14.11

Lecture 1 – Functions and continuity

1.5: ~~10~~, ~~21~~, ~~24~~, ~~27~~, ~~57~~, ~~64~~, ~~68~~, ~~69~~, ~~71~~, ~~74~~
2.5: ~~20~~, ~~41~~

10. $f(x) = x^4 - 16$

$y = x^4 - 16$

$x^4 = y + 16$

$x = \sqrt[4]{y+16}$ — depends on domain

$f(1) = f(-1) = -15$

$f: [0;\infty) \to \mathbb{R}$ one-to-one/bijective

$f: \mathbb{R} \to \mathbb{R}$ surjective

21. $f(x) = 1 + \sqrt{2+3x}$

$y - 1 = \sqrt{2+3x}$

$(y-1)^2 = 2 + 3x$

$f^{-1}(x) = \dfrac{(x-1)^2 - 2}{3}$

24. $y = x^2 - x \qquad x \geq \frac{1}{2}$

$0 = x^2 - x - y$

$D = 1 + 4y$

$x_{1,2} = \dfrac{1 \pm \sqrt{1+4y}}{2}$

27. $f(x) = \sqrt{4x+3} \qquad x \geq -\frac{3}{4}$

$y^2 = 4x + 3$

$f^{-1}(x) = \dfrac{x^2 - 3}{4}$

x	$f(x)$	$f^{-1}(x)$
0	$\sqrt{3}$	$-\frac{3}{4}$
1	$\sqrt{7}$	$-\frac{1}{2}$
$-\frac{3}{4}$	0	$-\frac{39}{64}$
$\frac{3}{4}$	$\sqrt{6}$	$-\frac{39}{64}$
$\sqrt{3}$	$\sqrt{4\sqrt{3}+3}$	0
7	$\sqrt{31}$	11.5
$2+\sqrt{7}$	$2+\sqrt{7}$	$2+\sqrt{7}$

$\sqrt{4x+3} = \dfrac{x^2-3}{4}$

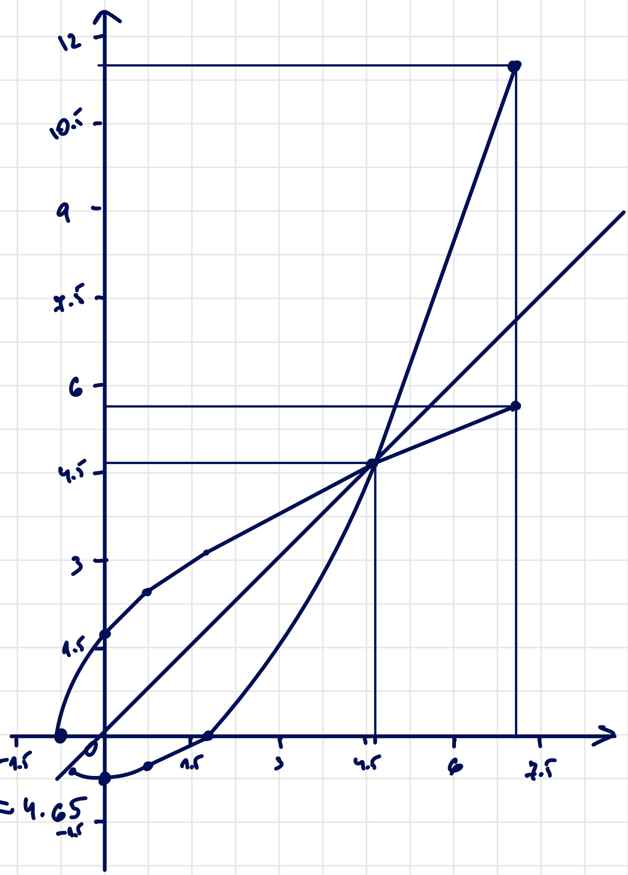
$16(4x+3) = (x^2-3)^2$

$64x + 48 = x^4 - 6x^2 + 9$

$x^4 - 6x^2 - 64x - 39 = 0$

$x^2 - 4x - 3 = 0 \qquad x_{1,2} = 2 \pm \sqrt{7}$

$x^2 + 4x + 13 = 0$ ✗



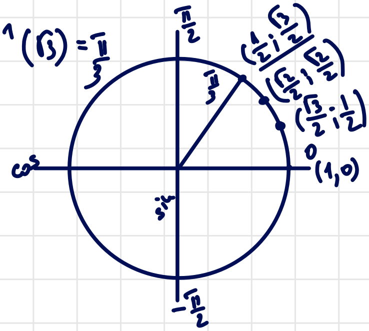
check for real solutions: $2+\sqrt{7} \approx 4.65$

57. $f(x)=\ln(e^x-3)$ $\quad f:(\ln(3);\infty)\to\mathbb{R}$ $\quad f^{-1}(x)=\ln(e^x+3)$ $\quad f:\mathbb{R}\to(\ln(3);\infty)$

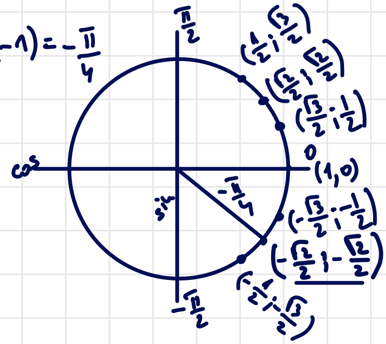
$e^x-3>0 \quad e^x>3 \quad x>\ln(3)\sim 1.1 \quad e^x+3>0 \quad e^x>-3 \quad x\in\mathbb{R}$

$e^y=e^x-3 \quad e^x=e^y+3 \quad \lim_{x\to-\infty}\ln(e^x+3)=\ln(3) \Rightarrow f(x)>\ln(3)$

64. a) $\tan^{-1}(\sqrt{3})=\frac{\pi}{3}$



b) $\arctan(-1)=-\frac{\pi}{4}$

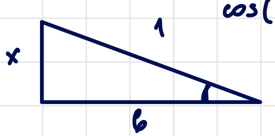


68. a) $\arcsin\left(\sin\left(\frac{5\pi}{4}\right)\right)=\frac{5\pi}{4}-\pi=\frac{\pi}{4}$

b) $\cos\left(2\sin^{-1}\left(\frac{5}{13}\right)\right)=-1+2\cos^2\left(\arcsin\left(\frac{5}{13}\right)\right)=-1+2\left(\sqrt{1-\left(\frac{5}{13}\right)^2}\right)^2=2\,\frac{169-25}{169}-1=$

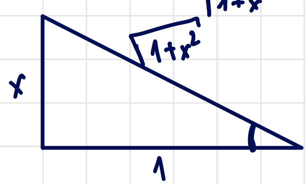
$\cos(2x)=-1+2\cos^2(x) \qquad \cos(\arcsin(x))=\sqrt{1-x^2} \qquad =\frac{119}{169}$

69. $\cos(\sin^{-1}x)=\sqrt{1-x^2}$

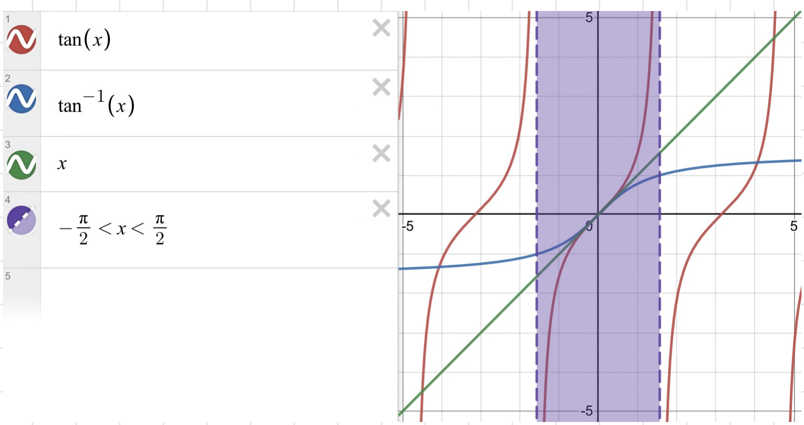


$\cos(\sin^{-1}x)=b$
$=\sqrt{1-x^2}$

71. $\sin(\tan^{-1}x)=\frac{x}{\sqrt{1+x^2}}$



74. $y=\tan x$
$y=\tan^{-1}x$
$y=x$
$-\frac{\pi}{2}<x<\frac{\pi}{2}$



40. $$f(x)=\begin{cases} \dfrac{x^2-x}{x^2-1} & x \neq 1 \\ 1 & x=1 \end{cases}$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x}{x+1} = \frac{1}{2} = \lim_{x\to 1^+}$$

but $f(1) = 1 \neq \lim_{x\to 1} f(x) = \frac{1}{2}$

41. $$f(x)=\begin{cases} x^2 & x < -1 \\ x & -1 \leq x < 1 \\ \dfrac{1}{x} & x \geq 1 \end{cases}$$

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} x^2 = 1 \neq \lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} x = \underline{-1}$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = \underline{1} = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{1}{x} = \underline{1}$$

$\Rightarrow$ discontinuous at $x = \underline{-1}$.

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Lecture 2 – Limits of functions

2.2: ~~5~~, ~~6~~
2.3: ~~10~~, ~~14~~, ~~17~~, ~~21~~, ~~23~~, ~~25~~, ~~27~~, ~~36~~
2.4: 4, 11, 36, 37, 38

5. $\lim_{x\to 1} f(x) = 2$ $\lim_{x\to 3^-} f(x) = 1$

$\lim_{x\to 3^+} f(x) = 4$ $\lim_{x\to 3} f(x) = DNE$

$f(3) = 3$

6. $\lim_{x\to -3^-} h(x) = 4$ $\lim_{x\to 0^-} h(x) = 1$

$h(2) = DNE$

$\lim_{x\to -3^+} h(x) = 4$ $\lim_{x\to 0^+} h(x) = -1$ $\lim_{x\to 5^+} h(x) = 3$

$\lim_{x\to 2} h(x) = 2$

$\lim_{x\to -3} h(x) = 4$ $\lim_{x\to 0} h(x) = DNE$ $\lim_{x\to 5^-} h(x) = 3$

$h(-3) = DNE$ $h(0) = 1$

10. a) $\frac{x^2+x-6}{x-2} = \frac{(x+3)(x-2)}{x-2} \neq x+3 \quad x=2$

b) $\lim_{x\to 2} \frac{x^2+x-6}{x-2} = \lim_{x\to 2} \frac{(x+3)\cancel{(x-2)}}{\cancel{x-2}} = \lim_{x\to 2} x+3 = 5$ $\quad x\neq 2$

14. $\lim_{x\to 4} \frac{x^2+3x}{x^2-x-12} = \lim_{x\to 4} \frac{x\cancel{(x+3)}}{\cancel{(x+3)}(x-4)} = \frac{4}{0} = DNE$

17. $\lim_{h\to 0} \frac{(h-5)^2-25}{h} = \lim_{h\to 0} \frac{h^{\cancel{2}}-10\cancel{h}+\cancel{25}-\cancel{25}}{\cancel{h}} = \lim_{h\to 0} h-10 = -10$

21. $\lim_{h\to 0} \frac{\sqrt{9+h}-3}{h} \cdot \frac{\sqrt{9+h}+3}{\sqrt{9+h}+3} = \lim_{h\to 0} \frac{\cancel{9}+\cancel{h}-\cancel{9}}{\cancel{h}(\sqrt{9+h}+3)} = \frac{1}{6}$

23. $\lim_{x\to 3} \frac{\frac{1}{x}-\frac{1}{3}}{x-3} = \lim_{x\to 3} \frac{\cancel{x-3}}{3x\cancel{(x-3)}} = \frac{1}{9}$

25. $\lim_{t\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t} \cdot \frac{\sqrt{1+t}+\sqrt{1-t}}{\sqrt{1+t}+\sqrt{1-t}} = \lim_{t\to 0} \frac{\cancel{1}+t-\cancel{1}+t}{t(\sqrt{1+t}+\sqrt{1-t})} = \frac{2}{1+1} = 1$

27. $\lim_{x\to 16} \frac{4-\sqrt{x}}{16x-x^2} \cdot \frac{4+\sqrt{x}}{4+\sqrt{x}} = \lim_{x\to 16} \frac{\cancel{16-x}}{x\cancel{(16-x)}(4+\sqrt{x})} = \frac{1}{128}$

30. $\lim_{x\to 0} \sqrt{x^3+x^2}\sin\frac{\pi}{x} = 0$ $\quad -\sqrt{x^3+x^2} \le \sqrt{x^3+x^2}\sin\frac{\pi}{x} \le \sqrt{x^3+x^2}$

$\lim_{x\to 0}$

$-1 \le \sin\frac{\pi}{x} \le 1 \;\Big|\cdot \sqrt{x^3+x^2}$ $\quad 0 \le \lim_{x\to 0} \sqrt{x^3+x^2}\sin\frac{\pi}{x} \le 0$

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Lecture 3 - Asymptotes

2.2: ~~13~~, ~~31~~, ~~33~~, ~~41~~

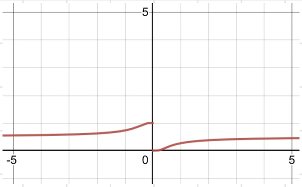
2.6: ~~17~~, ~~19~~, ~~27~~, ~~30~~, ~~35~~, ~~52~~, ~~65~~

13. $f(x) = \frac{1}{1+e^{\frac{1}{x}}}$

$\lim_{x\to 0^-} f(x) = 1$

$\lim_{x\to 0^+} f(x) = 0$

$\lim_{x\to 0} f(x) = DNE$



31. $\lim_{x\to 5^+} \frac{x+1}{x+5} = \infty$

33. $\lim_{x\to 1} \frac{2-x}{(x-1)^2} = \infty$

17. $\lim_{x\to -\infty} \frac{x-2}{x^2+1} = 0$

41. $\lim_{x\to 2^+} \frac{(x-4)(x+2)}{(x-2)(x-3)} = \frac{-\cdot +}{+\cdot -} = \infty$

19. $\lim_{t\to\infty} \frac{\sqrt{t}+t^2}{2t-t^2} = -1$

27. $\lim_{x\to\infty} \left(\sqrt{9x^2+x}-3x\right)\cdot\frac{\sqrt{9x^2+x}+3x}{\sqrt{9x^2+x}+3x} = \lim_{x\to\infty} \frac{\cancel{9x^2}+x-\cancel{9x^2}}{\sqrt{9x^2+x}+3x} = \frac{1}{6}$

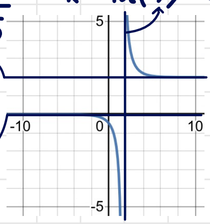
30. $\lim_{x\to\infty} \sqrt{x^2+1} = \infty$

35. $\lim_{x\to\infty} \arctan(e^x) = \frac{\pi}{2}$

52. $y = \frac{2e^x}{e^x-5}$ $\quad x = \ln(5)$ vertical asymptote

$\lim_{x\to\infty} y = 2$

$\lim_{x\to-\infty} y = 0$



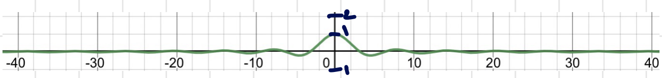
65. $\lim_{x\to\infty} \frac{\sin(x)}{x} = 0$

Squeeze Theorem:

$-1 \le \sin(x) \le 1 \quad \Big| : x$

$\lim_{x\to\infty} \quad -\frac{1}{x} \le \frac{\sin(x)}{x} \le \frac{1}{x}$

$0 \le \lim_{x\to\infty} \frac{\sin(x)}{x} \le 0$



∞ crossings of asymptote

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Lecture 4 – Lineralization and Extreme Values

2.7: ~~8~~, ~~16~~, ~~21~~, ~~41~~

3.10: ~~1~~, ~~2~~, ~~9~~, ~~15~~, ~~25~~, 27, ~~31~~, ~~33~~, ~~35~~

4.1: ~~27~~, ~~46~~, ~~49~~, ~~53~~

42: ~~12~~, ~~20~~, ~~33~~

8. $y = \dfrac{2x+1}{x+2} \quad (1, 1)$

$y' = \dfrac{2(x+2) - (2x+1)}{(x+2)^2} = \dfrac{\cancel{2x} + 4 - \cancel{2x} - 1}{(x+2)^2} = \dfrac{3}{(x+2)^2}$

$y'(1) = \dfrac{1}{3} \qquad y = \dfrac{1}{3}(x-1) + 1$

16. $s(t) = \dfrac{t^2}{2} - 6t + 23$

$v(t) = s'(t) = t - 6$

a) avg. $v(t)$

i) $[4, 8]$

$\dfrac{s(8) - s(4)}{8-4} = 0$

ii) $[6, 8]$

$\dfrac{s(8) - s(6)}{8-6} = 1$

iii) $[8, 10]$

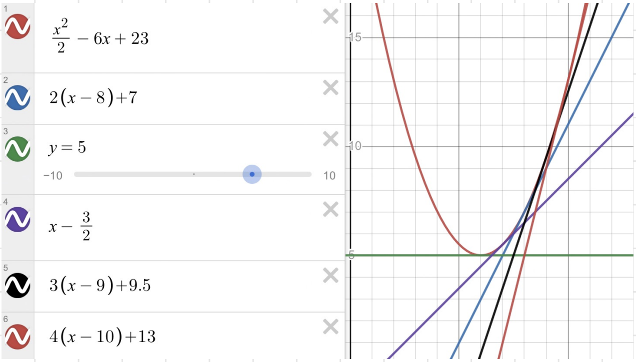
$\dfrac{s(10) - s(8)}{10-8} = 3$

iv) $[8, 12]$

$\dfrac{s(12) - s(8)}{12-8} = 4$

b) $v(8) = 2$

c)



21. $y = f(x) \quad (2, 4x-5)$

$f(2) = 4(2) - 5 = 3$

$f'(2) = 4$

41. $\lim\limits_{h \to 0} \dfrac{\cos(\pi + h) + 1}{h}$ $\quad -\cos(\pi)$

$f(x) = \cos(x)$

$a = \pi$

1. $f(x) = x^3 - x^2 + 3 \quad a = -2$

$f(a) = -8 - 4 + 3 = -9$

$f'(x) = 3x^2 - 2x$

$f'(a) = 12 + 4 = 16$

$L(x) = f(a) + f'(a)(x-a)$

$L(x) = -9 + 16(x+2)$

2. $f(x) = \sin x \quad a = \dfrac{\pi}{6}$

$f(a) = \dfrac{1}{2}$

$f'(x) = \cos x$

$f'(a) = \dfrac{\sqrt{3}}{2}$

$L(x) = f(a) + f'(a)(x-a)$

$L(x) = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\left(x - \dfrac{\pi}{6}\right)$

9. $\sqrt[4]{1+2x} \quad a=0$

$f(a)=1$

$f'(x)=\dfrac{1}{2\sqrt{(1+2x)^3}}$

$f'(a)=\dfrac{1}{2}$

$L(x)=f(a)+f'(a)(x-a)$

$L(x)=1+\frac{x}{2}$

$|dL(x)|=|L'(x)\,dx|<0.1$

$\left|\frac{dx}{2}\right|<0.1 \quad |\cdot 2$

$|dx|<0.2$

15. $y=e^{\frac{x}{10}}, \quad x=0, \quad dx=0.1$

a) $dy=y'dx=\dfrac{e^{\frac{x}{10}}}{10}dx$

b) $dy=\dfrac{e^0}{10}dx=0.01$

25. $\sqrt[3]{1001}$

$f(x)=\sqrt[3]{1000+x}$

$a=0 \quad f(a)=10$

$f'(x)=\dfrac{1}{3\sqrt[3]{(1000+x)^2}}$

$f'(a)=\dfrac{1}{300}$

$L(x)=10+\dfrac{x}{300}$

$\sqrt[3]{1001}=L(1)=10\frac{1}{300}$

27. $e^{0.1}$

$f(x)=e^{1-x} \quad f'(x)=-e^{1-x}$

$a=1 \quad f(a)=1 \quad f'(a)=-1$

$L(x)=1-(x-1)=2-x$

$e^{0.1}=L(0.9)=1.1$

31. $\dfrac{1}{9.98}$

$f(x)=\frac{1}{x} \quad f'(x)=-\frac{1}{x^2}$

$a=10 \quad f(a)=\frac{1}{10} \quad f'(a)=-\frac{1}{100}$

$L(x)=\frac{1}{10}-\frac{1}{100}(x-10)=\frac{1}{10}+\frac{1}{10}-\frac{x}{100}=\frac{1}{5}-\frac{x}{100}$

$\dfrac{1}{9.98}\approx L(9.98)=0.2-\dfrac{9.98}{100}=0.2-0.0998=0.1002$

33. $a=30\text{ cm} \quad da=0.1\text{ cm} \quad \dfrac{da}{a}=\dfrac{1}{300}$

a) $V=a^3$

$dV=3a^2\,da=3(30)^2\,0.1=270$

$\dfrac{dV}{V}=\dfrac{3\cancel{a^2}da}{a^{\cancel{3}}}=3\dfrac{da}{a}=0.01=1\%$

b) $S=6a^2$

$dS=12a\,da=12(30)\,0.1=36$

$\dfrac{dS}{S}=\dfrac{\overset{2}{\cancel{12}}\cancel{a}da}{\cancel{6}a^{\cancel{2}}}=2\dfrac{da}{a}=\dfrac{2}{300}=0.67\%$

35. $P=2\pi r=84 \quad dP=0.5$

a) $S=4\pi r^2=\dfrac{P^2}{\pi}$

$dS=\dfrac{2P\,dP}{\pi}=\dfrac{2(84)0.5}{\pi}=\dfrac{84}{\pi}$

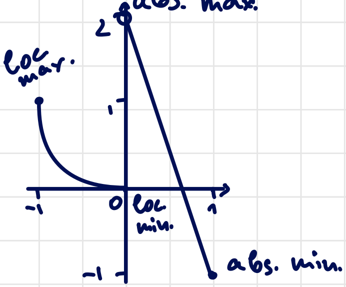
$\dfrac{dS}{S}=\dfrac{2\cancel{P}dP}{P^{\cancel{2}}}=\dfrac{2dP}{P}=\dfrac{1}{84}$

b) $V=\frac{4}{3}\pi r^3=\dfrac{P^3}{6\pi^2}$

$dV=\dfrac{P^2}{2\pi^2}dP=\dfrac{84}{2\pi^2}\,0.5=\dfrac{21}{\pi^2}$

$\dfrac{dV}{V}=\dfrac{\overset{3}{\cancel{6}}\cancel{P^2}dP}{\cancel{2}P^{\cancel{3}}}=3\dfrac{dP}{P}=\dfrac{1}{56}$

27. $f(x) = \begin{cases} x^2 & -1 \le x \le 0 \\ 2-3x & 0 < x \le 1 \end{cases}$

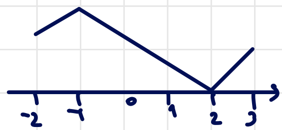


46. $f'(x) = \frac{100\cos^2 x}{10+x^2} - 1 \qquad 100\cos^2 x = 10+x^2$

$x_{max} = \sqrt{90} \Rightarrow 2\left\lceil \frac{\sqrt{90}}{\frac{\pi}{2}} \right\rceil = 2 \cdot 7 = 14$ solutions

49. $f(x) = 2x^3 - 3x^2 - 12x + 1 \qquad [-2, 3]$

$f'(x) = 6x^2 - 6x - 12 = 6(x-2)(x+1)$



$f(-2) = -3$

$f(-1) = 8 \to$ abs. max

$f(2) = -19 \to$ abs. min

$f(3) = -8$

53. $f(x) = x + \frac{1}{x} \qquad [0.2, 4]$

$f'(x) = \frac{(x+1)(x-1)}{x^2} \to 0$

$f(0.2) = 5.2$ abs. max. $\qquad f(4) = 4.25 \qquad f(1) = 2$ abs. min.

12. $f(x) = x^3 - 3x + 2 \qquad [-2, 2]$

$f'(c) = \frac{f(b) - f(a)}{b-a} = \frac{4-0}{2-(-2)} = 1$

$3c^2 - 3 = 1 \qquad 3(c^2-1) = 1$

$c^2 - 1 = \frac{1}{3} \qquad c^2 = \frac{4}{3} \qquad c = \pm\frac{2\sqrt{3}}{3}$

20. $x^3 + e^x = 0 \to$ no asymptotes

$3x^2 + e^x > 0 \to$ always increasing

$\Rightarrow$ pass through y-axis once

33. $f(x) = \frac{1}{x} \qquad g(x) = \begin{cases} \frac{1}{x} & x > 0 \\ 1 + \frac{1}{x} & x < 0 \end{cases}$

$f'(x) = g'(x) = -\frac{1}{x^2}$

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Lecture 5 - Implicit differentiation & Newton's method

3.5: ~~11~~, ~~27~~, ~~29~~, ~~49~~, ~~51~~, ~~57~~

4.8: ~~5~~, ~~7~~, ~~11~~, ~~13~~, ~~29~~

11. $y\cos x = x^2 + y^2$

$$\frac{dy}{dx}\cos x - y\sin x = 2x + 2y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2x + y\sin x}{\cos x - 2y}$$

27. $x^2 - xy - y^2 = 1$

$$2x - y - x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{2x-y}{2y+x} \quad (2,1)$$

$$\frac{dy}{dx} = \frac{3}{4} \qquad y = \frac{dy}{dx}x + k$$

$$1 = \frac{3}{2} + k \quad k = -\frac{1}{2} \quad y = \frac{3}{4}x - \frac{1}{2}$$

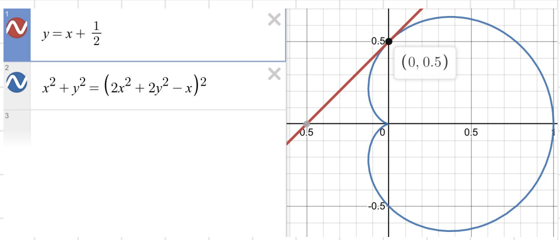
29. $x^2 + y^2 = (2x^2 + 2y^2 - x)^2 \quad (0, \frac{1}{2})$

$$\cancel{x^2} + y^2 = 4x^4 + 4y^4 + \cancel{x^2} + 8x^2y^2 - 4x^3 - 4xy^2$$

$$0 = 16x^3 + 16y^3\frac{dy}{dx} + 16xy^2 + 16x^2y\frac{dy}{dx} - 12x^2 - 4y^2 - 8xy\frac{dy}{dx} - 2y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{16x^3 + 16xy^2 - 12x^2 - 4y^2}{-16y^3 - 16x^2y + 8x + 2y}$$

$$\frac{dy}{dx} = 1 \qquad y = x + \frac{1}{2}$$



49. $y = (\arctan x)^2$

$$y' = \frac{2\arctan x}{1+x^2}$$

51. $y = \arcsin(2x+1)$

$$y' = \frac{1}{\sqrt{-x^2-x}}$$

57. $y = x\arcsin x + \sqrt{1-x^2}$

$$y' = \arcsin x + \cancel{\frac{x}{\sqrt{1-x^2}}} - \cancel{\frac{2x}{2\sqrt{1-x^2}}}$$

5. any of them without d

$f'(d) \to$ close to 0

7. $f(x) = \frac{2}{x} - x^2 + 1 = 0 \qquad f'(x) = -\frac{2}{x^2} - 2x$

$x_1 = 2 \quad f(x_1) = -2 \quad f'(x_1) = -4.5$

$x_2 = \frac{14}{9} \quad f(x_2) = -0.134 \quad f'(x_2) = -3.94$

$x_3 = 1.522$

11. $\sqrt[4]{75} = 2.94283096$

$f(x) = x^4 - 75 \quad f'(x) = 4x^3$

	$f(x)$	$f'(x)$
$x_0 = 3$	6	108
$x_1 = 2.9\overline{4}$	0.165	102.110

$x_2 = 2.94282857 \quad f(x_2) = -0.0002463 \quad f'(x_2) = 101.9424$

$x_3 = 2.94283096$

13. $f(x) = 3x^4 - 8x^3 + 2 = 0 \quad [2,3]$ $\qquad f'(x) = 12x^3 - 24x^2$

$\left.\begin{array}{l} f(2) = -14 \\ f(3) = 29 \end{array}\right\} \begin{array}{l} f(x)=0 \\ x\in(2,3) \end{array}$ mean value theorem

	$f(x)$	$f'(x)$
$x_0 = 2$	-14	0
$x_0 = 3$	29	108
$x_1 = 2.7315$	5.9639	65.4935
$x_2 = 2.64044$	0.55158	53.58118
$x_3 = 2.630146$	0.0065767	52.30969
$x_4 = 2.630020$		

29. $f(x) = x^2 - a = 0 \qquad f'(x) = 2x$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^2 - a}{2x_n}$$

$$= \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

$\sqrt{1000}$

$x_0 = 32$

$x_1 = 31.625$

$x_2 = 31.622777$

23.11

Lecture 6 – L'Hospital's Rule

4.4: ~~9~~, ~~19~~, ~~25~~, ~~35~~, ~~41~~, ~~53~~, ~~73~~, ~~74~~, ~~75~~

9. $\lim\limits_{x\to 4} \dfrac{x^2-2x-8}{x-4} = \lim\limits_{x\to 4} 2x-2 = 6$

19. $\lim\limits_{x\to\infty} \dfrac{\ln x}{\sqrt{x}} = \lim\limits_{x\to\infty} \dfrac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim\limits_{x\to\infty} \dfrac{2\sqrt{x}}{x} = \lim\limits_{x\to\infty} \dfrac{2}{\sqrt{x}} = 0$

25. $\lim\limits_{x\to 0} \dfrac{\sqrt{1+2x} - \sqrt{1-4x}}{x} = \lim\limits_{x\to 0} \dfrac{\cancel{2}}{\cancel{2}\sqrt{1+\cancel{2x}}\to 0} + \dfrac{2\cancel{4}}{\cancel{2}\sqrt{1-\cancel{4x}}\to 0} = 1+2 = 3$

35. $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{\cos x + e^x - 1} = \dfrac{0}{\cancel{1}+1-\cancel{1}} = 0$

41. $\lim\limits_{x\to 0} \dfrac{\cos x - 1 + \frac{1}{2}x^2}{x^4} = \lim\limits_{x\to 0} \dfrac{x - \sin x}{4x^3} = \lim\limits_{x\to 0} \dfrac{1-\cos x}{12x^2} = \lim\limits_{x\to 0} \dfrac{\sin x}{24x} = \dfrac{1}{24}$

53. $\lim\limits_{x\to 0^+} \dfrac{1}{x} - \dfrac{1}{e^x-1} = \lim\limits_{x\to 0^+} \dfrac{e^x-1-x}{x(e^x-1)} = \lim\limits_{x\to 0^+} \dfrac{e^x-1}{e^x-1+xe^x} = \lim\limits_{x\to 0^+} \dfrac{e^x}{e^x+e^x+\cancel{xe^x}\to 0} = \dfrac{1}{2}$

73. $\lim\limits_{x\to\infty} \dfrac{e^x}{x^n} = \lim\limits_{x\to\infty} \dfrac{e^x}{n!} = \infty$

74. $\lim\limits_{x\to\infty} \dfrac{\ln x}{x^p} = \lim\limits_{x\to\infty} \dfrac{1}{p x^p} = 0$

75. $\lim\limits_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}} = \lim\limits_{x\to\infty} \dfrac{1}{\sqrt{1+\cancel{\frac{1}{x^2}}}\to 0} = 1$

28.11

Lecture 7 – Integrals & Substitution rule

5.2: ~~55~~, ~~56~~, ~~59~~, ~~63~~

5.3: ~~59~~, ~~61~~

5.5: ~~3~~, ~~5~~, ~~7~~, ~~17~~, ~~19~~, ~~24~~, ~~43~~, ~~45~~, 55, 56, 59, 60, 66

55. $\int_0^4 x^2-4x+4 \, dx \geq 0$

$x^2-4x+4 = (x-2)^2 \Rightarrow f(x) \geq 0$

56. $\int_0^1 \sqrt{1+x^2}\, dx \leq \int_0^1 \sqrt{1+x}\, dx$

$[0,1]$: $x^2 \leq x \Rightarrow x^2+1 \leq x+1 \Rightarrow \sqrt{x^2+1} \leq \sqrt{x+1}$

59. $\int_0^1 x^3 dx = \frac{x^4}{4}\Big|_0^1 = \frac{1}{4}$ 63. $\int_0^2 x e^{-x} dx = -(x+1)e^{-x}\Big|_0^2 = 1-3e^{-2}$

59. $g(x) = \int_{2x}^{3x} \frac{u^2-1}{u^2+1} du$ $g'(x) = 3\left(\frac{9x^2-1}{9x^2+1}\right) - 2\left(\frac{4x^2-1}{4x^2+1}\right)$

63. $y = \int_{\cos(x)}^{\sin(x)} \ln(1+2v)\, dv = \cos(x)\ln(1+2\sin(x)) + \sin(x)\ln(1+2\cos(x))$

3. $\int x^2\sqrt{x^3+1}\, dx = \int \frac{u}{3} du = \frac{u^2}{6}+C = \frac{(x^3+1)^2}{6}+C$ $u = x^3+1$, $du = 3x^2 dx$

5. $\int \frac{x^3}{x^4-5} dx = \int \frac{1}{4u} du = \frac{\ln(u)}{4}+C = \frac{\ln(x^4-5)}{4}+C$ $u = x^4-5$, $du = 4x^3 dx$

7. $\int x\sqrt{1-x^2}\, dx = \int -\frac{\sqrt{u}}{2} du = -\frac{u^{\frac{3}{2}}}{3}+C = -\frac{(1-x^2)^{\frac{3}{2}}}{3}+C$ $u = 1-x^2$, $du = -2x\, dx$

17. $\int \frac{e^u}{(1-e^u)^2} du = \int -\frac{1}{x^2} dx = \frac{1}{x}+C = \frac{1}{1-e^u}+C$ $x = 1-e^u$, $dx = -e^u du$

19. $\int \frac{a+bx^2}{\sqrt{3ax+bx^3}} dx = \int \frac{1}{3\sqrt{u}} du = \frac{2\sqrt{u}}{3}+C = \frac{2\sqrt{3ax+bx^3}}{3}+C$ $u = 3ax+bx^3$, $du = (3a+3bx^2)dx$

24. $\int x\sqrt{x+2} = \frac{2}{3}(x+2)^{\frac{3}{2}}x - \int \frac{2}{3}(x+2)^{\frac{3}{2}} = \frac{2}{3}(x+2)^{\frac{3}{2}}\left(\frac{3}{5}x-\frac{4}{5}\right)+C$ $u = x$, $du = 1$, $dv = \sqrt{x+2}$, $v = \frac{2(x+2)^{\frac{3}{2}}}{3}$

43. $\int \frac{1}{\text{arsin}(x)\sqrt{1-x^2}} dx = \int \frac{1}{u} du = \ln(|u|)+C = \ln(|\arcsin(x)|)+C$ $u = \arcsin(x)$, $du = \frac{1}{\sqrt{1-x^2}} dx$

45. $\int \frac{1+x}{1+x^2} dx = \int \frac{1}{1+x^2} dx + \int \frac{x}{1+x^2} dx = \arctan(x) + \int \frac{1}{2u} du = \arctan(x) + \frac{\ln(u)}{2} + C =$

$u = 1+x^2$ $du = 2x$ $= \arctan(x) + \frac{\ln(x^2+1)}{2} + C$

55. $\int_0^1 \sqrt[3]{1+7x}\,dx = \int_1^8 \frac{u^{\frac{1}{3}}}{7}\,du = \frac{3}{28}u^{\frac{4}{3}}\Big|_1^8 = \frac{3}{28}\cdot(16-1) = \frac{45}{28} = 1\frac{17}{28}$ $\quad u = 1+7x$, $du = 7\,dx$

56. $\int_0^3 \frac{dx}{5x+1} = \int_1^{16} \frac{1}{5u}\,du = \frac{\ln(u)}{5}\Big|_1^{16} = \frac{\ln(16)}{5}$ $\quad u = 5x+1$, $du = 5\,dx$

59. $\int_1^2 \frac{e^{1/x}}{x^2}\,dx = \int_1^{\frac{1}{2}} -e^u\,du = -e^u\Big|_1^{\frac{1}{2}} = e^u\Big|_{\frac{1}{2}}^1 = e - \sqrt{e}$ $\quad u = \frac{1}{x}$, $du = -\frac{1}{x^2}\,dx$

60. $\int_0^1 x e^{-x^2}\,dx = \int_0^1 \frac{1}{2e^u}\,du = -\frac{1}{2e^u}\Big|_0^1 = \frac{1}{2} - \frac{1}{2e}$ $\quad u = x^2$, $du = 2x\,dx$

66. $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} x^4 \sin x\,dx = (4x^3+24x)\sin x + (-x^4+12x^2-24)\cos x\Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}} = 0$ $\quad \left.\begin{matrix} x^4 - \text{even} \\ \sin x - \text{odd} \end{matrix}\right\}\text{odd}$

29.11

Lecture 8 – Integration by parts

7.1: $\cancel{5}, \cancel{7}, \cancel{15}, \cancel{27}, \cancel{30}, \cancel{32}, \cancel{33}, \cancel{36}, \cancel{39}, \cancel{47}, \cancel{52}$

5. $\int t e^{-3t} dt = -t\frac{e^{-3t}}{3} + \int \frac{e^{-3t}}{3} dt = -\frac{e^{-3t}}{3}\left(t+\frac{1}{3}\right) + C$ $\quad dv = e^{-3t} \quad v = -\frac{e^{-3t}}{3}$ $\quad u = t \quad du = 1$

7. $\int (x^2+2x)\cos x \, dx = (x^2+2x)\sin x - \int (2x+2)\sin x \, dx = (x^2+2x-2)\sin x + (2x+2)\cos x + C$

$u = x^2+2x \quad dv = \cos x \qquad u = 2x+2 \quad dv = \sin x$

$du = 2x+2 \quad v = \sin x \qquad du = 2 \quad v = -\cos x \qquad u = (\ln x)^2 \quad dv = 1$

15. $\int (\ln x)^2 dx = x(\ln x)^2 - \int 2\ln x \, dx = x(\ln x)^2 - 2x\ln x + 2x + C \quad du = \frac{2\ln x}{x} \quad v = x$

27. $\int_1^5 \frac{\ln R}{R^2} dR = -\frac{\ln R}{R} + \int \frac{1}{R^2} dR = -\frac{\ln R + 1}{R} + C \quad u = \ln R \quad dv = R^{-2}$ $\quad du = \frac{1}{R} \quad v = -\frac{1}{R}$

30. $\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right) dx = \int_1^{\sqrt{3}} \operatorname{arccot}(x)\, dx = x\operatorname{arccot}(x) + \int \frac{x}{x^2+1} dx = x\operatorname{arccot}(x) + \frac{\ln(x^2+1)}{2}\Big|_1^{\sqrt{3}}$

$u = \operatorname{arccot}(x) \quad dv = 1 \qquad u = x^2+1$

$du = -\frac{1}{x^2+1} \quad v = x \qquad du = 2x\,dx \quad \int \frac{1}{2u} du = \frac{\ln(u)}{2}$

32. $\int_1^2 \frac{(\ln x)^2}{x^3} dx = -\frac{(\ln x)^2}{2x^2} + \int \frac{2\ln x}{2x^3} dx = -\frac{(\ln x)^2}{2x^2} - \frac{\ln x}{2x^2} + \int \frac{1}{2x^3} dx =$

$u = (\ln x)^2 \quad dv = x^{-3} \qquad u = \ln x \qquad = -\frac{\ln x}{2x^2}(\ln x + 1) - \frac{1}{4x^2}$

$du = \frac{2\ln x}{x} \quad v = -\frac{1}{2x^2} \qquad du = \frac{1}{x}$

33. $\int_0^{\frac{\pi}{3}} \sin(x)\ln(\cos x)\, dx = -\cos(x)\ln(\cos x) - \int \sin x \, dx = -\cos(x)\ln(\cos x) + \cos(x) + C$

$u = \ln(\cos x) \quad du = \frac{-\sin x}{\cos x} = -\tan x \qquad dv = \sin(x) \quad v = -\cos(x)$

36. $\int_0^t e^s \sin(t-s)\, ds = -\int_0^t e^s \sin(s-t)\, ds = \sin(t-s)e^s + \int \cos(s-t)e^s ds =$ $\quad du = \sin(t-s) \quad u = \cos(s-t)$

$u = \sin(s-t) \quad dv = e^s \qquad = \sin(t-s)e^s - \cos(t-s)e^s - \int \sin(t-s)e^s ds \quad v = dv = e^s$

$du = \cos(s-t) \quad v = e^s \qquad \int_0^t \sin(t-s)e^s ds = \frac{1}{2}\left(\sin(t-s)e^s - \cos(t-s)e^s\right)\Big|_0^t = \frac{\sin(t)+\cos(t)-e^t}{2}$

39. $\int_{\sqrt{\frac{\pi}{2}}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2)\, d\theta = \int_{\frac{\pi}{2}}^{\pi} \frac{x\cos(x)}{2} dx = \frac{x\sin(x)}{2} - \int \frac{\sin(x)}{2} dx = \frac{x\sin(x) + \cos(x)}{2}\Big|_{\frac{\pi}{2}}^{\pi} =$

$x = \theta^2 \quad u = x \quad dv = \cos(x) \quad = \frac{\pi\sin\pi + \cos\pi}{2} - \frac{\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2}}{2} = -\frac{\pi}{4} - \frac{1}{2}$

$dx = 2\theta\, d\theta \quad du = 1 \quad v = \sin(x)$

30.11

Lecture 9 - Improper Integrals

7.8: ~~1~~, ~~3~~, ~~5~~, ~~7~~, ~~13~~, ~~15~~, ~~28~~, ~~31~~, ~~41~~, ~~50~~, ~~51~~, ~~52~~, ~~55~~

1. a) $\int_1^2 \frac{x}{x-1}\,dx$ discontinuity at integrand

b) $\int_0^{\infty} \frac{1}{1+x^3}\,dx$ infinite interval of integration

c) $\int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx$ infinite interval of integration

d) $\int_0^{\frac{\pi}{2}} \cot(x)\,dx$ discontinuity at integrand

3. $\int x^{-3}\,dx = -\frac{1}{2x^2}$ $\quad \int_1^{10} x^{-3}\,dx = \frac{1}{200} - \frac{1}{2} = -\frac{99}{200}$ $\quad \int_1^{1000} x^{-3}\,dx = \frac{1}{2\,000\,000} - \frac{1}{2} = -\frac{999\,999}{2\,000\,000}$

$\int_1^{100} x^{-3}\,dx = \frac{1}{20\,000} - \frac{1}{2} = -\frac{9999}{20000}$ $\quad \int_1^{\infty} x^{-3}\,dx = \lim_{x\to\infty} \cancel{\frac{1}{2x^2}}^{\,0} - \frac{1}{2} = -\frac{1}{2}$

5. $\int_3^{\infty} \frac{1}{(x-2)^{\frac{3}{2}}}\,dx = -2(x-2)^{-\frac{1}{2}}\Big|_3^{\infty} = \lim_{x\to\infty} \cancel{\frac{-2}{\sqrt{x-2}}}^{\,0} - \frac{2}{\sqrt{3-2}} = -2$ convergent

7. $\int_{-\infty}^{0} \frac{1}{3-4x}\,dx = -\frac{\ln(3-4x)}{4}\Big|_{-\infty}^{0}$ divergent

13. $\int_{-\infty}^{\infty} x e^{-x^2}\,dx = \int_{-\infty}^{-\infty} -\frac{e^u}{2}\,du = -\frac{e^u}{2}\Big|_{-\infty}^{-\infty} = 0$ convergent

$u=-x^2 \quad du=-2x\,dx$

15. $\int_0^{\infty} \sin^2\alpha\,d\alpha = \frac{2\alpha - \sin(2\alpha)}{4}\Big|_0^{\infty}$ divergent

28. $\int_0^5 (5-x)^{-\frac{1}{3}}\,dx = -\frac{3}{2}(5-x)^{\frac{2}{3}}\Big|_0^5 = \frac{3 \cdot 5^{\frac{2}{3}}}{2}$ convergent

31. $\int_{-2}^{3} \frac{1}{x^4}\,dx = -\frac{x^{-3}}{3}\Big|_{-2}^{3}$ divergent

41. $\int_1^{\infty} e^{-x} = -\frac{1}{e^x}\Big|_1^{\infty} = \frac{1}{e} - \lim_{x\to\infty} \cancel{\frac{1}{e^x}}^{\,0} = \frac{1}{e}$

50. $\int_1^{\infty} \frac{1+\sin^2 x}{\sqrt{x}}\,dx \quad x \geq 1 \quad 0 \leq f(x) < 2$ divergent

51. $\int_1^{\infty} \frac{x+1}{\sqrt{x^4-x}}\,dx$ divergent

52. $\int_0^{\infty} \frac{\arctan x}{2+e^x}\,dx$ divergent

55. $\int_0^{\infty} \frac{1}{\sqrt{x}(1+x)}\,dx = \int_0^{\infty} \frac{2}{u^2+1}\,du = 2\arctan(u)\Big|_0^{\infty} = \pi$

$u=\sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$= \int_0^1 \frac{1}{\sqrt{x}(1+x)}\,dx + \int_1^{\infty} \frac{1}{\sqrt{x}(1+x)}\,dx$

$\hookrightarrow \frac{\pi}{2} \qquad \hookrightarrow \frac{\pi}{2}$

05.12

Lecture 10 – Sequences

11.1: 7, 11, 13, 17, 23, 27, 35, 43, 53, 54, 63, 74, 79, 80, 81

7. $a_n = \frac{1}{(n+1)!}$ $\quad \left\{1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \dots\right\}$

11. $a_0 = 2$ $\quad a_{n+1} = \frac{a_n}{1+a_n}$ $\quad \left\{2, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \frac{2}{9}, \dots\right\}$

13. $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \dots\right\}$ $\quad a_0 = \frac{1}{2}$ $\quad a_{n+1} = \frac{1}{a_n^{-1} + 2}$ $\quad / a_n = \frac{1}{2(n+1)}$

17. $\left\{\frac{1}{2}, -\frac{4}{3}, \frac{9}{4}, -\frac{16}{5}, \frac{25}{6}, \dots\right\}$ $\quad f(n) = \frac{(-1)^n (n+1)^2}{n+2}$

54. $\left\{1, \frac{1}{3}, \frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \frac{1}{5}, \frac{1}{4}, \frac{1}{6}\right\}$ diverges

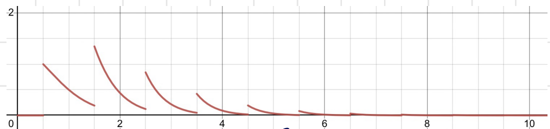
23. $a_n = \frac{3+5n^2}{n+n^2}$ $\quad \lim\limits_{n\to\infty} a_n = 5$

27. $a_n = \left(\frac{3}{7}\right)^n$ $\quad \lim\limits_{n\to\infty} a_n = 0$

35. $a_n = \frac{(-1)^n}{2\sqrt{n}}$ $\quad \lim\limits_{n\to\infty} a_n = 0$

43. $a_n = \frac{\cos^2 n}{2^n}$ $\quad \lim\limits_{n\to\infty} a_n = 0$

53. $\{0, 1, 0, 0, 1, 0, 0, 0, 1, \dots\}$ diverges



63. $a_n = \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{(2n)^n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cancel{\frac{1}{2n}}^{\to 0} \cdot \cancel{\frac{3}{2n}}^{\to 0} \cdot \cancel{\frac{5}{2n}}^{\to 0} \cdots \cancel{\frac{2n-1}{2n}}^{\to 1} = 0$$

74. $a_n = \frac{1-n}{2+n}$ $\quad a_{n+1} \overset{?}{<} a_n$ $\quad \frac{1-n-1}{2+n+1} \overset{?}{<} \frac{1-n}{2+n}$ $\quad \frac{n}{n+3} \overset{?}{>} \frac{n-1}{n+2}$

$n(n+2) \overset{?}{>} (n-1)(n+3)$

$\cancel{n^2} + \cancel{2n} \overset{?}{>} \cancel{n^2} + \cancel{2n} - 3$

$0 > -3$ ✓

$\Rightarrow$ decreasing $\quad \lim\limits_{n\to\infty} a_n = -1$

79. $\sqrt{2\sqrt{2\sqrt{2}\dots}}$ $\quad a_{n+1} > a_n$ $\quad 0 > a_n(a_n - 2)$

$a_0 = \sqrt{2}$ $\quad \sqrt{2a_n} > a_n \,|^2$ $\quad a_0 \in (0, 2)$

$a_{n+1} = \sqrt{2a_n}$ $\quad 2a_n > a_n^2$ $\quad \Rightarrow$ increasing $\quad \lim\limits_{n\to\infty} a_n = 2$

fixed, fixed; 0, 2

80. $a_0 = \sqrt{2}$

$a_{n+1} = \sqrt{2 + a_n}$ $\quad \sqrt{2 + a_n} > a_n \,|^2$

$a_{n+1} > a_n$ $\quad 2 + a_n > a_n^2$

$(a_n - 2)(a_n + 1) < 0$

fixed, fixed; -1, 2 $\quad \lim\limits_{n\to\infty} a_n = 2$

$a_0 \in (-1, 2) \Rightarrow$ increasing

81. $a_0 = 1$ $\quad a_{n+1} > a_n$ $\quad a_n^2 - 3a_n + 1 < 0$

$a_{n+1} = 3 - \frac{1}{a_n}$ $\quad 3 - \frac{1}{a_n} > a_n \,|\cdot a_n$ $\quad a_n = \frac{3 \pm \sqrt{5}}{2}$

fixed, fixed; $\frac{3-\sqrt{5}}{2}$, $\frac{3+\sqrt{5}}{2}$

$a_0 \in \left(\frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}\right) \Rightarrow a_n$ – increasing $\quad \lim\limits_{n\to\infty} a_n = \frac{3+\sqrt{5}}{2} < 3$

06.12

Lecture 11 – Series

11.2: ~~1~~, ~~3~~, ~~9~~, ~~15~~, ~~17~~, ~~19~~, ~~23~~, ~~27~~, ~~39~~, ~~53~~, ~~54~~, ~~58~~, ~~61~~

11.3: ~~6~~, ~~7~~, ~~9~~, ~~11~~, ~~15~~, ~~29~~, ~~34~~, ~~35~~, ~~43~~

3. $\lim_{n\to\infty} s_n = \lim_{n\to\infty} 2-3\cancel{(0.8)^n}^{\to 0} = 2$

9. $\sum_{n=1}^{\infty} \frac{12}{(-5)^n} = \frac{-\frac{12}{5}}{1-(-\frac{1}{5})} = -2$

15. $a_n = \frac{2n}{3n+1}$ $\quad \lim_{n\to\infty} a_n = \frac{2}{3} \Rightarrow$ divergent

17. $3-4+\frac{16}{3}-\frac{64}{9}+\ldots \Rightarrow$ divergent $\quad r=-\frac{4}{3}$

19. $10-2+0.4-0.08+\ldots$ $r=-\frac{1}{5} \Rightarrow$ convergent

$\sum a_n = \frac{10}{1-(-\frac{1}{5})} = \frac{25}{3}$

23. $\sum_{n=1}^{\infty} \frac{1}{4}\left(-\frac{3}{4}\right)^{n-1} = \frac{1/4}{1-(-\frac{3}{4})} = \frac{1}{7}$

27. $\frac{1}{3}+\frac{1}{6}+\frac{1}{9}+\frac{1}{12}+\frac{1}{15}+\ldots = \sum_{n=1}^{\infty}\frac{1}{3n}$ divergent $\quad \lim_{n\to\infty} \frac{\frac{1}{3n}}{\frac{1}{n}} = \frac{1}{3}$

58. $\sum_{n=1}^{\infty} (x+2)^n$ $\quad -3 < x < -1$

$\sum_{n=1}^{\infty} (x+2)^n = \frac{1}{1-(x+2)} = -\frac{1}{x+1}$

39. $\sum_{n=1}^{\infty} \arctan n$ $\quad \lim_{n\to\infty} \arctan n = \frac{\pi}{2} \Rightarrow$ divergent

61. $\sum_{n=0}^{\infty} \frac{2^n}{x^n}$ $\quad x>2$ or $x<-2$

$\sum_{n=0}^{\infty} \frac{2^n}{x^n} = \frac{1}{1-\frac{2}{x}} = \frac{x-2}{2}$

53. $2.\overline{516} = 2+\frac{516}{10^3}\cdot\frac{\cancel{10^3}}{999} = 2\frac{172}{333}$

$0.\overline{516}$ $\quad a=\frac{516}{10^3}$ $\quad r=\frac{1}{10^3}$

54. $10.1\overline{35} = \frac{101}{10}+\frac{35}{10^3}\cdot\frac{10^2}{99} = \frac{10034}{990}$

$0.0\overline{35}$ $\quad a=\frac{35}{10^3}$ $\quad r=\frac{1}{10^2}$

6. $\int_1^{\infty}(3n-1)^{-4}dx = -\frac{(3n-1)^{-3}}{9}\Big|_1^{\infty} = \frac{1}{72}$

9. $\sum_{n=1}^{\infty}\frac{1}{n^{\pi}}$ $\quad p>1$ convergent

7. $\int_1^{\infty}\frac{n}{n^2+1} = \int_2^{\infty}\frac{1}{2u}du = \frac{\ln(u)}{2}\Big|_2^{\infty} =$ divergent

$u=n^2+1$ $\quad du=2n\,dx$

15. $\sum_{n=1}^{\infty}\frac{\sqrt{n}+4}{n^2} = \sum_{n=1}^{\infty}\left(\frac{1}{n^{\frac{3}{2}}}+\frac{4}{n^2}\right)$ $\quad p_1, p_2 > 1$ convergent

11. $1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\frac{1}{125}+\ldots = \sum_{n=1}^{\infty}\frac{1}{n^3}$ $\quad p>1$ convergent

29. $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^p}$ $\quad p>1$

35. $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

34. $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

a) $\sum_{n=1}^{\infty}\left(\frac{3}{n}\right)^4 = 81\sum_{n=1}^{\infty}\frac{1}{n^4} = 81\cdot\frac{\pi^4}{90} = \frac{9\pi^4}{10}$

a) $\sum_{n=2}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}-1 = \frac{\pi^2-6}{6}$

b) $\sum_{k=5}^{\infty}\frac{1}{(k-2)^4} = \sum_{k=3}^{\infty}\frac{1}{k^4} = \frac{\pi^4}{90}-1-\frac{1}{16} = \frac{8\pi^4-765}{720}$

b) $\sum_{n=3}^{\infty}\frac{1}{(n+1)^2} = \sum_{n=4}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}-1-\frac{1}{4}-\frac{1}{9} = \frac{6\pi^2-49}{36}$

43. $\sum_{i=1}^{n} a_i \le \int_1^n f(x)\,dx$

c) $\sum_{n=1}^{\infty}\frac{1}{(2n)^2} = \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{24}$

a) $s_n = \sum_{i=1}^{n}\frac{1}{i} \le \int_1^n \frac{1}{x}dx = \ln(x)\Big|_1^n = \ln n$

b) $s_{10^6} \le \ln 10^6 = 13.8$ $\quad s_{10^9} \le \ln 10^9 = 20.7$

07.12 # Lecture 12 – Alternating series and comparison test

11.3: ~~37, 39~~

11.4: ~~5, 6, 9, 13, 23, 29, 31, 34, 35, 37, 43~~

11.5: ~~2, 3~~, 7, 11, 16, 21, 23, 29, 35

37. $\sum_{n=1}^{\infty} = \frac{1}{n^2}$ $\quad s_{10} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \frac{1}{64} + \frac{1}{81} + \frac{1}{100}$
$= 1.522$

$|s - s_N| \leq \int_N^{\infty} f(x)\,dx \Rightarrow \int_{10}^{\infty} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{10}^{\infty} = 0.1$

$\int_{11}^{\infty} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{11}^{\infty} = \frac{1}{11}$ $\quad \frac{1}{n} \leq 0.001$ $\quad n = 1000$

39. $\sum_{n=1}^{\infty} (2n+1)^{-6}$

$\int_n^{\infty} (2x+1)^{-6}\,dx \leq \frac{1}{10^5}$ $\quad \Rightarrow s_1 - \frac{1}{3^6} = \frac{1}{729} = 0.00137$

$\frac{(2n+1)^{-5}}{10} \leq \frac{1}{10^5}$ $\quad 10^4 \leq (2n+1)^5$

$n = 2$ $\quad 1000 \leq 3125$

5. $\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} + \frac{1}{n\sqrt{n}}$ $\quad p = \frac{1}{2} < 1 \to$ divergent; $p = \frac{3}{2} > 1 \to$ convergent $\Big\}$ divergent

6. $\sum_{n=1}^{\infty} \frac{n-1}{n^3+1} = \sum_{n=1}^{\infty} \frac{n}{n^3+1} - \frac{1}{n^3+1}$ $\quad p = 2 > 1$; $p = 3 > 1$ $\Big\}$ convergent

$2 + \cos x + \sin x$

9. $\sum_{k=1}^{\infty} \frac{\ln k}{k} \geq \sum_{k=1}^{\infty} \frac{1}{k} \to$ divergent $\Rightarrow$ divergent
$p = 1$

13. $\sum_{n=1}^{\infty} \frac{1+\cos n}{e^n}$ $\quad \int_1^{\infty} (1+\cos x)e^{-x}\,dx = \int_1^{\infty} e^{-x}\,dx + \int_1^{\infty} \cos x\, e^{-x} = \frac{\sin x - \cos x - 2}{2e^x}\Big|_1^{\infty} =$
convergent $\longleftarrow$ $\frac{2 + \cos 1 - \sin 1}{2e}$

$\int \cos x\, e^{-x}\,dx = -\cos x\, e^{-x} - \int \sin x\, e^{-x}\,dx = -\cos x\, e^{-x} + \sin x\, e^{-x} - \int \cos x\, e^{x}\,dx$

$u = \cos x$ $\quad dv = e^{-x}$ $\quad u = \sin x$ $\quad dv = e^{-x}$ $\quad \int \cos x\, e^{x}\,dx = \frac{\sin x - \cos x}{2e^x}$

$du = -\sin x$ $\quad v = -e^{-x}$ $\quad du = \cos x$ $\quad v = -e^{-x}$

23. $\sum_{n=1}^{\infty} \frac{5+2n}{(1+n^2)^2} = \sum_{n=1}^{\infty} \frac{5+2n}{n^4+2n^2+1}$ $\quad \int_1^{\infty} \frac{2x}{(x^2+1)^2}\,dx = \int_1^{\infty} \frac{1}{(u+1)^2}\,du = -\frac{1}{u+1}\Big|_1^{\infty} = \frac{1}{2}$
$u = x^2$ $\quad du = 2x$ $\quad$ convergent

29. $\sum_{n=1}^{\infty} \frac{1}{n!} \leq \sum_{n=1}^{\infty} \frac{1}{n^n}$ $\quad p = n > 1 \Rightarrow$ convergent

31. $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ $\quad \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \frac{\sin(0)}{0} = 1 \Rightarrow$ divergent

34. $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^4}$ $\quad \int_{10}^{\infty} e^{1/x} x^4\,dx$

$u = x^4$ $\quad dv = e^{1/x}$

$du = 4x^3$ $\quad v =$

35. $\sum_{n=1}^{\infty} 5^{-n} \cos^2 n$ $\int_{10}^{\infty} 5^{-x} \cos^2 x \, dx$

$u = 5^{-x}$ $dv = \cos^2 x$

$du = -\ln 5 \cdot 5^{-x}$ $v =$

42. $a_n > 0$, $\lim_{n\to\infty} n a_n \neq 0$ $\Rightarrow \lim_{n\to\infty} a_n \neq 0$ $\Rightarrow$ divergent

2. $\frac{2}{3} - \frac{2}{5} + \frac{2}{7} - \frac{2}{9} + \frac{2}{11} - \dots$ convergent

3. $-\frac{2}{5} + \frac{4}{6} - \frac{6}{7} + \frac{8}{8} - \frac{10}{9} + \dots$ divergent

7. $\sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1}$ $|a_{n+1}| > |a_n|$ divergent

11. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}$ $|a_{n+1}| < |a_n|$ convergent

16. $\sum_{n=1}^{\infty} \frac{n \cos n\pi}{2^n}$ $\cos(\pi) = 0 \Rightarrow$ convergent

21. $\sum_{n=1}^{\infty} \frac{(-0.8)^n}{n!}$ $|a_{n+1}| \leq 10^{-4}$ $s_6 = -0.8 + \frac{0.8^2}{2} - \frac{0.8^3}{6} + \frac{0.8^4}{24} - \frac{0.8^5}{120} + \frac{0.8^6}{720} =$

$\frac{0.8^k}{k!} \leq 10^{-4}$ $k = 7$

23. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$ $|a_{n+1}| \leq |a_n| \Rightarrow$ convergent $\frac{1}{k^6} < 0.00005$

$|a_{n+1}| < 0.00005 = 2 \times 10^{-4}$ $k = 6 \Rightarrow s_5$

29. $\sum_{n=1}^{\infty} (-1)^n n e^{-2n}$ $|a$