ODE: $\frac{\partial x}{\partial t} = f(x,t)$, $x(t_0) = x_0$

SDE: $\frac{\partial x}{\partial t} = f(x,t) + g(x,t)\underbrace{\xi(t)}_{\text{white noise}}$, $x(t_0) = x_0$

$\mathbb{E}[\xi(t)] = 0$

$\mathbb{E}[\xi(t_1)\xi(t_2)] = \delta(t_1 - t_2)$

Ex. Biochemical-Oxygen Demand (BOD)

$\frac{\partial b}{\partial t} = -k_1 b + s_1$, b - BOD $\left[\frac{mg}{l}\right]$, k_1 - reaction rate $\left[\frac{1}{day}\right]$, s_1 - source/sink

$\mathbb{E}(N_t) = 0$, $\mathbb{E}[N_t N_s] = \delta(t-s)$, N_t - Gaussian white noise

$\Rightarrow \frac{\partial B_t}{\partial t} = -k_1 B_t + s_t + \sigma N_t$, $B_{t_0} = B_0$, uncertainty on s_1

$\frac{\partial B_t}{\partial t} = -(k_1 + \sigma N_t) B_t + s_t = -k_1 B_t + s_t - B_t \sigma N_t$, $B_{t_0} = B_0$, uncertainty on k_1

Markovian process - information on the probability density of the state X_t at time t is sufficient for computing model predictions for times $\geq t$

Ex. 1

$X_k = aX_{k-1} + bX_{k-2} + W_k$, W_k - RV

X_k is not Markovian, $b \neq 0$, X_{k-2} is useful for predicting X_k

Define state vector: $Y_k = [X_k \;\; X_{k-1}]$

Markov process: $Y_k = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} Y_{k-1} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} W_k$

W_k now depends on $W_i, i \neq k \Rightarrow Y_k$ is not Markovian, since Y_{k-2} provides information for W_{k-1}, useful for predicting Y_k

Ex. 2

$X_t = a + bt$, $X_0 = a$, a, b - RV

X_t is Gaussian process, since linear on Gaussian variables.

X_t is not Markovian.

$X_{t_n} = X_{t_{n-1}} + b(t_n - t_{n-1})$

$\Rightarrow X_{t_n} = X_{t_{n-1}} + \frac{t_n - t_{n-1}}{t_{n-1} - t_{n-2}}(X_{t_{n-1}} - X_{t_{n-2}})$

$$\frac{\partial x}{\partial t} = f(x,t) + g(x,t)\xi(t)$$

Wiener process/Brownian motion: $\frac{\partial W}{\partial t} = \xi(t)$

$$dx = f(x,t)\,dt + g(x,t)\,dW$$

$$x(t) = x_0 \int_0^t f(x,s)\,ds + \int_0^t g(x,s)\,dW(s)$$

Def. Wiener process, W_t, $t \geq 0$ - process with $W_0 = 0$, increments $W_t - W_s$ are Gaussian RV

$\Rightarrow \mathbb{E}[W_t - W_s] = 0$, $\mathrm{Var}[W_t - W_s] = t - s$

non-overlapping increments are independent

$W_{t+1} - W_t = \Delta W_t$, $\mathbb{E}[\Delta W_t] = 0$, $\mathbb{E}[\Delta W_t^2] = \Delta t$, $W_{t+1} = W_t + \sqrt{\Delta t} \cdot N(0,1)$, $W_0 = 0$

$x(w) = e^{bw}$, $X_t = e^{bW_t}$

vector stochastic process $\{X_t, t \in T\}$ - a family of random variables/vectors indexed by parameter set T

1. $X_t \in \mathbb{Z}$ - discrete state space process } discrete parameter chain
   $t \in \mathbb{N}$ - discrete parameter set
2. $X_t \in \mathbb{R}$ - continuous state space process } random sequence
   $t \in \mathbb{N}$ - discrete parameter set
3. $X_t \in \mathbb{Z}$ - discrete state space process
   $t \in \mathbb{R}$ - continuous parameter set
4. $X_t \in \mathbb{R}$ - continuous state space process } Stochastic Random Process
   $t \in \mathbb{R}$ - continuous parameter set

Stochastic depends on time, t and $w \in \mathbb{R}$ (realization).

Ex. Random Walk (RW)

$\Delta x > 0$, $\Omega = \{H, T\}$, $p(H) = p$, $p(T) = q = 1-p$

η - random variable $= \begin{cases} \Delta x & \text{if } w = H \\ -\Delta x & \text{if } w = T \end{cases}$

$\Pr[\eta(w) = \Delta x] = p$, $\Pr[\eta(w) = -\Delta x] = q = 1-p$

$X_n = \sum_{i=0}^{n-1} \eta_i = \sum_{i=0}^{n-2} \eta_i + \eta_{n-1} = X_{n-1} + \eta_{n-1} \rightarrow$ difference equation

↳ any distribution

Stationarity of processes

$p(X_1,\dots,X_n,t_1,\dots,t_n)=p(X_1,\dots,X_n,t_1+s,\dots,t_n+s)$, $\forall n, \forall s\in\mathbb{R}$ - strict sense stationary process (SSS)

$n=1$ ① $p(X_1,t_1)=p(X_1,t_1+s)=p(X_1)$

$n=2$ $p(X_1,X_2,t_1,t_2)=p(X_1,X_2,t_1+s,t_2+s) \Rightarrow$ ② $p(X_1,X_2,t_1,t_2)=p(X_1,X_2,t_2-t_1)$

① + ② = wide sense (weak) stationary process (WSS)

Ex. $X_t=a+bt$, a,b - RV

$$\mathbb{E}[X_t]=\mathbb{E}[a+bt]=\mathbb{E}[a]+t\,\mathbb{E}[b]=t$$

$$R(t,\tau)=\mathbb{E}[X_tX_\tau]=\mathbb{E}[a^2]+\mathbb{E}[ab]t+\mathbb{E}[ab]\tau+\mathbb{E}[b^2]t\tau$$

$$=var(a)+cov(a,b)(t+\tau)+var(b)t\tau \quad (\mathbb{E}[a]=\mathbb{E}[b]=0)$$

$$=var(a)+var(b)t\tau \quad (cov(a,b)=0)$$

weak stationary process: $R(t_1,t_2)=R(t_1-t_2)$, $R(t,t)=\mathbb{E}[X_tX_t]=\mathbb{E}[X_t^2]=R(0)$ - constant

Wiener process: $\mathbb{E}[W(t)]=0<\int_{-\infty}^{+\infty}p(x,t)x\,dx$, $R(t,t+s)=\sigma^2 t$ - not stationary

Convergence:

i) $\{X_n\}\to x$ with probability

ii) $\{X_n\}\to x$ in probability $\quad \lim_{n\to\infty} X_n(\omega)=x$ almost $\forall\omega$

iii) $\{X_n\}\to x$ in mean square sense, iii) $\Rightarrow$ ii) $\quad \underset{n\to\infty}{l.i.m.}\, X_n=X$

$\lim_{n\to\infty}\mathbb{E}[|X_n-X|^2]=0$, $\mathbb{E}[|X_n|^2]<\infty$, $\mathbb{E}[|X|^2]<\infty$

$$\mathbb{E}[|y|^2]=\int_{-\infty}^{+\infty}|y|^2p(y)\,dy\geq\int_{-\infty}^{-\varepsilon}|y|^2p(y)\,dy+\int_{\varepsilon}^{+\infty}|y|^2p(y)\,dy\geq\varepsilon^2\left[\int_{-\infty}^{-\varepsilon}p(y)dy+\int_{\varepsilon}^{+\infty}p(y)\,dy\right]=\varepsilon^2 Pr[|y|>\varepsilon]$$

$y=X_n-X \Rightarrow \mathbb{E}[|X_n-X|^2]\geq Pr(|X_n-X|>\varepsilon)$

$X=\underset{n\to\infty}{l.i.m.}\,X_n$, $Y=\underset{n\to\infty}{l.i.m.}\,Y_n$

Th 1) $\underset{n\to\infty}{l.i.m.}(aX_n+bY_n)=aX+bY$

2) $\mathbb{E}[X]=\lim_{n\to\infty}\mathbb{E}[X_n]=\mathbb{E}[\underset{n\to\infty}{l.i.m.}\,X_n]$

3) $\mathbb{E}[XY]=\lim_{n\to\infty}\mathbb{E}[X_nY_n]=\mathbb{E}[\underset{n\to\infty}{l.i.m.}\,X_nY_n]=\mathbb{E}[\underset{n\to\infty}{l.i.m.}\,X_n\,\underset{n\to\infty}{l.i.m.}\,Y_n]$

$$\mathbb{E}[(a|x|-|y|)^2]=a^2\mathbb{E}[|x|^2]+\mathbb{E}[|y|^2]-2a\,\mathbb{E}[|x||y|]>0\Rightarrow D=4a^2\,\mathbb{E}[|x||y|]-4a^2\,\mathbb{E}[|x|^2]\,\mathbb{E}[|y|^2]\leq 0$$

Cauchy-Schwarz: $\mathbb{E}[|x||y|]\leq\sqrt{\mathbb{E}[|x|^2]\,\mathbb{E}[|y|^2]}$

Proof of Property I: $\underset{n\to\infty}{\text{l.i.m.}}(aX_n+bY_n)=aX+bY$

To prove: $\mathbb{E}\left[|aX_n+bY_n-(aX+bY)|^2\right]=0 \quad (n\to\infty)$

$$=\mathbb{E}\left[|a(X_n-X)+b(Y_n-Y)|^2\right]\le 2a^2\,\mathbb{E}\left[|X_n-X|^2\right]+2b^2\,\mathbb{E}\left[|Y_n-Y|^2\right]=0$$

$$\mathbb{E}\left[|X|^2\right]+\mathbb{E}\left[|Y|^2\right]+2\mathbb{E}\left[|XY|\right]\le 2\mathbb{E}\left[|X|^2\right]+2\mathbb{E}\left[|Y|^2\right]$$

$$\mathbb{E}\left[|X|^2\right]+\mathbb{E}\left[|Y|^2\right]-2\mathbb{E}\left[|XY|\right]\ge \mathbb{E}\left[|X|^2\right]+\mathbb{E}\left[|Y|^2\right]$$

$$\mathbb{E}\left[|X|^2\right]+\mathbb{E}\left[|Y|^2\right]-2\sqrt{\mathbb{E}\left[|X|^2\right]\mathbb{E}\left[|Y|^2\right]}\ge\mathbb{E}\left[|X|^2\right]+\mathbb{E}\left[|Y|^2\right]$$

$$0\le\left(\sqrt{\mathbb{E}\left[|X|^2\right]}-\sqrt{\mathbb{E}\left[|Y|^2\right]}\right)^2\ge\mathbb{E}\left[|X|^2\right]+\mathbb{E}\left[|Y|^2\right]=0$$

Proof of Property II: $\mathbb{E}[X]=\lim_{n\to\infty}\mathbb{E}[X_n]=\mathbb{E}\left[\underset{n\to\infty}{\text{l.i.m.}}X_n\right]$

To prove: $\mathbb{E}[X]=\lim_{n\to\infty}\mathbb{E}[X_n]$

$$\left|\mathbb{E}[X_n-X]\right|^2\le\mathbb{E}\left[|X_n-X|^2\right]\to 0\quad(n\to\infty)$$

Ex. 5

1) To prove: $\underset{n\to\infty}{\text{l.i.m.}}\; aX_n=aX$

$$\underset{n\to\infty}{\text{l.i.m.}}\;X_n=X \Rightarrow \lim_{n\to\infty}\mathbb{E}\left[|X_n-X|^2\right]=0$$

$$\Rightarrow a^2\lim_{n\to\infty}\mathbb{E}\left[|X_n-X|^2\right]=\lim_{n\to\infty}\mathbb{E}\left[a^2|X_n-X|^2\right]=\lim_{n\to\infty}\mathbb{E}\left[|aX_n-aX|^2\right]=0\Rightarrow\underset{n\to\infty}{\text{l.i.m.}}\;aX_n=aX$$

2) To prove: $\underset{n\to\infty}{\text{l.i.m.}}\;X_n+Y_n=X+Y$

$$\lim_{n\to\infty}\mathbb{E}\left[|X_n+Y_n-X-Y|^2\right]=\lim_{n\to\infty}\mathbb{E}\left[|X_n-X+Y_n-Y|^2\right]$$

$$\le\lim_{n\to\infty}\mathbb{E}\left[2|X_n-X|^2+2|Y_n-Y|^2\right]$$

$$=\lim_{n\to\infty}2\mathbb{E}\left[|X_n-X|^2\right]+\lim_{n\to\infty}2\mathbb{E}\left[|Y_n-Y|^2\right]=0$$

3) To prove: $\mathbb{E}\left(\underset{n\to\infty}{\text{l.i.m.}}\;X_n\right]=\lim_{n\to\infty}\mathbb{E}[X_n]=\mathbb{E}[X]$

$$\lim_{n\to\infty}\left|\mathbb{E}[X_n]-\mathbb{E}[X]\right|=\lim_{n\to\infty}\left|\mathbb{E}[X_n-X]\right|$$

$$\le\lim_{n\to\infty}\mathbb{E}\left[|X_n-X|\right]$$

$$\le\lim_{n\to\infty}\sqrt{\mathbb{E}\left[|X_n-X|^2\right]}=0$$

4) To prove: $\mathbb{E}\left[\underset{n\to\infty}{\text{l.i.m.}}\;X_nY_n\right]=\mathbb{E}\left[\underset{n\to\infty}{\text{l.i.m.}}\;X_n\;\underset{n\to\infty}{\text{l.i.m.}}\;Y_n\right]=\lim_{n\to\infty}\mathbb{E}[X_nY_n]=\mathbb{E}[XY]$

$$\lim_{n\to\infty}\left|\mathbb{E}[X_nY_n]-\mathbb{E}[XY]\right|=\lim_{n\to\infty}\left|\mathbb{E}[(X-X_n)Y]+\mathbb{E}[(Y-Y_n)X]-\mathbb{E}[(X-X_n)(Y-Y_n)]\right|$$

$$\le\lim_{n\to\infty}\left|\mathbb{E}[(X_n-X)Y]\right|+\left|\mathbb{E}[(Y-Y_n)X]\right|+\left|\mathbb{E}[(X-X_n)(Y-Y_n)]\right|$$

$$\le\lim_{n\to\infty}\mathbb{E}[|X-X_n|]\,\mathbb{E}[|Y|]+\mathbb{E}[|Y-Y_n|]\,\mathbb{E}[|X|]+\mathbb{E}[|X-X_n|]\,\mathbb{E}[|Y-Y_n|]$$

$$\le\lim_{n\to\infty}\sqrt{\mathbb{E}[|X-X_n|^2]}\,\mathbb{E}[|Y|]+\sqrt{\mathbb{E}[|Y-Y_n|^2]}\,\mathbb{E}[|X|]+\sqrt{\mathbb{E}[|X-X_n|^2]}\sqrt{\mathbb{E}[|Y-Y_n|^2]}$$

$$=0$$

$$\dot{X}_t = \frac{\partial X_t}{\partial t} = \underset{h\to 0}{\text{l.i.m.}} \frac{X_{t+h}-X_t}{h}$$

X_t is continuous at $t \in T$. $\Leftrightarrow \underset{h\to 0}{\text{l.i.m.}} X_{t+h} = X_t \Rightarrow \mathbb{E}\left[|X_{t+h}-X_t|^2\right] \to 0 \ (h\to 0)$

$$\mathbb{E}\left[|X_{t+h}-X_t|^2\right] = \mathbb{E}\left[(X_{t+h}-X_t)(X_{t+h}-X_t)\right]$$
$$= \mathbb{E}\left[|X_{t+h}|^2\right] - 2\mathbb{E}\left[(X_{t+h})(X_t)\right] + \mathbb{E}\left[|X_t|^2\right] \qquad \text{if } R_{xx}(t,\tau) - \text{cont.}$$
$$= R_{xx}(t+h,t+h) - 2R_{xx}(t+h,t) + R_{xx}(t,t) \to 0 \ (h \to 0) \qquad \text{in } (t,t),\ \forall t \in T$$

$\boxed{\text{Th}}$ X_t - mean square continuous in $t \Leftrightarrow R_{xx}(t,\tau)$ - continuous in (t,t)

$$\lim_{h,h'\to 0} \mathbb{E}\left[X_{t+h} X_{t+h'}\right] = \mathbb{E}\left[X_t X_t\right]$$

Corollary $R_{xx}(t,\tau)$ - continuous on (t,t)

$\Leftrightarrow m_x(t) = \mathbb{E}\left[X_t\right]$ - continuous at t, $C_{xx}(t,\tau)$ - continuous on (t,t)

$\lim_{h,h'\to 0} R_{xx}(t+h,\tau+h') = R_{xx}(t,\tau)$ - continuous

$\boxed{\text{Th}}$ X_t is mean square differentiable iff $\exists \left.\frac{\partial^2 R_{xx}(t,\tau)}{\partial t\, \partial \tau}\right|_{t,t}$

Proof

To prove: $\lim_{h\to 0} \mathbb{E}\left[\left|\frac{X_{t+h}-X_t}{h} - \dot{X}_t\right|^2\right] = 0$

Cauchy: $\{X_n\} \to X \Rightarrow |X_n - X_m| \to 0 \ (n,m\to\infty)$

$$\lim_{h,h'\to 0} \frac{1}{hh'} \mathbb{E}\left[(X_{t+h}-X_t)(X_{t+h'}-X_t)\right] = \lim_{h,h'\to 0} \frac{1}{hh'}\left(\mathbb{E}\left[X_{t+h}X_{t+h'}\right] + \mathbb{E}\left[X_t X_t\right] - \mathbb{E}\left[X_t X_{t+h'}\right] - \mathbb{E}\left[X_{t+h} X_t\right]\right)$$
$$= \lim_{h,h'\to 0} \frac{1}{hh'}\left\{\mathbb{E}\left[X_{t+h}X_{t+h'}\right] - \mathbb{E}\left[X_{t+h}X_t\right] - \left(\mathbb{E}\left[X_t X_{t+h'}\right] - \mathbb{E}\left[X_t X_t\right]\right)\right)$$
$$= \frac{R_{xx}(t+h,t+h') - R_{xx}(t+h,t)}{hh'} - \frac{R_{xx}(t,t+h') - R_{xx}(t,t)}{hh'}$$
$$= \frac{1}{h}\frac{\partial R_{xx}(t+h',\tau)}{\partial \tau} - \frac{1}{h}\frac{\partial R_{xx}(t,\tau)}{\partial \tau} = \left.\frac{\partial^2 R_{xx}(t,\tau)}{\partial t\,\partial\tau}\right|_{t,t}$$

Properties: X_t - mean square differentiable

1) $m_{\dot{x}}(t) = \frac{\partial}{\partial t}\mathbb{E}\left[X_t\right] = \frac{\partial}{\partial t} m_x(t)$
2) $R_{\dot{x}x}(t,\tau) = \frac{\partial}{\partial t}\mathbb{E}\left[X_t X_\tau\right] = \mathbb{E}\left[\dot{X}_t X_\tau\right] = \frac{\partial}{\partial t} R_{xx}(t,\tau)$
3) $R_{\dot{x}\dot{x}}(t,\tau) = \frac{\partial}{\partial t}\frac{\partial}{\partial \tau}\mathbb{E}\left[X_t X_\tau\right] = \mathbb{E}\left[\dot{X}_t \dot{X}_\tau\right] = \frac{\partial^2}{\partial t\,\partial\tau} R_{xx}(t,\tau)$
4) $C_{\dot{x}\dot{x}}(t,\tau) = \frac{\partial^2}{\partial t\,\partial \tau} C_{xx}(t,\tau)$

Ex. $X_t = at+b,\ a,b - RV \Rightarrow \dot{X}_t = a$

$$\lim_{h\to 0} \mathbb{E}\left[\left|\frac{X_{t+h}-X_t}{h} - \dot{X}_t\right|^2\right] = \mathbb{E}\left[\left|\frac{a(t+h)-at}{h} - a\right|^2\right] = 0,\ \forall h$$

Ex. $Y_t = X_t^2 = a^2t^2 + 2atb + b^2 \Rightarrow \dot{Y}_t = 2a^2t + 2ab = 2a(at+b) = 2aX_t$

$$\frac{\partial Y_t}{\partial t} = 2X_t \frac{\partial X_t}{\partial t} = 2X_t \cdot a = 2a^2t + 2ab$$

$$\lim_{h\to 0} \mathbb{E}\left[\left|\frac{Y_{t+h}-Y_t}{h} - \dot{Y}_t\right|^2\right] = \lim_{h\to 0} \mathbb{E}\left[\frac{(a(t+h)+b)^2-(at+b)^2}{h} - (2a^2t+2ab)\right|^2\Big] = \mathbb{E}\left[a^4\right]h^2 \to 0$$

Ex. $W_t' = \frac{\partial W_t}{\partial t} = \underset{h\to 0}{\text{l.i.m.}} \frac{W_{t+h}-W_t}{h}$

$$\mathbb{E}\left[W_t'\right] = \mathbb{E}\left[\underset{h\to 0}{\text{l.i.m.}} \frac{W_{t+h}-W_t}{h}\right] = \lim_{h\to 0} \mathbb{E}\left[\frac{W_{t+h}-W_t}{h}\right] = 0$$

$$\mathbb{E}\left[W_t' W_s'\right] = \mathbb{E}\left[\underset{h\to 0}{\text{l.i.m.}} \frac{W_{t+h}-W_t}{h} \underset{h\to 0}{\text{l.i.m.}} \frac{W_{s+h}-W_s}{h}\right] = \lim_{h\to 0} \mathbb{E}\left[\frac{W_{t+h}-W_t}{h} \frac{W_{s+h}-W_s}{h}\right] = \delta(t-s)$$

Integration

$[a,b]\quad a = t_0 < t_1 < \ldots < t_N = b,\quad t_i \le t_i' \le t_{i+1},\ \rho_i = \max(t_{i-1} - t_i)$

$$\underset{\rho\to 0}{\text{l.i.m.}} \sum_{i=0}^{n-1} X_{t_i'}(t_{i+1}-t_i) = \int_a^b X_t\, dt$$

Th X_t - mean square Riemann integral on $[a,b] \Leftrightarrow R_{xx}(t,\tau)$ - integrable on $[a,b]\times[a,b]$

Properties:

1) $\mathbb{E}\left[\int_a^b X_t\, dt\right] = \int_a^b m_x(t)\, dt = \int_a^b \mathbb{E}[X_t]\, dt$

2) $\mathbb{E}\left[\int_a^b X_t\, dt \int_c^d X_\tau\, d\tau\right] = \int_a^b \int_c^d \mathbb{E}[X_t X_\tau]\, d\tau\, dt$

3) $cov\left(\int_a^b X_t\, dt, \int_c^d X_\tau\, d\tau\right) = \int_a^b \int_c^d cov(X_t, X_\tau)\, d\tau\, dt$

Th Fundamental Theorem of Calculus

$\dot{X}_t$ - mean square integrable $\Leftrightarrow X_t = X_0 + \int_0^t \dot{X}_s\, ds$ with probability 1

Wiener-Lévy (Brownian motion) process $\{W_t, t\ge 0\}$

1) $Pr(W(0)=0) = 1$

2) $\{W_t, t\ge 0\}$ has stationary independent increments, (Gaussian) $W_{t+h} - W_t$ is independent

$W_t - W_\tau,\ W_{t+h} - W_{\tau+h}$ has same distribution $N(0,1)$

3) $\mathbb{E}[W_t] = 0$

4) $\forall t > 0$, W_t is normally distributed.

$$W_t' = \frac{\partial W_t}{\partial t} = N_t \quad \text{or} \quad dW_t = N_t\, dt$$

$R_{ww}(t,\tau) = \mathbb{E}[W_t W_\tau] = \mathbb{E}[((W_t - W_\tau) + W_\tau) W_\tau]$

$\mathbb{E}[W_t - W_\tau] = \mathbb{E}[W_t] - \mathbb{E}[W_\tau] = 0$

$var(W_t - W_\tau) = \mathbb{E}[|W_t - W_\tau|^2] = \sigma^2(t-\tau)$

$R_{ww}(t,\tau) = \mathbb{E}[(W_t - W_\tau)W_\tau] + \mathbb{E}[W_\tau^2] = \sigma^2 \tau$

$\Rightarrow R_{ww}(t,\tau) = \sigma^2 \min\{t,\tau\}$ - continuous $\Rightarrow W_t$ — mean square continuous

$\frac{\partial^2}{\partial t \partial \tau} R_{ww}(t,\tau)$ - doesn't exist at $t=\tau \Rightarrow W_t$ - not differentiable

Markov process $\{X_t, t \in T\}$

$\frac{\partial x}{\partial t} = f(x(t)) \Longleftrightarrow \frac{\partial x}{\partial t}$ depends only on $x(t)$, not on $x(\tau)$, $\tau < t$

Def. $Pr\{X_{t_n}(\omega) \le \lambda \mid X_{t_1}, \dots, X_{t_{n-1}}\} = Pr\{X_{t_n}(\omega) \le \lambda \mid X_{t_{n-1}}\}$, $\forall \lambda, t_1, \dots, t_n$

probability density function: $p(X_{t_n} \mid X_{t_1} \dots X_{t_{n-1}}) = p(X_{t_n} \mid X_{t_{n-1}})$

$p(X_{t_n}, X_{t_{n-1}}, \dots, X_{t_1}) = p(X_{t_n} \mid X_{t_{n-1}}, \dots, X_{t_1}) = p(X_{t_n} \mid X_{t_{n-1}})\, p(X_{t_{n-1}}, \dots, X_{t_1}) = \dots = p(X_{t_n} \mid X_{t_{n-1}}) \dots p(X_{t_2} \mid X_{t_1})\, p(X_{t_1})$

$\frac{\partial X_t}{\partial t} = f(X_t, t) + g(X_t, t)\xi(t)$, $\xi(t) = \frac{\partial W_t}{\partial t} = N_t$

$\Rightarrow dX_t = f(X_t, t)\,dt + g(X_t, t)\,dW_t$

$X_t = X_{t_0} + \int_{t_0}^{t} f(X_s, s)\,ds + \int_{t_0}^{t} g(X_s, s)\,dW_s$, $\{X_t, t \ge 0\}$ - Markov process

$X_{t+\delta t} = X_t + \int_t^{t+\delta t} f(X_s, s)\,ds + \int_t^{t+\delta t} g(X_s, s)\,dW_s \approx X_t + f(X_t, t)\delta t + g(X_t, t)\,\underbrace{dW_t}_{\to W_{t+\delta t} - W_t}$

$X_t = e^{W_t} \Rightarrow \cancel{\frac{\partial X_t}{\partial t} = e^{W_t} \frac{\partial W_t}{\partial t}}$ $\quad f(t) = e^{at} \to$ deterministic $\Rightarrow \frac{\partial f}{\partial t} = ae^{at} = af(t)$

$\delta X_t = X_{t+\delta t} - X_t = e^{W_{t+\delta t}} - e^{W_t} = e^{W_t + \delta W_t} - e^{W_t} = e^{W_t}(e^{\delta W_t} - 1) = X_t(\cancel{1} + \delta W_t + \frac{\delta W_t^2}{2} \cancel{- 1})$

$\mathbb{E}[\delta X_t - X_t \delta W_t] = \mathbb{E}[X_t \frac{\delta W_t^2}{2}] = \mathbb{E}[X_t]\,\mathbb{E}[\frac{\delta W_t^2}{2}] = \mathbb{E}[X_t]\frac{\delta t}{2}$, $\delta W_t = W_{t+\delta t} - W_t$

$\Rightarrow \mathbb{E}[\delta X_t - X_t \delta W_t - X_t \frac{\delta W_t^2}{2}] = \theta(\delta t)$ $\qquad \mathbb{E}[\delta W_t^2] = \mathbb{E}[|W_{t+\delta t} - W_t|^2] = \delta t$

$dX_t = X_t\,dW_t + \frac{1}{2} X_t (dW_t)^2$, $X_t = 1 + \int_0^t X_s\,dW_s + \int_0^t \frac{X_s}{2}\,dW_s^2$, $X_t = e^{W_t}$

Itô Stochastic Integral, $\int_{t_0}^{t} G_s\,dW_s$

Ex. $\int_{t_0}^{t} s\,ds = \lim_{h \to 0} \sum t_i' (t_{i+1} - t_i) = \frac{t^2 - t_0^2}{2}$

$\Rightarrow \int_{t_0}^{t} W_s\,dW_s = \text{l.i.m.}_{h \to 0} \sum W_{t_i'} (W_{t_{i+1}} - W_{t_i}) =$

$= \frac{W_t^2 - W_{t_0}^2}{2} - \frac{1}{2}\, \text{l.i.m.}_{h \to 0} \sum (W_{t_{i+1}} - W_{t_i})^2 + \text{l.i.m.}_{h \to 0} \sum (W_{t_i'} - W_{t_i})^2 + \text{l.i.m.}_{h \to 0} \sum (W_{t_{i+1}} - W_{t_i'})(W_{t_i'} - W_{t_i})$

$$\underset{h\to 0}{\text{l.i.m}} \sum (W_{t_{i+1}} - W_{t_i})^2 = t - t_0$$
$$\underset{h\to 0}{\text{l.i.m.}} \sum (W_{t_i'} - W_{t_i})^2 = \sum (t_i' - t_i)$$
$$\underset{h\to 0}{\text{l.i.m.}} \sum (W_{t_{i+1}} - W_{t_i'})(W_{t_i'} - W_{t_i}) = 0$$
$$\Rightarrow \int_{t_0}^{t} W_s \, dW_s = \frac{W_t^2 - W_{t_0}^2}{2} - \frac{t - t_0}{2} + \sum (t_i' - t_i)$$

Ex. 7

1) To prove: $\underset{h\to 0}{\text{l.i.m.}} \sum (W_{t_{i+1}} - W_{t_i})^2 = t - t_0$

$$\mathbb{E}\left[\left| \sum (W_{t_{i+1}} - W_{t_i})^2 - (t - t_0) \right|^2 \right] = \mathbb{E}\left[\left| \sum (W_{t_{i+1}} - W_{t_i})^2 \right|^2 - 2(t - t_0) \sum (W_{t_{i+1}} - W_{t_i})^2 + (t - t_0)^2 \right]$$
$$= \mathbb{E}\left[\left| \sum (W_{t_{i+1}} - W_{t_i})^2 \right|^2 \right] - 2(t - t_0)\, \mathbb{E}\left[\sum (W_{t_{i+1}} - W_{t_i})^2 \right] + (t - t_0)^2$$
$$= \underbrace{\mathbb{E}\left[\sum (W_{t_{i+1}} - W_{t_i})^2 \right] \mathbb{E}\left[\sum (W_{t_{i+1}} - W_{t_i})^2 \right]}_{=(t - t_0)^2} \underbrace{- \sum \left(\mathbb{E}\left[|W_{t_{i+1}} - W_{t_i}|^2 \right] \right)^2 + \sum \mathbb{E}\left[|W_{t_{i+1}} - W_{t_i}|^4 \right]}_{\text{non overlapping increments } (W_{t_{i+1}} - W_{t_i})} - (t - t_0)^2$$
$$\Rightarrow = - \sum \left(\mathbb{E}\left[|W_{t_{i+1}} - W_{t_i}|^2 \right] \right)^2 + \sum \mathbb{E}\left[|W_{t_{i+1}} - W_{t_i}|^4 \right] = O(\Delta t) = O(h)$$

Wiener: $\lim_{h\to 0} \mathbb{E}\left[\left| \sum (W_{t_{i+1}} - W_{t_i})^2 - (t - t_0) \right|^2 \right] = 0 \Rightarrow \underset{h\to 0}{\text{l.i.m.}} \sum (W_{t_{i+1}} - W_{t_i})^2 = t - t_0$

2) To prove: $\underset{h\to 0}{\text{l.i.m.}} \sum (W_{t_{i+1}} - W_{t_i'})(W_{t_i'} - W_{t_i}) = 0$

$$\mathbb{E}\left[\left| \sum (W_{t_{i+1}} - W_{t_i'})(W_{t_i'} - W_{t_i}) \right|^2 \right] = \sum \mathbb{E}\left[|W_{t_{i+1}} - W_{t_i'}|^2 |W_{t_i'} - W_{t_i}|^2 \right] = \mathbb{E}\left[|W_{t_{i+1}} - W_{t_i'}|^2 \right] \mathbb{E}\left[|W_{t_i'} - W_{t_i}|^2 \right]$$
$$= \sum (t_{i+1} - t_i')(t_i' - t_i) = \sum O(\Delta t)\, O(\Delta t) = O(\Delta t) = O(h)$$

Def. Itô Integral: $\int_{t_0}^{t} G_s \, dW_s = \underset{h\to 0}{\text{l.i.m.}} \sum G_{t_i} (W_{t_{i+1}} - W_{t_i})$

Properties:

1) $\mathbb{E}\left[\int_{t_0}^{t} G_s \, dW_s \right] = 0$
2) $\mathbb{E}\left[\int_{t_0}^{t} G_s \, dW_s \int_{t_0}^{t} H_s \, dW_s \right] = \int_{t_0}^{t} \mathbb{E}[G_s H_s] \, ds$

Ex. 8

1) To prove: $\mathbb{E}\left[\int_{t_0}^{t} G_s \, dW_s \right] = 0$

$$\mathbb{E}\left[\int_{t_0}^{t} G_s \, dW_s \right] = \mathbb{E}\left[\underset{h\to 0}{\text{l.i.m.}} \sum G_{t_i} (W_{t_{i+1}} - W_{t_i}) \right]$$
$$= \lim_{h\to 0} \sum \mathbb{E}\left[G_{t_i} (W_{t_{i+1}} - W_{t_i}) \right]$$
$$= \lim_{h\to 0} \sum \mathbb{E}[G_{t_i}]\, \mathbb{E}[W_{t_{i+1}} - W_{t_i}] = 0$$

2) To prove: $\mathbb{E}\left[\int_{t_0}^{t} G_s \, dW_s \int_{t_0}^{t} H_s \, dW_s \right] = \int_{t_0}^{t} \mathbb{E}[G_s H_s] \, ds$

$$\mathbb{E}\left[\int_{t_0}^{t} G_s\, dW_s \int_{t_0}^{t} H_s\, dW_s\right] = \mathbb{E}\left[\underset{k\to 0}{\text{l.i.m.}} \sum G_{t_i}(W_{t_{i+1}}-W_{t_i})\ \underset{k\to 0}{\text{l.i.m.}} \sum H_{t_i}(W_{t_{i+1}}-W_{t_i})\right]$$

$$= \lim_{k\to 0} \mathbb{E}\left[\sum G_{t_i}(W_{t_{i+1}}-W_{t_i}) \sum H_{t_i}(W_{t_{i+1}}-W_{t_i})\right]$$

$$= \lim_{k\to 0} \sum \mathbb{E}\left[G_{t_i}H_{t_i}(W_{t_{i+1}}-W_{t_i})^2\right]$$

$$= \lim_{k\to 0} \sum \mathbb{E}\left[G_{t_i}H_{t_i}\right]\mathbb{E}\left[|W_{t_{i+1}}-W_{t_i}|^2\right]$$

$$= \lim_{k\to 0} \sum \mathbb{E}\left[G_{t_i}H_{t_i}\right]\Delta t = \lim_{k\to 0} \sum \mathbb{E}\left[G_{t_i}H_{t_i}\right]k = \int_{t_0}^{t} \mathbb{E}\left[G_s H_s\right] ds$$

Ex. $\int_{t_0}^{t} W_s\, dW_s = \frac{W_t^2 - W_{t_0}^2}{2} - \frac{t-t_0}{2}$ (Itô)

Def. Stratonovitch Integral: $\int_{t_0}^{t} G_s\, dW_s = \underset{k\to 0}{\text{l.i.m.}} \sum G_{\frac{t_{i+1}+t_i}{2}}(W_{t_{i+1}}-W_{t_i})$, $t_i' = \frac{t_{i+1}+t_i}{2}$

Ex. $\int_{t_0}^{t} W_s\, dW_s = \frac{W_t^2 - W_{t_0}^2}{2}$ (Stratonovitch)

Stochastic Differential Equation

$dX_t = f(X_t,t)dt + g(X_t,t)\,dW_t$

$X_t = X_{t_0} + \int_{t_0}^{t} f(X_s,s)\,ds + \int_{t_0}^{t} g(X_s,s)\,dW_s$

$X_{t+\Delta t} = X_t + \int_t^{t+\Delta t} f(X_s,s)\,ds + \int_t^{t+\Delta t} g(X_s,s)\,dW_s \approx X_t + \int_t^{t+\Delta t} f(X_t,t)\,ds + \int_t^{t+\Delta t} g(X_t,t)\,dW_s = X_t + f(X_t,t)\Delta t + g(X_t,t)(\overbrace{W_{t+\Delta t}-W_t}^{\sqrt{\Delta t}})$

↳ Euler scheme for Itô differential equation, Markov process

$\Rightarrow X_t$ - Markov process

ODE: $\frac{\partial x}{\partial t} = f(x)$, $\frac{\partial}{\partial t} v(x(t)) = \frac{\partial v(x(t))}{\partial x}\cdot\frac{\partial x}{\partial t} = f(x(t))\frac{\partial v(x(t))}{\partial x}$

SDE: $dX_t = f(X_t)\,dt + g(X_t)\,dW_t$

$\Rightarrow dv = \frac{\partial v}{\partial x}\cdot dX_t = \frac{\partial v}{\partial x}\cdot(f(X_t)\,dt + g(X_t)\,dW_t)$ is wrong.

$dv = v'(x(t))\,dX_t + \underbrace{\tfrac{1}{2}g^2(X_t)\,v''(X_t)}_{\text{Itô correction}}dt = \left[v'(X_t)f(X_t) + \underbrace{\tfrac{1}{2}g^2(X_t)\,v''(X_t)}_{\text{Itô correction}}\right]dt + g(X_t)\,v'(X_t)\,dW_t$

$\int_{t_0}^{t} g(X_t)\,dW_t = \underset{\rho\to 0}{\text{l.i.m.}} \sum_{i=0}^{n-1} g(t_i)(W_{t_{i+1}}-W_{t_i})$ $\quad a = t_0 < t_1 < \ldots < t_N = b$

Ex.

$\frac{\partial B_t}{\partial t} = -K_1 B_t + S_t + \sigma N_t$

$dB_t = (-K_1 B_t + S_t)\,dt + \sigma\,dW_t$

$B_{t_0} = B_0$

$dS_t = A(t,S_t)\,dt + B(t,S_t)\,dW_t$

$S(t) = S(0) + \int_0^t A(z,S_z)\,dz + \int_0^t B(z,S_z)\,dz$

Itô's Differential Rule

$Y_t = Y(t) = u(t, S_t) \qquad u: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$

$dY_t = \frac{\partial u}{\partial t}\cdot dt + \frac{\partial u}{\partial x}\cdot dS_t + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} B^2 dt$ where $x = S_t$, let $u_t = \frac{\partial u}{\partial t}$

$\Rightarrow dY_t = [u_t(t,S_t) + u_x(t,S_t)A(t,S_t) + \frac{1}{2}u_{xx}(t,S_t)B^2(t,S_t)]dt + u_x B(t,S_t)\,dW_t \to$ Itô's Rule

Ex. $dS_t = dW_t,\ S_t = W_t,\ u(t,x) = y_0 e^{\mu t + \sigma x}$

$u_x = \sigma y_0^{\mu t+\sigma x} = \sigma u,\ u_{xx} = \sigma^2 y_0 e^{\mu t+\sigma x} = \sigma^2 u,\ u_t = \mu y_0 e^{\mu t+\sigma x} = \mu u$

$dY_t = [\mu Y_t + \frac{\sigma^2}{2}Y_t]dt + \sigma Y_t\,dW_t = [\mu + \frac{\sigma^2}{2}]Y_t\,dt + \sigma Y_t\,dW_t,\ Y_t = y_0 e^{\mu t + \sigma W_t},\ Y_0 = y_0$

$\delta t, \delta S_t \ll 1$

$u(t+\delta t, S_t + \delta S_t) = u(t,S_t) + \delta t\,\frac{\partial u}{\partial t}(t,S_t) + \delta S_t\,\frac{\partial u}{\partial x}(t,S_t) + \underbrace{\frac{(\delta t)^2}{2}}_{=0}\frac{\partial^2 u}{\partial t^2} + \underbrace{\frac{\partial^2 u}{\partial t\partial x}\delta t\,\delta S_t}_{=0} + \frac{(\delta S_t)^2}{2}\frac{\partial^2 u}{\partial x^2}$

$dt\,dt = 0,\ dW_t\,dt = 0,\ dt\,dW_t = 0,\ dW_t\,dW_t = dt$

Ex. $dS_t = dW_t,\ S_t = W_t,\ u(t,x) = \sinh(c+t+x)$

$dY_t = [\cosh(c+t+W_t) + \frac{1}{2}\sinh^2(c+t+W_t)]dt + \cosh(c+t+W_t)\,dW_t$

$\cosh^2 x - \sinh^2 x = 1 \Rightarrow \cosh(x) = \sqrt{1+\sinh^2(x)}$

$\Rightarrow dY_t = \sqrt{1+Y_t^2} + \frac{1}{2}Y_t^2 + \sqrt{1+Y_t^2}\,dW_t,\ Y_t = \sinh(c+t+W_t),\ Y_0 = \sinh(c)$

$dX_t = f(X_t,t)\,dt + g(X_t,t)\,dW_t$

$d\phi = \frac{\partial \phi}{\partial t}dt + \frac{\partial \phi}{\partial x}dX_t + \frac{1}{2}g^2\frac{\partial^2\phi}{\partial x^2}dt = d\Phi_t$ for stochastic process $\Phi_t = \phi(X_t,t)$

Taylor series expansion of Euler Scheme (Δt) $\quad \Delta X = f(X_t,t)\Delta t + g(X_t,t)\Delta W + \dots$

or $X_{t+\Delta t} = X_t + f(X_t,t)\Delta t + g(X_t,t)(W_{t+\Delta t} - W_t) + \dots$

$\Rightarrow \Delta\phi = \phi(X_t + \Delta X, t+\Delta t) - \phi(X_t,t) = \frac{\partial\phi}{\partial t}\Delta t + \underbrace{\frac{1}{2}\frac{\partial^2\phi}{\partial t^2}(\Delta t)^2}_{=0} + \frac{\partial\phi}{\partial x}\Delta X + \frac{1}{2}\frac{\partial^2\phi}{\partial x^2}(\Delta X)^2 + \dots =$

$= \frac{\partial\phi}{\partial t}\Delta t + \frac{\partial\phi}{\partial x}(f(X_t,t)\Delta t + g(X_t,t)\Delta W) + \frac{1}{2}\frac{\partial^2\phi}{\partial x^2}(f(X_t,t)\Delta t + g(X_t,t)\Delta W)^2 + \dots =$

$= \frac{\partial\phi}{\partial t}\Delta t + \frac{\partial\phi}{\partial x}(f(X_t,t)\Delta t + g(X_t,t)\Delta W) + \frac{1}{2}\frac{\partial^2\phi}{\partial x^2}g(X_t,t)^2\Delta W^2 + \dots =$

$= \frac{\partial\phi}{\partial t}\Delta t + \frac{\partial\phi}{\partial x}\Delta X + \frac{1}{2}\frac{\partial^2\phi}{\partial x^2}g(X_t,t)^2\Delta t + \dots$

Ex. $dX_t = \mu X_t\,dt + \sigma X_t\,dW_t,\ \phi = X^2,\ \frac{\partial\phi}{\partial t} = 0,\ \frac{\partial\phi}{\partial x} = 2x,\ \frac{\partial^2\phi}{\partial x^2} = 2$

$dX_t^2 = 0 + 2X_t(\mu X_t\,dt + \sigma X_t\,dW_t) + \frac{1}{2}\sigma^2 X_t^2\, 2\,dt = 2\mu X_t^2\,dt + 2\sigma X_t^2\,dW_t + X_t^2\sigma^2\,dt =$

$= (2\mu + \sigma^2)X_t^2\,dt + 2\sigma X_t^2\,dW_t$

$dX_t = f(X_t,t)\,dt + g(X_t,t)\,dW_t$ - Itô Equation

$\Rightarrow dX_t = \left(f(X_t,t) - \frac{1}{2} g(X_t,t) \frac{\partial g(X_t,t)}{\partial x}\right) dt + g(X_t,t)\,dW_t$ - Stratonovitz Equation

$dX_t = f(X_t,t)\,dt + g(X_t,t)\,dW_t$ - Stratonovitz Equation

$\Rightarrow dX_t = \left(f(X_t,t) + \frac{1}{2} g(X_t,t) \frac{\partial g(X_t,t)}{\partial x}\right) dt + g(X_t,t)\,dW_t$ - Itô Equation

<u>Ex.</u> Itô: $d\phi_t = \frac{b^2}{2}\phi_t\,dt + b\,\phi_t\,dW_t \Rightarrow$ Stratonovitz: $d\phi_t = b\,\phi_t\,dW_t$, $\phi_0 = 1$

<u>Ex.</u> $dM_t = -aM_t\,dt + a\,dW_t \Rightarrow \mathbb{E}[M_t] = 0$, $\mathbb{E}[M_t M_s] = \frac{a}{2} e^{-a|t-s|}$

$\frac{dX_t}{dt} = f(X_t,t) + g(X_t,t) M_t$ - not Markovian process

$\left.\begin{array}{l} dX_t = f(X_t,t)\,dt + g(X_t,t) M_t\,dt \\ dM_t = -aM_t\,dt + a\,dW_t \end{array}\right\} [X_t\ M_t]$ - Markovian process

<u>Ex. 13</u>

$Y_t = (W_t + \sqrt{Y_0})^2$, $t \geq 0$

To prove: $dY_t = dt + 2\sqrt{Y_t}\,dW_t$, $t \geq 0$ - Itô SDE

$X_t = W_t$, $\phi(x,t) = (x + \sqrt{Y_0})^2 \Rightarrow \frac{\partial\phi}{\partial t} = 0$, $\frac{\partial\phi}{\partial x} = 2(x + \sqrt{Y_0})$, $\frac{\partial^2\phi}{\partial x^2} = 2$

$d\phi_t = 2(X_t + \sqrt{Y_0})\,dX_t + \frac{1}{2}\cdot 2\cdot dt = 2(W_t + \sqrt{Y_0})\,dW_t + dt$

$\Rightarrow dY_t = dt + 2(W_t + \sqrt{Y_0})\,dW_t = dt + 2\sqrt{Y_t}\,dW_t$

Stratonovitz SDE: $dY_t = 2(W_t + \sqrt{Y_t})\,dW_t = 2\sqrt{Y_t}\,dW_t$, correction: $-2\sqrt{Y_t}\frac{\partial}{\partial y}[\sqrt{Y_t}] = -dt$

<u>Ex. 14</u>

Itô SDE: $dX_t = \mu X_t\,dt + \sigma X_t\,dW_t$, $X_0 = x_0$

$Z_t = \ln X_t$ or $\phi(x,t) = \ln x \Rightarrow \frac{\partial\phi}{\partial t} = 0$, $\frac{\partial\phi}{\partial x} = \frac{1}{x}$, $\frac{\partial\phi}{\partial x} = -\frac{1}{x^2}$

$d\phi_t = \frac{1}{X_t}dX_t - \frac{1}{2}\cdot\frac{1}{X_t^2}\cdot\sigma^2\cdot X_t^2\,dt = \mu dt + \sigma W_t - \frac{1}{2}\sigma^2 dt$

$\Rightarrow dZ_t = (\mu - \frac{1}{2}\sigma^2)\,dt + \sigma\,dW_t$

Analytical Solution $\left|\begin{array}{l} Z_t = (\mu - \frac{1}{2}\sigma^2)(t - t_0) + \sigma(W_t - W_{t_0}) + Z_0 \\ X_t = e^{Z_t} = X_0 e^{(\mu - \frac{1}{2}\sigma^2)(t-t_0) + \sigma(W_t - W_{t_0})} \end{array}\right.$

<u>Ex. 15</u>

$\int_{t_0}^{t} W_s\,dW_s$, $X_t = W_t$, $\phi(x,t) = x^2 \Rightarrow \frac{\partial\phi}{\partial t} = 0$, $\frac{\partial\phi}{\partial x} = 2x$, $\frac{\partial^2\phi}{\partial x^2} = 2$

$d\phi_t = 2X_t\,dt + \frac{1}{2}\cdot 2\cdot dt \Rightarrow \phi_t = \phi_{t_0} + 2\int_{t_0}^{t} X_s\,dX_s + \int_{t_0}^{t} dt$

$\Rightarrow W_t^2 = W_{t_0}^2 + 2\int_{t_0}^{t} W_s\,dW_s + (t - t_0) \Rightarrow \int_{t_0}^{t} W_s\,dW_s = \frac{1}{2}(W_t^2 - W_{t_0}^2) - \frac{1}{2}(t - t_0)$

Ex. 16

$\mathbb{E}[W_t^4]$, $X_t = W_t$ $(g=1)$, $\phi(x,t) = x^4 \Rightarrow \frac{\partial \phi}{\partial t} = 0$, $\frac{\partial \phi}{\partial x} = 4x^3$, $\frac{\partial^2 \phi}{\partial x^2} = 12x^2$

$d\phi_t = 4X_t^3 dX_t + \frac{1}{2}\cdot 12 X_t^2 dt = 4W_t^3 dW_t + 6W_t^2 dt$

$\Rightarrow W_t^4 = W_0^4 + 4\int_0^t W_s^3 ds + 6\int_0^t W_s^2 \, ds$

$\Rightarrow \mathbb{E}[W_t^4] = \mathbb{E}[W_0^4] + 4\,\mathbb{E}[\int_0^t W_s^3 dW_s] + 6\,\mathbb{E}[\int_0^t W_s^2 ds] = 6\int_0^t \mathbb{E}[W_s^2] ds = 6\int_0^t s \, ds = 3t^2$

Ex. 17

Itô SDE $\begin{cases} dX_t = aX_t dt - Y_t dW_t, & X_0 = x_0 \\ dY_t = aY_t dt + X_t dW_t, & Y_0 = y_0 \end{cases}$

$R_t = X_t^2 + Y_t^2$, $t \geq 0$

$dX_t = aX_t dt - Y_t dW_t$, $\phi_1(x,t) = x^2 \Rightarrow \frac{\partial \phi_1}{\partial t} = 0$, $\frac{\partial \phi_1}{\partial x} = 2x$, $\frac{\partial^2 \phi_1}{\partial x^2} = 2$, $g = -y$

$d\phi_{1_t} = 2X_t dX_t + \frac{1}{2}\cdot 2\cdot Y_t^2 dt = 2X_t(aX_t dt - Y_t dW_t) + Y_t^2 dt = 2aX_t^2 dt + Y_t^2 dt - 2X_t Y_t dW_t$

$dY_t = aY_t dt + X_t dW_t$, $\phi_2(y,t) = y^2 \Rightarrow \frac{\partial \phi_2}{\partial t} = 0$, $\frac{\partial \phi_2}{\partial y} = 2y$, $\frac{\partial^2 \phi_2}{\partial y^2} = 2$, $g = x$

$d\phi_{2_t} = 2Y_t dY_t + \frac{1}{2}\cdot 2\cdot X_t^2 dt = 2Y_t(aY_t dt + X_t dW_t) + X_t^2 dt = 2aY_t^2 dt + X_t^2 dt + 2Y_t X_t dW_t$

$dR_t = d\phi_{1_t} + d\phi_{2_t} = 2aR_t dt + R_t dt = (2a+1)R_t \Rightarrow \frac{\partial R_t}{\partial t} = (2a+1)R_t$, $R_0 = x_0^2 + y_0^2 \Rightarrow$ deterministic

Estimation of Probability Density

Def. Kernel-function, s.t. $\int_{-\infty}^{\infty} K(x)dx = 1$

N - # realizations of X_t

$\Rightarrow \hat{p}(x,t) = \frac{1}{N}\sum_{u=1}^{N} K\left(\frac{x - X_t^u}{\lambda}\right)$ where λ - bandwidth of Kernel estimator

$\lambda_{opt} = O\left(\frac{1}{N^{1/(d+4)}}\right)$ where d - dimensionality of X_t $\Rightarrow \sigma_{error} = O\left(\frac{1}{N^{4/(d+4)}}\right)$

Fokker Planck Equation

$\frac{\partial p}{\partial t} = -\sum \frac{\partial(f_i p)}{\partial x_i} + \frac{1}{2}\sum\sum \frac{\partial^2((gg^T)_{ij} p)}{\partial x_i \partial x_j}$, $t \geq t_0$ where $p(x_1, x_2, \dots, t \mid x_{1_{t_0}}, x_{2_{t_0}}, \dots, t_0)$ - prob. dens. func.

if $X_{t_0} = x_0 \Rightarrow p(x_{1_{t_0}}, x_{2_{t_0}}, \dots, t_0) = \delta(X_{t_0} - x_0)$

$\phi(x,t) = K(x)$

$dK(X_t) = \frac{\partial K}{\partial x} dX_t + \frac{1}{2} g(X_t,t)^2 \frac{\partial^2 K}{\partial x^2} dt = \frac{\partial K}{\partial x}(f(X_t,t)dt + g(X_t,t)dW_t) + \frac{1}{2} g(X_t,t)^2 \frac{\partial^2 K}{\partial x^2} dt$

$d\,\mathbb{E}[K(X_t)] = \mathbb{E}[\frac{\partial K}{\partial x} f(X_t,t) + \frac{1}{2} g(X_t,t)^2 \frac{\partial^2 K}{\partial x^2}] dt$ or $\frac{\partial}{\partial t}\mathbb{E}[K(X_t)] = \mathbb{E}[\frac{\partial K}{\partial x} f(X_t,t) + \frac{1}{2} g(X_t,t)^2 \frac{\partial^2 K}{\partial x^2}]$

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} K(x)\, p(x,t)\, dx = \int_{-\infty}^{\infty} \frac{\partial K}{\partial x} f(x,t)\, p(x,t)\, dx + \int_{-\infty}^{\infty} \frac{1}{2} g(x,t)^2 \frac{\partial^2 K}{\partial x^2}\, p(x,t)\, dx$$

$$\int_{-\infty}^{\infty} K(x)\frac{\partial p}{\partial t}\, dx = \int_{-\infty}^{\infty} -K(x)\frac{\partial fp}{\partial x}\, dx + \int_{-\infty}^{\infty} K(x)\frac{1}{2}\frac{\partial^2 g^2 p}{\partial x^2}\, dx \quad \text{or} \quad \int_{-\infty}^{\infty} K(x)\left(\frac{\partial p}{\partial t} + \frac{\partial fp}{\partial x} - \frac{1}{2}\frac{\partial^2 g^2 p}{\partial x^2}\right) dx = 0$$

Ex. $dX_t = \mu\, dt + \sigma\, dW_t$, $X_{t_0} = x_0$

$$\frac{\partial p}{\partial t} = -\mu \frac{\partial p}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 p}{\partial x^2},\ t \geq t_0,\ p(x,t_0) = \delta(x-x_0) \Rightarrow p(x,t|x_0,t_0) = N(x_0+\mu t, \sigma^2)$$

Numerical Approximation

$\frac{\partial x}{\partial t} = f(x,t)$, $x(t_0) = x_0$

Euler scheme: $x_{n+1} = x_n + \Delta t\, f(x_n, n)$

$dX_t = f(X_t)\,dt + g(X_t)\,dW_t \quad 0 \leq t \leq T$, $X(0) = x_0$

$X_t = X_{t_0} + \int_{t_0}^{t} f(X_s)\, ds + \int_{t_0}^{t} g(X_s)\, dW_s$

$\Rightarrow X_{t_{n+1}} = X_{t_n} + \int_{t_n}^{t_{n+1}} f(X_s)\, ds + \int_{t_n}^{t_{n+1}} g(X_s)\, dW_s$ ← exact solution

Euler-Maruyama: $\tilde{X}_{n+1} = \tilde{X}_n + f(\tilde{X}_n)\Delta t + g(\tilde{X}_n)[W(t_{n+1}) - W(t_n)]$

Def. p-order of convergence if $\exists K, \delta > 0$ s.t. $|x(T) - x_N| \leq K(\Delta t)^p$, $\forall \Delta t$, $0 < \Delta t < \Delta$

local error of Euler scheme: $O(\Delta t^2)$

global error of Euler scheme: $E_N = O(\Delta t)$

$E_N = E_{N-1} + O(\Delta t^2) = E_{N-2} + 2\, O(\Delta t^2) = \dots = N\, O(\Delta t^2) = \frac{T}{\Delta t} O(\Delta t^2) = T\, O(\Delta t) = O(\Delta t)$

Def. p-strong order of convergence if $\exists K, \delta > 0$ s.t. $\mathbb{E}[|X_T - X_N|] \leq K(\Delta t)^p$, $\forall \Delta t$, $0 < \Delta t < \delta$

Def. p-weak order of convergence if $\exists K, \delta > 0$ s.t. $|\mathbb{E}[h(X_T,T)] - \mathbb{E}[h(X_N,N)]| \leq K(\Delta t)^p$, $\forall \Delta t$, $0 < \Delta t < \delta$, $\forall h$-polynomial

Euler-Maryama $\begin{cases} \text{weak: } 1 \\ \text{strong: } \frac{1}{2} \end{cases}$

$$d\phi = \frac{\partial \phi}{\partial t} dt + \frac{\partial \phi}{\partial x} dX_t + \frac{1}{2} g^2 \frac{\partial^2 \phi}{\partial x^2} dt = \frac{\partial \phi}{\partial t} dt + \frac{\partial \phi}{\partial x} f\, dt + \frac{1}{2} g^2 \frac{\partial^2 \phi}{\partial x^2} dt + \frac{\partial \phi}{\partial x} g\, dW_t = \mathcal{L}^0 \phi\, dt + \mathcal{L}^1 \phi\, dW_t$$

where $\mathcal{L}^0 \phi = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x} f + \frac{1}{2} g^2 \frac{\partial^2 \phi}{\partial x^2}$

$\mathcal{L}^1 \phi = \frac{\partial \phi}{\partial x} g$

$$df = \mathcal{L}^0 f\, dt + \mathcal{L}^1 f\, dW_t \Rightarrow f(X_s, s) = f(X_{t_0}, t_0) + \int_{t_0}^{s} \mathcal{L}^0 f\, d\tau + \int_{t_0}^{s} \mathcal{L}^1 f\, dW_\tau$$

$$dg = \mathcal{L}^0 g\, dt + \mathcal{L}^0 g\, dW_t \Rightarrow g(X_s, s) = g(X_{t_0}, t_0) + \int_{t_0}^{s} \mathcal{L}^0 g\, d\tau + \int_{t_0}^{s} \mathcal{L}^1 g\, dW_\tau$$

$$\Rightarrow X_t = X_{t_0} + \int_{t_0}^{t} \left[f(X_{t_0}, t_0) + \int_{t_0}^{s} \mathcal{L}^0 f\, d\tau + \int_{t_0}^{s} \mathcal{L}^1 f\, dW_\tau \right] ds + \int_{t_0}^{t} \left[g(X_{t_0}, t_0) + \int_{t_0}^{s} \mathcal{L}^0 g\, d\tau + \int_{t_0}^{s} \mathcal{L}^1 g\, dW_\tau \right] dW_s$$

$$= X_{t_0} + f(X_{t_0}, t_0)(t - t_0) + g(X_{t_0}, t_0)(W_t - W_{t_0}) + \text{error terms}$$

Euler scheme: $X_{t+\Delta t} = X_t + f(X_t, t)\Delta t + g(X_t, t)(W_{t+\Delta t} - W_t)$ or $X_{u+1} = X_u + f(X_u, u)\Delta t + g(X_u, u)\Delta W_u$

Error terms:

$\rightarrow \int_{t_0}^{t}\int_{t_0}^{s} \mathcal{L}^1 g\, dW_\tau dW_s < K_4 \int_{t_0}^{t}\int_{t_0}^{s} dW_\tau dW_s = \frac{K_4}{2}(\Delta W_u^2 - \Delta t) = O(\Delta t)$ - local, $\sqrt{var(E_N)} = O(\Delta t^{0.5})$ - global } Wiener

$var(E_N) = var(E_{N-1}) + O(\Delta t^2) = var(E_{N-2}) + 2O(\Delta t^2) = \dots = N\,O(\Delta t^2) = \frac{T}{\Delta t} O(\Delta t^2) = T\,O(\Delta t) = O(\Delta t)$

$\rightarrow \int_{t_0}^{t}\int_{t_0}^{s} \mathcal{L}^0 f d\, d\tau ds < K_1 \int_{t_0}^{t}\int_{t_0}^{s} d\tau ds = O(\Delta t^2)$ - local $\Rightarrow O(\Delta t)$ - global

$\rightarrow \int_{t_0}^{t}\int_{t_0}^{s} \mathcal{L}^1 f\, dW_\tau\, ds < K_2 \int_{t_0}^{t}\int_{t_0}^{s} dW_\tau ds = O(\Delta t \Delta W_u) = O(\Delta t^{1.5})$ - local $\Rightarrow O(\Delta t)$ - global (Wiener)

$\rightarrow \int_{t_0}^{t}\int_{t_0}^{s} \mathcal{L}^0 g\, d\tau dW_s < K_3 \int_{t_0}^{t}\int_{t_0}^{s} d\tau dW_s = O(\Delta t \Delta W_u) = O(\Delta t^{1.5})$ - local $\Rightarrow O(\Delta t)$ - global (Wiener)

$\Rightarrow$ Euler-Maruyama - dominated by $\int_{t_0}^{t}\int_{t_0}^{s} \mathcal{L}^1 g\, dW_\tau dW_s$

$$d\mathcal{L}^1 g = \mathcal{L}^0 \mathcal{L}^1 g\, d\tau + \mathcal{L}^1 \mathcal{L}^1 g\, dW_\tau \Rightarrow \mathcal{L}^1 g(X_\tau, \tau) = \mathcal{L}^1 g(X_{t_0}, t_0) + \int_{t_0}^{\tau} \mathcal{L}^0 \mathcal{L}^1 g\, dr + \int_{t_0}^{\tau} \mathcal{L}^0 \mathcal{L}^1 g\, dW_r$$

$$\Rightarrow X_t = X_{t_0} + \int_{t_0}^{t} \left[f(X_{t_0}, t_0) + \int_{t_0}^{s} \mathcal{L}^0 f\, d\tau + \int_{t_0}^{s} \mathcal{L}^1 f\, dW_\tau \right] ds + \int_{t_0}^{t} \left[g(X_{t_0}, t_0) + \int_{t_0}^{s} \mathcal{L}^0 g\, d\tau + \int_{t_0}^{s} \mathcal{L}^1 g\, dW_\tau \right] dW_s$$

$$= X_{t_0} + f(X_{t_0}, t_0)(t - t_0) + g(X_{t_0}, t_0)(W_t - W_{t_0}) + \mathcal{L}^1 g(X_{t_0}, t_0) + \mathcal{L}^1 g(X_{t_0}, t_0) \int_{t_0}^{t}\int_{t_0}^{s} dW_\tau dW_s + \text{error terms}$$

$$= X_{t_0} + f(X_{t_0}, t_0)(t - t_0) + g(X_{t_0}, t_0)(W_t - W_{t_0}) + \frac{1}{2} g(X_{t_0}, t_0) \frac{\partial g}{\partial x}\left((W_t - W_{t_0})^2 - (t - t_0)\right) + \text{error terms}$$

Milstein scheme: $X_{u+1} = X_u + f(X_u, u)\Delta t + g(X_u, u)\Delta W_u + \frac{1}{2} g(X_u, u) \frac{\partial g}{\partial x}(\Delta W_u^2 - \Delta t) \Rightarrow$ weak: 1, strong: 1

Pr. 1

$\mathbb{E}\left[\int_{t_0}^{t} G_s\, dW_s\right] = 0$ does not hold for Stratonovitz integrals.

$\int_{t_0}^{t} W_s\, dW_s \overset{str}{=} \frac{W_t^2}{2} - \frac{W_{t_0}^2}{2} \Rightarrow \mathbb{E}\left[\frac{W_t^2}{2} - \frac{W_{t_0}^2}{2}\right] = \frac{1}{2}(t - t_0)$

$\mathbb{E}$[Stratonovitz] = Itô correction $\Rightarrow \mathbb{E}$[Itô] = 0

Pr. 2

$\Phi_t = \cos(aW_t + b)$, $X_t = W_t$, $dX_t = dW_t$, $g = 1$, $\phi(x) = \cos(ax+b) \Rightarrow \frac{\partial\phi}{\partial t} = 0$, $\frac{\partial\phi}{\partial x} = -a\sin(ax+b)$, $\frac{\partial^2\phi}{\partial x^2} = -a^2\cos(ax+b)$

$d\Phi_t \overset{Itô}{=} -a\sin(aX_t+b)\,dX_t - \frac{1}{2}a^2\cos(aX_t+b)\,dt$

$= -a\sin(aW_t+b)\,dW_t - \frac{1}{2}a^2\cos(aW_t+b)\,dt$

$= -a\sqrt{1-\cos^2(aW_t+b)}\,dW_t - \frac{1}{2}a^2\cos(aW_t+b)\,dt$

$= -\frac{1}{2}a^2\Phi_t\,dt + a\sqrt{1-\Phi_t^2}\,dW_t$

$\Phi_0 = \cos(aW_0+b) = \cos(b)$

Pr. 3

$\Phi_t = (W_t)^4$, $X_t = W_t$, $dX_t = dW_t$, $g = 1$, $\phi(x) = x^4 \Rightarrow \frac{\partial\phi}{\partial t} = 0$, $\frac{\partial\phi}{\partial x} = 4x^3$, $\frac{\partial^2\phi}{\partial x^2} = 12x^2$

$d\Phi_t \overset{Itô}{=} 4X_t^3\,dX_t + \frac{1}{2}\cdot g\cdot 12X_t^2\,dt = 6X_t^2\,dt + 4X_t^3\,dX_t$

$= 6W_t^2\,dt + 4W_t^3\,dW_t \qquad W_t = \sqrt[4]{\Phi_t}$

$= 6\sqrt{\Phi_t}\,dt + 4\sqrt[4]{\Phi_t^3}\,dW_t$

$\Phi_0 = W_0 = 0$

Pr. 4

$\Phi_t = (W_t)^4$, $d\Phi_t \overset{Itô}{=} 6\sqrt{\Phi_t}\,dt + 4\sqrt[4]{\Phi_t^3}\,dW_t \Rightarrow g(\Phi_t) = 4\sqrt[4]{\Phi_t^3}$, $\frac{\partial g}{\partial\phi} = 3\sqrt[4]{\Phi_t^{-1}}$

Itô correction term: $-\frac{1}{2}g(\Phi_t)\frac{\partial g}{\partial\phi} = -\frac{1}{2}\cdot 4\cdot\sqrt[4]{\Phi_t^3}\cdot 3\cdot\sqrt[4]{\Phi_t^{-1}} = -6\sqrt{\Phi_t}$

$d\Phi_t \overset{str}{=} 4\sqrt[4]{\Phi_t^3}\,dW_t$

Pr. 5

$\frac{\partial x}{\partial t} = ax^2 - bx^3 + c\sin(x)$, $x(0) = 1$, $t \geq 1$, $A_t = a + N_t$

$\frac{\partial X_t}{\partial t} \overset{str}{=} (a+N_t)X_t^2 + bX_t^3 + c\sin(X_t)$

$= aX_t^2 - bX_t^3 + c\sin(X_t) + X_t^2 N_t$

$dX_t \overset{str}{=} (aX_t^2 - bX_t^3 + c\sin(X_t))\,dt + X_t\,dW_t \Rightarrow g(X_t) = X_t^2 \quad \frac{\partial g}{\partial x} = 2X_t$

Itô correction term: $-\frac{1}{2}g(X_t)\frac{\partial g}{\partial x} = \frac{1}{2}\cdot X_t^2\cdot 2X_t = X_t^3$

$dX_t \overset{Itô}{=} (aX_t^2 - bX_t^3 + c\sin(X_t) + X_t^3)\,dt + X_t^2\,dW_t$, $X_0 = 1$

Pr. 6

$dX_t = u\,dt + g\,dW_t$, $X_0 = 0$, $t \geq 0$

$dM_t = -aM_t\,dt$, $M_0 = 1$, $t \geq 0$

X-position, M-mass, u-velocity of water, decaying matter

$$d\begin{pmatrix} X_t \\ M_t \end{pmatrix} \overset{\text{Itô}}{=} \begin{pmatrix} u \\ -aM_t \end{pmatrix} dt + \begin{pmatrix} g \\ 0 \end{pmatrix} dW_t, \quad \begin{pmatrix} X_0 \\ M_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\vec{f}(X_t, M_t, t)$ $\quad$ $\vec{g}(X_t, M_t, t)$

Fokker-Planck: $\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}(up) - \frac{\partial}{\partial x}(-amp) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(g^2 p) = -u\frac{\partial p}{\partial x} + a\frac{\partial mp}{\partial m} + \frac{g^2}{2}\frac{\partial^2 p}{\partial x^2}$

$p(x, m, 0) = \delta(x)\,\delta(m-1)$