

# System $F_i$

## a higher-order polymorphic $\lambda$ -calculus with erasable term indices

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### Abstract

We describe System  $F_i$ , a polymorphic lambda calculus, which supports term-indexed datatypes, such as length-indexed lists. System  $F_i$  is the smallest extension to System  $F_\omega$  we could devise that can express term indices. Term-indexed datatypes are expressible in richer systems, such as the Implicit Calculus Constructions (ICC), but such systems come coupled with orthogonal features such as large eliminations and dependent types. We argue there are important pedagogical benefits of isolating the minimal features to support term-indexing. We relate System  $F_i$  to other known systems. We prove erasure properties of  $F_i$  which captures the idea that term indices are erasable, since they are irrelevant for reduction. Erased  $F_i$  terms are  $F_\omega$  terms, so  $F_i$  inherits the strong normalization property from System  $F_\omega$ . We also show that System  $F_i$  is logically consistent by observing that System  $F_i$  is embeddable into a subset of ICC, which is known to be logically consistent. We argue that System  $F_i$  provides a theory for analyzing term-indexed Generalized Algebraic Data Structures and could provide the basis for the design of a light-weight logically-sound system, providing some of the benefits of logical systems based upon dependent calculi, but with much less complexity.

**Categories and Subject Descriptors** D.3.3 [Programming Languages]: Language Constructs and Features—data types and structures; F.4.1 [Mathematical Logic and Formal Languages]: Mathematical Logic—lambda calculus and related systems

**General Terms** Languages, Theory

**Keywords** term-indexed data types, generalized algebraic data types, higher-order polymorphism, type constructor polymorphism, higher-kinded types, impredicative encoding

### 1. Introduction

The introduction of Generalized Algebraic Datatypes (GADTs) to functional languages such as Haskell and OCaml has popularized the use of indexed datatypes as a light-weight, type-based mechanism to raise users' confidence that software systems maintain important properties. A classic example is Louis-Julien Guillemette's thesis [6] which encodes the classic paper by Morrisett et al. [11] completely in Haskell. The system embeds a multi-stage compiler,

compiling from System F, all the way to typed assembly language, using indexed types to show that every stage preserves type information.

This impressive system provides confidence, but no guarantees, since it uses types to enforce type preservation between the stages of the compiler. In Haskell, the non-terminating computation can be assigned any type, so in practice, it is possible that this property is a consequence of such a non-terminating computation in the program code.

On the other hand, the higher-order polymorphic lambda calculus, System  $F_\omega$ , is known to be strongly normalizing, and through the use of Church encodings or Scott encodings of data structures is rich enough to express systems with a rich collection of data structures. Unfortunately, it is not known how to express the kind of *term-indexed datatypes* in  $F_\omega$  that are necessary to support Guillemette's system.

Xavier Leroy, in his CompCert system [7], showed that a much richer language, the Calculus of Inductive Constructions (CIC), which is the basis of the Coq theorem prover, is expressive enough to guarantee type preservation between compiler stages (and quite a bit more). Unfortunately, this expressivity comes with a cost. Programmers must learn to use both dependent types, and a new programming paradigm, programming by code extraction.

Is there an expressive system supporting non-dependent indexed types, somewhere in between  $F_\omega$  and dependent calculi? Can we use non-dependent indexed types when they are all we need? Can we program, rather than extract code? The goal of this paper is to develop the theory necessary to begin to answer these and related questions.

To approach this goal, we have designed a new calculus, System  $F_i$ , which extends System  $F_\omega$  with just a few new features. Despite the paucity of new features,  $F_i$  is expressive enough to embed non-dependent *term-indexed datatypes* and their eliminators. Our contributions are:

- Identifying which features are needed for a  $\lambda$ -calculus to embed term-indexed datatypes (§2), in isolation from other features normally associated with such calculi (e.g., general recursion, large elimination, dependent types).
- The design of the calculus, System  $F_i$  (§3), which we use to studying the properties of languages with term-indexed datatypes, by embedding such datatypes into the calculus (§4). For example, we can use  $F_i$  to give a formal proof that the Mendler-style eliminators for GADTs appearing in last year's ICFP paper [3] are indeed normalizing).
- Showing that  $F_i$  enjoys a simple erasure property (§5.2) and inherits metatheoretic results (e.g., strong normalization, logical consistency) from well-known calculi (such as  $F_\omega$ ) related to System  $F_i$  (§5.3).

A functional language supporting term-indexed datatypes:

```
data Nat = Z | S n
data Vec (a:*) {i:N} where
  VNil   : Vec a {Z}
  VCons  : a -> Vec a {i} -> Vec a {S i}
```

System  $F_i$ :

```
Vec  $\triangleq$   $\lambda A^* . \lambda i^{\text{Nat}} . \forall X^{\text{Nat} \rightarrow *}$ .
   $X\{Z\} \rightarrow (\forall i^{\text{Nat}} . A \rightarrow X\{i\} \rightarrow X\{S i\}) \rightarrow X\{i\}$ 
```

**Figure 1.** A Motivating Example: length-indexed lists

## 2. Motivation

It is well known that datatypes can be embedded into polymorphic lambda calculi (e.g., [1]) using functional encodings such as the Church encoding. In System  $F$ , we can embed *regular datatypes*, such as homogeneous lists:

```
Haskell: data List a = Nil | Cons a (List a)
System F: List A  $\triangleq$   $\forall X . X \rightarrow (A \rightarrow X \rightarrow X) \rightarrow X$ 
```

Note the use of the universally quantified type variable  $X$  and the regularity of  $(\text{List } a)$  in the datatype definition.

In System  $F_\omega$ , we can embed *type-indexed datatypes*, which include datatypes that are not regular. For example, we can embed powerlists with heterogeneous elements where an element of type  $a$  is followed by an element of type  $(a, a)$ :

```
Haskell: data Powl a = PNil | PCons a (Powl(a,a))
System  $F_\omega$ : Powl  $\triangleq$   $\lambda A^* . \forall X^{* \rightarrow *}$ .
   $XA \rightarrow (A \rightarrow X(A \times A) \rightarrow XA) \rightarrow XA$ 
```

Note the non-regular occurrence  $(\text{Powl}(a, a))$  and the use of the type constructor variable  $X$  universally quantifying over type constructors of kind  $* \rightarrow *$ .

What extensions to  $F_\omega$  do we need to embed datatypes that are indexed by terms (such as  $Z$ ,  $i$ , and  $S i$ ), as well as indexed by types ( $a$ )? The classical example of such a type is the length-indexed lists  $(\text{Vec } a \{i\})$  outlined in Figure 1. From this motivating example, we learn that the calculus would need four additional constructs:

- index arrow kinds  $(\text{Nat} \rightarrow *)$ ,
- index abstraction  $(\lambda i^{\text{Nat}} . \dots)$ ,
- index application  $(X\{i\})$ , and
- index polymorphism  $(\forall i^{\text{Nat}} . \dots)$ .

## 3. System $F_i$

System  $F_i$  is a higher-order polymorphic lambda calculus with term indices. System  $F_i$  was designed to extend System  $F_\omega$  by the inclusion of term indices. The complete syntax and rules of  $F_i$  are described in Figure 2 and Figure 3. The syntax and rules are highlighted by **grey boxes**. The extensions new to  $F_i$ , which are not originally part of  $F_\omega$ , appear in the boxes. If one excludes all the grey boxes from Figure 2 and Figure 3, one obtains a version of  $F_\omega$ . In particular, it is a version of  $F_\omega$  with Curry-style terms and typing contexts separated into two parts (type level and term level contexts). We first discuss the rationale for these design choices (§3.1), and then introduce the new constructs of  $F_i$ , which are not found in  $F_\omega$  (§3.2).

### 3.1 Rationale for the design choices

Terms in  $F_i$  are Curry style. That is, term level abstractions are unannotated  $(\lambda x . t)$ , and type generalizations  $(\forall I)$  and type instan-

tiations  $(\forall E)$  are implicit at term level. A Curry-style calculus generally has an advantage over its Church-style counterpart when reasoning about properties of reduction. For instance, the Church-Rosser property naturally holds for  $\beta$ -,  $\eta$ -, and  $\beta\eta$ -reduction in the Curry style, but may not hold in the Church style. This is due to the presence of annotations in abstractions [9].<sup>1</sup>

Type constructors, on the other hand, remain Church style in  $F_i$ . That is, type level abstractions are annotated by kinds  $(\lambda X^k . F)$ . Choosing type constructors to be Church style makes the kind of a type constructor visually explicit. The choice of style for type constructors is not as crucial as the choice of style for terms, since the syntax and kinding rules at type level are essentially a simply typed lambda calculus. Annotating the type level abstractions with kinds makes kinds explicit in the type syntax. Since  $F_i$  is essentially an extension of  $F_\omega$  with a new formation rule for kinds, making kinds explicit is a pedagogical tool to emphasize the consequences of this new formation rule. As a notational convention, we write  $A$  and  $B$ , instead of  $F$  and  $G$ , where  $A$  and  $B$  are expected to be types (i.e., nullary type constructors) of kind  $*$ .

In a language with term indices, terms appear in types (e.g., the length index  $(n + m)$  in the type  $\text{Vec } \text{Nat } \{n + m\}$ ). Such terms contain variables. The binding sites of these variables matter. In  $F_i$ , we expect such variables to be statically bound. Dynamically bound index variables would require a dependently typed calculus, such as the calculus of constructions. To reflect this design choice, typing contexts are separated into type level contexts  $(\Delta)$  and term level contexts  $(\Gamma)$ . Type level (static) variables  $(X, i)$  are bound in  $\Delta$  and term (dynamic) variables  $(x)$  are bound in  $\Gamma$ . Type level variables are either type constructor variables  $(X)$  or term variables to be used as indices  $(i)$ . As a notational convention, we write  $i$ , instead of  $x$ , when term variables are to be used as indices (i.e., introduced by either index abstraction or index polymorphism).

In contrast to our design choice, System  $F_\omega$  is most often formalized using a single context, which binds both type variables  $(X)$  and term variables  $(x)$ . In such a formalization, the free type variables in the typing of the term variable must be bound earlier in the context. For example, if  $X_1$  and  $X_2$  appear free in the type of  $x$ , they must appear earlier in the single context  $(\Gamma)$  as below:

$$\Gamma = \dots, X_1^*, \dots, X_2^*, \dots, (x : \forall X^* . X_1 \rightarrow X_2 \rightarrow X), \dots$$

In such a formalization, the side condition  $(X \notin \Gamma)$  in the  $(\forall I)$  rule of Figure 2 is not necessary, since such a condition is already a part of the well-formedness condition for the context (i.e.,  $\Gamma, X^k$  is well-formed when  $X \notin \text{FV}(\Gamma)$ ). Thus, for  $F_\omega$ , it is only a matter of taste whether to formalize the system using a single context or two contexts, since they are equivalent formalizations with comparable complexity.

However, in  $F_i$ , we separate the context into two parts to distinguish term variables used in types (which we call index variables, or indices, and are bound as  $\Delta, i^A$ ) from the ordinary use of term variables (which are bound as  $\Gamma, x : A$ ). The expectation is that indices should have no effect on reduction at the term level. Although it is imaginable to formalize  $F_i$  with a single typing context and distinguish index variables from ordinary term variables using more general concepts (e.g., capability, modality), we think that splitting the typing context into two parts is the simplest solution.

### 3.2 The constructs new to $F_i$ compared to $F_\omega$

We expect readers to be familiar with  $F_\omega$  and focus on describing the new constructs of  $F_i$ , which appear in the grey boxes.

<sup>1</sup> The Church-Rosser property, in its strictest sense (i.e.,  $\alpha$ -equivalence over terms), generally does not hold in Church-style calculi, but may hold under certain approximations, such as modulo ignoring the annotations in abstractions.

**Syntax:**

Term Variables	$x, i$
Type Constructor Variables	$X$
Kinds	$\kappa ::= * \mid \kappa \rightarrow \kappa \mid A \rightarrow \kappa$
Type Constructors	$A, B, F, G ::= X \mid A \rightarrow B \mid \lambda X^\kappa. F \mid F G \mid \forall X^\kappa. B \mid \lambda i^A. F \mid F \{s\} \mid \forall i^A. B$
Terms	$r, s, t ::= x \mid \lambda x. t \mid r s$
Typing Contexts	$\Delta ::= \cdot \mid \Delta, X^\kappa \mid \Delta, i^A$ $\Gamma ::= \cdot \mid \Gamma, x : A$

**Well-formed typing contexts:**

$$\boxed{\vdash \Delta} \quad \frac{}{\vdash \cdot} \quad \frac{\vdash \Delta \quad \vdash \kappa : \square}{\vdash \Delta, X^\kappa} (X \notin \text{dom}(\Delta)) \quad \frac{\vdash \Delta \quad \vdash A : *}{\vdash \Delta, i^A} (i \notin \text{dom}(\Delta))$$

$$\boxed{\Delta \vdash \Gamma} \quad \frac{\vdash \Delta}{\Delta \vdash \cdot} \quad \frac{\Delta \vdash \Gamma \quad \Delta \vdash A : *}{\Delta \vdash \Gamma, x : A} (x \notin \text{dom}(\Gamma))$$

**Sorting:**  $\boxed{\vdash \kappa : \square}$  (A)  $\frac{}{\vdash * : \square}$  (R)  $\frac{\vdash \kappa : \square \quad \vdash \kappa' : \square}{\vdash \kappa \rightarrow \kappa' : \square}$  (Ri)  $\frac{\vdash A : * \quad \vdash \kappa : \square}{\vdash A \rightarrow \kappa : \square}$

**Kinding:**  $\boxed{\Delta \vdash F : \kappa}$  (Var)  $\frac{X^\kappa \in \Delta \quad \vdash \Delta}{\Delta \vdash X : \kappa}$  ( $\rightarrow$ )  $\frac{\Delta \vdash A : * \quad \Delta \vdash B : *}{\Delta \vdash A \rightarrow B : *}$

$$(\lambda) \frac{\vdash \kappa : \square \quad \Delta, X^\kappa \vdash F : \kappa'}{\Delta \vdash \lambda X^\kappa. F : \kappa \rightarrow \kappa'} \quad (@) \frac{\Delta \vdash F : \kappa \rightarrow \kappa' \quad \Delta \vdash G : \kappa}{\Delta \vdash F G : \kappa'} \quad (\forall) \frac{\vdash \kappa : \square \quad \Delta, X^\kappa \vdash B : *}{\Delta \vdash \forall X^\kappa. B : *}$$

$$(\lambda i) \frac{\vdash A : * \quad \Delta, i^A \vdash F : \kappa}{\Delta \vdash \lambda i^A. F : A \rightarrow \kappa} \quad (@i) \frac{\Delta \vdash F : A \rightarrow \kappa \quad \Delta; \Delta \vdash s : A}{\Delta \vdash F \{s\} : \kappa} \quad (\forall i) \frac{\vdash A : * \quad \Delta, i^A \vdash B : *}{\Delta \vdash \forall i^A. B : *}$$

$$(Conv) \frac{\Delta \vdash A : \kappa \quad \Delta \vdash \kappa = \kappa' : \square}{\Delta \vdash A : \kappa'}$$

**Typing:**  $\boxed{\Delta; \Gamma \vdash t : A}$  ( $\cdot$ )  $\frac{x : A \in \Gamma \quad \Delta \vdash \Gamma}{\Delta; \Gamma \vdash x : A}$  ( $\cdot i$ )  $\frac{i^A \in \Delta \quad \Delta \vdash \Gamma}{\Delta; \Gamma \vdash i : A}$

$$(\rightarrow I) \frac{\Delta \vdash A : * \quad \Delta; \Gamma, x : A \vdash t : B}{\Delta; \Gamma \vdash \lambda x. t : A \rightarrow B} \quad (\rightarrow E) \frac{\Delta; \Gamma \vdash r : A \rightarrow B \quad \Delta; \Gamma \vdash s : A}{\Delta; \Gamma \vdash r s : B}$$

$$(\forall I) \frac{\vdash \kappa : \square \quad \Delta, X^\kappa; \Gamma \vdash t : B}{\Delta; \Gamma \vdash t : \forall X^\kappa. B} (X \notin \text{FV}(\Gamma)) \quad (\forall E) \frac{\Delta; \Gamma \vdash t : \forall X^\kappa. B \quad \Delta \vdash G : \kappa}{\Delta; \Gamma \vdash t : B[G/X]}$$

$$(\forall Ii) \frac{\vdash A : * \quad \Delta, i^A; \Gamma \vdash t : B}{\Delta; \Gamma \vdash t : \forall i^A. B} \left( \begin{array}{l} i \notin \text{FV}(t), \\ i \notin \text{FV}(\Gamma) \end{array} \right) \quad (\forall Ei) \frac{\Delta; \Gamma \vdash t : \forall i^A. B \quad \Delta; \cdot \vdash s : A}{\Delta; \Gamma \vdash t : B[s/i]}$$

$$(=) \frac{\Delta; \Gamma \vdash t : A \quad \Delta \vdash A = B : *}{\Delta; \Gamma \vdash t : B}$$

**Reduction:**  $\boxed{t \rightsquigarrow t'}$   $\frac{}{(\lambda x. t) s \rightsquigarrow t[s/x]} \quad \frac{t \rightsquigarrow t'}{\lambda x. t \rightsquigarrow \lambda x. t'} \quad \frac{r \rightsquigarrow r' \quad s \rightsquigarrow s'}{r s \rightsquigarrow r' s} \quad \frac{s \rightsquigarrow s'}{r s \rightsquigarrow r s'}$

**Figure 2.** Syntax, Typing rules, and Reduction rules of  $F_i$

**Kind equality:**  $\boxed{\vdash \kappa = \kappa' : \square}$   $\frac{}{\vdash * = * : \square}$   $\frac{\vdash \kappa_1 = \kappa'_1 : \square \quad \vdash \kappa_2 = \kappa'_2 : \square}{\vdash \kappa_1 \rightarrow \kappa_2 = \kappa'_1 \rightarrow \kappa'_2 : \square}$   $\frac{\cdot \vdash A = A' : * \quad \vdash \kappa = \kappa' : \square}{\vdash A \rightarrow \kappa = A' \rightarrow \kappa' : \square}$

$\frac{\vdash \kappa = \kappa' : \square}{\vdash \kappa' = \kappa : \square}$   $\frac{\vdash \kappa = \kappa' : \square \quad \vdash \kappa' = \kappa'' : \square}{\vdash \kappa = \kappa'' : \square}$

**Type constructor equality:**  $\boxed{\Delta \vdash F = F' : \kappa}$   $\frac{\Delta, X^\kappa \vdash F : \kappa' \quad \Delta \vdash G : \kappa}{\Delta \vdash (\lambda X^\kappa. F) G = F[G/X] : \kappa'}$   $\frac{\Delta, i^A \vdash F : \kappa \quad \Delta; \cdot \vdash s : A}{\Delta \vdash (\lambda i^A. F) \{s\} = F[s/i] : \kappa}$

$\frac{\Delta \vdash X : \kappa}{\Delta \vdash X = X : \kappa}$   $\frac{\Delta \vdash A = A' : * \quad \Delta \vdash B = B' : *}{\Delta \vdash A \rightarrow B = A' \rightarrow B' : *}$

$\frac{\vdash \kappa : \square \quad \Delta, X^\kappa \vdash F = F' : \kappa'}{\Delta \vdash \lambda X^\kappa. F = \lambda X^\kappa. F' : \kappa \rightarrow \kappa'}$   $\frac{\Delta \vdash F = F' : \kappa \rightarrow \kappa' \quad \Delta \vdash G = G' : \kappa}{\Delta \vdash F G = F' G' : \kappa'}$   $\frac{\vdash \kappa : \square \quad \Delta, X^\kappa \vdash B = B' : *}{\Delta \vdash \forall X^\kappa. B = \forall X^\kappa. B' : *}$

$\frac{\cdot \vdash A : * \quad \Delta, i^A \vdash F = F' : \kappa}{\Delta \vdash \lambda i^A. F = \lambda i^A. F' : A \rightarrow \kappa}$   $\frac{\Delta \vdash F = F' : A \rightarrow \kappa \quad \Delta; \cdot \vdash s = s' : A}{\Delta \vdash F \{s\} = F' \{s'\} : \kappa}$   $\frac{\cdot \vdash A : * \quad \Delta, i^A \vdash B = B' : *}{\Delta \vdash \forall i^A. B = \forall i^A. B' : *}$

$\frac{\Delta \vdash F = F' : \kappa}{\Delta \vdash F' = F : \kappa}$   $\frac{\Delta \vdash F = F' : \kappa \quad \Delta \vdash F' = F'' : \kappa}{\Delta \vdash F = F'' : \kappa}$

**Term equality:**  $\boxed{\Delta; \Gamma \vdash t = t' : A}$   $\frac{\Delta; \Gamma, x : A \vdash t : B \quad \Delta; \Gamma \vdash s : A}{\Delta; \Gamma \vdash (\lambda x. t) s = t[s/x] : B}$   $\frac{\Delta; \Gamma \vdash x : A}{\Delta; \Gamma \vdash x = x : A}$

$\frac{\Delta \vdash A : * \quad \Delta; \Gamma, x : A \vdash t = t' : B}{\Delta; \Gamma \vdash \lambda x. t = \lambda x. t' : B}$   $\frac{\Delta; \Gamma \vdash r = r' : A \rightarrow B \quad \Delta; \Gamma \vdash s = s' : A}{\Delta; \Gamma \vdash r s = r' s' : B}$

$\frac{\vdash \kappa : \square \quad \Delta, X^\kappa; \Gamma \vdash t = t' : B}{\Delta; \Gamma \vdash t = t' : \forall X^\kappa. B} (X \notin \text{FV}(\Gamma))$   $\frac{\Delta; \Gamma \vdash t = t' : \forall X^\kappa. B \quad \Delta \vdash G : \kappa}{\Delta; \Gamma \vdash t = t' : B[G/X]}$

$\frac{\cdot \vdash A : * \quad \Delta, i^A; \Gamma \vdash t = t' : B}{\Delta; \Gamma \vdash t = t' : \forall i^A. B} \left( \begin{smallmatrix} i \notin \text{FV}(t), \\ i \notin \text{FV}(t'), \\ i \notin \text{FV}(\Gamma) \end{smallmatrix} \right)$   $\frac{\Delta; \Gamma \vdash t = t' : \forall i^A. B \quad \Delta; \cdot \vdash s : A}{\Delta; \Gamma \vdash t = t' : B[s/x]}$

$\frac{\Delta; \Gamma \vdash t = t' : A}{\Delta; \Gamma \vdash t' = t : A}$   $\frac{\Delta; \Gamma \vdash t = t' : A \quad \Delta; \Gamma \vdash t' = t'' : A}{\Delta; \Gamma \vdash t = t'' : A}$

**Figure 3.** Equality rules of  $F_i$

**Kinds.** The key extension to  $F_\omega$  is the addition of term-indexed arrow kinds of the form  $A \rightarrow \kappa$ . This allows type constructors to have terms as indices. The rest of the development of  $F_i$  flows naturally from this single extension.

**Sorting.** The formation of indexed arrow kinds is governed by the sorting rule  $(Ri)$ . The rule  $(Ri)$  specifies that an indexed arrow kind  $A \rightarrow \kappa$  is well-sorted when  $A$  has kind  $*$  under the empty type level context  $(\cdot)$  and  $\kappa$  is well-sorted.

Requiring the use of the empty context avoids dependent kinds (i.e., kinds depending on type level or value level bindings). The type  $A$  appearing in the index arrow kind  $A \rightarrow \kappa$  must be well-kinded under the empty type level context  $(\cdot)$ . That is,  $A$  should to be a closed type of kind  $*$ , which does not contain any free type variables or index variables. For example,  $(\text{List } X \rightarrow *)$  is not a well-sorted kind, while  $(\forall X^*. \text{List } X \rightarrow *)$  is a well-sorted kind.

**Typing contexts.** Typing contexts are split into two parts. Type level contexts  $(\Delta)$  for type level (static) bindings, and term level contexts  $(\Gamma)$  for term level (dynamic) bindings. A new form of index variable binding  $(i^A)$  can appear in type level contexts in addition to the traditional type variable bindings  $(X^\kappa)$ . There is

only one form of term level binding  $(x : A)$  that appears in term level contexts.

**Well formed typing contexts.** A type level context  $\Delta$  is well-formed if (1) it is either empty, or (2) extended by a type variable binding  $X^\kappa$  whose kind  $\kappa$  is well-sorted under  $\Delta$ , or (3) extended by an index binding  $i^A$  whose type  $A$  is well-kinded under the empty type level context at kind  $*$ . This restriction is similar to the one that occurs in the sorting rule  $(Ri)$  for term-indexed arrow kinds (see the paragraph **Sorting**). The consequence of this is that, in typing contexts and in sorts,  $A$  must be closed type (not a type constructor!) without free variables.

A term level context  $\Gamma$  is well-formed under a type level context  $\Delta$  when it is either empty or extended by a term variable binding  $x : A$  whose type  $A$  is well-kinded under  $\Delta$ .

**Type constructors and their kinding rules.** We extend the type constructor syntax by three constructs, and extend the kinding rules accordingly for these new constructs.

$\lambda i^A. F$  is the type level abstraction over an index (or, index abstraction). Index abstractions introduce indexed arrow kinds by the kinding rule  $(\lambda i)$ . Note, the use of the new form of context extension,  $i^A$ , in the kinding rule  $(\lambda i)$ .

$F\{s\}$  is the type level index application. In contrast to the ordinary type level application ( $F\ G$ ) where the argument ( $G$ ) is a type constructor, the argument of an index application ( $F\{s\}$ ) is a term ( $s$ ). We use the curly bracket notation around an index argument in a type to emphasize the transition from ordinary type to term, and to emphasize that  $s$  is an index term, which is erasable. Index applications eliminate indexed arrow kinds by the kinding rule  $(@i)$ . Note, we type check the index term ( $s$ ) under the current type level context paired with the empty term level context  $(\Delta; \cdot)$  since we do not want the index term ( $s$ ) to depend on any term level bindings. Allowing such a dependency would admit true dependent types.

$\forall i^A. B$  is an index polymorphic type. The formation of indexed polymorphic types is governed by the kinding rule  $\forall i$ , which is very similar to the formation rule  $(\forall)$  for ordinary polymorphic types.

In addition to the rules  $(\lambda i)$ ,  $(@i)$ , and  $(\forall i)$ , we need a conversion rule  $(Conv)$  at kind level. This is because the new extension to the kind syntax  $A \rightarrow \kappa$  involves types. Since kind syntax involves types, we need more than simple structural equality over kinds. The new equality over kinds is the usual structural equality extended by type constructor equality when comparing indexed arrow kinds (see Figure 3).

**Terms and their typing rules** The term syntax is exactly the same as other Curry-style calculi. We write  $x$  for ordinary term variables introduced by term level abstractions  $(\lambda x. t)$ . We write  $i$  for index variables introduced by index abstractions  $(\lambda i^A. F)$  and by index polymorphic types  $(\forall i^A. B)$ . As discussed earlier, the distinction between  $x$  and  $i$  is for the convenience of readability.

Since  $F_i$  has index polymorphic types  $(\forall i^A. B)$ , we need typing rules for index polymorphism:  $(\forall Ii)$  for index generalization and  $(\forall Ei)$  for index instantiation.

The index generalization rule  $(\forall Ii)$  is similar to the type generalization rule  $(\forall I)$ , but generalizes over index variables ( $i$ ) rather than type constructor variables ( $X$ ). The rule  $(\forall Ii)$  has two side conditions while the rule  $(\forall I)$  has only one side conditions. The additional side condition  $i \notin FV(t)$  in the  $(\forall Ii)$  rule prevents terms from accessing the type level index variables introduced by index polymorphism. Without this side condition,  $\forall$ -binder would no longer behave polymorphically, but instead would behave as a dependent function, which are usually denoted by the  $\Pi$ -binder in dependent type theories. The rule  $(\forall I)$  for ordinary type generalization does not need such additional side condition because type variables cannot appear in the syntax of terms. The side conditions on generalization rules for polymorphism is fairly standard in dependently typed languages supporting distinctions between polymorphism (or, erasable arguments) and dependent functions (e.g., IPTS[10], ICC[9]).

The index instantiation rule  $(\forall Ei)$  is similar to the type instantiation rule  $(\forall E)$ , except that we type check the index term  $s$  to be instantiated for  $i$  in the current type level context paired with the empty term level context  $(\Delta; \cdot)$  rather than the current term level context. Since index terms are at type level, they should not depend on term level bindings.

In addition to the rules  $(\forall Ii)$  and  $(\forall Ei)$  for index polymorphism, we need an additional variable rule  $(: i)$  to be able to access the index variables already in scope. Terms ( $s$ ) used at type level in index applications ( $F\{s\}$ ) should be able to access index variables already in scope. For example,  $\lambda i^A. F\{i\}$  should be well-kinded under a context where  $F$  is well-kinded, justified by the derivation in Figure 4.

## 4. Embedding datatypes and their eliminators

We demonstrate some examples of embedding datatypes into System  $F_i$ . We first illustrate the embedding of both non-recursive datatypes and recursive datatypes as Church-encoded terms (§4.1). Then, we illustrate a more involved embedding for the recursive datatypes based on two-level types (§4.2).

### 4.1 Embedding datatypes of Church-encoded terms

boolean type	$Bool = \forall X. X \rightarrow X \rightarrow X$
constructors	$True = \lambda x_1. \lambda x_2. x_1$ $False = \lambda x_1. \lambda x_2. x_2$
eliminator	$\lambda x. \lambda x_1. \lambda x_2. x\ x_1\ x_2$ (if $x$ then $x_1$ else $x_2$ )
pair type	$A_1 \times A_2 = \forall X. (A_1 \rightarrow A_2 \rightarrow X) \rightarrow X$
constructor	$pair = \lambda x_1. \lambda x_2. \lambda x'. x'\ x_1\ x_2$
eliminator	$\lambda x. \lambda x'. x\ x'$ (by passing appropriate values to $x'$ , we get $fst = \lambda x. x(\lambda x_1. \lambda x_2. x_1)$ , $snd = \lambda x. x(\lambda x_1. \lambda x_2. x_2)$ )
sum type	$A_1 + A_2 = \forall X^*. (A_1 \rightarrow X) \rightarrow (A_2 \rightarrow X) \rightarrow X$
constructors	$inl = \lambda x. \lambda x_1. \lambda x_2. x_1\ x$ $inr = \lambda x. \lambda x_2. \lambda x_1. x_2\ x$
eliminator	$\lambda x. \lambda x_1. \lambda x_2. x\ x_1\ x_2$ (case $x$ of { $inl\ x' \rightarrow x_1\ x'$ ; $inr\ x' \rightarrow x_2\ x'$ })
list type	$List = \lambda A^*. \forall X^*. (A \rightarrow X \rightarrow X) \rightarrow X \rightarrow X$
constructors	$cons = \lambda x_a. \lambda x. \lambda x_c. \lambda x_n. x_c\ x_a\ (x\ x_c\ x_n)$ $nil = \lambda x_c. \lambda x_n. \lambda x_n$
eliminator	$\lambda x. \lambda x_c. \lambda x_n. x\ x_c\ x_n$ ( $foldr\ x_z\ x_c\ x$ in Haskell)
lists encoding of type $List\ A = \forall X. (A \rightarrow X \rightarrow X) \rightarrow X \rightarrow X$	
constructors	$cons = \lambda x_a. \lambda x. \lambda x_c. \lambda x_n. x_c\ x_a\ (x\ x_c\ x_n)$ , $nil = \lambda x_c. \lambda x_n. \lambda x_n$
eliminator	$\lambda x. \lambda x_c. \lambda x_n. x\ x_c\ x_n$ ( $foldr\ x_z\ x_c\ x$ in Haskell)

$$Powl = \lambda A^*. \forall X^{* \rightarrow *}. (A \rightarrow X(A \times A) \rightarrow XA) \rightarrow XA \rightarrow XA$$

$$Powl = \lambda A^*. \forall X^{* \rightarrow *}. (A \rightarrow X(A \times A) \rightarrow XA) \rightarrow XA \rightarrow XA \rightarrow XA$$

$$List = \lambda A^*. \forall X^*. (A \rightarrow X \rightarrow X) \rightarrow X \rightarrow X \rightarrow X$$

### 4.2 Embedding the recursive type operator and the Mendler-style iterators

We can divide a recursive datatype definition into two levels, by factoring out the recursive type operator, which ties the knot of the recursive definition, and a non-recursive base structure, which describes the shape (i.e., number of data constructors and their types) of the recursive datatype.

For the non-recursive base structures, or non-recursive datatypes, we can use the same impredicative encodings in the previous subsection. That is, booleans, sums, and products are encoded as in Figure TODO.

Encoding of two-level types are more involved than the encoding, but iterator definitions become more general and uniform.

## 5. Metatheory

The expectation is that System  $F_i$  has all the nice properties of System  $F_\omega$ , yet is more expressive because of the addition of term-indexed types.

We show some basic well-formedness properties for the judgments of  $F_i$  in §5.1. We prove erasure properties of  $F_i$ , which captures the idea that indices are erasable since they are irrelevant for reduction in §5.2. We show strong normalization, logical consis-

$$\begin{array}{c}
(\lambda i) \cdot \vdash A : * \\
\hline
(\text{@}i) \frac{\Delta, i^A \vdash F : A \rightarrow \kappa \quad (\text{: } i) \frac{i^A \in \Delta, i^A \quad \Delta \vdash \cdot}{\Delta, i^A; \cdot \vdash i : A}}{\Delta, i^A \vdash F\{i\} : \kappa} \\
\hline
\Delta \vdash \lambda i^A. F\{i\} : A \rightarrow \kappa
\end{array}$$

**Figure 4.** Kinding derivation for an index abstraction

```

newtype Mu* (f :: * -> *)
  = In* (f (Mu* f))

data ListF (a :: *) (r :: *)
  = Nil
  | Cons a r

type List a = Mu* (ListF a)
nil         = In* Nil
cons x xs   = In* (Cons x xs)

mit* :: (∀ r. (r -> x) -> f r -> x) -> Mu0 f -> x
mit* phi (In* z) = phi (mit* phi) z

newtype Mu(*->*) (f :: (*->*) -> (*->*)) (a :: *)
  = In(*->*) (f (Mu(*->*) f)) a

data PowlF (r :: *->*) (a :: *)
  = PNil
  | PCons a (r(a,a))

type Powl a = Mu(*->*) PowlF a
pnil        = In(*->*) PNil
pcons x xs  = In(*->*) (PCons x xs)

mit(*->*) :: (∀ r a. (∀ a. r a -> x a) -> f r a -> x a)
          -> Mu(*->*) f a -> x a
mit(*->*) phi (In(*->*) z) = phi (mit(*->*) phi) z

-- above is Haskell (with some GHC extensions)
-- below is Haskell-ish pseudocode

newtype Mu(Nat->*) (f :: (Nat->*) -> (Nat->*)) {n :: Nat}
  = In(Nat->*) (f (Mu(Nat->*) f)) {n}

data VecF (a :: *) (r :: Nat->*) {n :: Nat} where
  VNil :: VecF a r {Z}
  VCons :: a -> r n -> VecF a r {S n}

type Vec a {n :: Nat} = Mu(Nat->*) (VecF a) {n}
vnil          = In(Nat->*) VNil
vcons x xs    = In(Nat->*) (VCons x xs)

mit(Nat->*) :: (∀ r n. (∀ n. r {n} -> x {n}) -> f r {n} -> x {n})
            -> Mu(Nat->*) f {n} -> x {n}
mit(Nat->*) phi (In(Nat->*) z) = phi (mit(Nat->*) phi) z

```

**Figure 5.** 2-level types and their Mendler-style iterators in Haskell

tence, and subject reduction for  $F_i$  by reasoning about well-known calculi related to  $F_i$  in §5.3.

### 5.1 Well-formedness properties and substitution lemmas

We want to show that the sorting, kinding, and typing derivations give well-formed results under well-formed contexts. That is, sorting derivations result in well-formed sorts (Proposition 1), kinding derivations result in well-sorted kinds under well-formed type level contexts (Proposition 2), and typing derivations result in well-kinded types under well-formed type and term level contexts (Proposition 3).

Since the definitions of sorting, kinding, and typing rules are mutually recursive, these three properties are considered as one big property (illustrated below) in order to be more rigorous about the induction principle used in the proof.

**Proposition** (The big well-formedness property of  $F_i$ , roughly<sup>2</sup>).

$$\begin{array}{l}
\text{case } \boxed{\vdash \kappa : \square} \quad \frac{\vdash \kappa : \mathfrak{s}}{\mathfrak{s} = \square} \\
\text{(Proposition 1)} \\
\text{case } \boxed{\Delta \vdash F : \kappa} \quad \frac{\vdash \Delta \quad \Delta \vdash F : \kappa}{\vdash \kappa : \square} \\
\text{(Proposition 2)} \\
\text{case } \boxed{\Delta; \Gamma \vdash t : A} \quad \frac{\Delta \vdash \Gamma \quad \Delta; \Gamma \vdash t : A}{\Delta \vdash A : *} \\
\text{(Proposition 3)}
\end{array}$$

The big well-formedness property has one of the three forms –  $\boxed{\vdash \kappa : \square}$  (sorting),  $\boxed{\Delta \vdash F : \kappa}$  (kinding), and  $\boxed{\Delta; \Gamma \vdash t : A}$  (typing). That is, a derivation for a judgment of either sorting, kinding, or typing results in either a well-formed sort (when it is a sorting judgment), a well-sorted kind (when it is a kinding judgment), or a well-kinded type (when it is a typing judgment), under well-formed contexts for the judgment (no context for sorting judgments,  $\Delta$  for kinding judgments, and  $\Delta; \Gamma$  for typing judgments).

We can prove the big well-formedness property of  $F_i$  by induction on the derivation of a judgment, which can be any one of the three forms. Here, we illustrate the proof for the three propositions as if they were separate proofs. Because it provides a more intuitive proof sketch, during the proof description, the proof for each proposition references the other properties (which are yet another application of the induction hypothesis of the big well-formedness property). So, when we say “by induction” during the proofs, what we really mean is the induction hypothesis of the big well-formedness property.

**Proposition 1** (sorting derivations result in well-formed sorts).

$$\frac{\vdash \kappa : \mathfrak{s}}{\mathfrak{s} = \square}$$

<sup>2</sup>Technically, this is not yet completely rigorous since there are three more forms of judgments in the mutually recursive definition. The *kind equality*, *type considered equality*, and *term equality* rules are part of the mutually recursive definition along with the sorting, kinding, and typing rules. So, the complete description of the big well-formedness property will consist of six cases, which correspond to Proposition 1, Proposition 2, Proposition 3, Lemma 1, Lemma 2, and Lemma 3.

*Proof.* Obvious since  $\square$  is the only sort in  $F_i$ .  $\square$

**Proposition 2** (kinding derivations under well-formed contexts result in well-sorted kinds).

$$\frac{\vdash \Delta \quad \Delta \vdash F : \kappa}{\vdash \kappa : \square}$$

*Proof.* By induction on the derivation.

case (*Var*) Trivial by the second well-formedness rule of  $\Delta$ .

case (*Conv*) By induction and Lemma 1.

case ( $\lambda$ )

By induction and Proposition 1 we know that  $\vdash \kappa : \square$ .

By the second well-formedness rule of  $\Delta$ , we know that  $\vdash \Delta, X^\kappa$  since we already know that  $\vdash \kappa : \square$  and  $\vdash \Delta$  from the property statement.

By induction, we know that  $\vdash \kappa' : \square$  since we already know that  $\vdash \Delta, X^\kappa$  and that  $\Delta, X^\kappa \vdash F : \kappa'$  from induction hypothesis.

By the sorting rule (*R*), we know that  $\vdash \kappa \rightarrow \kappa' : \square$  since we already know that  $\vdash \kappa : \square$  and  $\vdash \kappa' : \square$ .

case ( $\otimes$ ) By induction, easy.

case ( $\lambda i$ )

By induction and Proposition 3 we know that  $\vdash A : *$ . By the third well-formedness rule of  $\Delta$ , we know that  $\vdash \Delta, i^A$  since we already know that  $\vdash A : *$  and that  $\vdash \Delta$  from the property statement.

By induction, we know that  $\vdash \kappa : \square$  since we already know that  $\vdash \Delta, i^A$  and that  $\Delta, i^A \vdash F : \kappa$  from the induction hypothesis.

By the sorting rule (*Ri*), we know that  $\vdash A \rightarrow \kappa : \square$  since we already know that  $\vdash A : *$  and  $\vdash \kappa : \square$ .

case ( $\otimes i$ ) By induction and Proposition 3, easy.

case ( $\rightarrow$ ) Trivial since  $\vdash * : \square$ .

case ( $\forall$ ) Trivial since  $\vdash * : \square$ .

case ( $\forall i$ ) Trivial since  $\vdash * : \square$ .  $\square$

The basic structure of the proof for the following proposition on typing derivations is similar to above. So, we illustrate the proof for most of the cases, which can be done by applying the induction hypothesis, rather bravely. We elaborate more on interesting cases ( $\forall E$ ) and ( $\forall Ei$ ) which involve substitutions in the types resulting from the typing judgments.

**Proposition 3** (typing derivations under well-formed contexts result in well-kinded types).

$$\frac{\Delta \vdash \Gamma \quad \Delta; \Gamma \vdash t : A}{\Delta \vdash A : *}$$

*Proof.* By induction on the derivation.

case ( $:$ ) Trivial by the second well-formedness rule of  $\Gamma$ .

case ( $: i$ ) Trivial by the third the well-formedness rule of  $\Delta$ .

case ( $=$ ) By induction and Lemma 2.

case ( $\rightarrow I$ ) By induction and well-formedness of  $\Gamma$ .

case ( $\rightarrow E$ ) By induction.

case ( $\forall I$ ) By induction and well-formedness of  $\Delta$ .

case ( $\forall E$ )

By induction we know that  $\Delta \vdash \forall X^\kappa. B : *$ .

By the kinding rule ( $\forall$ ), which is the only kinding rule able to derive  $\Delta \vdash \forall X^\kappa. B : *$ , we know that  $\Delta, X^\kappa \vdash B : *$ .

Then, we use the type substitution lemma (Lemma 4).

case ( $\forall Ei$ ) By induction and well-formedness of  $\Delta$ .

case ( $\forall Ei$ )

By induction we know that  $\Delta \vdash \forall i^A. B : *$ .

By the kinding rule ( $\forall i$ ), which is the only kinding rule able to derive  $\Delta \vdash \forall i^A. B : *$ , we know that  $\Delta, i^A \vdash B : *$ .

Then, we use the index substitution lemma (Lemma 5).  $\square$

**Lemma 1** (kind equality is well-sorted).  $\frac{\vdash \kappa = \kappa' : \square}{\vdash \kappa : \square \quad \vdash \kappa' : \square}$

*Proof.* By induction on the derivation of kind equality and using the sorting rules.  $\square$

**Lemma 2** (type constructor equality is well-kinded).

$$\frac{\Delta \vdash F = F' : \kappa}{\Delta \vdash F : \kappa \quad \Delta \vdash F' : \kappa}$$

*Proof.* By induction on the derivation of type constructor equality and using the kinding rules. Also use the type substitution lemma (Lemma 4) and the index substitution lemma (Lemma 5).  $\square$

**Lemma 3** (term equality is well-typed).

$$\frac{\Delta, \Gamma \vdash t = t' : A}{\Delta, \Gamma \vdash t : A \quad \Delta, \Gamma \vdash t' : A}$$

*Proof.* By induction on the derivation of term equality and using the typing rules. Also use the term substitution lemma (Lemma 6).  $\square$

The proofs for the three lemmas above are straightforward once we have dealt with the interesting cases for the equality rules involving substitution. We can prove those interesting cases by applying the substitution lemmas. The other cases fall into two categories: firstly, the equality rules following the same structure of the sorting, kinding, and typing rules; and secondly, the reflexive rules and the transitive rules. The proof for the equality rules following the same structure of the sorting, kinding, and typing rules can be proved by induction and applying the corresponding sorting, kinding, and typing rules. The proof for the reflexive rules and the transitive rules can be proved simply by induction.

**Lemma 4** (type substitution).  $\frac{\Delta, X^\kappa \vdash F : \kappa' \quad \Delta \vdash G : \kappa}{\Delta \vdash F[G/X] : \kappa'}$

**Lemma 5** (index substitution).  $\frac{\Delta, i^A \vdash F : \kappa \quad \Delta; \vdash s : A}{\Delta \vdash F[s/i] : \kappa}$

**Lemma 6** (term substitution).  $\frac{\Delta; \Gamma, x : A \vdash t : B \quad \Delta; \vdash s : A}{\Delta, \Gamma \vdash t[s/x] : B}$

## 5.2 Erasure properties

**Definition 1** (index erasure).

$$\boxed{\kappa^\circ} \quad *^\circ = * \quad (\kappa \rightarrow \kappa')^\circ = \kappa^\circ \rightarrow \kappa^\circ \quad (A \rightarrow \kappa)^\circ = \kappa^\circ$$

$$\boxed{F^\circ} \quad X^\circ = X \quad (A \rightarrow B)^\circ = A^\circ \rightarrow B^\circ$$

$$(\lambda X^\kappa. F)^\circ = \lambda X^{\kappa^\circ}. F^\circ \quad (\lambda i^A. F)^\circ = F^\circ$$

$$(F G)^\circ = F^\circ G^\circ \quad (F \{s\})^\circ = F^\circ$$

$$(\forall X^\kappa. B)^\circ = \forall X^{\kappa^\circ}. B^\circ \quad (\forall i^A. B)^\circ = B^\circ$$

$$\boxed{\Delta^\circ} \quad \cdot^\circ = \cdot \quad (\Delta, X^\kappa)^\circ = \Delta^\circ, X^{\kappa^\circ} \quad (\Delta, i^A)^\circ = \Delta^\circ$$

$$\boxed{\Gamma^\circ} \quad \cdot^\circ = \cdot \quad (\Gamma, x : A)^\circ = \Gamma^\circ, x : A^\circ$$

**Theorem 1** (index erasure on well-sorted kinds).  $\frac{\vdash \kappa : \square}{\vdash \kappa^\circ : \square}$

*Proof.* By induction on the sorting derivation.  $\square$

*Remark 1.* For any well-sorted kind  $\kappa$  in  $F_i$ ,  $\kappa^\circ$  is a kind in  $F_\omega$ .

**Theorem 2** (index erasure on well-formed type level contexts).

$$\frac{\vdash \Delta}{\vdash \Delta^\circ}$$

*Proof.* By induction on the derivation for well-formed type level context and using Theorem 1.  $\square$

*Remark 2.* For any well-formed type level context  $\Delta$  in  $F_i$ ,  $\Delta^\circ$  is a well-formed type level context in  $F_\omega$ .

**Theorem 3** (index erasure on kind equality).  $\frac{\vdash \kappa = \kappa' : \square}{\vdash \kappa^\circ = \kappa'^\circ : \square}$

*Proof.* By induction on the kind equality judgement.  $\square$

*Remark 3.* For any well-sorted kind equality  $\vdash \kappa = \kappa' : \square$  in  $F_i$ ,  $\vdash \kappa^\circ = \kappa'^\circ : \square$  is a well-sorted kind equality in  $F_\omega$ .

The three theorems above on kinds are rather simple to prove since there is no need to consider mutual recursion in the definition of kinds due to the erasure operation on kinds. Recall that the erasure operation on kinds discards the type ( $A$ ) appearing in the index arrow type ( $A \rightarrow \kappa$ ). So, there is no need to consider the types appearing in kinds and the index terms appearing in those types, after the erasure.

**Theorem 4** (index erasure on well-kinded type constructors).

$$\frac{\vdash \Delta \quad \Delta \vdash F : \kappa}{\Delta^\circ \vdash F^\circ : \kappa^\circ}$$

*Proof.* By induction on the kinding derivation.

case (*Var*) Use Theorem 2.

case (*Conv*) By induction and using Theorem 3.

case ( $\lambda$ ) By induction and using Theorem 1.

case ( $\textcircled{\text{A}}$ ) By induction.

case ( $\lambda i$ )

We need to show that  $\Delta^\circ \vdash (\lambda i^A.F)^\circ : (A \rightarrow \kappa)^\circ$ , which simplifies to  $\Delta^\circ \vdash F^\circ : \kappa^\circ$  by Definition 1.

By induction, we know that  $(\Delta, i^A)^\circ \vdash F^\circ : \kappa^\circ$ , which simplifies  $\Delta^\circ \vdash F^\circ : \kappa^\circ$  by Definition 1.

case ( $\textcircled{\text{A}}i$ )

We need to show that  $\Delta^\circ \vdash (F \{s\})^\circ : \kappa^\circ$ , which simplifies to  $\Delta^\circ \vdash F^\circ : \kappa^\circ$  by Definition 1.

By induction we know that  $\Delta^\circ \vdash F^\circ : (A \rightarrow \kappa)^\circ$ , which simplifies to  $\Delta^\circ \vdash F^\circ : \kappa^\circ$  by Definition 1.

case ( $\rightarrow$ ) By induction.

case ( $\forall$ )

We need to show that  $\Delta^\circ \vdash (\forall X^\kappa.B)^\circ : *^\circ$ , which simplifies to  $\Delta^\circ \vdash \forall X^{\kappa^\circ}.B^\circ : *$  by Definition 1.

Using Theorem 1, we know that  $\vdash \kappa^\circ : \square$ .

By induction we know that  $(\Delta, X^\kappa)^\circ \vdash B^\circ : *^\circ$ , which simplifies to  $\Delta^\circ, X^{\kappa^\circ} \vdash B^\circ : *$  by Definition 1.

Using the kinding rule ( $\forall$ ), we get exactly what we need to show:  $\Delta^\circ \vdash \forall X^{\kappa^\circ}.B^\circ : *$ .

case ( $\forall i$ )

We need to show that  $\Delta^\circ \vdash (\forall i^A.B)^\circ : *^\circ$ , which simplifies to  $\Delta^\circ \vdash B^\circ : *$  by Definition 1.

By induction we know that  $(\Delta, i^A)^\circ \vdash B^\circ : *^\circ$ , which simplifies  $\Delta^\circ \vdash B^\circ : *$  by Definition 1.  $\square$

**Theorem 5** (index erasure on type constructor equality).

$$\frac{\Delta \vdash F = F' : \kappa}{\Delta^\circ \vdash F^\circ = F'^\circ : \kappa^\circ}$$

*Proof.* By induction on the derivation of type constructor equality.

Most of the cases are done by applying the induction hypothesis and sometimes using Proposition 2.

The only interesting cases, which are worth elaborating, are the equality rules involving substitution. There are two such rules.

$$\frac{\Delta, X^\kappa \vdash F : \kappa' \quad \Delta \vdash G : \kappa}{\Delta \vdash (\lambda X^\kappa.F) G = F[G/X] : \kappa'}$$

We need to show  $\Delta^\circ \vdash ((\lambda X^\kappa.F) G)^\circ = (F[G/X])^\circ : \kappa'^\circ$ , which simplifies to  $\Delta^\circ \vdash (\lambda X^{\kappa^\circ}.F^\circ) G^\circ = (F[G/X])^\circ : \kappa'^\circ$  by Definition 1.

By induction, we know that  $(\Delta, X^\kappa)^\circ \vdash F^\circ : \kappa'^\circ$ , which simplifies to  $\Delta^\circ, X^{\kappa^\circ} \vdash F^\circ : \kappa'^\circ$  by Definition 1.

Using the kinding rule ( $\lambda$ ), we get  $\Delta^\circ \vdash \lambda X^{\kappa^\circ}.F^\circ : \kappa^\circ \rightarrow \kappa'^\circ$ .

Using the kinding rule ( $\textcircled{\text{A}}$ ), we get  $\Delta^\circ \vdash (\lambda X^{\kappa^\circ}.F^\circ) G^\circ : \kappa'^\circ$ .

Using the very equality rule of this case,

we get  $\Delta^\circ \vdash (\lambda X^{\kappa^\circ}.F^\circ) G^\circ = F^\circ[G^\circ/X] : \kappa'^\circ$ .

All we need to check is  $([G/X]F)^\circ = F^\circ[G^\circ/X]$ , which is easy to see.

$$\frac{\Delta, i^A \vdash F : \kappa \quad \Delta; \cdot \vdash s : A}{\Delta \vdash (\lambda i^A.F) \{s\} = F[s/i] : \kappa}$$

By induction we know that  $\Delta^\circ \vdash F^\circ : \kappa^\circ$ .

The erasure of the left hand side of the equality is

$((\lambda i^A.F) \{s\})^\circ = (\lambda i^A.F)^\circ = F^\circ$ .

All we need to show is  $(F[s/i])^\circ = F^\circ$ , which is obvious since index variables can only occur in index terms and index terms are always erased. Recall the index erasure over type constructors in Definition 1, in particular,  $(\lambda i^A.F)^\circ = F^\circ$ ,  $(F\{s\})^\circ = F^\circ$ , and  $(\forall i^A.B)^\circ = B^\circ$ .  $\square$

*Remark 4.* For any well-kinded type constructor equality  $\Delta \vdash F = F' : \kappa$  in  $F_i$ ,  $\Delta^\circ \vdash F^\circ = F'^\circ : \kappa^\circ$  is a well-kinded type constructor equality in  $F_\omega$ .

The proofs for the two theorems above on type constructors need not consider mutual recursion in the definition of type constructors due to the erasure operation. Recall that the erasure operation on type constructors discards the index term ( $s$ ) appearing in the index application ( $F \{s\}$ ). So, there is no need to consider the index terms appearing in the types after the erasure.

**Theorem 6** (index erasure on well-formed term level contexts).

$$\frac{\Delta \vdash \Gamma}{\Delta^\circ \vdash \Gamma^\circ}$$

*Proof.* By induction on  $\Gamma$ .

case ( $\Gamma = \cdot$ ) It trivially holds.

case ( $\Gamma = \Gamma', x : A$ ), we know that  $\Delta \vdash \Gamma'$  and  $\Delta \vdash A : *$  by the well-formedness rules and that  $\Delta^\circ \vdash \Gamma'^\circ$  by induction.

From  $\Delta \vdash A : *$ , we know that  $\Delta^\circ \vdash A^\circ : *$  by Theorem 4.

We know that  $\Delta^\circ \vdash \Gamma'^\circ, x : A^\circ$  from  $\Delta^\circ \vdash \Gamma'^\circ$  and  $\Delta^\circ \vdash A^\circ : *$  by the well-formedness rules.

Since  $\Gamma'^\circ, x : A^\circ = (\Gamma', x : A)^\circ = \Gamma^\circ$  by definition, we know that  $\Delta^\circ \vdash \Gamma^\circ$ .  $\square$

**Theorem 7** (index erasure on index-free well-typed terms).

$$\frac{\Delta \vdash \Gamma \quad \Delta; \Gamma \vdash t : A \quad \text{dom}(\Delta) \cap \text{FV}(t) = \emptyset}{\Delta^\circ; \Gamma^\circ \vdash t : A^\circ}$$



*Proof.* By induction on the typing derivation. Interesting cases are the index related rules  $(: i)$ ,  $(\forall I i)$ , and  $(\forall E i)$ . Proofs for the other cases are straightforward by induction and applying other erasure theorems corresponding to the judgment forms.

- case  $(:)$  By Theorem 6, we know that  $\Delta^\circ \vdash \Gamma^\circ$  when  $\Delta \vdash \Gamma$ . By definition of erasure on term-level context, we know that  $x : A^\circ \in \Gamma^\circ$  when  $x : A \in \Gamma$ .
- case  $(: i)$  Vacuously true since  $t$  does not contain any index variables (i.e.,  $\text{dom}(\Delta) \cap \text{FV}(t) = \emptyset$ ).
- case  $(\rightarrow I)$  By Theorem 4, we know that  $\circ \vdash A^\circ : *$ . By induction, we know that  $\Delta^\circ; \Gamma^\circ, x : A^\circ \vdash t^\circ : B^\circ$ . Applying the  $(\rightarrow I)$  rule to what we know, we have  $\Delta^\circ; \Gamma^\circ \vdash \lambda x. t^\circ : A^\circ \rightarrow B^\circ$ .
- case  $(\rightarrow E)$  Straightforward by induction.
- case  $(\forall I)$  By Theorem 1, we know that  $\vdash \kappa^\circ : \square$ . By induction, we know that  $\Delta^\circ, X^{\kappa^\circ}; \Gamma^\circ \vdash t : B^\circ$ . Applying the  $(\forall I)$  rule to what we know, we have  $\Delta^\circ; \Gamma^\circ \vdash t : \forall X^{\kappa^\circ}. B^\circ$ .
- case  $(\forall E)$  By induction, we know that  $\Delta^\circ; \Gamma^\circ \vdash t : \forall X^{\kappa^\circ}. B^\circ$ . By Theorem 4, we know that  $\Delta^\circ \vdash G^\circ : \kappa^\circ$ . Applying the  $(\forall E)$  rule, we have  $\Delta^\circ; \Gamma^\circ \vdash t : B^\circ[G^\circ/X]$ .
- case  $(\forall I i)$  By Theorem 4, we know that  $\circ \vdash A^\circ : *$ . By induction, we know that  $\Delta^\circ; \Gamma^\circ \vdash t : B^\circ$ , which is what we want since  $(\forall i^A. B)^\circ = B^\circ$ .
- case  $(\forall E i)$  By induction, we know that  $\Delta^\circ; \Gamma^\circ \vdash t : B^\circ$ , which is what we want since  $(B[s/i])^\circ = B^\circ$ .
- case  $(=)$  By Theorem 5 and induction.  $\square$

**Definition 2** (index variable selection).

$$\circ^\bullet = \circ \quad (\Delta, X^A)^\bullet = \Delta^\bullet \quad (\Delta, i^A)^\bullet = \Delta^\bullet, i^A$$

The index variable selection operation  $(\bullet)$  selects all the index variable bindings from the type level context.

**Theorem 8** (index erasure on well-formed term level contexts prepended by index variable selection).

$$\frac{\Delta \vdash \Gamma}{\Delta^\circ \vdash (\Delta^\bullet, \Gamma^\circ)^\circ}$$

*Proof.* Straightforward by Theorem 6 and the typing rule  $(: i)$ .  $\square$

**Theorem 9** (index erasure on well-typed terms).

$$\frac{\Delta \vdash \Gamma \quad \Delta; \Gamma \vdash t : A}{\Delta^\circ; (\Delta^\bullet, \Gamma^\circ)^\circ \vdash t : A^\circ}$$

*Proof.* The proof is almost the same as as Theorem 7, except for the  $(: i)$  case. The proof for the  $(: i)$  case is easy since  $i^A \in \Delta^\bullet$  when  $i^A \in \Delta$  by definition of the index variable selection operation. The indices from  $\Delta$  being prepended to  $\Gamma$  do not affect the proof for the other cases.  $\square$

### 5.3 Strong normalization and logical consistency

Strong normalization is a corollary of the erasure property since we know that System  $F_\omega$  is strongly normalizing. Logical consistency is immediate since System  $F_i$  is a strict subset of the *restricted implicit calculus* [8], which is a restriction of ICC [9]. Subject reduction is also immediate since  $F_i$  is a strict subset of ICC.

### 5.4 TODO about void type instantiation TODO

Why did we bother to design  $F_i$ ? What is different from a Curry-style dependent calculus with implicit arguments such as ICC? The following rule is the instantiation

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash t : \forall x^A. B} \quad (x \notin \text{FV}(t))$$

Consider when  $A = \text{Void}$  and  $B = \forall i^{\text{Void}}. \text{NeverEverVoid } \{i\}$ . In the calculus above, we can instantiate  $i$  with  $y$  provided that  $(y : \text{Void}) \in \Gamma$ . It is a void type instantiation, Uh-Oh ...

In  $F_i$  we cannot instantiate  $B$  with any of the term variables since index instantiation can not refer to the term-level context but only refer to the type-level context  $\Delta$ . Recall

$$(\forall E i) \frac{\Delta; \Gamma \vdash t : \forall i^A. B \quad \Delta; \cdot \vdash s : A}{\Delta; \Gamma \vdash t : B[s/i]}$$

**Proposition 4** (anti-dependency on arrow kinds).

$$\frac{\vdash \Delta, X^\kappa \quad \Delta, X^\kappa \vdash F : \kappa'}{X \notin \text{FV}(\kappa')}$$

*Proof.* By Proposition 2,  $\vdash \kappa'$ . Note that  $\vdash \kappa'$  does not involve any type level context.

Therefore,  $X$  cannot appear free in  $\kappa'$ .  $\square$

**Proposition 5** (anti-dependency on indexed arrow kinds).

$$\frac{\vdash \Delta, i^A \quad \Delta, i^A \vdash F : \kappa}{i \notin \text{FV}(\kappa)}$$

*Proof.* By Proposition 2,  $\vdash \kappa'$ . Note that  $\vdash \kappa'$  does not involve any type level context. Therefore,  $i$  cannot appear free in  $\kappa'$ .  $\square$

**Proposition 6** (anti-dependency on arrow types).

$$\frac{\Delta \vdash \Gamma, x : A \quad \Delta; \Gamma, x : A \vdash t : B}{x \notin \text{FV}(B)}$$

*Proof.* By Proposition 3,  $\Delta \vdash B : *$ . Note that  $\Delta \vdash \kappa'$  does not involve any term level context. Therefore,  $x$  cannot appear free in  $B$ .  $\square$

*Remark 5.* Our system is more strong??? than anti-dependency on arrow types TODO

## 6. Related work

Among theoretical calculi,  $F_i$  is most closely related to Curry-style System  $F_\omega$  [1, 2, 5] and Implicit Calculus of Constructions (ICC) [9]. All terms typable in Curry-style System  $F_\omega$  are typable in System  $F_i$  with the same type, and all terms typable in  $F_i$  are typable in ICC with the same type.<sup>3</sup> So, the subject reduction and strong normalization of  $F_i$  can be automatically derived from ICC. ICC is more than just an extension of  $F_i$ , as described in our work, with dependent types and stratified universes, since ICC includes  $\eta$ -reduction and the extensionality typing rule. We do not foresee any problem of adding  $\eta$ -reduction and the extensionality typing rule to  $F_i$ . Although System  $F_i$  accepts less terms than ICC,  $F_i$  enjoys a stronger erasure property (Theorem ??), which ICC cannot provide due to its support for full dependent types. In System  $F_i$ , index terms appearing in types (e.g.,  $s$  in  $F\{s\}$ ) are always erasable. Mishra-Linger and Sheard [10] formalized a generic framework which describes the erasure on arbitrary Church-style calculi (EPTS) and Curry-style calculi (IPTS).

In §3.1, we have mentioned that Curry-style calculi enjoys better reduction properties (e.g.,  $\beta\eta$ -reduction is Church-Rosser) than Church-style calculi. Nederpelt [12] showed a counterexample to the Church-Rosser property for  $\beta\eta$ -reduction of Church-style terms. Geuvers [4] proved that  $\beta\eta$ -reduction is Church-Rosser in functional PTSS, which is a certain class of Church-style calculi. Seldin [13] discusses the relation between the Church-style typing and the Curry-style typing.

<sup>3</sup>The  $*$  kind in  $F_\omega$  and  $F_i$  corresponds to Set in ICC.

In a more practical setting for language implementation, Yorgey et al. [15] "Giving Haskell a Promotion" most closely related work would be this

Swamy et al. [14] value dependent types in F-star from MSR what others to discuss?

Translating Generalized Algebraic Data Types to System F Martin Sulzmann and Meng Wang

Stephanie's Rw and related work. they prove parametricity in the presense of indices of GADTs

## 7. Conclusion and Future work

TODO

mention nax

We are exploring whether Leibniz equality over indices (i.e.,  $s_1 = s_2$  encoded as  $\forall X^{A \rightarrow *}. X\{s_1\} \rightarrow X\{s_2\}$ ) may help us express functions whose domains are restricted by term-indices (e.g.,  $\text{vtail} :: \text{Vec } a \{S \ n\} \rightarrow \text{Vec } a \ n$ ). We wonder what extension we need to enable large eliminations (i.e., computing types from term-indices). We are also developing a programming language Nax, which supports type inference with little annotation, based on System  $F_i$ .

## A. Appendix Title

This is the text of the appendix, if you need one.

## Acknowledgments

Acknowledgments, if needed.

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