19/12

Lecture 13 – Ratio Test & Power Series

11.6: ~~1~~, ~~2~~, ~~13~~, ~~14~~, ~~19~~, ~~21~~, ~~32~~, ~~33~~, ~~34~~, ~~36~~, ~~40~~

11.8: 1, 2, 3, 4, 7, 11, 19, 25, 30, 35

1. a) D b) AC c) inconclusive

2. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ $|a_n| = \frac{1}{\sqrt{n}}$ $p = \frac{1}{2} \to$ divergent $\Rightarrow$ CC

13. $\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$ $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{10^{n+1}\cancel{(n+1)}\cancel{4^{2n+1}}}{\cancel{10^n}\cancel{(n+2)}4^{2n+3}\,2} = \frac{5}{8} \Rightarrow$ AC

14. $\sum_{n=1}^{\infty} \frac{n!}{100^n}$ $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)\cancel{!}\,\cancel{100^n}}{\cancel{n!}\,100^{n+1}} = \infty \Rightarrow$ D

19. $\sum_{n=1}^{\infty} \frac{n^{100}100^n}{n!}$ $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^{100}100^{n+1}\,n!}{n^{100}\cancel{100^n}(n+1)!} = \cancel{\left(\frac{n+1}{n}\right)}^{100}\cdot\cancel{\frac{100}{n+1}} = 0 \Rightarrow$ AC

21. $\sum_{n=1}^{\infty} \frac{(-1)^n(n+1)!}{\prod_{i=1}^{n} 2n-1}$ $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{(n+2)\cancel{!}\prod_{i=1}^{n}(2n-1)}{\cancel{(n+1)!}\prod_{i=1}^{n+1} 2n-1} = \frac{n+2}{2n+1} = \frac{1}{2} \Rightarrow$ AC

32. $\sum_{n=1}^{\infty} \left(\frac{1-n}{2+3n}\right)^n$ $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \left|\frac{1-n}{2+3n}\right| = \frac{1}{3} \Rightarrow$ AC

33. $\sum_{n=1}^{\infty} \frac{(-9)^n}{n10^{n+1}}$ $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \left|\frac{-9}{10\sqrt[n]{10n}}\right| = 0 \Rightarrow$ AC

34. $\sum_{n=1}^{\infty} \frac{n5^{2n}}{10^{n+1}}$ $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \left|\frac{25\sqrt[n]{n}}{10\sqrt[n]{10}}\right| = \infty \Rightarrow$ D

36. $\sum_{n=1}^{\infty} \frac{\sin(\frac{n\pi}{6})}{1+n\sqrt{n}}$ $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{\sin\left(\frac{(n+1)\pi}{6}\right) n^{\frac{3}{2}}}{\sin\left(\frac{n\pi}{6}\right)(n+1)^{\frac{3}{2}}} = 1 \Rightarrow$ inconclusive

40. $a_n = \prod_{i=1}^{n-1} \frac{2+\cos i}{\sqrt{i}}$ $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{2+\cos n}{\sqrt{n}} = 0 \Rightarrow$ AC

1. series of type: $\sum_{n=0}^{\infty} c_n(x-a)^n$

2. a) values of $(x-a)$ for which the series converges
   b) values of $|x-a|$ for which the series converges

3. $\sum_{n=1}^{\infty} (-1)^n n x^n$ $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{\cancel{(n+1)}x^{n+1}}{\cancel{n}\,\cancel{x^n}} = x \Rightarrow R=1$ $(-1,1)$

4. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}}$ $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{x^{n+1}\cancel{\sqrt[3]{n}}}{\cancel{x^n}\cancel{\sqrt[3]{n+1}}} = x \Rightarrow R=1$ $[-1,1]$

7. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{x^{n+1}\, \cancel{n!}}{\cancel{x^n}\,(n+1)\cancel{!}} = 0 \Rightarrow R=\infty$ $(-\infty,+\infty)$

11. $\sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{\sqrt{n}} x^n$ $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{4^{n+1}\, \cancel{x^{n+1}}\, \cancel{\sqrt{n}}}{4^n x^n \cancel{\sqrt{n+1}}} = \frac{1}{4n} \Rightarrow R=\frac{1}{4}$ $(-\frac{1}{4},\frac{1}{4}]$

19. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^n}$ $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{(x-2)^{n+1}\, \cancel{n^n}}{\cancel{(x-2)^n}\,(n+1)^{n+1}} = \frac{x-2}{n+1} \Rightarrow R=\infty$ $(-\infty,+\infty)$

25. $\sum_{n=1}^{\infty} \frac{(5x-4)^n}{n^3}$ $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{(5x-4)^{n+1}\, \cancel{n^3}}{\cancel{(5x-4)^n}\,\cancel{(n+1)^3}} = 5x-4 \Rightarrow R=\frac{1}{5}$ $[\frac{3}{5}, 1]$

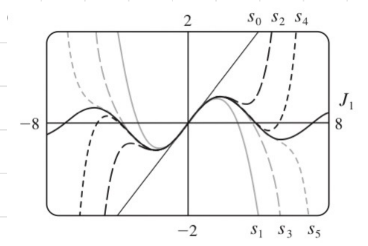
30. $\sum_{n=0}^{\infty} c_n x^n$ $x=-4 \to C$ $x=6 \to D$

a) $\sum_{n=0}^{\infty} c_n \to$ inconclusive

b) $\sum_{n=0}^{\infty} c_n 8^n \to D$

c) $\sum_{n=0}^{\infty} c_n (-3)^n \to C$

d) $\sum_{n=0}^{\infty} (-1)^n c_n 9^n \to D$



35. $J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(n+1)!\, 2^{2n+1}}$ a) $x \in (-\infty,\infty)$ b,c)

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Lecture 14 – Functions as Power Series

11.9: ~~1~~, ~~3~~, ~~4~~, ~~7~~, ~~13~~, ~~15~~, ~~19~~, ~~23~~, ~~25~~, ~~27~~

1. $R=10$ $\qquad \frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n x^n\right] = \sum_{n=1}^{\infty} n c_n x^{n-1}$

11.10: ~~5~~, ~~11~~, ~~15~~

3. $f(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n \quad (-1,1)$

4. $f(x) = \frac{5}{1-4x^2} = 5\sum_{n=0}^{\infty} 4^n x^{2n} \quad \left(-\frac{1}{2},\frac{1}{2}\right)$

7. $f(x) = \frac{x^2}{x^4+16} = \sum_{n=0}^{\infty} \frac{(-x^4)^n}{16^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{16^{n+1}} \quad (-2,2)$

13. a) $f(x) = \frac{1}{(1+x)^2} = \frac{d}{dx}\left[\sum_{n=0}^{\infty}(-x)^n\right] = \sum_{n=1}^{\infty} n(-x)^{n-1} \quad R=1 \qquad \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n$

b) $f(x) = \frac{1}{(1+x)^3} = \frac{d}{dx}\left[\sum_{n=1}^{\infty} n(-x)^{n-1}\right] = \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)(-x)^{n-2} \quad R=1$

c) $f(x) = \frac{x^2}{(1+x)^3} = \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)(-x)^n \quad R=1$

15. $f(x) = \ln(5-x) = \ln 5 - \int \frac{1}{5-x}dx = \ln 5 - \int \sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}}dx = \ln 5 - \sum_{n=0}^{\infty} \frac{x^{n+1}}{5^{n+1}(n+1)} \quad R=5$

19. $f(x) = \frac{1+x}{(1-x)^2} = (1+x)\frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty}(n+1)x^n(1+x) \quad R=1$

23. $f(x) = \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = \int\sum_{n=0}^{\infty}(-x)^n dx - \int\sum_{n=0}^{\infty} x^n dx = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1} - x^{n+1}}{n+1} =$

$= \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1} \quad R=1$

25. $\int \frac{t}{1-t^8}dt = \int \sum_{n=0}^{\infty} t^{8n+1}dt = \sum_{n=0}^{\infty}\frac{t^{8n+2}}{8n+2} \quad R=1$

27. $\int x^2 \ln(1+x)dx = \int x^2 \int \frac{1}{1+x}dx\,dx = \int x^2 \int \sum_{n=0}^{\infty}(-x)^n dx\,dx = \int \sum_{n=0}^{\infty}\frac{(-x)^{n+3}}{n+1}dx = \sum_{n=0}^{\infty}\frac{(-1)^{n+3}x^{n+4}}{(n+1)(n+3)} \quad R=1$

5. $f(x) = xe^x = x\sum_{n=0}^{\infty}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{x^{n+1}}{n!} = x + x^2 + \frac{x^3}{2} + \frac{x^4}{6}$

10. $f(x) = \cos^2 x = 1 - \frac{2x^2}{2!} + \frac{4x^4}{4!} - \frac{8x^6}{6!} + \dots$

$f(0)=1 \quad f'(x) = -2\cos x \sin x = 0 \quad f''(x) = -2\cos^2 x + 2\sin^2 x = -2$

$f'''(x) = 4\sin x\cos x = 0 \quad f''''(x) = 4\cos^2 x - 4\sin^2 x = 4 \quad f^5(x) = -8\sin x\cos x = 0$

15. $f(x) = 2^x = e^{x\ln(2)} = \sum_{n=0}^{\infty}\frac{(x\ln(2))^n}{n!} \quad R=\infty \qquad f^6(x) = -8\cos^2 x + 8\sin^2 x = -8$

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Lecture 15 — Taylor Series

11.10: ~~2~~, ~~19~~, ~~21~~, ~~33~~, ~~41~~, ~~52~~, ~~54~~, ~~57~~, ~~60~~, ~~63~~, ~~65~~, ~~75~~, ~~77~~

11.11: ~~5~~, ~~7~~, ~~15~~, ~~19~~, ~~21~~, ~~27~~

2. $f(1)=1$ und 1.6
$f(1)=1$ und $2.8-0.6+1.5+0.1\ldots$

19. $f(x)=x^5+2x^3+x = 50+105(x-2)+\frac{\cancel{184}^{92}}{\cancel{2!}}(x-2)^2+\frac{\cancel{252}^{42}}{\cancel{3!}}(x-2)^3+\frac{\cancel{240}^{10}}{\cancel{4!}}(x-2)^4+\frac{\cancel{120}}{\cancel{5!}}(x-2)^5$

$f(2)=32+2.8+2=50$

$f'(2)=5x^4+6x^2+1=80+24+1=105$ $\qquad R=\infty$

$f''(2)=20x^3+12x=184$

$f'''(2)=60x^2+12=252$

$f''''(2)=120x=240$

$f^5(2)=120$

21. $f(x)=\ln x=\ln 2+\sum_{n=1}^{\infty}\frac{(x-2)^n}{n!}(-1)^{n+1}\frac{\cancel{(n-1)!}}{2^n}$

$f(2)=\ln 2$

$f'(2)=\frac{1}{x}=\frac{1}{2}$

$f''(2)=\frac{1}{x^2}=-\frac{1}{4}$

$f'''(2)=\frac{2}{x^3}=\frac{1}{4}$

33. $\frac{1}{(2+x)^3}=\frac{1}{2}\frac{d^2}{dx^2}\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^{n+1}}\,dx=\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)(n+2)\,x^n}{2^{n+4}}$ $\quad R=2$

41. $f(x)=\frac{x}{\sqrt{4+x^2}}=\frac{x}{2\sqrt{1+\frac{x^2}{4}}}=\frac{x}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\frac{x^2}{4}\right)^n=\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\frac{x}{2}\right)^{2n+1}$ $\quad R=2$

52. $\frac{1}{\sqrt[4]{1+x}}=(1+x)^{-\frac{1}{4}}=\sum_{n=0}^{\infty}\binom{-\frac{1}{4}}{n}x^n$ $\qquad x=0.1$

$\to 1-\frac{1}{4}x+\frac{5}{32}x^2=1-0.025+0.002=0.977$

54. $\int x^2\sin(x^2)\,dx=\int\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+4}}{(2n+1)!}\,dx=\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+5}}{(4n+4)(2n+1)!}$

57. $\int_0^{\frac{1}{2}}x^3\arctan x\,dx=\int_0^{\frac{1}{2}}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+4}}{2n+1}\,dx=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+5}}{(2n+5)(2n+1)}\Big|_0^{\frac{1}{2}}=\frac{1}{2^5\cdot 5}-\frac{1}{2^7\cdot 3\cdot 7}=0.0059$

60. $\int_0^{\frac{1}{2}}x^2e^{-x^2}\,dx=\int_0^{\frac{1}{2}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{n!}\,dx=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+3}}{(2n+3)\,n!}\Big|_0^{\frac{1}{2}}=\frac{1}{2^3\cdot 3}-\frac{1}{2^5\cdot 5}=0.035$

63. $\lim_{x\to 0}\frac{\sin x-x+\frac{1}{6}x^3}{x^5}=\lim_{x\to 0}\frac{(\cancel{x}-\cancel{\frac{x^3}{6}}+\frac{\cancel{x^5}}{5!}-\cancel{\frac{x^7}{7!}+\ldots})-\cancel{x}+\cancel{\frac{x^3}{6}}}{\cancel{x^5}}=\frac{1}{120}$

65. $\lim\limits_{x\to 0} \dfrac{3\tan^{-1}x - 3x + x^3}{x^5} = \lim\limits_{x\to 0} \dfrac{3\left(\cancel{x} - \cancel{\frac{x^3}{3}} + \frac{x^5}{5} - \cancel{\frac{x^7}{7} + \ldots}^{\,0}\right) - \cancel{3x} + \cancel{x^3}}{\cancel{x^5}} = \dfrac{3}{5}$

75. $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \dfrac{3^n}{n5^n} = \ln\left(1+\frac{3}{5}\right) = \ln(1.6)$

77. $\sum\limits_{n=0}^{\infty} \dfrac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!} = \sin\left(\frac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$

5. $f(x) = \cos x \quad a = \frac{\pi}{2} \quad T_3(x) = -\left(x-\frac{\pi}{2}\right) + \dfrac{\left(x-\frac{\pi}{2}\right)^3}{6}$

$f(a) = 0 \quad f'(a) = -\sin a = -1 \quad f''(a) = -\cos a = 0 \quad f'''(a) = \sin a = 1$

7. $f(x) = \ln x \quad a = 1 \quad T_3(x) = x - 1 - \dfrac{(x-1)^2}{2} - \dfrac{(x-1)^3}{3}$

$f(a) = 0 \quad f'(a) = \frac{1}{a} = 1 \quad f''(a) = -\frac{1}{a^2} = -1 \quad f'''(a) = \dfrac{2}{a^3} = 2$

15. $f(x) = x^{\frac{2}{3}} \quad a = 1 \quad T_3(x) = 1 + \frac{2}{3}(x-1) - \dfrac{(x-1)^2}{9} - \dfrac{4(x-1)^3}{81}$

$f(a) = 1 \quad f'(a) = \frac{2}{3} a^{-\frac{1}{3}} = \frac{2}{3} \quad f''(a) = -\frac{2}{9} a^{-\frac{4}{3}} = -\frac{2}{9} \quad f'''(a) = \frac{8}{27} a^{-\frac{7}{3}} = \frac{8}{27}$

19. $f(x) = e^{x^2} \quad a = 0 \quad T_3(x) = 1 + x^2$

$f(a) = 1 \quad f'(a) = 2ae^{a^2} = 0 \quad f''(a) = 2e^{a^2} + 4a^2e^{a^2} = 2$

$f'''(a) = 4ae^{a^2} + 8ae^{a^2} + 8a^3e^{a^2} = 0$

21. $f(x) = x\sin x \quad a = 0 \quad T_4(x) = x^2 - \dfrac{x^4}{6}$

$f(a) = 0 \quad f'(a) = \sin a + a\cos a = 0 \quad f''(a) = 2\cos a - a\sin a = 2$

$f'''(a) = -3\sin a - a\cos a = 0 \quad f^{4}(a) = -4\cos a + a\sin a = -4$

27. $\sin x \approx x - \dfrac{x^3}{6} \quad 0.01 = \dfrac{1}{5!} x^5 \quad x = \sqrt[5]{1.2} = 1.037 \Rightarrow x \in (-1.037, 1.037)$

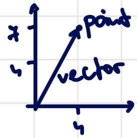
09/01

Lecture 16 – Vectors & Functions of several variables

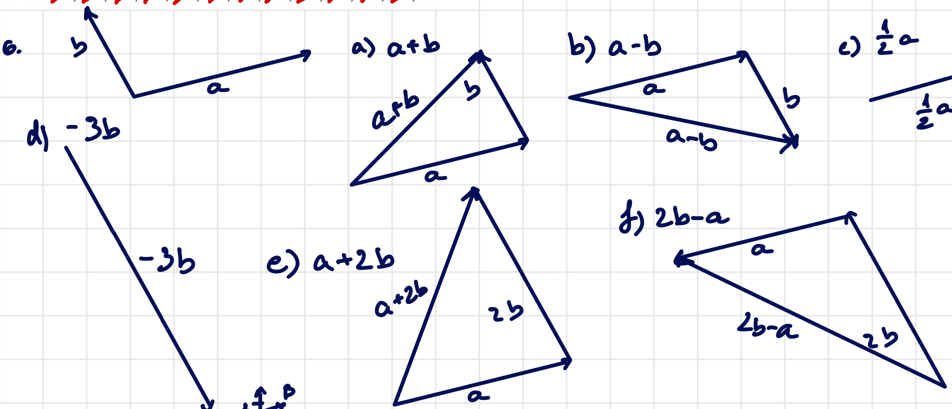
12.2: ~~2~~, ~~4~~, ~~6~~, ~~9~~, ~~19~~, ~~21~~

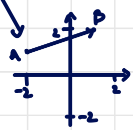
12.3: ~~3~~, ~~5~~, ~~13~~, ~~15~~, ~~17~~, ~~19~~

14.1: ~~15~~, ~~21~~, ~~32~~, ~~46~~, ~~47~~, ~~49~~, ~~61~~, ~~62~~, ~~63~~, ~~64~~, ~~65~~, ~~66~~, ~~68~~

2. 

4. a) $\overrightarrow{AC}$ b) $\overrightarrow{CB}$
c) $\overrightarrow{DA}$ d) $\overrightarrow{DB}$

6. 

9. $A(-2,1)$ 

$B(1,2)$

$a = \langle 3, 1\rangle$

19. $a = \langle -3, 4\rangle$ $\quad 4a+2b = \langle 6, 14\rangle$

$b = \langle 9, -1\rangle$ $\quad |a| = 5$

$a+b = \langle 6, 3\rangle$ $\quad |a-b| = |\langle -12, 5\rangle| = 13$

21. $a = \langle 4, -3, 2\rangle$ $\quad 4a+2b = \langle 20, -12, 0\rangle$

$b = \langle 2, 0, -4\rangle$ $\quad |a| = \sqrt{29}$

$a+b = \langle 6, -3, -2\rangle$ $\quad |a-b| = |\langle 2, -3, 6\rangle| = 7$

3. $a = \langle 1.5, 0.4\rangle$

$b = \langle -4, 6\rangle$

$a \cdot b = -6 + 2.4 = -3.6$

6. $a = \langle 4, 1, \frac{1}{4}\rangle$

$b = \langle 6, -3, -8\rangle$

$a \cdot b = 24 - 3 - 2 = 19$

13. a) $\theta = \frac{\pi}{2}$ $\quad \cos\theta = 0$

b) $|i| = |j| = |k| = \underline{1}$

$\theta = 0$ $\quad \cos\theta = \underline{1}$

15. $a = \langle 4, 3\rangle$

$b = \langle 2, -1\rangle$

$$\cos\theta = \frac{a \cdot b}{|a||b|} = \frac{8-3}{\cancel{5}\sqrt{5}} = \frac{1}{\sqrt{5}}$$

$$\theta = \arccos\left(\frac{1}{\sqrt{5}}\right) = 1.11 / 63^\circ$$

17. $a = \langle 1, -4, 1\rangle$

$b = \langle 0, 2, -2\rangle$

$$\cos\theta = \frac{a b}{|a||b|} = \frac{-8-2}{12} = -\frac{5}{6}$$

$$\theta = \arccos\left(-\frac{5}{6}\right) = 2.56 / 146^\circ$$

19. $a = \langle 4, -3, 1\rangle$

$b = \langle 2, 0, -1\rangle$

$$\cos\theta = \frac{a \cdot b}{|a||b|} = \frac{8-1}{\sqrt{26}\sqrt{5}} = \frac{7}{\sqrt{130}}$$

$$\theta = \arccos\theta = 0.91 / 52^\circ$$

15. $f(x,y) = \ln(9 - x^2 - 9y^2)$

D: $x^2 + 9y^2 < 9$

21. $f(x,y,z) = \sqrt{4-x^2} + \sqrt{9-y^2} + \sqrt{1-z^2}$

D: $x \in [-2,2]$, $y \in [-3,3]$, $z \in [-1,1]$

32. a) ✓ b) ✓ c) ✓ d) ✓ e) ✓ f) ✓

61. $z = \sin(xy)$, C, ✓

62. $z = e^x \cos(y)$, A, ✓

63. $z = \sin(x-y)$, F, ✓

64. $z = \sin x - \sin y$, E, ✓

65. $z = (1-x^2)(1-y^2)$, B, ✓

66. $z = \frac{x-y}{1+x^2+y^2}$, D, ✓

65. $f(x,y,z) = x^2 + 3y^2 + 5z^2$

sphere

elonged along x-axis $x = \sqrt{5}\, z$

elonged along y-axis $y = \sqrt{\frac{5}{3}}\, z$

10/01 # Lecture 17 – Partial Derivatives & Linearization

14.3: ~~5~~, ~~6~~, ~~10~~, ~~15~~, ~~21~~, ~~25~~, ~~35~~, ~~53~~, ~~55~~ ~~59~~, ~~60~~, ~~65~~

5. a) + b) − 6. a) − b) −

10. $f_x(2,1) \sim 3$ $f_y(2,1) \sim -2$

14.4: ~~1~~, ~~3~~, ~~5~~, ~~11~~, ~~13~~, ~~19~~, ~~21~~

15. $f(x,y) = x^4 + 5xy^3$
$f_x = 4x^3 + 5y^3$
$f_y = 15xy^2$

21. $f(x,y) = \frac{x}{y}$
$f_x = \frac{1}{y}$
$f_y = -\frac{x}{y^2}$

25. $f(u,v) = (u^2v - v^3)^5$
$f_u = 10uv(u^2v - v^3)^4$
$f_v = 5(u^2 - 3v^2)(u^2v - v^3)^4$

35. $p = \sqrt{t^4 + u^2\cos v}$
$p_t = \frac{2t^3}{\sqrt{t^4+u^2\cos v}}$
$p_u = \frac{u\cos v}{\sqrt{t^4+u^2\cos v}}$
$p_v = \frac{-u^2\sin v}{\sqrt{t^4+u^2\cos v}}$

53. $f(x,y) = x^4y - 2x^3y^2$
$f_x = 4x^3y - 6x^2y^2$
$f_y = x^4 - 4x^3y$
$f_{xy} = 4x^3 - 12x^2y$
$f_{xx} = 12x^2y - 12xy^2$
$f_{yy} = -4x^3$

55. $z = \frac{y}{2x+3y}$
$z_y = \frac{2x}{(2x+3y)^2}$ $z_{yy} = \frac{-12x}{(2x+3y)^3}$
$z_x = \frac{-2y}{(2x+3y)^2}$ $z_{xx} = \frac{8y}{(2x+3y)^3}$
$z_{xy} = \frac{-4x+6y}{(2x+3y)^3}$

59. $u = x^4y^3 - y^4$
$u_x = 4x^3y^3$ $u_{xy} = 12x^3y^2$
$u_y = 3x^4y^2 - 4y^3$ $u_{yx} = 12x^3y^2$

60. $u = e^{xy}\sin y$
$u_x = ye^{xy}\sin y$
$u_y = xe^{xy}\sin y + e^{xy}\cos y$
$u_{xy} = e^{xy}\sin y + xye^{xy}\sin y + ye^{xy}\cos y$
$u_{yx} = e^{xy}\sin y + xye^{xy}\sin y + ye^{xy}\cos y$

65. $f(x,y,z) = e^{xyz^2}$
$f_x = yz^2e^{xyz^2}$
$f_{xy} = z^2e^{xyz^2} + xyz^4e^{xyz^2}$
$f_{xyz} = 2ze^{xyz^2} + 2xyz^3e^{xyz^2} + 4xyz^3e^{xyz^2} + 2x^2y^2z^5e^{xyz^2}$

1. $z = 2x^2 + y^2 - 5y$
$z_x = 4x$
$z_y = 2y - 5$
$(1, 2, -4)$
$z + 4 = 4(x-1) - (y-2)$
$z + 4 = 4x - 4 - y + 2$
$z = 4x - y - 6$

3. $z = e^{x-y}$
$z_x = e^{x-y}$
$z_y = -e^{x-y}$
$(2, 2, 1)$
$z - 1 = (x-2) - (y-2)$
$z = x - y + 1$

5. $z = x\sin(x+y)$
$z_x = \sin(x+y) + x\cos(x+y)$
$z_y = x\cos(x+y)$
$(-1, 1, 0)$
$z = -(x+1) - (y-1)$
$z = -x - y$
$x + y + z = 0$

11. $f(x,y) = 1 + x\ln(xy-5)$

$f(2,3) = 1$

$f_x = \ln(xy-5) + \frac{y}{xy-5} = 6$

$f_y = \frac{x^2}{xy-5} = 4$

$L(x,y) = 1 + 6(x-2) + 4(y-3) = 6x + 4y - 23$

13. $f(x,y) = x^2 e^y$

$f(1,0) = 1$

$f_x = 2xe^y = 2$

$f_y = x^2 e^y = 1$

$L(x,y) = 1 + 2(x-1) + y$

$= 2x + y - 1$

19. $f(2,5) = 6$

$f_x(2,5) = 1$

$f_y(2,5) = -1$

$L(x,y) = 6 + (x-2) - (y-5)$

$L(2.2, 4.9) = 6 + 0.2 + 0.1 = 6.3$

21. $f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$

$f(3,2,6) = 7$

$f_x = \frac{3}{7}$

$f_y = \frac{2}{7}$

$f_z = \frac{3}{7}$

$$\begin{cases} L(x,y,z) = 7 + \frac{3}{7}(x-3) + \frac{2}{7}(y-2) + \frac{6}{7}(z-6) \\ L(3.02, 1.97, 5.99) = 7 + \cancel{\frac{3}{7} \cdot 0.02} - \cancel{\frac{2}{7} \cdot 0.03} - \frac{6}{7} \cdot 0.01 = 6.9914 \end{cases}$$

11/01 Lecture 18 – The chain rule & the directional derivative

14.5: ~~1~~, ~~2~~, ~~3~~, ~~7~~, ~~11~~, ~~21~~, ~~23~~

14.6: ~~4~~, ~~5~~, ~~7~~, ~~9~~, ~~11~~, ~~17~~, ~~19~~, ~~21~~, ~~23~~

1. $z = xy^3 - x^2y \quad x = t^2+1 \quad y = t^2-1$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = (y^3 - 2xy)2t + (3xy^2 - x^2)2t$$

2. $z = \frac{x-y}{x+2y} \quad x = e^{\pi t} \quad y = e^{-\pi t}$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = \frac{y}{(x+2y)^2}\cdot\pi e^{\pi t} + \frac{3x}{(x+2y)^2}\cdot\pi e^{\pi t}$$

3. $z = \sin x \cos y \quad x = \sqrt{t} \quad y = \frac{1}{t}$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = \frac{\cos x\cos y}{2\sqrt{t}} + \frac{\sin x \sin y}{t^2}$$

7. $z = (x-y)^5 \quad x = s^2t \quad y = st^2$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = 5(x-y)^4s^2 - 10(x-y)^4st$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} = 10(x-y)^4st - 5(x-y)^4t^2$$

11. $z = e^r\cos\theta \quad r = st \quad \theta = \sqrt{s^2+t^2}$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial s} = e^r\cos\theta . t - \frac{e^r\sin\theta}{\sqrt{s^2+t^2}}.s$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial t} = e^r\cos\theta . s - \frac{e^r\sin\theta}{\sqrt{s^2+t^2}}.t$$

21. $z = x^4 + x^2y \quad x = s+2t-u = 7 \quad y = stu^2 = 8 \quad s=4 \quad t=2 \quad u=1$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} = 4x^3 + 2xy + x^2tu^2 = 4\cdot 343 + 2\cdot 56 + 49\cdot 2 = 1582$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = 2(4x^3+2xy) + x^2su^2 = 8\cdot 343 + 4\cdot 56 + 49\cdot 4 = 3164$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = -4x^3 - 2xy + 2x^2stu = -4\cdot 343 - 2\cdot 56 + 49\cdot 16 = -700$$

23. $w = xy + yz + zx \quad x = r\cos\theta = 0 \quad y = r\sin\theta = 2 \quad z = r\theta = \pi \quad r = 2 \quad \theta = \frac{\pi}{2}$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial r} = (y+z)\cos\theta + (x+z)\sin\theta + (x+y)\theta = 2\pi$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial \theta} = -(y+z)r\sin\theta + (x+z)r\cos\theta + (x+y)r = -2\pi$$

4. $f(x,y) = xy^3 - x^2 \quad (1,2) \quad \theta = \frac{\pi}{3}$

$$D_uf = |\nabla f|\cos\theta = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\cos\theta = \frac{\sqrt{180}}{2} = \sqrt{45}$$

5. $f(x,y) = y\cos(xy) \quad (0,1) \quad \theta = \frac{\pi}{4}$

$$D_uf = |\nabla f|\cos\theta = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\cos\theta = \frac{\sqrt{2}}{2}$$

7. $f(x,y) = \frac{x}{y} \quad P(2,1) \quad u = \langle \frac{3}{5}, \frac{4}{5}\rangle$

$$\nabla f(x,y) = \langle \frac{1}{y}, -\frac{x}{y^2}\rangle = \langle 1, -2\rangle$$

$$D_uf(x,y) = -1$$

9. $f(x,y,z) = x^2yz - xyz^3 \quad P(2,-1,1) \quad u = \langle 0, \frac{4}{5}, -\frac{3}{5} \rangle$

$\nabla f(x,y,z) = \langle 2xyz - yz^3, x^2z - xz^3, x^2y - 3xyz^2 \rangle = \langle -3, 2, 2 \rangle$

$D_u f(x,y,z) = \frac{2}{5}$

11. $f(x,y) = e^x \sin y \quad (0, \frac{\pi}{3}) \quad v = \langle -6, 8 \rangle \quad |v| = 10$

$D_u f(x,y) = \frac{f_x \cdot (-6) + f_y \cdot 8}{|v|} = \frac{-6e^x \sin y + 8e^x \cos y}{10} = \frac{4 - 3\sqrt{3}}{10}$

17. $h(r,s,t) = \ln(3r + 6s + 9t) \quad (1,1,1) \quad v = \langle 4, 12, 6 \rangle \quad |v| = 14$

$D_u h(r,s,t) = \frac{4h_r + 12h_s + 6h_t}{14} = \frac{4 + 24 + 18}{14(r + 2s + 3t)} = \frac{23}{42}$

19. $f(x,y) = \sqrt{xy} \quad P(2,8) \quad Q(5,4) \quad \overrightarrow{PQ} = \langle 3, -4 \rangle \quad u = \langle \frac{3}{5}, -\frac{4}{5} \rangle$

$D_u f(x,y) = \frac{3f_x - 4f_y}{5} = \frac{3y - 4x}{2\sqrt{xy} \cdot 5} = \frac{\cancel{24 - 8}\ 3-1}{\cancel{2 \cdot 4} \cdot 5} = \frac{2}{5}$

21. $f(x,y) = 4y\sqrt{x} \quad (4,1) \quad |\nabla f| = \sqrt{65}$

$\nabla f(x,y) = \langle f_x, f_y \rangle = \langle \frac{2y}{\sqrt{x}}, 4\sqrt{x} \rangle = \langle 1, 8 \rangle$

23. $f(x,y) = \sin(xy) \quad (1,0) \quad |\nabla f| = 1$

$\nabla f(x,y) = \langle f_x, f_y \rangle = \langle y\cos(xy), x\cos(xy) \rangle = \langle 0, 1 \rangle$

16/01 **Lecture 19 – Minimum and maximum value**

14.7: 3, 7, 9, 15, 21, 23, 31, 33, 41, 43

3. $f(1,1)=3 \to$ local minimum

$f(0,0)=4 \to$ saddle point

$f(x,y)=4+x^3+y^3-3xy$

$f_x=3x^2-3y \qquad y=x^2 \qquad 3y^4-3y=3y(y^3-1)=3y(y-1)(y^2+y+1)$

$f_y=3y^2-3x \qquad x=y^2 \qquad 3x^4-3x=3x(x^3-1)=3x(x-1)(x^2+x+1)$

$$\left.\begin{aligned} f_{xx}&=6x\\ f_{yy}&=6y\\ f_{xy}&=-3\end{aligned}\right| D=f_{xx}f_{yy}-f_{xy}^2=36xy-9 \begin{cases} (0,0) & D=-9 \to \text{saddle point}\\ & \left.\begin{aligned} &f_{xx}=6\\ (1,1)\quad &D=27\end{aligned}\right\} \to \text{local minimum}\end{cases}$$

7. $f(x,y)=(x-y)(1-xy)=x-y-x^2y+xy^2$

$f_x=1-2xy+y^2 \qquad x=-\frac{y^2+1}{2y} \qquad \cancel{1}-x^2\cancel{-1}+\left(\frac{x^2+1}{2x}\right)^2=0 \qquad \frac{-4x^4+x^4+2x^2+1}{4x^2}=0$

$f_y=-1-x^2+2xy \qquad y=\frac{x^2+1}{2x} \qquad -1-\left(\frac{y^2+1}{2y}\right)^2-y^2-1=0 \qquad \frac{-4y^4-y^4-8y^2-2y^2-1}{4y^2}=0$

$$\left.\begin{aligned} f_{xx}&=-2y\\ f_{yy}&=2x\\ f_{xy}&=2(y-x)\end{aligned}\right\} \begin{aligned} D&=-4xy+4(y-x)^2=\\ &=4y^2+4x^2-12xy\end{aligned}$$

$\frac{-(x-1)(x+1)(3x^2+1)}{4x^2}=0$

$(0,0)\ D=0$

$(1,1)\ D=-4$

$(-1,-1)\ D=-4$

9. $f(x,y)=x^2+y^4+2xy$

$f_x=2x+2y \qquad x=-y$

$f_y=4y^3+2x \qquad 4y^3-2y=2y(2y^2-1)=2y(\sqrt{2}y-1)(\sqrt{2}y+1)$

$$\left.\begin{aligned} f_{xx}&=2\\ f_{yy}&=12y^2\\ f_{xy}&=2\end{aligned}\right| D=24y^2-4 \begin{cases} \left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right) & D=8 \to \text{local minimum}\\ (0,0) & D=-4 \to \text{saddle point}\\ \left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right) & D=8 \to \text{local minimum}\end{cases}$$

15. $f(x,y)=e^x\cos y$

$$\left.\begin{aligned} f_x&=e^x\cos y\\ f_y&=-e^x\sin y\end{aligned}\right\} \text{no critical points}$$

$$\left.\begin{aligned} f_{xx}&=e^x\cos y\\ f_{yy}&=-e^x\cos y\\ f_{xy}&=-e^x\sin y\end{aligned}\right| D=-(e^x\cos y)^2-(e^x\sin y)^2=-e^{2x}$$

21. $f(x,y) = x^2 + 4y^2 - 4xy + 2$

$f_x = 2x - 4y = 2(x-2y)$ $\quad x = 2y \to$ infinite critical points

$f_y = 8y - 4x = 4(2y-x)$

$\left.\begin{array}{l} f_{xx} = 2 \\ f_{xy} = -4 \\ f_{yy} = 8 \end{array}\right\} D = 0$ $\quad f(0,0) = 2$

$f(x,y) = (2y-x)^2 + 2$

$\Rightarrow$ critical points = absolute minimums

22. $f(x,y) = x^2 + y^2 + \frac{1}{x^2y^2}$

$f_x = 2x - \frac{2}{x^3}$ $\quad x^4 = 1$ $\quad (\pm1, \pm1)$

$f_y = 2y - \frac{2}{y^3}$ $\quad y^4 = 1$ $\quad \left.\begin{array}{l} f_{xx} = 8 \\ D = 16 \end{array}\right\}$ local minimums

$\left.\begin{array}{l} f_{xx} = 2 + \frac{6}{x^4} \\ f_{yy} = 2 + \frac{6}{y^4} \\ f_{xy} = 0 \end{array}\right\} D = 4 + \frac{6}{x^4} + \frac{6}{y^4}$

31. $f(x,y) = x^2 + y^2 - 2x$ $\quad D = \{(x,y) \mid x \in [0,2],\ y \in [-2,2],\ |x+y| \le 2\}$

$f_x = 2x - 2$ $\quad (1,0)$ $\quad (0, \pm 2) \to f = 4 \to$ absolute max

$f_y = 2y$ $\quad f_{xx} = 2$ $\quad f_{yy} = 2$ $\quad f_{xy} = 0$ $\quad D = 4 \Rightarrow$ absolute minimum

33. $f(x,y) = x^2 + y^2 + x^2y + 4$ $\quad D = \{(x,y) \mid |x| \le 1, |y| \le 1\}$ $\quad f(\pm1, 1) = 7 \to$ absolute maximum

$f_x = 2x + 2xy$ $\quad y = -1$ $\quad x = 0$ $\quad (\sqrt{2}, -1)$ $\quad D = -8$ saddle points

$f_y = 2y + x^2$ $\quad y = -\frac{x^2}{2}$ $\quad x = \sqrt{-2y}$ $\quad (-\sqrt{2}, -1)$ $\quad D = -8$

$f_{xx} = 2(1+y)$ $\quad f_{xy} = 2x$ $\quad (0,0)$ $\quad D = 4$ absolute minimum $\quad f(0,0) = 4$

$f_{yy} = 2$ $\quad D = 4(1+y) - 4x^2$ $\quad f_{xx} = 2$

41. $(2, 0, -3)$ $\quad x + y + z = 1$

$f = (x-2)^2 + (1-x-z)^2 + (z+3)^2$

$f = x^2 - 4x + 4 + x^2 + z^2 + 1 - 2x - 2z + 2xz + z^2 + 6z + 9$

$f = 2x^2 - 6x + 2z^2 + 4z + 2xz + 14$

$f_x = 4x - 6 + 2z$ $\quad 4x - 6 + 2z = 4z + 4 + 2x$

$f_z = 4z + 4 + 2x$ $\quad 2x = 10 + 2z$ $\quad x = 5 + z$

$\left.\begin{array}{l} f_{xx} = 4 \\ f_{zz} = 4 \\ f_{xz} = 2 \end{array}\right\} D = 12$ $\quad 4z + 4 + 10 + 2z = 0$ $\quad z = -\frac{7}{3}$

$x = \frac{8}{3}$ $\quad y = \frac{2}{3}$

$f = \sqrt{(x-2)^2 + y^2 + (z+3)^2} = \sqrt{\left(\frac{8}{3}-2\right)^2 + \left(\frac{2}{3}\right)^2 + \left(-\frac{7}{3}+3\right)^2} = \sqrt{\left(\frac{2}{3}\right)^2 \cdot 3} = \frac{2}{\sqrt{3}}$

43. $z^2 = x^2 + y^2$ $\quad (4, 2, 0)$

$f = (x-4)^2 + (y-2)^2 + z^2$

$f = x^2 - 8x + 16 + y^2 - 4y + 4 + x^2 + y^2$

$f_x = 4x - 8$ $\quad x = 2$ absolute minimum

$f_y = 4y - 4$ $\quad y = 1$

$\left.\begin{array}{l} f_{xx} = 4 \\ f_{yy} = 4 \\ f_{xy} = 0 \end{array}\right\} D = 16$ $\quad z = \pm\sqrt{5}$ $\quad (2, 1, \pm\sqrt{5})$

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Lecture 20 - complex numbers

Appendix H:

1, 3, 5, 7, 9, 11, 12, 14, 15, 17, 18, 19, 21, 23, 24, 26, 27, 29, 31, 33, 35, 47

1. $(5-6i)+(3+2i)=8-4i$

3. $(2+5i)(4-i)=13+18i$

5. $\overline{12+7i}=12-7i$

7. $\frac{1+4i}{3+2i}=\frac{(1+4i)(3-2i)}{13}=\frac{11+10i}{13}$

9. $\frac{1}{1+i}=\frac{1-i}{2}$

11. $i^3=-i$

12. $i^{100}=1$

14. $\sqrt{-3}\sqrt{-12}=6$

15. $\overline{12-5i}=12+5i \quad |12-5i|=13$

17. $\overline{-4i}=4i \quad |-4i|=4$

19. $4x^2+9=0 \quad x=\pm\frac{3}{2}i$

21. $x^2+2x+5=0 \quad D=b^2-4ac=-16 \quad x_{1,2}=\frac{-b\pm\sqrt{D}}{2a}=\frac{-2\pm4i}{2}=-1\pm2i$

23. $z^2+z+2=0 \quad D=b^2-4ac=-7 \quad z_{1,2}=\frac{-b\pm\sqrt{D}}{2a}=\frac{-1\pm\sqrt{7}i}{2}$

24. $z^2+\frac{1}{2}z+\frac{1}{4}=0 \quad 4z^2+2z+1=0 \quad D=b^2-4ac=-12 \quad z_{1,2}=\frac{-b\pm\sqrt{D}}{2a}=\frac{-2\pm2\sqrt{3}i}{8}=\frac{-1\pm\sqrt{3}i}{4}$

26. $1-\sqrt{3}i \quad |z|=2 \quad \tan\theta=-\sqrt{3} \quad \theta=\frac{5\pi}{3} \quad z=2(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3})$

27. $3+4i \quad |z|=5 \quad \tan\theta=\frac{4}{3} \quad \theta=\tan^{-1}(\frac{4}{3}) \quad z=5(\cos(\tan^{-1}\frac{4}{3})+i\sin(\tan^{-1}\frac{4}{3}))$

29. $z=\sqrt{3}+i \quad |z|=2 \quad \tan(\theta)=\frac{1}{\sqrt{3}} \quad \theta=\frac{\pi}{6} \Rightarrow z=2(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})$

$w=1+\sqrt{3}i \quad |w|=2 \quad \tan(\theta)=\sqrt{3} \quad \theta=\frac{\pi}{3} \Rightarrow w=2(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})$

$zw=4(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}) \qquad 1=1+0i=1(\cos 0+i\sin 0)$

$\frac{z}{w}=\cos-\frac{\pi}{6}+i\sin-\frac{\pi}{6} \qquad \frac{1}{z}=\frac{1}{2}(\cos-\frac{\pi}{6}+i\sin-\frac{\pi}{6})$

31. $z=2\sqrt{3}-2i \quad |z|=4 \quad \tan(\theta)=\frac{-1}{\sqrt{3}} \quad \theta=\frac{5\pi}{6} \quad z=4(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6})$

$w=-1+i \quad |w|=\sqrt{2} \quad \tan(\theta)=-1 \quad \theta=\frac{3\pi}{4} \quad w=\sqrt{2}(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4})$

$zw=4\sqrt{2}(\cos\frac{7\pi}{12}+i\sin\frac{7\pi}{12})$

$\frac{z}{w}=\frac{4}{\sqrt{2}}(\cos-\frac{11\pi}{12}+i\sin\frac{-11\pi}{12})$

$\frac{1}{z}=\frac{1}{4}(\cos-\frac{\pi}{6}+i\sin-\frac{\pi}{6})$

33. $(1+i)^{20}=2^{10}(\cancel{\cos 5\pi}^{-1}+i\,\cancel{\sin 5\pi}^{0})=-2^{10}$

$1+i \quad |z|=\sqrt{2} \quad \tan\theta=1 \quad \theta=\frac{\pi}{4} \quad z=\sqrt{2}(\cos\frac{\pi}{4}+i\sin(\frac{\pi}{4}))$

35. $(2\sqrt{3}+2i)^5 \Rightarrow 2^{10}(\cos\frac{5\pi}{6}-i\sin\frac{5\pi}{6})=2^9(i-\sqrt{3})$

$2\sqrt{3}+2i \quad |z|=4 \quad \tan\theta=\frac{1}{\sqrt{3}} \quad \theta=\frac{\pi}{6} \Rightarrow z=4(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6})$

47. $n=3 \quad r=1$

$(\cos\theta+i\sin\theta)^3=\cos 3\theta+i\sin 3\theta$

$=\cos^3\theta+3\cos^2\theta(i\sin\theta)+3\cos\theta(i\sin\theta)^2+(i\sin\theta)^3$

$=\cos^3\theta+(3\cos^2\theta\sin\theta)i-3\cos\theta\sin^2\theta-(\sin^3\theta)i$

$=(\cos^3\theta-3\cos\theta\sin^2\theta)+(3\sin\theta\cos^2\theta-\sin^3\theta)i$

$\cos 3\theta=\cos^3\theta-3\sin^2\theta\cos\theta$

$\sin 3\theta=3\sin\theta\cos^2\theta-\sin^3\theta$

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Lecture 21 - Complex numbers

Appendix H: 37, 39, 41, 43, 45, 48, 49, 50

37. $z^8 = 1$

$|z|^8 e^{i8\varphi} = e^0$

$|z| = 1 \quad 8\varphi = 2k\pi \quad \varphi = \frac{k\pi}{4}$

$$\begin{cases} \varphi = \frac{\pi}{4} & z = e^{i\frac{\pi}{4}} & \varphi = \frac{5\pi}{4} & z = e^{\frac{5\pi}{4}i} \\ \varphi = \frac{\pi}{2} & z = e^{\frac{\pi}{2}i} & \varphi = \frac{3\pi}{2} & z = e^{\frac{3\pi}{2}i} \\ \varphi = \frac{3\pi}{4} & z = e^{\frac{3\pi}{4}i} & \varphi = \frac{7\pi}{4} & z = e^{\frac{7\pi}{4}i} \\ \varphi = \pi & z = e^{\pi i} & \varphi = 2\pi & z = e^{2\pi i} = e^0 \end{cases}$$

39. $z^3 = i$

$|z|^3 e^{3i\varphi} = e^{\frac{\pi}{2}i}$

$|z| = 1 \quad 3\varphi = \frac{\pi}{2} + 2k\pi$

$\varphi = \frac{\pi}{6} + \frac{2k\pi}{3}$

$$\begin{cases} \varphi = \frac{\pi}{6} & z = e^{\frac{\pi}{6}i} \\ \varphi = \frac{5\pi}{6} & z = e^{\frac{5\pi}{6}i} \\ \varphi = \frac{3\pi}{2} & z = e^{\frac{3\pi}{2}i} \end{cases}$$

41. $e^{\frac{\pi}{2}i} = \overset{0}{\cancel{\cos\frac{\pi}{2}}} + i\overset{1}{\cancel{\sin\frac{\pi}{2}}} = i$

43. $e^{\frac{\pi}{3}i} = \overset{\frac{1}{2}}{\cancel{\cos\frac{\pi}{3}}} + i\overset{\frac{\sqrt{3}}{2}}{\cancel{\sin\frac{\pi}{3}}} = \frac{1+\sqrt{3}i}{2}$

45. $e^{2+\pi i} = e^2(\overset{-1}{\cancel{\cos\pi}} + i\overset{0}{\cancel{\sin\pi}}) = -e^2$

48. $e^{xi} = \cos x + i\sin x$

$\overline{e^{xi}} = e^{-xi} = \cos x - i\sin x$

$\cos x = \frac{e^{xi} + e^{-xi}}{2} \qquad \sin(x) = \frac{e^{xi} - e^{-xi}}{2i}$

49. $u(x) = f(x) + ig(x)$

$u'(x) = f'(x) + ig'(x)$

$F(x) = e^{rx} \quad r = a+bi$

$F'(x) = re^{rx}$

50. a) $\int e^{(1+i)x}dx = \frac{e^{(1+i)x}}{1+i}$

b) $\int e^x\cos x\,dx = e^x\cos x + \int e^x\sin x\,dx = e^x\cos x + e^x\sin x - \int e^x\cos x\,dx$

$\Rightarrow \int e^x\cos x\,dx = e^x\frac{\cos x + \sin x}{2}$

$\int e^x\sin x\,dx = e^x\sin x - \int e^x\cos x\,dx = e^x\sin x - e^x\cos x - \int e^x\sin x\,dx$

$\Rightarrow \int e^x\sin x\,dx = e^x\frac{\sin x - \cos x}{2}$

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Lecture 22 – Double integrals over rectangles

15.1: ~~5~~, ~~6~~, ~~9~~, ~~10~~, ~~15~~, ~~17~~, ~~19~~, ~~21~~, ~~23~~, ~~29~~, ~~31~~, ~~33~~, ~~37~~, ~~38~~, ~~39~~, ~~41~~

9. $\iint_R \sqrt{2}\, dA = 24\sqrt{2} \quad R = \{(x,y) \mid 2 \le x \le 6,\ -1 \le y \le 5\}$

10. $\iint_R (2x+1)\, dA = \int_0^4 \int_0^2 2x+1\, dx\, dy = \int_0^4 x^2+x \Big|_0^2 dy = \int_0^4 6\, dy = 24$
$R = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le 4\}$

15. $\int_1^4 \int_0^2 (6x^2y - 2x)\, dy\, dx = \int_1^4 3x^2y^2 - 2xy \Big|_0^2 dx = \int_1^4 12x^2 - 4x\, dx = 4x^3 - 2x^2 \Big|_1^4 = 256 - 32 - 4 + 2$
$= 222$

17. $\int_0^1 \int_1^2 (x + e^{-y})\, dx\, dy = \int_0^1 \frac{x^2}{2} + \frac{x}{e^y} \Big|_1^2 dy = \int_1^2 2 + \frac{2}{e^y} - \frac{1}{2} - \frac{1}{e^y}\, dy = \frac{3}{2}y - e^{-y} \Big|_0^1 =$
$= \frac{3}{2} - \frac{1}{e} + 1 = \frac{5}{2} - e^{-1}$

19. $\int_{-3}^3 \int_0^{\frac{\pi}{2}} y + y^2 \cos x\, dx\, dy = \int_{-3}^3 yx + y^2 \sin(x) \Big|_0^{\frac{\pi}{2}} dy = \int_{-3}^3 \frac{\pi}{2}y + y^2\, dy = \frac{\pi y^2}{4} + \frac{y^3}{3} \Big|_{-3}^3 = 18$

21. $\int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right) dy\, dx = \int_1^4 x\ln(y) + \frac{y^2}{2x} \Big|_1^2 dx = \int_1^4 x\ln(2) + \frac{3}{2x}\, dx = \frac{x^2\ln(2)}{2} + \frac{3}{2}\ln(x) \Big|_1^4 =$
$= \frac{1}{2}(15\ln(2) + 3\ln(4))$

23. $\int_0^3 \int_0^{\frac{\pi}{2}} t^2 \sin^3\phi\, d\phi\, dt = \int_0^3 \left(\frac{\cos^3\phi}{3} - \cos\phi\right) t^2 \Big|_0^{\frac{\pi}{2}} dt = \int_0^3 \frac{2}{3}t^2\, dt = \frac{2t^3}{9} \Big|_0^3 = 6$

29. $\int_0^1 \int_{-3}^3 \frac{xy^2}{x^2+1}\, dy\, dx = \int_0^1 \frac{xy^3}{3(x^2+1)} \Big|_{-3}^3 dx = \int_0^1 \frac{18x}{x^2+1}\, dx = 9\ln(x^2+1) \Big|_0^1 = 9\ln(2)$

31. $\int_0^{\frac{\pi}{6}} \int_0^{\frac{\pi}{3}} x\sin(x+y)\, dy\, dx = \int_0^{\frac{\pi}{6}} -x\cos(x+y) \Big|_0^{\frac{\pi}{3}} dx = \int_0^{\frac{\pi}{6}} -x\cos\left(x+\frac{\pi}{3}\right) + x\cos(x)\, dx =$
$= \frac{(x+\sqrt{3})\sin x + (1-\sqrt{3}x)\cos x}{2} \Big|_0^{\frac{\pi}{6}} = \frac{1}{2}(\sqrt{3}-1) - \frac{\pi}{12}$

33. $\int_0^3 \int_0^2 ye^{-xy}\, dx\, dy = \int_0^3 -e^{-xy} \Big|_0^2 dy = \int_0^3 1 - e^{-2x}\, dx = x + \frac{e^{-2x}}{2} \Big|_0^3 = 3 + \frac{1}{2e^6} - \frac{1}{2} = \frac{1}{2}(5 + e^{-6})$

37. $\int_{-1}^2 \int_{-1}^1 \frac{15+4x+6y}{2}\, dy\, dx = \int_{-1}^2 \frac{15y+4xy+3y^2}{2} \Big|_{-1}^1 dx = \int_{-1}^2 18 + 4x\, dx = 18x + 2x^2 \Big|_{-1}^2 = 36 + 8$
$+ 18 - 2 =$
$= 60$

38. $\int_{-1}^1 \int_1^2 3y^2 - x^2 + 2\, dy\, dx = \int_{-1}^1 y^3 - x^2y + 2y \Big|_1^2 dx = \int_{-1}^1 9 - x^2\, dx = 9x - \frac{x^3}{3} \Big|_{-1}^1 = \frac{52}{3}$

39. $\int_{-1}^1 \int_{-2}^2 1 - \frac{x^2}{4} - \frac{y^2}{9}\, dy\, dx = \int_{-1}^1 y - \frac{x^2y}{4} - \frac{y^3}{27} \Big|_{-2}^2 dx = \int_{-1}^1 4 - x^2 - \frac{16}{27}\, dx = 4x - \frac{x^3}{3} - \frac{16}{27}x \Big|_{-1}^1 = \frac{166}{27}$

41. $\int_0^1 \int_{-1}^1 1 + x^2ye^y\, dx\, dy = \int_0^1 x + \frac{x^3ye^y}{3} \Big|_{-1}^1 dy = \int_0^1 2 + \frac{2}{3}ye^y\, dy = 2y + \frac{2}{3}(y-1)e^y \Big|_0^1 = \frac{8}{3}$

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Lecture 23 – Double integrals over general regions

15.2: ~~1~~, ~~3~~, ~~5~~, ~~7~~, ~~11~~, ~~15~~, ~~16~~, ~~17~~, ~~19~~, ~~21~~, ~~32~~, ~~37~~, ~~45~~, ~~48~~, ~~51~~, ~~53~~, ~~57~~, ~~58~~

1. $\int_1^5\int_0^x 8x-2y\,dy\,dx = \int_1^5 8xy-y^2\Big|_0^x dx = \int_1^5 7x^2\,dx = \frac{7}{3}x^3\Big|_1^5 = \frac{868}{3}$

3. $\int_0^1\int_0^y xe^{y^3}\,dx\,dy = \int_0^1 \frac{x^2e^{y^3}}{2}\Big|_0^y dy = \int_0^1 \frac{y^2e^{y^3}}{2}\,dy = \frac{e^{y^3}}{6}\Big|_0^1 = \frac{1}{6}(e-1)$

5. $\int_0^1\int_0^{s^2} \cos(s^3)\,dt\,ds = \int_0^1 s^2\cos(s^3)\,ds = \frac{\sin(s^3)}{3}\Big|_0^1 = \frac{\sin 1}{3}$

7. $\int_0^4\int_0^{\sqrt{x}} \frac{y}{x^2+1}\,dy\,dx = \int_0^4 \frac{y^2}{2x^2+2}\Big|_0^{\sqrt{x}} dy = \int_0^4 \frac{x}{2x^2+2}\,dy = \frac{\ln(2x^2+2)}{4}\Big|_0^4 = \frac{\ln(17)}{4}$

15. $y=x-2 \quad x=y+2$
$x=y^2 \quad y^2-y-2=0 \quad \begin{cases} y=-1 \\ y=2 \end{cases}$
$\int_{-1}^2\int_{y^2}^{y+2} y\,dx\,dy = \int_{-1}^2 -y^3+y^2+2y\,dy = 2\frac{1}{4}$

17. $\int_0^1\int_0^{x^2} x\cos y\,dy\,dx = \int_0^1 x\sin y\Big|_0^{x^2} dx = \int_0^1 x\sin x^2\,dx = -\frac{\cos x^2}{2}\Big|_0^1 = \frac{1}{2}(1-\cos 1)$

19. $\int_0^1\int_1^{x+1} y^2\,dy\,dx + \int_1^4\int_1^{\frac{7-x}{3}} y^2\,dy\,dx = \int_0^1 \frac{y^3}{3}\Big|_1^{x+1} dx + \int_1^4 \frac{y^3}{3}\Big|_1^{\frac{7-x}{3}} dx =$

$= \int_0^1 \frac{(x+1)^3-1}{3}\,dx + \int_1^4 \frac{\left(\frac{7-x}{3}\right)^3-1}{3}\,dx = \int_0^1 \frac{x^3}{3}+x^2+x\,dx + \int_1^4 \frac{343}{81} - \frac{49x}{27} + \frac{7x^2}{27} - \frac{x^3}{81} - \frac{1}{3}\,dx =$

$= \frac{x^4}{12} + \frac{x^3}{3} + \frac{x^2}{2}\Big|_0^1 + \frac{343}{81}x - \frac{49x^2}{54} + \frac{7x^3}{81} - \frac{x^4}{324} - \frac{x}{3}\Big|_1^4 = \frac{11}{3}$

21. $\iint_D (2x-y)\,dA = 0$ D-circle around origin

45. $\int_0^1\int_0^y f(x,y)\,dx\,dy \quad D=\{(x,y)\mid 0\le x\le y,\ 0\le y\le 1\}$
$\int_0^1\int_x^1 f(x,y)\,dy\,dx \quad D=\{(x,y)\mid x\le y\le 1,\ 0\le x\le 1\}$

48. $\int_{-2}^2\int_0^{\sqrt{4-y^2}} f(x,y)\,dx\,dy \quad D=\{(x,y)\mid 0\le x\le\sqrt{4-y^2},\ -2\le y\le 2\}$
$\int_0^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\,dy\,dx \quad D=\{(x,y)\mid -\sqrt{4-x^2}\le y\le\sqrt{4-x^2},\ 0\le x\le 2\}$

51. $\int_0^1\int_{3y}^3 e^{x^2}\,dx\,dy = \int_0^3\int_0^{\frac{x}{3}} e^{x^2}\,dy\,dx = \int_0^3 \frac{xe^{x^2}}{3}\,dx = \frac{e^{x^2}}{6}\Big|_0^3 = \frac{1}{6}(e^9-1)$

53. $\int_0^1\int_{\sqrt{x}}^1 \sqrt{y^3+1}\,dy\,dx = \int_0^1\int_0^{y^2} \sqrt{y^3+1}\,dx\,dy = \int_0^1 \sqrt{y^3+1}\,y^2\,dy = \frac{2(y^3+1)^{\frac{3}{2}}}{9}\Big|_0^1 = \frac{2}{9}(2\sqrt{2}-1)$