05/03

Academic Reasoning 1

① $T: \mathbb{R}^n \to \mathbb{R}^m$ $\quad \{v_1, v_2, v_3\}$ - linearly independent $\to$ $\{T(v_1), T(v_2), T(v_3)\}$ - linearly independent

$n \geq 3$, $m \geq 3$

$T(x) = Ax \Rightarrow T(v_1) = Av_1, \; T(v_2) = Av_2, \; T(v_3) = Ax_3$

$\Rightarrow$ Dependency of $\{T(v_1), T(v_2), T(v_3)\}$ = dependency of columns of A

$\Rightarrow$ false, $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad \left.\begin{matrix} v_1 = e_1 \\ v_2 = e_2 \\ v_3 = e_3 \end{matrix}\right\} \Rightarrow \begin{matrix} T(v_1) = e_1 \\ T(v_2) = 2e_1 \\ T(v_3) = 3e_1 \end{matrix}$

$\{v_1, v_2, v_3\}$ - linearly independent
columns of A - linearly dependent
$\{T(v_1), T(v_2), T(v_3)\}$ - lin. dependent

② $\left.\begin{matrix} \{v_1, v_2, v_3\} \text{ - linearly independent} \\ \{v_3, v_4\} \text{ - linearly independent} \end{matrix}\right\} \to \{v_1, v_2, v_3, v_4\}$ - linearly independent

False, counter example - v_1 or v_2 - linearly dependent with v_4

$v_1 = e_1 \quad v_2 = e_2 \quad v_3 = e_3 \quad v_4 = c(v_1 \text{ or } v_2)$

$\Rightarrow \left.\begin{matrix} \{v_1, v_2, v_3\} \text{ - linearly independent} \\ \{v_3, v_4\} \text{ - linearly independent} \end{matrix}\right\}$ $\{v_1, v_2, v_3, v_4\}$ - linearly dependent
v_4 - linearly dependent with v_1 or v_2

concrete example: $v_1 = e_1, \; v_2 = e_2, \; v_3 = e_3, \; v_4 = e_1$

③ A - $n \times n$ matrix

$\exists b \in \mathbb{R}^n$ s.t. $Ax = b$ - indefinitely many solutions

$\to \exists c \in \mathbb{R}^n$ s.t. $Ax = c$ - no solutions

$\exists b \in \mathbb{R}^n$ s.t. $Ax = b$ - ∞ solutions

$\Rightarrow A$ - at least one column without pivot (Invertible Matrix Theorem)

$\Rightarrow \exists c \in \mathbb{R}^n$ s.t. $Ax = c$ - 0 solutions

④ A - reduced echelon form $= I$

$\to A^2$ - reduced echelon form $= I$

A - reduced echelon form $= I$ (Invertible Matrix Theorem)

$\Rightarrow A$ - invertible (Invertible Matrix Theorem)

$\Rightarrow A^2$ - invertible (Invertible Matrix Theorem)

$\Rightarrow A^2$ - reduced echelon form $= I$