# Functional Completeness (2.1.8.–2.1.9.)

**Def.** functionally complete set of logical operators → all formulas can be rewritten to an equivalent form using only operators from the set

2.1./Ex. 9

$p \downarrow q \equiv \neg(p \vee q) \equiv \neg p \wedge \neg q$

$\neg p \equiv p \downarrow 0$

$p \wedge q \equiv \neg p \downarrow \neg q$

$p \vee q \equiv \neg(p \downarrow q)$

$p \rightarrow q \equiv \neg p \vee q \equiv \neg(\neg p \downarrow q)$

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p) \equiv (\neg p \vee q) \wedge (\neg q \vee p) \equiv$

$\equiv \neg\big((\neg p \downarrow q) \vee (p \downarrow \neg q)\big)$

$p \otimes q \equiv \neg(p \leftrightarrow q) \equiv (\neg p \downarrow q) \vee (p \downarrow \neg q)$

2.1./Ex. 10

a) two atoms $= (2^2 =)\ 4$ columns $= (2^{2^2} = 2^4 =)\ 16$ truth tables

b) $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$

| $p$ | $q$ | $p \wedge q$ | $p \vee q$ | $p \rightarrow q$ | $p \leftrightarrow q$ | $p \otimes q \equiv \neg(p \leftrightarrow q)$ | $\neg p \wedge q$ | $p \wedge \neg q$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 |
| 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |

| $p$ | $q$ | $\neg(p \vee q)$ | $p \vee \neg q$ | $\neg(p \wedge q)$ | $p \leftrightarrow p$ | $p \wedge \neg p$ | $\neg p$ | $\neg q$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 |
| 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 |

# Predicates and quantifiers (2.4.–2.4.3)

**Def.** $P(a) \to$ applying $P$ to $a$
one-place predicate entity (in the domain of discourse)

Charles Sanders Pierce

**Def.** $\forall$ – universal quantifier = every, all
$\exists$ – existential quantifier = some

$P(x) \to$ open statement
↳ free variable
[quantifier]$x\, P(x) \to x$ becomes bound

2.4./Ex. 5 $H(x) = x$ is happy
There are exactly three happy people.
$$\exists a\,(H(a) \wedge \exists b\,(\neg(a=b) \wedge H(b) \wedge \exists c\,(\neg(a=c) \wedge \neg(b=c) \wedge H(c) \wedge$$
$$\forall d\,(\neg(a=d) \wedge \neg(b=d) \wedge (c=d) \leftrightarrow \neg H(d)))$$

2.4./Ex. 8 $T(x,y) = x$ takes $y$
$x$ – students
$y$ – CS courses
a) $\forall x \forall y\, T(x,y)$ – All students take all CS courses.
b) $\forall x \exists y\, T(x,y)$ – All students take a CS course.
c) $\forall y \exists x\, T(x,y)$ – All courses are taken by a student.
d) $\exists x \exists y\, T(x,y)$ – There is a student that takes a CS course.
e) $\exists x \forall y\, T(x,y)$ – There is a student that takes all CS courses.
f) $\exists y \forall x\, T(x,y)$ – There is a course that is taken by all students.

2.4. / Ex. 10 $F(x,t)$ = can fool person $x$ at time $t$

You can fool some of the people all of the time, and you fool all of the people some of the time, but you can't fool all of the people all of the time.

$\exists x \forall t (F(x,t)) \land \exists t \forall x (F(x,t)) \land \forall x \forall t (\neg F(x,t))$

2.4. / Ex. 11

a) All crows are black. $\forall c (B(c))$

b) Any white bird is not a crow. $\exists b (W(b) \rightarrow \neg (b=c))$

c) Not all politicians are honest. $\exists p (\neg H(p))$

d) All green elephants have purple feet. $\forall e (G(e) \rightarrow P(f))$

e) There is no one who does not like pizza. $\nexists a (\neg L(a,p))$

f) Anyone who passes the final exam will pass the course.
$\forall x (P(x,e) \rightarrow P(x,c))$

g) If $x$ is any positive number, then there is a number $y$ such that $y^2 = x$. $\forall x (P(x) \rightarrow \exists y (S(x,y))$

## Equivalence (2.4.5.)

**Def.** $P$ - predicate, always true $\rightarrow$ tautology
$P, Q$ - logically equivalent if $P \Leftrightarrow Q$ is tautology ($P \equiv Q$)

2.4. / Ex. 1

a) $\neg \forall x (\neg P(x)) \equiv \exists x (P(x))$

b) $\neg \exists x (P(x) \land Q(x)) \equiv \forall x (\neg P(x) \lor \neg Q(x))$

c) $\neg \forall z (P(z) \rightarrow Q(z)) \equiv \exists z (P(z) \land \neg Q(z))$

d) $\neg ((\forall x P(x)) \land \forall y (Q(y)) \equiv \exists x (\neg P(x)) \lor \exists y (\neg Q(y))$

e) $\neg \forall x \exists y P(x,y) \equiv \exists x \forall y \neg P(x,y)$

f) $\neg \exists x (R(x) \land \forall y\, S(x,y)) \equiv \forall x (\neg R(x) \lor \exists y \neg S(x,y))$

g) $\neg \exists y (P(y) \Leftrightarrow Q(y)) \equiv \forall y (P(y) \otimes Q(y))$

h) $\neg (\forall x (P(x) \rightarrow (\exists y\, Q(x,y)))) \equiv \exists x (P(x) \land \forall y \neg Q(x,y))$

2.4. / Ex. 2 $\neg(\forall x\, P(x)) \equiv \exists x(\neg P(x))$

I don't hate everyone

There is no person I hate

2.4. / Ex. 3 a) $\neg \exists u(\forall s\, C(s,u)) \equiv \forall u(\exists s\, \neg C(s,u))$

b) $\neg \exists u(\forall s(L(s,u) \rightarrow P(s))) \equiv \forall u(\exists s(L(s,u) \wedge \neg P(s)))$

c) $\neg \exists u(\forall s(L(s,u) \rightarrow (\exists x \exists y \exists z\, Q(x,y,z)))) \equiv \forall u(\exists s(L(s,u) \wedge \forall x \forall y \forall z\, \neg Q(x,y,z)$

d) $\neg \exists u(\forall s(L(s,u) \rightarrow (\exists x \exists y \exists z(s = xyz \wedge R(x,y) \wedge T(y) \wedge U(x,y,z))))$

$\equiv \forall u(\exists s(L(s,u) \wedge (\forall x \forall y \forall z(s \neq xyz \vee \neg R(x,y) \vee \neg T(y) \vee \neg U(x,y,z)))))$

2.4. / Ex. 4 $x = \{x_1, x_2\}$

$\forall x\, P(x) \equiv P(x_1) \wedge P(x_2)$ $\quad \neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$

$\exists x\, P(x) \equiv P(x_1) \vee P(x_2)$ $\quad \neg P(x_1) \vee \neg P(x_2) \equiv \neg P(x_1) \vee \neg P(x_2)$

2.4. / Ex. 13 Jane is looking for a dog.

↳ particular one $\exists x(\forall y(\neg(x=y) \wedge LookingFor(jane, x) \wedge Dog(x))$

## Tarksi's world (2.4.4.)

2.4. / Ex. 6 $\exists x\, Red(x) \wedge \exists x\, Square(x)$ There is a red shape and a square.

$\exists x(Red(x) \wedge Square(x))$ There is a red square.

2.4. / Ex. 7 There is only one ball, so you need to have it.

$\exists x(Ball(x) \wedge Have(you, x))$

2.4. / Ex. 12 Someone has the answer to every questions.

$\exists x \forall y(A(x,y))$