Mendler-style Recursion Schemes for Mixed-Variant Datatypes

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Outline

Mendler-style Recursion Schemes
    mit (iteration)
    msfit (iteration with syntactic Inverse)

Untyped HOAS (Regular Mixed-Variant Datatype)
Formatting Untyped HOAS (using msfit at kind $*$)

Simply-Typed HOAS (Indexed Mixed-Variant Datatype)
Formatting Untyped HOAS (using msfit at kind $* \longrightarrow *$)

The Nax language

Need for yet another Mendler-style Recursion Scheme

(iso-) Recrusive Types in Functional Languages

$$\text{kinding:}\quad (\mu\text{-form})\frac{\Gamma \vdash F : * \to *}{\Gamma \vdash \mu F : *}$$

$$\text{typing:}\quad (\mu\text{-intro})\frac{\Gamma \vdash t : F(\mu F)}{\Gamma \vdash \mathsf{In}\ t : \mu F} \qquad (\mu\text{-elim})\frac{\Gamma \vdash t : \mu F}{\Gamma \vdash \mathsf{unIn}\ t : F(\mu F)}$$

$$\text{reduction:}\ (\mathsf{unIn\text{-}In})\frac{}{\mathsf{unIn}\ (\mathsf{In}\ t) \rightsquigarrow t}$$

already inconsistent as a logic (via Curry--Howard correspondence)
without even having to introduce any term-level recursion
since we can already define general recursion using this

Having both **unrestircted formation** and **unristricted elimination** of μ leads to inconsistency

```
data T a r = C (r → a)

w : μ (T a) → a   -- encoding of (λx. x x) in the untyped λ-calc
w = λv. case (unIn v) of  (C x) → f (C x)

-- w (C w)   amounts to a well-known diverging term in λ-calculus

fix : (a → a) → a     -- the Y combinator (λf.(λx.f(xx))(λx.f(xx)))
fix = λf. (λx. f (w x)) (In (C (λx. f (w x))))
```

To recover consistency, one could

either restrict formation (conventional approach)

or restrict elimination (Mendler-style approach)

Conventional Iteration over Recursive Types

kinding: $$(\mu\text{-form}^+)\frac{\Gamma \vdash F : * \to * \qquad \mathsf{positive}(F)}{\Gamma \vdash \mu F : *}$$

allow formation of only positive recursive types (i.e., when F has a map)

typing: (μ-intro) and (μ-elim) same as functional language

$$(\mathbf{It})\frac{\Gamma \vdash t : \mu F \qquad \Gamma \vdash \varphi : FA \to A}{\Gamma \vdash \mathbf{It}\ \varphi\ t : A}$$

reduction: (unIn-In) same as functional language

freely eliminate (i.e. use unIn) recursive values

$$(\mathbf{It}\text{-}\mathsf{In})\frac{}{\mathbf{It}\ \varphi\ (\mathsf{In}\ t) \rightsquigarrow \varphi\ (\mathsf{map}_F\ (\mathbf{It}\ \varphi)\ t)}$$

Mendler-style Iteration over Recursive Types

kinding: (μ-form) same as functional language

freely form ANY recursive type!!

typing: (μ-intro) same as functional language

note, no (μ-elim) rule

$$(\textbf{mit})\frac{\Gamma \vdash t : \mu F \qquad \Gamma \vdash \varphi : \forall X.(X \to A) \to FX \to A}{\Gamma \vdash \textbf{mit}\ \varphi\ t : A}$$

elimination is possible only through mit

reduction: $(\textbf{mit-In})\dfrac{}{\textbf{mit}\ \varphi\ (\mathsf{In}\ t) \rightsquigarrow \varphi\ (\textbf{mit}\ \varphi)\ t}$

Mendler-style Iteration

```
-- Mu at different kinds (there are many more, one at each kind)
data Mu0 (f :: * → *)                  = In0 (f (Mu0 f ))
data Mu1 (f :: (*→*)→(*→*)) i= In1 (f (Mu1 f ) i)

-- Mendler-style iterators at different kinds
mit0 :: Phi0 f a → Mu f → a
mit0 phi (In0 x) = phi (mit0 phi) x

mit1 :: Phi1 f a i → Mu f i → a i
mit1 phi (In1 x) = phi (mit1 phi) x

                                recursive call
type Phi0 = ∀ r.          (r   →  a ) →       (f r   → a  )
type Phi1 = ∀ r. (∀ i. r i → a i) → (∀ i. f r i → a i)
```

- ✔ Uniformly defined at different kinds (can handle non-regular datatypes quite the same way as handling regular datatypes)
- ✔ Terminate for ANY datatype (can embed Mu and mit into Fw. see Abel, Matthes and Uustallu TCS'04)
- ✔ However, not very useful for mixed-variant datatypes

eliminating In0 or In1 by pattern matching is only allowed in Mendler-style iterators

Mendler-style Iteration with a syntactic inverse

```
-- Mu' at different kinds (we only use two of them in this talk)
data Mu'0 (f :: * → *)             a     = In'0 (f (Mu'0 f a))   | Inverse0 a
data Mu'1 (f :: (*→*)→(*→*)) a i  = In'1 (f (Mu'1 f a) i)| Inverse1 (a i)

-- msfit at different kinds
msfit0 :: Phi'0 f a → Mu'0 f → a
msfit0 phi (In'0 x) = phi Inverse0 (msfit0 phi) x

msfit1 :: Phi'1 f a i → Mu'1 f i → a i
msfit1 phi (In'1 x) = phi Inverse1 (msfit1 phi) x
```

- ✔ Terminate for ANY datatype (can embed Mu and msfit into Fw. see Ahn and Sheard ICFP'11)
- ✔ msfit is quite useful for mixed-variant datatypes, due to "inverse" operatation in addition to "recursive call"

```
                                    inverse                    recursive call
type Phi'0 f a = ∀ r.          (a   → r a  ) →        (r a   → a  ) →        (f r a     → a  )
type Phi'1 f a = ∀ r. (∀ i. a i → r a i) → (∀ i. r a i → a i) → (∀ i. f (r a) i → a i)
```

eliminating In'0 or In'1 by pattern matching is only allowed in Mendler-style iterators

Untyped HOAS

(a regular mixed-variant datatype)

```
-- using general recursion at type level
data Exp = Lam (Exp → Exp) | App Exp Exp

-- using fixpoint (Mu'0) over non-recursive base structure (ExpF)
data ExpF r = Lam (r → r) | App r r
type Exp' a = Mu'0 ExpF a          -- (Exp' a) may contain Inverse
type Exp =  ∀ a . Exp' a   -- Exp does not contain Inverse

lam :: (∀ a. Exp' a → Exp' a) -> Exp        -- it's not  (Exp → Exp) → Exp
lam  f  = In'0 (Lam f )                     -- f can handle Inverse containg values
                                            -- but can never examine its content

app :: Exp → Exp → Exp
app e1 e2 = In'0 (App e1 e2)
```

Formatting Untyped HOAS using **msfit0** at kind *

```
showExp :: Exp → String
showExp e = msfit0  phi e vars   where
  phi :: Phi'0 ExpF ([String] → String)
  phi inv showE (Lam z)     =
          λ(v:vs) → "(λ"++v++"→"++ showE (z (inv (const v))) vs ++")"
  phi inv showE (App x y)  =
          λvs      → "("++ showE x vs ++" "++ showE y vs ++")"

                          inverse          recursive call
type Phi0 f a =  ∀  r. (a → r a) -> (r a -> a) → (f r a → a)

vars = [ "x"++show n | n<-[0..] ] :: [String]
```

Simply-Typed HOAS

(a non-regular mixed-variant datatype)

```
-- using general recursion at type level
data Exp t where                -- Exp :: * → *
  Lam :: (Exp t1 → Exp t2) → Exp (t1 → t2)
  App :: Exp (t1 → t2) → Exp t1 → Exp t2

-- using fixpoint (Mu'1) over non-recursive base structure (ExpF)
data ExpF r t where             -- ExpF :: (* → *) → (* → *)
  Lam :: (r t1 → r t2) → ExpF r (t1 → t2)
  App :: r (t1 → t2) → r t1 → ExpF r t2
type Exp' a t = Mu'1 ExpF a t   -- (Exp' a t) might contain Inverse
type Exp t =  ∀ a . Exp' a t    -- (Exp t) does not contain Inverse

lam :: (forall a . Exp' a t1 -> Exp' a t2) -> Exp (t1 → t2)
lam  f  = In'1 (Lam f )         -- f can handle Inverse containg values
                                -- but can never examine its content
app :: Exp (t1 → t2) → Exp t1 → Exp t2
app e1 e2  = In'1 (App e1 e2)
```

Evaluating Simply-Typed HOAS using msfit1 at kind $*\to*$

```
newtype Id a = MkId { unId :: a }

evalHOAS :: Exp t → Id t
evalHOAS e = msfit1 phi e   where
  phi :: Phi'1 ExpF Id          {- v :: t1 -}      {- f :: r Id t1 -> r Id t2 -}
  phi inv ev (Lam f ) = MkId(λv → unId(ev (f (inv (MkId v)))))
  phi inv ev (App e1 e2) = MkId(unId(ev e1) (unId(ev e2)))
```

inverse — recursive call

$$\text{type Phi'1 f a} = \forall\, r.\ (\forall\, i.\ a\ i \to r\ a\ i) \to (\forall\, i.\ r\ a\ i \to a\ i) \to (\forall\, i.\ f\ (r\ a)\ i \to a\ i)$$

This is example is a super awesome cooooool discovery that System Fw can express Simply-Typed HOAS evaluation!!!

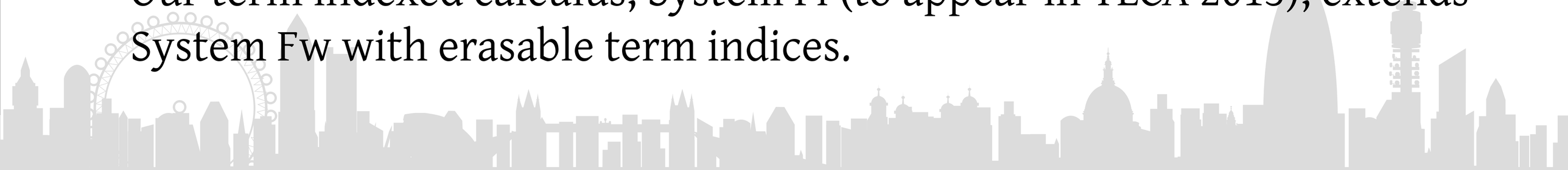
The Nax language

- ✔ Built upon the idea of Mendler-style recursion schemes
- ✔ Supports an extention of Hindley--Milner type inference to make it easier to use Mendler-style recursion schemes and indexed datatypes
- ✔ Handling differnt notions of recursive type operators (Mu, Mu') still needs more rigourous clarification in the theory, but intuitively,

  Mu0 f = ∀ a. Mu'0 f a
  Mu1 f = ∀ a. Mu'1 f a i
  ...

  So, we hide Mu' from users as if there is one kind of Mu and Mu' is only used during the computation of msfit

- ✔ Supports **term-indexed types** as well as type-indxed types.
  Our term indexed calculus, System Fi (to appear in TLCA 2013), extends System Fw with erasable term indices.



Need for yet another Mendler-style recursion scheme

There are more recursion schemes other than **mit** and **msfit**

E.g., Mendler-style primitive recursion (**mpr**) can **cast** from abstract recursive type (r) to concrete recursive type (Mu f).

Phi0 f a = ∀ r. (r → Mu f) → (r → a) → (f r → a)

Phi1 f a = ∀ r. (∀ i. r i → Mu f i) → (∀ i. r i → a i) → (∀ i. f r i → a i)

With **mpr**, one can write constant time predecessor for natrual numbers and tail function for lists by using the cast operation.

Next example motivates an extension to mpr that can **uncast** from concrete recursive type (Mu f i) to abstract recursive type (r i)

Phi1 f a = ∀ r. (∀ i. Mu f i -> r i) →
(∀ i. r i → Mu f i) → (∀ i. r i → a i) → (∀ i. f r i → a i)

Recursion scheme with above Phi1 type does not terminate for mixed-varaint datatypes -- needs some additional restriction on uncast.

Evaluating Simply-Typed HOAS into a user defined value domain

```
data V r t where V :: (r t1 → r t2) → V r (t1 → t2)
type Val t = Mu V t
val f = In1 (V f )

evalHOAS :: Exp t → Val t
evalHOAS e = msfit1 phi e   where
  phi :: Phi'1 ExpF (Mu1 V)          -- f :: r Val t1 -> r Val t2 ,    v :: Val t1
  phi inv ev (Lam f ) = val(λv → ev (f (inv v)))
  phi inv ev (App e1 e2) =  unVal (ev e1) (ev e2)
```

Only if we had unVal :: Val (t1 → t2) → (Val t1 → Val t2) it would be possible to write this, but unVal does not seem to be definable using any known Mendler-style recursion scheme.

New recursion scheme to write unVal

```
data V r t where V :: (r t1 → r t2) → V r (t1 → t2)
type Val t = Mu V t
val f = In1 (V f )
unVal v = unId(mprsi phi v) where
                    phi :: Phi1 V Id
                    phi uncast cast (V f) = Id(λv. cast (f (uncast v)))

mprsi :: Phi1 f a → Mu f i → a i
mprsi phi (In1 x) = phi  id  id  (mrpsi phi)  x
-- size restriction over indices over indices in both uncast and cast
type Phi1 f a =  ∀  r  j.    (∀ i. (i < j) ⇒ Mu f i → r i) →      -- uncast
                             (∀ i. (i < j) ⇒ r i → Mu f i) →      -- cast
                             (∀ i. r i → a i) → (r j → a j)
-- or, maybe without the size restriction over indices in cast
type Phi1 f a =  ∀  r  j.    (∀ i. (i < j) ⇒ Mu f i → r i) →      -- uncast
                             (∀ i.           r i → Mu f i) →      -- cast
                             (∀ i. r i → a i) → (f r j → a j)
```

Preliminary idea that still need further studies for the termination proof

Questions or Suggestions?

Thanks for listening.

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