19/11

Ratio test & Power series [11.5, 11.6, 11.8]

$\sum_{n=1}^{\infty} \frac{(-1)^{\frac{n^2+n}{2}}}{4^n}$ $\quad \frac{n^2+n}{2} = 1, 3, 6, 10, 15$ (for $n = 1, 2, 3, 4, 5$) $\Rightarrow$ alternating $r = \frac{1}{4} \Rightarrow$ convergent

Absolute convergence

$\sum a_n$ – absolutely convergent if $\sum |a_n|$ – convergent, otherwise conditionally

Riemann's rearrangement – produces absolutely convergent series with same sum

$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$ $\quad \frac{n+1}{(n+1)^2+1} \overset{?}{\leq} \frac{n}{n^2+1}$ $\quad \int_1^{\infty} \frac{x}{x^2+1}\,dx = \frac{\ln(u)}{2} + C \Big|_{\infty}^{2} \rightarrow$ divergent

$\lim_{n\to\infty} \frac{n}{n^2+1} = 0$ $\quad \frac{n+1}{n^2+2n+2} \overset{?}{\leq} \frac{n}{n^2+1}$ $\quad u = x^2+1 \quad du = 2x$

$\cancel{x^3}+\cancel{x^2}+x+1 \overset{?}{\leq} \cancel{n^3}+2n^2+2n \quad 1 < n^2+n \Rightarrow$ convergent

conditionally convergent

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{\frac{2}{3}}}$ $\quad \left.\begin{array}{l}\lim_{n\to\infty} a_n = 0 \\ |a_{n+1}| < |a_n|\end{array}\right\}$ convergent $\quad \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{3}}}$ $\quad p = \frac{2}{3} < 1 \Rightarrow$ divergent

$\Rightarrow$ conditionally convergent

$\sum_{n=1}^{\infty} \frac{-n}{n^2+1}$ $\quad \int_1^{\infty} \frac{-x}{x^2+1}\,dx = -\frac{\ln(x^2+1)}{2}\Big|_1^{\infty} + C \Rightarrow$ divergent

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$ $\quad \begin{array}{l}\lim_{n\to\infty} a_n = 0 \\ |a_{n+1}| < |a_n|\end{array}$ $\quad \sum \frac{1}{n^2+1} < \sum \frac{1}{n^2} \quad p = 2 > 1 \Rightarrow$ convergent

$\Rightarrow$ absolutely convergent

Ratio test

$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ $\quad \lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} \cdot \frac{\cancel{2^n}}{n^2} = \lim_{n\to\infty} \frac{n^2+2n+1}{2n^2} = \frac{1}{2} < 1 \Rightarrow$ absolute convergence

$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ $\quad \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{\frac{3}{2}} = 1 \Rightarrow$ inconclusive

$\sum_{n=1}^{\infty} \frac{10^n}{(n-1)4^{2n+1}}$ $\quad \lim_{n\to\infty} \frac{10^{\cancel{n+1}}}{n\,\underset{16}{\cancel{4^{2n+3}}}} \cdot \frac{(n-1)\cancel{4^{2n+1}}}{\cancel{10^n}} = \lim_{n\to\infty} \frac{5(n-1)}{8n} = \frac{5}{8} < 1 \Rightarrow$ absolute convergence

$\sum_{n=1}^{\infty} \frac{n!}{100^n}$ $\quad \lim_{n\to\infty} \frac{n+1\cancel{!}}{100^{\cancel{n+1}}} \cdot \frac{\cancel{100^n}}{\cancel{n!}} = \lim_{n\to\infty} \frac{n+1}{100} = \infty \Rightarrow$ divergent

$\sum_{n=1}^{\infty} \frac{n^{100}100^n}{n!}$ $\quad \lim_{n\to\infty} \frac{(n+1)^{100} \cdot 100^{\cancel{n+1}}}{n+1\cancel{!}} \cdot \frac{\cancel{n!}}{n^{100} \cdot \cancel{100^n}} = \lim_{n\to\infty} \frac{100}{n+1}\left(\frac{n+1}{n}\right)^{100} = 0 \Rightarrow$ absolute conv.

Power series

$\sum_{n=0}^{\infty} c_n (x-a)^n \rightarrow$ power series in $(x-a)$

$R \in [0, \infty)$ – radius of convergence (a – center of convergence)

$|x-a| < R \rightarrow$ (absolutely) convergent $\quad |x-a| > R \rightarrow$ divergent

interval of convergence: $(-\infty, +\infty)$ or $[/(a-R, a+R)/]$

$\sum_{n=1}^{\infty} \frac{2}{n}\left(\frac{x+1}{3}\right)^n \quad R=3 \qquad \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad R=\infty$

$\sum_{n=1}^{\infty} \left(\frac{5x+1}{4}\right)^n \cdot \frac{1}{4} \quad R=\frac{4}{5} \qquad \sum_{n=1}^{\infty} c_n (x-3)^n$

$x=5 \rightarrow$ convergent $\quad x=-1$ – divergent $\quad a=3$

$\Rightarrow |x-3| \le 2$ – convergent $\quad |x-3| \ge 4$ – divergent

$\sum_{n=1}^{\infty} (-1)^n n x^n \quad R=1 \quad a=0 \quad (-1, 1)$

$\sum_{n=1}^{\infty} \frac{(-4x)^n}{\sqrt{n}} \quad R=\frac{1}{4} \quad a=0 \quad \left(-\frac{1}{4}, \frac{1}{4}\right]$

$\sum_{n=2}^{\infty} \left(\frac{x+2}{2}\right)^n \frac{1}{\ln(n)} \quad R=2 \quad a=-2 \quad [-4, 0)$

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Functions as power series [11.9, 11.10]

$$\frac{1}{5-x} \to \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n \quad a=0 \quad R=5 \quad (-5,5)$$

$$\to \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x-3}{2}\right)^n \quad a=3 \quad R=2 \quad (1,5)$$

$$\to \sum_{n=0}^{\infty}(x-4)^n \quad a=4 \quad R=1 \quad (3,5)$$

Manipulations with power series

$$f(x)=\sum_{n=0}^{\infty}a_n x^n \quad |x|<R_1 \qquad g(x)=\sum_{n=0}^{\infty}b_n x^n \quad |x|<R_2$$

$$cx^k f(x)=\sum_{n=0}^{\infty}c a_n x^{n+k} \quad |x|<R_1,\ k\in\mathbb{N}$$

$$f(cx^k)=\sum_{n=0}^{\infty}a_n c^n x^{nk} \quad |cx^k|<R_1,\ k\in\mathbb{N}$$

$$f(x-c)=\sum_{n=0}^{\infty}a_n(x-c)^n \quad |x-c|<R_1$$

$$f(x)+g(x)=\sum_{n=0}^{\infty}(a_n+b_n)x^n \quad |x|<\min\{R_1,R_2\}$$

$$f(x)=\sum_{n=0}^{\infty}c_n(x-a)^n$$

$$f'(x)=\sum_{n=1}^{\infty}nc_n(x-a)^{n-1}$$

$$\int f(x)\,dx = C+\sum_{n=0}^{\infty}\frac{c_n}{n+1}(x-a)^{n+1}$$

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n \quad x\in(-1,1)$$

$$\ln(1+x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1} \quad x\in(-1,1]$$

$$\arctan(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1} \quad x\in[-1,1]$$

$$\frac{x}{16-x^4}=\sum_{n=0}^{\infty}\frac{x^{4n+1}}{16^{n+1}}$$

$$\frac{\arctan(x^2)}{x} \quad x=0 \quad > \quad x-\frac{x^5}{3}+\frac{x^9}{5}$$

$$1+2x+x^2+2x^3+x^4+2x^5+x^6+\ldots=\frac{1+2x}{1-x^2}$$

$$f(x)=\frac{5}{1-4x^2}=\sum_{n=0}^{\infty}5\cdot(4x^2)^n \quad \underset{R=\frac{1}{2}}{a=0} \quad \left(-\frac{1}{2},\frac{1}{2}\right)$$

$$g(x)=\frac{x^2}{x^4+16}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+2}}{16^{n+1}} \quad a=0 \; R=2 \quad (-2,2)$$

$$h(x)=\ln(5-x)=\ln(5)-\sum_{n=1}^{\infty}\frac{x^n}{n5^n} \quad a=0 \quad R=5 \quad (-5,5)$$

Taylor series

$\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n \to$ Taylor series of f at a /$a=0 \to$ Maclaurin series/

$(1+x)^r=\sum_{n=0}^{\infty}\binom{r}{n}x^n$ $\quad x\in(-1,1)$

binomial coefficient: $\binom{r}{0}=1$, $\binom{r}{1}=r$, $\binom{r}{n}=\frac{r\cdot\ldots\cdot(r-n+1)}{n!}$

$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n \quad R=1$

$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!} \quad R=\infty$

$\sin(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} \quad R=\infty$

$\cos(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} \quad R=\infty$

$\arctan(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1} \quad R=1$

$\ln(1+x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1} \quad R=1$

$(1+x)^r=\sum_{n=0}^{\infty}\binom{r}{n}x^n \quad R=1$

$$f(x)=x^3\cos(x^2)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+3}}{(2n)!}$$

$$f(x)=xe^x=\sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}=x+x^2+\frac{x^3}{2}+\frac{x^4}{6}+\ldots$$

$$g(x)=(1-x)^{-2}=\sum_{n=0}^{\infty}x^{2n} \quad R=1 \quad a=0 \quad (-1,1)$$

$$h(x)=2^x=e^{x\ln 2}=\sum_{n=0}^{\infty}\frac{(x\ln 2)^n}{n!} \quad R=\infty \; a=0 \quad (-\infty,+\infty)$$

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Taylor series [11.10]

Analytic functions

f - power series near a and $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$ on open interval containing a

$T_n(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k \quad R_n(x)=f(x)-T_n(x) \quad \lim_{n\to\infty}R_n(x)=0$

$\sum_{n=0}^{\infty}\frac{(-3)^{n+2}}{n!}=9e^{-3}$

$f(x)=\arctan(x) \quad f^{(7)}(0)=-6!=-720$

$\lim_{x\to 0}\frac{x^3}{\ln(1+x)+e^{-x}-1}=\lim_{x\to 0}\frac{3x^2}{\frac{1}{1+x}-e^{-x}}=\lim_{x\to 0}\frac{6x}{-\frac{1}{(1+x)^2}+e^{-x}}=\lim_{x\to 0}\frac{6}{\frac{2}{(1+x)^3}-e^{-x}}=\frac{6}{2-1}=6$

$\int_0^1 x^2\cos(\sqrt{x})\,dx=2x^2\sin\sqrt{x}-\int_0^1 4x\sin\sqrt{x}\,dx=2x^2\sin\sqrt{x}+8x\cos\sqrt{x}-\int_0^1 8\cos\sqrt{x}\,dx=$

$u=x^2 \quad du=2x \qquad\qquad u=4x \quad du=4 \qquad =2x^2\sin\sqrt{x}+8x\cos\sqrt{x}-16\sin\sqrt{x}\Big|_0^1=0.21$

$dv=\cos\sqrt{x} \quad v=2\sin\sqrt{x} \qquad\qquad dv=\sin\sqrt{x} \quad v=-2\cos\sqrt{x}$

$\lim_{x\to 0}\frac{\sin(x)-x+\frac{1}{6}x^3}{x^5}=\lim_{x\to 0}\frac{\sin(x)}{120x}=\frac{1}{120}$

Taylor polynomials

$f(x)=\sqrt{x} \quad a=4 \quad 2^{nd}$ order $\quad f(x)=\sqrt{x}=2 \quad f'(x)=\frac{1}{2\sqrt{x}}=\frac{1}{4} \quad f''(x)=-\frac{1}{4\sqrt{x^3}}=-\frac{1}{32}$

$\Rightarrow T_2(x)=f(x)+f'(x)(x-a)+\frac{f''(x)(x-a)^2}{2!}=2+\frac{x-4}{4}-\frac{(x-4)^2}{64}$

$f^{4}(x)=0$

$f(x)=\sin(2x) \quad a=0 \quad 4^{th}$ order $\quad f(x)=0 \quad f'(x)=2\cos(2x)=2 \quad f''(x)=0 \quad f'''(x)=-8$

$T_4(x)=\cancel{f(x)}^{\to 0}+f'(x)(x-a)+\cancel{\frac{f''(x)(x-a)^2}{2!}}^{\to 0}+\frac{f'''(x)(x-a)^3}{3!}+\cancel{\frac{f^{4}(x)(x-a)^4}{4!}}^{\to 0}=2x-\frac{4}{3}x^3$

$f(x)=T_k(x)+h_k(x)(x-a)^k \quad \lim_{x\to a}h_k(x)=0$

$f(x)=T_k(x)+\frac{f^{k+1}(c)}{(k+1)!}(x-a)^{k+1} \quad c\in(x,a)$

$\left.\begin{array}{l}|f^{k+1}(x)|\le M\\ \text{for } |x-a|\le d\end{array}\right\} \Rightarrow |f(x)-T_k(x)|\le\frac{M}{(k+1)!}|x-a|^{k+1}$

09/01

Vectors & Functions of several variables [12.1-12.3, 14.1]

$x \in \mathbb{R}^n$ $\langle x_1, x_2 \dots x_n \rangle$ - components of the vector

$x = \langle x_1, x_2 \rangle$ $y = \langle y_1, y_2 \rangle$

$x + y = \langle x_1 + y_1, x_2 + y_2 \rangle$

$cx = \langle cx_1, cx_2 \rangle$

$x \cdot y = x_1 y_1 + x_2 y_2$

$|x| = \sqrt{x \cdot x} = \sqrt{x_1^2 + x_2^2}$

$i = \langle 1, 0 \rangle$ $j = \langle 0, 1 \rangle$

$x = \langle x_1, x_2, x_3 \rangle$ $y = \langle y_1, y_2, y_3 \rangle$

$x + y = \langle x_1 + y_1, x_2 + y_2, x_3 + y_3 \rangle$

$cx = \langle cx_1, cx_2, cx_3 \rangle$

$x \cdot y = x_1 y_1 + x_2 y_2 + x_3 y_3$

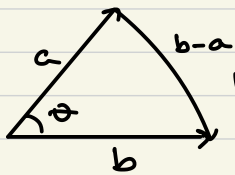
$|x| = \sqrt{x \cdot x} = \sqrt{x_1^2 + x_2^2 + x_3^2}$

$i = \langle 1, 0, 0 \rangle$ $j = \langle 0, 1, 0 \rangle$ $k = \langle 0, 0, 1 \rangle$

a, b - vectors $\in \mathbb{R}^n$

θ - angle between vectors

$a \cdot b = |a||b| \cos\theta$



$|b-a|^2 = |a|^2 + |b|^2 - 2|a||b|\cos\theta$

$\cancel{|b|^2} - 2a \cdot b + \cancel{|a|^2} = \cancel{|a|^2} + \cancel{|b|^2} - 2|a||b|\cos\theta$

$\cos\theta = \dfrac{a \cdot b}{|a||b|}$

$a, b \neq 0$ $a \cdot b = 0 \Leftrightarrow a$ orthogonal to b (perpendicular)

Equations

$ax + by = c$ line orthogonal to $\langle a, b \rangle$

$(x-a)^2 + (y-b)^2 = r^2$ circle with center (a, b), radius r

$ax + by + cz = d$ plane orthogonal to $\langle a, b, c \rangle$

$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$ sphere with center (a, b, c), radius r

$(-2, 1) \longrightarrow (1, 2)$ $\langle 1 - (-2), 2 - 1 \rangle = \langle 3, 1 \rangle$

$a = \langle -3, 4 \rangle$ $a + b = \langle 6, 3 \rangle$ $|a| = 5$

$b = \langle 9, -1 \rangle$ $4a + 2b = \langle 6, 14 \rangle$ $|b - a| = |\langle -12, -5 \rangle| = 13$

$a = 4i - 3j + 2k = \langle 4, -3, 2 \rangle$ $a + b = \langle 6, -3, -2 \rangle$ $|a| = \sqrt{29}$

$b = 2i - 4k = \langle 2, 0, -4 \rangle$ $4a + 2b = \langle 20, -12, 0 \rangle$ $|b - a| = |\langle -2, 3, -6 \rangle| = 7$

$a = \langle 4, 1, \frac{1}{4} \rangle$ $b = \langle 6, -3, -8 \rangle$ $a \cdot b = 24 - 3 - 2 = 19$

$u = \langle 5, 1 \rangle \quad v = \langle 3, 2 \rangle \qquad \cos\theta = \dfrac{u \cdot v}{|u|\,|v|} = \dfrac{15+2}{\sqrt{26}\sqrt{13}} = \dfrac{17}{13\sqrt{2}} \qquad \theta = \arctan\left(\dfrac{17}{13\sqrt{2}}\right)$

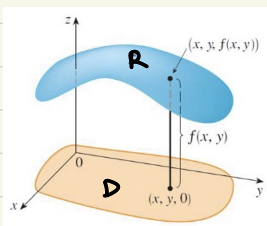
$a = \langle 1, -4, 1 \rangle \quad b = \langle 0, 2, -2 \rangle \quad \cos\theta = \dfrac{a \cdot b}{|a|\,|b|} = \dfrac{-8-2}{\sqrt{18}\sqrt{8}} = \dfrac{-10}{12} = \dfrac{-5}{6} \quad \theta = \arctan\left(-\dfrac{5}{6}\right)$

Functions of several variables

f - function of n variables
$(x_1, \dots, x_n) \in \mathbb{R}^n$
$f(x)$ where $x = \langle x_1, \dots, x_n \rangle$
maximal (natural) D — set of all points x for which $f(x)$ exists
range R — set of all possible values of $f(x)$ $\quad R = \{ f(x) \mid x \in D \}$

$f(x,y) = \dfrac{\ln(2-y)}{1-x} \quad D = \{ (x,y) \in \mathbb{R}^2 : y < 2 \text{ and } x \neq 1 \}$

graph of $f(x,y)$ is set $\mathbb{R}^3$ of $(x, y, f(x,y))$, $(x,y) \in D$

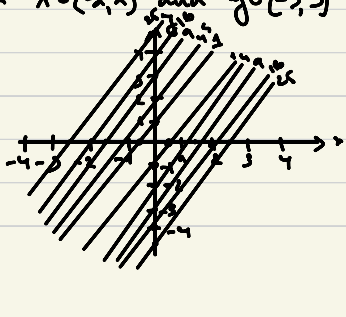


Level curve

level curves of $f(x,y) \in \mathbb{R}^2 \quad f(x,y) = c$
several level curves = contour map
level surfaces of $f(x,y,z) \in \mathbb{R}^3 \quad f(x,y,z) = c$
$f(x,y,z) = x^2 + y^2 + z^2 \rightarrow$ spheres centered at $(0,0,0)$

$f(x,y,z) = \sqrt{4 - x^2} + \sqrt{9 - y^2} + \sqrt{1 - z^2}$
$D = \{ (x,y,z) \in \mathbb{R}^3 : x \in [-2, 2] \text{ and } y \in [-3, 3] \text{ and } z \in [-1, 1] \}$

$f(x,y) = (y - 2x)^2$
$f(x,y) = 1$
$f(x,y) = 4$
$f(x,y) = 9$
$f(x,y) = 16$
$f(x,y) = 25$



$f(x,y,z) = x^2 + 2y^2 + 3z^2$
sphere
elongated along x-axis, $x = \sqrt{3}\, z$
elongated along y-axis, $y = \sqrt{2}\, z$

10/9

Partial derivatives & Linearization [14.2–14.4]

$\lim_{\vec{x}\to\vec{a}} f(x) = L \leftarrow \forall \varepsilon > 0,\ \exists \delta > 0$ s.t. $0 < |x-a| < \delta \longrightarrow |f(x) - L| < \varepsilon$

$$f(x, y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$$

Table 1 Values of $f(x, y)$

$x \backslash y$	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455
−0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
−0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0	0.841	0.990	1.000	1	1.000	0.990	0.841
0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455

$$g(x, y) = \frac{x^2-y^2}{x^2+y^2}$$

Table 2 Values of $g(x, y)$

$x \backslash y$	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000
0.5	0.600	0.000	0.724	1.000	0.724	0.000	0.600
−0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0	−1.000	−1.000	−1.000		−1.000	−1.000	−1.000
0.2	0.923	0.724	0.000	1.000	0.000	0.724	0.923
0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000

$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{\sin(0)}{0} = 1$

$\lim_{(x,y)\to(0,0)} f(x,y) = \text{DNE}$

$$f(x,y) = \frac{xy}{x^2+y^2} \begin{cases} \text{along } x=0 \to 0 \\ \text{along } y=0 \to 0 \\ \text{along } x=y \to \frac{1}{2} \end{cases}$$

$\Rightarrow$ limit DNE

$f(x,y) = \dfrac{x^2y}{x^4+y^2}$ $\quad x=0 \to 0$

$y = 0 \to 0$

$y = kx \to 0$

$y = x^2 \to \frac{1}{2}$

$\lim_{\vec{x}\to\vec{a}} f(x) = f(a) \longrightarrow f$ continuous at $x = a$

Partial derivatives

$$f_x(a,b) = \frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$f_y(a,b) = \frac{\partial f}{\partial y}(x,y) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

$f(x,y) = x^3 + x^2y^3 - 2y^2$

$f_x(x,y) = 3x^2 + 2xy^3$

$f_y(x,y) = 3x^2y^2 - 4y$

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$f(x,y) = x^3 + x^2y^3 - 2y^2$

$f_{xx}(x,y) = 6x + 2y^3$

$f_{yy}(x,y) = 6yx^2 - 4$

$f_{xy}(x,y) = 6xy^2$

$f_{yx}(x,y) = 6xy^2$

$f(x,y) = \sin(xy)$

$f_x(x,y) = y\cos(xy)$

$f_{xx}(x,y) = -y^2\sin(xy)$

$f_y(x,y) = x\cos(x)$

$f_{yy}(x,y) = -x^2\sin(xy)$

$f_{xy}(x,y) = \cos(xy) - xy\sin(xy)$

$f_{yx}(x,y) = \cos(xy) - xy\sin(xy)$

f - defined on disk D, $(a,b) \in D$
f_{xy}, f_{yx} - continuous on $D \to f_{xy}(a,b) = f_{yx}(a,b)$

$f(x,y) = \frac{x}{y}$ $\quad g(x,y) = (2x+3y)^{10}$ $\quad h(x,y) = \frac{y}{2x+3y}$ $\quad h_{yy}(x,y) = \frac{18y}{(2x+3y)^3} - \frac{6}{(2x+3y)^2}$

$f_x(x,y) = \frac{1}{y}$ $\quad g_x(x,y) = 20(2x+3y)^9$ $\quad h_{xy}(x,y) = \frac{6y-4x}{(2x+3y)^3}$

$f_y(x,y) = -\frac{x}{y^2}$ $\quad g_y(x,y) = 30(2x+3y)^9$ $\quad h_{xx}(x,y) = \frac{8y}{(2x+3y)^3}$

Tangent planes

tangent plane at $P \to$ plane through T_1, T_2
T_1, T_2 - tangent line to curves C_1, C_2 through P on $z = f(x,y)$
$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$ — linearization at (a,b)
$x = (x_1, \dots, x_n)$ $\quad a = (a_1, \dots, a_n)$ $\quad L(x) = f(a) + f_{x_1}(a)(x_1 - a_1) + \dots + f_{x_n}(a)(x_n - a_n)$

$f(x,y) = \ln(x^2 - y)$ $\quad (2,1)$ $\quad z = \ln(3) + \frac{4}{3}(x-2) - \frac{1}{3}(y-1)$
$(1.01, 1.94, 2.01)$
$f(x,y,z) = \sqrt{x^2+y^2+z^2}$ $\quad L(x,y,z) = f(1,2,2) + f_x(1,2,2)(x-1) + f_y(1,2,2)(y-2) + f_z(1,2,2)(z-2)$
$f_x(x,y,z) = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{1}{3}$ $\quad L(1.01, 1.94, 2.01) = 3 + \frac{1}{3} \cdot 0.01 - \frac{2}{3} 0.06 + \frac{2}{3} \cdot 0.01 =$
$f_y(x,y,z) = f_z(x,y,z) = \frac{2}{3}$ $\quad = 3 + 0.01 - 0.04 = 2.97$

Differentiability

$\lim_{x \to a} \frac{f(x) - L(x)}{|x-a|} = 0 \to$ differentiable at a
$\exists f_x, f_y$ near (a,b) and continuous $\to$ differentiable at (a,b)

$z = e^{x-y}$ $\quad f_x(z) = e^{x-y}$ $\quad L(x,y) = f(2,2) + f_x(2,2)(x-2) + f_y(2,2)(y-2)$
$(2,2,1)$ $\quad f_y(z) = -e^{x-y}$ $\quad L(x,y) = 1 + (x - \cancel{2}) - (y - \cancel{2}) = x - y + 1$

$z = \frac{2\sqrt{y}}{x}$ $\quad f_y(z) = \frac{1}{x\sqrt{y}}$ $\quad L(x,y) = f(-1,1) + f_x(-1,1)(x+1) + f_y(-1,1)(y-1)$
$(-1,1,-2)$ $\quad f_x(z) = -\frac{2\sqrt{y}}{x^2}$ $\quad L(x,y) = -2 - 2(x+1) - (y-1) = -2x - y - 3$

11/10 **The chain rule and the directional derivative** [14.5, 14.6]

$f(x_1, x_2, \dots, x_n)$ and $\forall x\ (t_1, t_2, \dots, t_m)$

$$\frac{\partial f}{\partial t_i} = \frac{\partial f}{\partial x_1}\cdot\frac{\partial x_1}{\partial t_i} + \dots + \frac{\partial f}{\partial x_n}\cdot\frac{\partial x_n}{\partial t_i}$$

$z = e^x \sin y$ $\quad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} = e^x \sin y \cdot t^2 + e^x \cos y \cdot 2st$

$x = st^2 \quad y = s^2 t$ $\quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = e^x \sin y \cdot 2st + e^x \cos y \cdot s^2$

$f(x,y) = \sin(x)\cos(y)$ $\quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t} = \frac{\cos(x)\cos(y)}{2\sqrt{t}} + \frac{\sin(x)\sin(y)}{t^2}$

$x = \sqrt{t} \quad y = \frac{1}{t}$

$f(x,y) = x^2 + y^2$ $\quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t} = 6x + 2y$

$x = 2s + 3t \quad y = s + t$ $\quad \frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s} = 4x + 2y$ $\qquad s = 4 \quad t = 2 \quad u = 1$

$h(x,y) = x^4 + x^2 y$ $\quad \frac{\partial h}{\partial s} = \frac{\partial h}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial h}{\partial y}\cdot\frac{\partial y}{\partial s} = 4x^3 + 2xy + x^2 tu^2 = 4x^3 + 2xy + 2x^2$

$x = s + 2t - u$ $\quad \frac{\partial h}{\partial t} = \frac{\partial h}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial h}{\partial y}\cdot\frac{\partial y}{\partial t} = 8x^3 + 4xy + 2xsu^2 = 8x^3 + 4xy + 8x$

$y = stu^2$ $\quad \frac{\partial h}{\partial u} = \frac{\partial h}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial h}{\partial y}\cdot\frac{\partial y}{\partial u} = -4x^3 - 2xy + 2x^2 stu = -4x^3 - 2xy + 16x^2$

Directional derivatives

$f(x,y) = 10 - 2x + 7y$ $\quad \frac{\partial f}{\partial x} = -2$ $\quad D_u f(\underline{a}) = \frac{\partial f}{\partial u}(\underline{a}) = \lim_{h\to 0} \frac{f(\underline{a} + h\underline{u}) - f(\underline{a})}{h}$

$\langle \frac{4}{5}, \frac{3}{5} \rangle, \ (2,1)$ $\quad \frac{\partial f}{\partial y} = 7$ $\quad |\underline{u}| = \sqrt{(\frac{3}{5})^2 + (\frac{4}{5})^2} = 1$ $\quad \underline{a} + h\underline{u} = (2 + \frac{4}{5}h,\ 1 + \frac{3}{5}h)$

$D_u f(\underline{a}) = \lim_{h\to 0} \frac{\cancel{10} - 2(\cancel{2} + \frac{4}{5}h) + 7(\cancel{1} + \frac{3}{5}h) - \cancel{10} + \cancel{4} - \cancel{7}}{h} = \frac{13}{5}$ $\quad \frac{\partial f}{\partial u}(\underline{a}) = \frac{4}{5}(-2) + \frac{3}{5}(7)$

$f, \ (a,b), \ u = \langle u_1, u_2 \rangle$ $\qquad \hat{u} = \frac{u}{|u|}$

$$D_{\vec{u}} f(a,b) = \lim_{h\to 0} \frac{f(a + u_1 h, b + u_2 h) - f(a,b)}{h} = f_x(a,b)\,u_1 + f_y(a,b)\,u_2$$

$$D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \frac{u}{|u|} \qquad \nabla f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle$$

$$\nabla f(x) = \langle f_{x_1}(x), f_{x_2}(x), \dots, f_{x_n}(x) \rangle$$

$f(x,y) = 4 - x^2 - 2y^2 \quad (1,1) \quad u = \langle 3, 1 \rangle$ $\quad D_{\vec{u}} f(1,1) = \frac{-2x \cdot 3 - 4y \cdot 1}{|u|} = \frac{-10}{\sqrt{10}} = -\sqrt{10}$

$\sin(xy^2)$ $\quad \nabla f(\frac{\pi}{3}, -1) = \langle y^2 \cos(xy^2), 2xy\cos(xy^2) \rangle = \langle \frac{1}{2}, -\frac{\pi}{3} \rangle$

$f(x,y) = ye^{xy}$ $\quad \nabla f(0,2) = \langle y^2 e^{xy}, e^{xy} + xye^{xy} \rangle = \langle 4, 1 \rangle$ $\qquad \langle \frac{4}{\sqrt{17}}, \frac{1}{\sqrt{17}} \rangle$

$D_u f(x,y) = \nabla f(x,y) \cdot u = |\nabla f(x,y)|\,|u| \cos\theta = |\nabla f(x,y)| \cos\theta$

$f(x,y) = xy^3 - x^2 \quad (1,2) \quad \theta = \frac{\pi}{3} \quad 8-2=6 \quad 12 \qquad \sqrt{180} = 6\sqrt{5}$

$D_u f(x,y) = |\nabla f(x,y)| \cdot \cos\theta = |\langle y^3 - 2x, 3xy^2 \rangle| \frac{1}{2} = 3\sqrt{5}$

$g(x,y) = \frac{x}{y} \quad (3,0) \quad u = \langle \frac{3}{5}, \frac{4}{5} \rangle \quad |u| = 1$

$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle = \langle \frac{1}{y}, -\frac{x}{y^2} \rangle = \langle \infty, \infty \rangle \quad D_u f(x,y) = \infty$

$h(x,y) = e^x \sin(y) \quad (0, \frac{\pi}{3}) \quad u = \langle -6, 8 \rangle \quad |u| = 10$

$D_u f(x,y) = \nabla f(x,y) \cdot u = \langle e^x \sin(y), e^x \cos(y) \rangle \cdot u = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle \cdot \langle -6, 8 \rangle = \dfrac{\frac{\sqrt{3}}{2} \cdot -6 + \frac{8}{2}}{10} = \dfrac{4 - 3\sqrt{3}}{10}$

$f(x,y) = 5xy^2 \quad (3,-2) \quad \langle 1, -3 \rangle \quad \cos\theta = \frac{1+9}{\sqrt{10}} = \sqrt{10}$

$\nabla f(x,y) = \langle 5y^2, 10xy \rangle = \langle 20, -60 \rangle$

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Minimum and Maximum values [14.7]

$c \in D(f)$

$f(c)$ $\begin{cases} \text{absolute maximum value of } f \text{ on } D \text{ if } f(c) \geq f(x),\ \forall x \in D \\ \text{absolute minimum value of } f \text{ on } D \text{ if } f(c) \leq f(x),\ \forall x \in D \\ \text{local maximum value of } f \text{ if } f(c) \geq f(x),\ x \text{ near } c \\ \text{local minimum value of } f \text{ if } f(c) \leq f(x),\ x \text{ near } c \end{cases}$

$f(x)$ – continuous on closed, bounded $D \subset \mathbb{R}^n$

↳ attains both absolute extrema on that domain

region $D \subset \mathbb{R}^n$, $\exists M > 0$ s.t. $|x| < M$, $\forall x \in D$ → bounded

region $D \subset \mathbb{R}^n$, $\exists M > 0$ s.t. $|x| \leq M$, $\forall x \in D$ → closed

f – (local) extremum at c and $\exists f_{x_i}(c) \rightarrow f_{x_i}(c) = 0$, $\forall x_i$

Critical points

$f(x,y) = 12xy - x^3 - y^3$ $(0,0)$ $(4,4)$

$\frac{\partial}{\partial x} f(x,y) = 12y - 3x^2 = 0$ $y = \frac{x^2}{4}$

$\frac{\partial}{\partial y} f(x,y) = 12x - 3y^2 = 0$ $x = \frac{y^2}{4}$

$f(x,y) = 10 - x^2 - 3y^2$

$\left.\begin{array}{l} \frac{\partial}{\partial x} f(x,y) = -2x \\ \frac{\partial}{\partial y} f(x,y) = -6y \end{array}\right\}$ $(0,0)$ → local maximum

$f(x,y) = 10 - x^2 + 3y^2$

$\left.\begin{array}{l} \frac{\partial}{\partial x} f(x,y) = -2x \\ \frac{\partial}{\partial y} f(x,y) = 6y \end{array}\right\}$ $(0,0)$ → not extremum

point (a,b) – critical if $f_x(a,b) / f_y(a,b) = 0 / DNE$

Second Derivative Test

$D = f_{xx}(a,b)\, f_{yy}(a,b) - f_{xy}(a,b)^2$

$D<0 \to$ saddle point

$D>0 \begin{cases} f_{xx}(a,b)>0 \to \text{local min} \\ f_{xx}(a,b)<0 \to \text{local max.} \end{cases}$

$f(x,y) = 2x^2+2xy+2y^2-6x$

$\left.\begin{array}{l} f_{xx}(x,y)=4 \\ f_{yy}(x,y)=4 \\ f_{xy}(x,y)=2 \end{array}\right\} D=12 \Rightarrow (2,-1) \to$ local minimum

$f(x,y)=2x^2-8xy+y^4-4y^3$

$f_x(x,y)=4x-8y \qquad x=2y \qquad$ critical points: $(0,0)$, $(8,4)$, $(-2,-1)$

$f_y(x,y)=-8x+4y^3-12y^2$

$4y^3-12y^2-16y=4y(y^2-3y-4)=4y(y-4)(y+1)$

$\left.\begin{array}{l} f_{xx}(x,y)=4 \\ f_{yy}(x,y)=12y^2-24y \\ f_{xy}(x,y)=-8 \end{array}\right\} D=48(y^2-2y)-64 \begin{cases} D=-64 \Rightarrow (0,0) \to \text{saddle} \\ D=80 \Rightarrow (-2,-1) \to \text{local min.} \\ D=320 \Rightarrow (8,4) \to \text{local min.} \end{cases}$

$f(x,y)=e^x\cos(y)$

$\left.\begin{array}{l} f_x(x,y)=e^x\cos(y) \\ f_y(x,y)=-e^x\sin(y) \end{array}\right\} \neq 0$

$f(x,y)=x^2+y^2-2x$

$D:\ x\in[0,2],\ y\in[-2,2]$

$f_x(x,y)=2x-2$

$f_y(x,y)=2y$

$(1,0) \to$ absolute min

$\left.\begin{array}{l} (0,2) \\ (0,-2) \end{array}\right\}$ absolute max

$(2,0,-3)$ to $x+y+z=1$

$f=(x-2)^2+(1-x-z)^2+(z+3)^2$

$f=x^2-4x+4+x^2+z^2+1-2x-2z+2xz+z^2+6z+9$

$f=2x^2-6x+2z^2+4z+2xz+14$

$f_x = 4x-6+2z \qquad 4x-6+2z=4z+4+2x$

$f_z = 4z+4+2x \qquad 2x=10+2z \quad x=5+z$

$\left.\begin{array}{l} f_{xx}=4 \\ f_{zz}=4 \\ f_{xz}=2 \end{array}\right\} D=12 \qquad 4z+4+10+2z=0 \quad z=-\frac{7}{3}$

$x=\frac{8}{3} \quad y=\frac{2}{3}$

$f=\sqrt{(x-2)^2+y^2+(z+3)^2}=\sqrt{\left(\frac{8}{3}-2\right)^2+\left(\frac{2}{3}\right)^2+\left(-\frac{7}{3}+3\right)^2}$

$=\sqrt{\left(\frac{2}{3}\right)^2\cdot 3}=\frac{2}{\sqrt{3}}$

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Complex numbers

$i^2 = -1$

Def. complex number $z = a + bi$, $a, b \in \mathbb{R}$

real part: $\mathrm{Re}(z) = a$ $\qquad |z| = \sqrt{a^2+b^2}$

imaginary part: $\mathrm{Im}(z) = b$

$z = a+bi$, $w = c+di$ $\Rightarrow z+w,\ z-w,\ zw,\ \frac{z}{w}$ ($w \neq 0$), $\bar{z} = a - bi$ → complex numbers

$\bar{z}$ → complex conjugate

$$(1+i)+(3-2i) = 4-i$$
$$(2-i)(1+3i) = 2-i+6i-3i^2 = 5+5i$$
$$\frac{1-2i}{4+i} = \frac{(1-2i)(4-i)}{17} = \frac{2-9i}{17}$$

z - complex number

$|z| = \sqrt{z\bar{z}}$ → real number

$\mathrm{Re}(z) = \frac{z+\bar{z}}{2}$

$\mathrm{Im}(z) = \frac{z-\bar{z}}{2i}$

$\bar{z} = z$ → only for real numbers

z, w - complex numbers

$\overline{z+w} = \bar{z} + \bar{w}$

$\overline{z-w} = \bar{z} - \bar{w}$

$\overline{zw} = \bar{z}\,\bar{w}$

$\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$

$\bar{\bar{z}} = z$

Quadratic equations

$z^2+3=0 \qquad z^2=-3 \qquad z = 3i$

$z^2 - 6z + 13 = 0$

$D = b^2 - 4ac = 36 - 52 = -16$

$$z_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{6 \pm 4i}{2} = 3 \pm 2i$$

$z^2+z+1=0$

$D=b^2-4ac=1-4=-3$

$z_{1,2}=\frac{-b\pm\sqrt{D}}{2a}=\frac{-1\pm\sqrt{3}i}{2}$

$x^2+2x+5=0$

$D=b^2-4ac=-16$

$x_{1,2}=\frac{-b\pm\sqrt{D}}{2a}=\frac{-2\pm 4i}{2}=-1\pm 2i$

$(2+5i)(4-i)=13+18i$

$\frac{1}{1+i}=\frac{1-i}{2}=\frac{1}{2}-\frac{1}{2}i$

$\overline{12+7i}=12-7i$

$\sqrt{-3}\sqrt{-12}=6$

$\overline{12-5i}=12+5i$ $\quad |12-5i|=13$

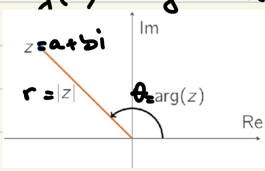
$z^2+z+2=0$

$D=b^2-4ac=-7$

$z_{1,2}=\frac{-b\pm\sqrt{D}}{2a}=\frac{-1\pm\sqrt{7}i}{2}$

Polar form

$\arg(z)$ – argument of z



$|vw|=|v||w|$

$\arg(vw)=\arg(v)+\arg(w)$

$r=\sqrt{a^2+b^2}$ $\quad \tan(\theta)=\frac{b}{a}$

$a=r\cos(\theta)$ $\quad b=r\sin(\theta)$

polar form $z=r(\cos(\theta)+i\sin(\theta))=re^{i\theta}$

$z=r(\cos(\theta)+i\sin(\theta))$

$w=s(\cos(\phi)+i\sin(\phi))$

$zw=rs(\cos(\theta+\phi)+i\sin(\theta+\phi))$

$z=-1+i$ $\quad |z|=\sqrt{2}$ $\quad \arg(z)=\frac{3\pi}{4}$

$(1+i)^{20}=((1+i)^2)^{10}=(\cancel{1}+\cancel{i^2}+2i)^{10}=-1024$

$z=\frac{1}{2}(\cos(\frac{5}{12}\pi)+i\sin(\frac{5}{12}\pi))$

$w=3(\cos(\frac{13}{12}\pi)+i\sin(\frac{13}{12}\pi))$

$zw=\frac{3}{2}(\underbrace{\cos(\frac{3}{2}\pi)}_{0}+i\underbrace{\sin(\frac{3}{2}\pi)}_{-1})=-\frac{3}{2}i$

$z=1+i$ $\quad z^{10}=(z^2)^5=(\cancel{1}+2i+\cancel{i^2})^5=32i$

$1-i\sqrt{3}$ $\quad |z|=2$ $\quad \tan(\theta)=-\sqrt{3}$ $\quad \theta=\frac{5\pi}{3}$ $\quad \Rightarrow z=2(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3})$

$z=\sqrt{3}+i$ $\quad |z|=2$ $\quad \tan(\theta)=\frac{1}{\sqrt{3}}$ $\quad \theta=\frac{\pi}{6}$ $\quad \Rightarrow z=2(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})$

$w=1+\sqrt{3}i$ $\quad |w|=2$ $\quad \tan(\theta)=\sqrt{3}$ $\quad \theta=\frac{\pi}{3}$ $\quad \Rightarrow w=2(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})$

$zw=4(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})$

$1=1+0i=1(\cos 0+i\sin 0)$

$\frac{2}{w}=\cos-\frac{\pi}{6}+i\sin-\frac{\pi}{6}$

$\frac{1}{z}=\frac{1}{2}(\cos-\frac{\pi}{6}+i\sin-\frac{\pi}{6})$

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Complex numbers

$$(\sqrt{3}+i)^{-9} = 2^{-9}\left(\overset{0}{\cancel{\cos\frac{\pi}{2}}} + i\,\overset{1}{\cancel{\sin\frac{\pi}{2}}}\right) = \frac{i}{512}$$

Complex exponentials

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$w = s(\cos\phi + i\sin\phi) = se^{i\phi}$$

$$zw = rs(\cos(\theta+\phi) + i\sin(\theta+\phi)) = rs\,e^{i(\theta+\phi)}$$

$$\frac{z}{w} = \frac{r}{s}(\cos(\theta-\phi) + i\sin(\theta-\phi)) = \frac{r}{s}e^{i(\theta-\phi)}, \quad s \neq 0$$

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)) = r^n e^{in\theta}$$

$$e^{it} = \cos(t) + i\sin(t)$$

$$e^{a+bi} = e^a \cdot e^{bi}$$

Roots

$z^3 = 27i$

$|z^3| = |27i| \qquad \arg(z^3) = \arg(27i) + 2k\pi$

$|z|^3 = 27 \qquad 3\arg(z) = \frac{\pi}{2} + 2k\pi$

$|z| = 3 \qquad \arg(z) = \frac{\pi}{6} + \frac{2}{3}k\pi$

$k=0$: $\arg(z) = \frac{\pi}{6} \qquad z_1 = 3e^{\frac{\pi}{6}i} = 3\left(\overset{\sqrt{3}/2}{\cancel{\cos\left(\frac{\pi}{6}\right)}} + i\,\overset{\frac{1}{2}}{\cancel{\sin\left(\frac{\pi}{6}\right)}}\right) = \frac{3\sqrt{3}}{2} + \frac{3}{2}i$

$k=1$: $\arg(z) = \frac{5\pi}{6} \qquad z_2 = 3e^{\frac{5\pi}{6}i} = 3\left(\overset{-\frac{\sqrt{3}}{2}}{\cancel{\cos\left(\frac{5\pi}{6}\right)}} + i\,\overset{\frac{1}{2}}{\cancel{\sin\left(\frac{5\pi}{6}\right)}}\right) = -\frac{3\sqrt{3}}{2} + \frac{3}{2}i$

$k=-1$: $\arg(z) = -\frac{\pi}{2} = \frac{3\pi}{2}\ (k=2) \qquad z_3 = 3e^{\frac{3\pi}{2}i} = 3\left(\overset{0}{\cancel{\cos\left(\frac{3\pi}{2}\right)}} + i\,\overset{-1}{\cancel{\sin\left(\frac{3\pi}{2}\right)}}\right) = -3i$

$z^4 = -16$

$|z|^4 e^{4i\varphi} = 16e^{i\pi}$

$|z| = 2 \qquad e^{4i\varphi} = e^{i\pi}$

$4\varphi = \pi + 2k\pi \qquad \varphi = \frac{\pi}{4} + \frac{k\pi}{2}$

$$\begin{cases} \varphi = \frac{\pi}{4} \\ \varphi = \frac{3\pi}{4} \\ \varphi = \frac{5\pi}{4} \\ \varphi = \frac{7\pi}{4} \end{cases}$$

$z = \pm\sqrt{2} \pm \sqrt{2}i$

Fundamental Theorem of Algebra

$p(z)$ - polynomial, degree $n \geq 1 \Rightarrow \exists a \in \mathbb{C}$ s.t. $p(a) = 0$

$p(z) = c(z-a_1)(z-a_2)\dots(z-a_n) \qquad a_1, \dots, a_n$ - roots of $p(z)$, $c \in \mathbb{C}$

$p(z)$ - polynomial with real coefficients

$p(a) = 0 \rightarrow p(\bar{a}) = 0$

$$z^6 = -8$$

$$|z|^6 e^{6i\varphi} = 8e^{i\pi}$$

$$|z| = \sqrt{2} \qquad e^{6i\varphi} = e^{i\pi}$$

$$6\cancel{i}\varphi = \cancel{i}\pi + 2k\pi \qquad \varphi = \frac{\pi}{6} + \frac{k\pi}{3}$$

$$\begin{cases} \varphi = \frac{\pi}{6} & z = \sqrt{2}e^{\frac{\pi}{6}i} = \sqrt{2}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \frac{\sqrt{3}+i}{\sqrt{2}} \\ \varphi = \frac{\pi}{2} & z = \sqrt{2}e^{\frac{\pi}{2}i} = \sqrt{2}(0 + 1i) = \sqrt{2}i \\ \varphi = \frac{5\pi}{6} & z = \sqrt{2}e^{\frac{5\pi}{6}i} = \sqrt{2}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \frac{i-\sqrt{3}}{\sqrt{2}} \\ \varphi = \frac{7\pi}{6} & z = \sqrt{2}e^{\frac{7\pi}{6}i} = \sqrt{2}\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \frac{-\sqrt{3}-i}{\sqrt{2}} \\ \varphi = \frac{3\pi}{2} & z = \sqrt{2}e^{\frac{3\pi}{2}i} = \sqrt{2}(0 - 1i) = -\sqrt{2}i \\ \varphi = \frac{11\pi}{6} & z = \sqrt{2}e^{\frac{11\pi}{6}i} = \sqrt{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \frac{\sqrt{3}-i}{\sqrt{2}} \end{cases}$$

$$z^3 + 6z + 20 = 0$$

$$z = -2 \qquad z = 1 - 3i$$

$$(z+2)(z-1+3i)(z-x) = z^3 + 6z + 20$$

$$\cancel{z^3} - xz^2 - z^2 + zx + 3iz^2 - 3izx + 2z^2 - 2zx - 2z + 2x + 6iz - 6ix = \cancel{x^3} + 6z + 20$$

$$z^2(-x - 1 + 3i + 2) + z(x - 3ix - 2x - 2 + 6i) + z^0(2x - 6ix) = 6z + 20$$

$$\left|\begin{array}{l} -x + 3i + 1 = 0 \\ -x - 3ix + 6i - 2 = 6 \\ 2x - 6ix = 20 \end{array}\right. \qquad \left|\begin{array}{l} x = 3i + 1 \\ -\cancel{3i} - 1 - \cancel{3i} - 9i^2 + \cancel{6i} - 2 = 6 \checkmark \\ \cancel{6i} + 2 - 18i^2 - \cancel{6i} = 20 \checkmark \end{array}\right.$$

$$z^8 = 1$$

$$|z|^8 e^{i8\varphi} = e^0$$

$$|z| = 1 \qquad 8\varphi = 2k\pi \qquad \varphi = \frac{k\pi}{4}$$

$$\begin{cases} \varphi = \frac{\pi}{4} & z = e^{i\frac{\pi}{4}} & \varphi = \frac{5\pi}{4} & z = e^{\frac{5\pi}{4}i} \\ \varphi = \frac{\pi}{2} & z = e^{\frac{\pi}{2}i} & \varphi = \frac{3\pi}{2} & z = e^{\frac{3\pi}{2}i} \\ \varphi = \frac{3\pi}{4} & z = e^{\frac{3\pi}{4}i} & \varphi = \frac{7\pi}{4} & z = e^{\frac{7\pi}{4}i} \\ \varphi = \pi & z = e^{\pi i} & \varphi = 2\pi & z = e^{2\pi i} = e^0 \end{cases}$$

$$e^{\frac{\pi}{2}i} = \cancel{\cos\frac{\pi}{2}}^{\,0} + i\cancel{\sin\frac{\pi}{2}}^{\,1} = i$$

$$e^{2+\pi i} = e^2(\cancel{\cos\pi}^{\,-1} + i\cancel{\sin\pi}^{\,0}) = -e^2$$

$$\left.\begin{array}{l} e^{xi} = \cos x + i\sin x \\ \overline{e^{xi}} = e^{-xi} = \cos x - i\sin x \end{array}\right\} \cos x = \frac{e^{xi} + e^{-xi}}{2} \qquad \sin(x) = \frac{e^{xi} - e^{-xi}}{2i}$$

23/01

Double integrals over rectangles [15.1]

Riemann sum $= \sum_{i=1}^{N} f(x_i, y_i)\Delta A_i$

$\iint_D f(x,y)\,dA = \lim_{\Delta A_i \to 0^+} \sum_i f(x_i,y_i)\Delta A_i \to$ volume

$\iint_D f(x,y) + g(x,y)\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$

$\iint_D c f(x,y)\,dA = c\iint_D f(x,y)\,dA$

$f \geq g \to \iint_D f(x,y)\,dA \geq \iint_D g(x,y)\,dA$

$\iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA = \iint_{D_1 \cup D_2} f(x,y)\,dA$

$\text{Area}(D) = \iint_D 1\,dA$

$R=[0,2]\times[0,3]$ $\quad \iint_R (4-y)\,dA = \int_0^2\int_0^3 4-y\,dy\,dx = \int_0^2 4y-\frac{y^2}{2}\Big|_0^3 dx = \int_0^2 \frac{15}{2}\,dx = \frac{15}{2}x\Big|_0^2 = 15$

$R=[0,2]\times[0,2]$ $\quad \iint_R (f(x,y)+2)\,dx\,dy = \iint_R f(x,y)\,dx\,dy + \iint_R 2\,dx\,dy = 4+2^3 = 12$

$R=[a,b]\times[c,d]$

$\iint_R f(x,y)\,dA = \int_a^b\left(\int_c^d f(x,y)\,dy\right)dx = \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$

$\iint_R (18-x^2-2y^2)\,dA = \int_0^3\int_0^2 18-x^2-2y^2\,dy\,dx = \int_0^3 18y-x^2y-\frac{2y^3}{3}\Big|_0^2 dx = \int_0^3 36-2x^2-\frac{16}{3}\,dx =$

$R=[0,3]\times[0,2]$ $\quad = 36x-\frac{2x^3}{3}-\frac{16}{3}x\Big|_0^3 = 98-18-16=64$

$f(x,y)=g(x)h(y) \quad R=[a,b]\times[c,d]$

$\iint_R f(x,y)\,dA = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right)$

$\int_1^3\int_0^2 ye^{xy}\,dx\,dy = \int_1^3 e^{xy}\Big|_0^2 dy = \int_1^3 e^{2y}-1\,dy$

$\iint_R y\sin(xy)\,dA = \int_0^{\frac{\pi}{2}}\int_0^2 y\sin(xy)\,dx\,dy$

$R=[0,2]\times[0,\frac{\pi}{2}]$

$\iint_R \frac{x^3}{1+y^4}\,dA = \int_0^2\int_{-2}^2 \frac{x^3}{1+y^4}\,dx\,dy$

$R=[-2,2]\times[0,2]$

$\iint_R \sqrt{2}\,dA = \sqrt{2}\times4\times6 = 24\sqrt{2}$

$R=[2,6]\times[-1,5]$

$\int_1^4\int_0^2 (6x^2y-2x)\,dy\,dx =$

$= \int_1^4 3x^2y^2-2xy\Big|_0^2 dx =$

$= \int_1^4 12x^2-4x\,dx = 4x^3-2x^2\Big|_1^4 =$

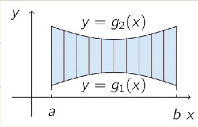
$= 256-32-4+2 = 222$

24/01

Double integrals over general regions [15.2]

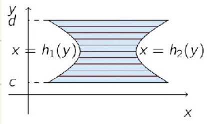
$$\left.\begin{matrix} y=0 \\ y=\sqrt{4-x^2} \end{matrix}\right\} \quad \left.\begin{matrix} 4-x^2=0 \\ x=\pm 2 \end{matrix}\right\} \Rightarrow \int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} y \, dy \, dx$$

Types of regions



Type I: $D=\{(x,y) \mid a\le x\le b,\ g_1(x)\le y\le g_2(x)\}$

$$\iint_D f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$



Type II: $D=\{(x,y) \mid c\le y\le d,\ h_1(y)\le x\le h_2(y)\}$

$$\iint_D f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$

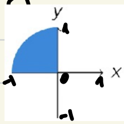
R - triangle with vertices $(0,0)$, $(0,2)$, $(2,2)$

$$\iint_R f(x,y)\, dA = \int_0^2 \int_x^2 f(x,y)\, dy\, dx$$

R - region between $y=2$ and $y=6-x^2$

$6-x^2=2 \quad x=\pm 2$

$$\iint_R f(x,y)\, dA = \int_{-2}^{2}\int_{2}^{6-x^2} f(x,y)\, dy\, dx$$

$$\int_{-1}^{0}\int_{0}^{\sqrt{1-x^2}} f(x,y)\, dy\, dx \rightarrow D =$$ 

$$E: \begin{cases} x\ge 0,\ y\ge 0,\ z\ge 0 \\ 3x+3y+z\le 3 \end{cases}$$

$$\mathrm{Vol}(E) = \iint_D (3-3x-3y)\, dA \qquad D=\{(x,y) \mid 0\le x\le 1,\ 0\le y\le 1-x\}$$

$$\int_0^1\int_x^1 \cos(y^2)\, dy\, dx = \int_0^1\int_0^y \cos(y^2)\, dx\, dy = \int_0^1 y\cos(y^2)\, dy = \frac{1}{2}\sin(y^2)\Big|_0^1 = \frac{1}{2}(\sin 1 - 1)$$

$D=\{(x,y) \mid 0\le x\le 1,\ x\le y\le 1\} = \{(x,y) \mid 0\le y\le 1,\ 0\le x\le y$

$$\int_1^5 \int_0^x 8x-2y \, dy \, dx = \int_1^5 8xy - y^2 \Big|_0^x dx = \int_1^5 7x^2 dx = \frac{7x^3}{3} \Big|_1^5 = \frac{7}{3} \cdot 124 = \frac{868}{3}$$

$$\int_0^1 \int_0^{s^2} \cos(s^3) \, dt \, ds = \int_0^1 s^2 \cos(s^3) \, ds = \frac{\sin(s^3)}{3} \Big|_0^1 = \frac{\sin(1)}{3}$$

$$\iint_D \frac{y}{x^2+1} \, dA = \int_0^4 \int_0^{\sqrt{x}} \frac{y}{x^2+1} \, dy \, dx = \int_0^4 \frac{y^2}{2x^2+2} \Big|_0^{\sqrt{x}} dx = \int_0^4 \frac{x}{2x^2+2} \, dx = \frac{\ln(2x^2+2)}{4} \Big|_0^4 = \frac{\ln(17)}{2}$$

$D = \{ (x,y) \mid 0 \leq x \leq 4, \ 0 \leq y \leq \sqrt{x} \}$

$D: \ y = x-2$

$x = y^2$

$$\iint_D y \, dA = \int_{-1}^{2} \int_{y^2}^{y+2} y \, dx \, dy = \int_1^4 \int_{x-2}^{\sqrt{x}} y \, dy \, dx$$

$\begin{cases} x = y^2 \\ x = y+2 \end{cases}$ $\begin{cases} y+2 = y^2 \\ y^2 - y - 2 = 0 \end{cases}$ $\begin{cases} (y-2)(y+1) = 0 \\ y = -1 \quad y = 2 \end{cases}$ $D = \{ (x,y) \mid -1 \leq y \leq 2, \ y^2 \leq x \leq y+2 \}$

$\begin{cases} y = x-2 \\ y = \sqrt{x} \end{cases}$ $\begin{cases} \sqrt{x} = x-2 \ \mid^2 \\ x^2 - 5x + 4 = 0 \end{cases}$ $\begin{cases} (x-4)(x-1) = 0 \\ x = 1 \quad x = 4 \end{cases}$ $D: \{ (x,y) \mid 1 \leq x \leq 4, \ x-2 \leq y \leq \sqrt{x} \}$

$$\int_{-1}^2 \int_{y^2}^{y+2} y \, dx \, dy = \int_{-1}^2 y(y+2-y^2) \, dy = \int_{-1}^2 -y^3 + y^2 + 2y \, dy = -\frac{y^4}{4} + \frac{y^3}{3} + y^2 \Big|_{-1}^2 =$$

$$= \cancel{-4} + \frac{8}{3} \cancel{+4} + \frac{1}{4} + \frac{1}{3} - 1 = 2\frac{1}{4}$$

25/01

Review

Taylor series

$f(x)=e^{2x^3}$ $\quad f^{(9)}(0)=?$ $\qquad e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots+\frac{x^n}{n!}=\sum_{n=0}^{\infty}\frac{x^n}{n!}$

$f^{(9)}(0) = \frac{4}{3}\cdot\frac{9!}{x^9}=\frac{4}{3}9!$ $\qquad f(x)=e^{2x^3}=1+2x^3+\frac{4x^6}{2!}+\frac{8x^9}{3!}+\dots+\frac{2^n x^{3n}}{n!}$

$\int x^3e^{-3x^3}dx = \int x^3-3x^6+\frac{9x^9}{2!}\,dx = \frac{x^4}{4}-\frac{3x^7}{7}+\frac{9x^{10}}{20}$

$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots+\frac{x^n}{n!}=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ $\qquad e^{-3x^3}=1-3x^3+\frac{9x^6}{2!}-\frac{27x^9}{3!}+\dots$

Directional derivatives

$f(x,y,z)=xyz-x^2-y^2-yz$ $\quad \langle yz-2x, xz-2y-z, xy-y\rangle$ $\qquad \langle 1,1,-1\rangle$

$D_uf = \frac{\langle f_x,f_y,f_z\rangle\cdot\langle 1,1,-1\rangle}{|u|} = \frac{-1-2-1-\cancel{1}+\cancel{1}+\cancel{1}-1}{\sqrt{3}} = -\frac{5}{\sqrt{3}}$

$f(x,y)=\frac{x^2+y^2}{xy}$ $\quad (1,2)$ $\qquad \left\langle \frac{2x^2y-y(x^2+y^2)}{x^2y^2}, \frac{2xy^2-x(x^2+y^2)}{x^2y^2}\right\rangle = \left\langle -\frac{3}{2},\frac{3}{4}\right\rangle$

$f(x,y)=x^3-xy-3y^2$ $\quad (1,1)$ $\qquad \langle 3x^2-y, -7y\rangle = \langle 2,-7\rangle$

perpendicular: $\langle 7,2\rangle \Rightarrow y=\frac{7}{2}x-\frac{5}{2}$

Power Series, closed form

$\sum_{n=0}^{\infty}\frac{5}{3}\cdot\frac{(x-3)^{3n+2}}{2^{3n}} = \frac{5}{3}(x-3)^2\sum_{n=0}^{\infty}\left(\frac{x-3}{2}\right)^{3n} = \frac{5}{3}\frac{(x-3)^2}{1-\left(\frac{x-3}{2}\right)^3}$

$\sum_{n=0}^{\infty}\frac{5(-1)^{n+1}}{2^{4n+2}(2n+1)!}(x-3)^{6n+3} = -5\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1!}\left(\frac{(x-3)^3}{4}\right)^{2n+1} = -5\sin\left(\left(\frac{x-3}{4}\right)^3\right)$

Implicit differentiation

$(-2,2)$ $\quad 2x^3+y^3=2xy$ $\qquad 6x^2+3y^2\frac{dy}{dx}=2y+2x\frac{dy}{dx}$ $\qquad \frac{dy}{dx}=\frac{6x^2-2y}{2x-3y^2}$

$\frac{dy}{dx}=-\frac{5}{4}$ $\quad y=-\frac{5}{4}x-\frac{1}{2}$ $\qquad 6x^2-2y=0$ $\quad 2(3x^2-y)=0$

$y=3x^2$ $\quad 2x^3+27x^6=6x^3$ $\quad x^3=\frac{4}{27}$ $\quad x=\frac{\sqrt[3]{4}}{3}$ $\quad y=\frac{\sqrt[3]{16}}{3}$ $\qquad (\arctan(x))'=\frac{1}{x^2+1}$

Convergence of series

$$1-\sqrt{\frac{\pi}{2}}+\sqrt{\frac{\pi^2}{4}}-\sqrt{\frac{\pi^3}{8}}+\ldots=\sum_{n=0}^{\infty}\left(-\frac{\pi}{2}\right)^{\frac{n}{2}} \qquad r=-\sqrt{\frac{\pi}{2}}>1 \Rightarrow \text{divergent}$$

$$\sum_{n=1}^{\infty}\frac{n^2\cos^2(\pi n)}{n^3-2} \qquad \text{comparison: } b_n=\sum\frac{1}{n} \rightarrow \text{divergent}$$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\sum\frac{n^3}{n^3-2}=\underline{1} \Rightarrow \text{divergent}$$

$$\sum_{n=1}^{\infty}\frac{-6^n}{3^n+4^{2n}} \qquad \lim_{n\to\infty}\frac{3^n+4^{2n}}{6^n}=\frac{1}{2^n}+\left(\frac{8}{3}\right)^n=\infty \Rightarrow \lim_{n\to\infty}|a_n|=0$$

$$\left(\frac{6^n}{3^n+4^{2n}}\right)'=\frac{\ln 6.6^n(3^n+16^n)-6^n(\ln 3.3^n+\ln 16.16^n)}{(3^n+16^n)^2}$$

$0.7 \qquad -0.98$ $\qquad \Rightarrow |a_{n+1}|<|a_n|$

$$6^n\left(\ln 2.3^n+\ln\left(\frac{3}{8}\right)16^n\right)<0 \qquad \Rightarrow \text{convergent}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n^4}{(n+2)(n^3-5)} \qquad \lim_{n\to\infty}\frac{n^4}{n^4+2n^3-5n-10}=\underline{1} \Rightarrow \text{divergent}$$

(Inverse) Functions

$$f(x)=\frac{1}{\sqrt{x}-1} \qquad x\geq 0,\ \neq 1$$

$$x=\frac{1}{\sqrt{y}-1} \qquad x\sqrt{y}-x=\underline{1} \qquad x\sqrt{y}=1+x \qquad \sqrt{y}=\frac{x+1}{x} \qquad y=\left(\frac{x+1}{x}\right)^2 \qquad f^{-1}(x)=\left(\frac{x+1}{x}\right)^2$$

$$f(x)=\frac{1}{x^2-1} \qquad x\neq\underline{\pm 1}$$

$$x=\frac{1}{y^2-1} \qquad xy^2-x=\underline{1} \qquad xy^2=1+x \qquad y^2=\frac{1+x}{x} \qquad y=\sqrt{\frac{1+x}{x}} \qquad f^{-1}(x)=\sqrt{\frac{1+x}{x}}$$

$$\tan\left(\arccos\left(\frac{2}{3}\right)\right)=\frac{\sin(\arccos(\frac{2}{3}))}{\cos(\arccos(\frac{2}{3}))}=\frac{\frac{\sqrt{5}}{3}}{\frac{2}{3}}=\frac{\sqrt{5}}{2}$$

Limits of functions

$$\lim_{x\to 2}\frac{2-\sqrt{x+2}}{x^2-5x+6}=\frac{-\cancel{4-x-2}}{\cancel{(x-2)}(x-3)(2+\sqrt{x+2})}=\frac{1}{4}$$

$$\lim_{x\to 0}\frac{x\cos(2x^2)-x}{x^5}=\frac{\cos(2x^2)}{x^4}=\frac{-4x\sin(2x^2)}{4x^3}=-2\,\frac{\sin(2x^2)}{2x^2}=-2$$

$$\lim_{x\to 1}\frac{\ln(x)+x-1}{x^2-3x+2}=\frac{\frac{1}{x}+1}{2x-3}=-2$$

$$\lim_{x\to\pi}(x-\pi)^2e^{\sin\frac{1}{x-\pi}}=0$$

$$\lim_{x\to 3}(x^2-6x+10)^{\frac{x+2}{x^2-6x+9}}=\underline{1}$$

Asymptotes

$$f(x)=\frac{\sqrt{8x^2+9}-3x}{x-3} \qquad \lim_{x\to\infty}f(x)=\frac{\sqrt{8}-3}{-3} \qquad \lim_{x\to-\infty}=\frac{\sqrt{8}+3}{-3} \qquad \text{vertical: } x=3$$

$$f(x)=\frac{\ln|5x+2|}{\ln\sqrt{3x+2}} \qquad \lim_{x\to\infty}f(x)=\text{indeterminate} \qquad \text{vertical: } x=-0.4\rightarrow+\infty$$

$$x=-\frac{1}{3}\rightarrow-\infty$$

Linearization

$f(x) = \dfrac{\sin\left(\frac{x^2}{\pi}\right)+1}{\cos x}$ $\quad f(a) = -1 \quad f'(x) = \dfrac{\frac{2x}{\pi}\cos\left(\frac{x^2}{\pi}\right)\cos x + \left(\sin\left(\frac{x^2}{\pi}\right)+1\right)\sin x}{\cos^2 x}$

$a = \pi \qquad f'(a) = 2$

$L(x) = -1 + 2(x - \pi)$

$f(x) = \sqrt{x} \quad f'(x) = \frac{1}{2\sqrt{x}} \quad L(x) = 3 + \frac{1}{6}(x-a)$

$a = 9 \quad f(a) = 3 \quad f'(a) = \frac{1}{6} \quad L(9.1) = 3 + \frac{1}{60} = 3.0167$

$f(x) = \ln(x) \quad f'(x) = \frac{1}{x} \quad L(x) = x - 1$

$a = 1 \quad f(a) = 0 \quad f'(a) = 1 \quad L(1.03) = 0.03$

$f(x) = \sqrt{x} \quad f'(x) = \frac{1}{2\sqrt{x}} \quad L(x) = 1.5 + \frac{x - 2.25}{3}$

$a = 2.25 \quad f(a) = 1.5 \quad f'(a) = \frac{1}{3} \quad L(2.22) = 1.5 - 0.01 = 1.49$

Differentials

$$\left.\begin{array}{l} l = 200 \\ r = 5 \\ dr = 0.05 \end{array}\right\} V = \pi r^2 l \qquad dV = 2\pi r l\, dr = 100\pi$$

Substitution

$\int_1^2 \frac{e^x}{e^{2x}+1}\,dx = \int_1^2 \frac{1}{u^2+1}\,du = \arctan u \Big|_1^2 = \arctan(e^2) - \arctan(e) \qquad u = e^x$

$\int_0^{\frac{\pi}{4}} \tan x\, dx = \int_{u(0)}^{\frac{\sqrt{2}}{2}} \frac{u}{1-u^2}\,du \qquad \tan x\, dx = \frac{\sin x}{\cos x}\,dx = \frac{u}{1-u^2}\,du$

$u = \sin x \quad du = \cos x\, dx = \sqrt{1-u^2}\,dx$

Integration by Parts

$\int \frac{\sin(3x)}{x} + \frac{\cos(3x)}{3x^2}\,dx = -\frac{\cos(3x)}{3x} - \int \frac{\cos(3x)}{3x^2}\,dx - \frac{\cos(3x)}{3x} - \int \frac{\sin(3x)}{x}\,dx =$

$dv = \sin(3x) \quad u = x^{-1} \qquad u = \cos(3x) \quad dv = \frac{1}{3x^2} \qquad = -\frac{\cos(3x)}{3x}$

$v = -\frac{\cos(3x)}{3} \quad du = -\frac{1}{x^2} \qquad du = -3\sin(3x) \quad v = -\frac{1}{3x}$

$\int x \sin^2(x)\,dx = \int x\left(\frac{1}{2} - \frac{\cos(2x)}{2}\right)dx = \frac{x^2}{4} - \int \frac{x\cos(2x)}{2}\,dx =$

$= \frac{x^2}{4} + \frac{x\sin(2x)}{2} - \int \frac{\sin(2x)}{2}\,dx = \frac{x^2}{4} - \frac{x\sin(2x)}{4} - \frac{\cos(2x)}{8} \qquad u = x \quad dv = \cos(2x)$

$u = 2x \qquad du = 1 \quad v = \frac{\sin(2x)}{2}$

Integration

$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \sin(x)\cos(x)e^{2\sin x}\,dx = \int_{-2}^{2} \frac{ue^u}{4}\,du = \frac{e^u}{4}\Big|_{-2}^{2} = \frac{e^2}{4} - \frac{1}{4e^2} = \frac{e^4-1}{4e^2}$$

$u = 2\sin x \quad du = 2\cos x$

$$\int_0^{\pi} x^3\sin(x^2)\,dx = \int_0^{\pi^2} \frac{u\sin u}{2}\,du = -\frac{u\cos u}{2} + \int \frac{\cos u}{2}\,du = -\frac{u\cos u}{2} + \frac{\sin u}{2}\Big|_0^{\pi^2} = -\frac{\pi^2\cos\pi^2}{2} + \frac{\sin\pi^2}{2}$$

$u = x^2 \quad du = 2x \qquad dv = \frac{\sin u}{2} \quad v = -\frac{\cos u}{2}$

$$\int_{-3}^{3} \frac{x}{1+x^4}\,dx = 0$$

Improper Integrals

$$\int_0^{\infty} \frac{e^{\frac{1}{x}}}{x^2}\,dx = -e^{\frac{1}{x}}\Big|_0^{\infty} \text{ divergent}$$

$$\int_1^{\infty} e^{-x}\,dx = -e^{-x}\Big|_1^{\infty} = \frac{1}{e}$$

$$\int_e^{\infty} \frac{1}{x(\ln x)^p}\,dx = \frac{\ln^{1-p}(x)}{1-p}\Big|_e^{\infty} = \frac{1}{p-1}, \quad p>1$$

$$\int_1^{\infty} \frac{1}{x(\ln x)^p}\,dx = \frac{\ln^{1-p}(x)}{1-p}\Big|_1^{\infty} \text{ divergent}, \quad p\in\emptyset$$

Sequences

$a_1 = 1 \quad a_{n+1} = \sqrt{7+2a_n} \quad 0 > a_n^2 - 2a_n - 7 \quad a_n = 1\pm2\sqrt{2}$

$a_{n+1} > a_n \quad \sqrt{7+2a_n} > a_n \quad 7+2a_n > a_n^2$ decr. & increasing ($1-2\sqrt{2}$) & decresing ($1+2\sqrt{2}$) $\quad \lim_{n\to\infty} a_n = 1+2\sqrt{2}$

$a_n = n\sin(n\pi) = 0 \quad \lim_{n\to\infty} a_n = 0$

$a_{n+1} = 10 - \frac{21}{a_n} \quad 0 > a_n^2 - 10a_n + 21$ decr. (3) & increasing (7) & decr. $\quad a_n < 0, \; a_{n+1} > 10 > 7$

$a_{n+1} > a_n \quad 10 - \frac{21}{a_n} > a_n \quad 0 > (a_n-3)(a_n-7) \quad \lim_{n\to\infty} a_n = 7$

Sum of a series

$$\sum_{n=1}^{\infty} \frac{3+4^n}{(-5)^n} = \sum_{n=1}^{\infty} 3\cdot\left(-\frac{1}{5}\right)^n + \left(-\frac{4}{5}\right)^n = 3\cdot\frac{1}{1+\frac{1}{5}} + \frac{1}{1+\frac{4}{5}} = \frac{5}{2} + \frac{5}{9} = \frac{55}{18} = 3\frac{1}{18}$$

Power Series, interval of convergence

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^n(n^2+1)}(x+4)^n = \sum_{n=0}^{\infty}\left(-\frac{1}{n^2+1}\right)\left(-\frac{x+4}{2}\right)^n \quad R=2 \quad x\in[-2,-6]$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^n n!}(x-e)^n = \sum_{n=0}^{\infty}\left(-\frac{1}{n!}\right)\left(\frac{e-x}{2}\right)^n \quad R=2 \quad x\in(e-2, e+6]$$

Multivariable functions

$$f(x,y) = \frac{e^{x+y}}{x^2+y^2-25} \qquad x^2+y^2 \neq 25$$
$$D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \neq 25\}$$

Multivariate Linearization

$f(x,y,z) = \frac{x^2-y^2}{2z^2}$ $\quad f_x = 2x = 4 \quad f_y = -2y = -2 \quad f_z = -\frac{1}{z^3} = -1$

$P(2,1,1) \quad f(P) = \frac{3}{2} \qquad L(x,y,z) = \frac{3}{2} + 4(x-2) - 2(y+2) - 1(z+1)$

$f(x,y) = x\sin\left(\frac{x}{y}\right) \quad f_x = \sin\left(\frac{x}{y}\right) - \frac{x}{y}\cos\left(\frac{x}{y}\right) = 1 \quad f_y = \frac{x}{y^2}\cos\left(\frac{x}{y}\right) = 0$

$P(\pi,2) \quad f(P) = \pi \qquad L(x,y) = \pi + 1(x-1)$

Critical points

$f(x,y) = (x^2+y^2)e^{-4x^2-3y^2} \qquad x=0 \ y=0 \to$ local min

$f_x = 2x \cdot e^{-4x^2-3y^2} - 8x(x^2+y^2)e^{-4x^2-3y^2}$

$f_y = 2y \cdot e^{-4x^2-3y^2} - 6y(x^2+y^2) \cdot e^{-4x^2-3y^2}$

Double Integrals

$f(x,y) = \cos(x^2) \quad \int_0^1\int_0^{3x} \cos(x^2)\,dy\,dx = \int_0^1 y\cos(x^2)\Big|_0^{3x} dx = \int_0^1 3x\cos(x^2)\,dx = \frac{3\sin(x^2)}{2}\Big|_0^1 = \frac{3\sin(1)}{2}$

$\left.\begin{array}{l} y = 1-x^2 \\ y = (x+1)^2 \end{array}\right\} \begin{array}{l} \cancel{1}-x^2 = x^2+2x+\cancel{1} \\ 2x^2+2x=0 \end{array} \begin{cases} x=0 \\ x=-1 \end{cases} \quad \int_{-1}^{0}\int_{(x+1)^2}^{1-x^2} \frac{x}{\sqrt{y}}\,dy\,dx$

$\int_{-2}^{2}\int_{\sqrt{1-x^2}}^{\sqrt{2-x^2}} xy\,dy\,dx$

Complex numbers

$\left(\sqrt{2}-i\sqrt{2}\right)^{-5} = \left(2e^{-\frac{\pi}{4}i}\right)^{-5} = \frac{1}{32}e^{\frac{5\pi}{4}i} \qquad \frac{1}{32}\cdot\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{64}$

$z^6 = -8 \qquad |z|^6 e^{6i\varphi} = 8e^{i\pi} \qquad |z| = \sqrt{2} \qquad \varphi = \frac{\pi}{6} + \frac{k\pi}{3}$

$\frac{1+2i}{1-i}\cdot\frac{1+i}{1+i} = \frac{1+3i+2i^2}{1-i^2} = \frac{1-2+3i}{2} = \frac{3}{2}i - \frac{1}{2}$

$\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)(2i) = \sqrt{3}i - 1 = 2e^{-\frac{\pi}{6}i}$

$z^4 - iz^2 + 3 = 1 \qquad D = b^2-4ac = -1-8 = -9$

$z^2 = x \quad x^2 - ix + 2 = 0 \quad x_{1,2} = \frac{-b\pm\sqrt{D}}{2a} = \frac{i\pm 3i}{2} \begin{cases} z^2 = -i \to z = \pm i\sqrt{i} \\ z^2 = 2i \to z = \pm\sqrt{2i} \end{cases}$