde{s} \\
&\leq \frac{2}{m} \|y\|_2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\hat{\phi}_0(s, \tilde{s})| \sum_{j=1}^n |s_j| ds d\tilde{s}.
\end{aligned}
$$

Since ϕ_0 is a Schwartz function, also the Fourier transform $\hat{\phi}_0$ is a Schwartz function. So the latter expression converges to 0. $\qquad\square$

Proposition 4.4 (see Lemma 9 of [18]). *Let ν be a finite measure on $\mathbb{R}^n \times \mathbb{R}^n$. Let $\phi \in L^1(\mathbb{R}^n \times \mathbb{R}^n, \nu)$. Define,*

$$
\phi_k(\xi, \tilde{\xi}) = \left(\frac{k}{\sqrt{\pi}}\right)^n \int_{\mathbb{R}^n} e^{-k\eta\cdot\eta} \phi(\xi - \eta, \tilde{\xi} - \eta) d\eta.
$$

Then, $\|\phi_k - \phi\|_1 \to 0$ as $k \to \infty$.

Proof. Since the bounded absolutely continuous functions are dense in $L^1(\mathbb{R}^n \times \mathbb{R}^n, \nu)$ we may assume that ϕ is bounded and absolutely continuous. Let $\epsilon > 0$ and choose $\delta > 0$ such that for every $(\xi, \tilde{\xi}), (\eta, \tilde{\eta}) \in \mathbb{R}^n \times \mathbb{R}^n$ with $\|(\xi, \tilde{\xi}) - (\eta, \tilde{\eta})\|_2 < \delta$, we have $|\phi(\xi, \tilde{\xi}) - \phi(\eta, \tilde{\eta})| < \epsilon$. Then,

$$
\begin{aligned}
\|\phi_k - \phi\|_\infty &= \sup_{(\xi, \tilde{\xi})} \left(\frac{k}{\sqrt{\pi}}\right)^n \left| \int_{\eta\in\mathbb{R}^n, \|\eta\|_2 \geq \delta} e^{-k\eta\cdot\eta} (\phi(\xi - \eta, \tilde{\xi} - \eta) - \phi(\xi, \tilde{\xi})) d\eta \right. \\
&\quad \left. + \int_{\eta\in\mathbb{R}^n, \|\eta\|_2 < \delta} e^{-k\eta\cdot\eta} (\phi(\xi - \eta, \tilde{\xi} - \eta) - \phi(\xi, \tilde{\xi})) d\eta \right| \\
&\leq 2\|\phi\|_\infty \left(\frac{k}{\sqrt{\pi}}\right)^n \int_{\eta\in\mathbb{R}^n, \|\eta\|_2 \geq \delta} e^{-k\eta\cdot\eta} d\eta + \epsilon.
\end{aligned}
$$

The latter expression converges to 0 as $k \to \infty$. Since ν is finite, this implies that $\|\phi_k - \phi\|_1 \to 0$. $\qquad\square$

5. COMMUTATOR ESTIMATES

This section contains the main result of this paper. We prove that the best constant for operator Lipschitz inequalities and commutator estimates in Schatten-von Neumann classes are of order $\frac{p^2}{p-1}$.

In this section, $\xi, \tilde{\xi}$ are vectors in $\mathbb{R}^n$. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function. We define ϕ_f, ψ_f and $\phi_j, \psi_j, 1 \leq j \leq n$ on $\mathbb{R}^n \times \mathbb{R}^n$ by

(5.1)
$$\psi_f(\xi, \tilde{\xi}) = f(\xi) - f(\tilde{\xi}), \qquad \phi_f(\xi, \tilde{\xi}) = \begin{cases} \frac{f(\xi) - f(\tilde{\xi})}{\|\xi - \tilde{\xi}\|_2} & \text{if } \xi \neq \tilde{\xi}, \\ 0 & \text{if } \xi = \tilde{\xi}, \end{cases}$$

$$\psi_j(\xi, \tilde{\xi}) = \xi_j - \tilde{\xi}_j, \qquad \phi_j(\xi, \tilde{\xi}) = \begin{cases} \frac{\xi_j - \tilde{\xi}_j}{\|\xi - \tilde{\xi}\|_2} & \text{if } \xi \neq \tilde{\xi}, \\ 0 & \text{if } \xi = \tilde{\xi}. \end{cases}$$

Let E be a spectral measure on $\mathbb{R}^n$. Since the functions defined in (5.1) are all bounded, the double operator integrals $\mathcal{I}_{\phi_f}, \mathcal{I}_{\phi_j}, \mathcal{I}_{\psi_f}, \mathcal{I}_{\psi_j}$ with respect to E exist as bounded operators on $\mathcal{S}_2$.

Theorem 5.1. *Let $p \in (1, \infty)$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function with $\|f\|_{\mathrm{Lip}} \leq 1$. Let $y \in \mathcal{S}_p \cap \mathcal{S}_2$ be off-diagonal. For every $1 \leq j \leq n$, we have*

(5.2)
$$\|\mathcal{I}_{\phi_f} \mathcal{I}_{\phi_j}(y)\|_p \leq \frac{Cp^2}{p-1} \|y\|_p,$$

for a constant C that is independent of p, the spectral measure E and the Lipschitz function f.

Proof. In order to prove the theorem, we first make three assumptions on y, E and f. We show that each assumption can be made without loss of generality. Firstly, note that we assumed that y is off-diagonal. The next assumption shows that we may in fact assume that y has no non-trivial part in a specific open neighbourhood of the diagonal.

Assumption 1. For $l \in \mathbb{N}^*$, let $U_l = \{(\xi, \tilde{\xi}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \|\xi - \tilde{\xi}\|_2 < \frac{1}{l}\}$. It suffices to prove (5.2) for $y \in \mathcal{S}_p \cap \mathcal{S}_2$ for which there exists a $l \in \mathbb{N}$ such that $\int_{U_l} dF(\xi, \tilde{\xi})(y) = 0$.

Let $y \in \mathcal{S}_p \cap \mathcal{S}_2$ be off-diagonal. Let

$$\varphi_l(\xi, \tilde{\xi}) = \begin{cases} 1 & \text{if } \|\xi - \tilde{\xi}\|_2 > \frac{1}{l}, \\ 0 & \text{if } \|\xi - \tilde{\xi}\|_2 \leq \frac{1}{l}, \end{cases} \qquad \varphi_\infty(\xi, \tilde{\xi}) = \begin{cases} 1 & \text{if } \xi \neq \tilde{\xi}, \\ 0 & \text{if } \xi = \tilde{\xi}. \end{cases}$$

Then, $\int_{U_l} dF(\xi, \tilde{\xi})(\mathcal{I}_{\varphi_l}(y)) = 0$. Furthermore, using respectively the definition of $\mathcal{I}_{\varphi_l}$, the Lebesgue dominated convergence theorem and the fact that y is off-diagonal, we find for every $z \in \mathcal{S}_2$, as $l \to \infty$,

(5.3)
$$\tau(z \mathcal{I}_{\phi_f} \mathcal{I}_{\phi_j} \mathcal{I}_{\varphi_l} y) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi_f \phi_j \varphi_l(\xi, \tilde{\xi}) d\nu_{z,y}(\xi, \tilde{\xi})$$
$$\to \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi_f \phi_j \varphi_\infty(\xi, \tilde{\xi}) d\nu_{z,y}(\xi, \tilde{\xi})$$
$$= \tau(z \mathcal{I}_{\phi_f} \mathcal{I}_{\phi_j} y).$$

Suppose that we have proved the (5.2) with y replaced by $\mathcal{I}_{\varphi_l}(y)$, in particular for a constant C that is independent of l. Then, it follows from (5.3) that also (5.2) holds for y. In all, this shows that we can make Assumption 1.

Assumption 2. Suppose that $y \in \mathcal{S}_p \cap \mathcal{S}_2$ satisfies Assumption 1. It suffices to prove Theorem 5.1 under the condition that E is a discrete spectral measure on $\mathbb{R}^n$ with support contained in $\frac{1}{m}\mathbb{Z}^n$ for some $m \in \mathbb{N}$.

We show that indeed Assumption 2 suffices to prove Theorem 5.1. Let f be an arbitrary Lipschitz function with $\|f\|_{\mathrm{Lip}} \leq 1$. Let $f_l, l \in \mathbb{N}$ be a sequence of Lipschitz functions with $\|f\|_{\mathrm{Lip}} \leq 1$, such that $f_l(\xi, \tilde{\xi}) = f(\xi, \tilde{\xi})$ for every $(\xi, \tilde{\xi}) \in [-l, l]^n$ and such that f_l has compact support. Suppose that we have proved Theorem 5.1 for all f_l.

Note that for every $\xi, \tilde{\xi} \in \mathbb{R}^n$ we have $|\phi_{f_l}(\xi, \tilde{\xi})\phi_j(\xi, \tilde{\xi})| \leq 1$ and $\phi_{f_l} \to \phi_f$ pointwise. The Lebesgue dominated convergence theorem hence entails that for every $z \in \mathcal{S}_2$ we have, as $l \to \infty$,

$$\tau(z\mathcal{I}_{\phi_{f_l}}\mathcal{I}_{\phi_j}y) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \phi_{f_l}(\xi, \tilde{\xi})\phi_j(\xi, \tilde{\xi})dv_{z,y}(\xi, \tilde{\xi})$$
$$\to \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \phi_f(\xi, \tilde{\xi})\phi_j(\xi, \tilde{\xi})dv_{z,y}(\xi, \tilde{\xi})$$
$$= \tau(z\mathcal{I}_{\phi_{f_l}}\mathcal{I}_{\phi_j}y).$$

From this limit, it follows that Theorem 5.1 also holds for f. Hence, we may assume that f has compact support.

Let

$$G_k(\xi) = \left(\frac{k}{\sqrt{\pi}}\right)^n e^{-k(\xi \cdot \xi)},$$

be a dilated Gaussian and put $f_k = G_k * f$. By Proposition 4.4,

(5.4)

$$\tau(z\mathcal{I}_{\phi_{f_k}}\mathcal{I}_{\phi_j}y) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \phi_{f_k}(\xi, \tilde{\xi})d\nu_{z,\mathcal{I}_{\phi_j}y}(\xi, \tilde{\xi})$$
$$\to \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \phi_f(\xi, \tilde{\xi})d\nu_{z,\mathcal{I}_{\phi_j}y}(\xi, \tilde{\xi}) = \tau(z\mathcal{I}_{\phi_f}\mathcal{I}_{\phi_j}y), \qquad \text{for every } z \in \mathcal{S}_{p'} \cap \mathcal{S}_2.$$

The function f_k is Schwartz since f has compact support. Furthermore, $\|f_k\|_{\mathrm{Lip}} \leq 1$. Suppose that Theorem 5.1 is proved for all f_k, in particular with C independent of k. Then, (5.4) implies that Theorem 5.1 also holds for f. In all, this proves that we may assume that f is a Schwartz function.

Now, assume that f is a Schwartz function. Let E be a spectal measure on $\mathbb{R}^n$. We define discretized spectral measures by setting

$$E_m(\Omega) = \sum_{k \in \mathbb{Z}^n, \text{s.t.} \frac{k}{m} \in \Omega} E\left(\left[\frac{k_1}{m}, \frac{k_1+1}{m}\right) \times \ldots \times \left[\frac{k_n}{m}, \frac{k_n+1}{m}\right)\right), \qquad \Omega \subseteq \mathbb{R}^n \text{ a Borel set.}$$

Let $\mathcal{I}_{\phi_f}^m$ and $\mathcal{I}_{\phi_j}^m$ be the double operator integrals of ϕ_f and respectively ϕ_j with respect to the spectral measure E_m. Let $U_l, l \in \mathbb{N}^*$ be the open neighbourhood of Assumption 1 for y. Since f is Schwartz, there is a Schwartz function $\phi_{f,0}$ on $\mathbb{R}^n \times \mathbb{R}^n$ such that $\phi_{f,0}(\xi, \tilde{\xi}) = \phi_f(\xi, \tilde{\xi})$ for every $(\xi, \tilde{\xi}) \in \mathbb{R}^n \times \mathbb{R}^n \setminus U_{l+1}$. It follows from Proposition 4.3 that

(5.5)
$$\lim_{m \to \infty} \tau(z\mathcal{I}_\phi^m \mathcal{I}_{\phi_j}^m y) = \tau(z\mathcal{I}_\phi \mathcal{I}_{\phi_j}y), \qquad \text{for every } z \in \mathcal{S}_{p'} \cap \mathcal{S}_2.$$

Suppose that we have proved (5.2) for $\mathcal{I}^m_{\phi_f}, \mathcal{I}^m_{\phi_j}$ and y as in Assumption 1. In particular, the sequence in m given by $\mathcal{I}^m_{\phi_f} \mathcal{I}^m_{\phi_j} y$ is bounded in $\mathcal{S}_p$. Then, it follows from (5.5) that also (5.2) holds for $\mathcal{I}_{\phi_f} \mathcal{I}_{\phi_j} y$. In all, this proves that without loss of generality we can make Assumption 2.

Assumption 3. Let y be as in Assumption 1 and let E be a spectral measure as in Assumption 2. So the support of E is contained in $\frac{1}{m}\mathbb{Z}^n$. It suffices to prove Theorem 5.1 under the condition that f is a Lipschitz function with $\|f\|_{\mathrm{Lip}} \leq 1$ and such that there exists a $N \in \mathbb{N}$ such that f maps $\frac{1}{m}\mathbb{Z}^n$ into $\frac{1}{mN}\mathbb{Z}^n$.

We prove that Assumption 3 is sufficient to conclude Theorem 5.1. Let $B_m : \mathbb{R}^n \to \mathbb{R}$ be a smooth function such that $B_m(0) = 1$ and the support of B_m is contained in $[-\frac{1}{2m}, \frac{1}{2m}]^n$. Put

$$f_N(\xi) = f(\xi) + \sum_{k \in \mathbb{Z}^n} \left(\frac{\lfloor Nf(\frac{k}{m}) \rfloor}{N} - f\left(\frac{k}{m}\right) \right) B_m\left(\xi - \frac{k}{m}\right).$$

Then, f_N maps $\frac{1}{m}\mathbb{Z}^n$ to $\frac{1}{mN}\mathbb{Z}^n$ and we have $\|f_N\|_{\mathrm{Lip}} \leq 1 + \frac{1}{N}\|B_m\|_{\mathrm{Lip}}$ and $\|f - f_N\|_{\mathrm{Lip}} \leq \frac{1}{N}\|B_m\|_{\mathrm{Lip}}$. Moreover, for $z \in \mathcal{S}_{p'} \cap \mathcal{S}_2$,

$$(5.6) \qquad \begin{aligned} \tau(z(\mathcal{I}_{\phi_{f_N}} - \mathcal{I}_{\phi_f})\mathcal{I}_{\phi_j} y) &\leq \|(\phi_{f_N} - \phi_f)\phi_j\|_\infty \|z\|_2 \|y\|_2 \\ &\leq \|f_N - f\|_{\mathrm{Lip}} \|\phi_j\|_\infty \|z\|_2 \|y\|_2, \end{aligned}$$

which converges to 0 as $N \to \infty$. Suppose that (4.6) is proved for all functions $g_N := (1 + \frac{1}{N}\|B_m\|_{\mathrm{Lip}})^{-1} f_N$, (so that $\|g_N\|_{\mathrm{Lip}} \leq 1$). Then, in particular $\mathcal{I}_{\phi_{f_N}} \mathcal{I}_{\phi_j} y \in \mathcal{S}_p$ is bounded in N. It follows from (5.6) that also $\mathcal{I}_{\phi_f} \mathcal{I}_{\phi_j} y$ is contained in $\mathcal{S}_p$ and satisfies the estimate (5.2). In all, we conclude that without loss of generality, we can make Assumption 3.

We now prove Theorem 5.1 under Assumptions 1, 2 and 3. For $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$, define the spectral projection

$$p_k = E\left(\left[\frac{k_1}{m}, \frac{k_1 + 1}{m}\right) \times \ldots \times \left[\frac{k_n}{m}, \frac{k_n + 1}{m}\right) \right).$$

For $\xi \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$, define the unitary operator acting $\mathcal{H}$ by

$$u(\xi, \mu) = \sum_{k \in \mathbb{Z}^n} e^{2\pi i m N(\frac{k}{m} \cdot \xi + f(\frac{k}{m})\mu)} p_k,$$

For y as in Assumption 1, put

$$h_y = u \cdot y \cdot u^*.$$

Naturally, $h_y \in L^p_{\mathcal{S}_p}(\mathbb{T}^{n+1})$. We consider $\mathbb{T}^{n+1}$ equipped with the normalized Lebesgue measure. Then, $\|h_y\|_{L^p_{\mathcal{S}_p}(\mathbb{T}^{n+1})} = \|y\|_p$.

Fix $k, \tilde{k} \in \frac{1}{m}\mathbb{Z}^n$. Let $y \in p_k \mathcal{S}_p p_{\tilde{k}}$ satisfy the condition of Assumption 1. Since $\{p_i\}_{i \in \mathbb{Z}^n}$ is a family of mutually orthogonal projections,

$$(5.7) \qquad h_y(\xi, \mu) = e^{2\pi i m N((\frac{k}{m} - \frac{\tilde{k}}{m}) \cdot \xi + (f(\frac{k}{m}) - f(\frac{\tilde{k}}{m}))\mu)} y.$$

Let $\delta_{s,t}$ with $s \in \mathbb{Z}^n, t \in \mathbb{Z}$ be the function on $\mathbb{Z}^{n+1}$ that attains the value 1 on (s,t) and vanishes everywhere else. Taking the Fourier transform of h_y, we find

$$\mathcal{F}_2\left(h_y\right) = \delta_{N(k-\tilde{k}),mN(f(\frac{k}{m})-f(\frac{\tilde{k}}{m}))} y \in L^2_{\mathcal{S}_p}(\mathbb{Z}^{n+1}).$$

Using $\bar{m}_j$, the discretized version of m_j, see Theorem 2.5 and Lemma 3.2,

$$
\begin{aligned}
T_{\bar{m}_j} h_y =& (\mathcal{F}_2^{-1} \circ \bar{m}_j \circ \mathcal{F}_2) h_y \\
=& (\mathcal{F}_2^{-1} \circ \bar{m}_j) \left(\delta_{N(k-\tilde{k}),mN(f(\frac{k}{m})-f(\frac{\tilde{k}}{m}))} y \right) \\
=& \frac{f(\frac{k}{m}) - f(\frac{\tilde{k}}{m})}{\|\frac{k}{m} - \frac{\tilde{k}}{m}\|_2} \frac{\frac{k_j}{m} - \frac{\tilde{k}_j}{m}}{\|\frac{k}{m} - \frac{\tilde{k}}{m}\|_2} \mathcal{F}_2^{-1}\left(\delta_{N(k-\tilde{k}),mN(f(\frac{k}{m})-f(\frac{\tilde{k}}{m}))} y \right) \\
=& \frac{f(\frac{k}{m}) - f(\frac{\tilde{k}}{m})}{\|\frac{k}{m} - \frac{\tilde{k}}{m}\|_2} \frac{\frac{k_j}{m} - \frac{\tilde{k}_j}{m}}{\|\frac{k}{m} - \frac{\tilde{k}}{m}\|_2} h_y.
\end{aligned}
$$

(5.8)

On the other hand, recalling that $y \in p_k \mathcal{S}_p p_{\tilde{k}}$,

$$\mathcal{I}_\phi \mathcal{I}_{\phi_j} y = \frac{f(\frac{k}{m}) - f(\frac{\tilde{k}}{m})}{\|\frac{k}{m} - \frac{\tilde{k}}{m}\|_2} \frac{\frac{k_j}{m} - \frac{\tilde{k}_j}{m}}{\|\frac{k}{m} - \frac{\tilde{k}}{m}\|_2} y. \tag{5.9}$$

It follows from (5.7), (5.8) and (5.9) that for every $y \in \text{span}\left\{ p_k \mathcal{S}_p p_{\tilde{k}} \mid k, \tilde{k} \in \frac{1}{m}\mathbb{Z} \right\}$ that satisfies Assumption 1,

$$T_{\bar{m}_j} h_y = u \cdot \mathcal{I}_\phi \mathcal{I}_{\phi_j} y \cdot u^*. \tag{5.10}$$

In particular, $\mathcal{I}_{\phi_f} \mathcal{I}_{\phi_j} y \in \mathcal{S}_p$ for all $1 \le j \le n$. Taking the norm in $L^p_{\mathcal{S}_p}(\mathbb{T}^{n+1})$ on both sides of (5.10), one obtains the inequality

$$
\begin{aligned}
\|\mathcal{I}_{\phi_f} \mathcal{I}_{\phi_j} y\|_p =& \|u \cdot \mathcal{I}_{\phi_f} \mathcal{I}_{\phi_j} y \cdot u^*\|_{L^p_{\mathcal{S}_p}(\mathbb{T}^{n+1})} \\
=& \|T_{\bar{m}_j} h_y\|_{L^p_{\mathcal{S}_p}(\mathbb{T}^{n+1})} \\
\le& \|T_{\bar{m}_j} : L^p_{\mathcal{S}_p}(\mathbb{T}^{n+1}) \to L^p_{\mathcal{S}_p}(\mathbb{T}^{n+1})\| \|h_y\|_{L^p_{\mathcal{S}_p}(\mathbb{T}^{n+1})} \\
=& \|T_{\bar{m}_j} : L^p_{\mathcal{S}_p}(\mathbb{T}^{n+1}) \to L^p_{\mathcal{S}_p}(\mathbb{T}^{n+1})\| \|y\|_p
\end{aligned}
$$

Using respectively Theorem 2.5, Theorem 3.3 and Theorem 2.7, we continue the inequality:

$$
\begin{aligned}
\|\mathcal{I}_{\phi_f} \mathcal{I}_{\phi_j} y\|_p \le& \|T_{\bar{m}_j} : L^p_{\mathcal{S}_p}(\mathbb{T}^{n+1}) \to L^p_{\mathcal{S}_p}(\mathbb{T}^{n+1})\| \|y\|_p \\
\le& \|T_{m_j} : L^p_{\mathcal{S}_p}(\mathbb{R}^{n+1}) \to L^p_{\mathcal{S}_p}(\mathbb{R}^{n+1})\| \|y\|_p \\
\le& C_1 \, \text{UMD}_p(\mathcal{S}_p) \|y\|_p \\
\le& \frac{C_2 \, p^2}{p-1} \|y\|_p,
\end{aligned}
$$

where C_1 and C_2 are constants that are independent of the Lipschitz function f, the spectral measure E and $p \in (1,\infty)$. Since $\text{span}\left\{ p_k \mathcal{S}_p p_{\tilde{k}} \mid k, \tilde{k} \in \frac{1}{m}\mathbb{Z} \right\}$ is dense in $\mathcal{S}_p$, this concludes the theorem. $\qquad\square$

Theorem 5.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function with $\|f\|_{\mathrm{Lip}} \leq 1$. Let E be a spectral measure on $\mathbb{R}^n$ and let $\mathcal{A} = (A_1, \ldots, A_n)$ be defined as in (4.1). Let $p \in (1, \infty)$. Let $x \in B(\mathcal{H})$ be such that for all $1 \leq j \leq n$ we have $[A_j, x] \in \mathcal{S}_p$. Then, also $[f(\mathcal{A}), x] \in \mathcal{S}_p$. Moreover, there exists a constant C that is independent of $p \in (1, \infty)$, the spectral measure E and the Lipschitz function f such that*

$$(5.11) \qquad \|[f(\mathcal{A}), x]\|_p \leq \frac{Cp^2}{p-1} \sum_{j=1}^n \|[A_j, x]\|_p.$$

Proof. First assume that $x \in \mathcal{S}_p \cap \mathcal{S}_2$. In that case,

$$(5.12) \qquad [f(\mathcal{A}), x] = \mathcal{I}_{\psi_f}(x) = \sum_{j=1}^n \mathcal{I}_{\phi_f} \mathcal{I}_{\phi_j} \mathcal{I}_{\psi_j}(x) = \sum_{j=1}^n \mathcal{I}_{\phi_f} \mathcal{I}_{\phi_j} [A_j, x].$$

Here, the first and third equality are an application of (4.2). The second equality is a consequence of the fact that $g \mapsto \mathcal{I}_g$ is an algebra homomorphism from the bounded Borel functions on $\mathbb{R}^n \times \mathbb{R}^n$ to the bounded operators acting on $\mathcal{S}_2$. By Theorem 5.1 we have for all $1 \leq j \leq n$ that $\mathcal{I}_{\phi_f} \mathcal{I}_{\phi_j} [A_j, x] \in \mathcal{S}_p$ and moreover,

$$(5.13) \qquad \|\mathcal{I}_{\phi_f} \mathcal{I}_{\phi_j} [A_j, x]\|_p \leq \frac{C p^2}{p-1} \|[A_j, x]\|_p.$$

for a constant C which is independent of p, E and the function f with $\|f\|_{\mathrm{Lip}} \leq 1$. Clearly, (5.12) and (5.13) imply Theorem 5.2.

Now, let $\mathfrak{X} = B(\mathcal{H})$ and $\mathfrak{N} = \mathcal{S}_p$. For $x \in \mathfrak{X}$, let $S_j(x) = [A_j, x], T(x) = [f(\mathcal{A}), x]$. For $x \in \mathfrak{N}$, let $R_j(x) = \mathcal{I}_{\phi_f} \mathcal{I}_{\phi_j}(x)$. We will show that [14, Lemma 5.1] is applicable. The proof is exactly the same as the final part of the proof of [14, Theorem 5.3]. We sketch it here. Firstly, for $x \in \cap_{j=1}^n \ker(S_j)$ we have for all $1 \leq j \leq n$ that $[A_j, x] = 0$. Hence, $[f(\mathcal{A}), x] = 0$ and condition (1) of [14, Lemma 5.1] is satisfied. Secondly, as explaind in the proof of [14, Theorem 5.3] for any self-adjoint $Z \in B(\mathcal{H})$ the mapping $T_Z : \mathfrak{X} \to \mathfrak{X} : x \mapsto [Z, x]$ is hermitian and hence satisisfies $\ker(T_Z) \cap \overline{T_Z(\mathfrak{X})} = \{0\}$. Hence, condition (2) of [14, Lemma 5.1] is satisfied. Finally,

$$R_j \mathfrak{N} \subseteq \overline{\mathcal{I}_{\phi_f} \mathcal{I}_{\phi_j} \mathfrak{N}} \subseteq \overline{\mathcal{I}_{\psi_f} \mathfrak{N}} \subseteq \overline{T\mathfrak{N}} \subseteq \overline{T\mathfrak{X}}.$$

Here, the second inclusion follows from [14, Lemma 2.4]. This proves that condition (3) of [14, Lemma 5.1] is satisfied. Applying [14, Lemma 5.1] yields that

$$(5.14) \qquad [f(\mathcal{A}), x] = \sum_{j=1}^n \mathcal{I}_{\phi_f} \mathcal{I}_{\phi_j} [A_j, x],$$

for every $x \in B(\mathcal{H})$ such that for all $1 \leq j \leq n$ we have $[A_j, x] \in \mathcal{S}_p$. Applying Theorem 5.1 to (5.14) yields Theorem 5.2. $\square$

Remark 5.3. In [14, Theorem 5.3], Theorem 5.2 was proved with the weaker estimate

$$C_p \leq \frac{C p^{16}}{(p-1)^8}.$$

In [14] the norms of the two double operator integrals W_f and V_j appearing the proof of [14, Theorem 5.3] are estimated separately with constants that do not give the same sharp result as in Theorem 5.2. The novelty of our proof is the fact that we use the main result of [6, Theorem 3.1] (see Theorem 3.3) to give a direct estimate of $\mathcal{I}_\phi \mathcal{I}_{\phi_j}$.

Remark 5.4. The estimate $C_p \leq \frac{C\,p^2}{1-p}$ given in Theorem 5.2 is the best possible in the sense that in fact

$$C_p \sim \frac{C\,p^2}{1-p}.$$

Corollary 5.5. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a Lipschitz function with* $\|f\|_{\mathrm{Lip}} \leq 1$. *Let* $p \in (1, \infty)$. *Let* $X, Y \in B(\mathcal{H})$ *be self-adjoint operators such that* $X - Y \in \mathcal{S}_p$. *Then* $f(X) - f(Y) \in \mathcal{S}_p$. *Moreover, there exists a constant* C *that is independent of* $p \in (1, \infty)$ *and the Lipschitz function* f *such that*

(5.15)
$$\|f(X) - f(Y)\|_p \leq \frac{Cp^2}{p-1}\|X - Y\|_p.$$

Proof. Apply Theorem 5.2 to the case $n = 1$ and with

$$A_1 = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}, \quad x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$\square$

Finally, we show that Theorem 5.2 implies some variant of a weak L^1-type inequality. For $A \in B(\mathcal{H})$ a compact operator and for $t \in [0, \infty)$, let

$$\mu_t(A) = \inf\{\|Ap\| \mid p \in B(\mathcal{H}) \text{ projection such that } \tau(p) \leq t\}.$$

denote the decreasing rearrangement of singular values. Let $\mathcal{L}^{1,\infty}$ be the weak L^1-space associated with $B(\mathcal{H})$. It is defined as the space of all compact operators $A \in B(\mathcal{H})$ for which

$$\|A\|_{1,\infty} = \sup_{t \in [0, \infty)} t\mu_t(A) < \infty.$$

Consider also the space $M_{1,\infty}$ consisting of all compact operators A such that

$$\|A\|_{M_{1,\infty}} = \sup_{t \in [0, \infty)} \log(1 + t)^{-1} \int_0^t \mu_s(A)ds < \infty.$$

We have a norm decreasing inclusion $\mathcal{L}^{1,\infty} \subseteq M_{1,\infty}$. The following corollary is now a multivariable version of the result obtained in [15, Theorem 2.5 (ii)].

Corollary 5.6. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a Lipschitz function with* $\|f\|_{\mathrm{Lip}} \leq 1$. *Let* E *be a spectral measure on* $\mathbb{R}^n$ *and let* $\mathcal{A} = (A_1, \ldots, A_n)$ *be defined as in (4.1). Let* $x \in B(\mathcal{H})$ *be such that for all* $1 \leq j \leq n$ *we have* $[A_j, x] \in \mathcal{S}_1$. *Then, also* $[f(\mathcal{A}), x] \in M_{1,\infty}$. *Moreover, there exists a constant* C *that is independent of* E *such that*

$$\|[f(\mathcal{A}), x]\|_{M_{1,\infty}} \leq C\sum_{j=1}^n \|[A_j, x]\|_1.$$

Proof. Put $T = [f(\mathcal{A}), x]$. Let $s \gg 1$, set $p = \log(s)$ and $q = \frac{p}{p-1}$. Then, using the Hölder inequality, Theorem 5.2 and the inclusion $\mathcal{S}_1 \subseteq \mathcal{S}_q$, we find,

$$\int_0^s \mu_t(T)ds \leq s^{\frac{1}{p}} \left(\int_0^s \mu_t(T)^q dt \right)^{\frac{1}{q}}$$

$$\leq s^{\frac{1}{p}} \|T\|_q \leq s^{\frac{1}{p}} Cq \sum_{j=1}^n \|[A_j, x]\|_q \leq s^{\frac{1}{p}} Cq \sum_{j=1}^n \|[A_j, x]\|_1.$$

Since $s^{\frac{1}{p}} q = e^{\frac{\log(s)}{\log(s)-1}} \leq \log(s)$ for large s, we find that

$$\int_0^s \mu_t(T)ds \leq C \log(s) \sum_{j=1}^n \|[A_j, x]\|_1,$$

which implies that $\|T\|_{M_{1,\infty}} \leq C \sum_{j=1}^n \|[A_j, x]\|_1$. $\qquad\square$

Remark 5.7. According to [4, Theorem 4.5] (see also [3, Theorem 2.1]) the $\|\cdot\|_{M_{1,\infty}}$-norm is equivalent to the norm

$$(5.16) \qquad\qquad \|A\|_\zeta = \limsup_{p \downarrow 1} (p-1)\|A\|_p.$$

Using this observation, one may obtain an alternative proof of Corollary 5.6.

Remark 5.8. The question whether the weak L^1-type inequality holds, that is whether

$$\|[f(\mathcal{A}), x]\|_{1,\infty} \leq C \sum_{j=1}^n \|[A_j, x]\|_1,$$

remains open.

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