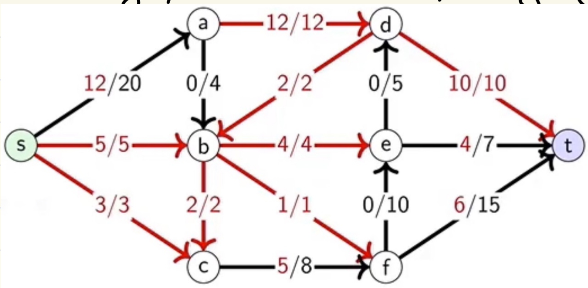
1.1. Network Maximum Flow

directed graph, source s, sink t, every edge – capacity $c(e)$



max flow $= 12+5+3=20$

$G=(V,E)$

capacity: $c: E \to \mathbb{N}$

s-t flow: $f: E \to \mathbb{N}$ s.t.

$\forall e \in E,\ f(e) \leq c(e)$

$\forall v \in (V-\{s,t\}),\ \sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$

value f: $v(f) = \sum_{e \text{ out of } s} f(e) = \sum_{e \text{ into } t} f(e)$

1.2 Minimum Cut

s-t cut: partition (A,B) of V s.t. $s \in A,\ t \in B$

cut capacity: $cap(A,B) = \sum_{e \text{ out of } A} c(e)$

net flow: across s-t cut (A,B) $\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e)$

Maximum Flow = Minimum Cut

2. Residual Graphs

original edge $e=(u,v) \in E$

residual edge $e^R = (v,u)$

residual capacity $c_f(e) = \begin{cases} c(e)-f(e) & \text{if } e \in E \\ f(e^R) & \text{if } e^R \in E \end{cases}$

residual graph $G_f = (V, E_f)$ s.t. $E_f = \{e \in E \mid f(e) < c(e)\} \cup \{e^R \mid e \in E \wedge f(e) > 0\}$

3.1 Ford-Fulkerson

Th| Flow value lemma

f – any flow, (A,B) – any s-t cut

$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e) = v(f)$

Proof| $v(f) = \sum_{e \text{ out of } s} f(e) + \sum_{v \in (A-\{s\})} \Big(\underbrace{\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ into } v} f(e)}_{0} \Big)$

$= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e)$

$\boxed{\text{Th}}$ Weak duality

f - any flow, (A,B) - any s-t cut

$v(f) \le cap(A,B)$

$\boxed{\text{Proof}}$ $v(f) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e) \overset{\text{(flow-value lemma)}}{\le} \sum_{e \text{ out of } A} f(e) \le \sum_{e \text{ out of } A} c(e) \overset{\text{(def of flow)}}{=} cap(A,B)$

```
function Ford-Fulkerson(G,s,t,c)
    for e in E do
        f(e) <- 0
    Gf <- residual graph of G
    while exists augmenting path P from s to t do
        f <- Augment(f,c,P)
        update Gf
    return f
```

finding path: DFS/BFS $O(n+m)$

for e in E do: $O(m)$

$C = cap(s, V - \{s\})$ - max possible flow

while exists augmenting path: $O(C)$

f <- Augment(f,c,P): $O(m)$

return f $\rightarrow$ total: $O(mC)$

```
function Augment(f,c,P)
    b <- Bottleneck(P,c)
    for e in P do
        if e in E then
            f(e) <- f(e) + b
        else
            f(eR) <- f(eR) - b
    return f
```

for e in P do: $O(m)$

return f $\rightarrow$ total: $O(m)$

3.2. Hospitals

Problem:

distribute n patients over k hospitals s.t. travel time $< 0.5h$, $\lceil \frac{n}{k} \rceil$ most patients

Solution:

$G = (V,E)$, $V = L \cup R \cup \{s,t\}$, L - hospitals ($|L| = k$), R - patients ($|R| = n$)

edge (u,v), $u \in L$, $v \in R$, if patient within range, capacity: 1

$\forall u \in L$: edge(s,u), capacity: $\lceil \frac{n}{k} \rceil$

$\forall v \in R$: edge(v,t), capacity: 1

Th| There is a valid assignment iff value of max flow in G = n.

Proof| Suppose there is a valid assignment A.

Flow f s.t.

$f(l,r)=1$ iff patient r assigned to hospital l in A

$f(r,t)=1$ for every patient r

$f(s,l)=$ number of patients assigned to hospital l in A

Flow is valid by capacity and flow conservation constraints.

$v(f)=n$

3.3. Bipartite matching

Problem:

sets A and B of n objects each, pair each item in A to 'suitable' one in B

Goal: find set of pairings that pairs every item in A to exactly one item in B

Solution:

$G=(V,E)$, $V=A\cup B\cup\{s,t\}$, $|A|=|B|=n$

$\forall a\in A$: edge (s,a), capacity: 1

$\forall b\in B$: edge (b,t), capacity: 1

edge (a,b), $a\in A$, $b\in B$, if (a,b) is 'suitable' pairing, capacity: 1

if $v(f)=n \rightarrow$ suitable matching is possible

3.4 Max-flow min-cut theorem

Th| Augmenting path theorem

Flow f is max flow iff there is no augmenting path.

Th| Max-flow min-cut theorem

Value of max flow is equal to capacity of min cut.

Proof|

1) $\exists(A,B)$ s.t. $v(f)=cap(A,B)$

$\downarrow$ weak duality lemma: $v(f)\le cap(A,B)$

2) Flow f is max flow.

$\downarrow$ contrapositive: $\neg 3\rightarrow\neg 2$

$\exists$ augmenting path $\rightarrow$ f is not max flow

3) There is no augmenting path relative to f.

To prove (3→1): If there is no augmenting path → $\exists(A,B)$ s.t. $v(f)=cap(A,B)$

Proof:

f-flow with no augmenting paths

A - set of nodes reachable from s in G_f

(A,B) - cut, $s\in A$, $t\notin A$ (no augmenting path to t in G_f)

$f(e)=c(e)$ for $\forall e=(u,v)$ out of A otherwise $v\in A$

$$\Rightarrow \sum_{e \text{ out of } A} f(e) = \sum_{e \text{ out of } A} c(e)$$

$f(e)=0$ for $\forall e=(u,v)$ into A otherwise $u\in A$

$$\Rightarrow v(f) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e) = \sum_{e \text{ out of } A} c(e) - 0 = cap(A,B)$$

3.5. Supply and Demand

Def | Circulation with Demand

$G=(V,E)$, capacity $c(e)$, $\forall e\in E$

node supply and demand $d(v)$, $\forall v\in V$, demand $d(v)>0$, supply $d(v)<0$

Def | Circulation

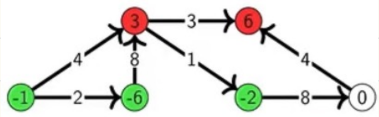
$\forall e\in E$: $0\le f(e)\le c(e)$

$$\forall v\in V: \sum_{e \text{ into } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$$

Problem: Circulation

Given (V,E,c,d) is there circulation?

Solution:



D=9

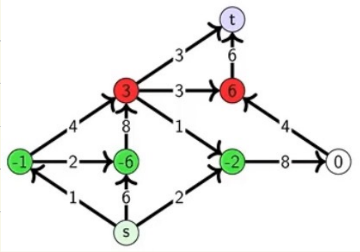
$\sum$ supply $=\sum$ demand $=D$

Add source s, sink t.

$\forall v$, $d(v)<0$, add edge $e=(s,v)$, $c(e)=-d(v)$

$\forall v$, $d(v)>0$, add edge $e=(v,t)$, $c(e)=d(v)$

Claim: G has circulation iff G' has max flow of D



3.6. Lower bounds

Problem: Hospitals with lower bound

+ each hospital gets at most C, at least c patients

<u>Def</u> Circulation with demand and lower bound

+ lower bounds $\delta(e), e \in E$

<u>Def</u> Circulation

* $\forall e \in E: \delta(e) \leq f(e) \leq c(e)$

Problem: Circulation

Given (V, E, δ, c, d) is there circulation?

Solution:

Remove lower bound k from edge $e = (u, v)$

$\Rightarrow d(u) += k, \; d(v) -= k, \; c(e) -= k$

3.7. Scaling max-flow

<u>Def</u> Scaling Ford-Fulkerson

1) $\forall e \in E, f(e) = 0$
2) Δ, smallest power of 2, larger than largest capacity leaving s, c^*
3) ignore edges if $c(e) < \Delta$
4) find s-t path P in G_f where $\forall e \in P, f(e) \leq c(e)$
5) augment flow along P by b
6) go to 4) while $\exists P$
7) half Δ while $\Delta > 1$, go to 2)

<u>Lemma 1</u> Outer loop (2-7) at most $1 + \lceil \log_2 c^* \rceil$ times

<u>Proof</u> $c^* \leq \Delta \leq 2c^*$, Δ halves each iteration.

<u>Lemma 2</u> At most $2m$ augmentations per outer loop iteration

<u>Proof</u> f'-flow after previous outer loop iteration ($\Delta' = 2\Delta$)

At most $\forall e$ used to increase flow, $v(f) \leq v(f') + m\Delta' = v(f') + 2m\Delta$

Each edge increases flow by at least Δ, at most $2m \Rightarrow O(m)$ per iteration

$\Rightarrow$ Runtime: $O(m \log c^*)$ augmentations, each augmentation $O(m)$ time

$\rightarrow$ total: $O(m^2 \log c^*)$

Scaling algorithm: use with large capacities.

Original bound still applies.

4. Projection Selection

Problem: Space shuttle equipment

Set E - possible space experiments, each with revenue p_j

Each experiment → set of instruments I_j

Cost of taking instrument i to space is c_i

Goal: Subset of experiments to maximize profit (revenue - costs)

Solution:

minimum (A,B) cut: A - experiments/instruments for space, B - not

$G = (V, E)$, $V = E' \cup I \cup \{s, t\}$

if $e' \in E'$ requires $i \in I$, add edge $e = (e', i)$, $c(e) = \infty$

$\forall j \in E'$: add edge $e = (s, j)$, $c(e) = p_j$

$\forall i \in I$: add edge $e = (i, t)$, $c(e) = c_i$

$cap(A,B) = \sum_{j \in B \wedge j \in E'} p_j + \sum_{i \in A \wedge i \in I} c_i$

Claim: Min-cut $(A,B) \Rightarrow A - \{s\}$ is optimal set of experiments

Proof profit $= \sum_{j \in A \wedge j \in E'} p_j - \sum_{i \in A \wedge i \in I} c_i = \underbrace{\sum_{j \in E'} p_j}_{\text{constant}} - \underbrace{\left(\sum_{j \in B \wedge j \in E'} p_j + \sum_{i \in A \wedge i \in I} c_i\right)}_{= cap(A,B) \text{ - minimal} \Rightarrow \text{maximal profit}}$ minimize to maximize profit