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Functions & continuity [1.5, 2.2, 2.5]

$a = \tilde{a} \to f(a) = f(\tilde{a})$ $\quad f(a) = b$

$\forall a \in A \to \exists b \in B$ $\quad f(\tilde{a}) = \tilde{b}$

$A, B \subseteq \mathbb{R}$ $\quad f: A \to B$

range: domain of $f(\tilde{a}) \subseteq B$

domain: domain of $a = A$

co-domain: domain of $b = B$

$f: A \to B \quad g: C \to D$

product $h(x) = f(x)g(x)$ $\quad h: A \cap C \to B \cap D$

composition $h(x) = (f \circ g)(x) = f(g(x))$ $\quad h: C \to B$

Inverse functions

$f, D, g = f^{-1}, R = \{r \in \mathbb{R} : \exists d \in D : f(d) = r\}$

$f(x) = y \leftrightarrow x = g(y)$

$f(f^{-1}(y)) = y \quad f^{-1}(f(x)) = x$

f^{-1}: f inverse

$f(x) = 3x + 5 \quad y = 3x + 5 \quad \frac{y-5}{3} = x \quad f^{-1}(x) = \frac{x-5}{3}$

$f(x) = x^2 \quad f: \mathbb{R} \to [0; \infty)$

$f(-1) = f(1) = 1$ → not well-defined inverse

$f(x) = \sqrt{x} \quad f: [0; \infty) \to \mathbb{R} \quad R = [0; \infty)$

$f^{-1}(x) = x^2 \quad f^{-1}: [0; \infty) \to [0; \infty)$

$f(x_1) \neq f(x_2) \quad x_1 \neq x_2$

f^{-1}: reflection of f on $y = x$

$f(x) = x^2 - 4x \quad x \leq 2$

$y = x^2 - 4x \quad x^2 - 4x - y = 0 \quad D = 16 + 4y$

$x = 2 \pm \sqrt{4+y} \quad x_{1,2} = \frac{4 \pm \sqrt{16+4y}}{2}$

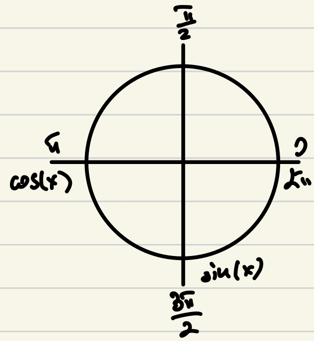
$f^{-1}(x) = 2 - \sqrt{4+x}$

Inverse trigonometric functions

$\arcsin(x) = y \Leftrightarrow \sin(y) = x \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$\arccos(x) = y \Leftrightarrow \cos(y) = x \quad 0 \leq y \leq \pi$

$\arctan(x) = y \Leftrightarrow \tan(y) = x \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$

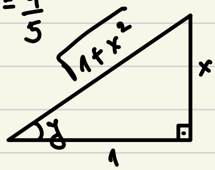


$\cos\left(\arcsin\left(\frac{3}{5}\right)\right) = \frac{4}{5}$

$\cos(\arctan(x))$

$y = \arctan(x)$

$\tan(y) = x$



$\cos(y) = \frac{1}{\sqrt{1+x^2}}$

Limits

$\lim_{x \to a} f(x) = L \qquad \lim_{x \to a} f(x) = L \Leftrightarrow \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

$\lim_{x \to a^-} f(x) = L$ (left)

$\lim_{x \to a^+} f(x) = L$ (right)

continuous at $x = a \Leftrightarrow \lim_{x \to a} f(x) = f(a)$

$f(x) = \begin{cases} \frac{x}{|x|} & x \neq 0 \\ a & x = 0 \end{cases} \qquad \left.\begin{array}{l} \lim_{x \to 0^-} f(x) = -1 \\ \lim_{x \to 0^+} f(x) = 1 \end{array}\right\}$ not continuous (can't be)

$f(x) = \begin{cases} x + a & x < 1 \\ 2^x & x \geq 1 \end{cases} \qquad \begin{array}{l} \lim_{x \to 1^-} f(x) = 1 + a \quad \Rightarrow a = 1 \\ \lim_{x \to 1^+} f(x) = 2^1 = 2 \end{array}$

$f(x) = \begin{cases} \frac{x^2 - x}{x^2 - 1} & x \neq 1 \\ 1 & x = 1 \end{cases} \qquad \begin{array}{l} \lim_{x \to 1^-} f(x) = \frac{x}{x+1} = \frac{1}{2} \\ \lim_{x \to 1^+} f(x) = \frac{x}{x+1} = \frac{1}{2} \end{array}$

$f(x) = \begin{cases} x^2 & x < -1 \\ x & -1 \leq x < 1 \\ \frac{1}{x} & x \geq 1 \end{cases} \qquad \left.\begin{array}{l} \lim_{x \to -1^-} f(x) = 1 \\ \lim_{x \to -1^+} f(x) = -1 \end{array}\right\}$ discontinuous at $x = -1$

$\lim_{x \to 1^-} f(x) = 1 = \lim_{x \to 1^+} f(x) \rightarrow$ continuous at $x = 1$

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Limits of functions [2.3, 2.4, 2.6]

Tolerance

$f(x) = \dfrac{x^2+4x-5}{-x^2+6x-5} = -\dfrac{(x+5)(x-1)}{(x-5)(x-1)} \Rightarrow \lim_{x\to a} f(x) = -\dfrac{x+5}{x-5}$ $\quad \lim_{x\to 1} = -\dfrac{6}{-4} = 1.5$

$\lim_{x\to a} f(x) = L \quad \overset{\text{take}}{\forall \varepsilon > 0} \to \overset{\text{find}}{\exists \delta > 0}$

$0 < |x-a| < \delta \to |f(x) - L| < \varepsilon$

$x \in (a-\delta; a+\delta) \quad f(x) \in (L-\varepsilon, L-\varepsilon)$

$\lim_{x\to 0} \sin\left(\frac{1}{x}\right) \to$ limit doesn't exist (DNE)

exam notation

$f(x) = x^2$

$\lim_{x\to a} f(x) = a^2$

$|f(x) - f(a)| < \varepsilon$

$|x^2 - a^2| = |(x-a)(x+a)| < \varepsilon$

$|x-a| < \dfrac{\varepsilon}{|x+a|} \leq \dfrac{\varepsilon}{|\delta + 2a|}$

$|x+a| = |x-a+2a| \leq |x-a| + |2a| < \delta + |2a|$

$|x-a| < \delta \Rightarrow |f(x) - f(a)| = |x-a|\,|x+a| \leq \delta(\delta + |2a|) < \varepsilon$

$\delta \leq 1 \leq \delta(1 + |2a|) \quad \delta \leq \dfrac{\varepsilon}{1+|2a|} \quad \min\left(1, \dfrac{\varepsilon}{1+|2a|}\right)$

$\varepsilon > 0, \ \delta = \min\left(1, \dfrac{\varepsilon}{1+2|a|}\right)$ $\quad |x^2 - a^2| = |x-a|\,|x+a| < \dfrac{\varepsilon}{1+2|a|} \cdot (1+2|a|) = \varepsilon$

$0 < |x-a| < \delta \to -\delta < x-a < \delta$

$-1 < x-a < 1$

$-1 + 2a < x+a < 1+2a \qquad |x+a| < 1+2|a|$

Left-limit

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\quad a-\delta < x < a \to |f(x) - L| < \varepsilon$

Right-limit

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\quad a < x < a+\delta \to |f(x) - L| < \varepsilon$

$f(x) = x^2 \qquad \delta = \dfrac{4\varepsilon}{1+2\varepsilon} \qquad |x-1| < \delta \to |x^2 - 1| < \frac{1}{2}$

$|x-1| < \delta \to |x^2 - 1| < \frac{1}{2} \qquad\qquad |x-1|\,|x+1| < \frac{1}{2}$

$a=2$

$|x-a|<\delta$ $\qquad |x-2|<\delta = \frac{4\varepsilon}{2\varepsilon+1}$

$|f(x)-L|<\varepsilon \qquad \left|\frac{1}{x}-\frac{1}{a}\right|<\varepsilon \qquad \frac{|x-a|}{|xa|}<\varepsilon \qquad \frac{|x-2|}{|2x|}<\varepsilon$

$f(x)=x^2$

$\varepsilon = \frac{1}{2} \to \delta \qquad |x-1|<\delta$

$|x+1| < |x-1+2| < |x-1|+2$

$|x^2-1|<\frac{1}{2} \qquad |x+1||x-1|<\frac{1}{2}$

$|x+1||x-1| < \delta(\delta+2) < \delta(1+2) < \frac{1}{2} \qquad \delta = \frac{1}{6}$

<u>Calculation rules</u>

$\lim_{x\to a} f(x) = L \qquad \lim_{x\to a} g(x) = M$

$\lim_{x\to a} cf(x) = cL \qquad \lim_{x\to a} f(x)g(x) = LM$

$\lim_{x\to a} f(x)+g(x) = L+M \qquad \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$

<u>Substitution rule</u>

$g: A \to B$

$f: B \to C$

$\lim_{x\to a} g(x) = L$

$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(L)$

$f(x) = \begin{cases} 1, & x \neq 0 \\ 0, & x=0 \end{cases} \qquad g(0)=0$

$\lim_{x\to 0} f(g(x)) = f(0) = 0$

$\lim_{x\to 0} f(x) = 1$

$\lim_{x\to 2} \sin\left(\frac{\pi x^2 - 3\pi x + 2\pi}{2x-4}\right) = \sin\left(\lim_{x\to 2} \frac{\cancel{(x-2)}(x-1)\pi}{2\cancel{(x-2)}}\right) = \sin\left(\frac{\pi}{2}\right) = 1$

Squeeze Theorem

$f(x) \le g(x) \le h(x)$

$\lim_{x\to a} f(x) = 2 = \lim_{x\to a} h(x) \Rightarrow \lim_{x\to a} g(x) = 2$

$\lim_{x\to 0} \sin\left(\frac{1}{x}\right) = DNE$

$\lim_{x\to 0} \sin\left(\frac{1}{x}\right) x = 1$

$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$

$-1 \le \sin\left(\frac{1}{x}\right) \le 1$

$\lim_{x\to 0} \quad -x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$

$0 \le \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) \le 0$

Indeterminate form $\frac{0}{0}$

$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$

$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ but may exist

$p(a) = q(a) = 0$

$\lim_{x\to a} \frac{p(x)}{q(x)}$ cancel out $(x-a)$

$\lim_{x\to 3} \frac{3x^2-10x+3}{x^2-2x-3} = \lim_{x\to 3} \frac{\cancel{(x-3)}(3x-1)}{\cancel{(x-3)}(x+1)} = \frac{8}{4} = 2$

$\lim_{x\to 4} \frac{a+b\sqrt{c}}{f(x)}$, multiply by $\frac{a-b\sqrt{c}}{a-b\sqrt{c}}$

$\lim_{x\to 0} \frac{2-\sqrt{4+x}}{x} \cdot \frac{2+\sqrt{4+x}}{2+\sqrt{4+x}} = \lim_{x\to 0} \frac{\cancel{4}-\cancel{4}-x}{2x+x\sqrt{4+x}} =$

$= \lim_{x\to 0} \frac{-\cancel{x}}{\cancel{x}(2+\sqrt{4+x})} = \frac{-1}{2+\sqrt{4}} = -\frac{1}{4}$

$f(x) = \frac{8x-24}{2x-6} = \frac{8(x-3)}{2(x-3)} \qquad \lim_{x\to 3} f(x) = 4$

$f(x) = \frac{\sqrt{4+x}-\sqrt{4-x}}{x} \cdot \frac{\sqrt{4+x}+\sqrt{4-x}}{\sqrt{4+x}+\sqrt{4-x}} = \frac{\cancel{4}+x-\cancel{4}+x}{x(\sqrt{4+x}+\sqrt{4-x})}$

$\lim_{x\to 0} f(x) = \frac{2\cancel{x}}{\cancel{x}(\sqrt{4+x}+\sqrt{4-x})} = \frac{1}{2}$

$$\lim_{h\to 0} \frac{(h-3)^2-9}{h} = \lim_{h\to 0} \frac{h^2-6h+\cancel{9}-\cancel{9}}{h} = \lim_{h\to 0} \frac{\cancel{h}(h-6)}{\cancel{h}} = -6$$

$$\lim_{h\to 0} \frac{\sqrt{9+h}-3}{h} \cdot \frac{\sqrt{9+h}+3}{\sqrt{9+h}+3} = \lim_{h\to 0} \frac{\cancel{9}+h-\cancel{9}}{h\left(\sqrt{9+h}+3\right)} = \frac{1}{6}$$

$$\lim_{x\to 3} \frac{\frac{1}{x}-\frac{1}{3}}{x-3} = \lim_{x\to 3} \frac{\cancel{x-3}}{3x\left(\cancel{x-3}\right)} = \frac{1}{9}$$

$$\lim_{t\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t} \cdot \frac{\sqrt{1+t}+\sqrt{1-t}}{\sqrt{1+t}+\sqrt{1-t}} = \lim_{t\to 0} \frac{\cancel{1}+t-\cancel{1}+t}{t\left(\sqrt{1+t}+\sqrt{1-t}\right)} = \frac{2}{2} = 1$$

$$\lim_{x\to 2} \frac{x^4-4x^3+3x^2+4x-4}{x^3-x^2-8x+12} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x^3-2x^2-x+2)}{\cancel{(x-2)}(x^2+x-6)} =$$

$$= \lim_{x\to 2} \frac{\cancel{(x-2)}(x-1)(x+1)}{\cancel{(x-2)}(x+3)} = \frac{3}{5}$$

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Asymptotes [2.2]

$$\lim_{x\to 2} \frac{1}{x} = \frac{1}{2}$$

$$\left|\frac{1}{x} - \frac{1}{2}\right| \le \varepsilon$$

$$\left|\frac{1}{x} - \frac{1}{2}\right| = \left|\frac{2-x}{2x}\right| = \frac{|2-x|}{|2x|}$$

$|x| = |-x| \quad \forall x \in \mathbb{R}$

$\delta = \frac{1}{2} \qquad \delta = \min\left(\frac{1}{2}, 3\varepsilon\right)$

$-\frac{1}{2} < x-2 < \frac{1}{2} \quad | +2$

$\frac{3}{2} < x < \frac{5}{2}$

$\frac{1}{5} < \frac{1}{2x} < \frac{1}{3}$

$\frac{1}{|2x|} < \frac{1}{3} \qquad \frac{|x-2|}{|2x|} < \frac{\delta}{3} = \varepsilon \Rightarrow \delta = 3\varepsilon$

$\delta = \frac{4\varepsilon}{1+2\varepsilon}$

$|x-2| < \delta$

$|f(x) - \frac{1}{2}| = \left|\frac{1}{x} - \frac{1}{2}\right| = \frac{|2-x|}{|2x|} < \frac{\delta}{|2x|}$

$\frac{4\varepsilon}{2(1+2\varepsilon)|x|} < \varepsilon \quad |x| > 1$

$|x-2| < \delta$

$-\delta < x-2 < \delta$

$2-\delta < x < 2+\delta$

$\frac{|2-x|}{|2x|} < \frac{\delta}{|2x|} = \frac{\varepsilon}{(\varepsilon+\frac{1}{2})|x|} = \frac{\varepsilon}{(\varepsilon+\frac{1}{2})(2-\delta)} = \frac{\varepsilon}{2(\varepsilon+\frac{1}{2})\frac{2}{1+2\varepsilon}} = \varepsilon$

$f(x) = \frac{2x^2 - 3x^3}{x + 3x^3} \qquad \lim_{x\to\pm\infty} f(x) = -1$

$f(x) = \arctan(x) \qquad \lim_{x\to-\infty} f(x) = -\frac{\pi}{2} \qquad \lim_{x\to\infty} f(x) = \frac{\pi}{2}$

Limits to infinity

$\left.\begin{array}{l} \lim_{x\to\infty} f(x) = L \\ \lim_{x\to-\infty} f(x) = L \end{array}\right\} \lim_{x\to\pm\infty} f(x) = L$

horizontal asymptote

$\forall \varepsilon > 0 \to \exists N > a \qquad \forall \varepsilon > 0 \to \exists N < b$

$x > N \to |f(x) - L| < \varepsilon \qquad x < N \to |f(x) - L| < \varepsilon$

$\lim_{x\to\infty} f(x) \quad (a, \infty) \qquad \lim_{x\to-\infty} f(x) \quad (-\infty, b)$

Evaluating Limits

$$\lim_{x\to\infty} \frac{1}{x^r} = 0, \quad r>0 \qquad \lim_{x\to\infty} \arctan(x) = \frac{\pi}{2} \qquad \lim_{x\to\infty} e^{-ax} = 0, \quad a>0$$

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0, \quad r>0 \qquad \lim_{x\to-\infty} \arctan(x) = -\frac{\pi}{2} \qquad \lim_{x\to\infty} \frac{1}{\ln(x)} = 0$$

$$\lim_{x\to\infty} f(x) = L, \quad \lim_{x\to\infty} g(x) = M$$

$$\lim_{x\to\infty} cf(x) = cL \qquad \lim_{x\to\infty} f(x)g(x) = LM$$

$$\lim_{x\to\infty} f(x)+g(x) = L+M \qquad \lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M\neq 0$$

$$\lim_{x\to\pm\infty} g(x) = b$$

f - continuous at b

$$\lim_{x\to\pm\infty} f(g(x)) = f\left(\lim_{x\to\pm\infty} g(x)\right) = f(b)$$

$$f(x) \leq g(x) \leq h(x)$$

$$\lim_{x\to\pm\infty} f(x) = L = \lim_{x\to\pm\infty} h(x) \Rightarrow \lim_{x\to\pm\infty} g(x) = L$$

$$\operatorname{sinc}(x) = \frac{\sin(x)}{x} = \frac{\sin(\pi x)}{\pi x} \qquad f(x) = \frac{1}{\pi x} \qquad h(x) = -\frac{1}{\pi x} \qquad h(x) \leq \frac{\sin(\pi x)}{\pi x} \leq f(x)$$

$$\lim_{x\to\infty} x - \sqrt{x^2+1} \cdot \frac{x+\sqrt{x^2+1}}{x+\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{\cancel{x^2} - \cancel{x^2} - 1}{x+\sqrt{x^2+1}} = 0 \qquad 0 \leq \lim_{x\to\pm\infty} \frac{\sin(\pi x)}{\pi x} \leq 0$$

$$\lim_{x\to\infty} \frac{2x+\sqrt{x}-x^3}{3x^3+x^2} \cdot \frac{x^3}{x^3} = \lim_{x\to\infty} \frac{\cancel{\frac{2}{x^2}}^{\to 0} + \cancel{\frac{\sqrt{x}}{x^3}}^{\to 0} - 1}{3 + \cancel{\frac{1}{x}}^{\to 0}} = -\frac{1}{3}$$

$$\lim_{x\to\infty} \frac{-x+3\sin(x^2)}{\sqrt{x+4x^2}} \cdot \frac{x}{x} = \lim_{x\to\infty} \frac{-1 + \cancel{\frac{3\sin(x^2)}{x}}^{\to 0}}{\sqrt{\cancel{\frac{1}{x}}^{\to 0}+4}} = \frac{-1}{\sqrt{4}} = -\frac{1}{2}$$

Vertical Asymptotes

$\lim_{x\to a} f(x) = \pm\infty$

$\lim_{x\to a} f(x) = \infty$

$\forall M>0 \to \exists \delta>0$

$0<|x-a|<\delta \to f(x)>M$

$\lim_{x\to a} f(x) = -\infty$

$\forall N>0 \to \exists \delta>0$

$0<|x-a|<\delta \to f(x)<N$

$$\lim_{x\to 2} \frac{x^2+2x+1}{x^2-x-2} = \lim_{x\to 2} \frac{(x+1)^x}{(\cancel{x+1})(x-2)} = \text{DNE}$$

$\to 2^-$ gives $-\infty$

2^+ gives $+\infty$

$$f(x) = \frac{x^2+2x+1}{x^2-x-2} = \frac{(x+1)^2}{(x+1)(x-2)} = \frac{x+1}{x-2}$$ at 2

$$\lim_{x\to 0^-} \arctan\left(\frac{1}{x}\right) = -\frac{\pi}{2}$$

$$\lim_{x\to\infty} x-\sqrt{x^2-3x}\cdot\frac{x+\sqrt{x^2-3x}}{x+\sqrt{x^2-3x}} = \lim_{x\to\infty} \frac{\cancel{x^2}-\cancel{x^2}+3x}{x+\sqrt{x^2-3x}} = \frac{3}{2}$$

$$\lim_{x\to-\infty} x-\sqrt{x^2-3x} = \lim_{x\to-\infty} \frac{3x}{x+\sqrt{x^2-3x}} = -\infty$$

Oblique asymptotes

$y = ax+b$

$\lim_{x\to\pm\infty} f(x)-ax-b=0$

$f(x) = \frac{P(x)}{Q(x)}$

$$f(x) = \frac{\frac{x^3}{2}+x^2+\frac{23x}{2}-24}{x^2-25} = \boxed{\frac{x}{2}+1} + \boxed{\frac{x+1}{x^2-25}} \to \lim_{x\to\pm\infty} = 0$$

oblique asymptote equation

$\deg(P) = \deg(Q)+1 \to a, b \in \mathbb{R}$

s.t. $f(x) = ax+b+\frac{R(x)}{Q(x)}$

$\deg(R)<\deg(Q)$

$y = ax+b$

Growth rates

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$

f grow faster than g iff $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$

g grow faster than f iff $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$

grow at comparable rate $\lim_{x\to\infty} \frac{f(x)}{g(x)} = c \neq 0$

$y = ax + b$

$\lim_{x\to\pm\infty} \left(f(x) - (ax+b) \right) = 0$

$\deg(P) = \deg(Q+1)$

$f(x) \sim ax + \frac{1}{x^{k+1}}$

$\sim ax$ at $\pm\infty$

$\lim_{x\to\pm\infty} \frac{f(x)}{x}$

$y = ax + b$

$\lim_{x\to\pm\infty} f(x) - (ax+b) = 0$

$\lim_{x\to 3^-} \frac{x^2+4x}{x^2-2x-3} = \frac{x(x+4)}{(x-3)(x+1)} = -\infty$

$\lim_{x\to 5^-} \frac{x+1}{x-5} = -\infty$

$\lim_{t\to -\infty} \frac{3t^2+t}{t^3-4t+1} = 0$

$\lim_{x\to 0^+} \arctan(\ln(x)) = \arctan(-\infty) = -\frac{\pi}{2}$

$f(x) = \frac{\sin x + 2x^2\sqrt{x} - x\sqrt{x}}{x\sqrt{x} - 2}$ $\qquad f(x) = \frac{\sin x + 4x - 2}{x\sqrt{x} - 2} + \boxed{2x - 1}$

$\lim_{x\to\infty} f(x) = \infty$ no horizontal $\qquad$ oblique asymptote

$x\sqrt{x} = 2 \quad x = \log_3 4$

$x^{\frac{3}{2}} = 2 \quad \lim_{x\to\log_3 4} f(x) = \pm\infty$ vertical asymptote

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Linearization and extreme values [2.7, 3.10, 4.1, 4.2]

$f(x)=x^2$

$\Delta x \neq 0$

$\Delta y = f(x+\Delta x)-f(x)$

$\frac{\Delta y}{\Delta x} = \frac{\cancel{x^2}+2x\cancel{\Delta x}+\Delta x^2-\cancel{x^2}}{\cancel{\Delta x}} = 2x+\Delta x$

$f(x)=|x-3|$

$\lim_{\Delta x\to 0} \frac{f(3+\Delta x)-f(3)}{\Delta x} = DNE$

Derivative

$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$

$f(x)=1-2x^2 \qquad a=-1$

$\frac{\Delta y}{\Delta x} = \frac{f(x)-f(-1)}{x-(-1)}$

$\frac{dy}{dx} = \lim_{x\to -1}\frac{f(x)-f(-1)}{x-(-1)}$

$f(x)=x^4-3x, \quad x=-1$

$y=f(-1)+f'(-1)(x+1)$

$y=4-7(x+1)$

f-differentiable at a iff $\exists f'(a)$

$f(x)=|x|$

↳ differentiable on $(-1,0)$

↳ not diff. on $(-1,1)$

$y=f(x), \ (a,f(a))$

$m=\lim_{x\to a}\frac{f(x)-f(a)}{x-a} \qquad y=f(a)+f'(a)(x-a)$

tangent line: $y=m(x-a)+b$

$(f(x)+g(x))'=f'(x)+g'(x)$

$(f(x)g(x))'=f'(x)g(x)+f(x)g'(x)$

$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$

$(f(g(x)))'=f'(g(x))g'(x)$

$\frac{d}{dx}x^r = r\cdot x^{r-1}$

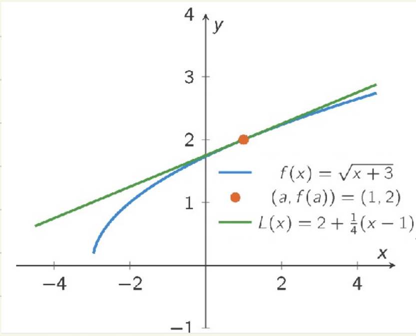
$\frac{d}{dx}\ln(x)=\frac{1}{x}$

$\frac{d}{dx}\sin(x)=\cos(x)$

$\frac{d}{dx}\cos(x)=-\sin(x)$

$\frac{d}{dx}e^x=e^x$

Linear approximations



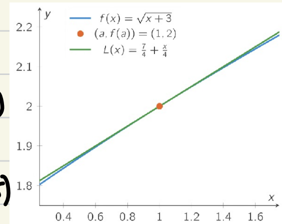
$$L(x) = f(a) + f'(a).(x-a)$$

$f(x) = \sqrt{x+3}$ $(1, 2)$

$L(x) = 2 + \frac{1}{4}(x-1)$

$\sqrt{3.98} = f(0.98) \approx L(0.98)$
$= 1.995$

$\sqrt{4.05} = f(1.05) \approx L(1.05)$
$= 2.0125$

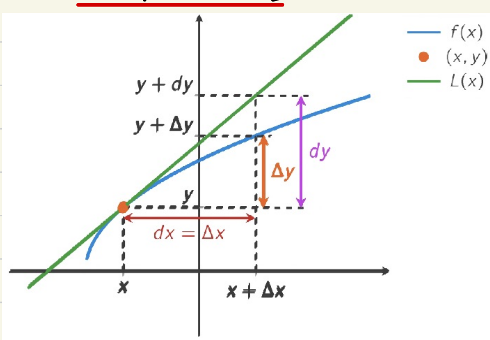


$f(x) = x\sqrt{x}$ $a=1$ $f(a)=1$
$f'(x) = \frac{d}{dx} x^{\frac{3}{2}} = \frac{3}{2}\sqrt{x}$ $f'(a) = \frac{3}{2}$
$\Big\}$ $L(x) = 1 + \frac{3}{2}(x-1) = \frac{3}{2}x - \frac{1}{2}$

$f(x) = \cos x$ $a=0$ $f(a)=1$
$f'(x) = -\sin x$ $f'(a)=0$
$\Big\}$ $L(x) = 1$

$f(x) = \sqrt{4-x}$ $a=0$ $f(a)=2$
$f'(x) = -\frac{1}{2\sqrt{4-x}}$ $f'(a) = -\frac{1}{4}$
$\Big\}$ $L(x) = 2 - \frac{x}{4}$
$\sqrt{3.94} = L(0.06) = 2 - \overset{0.015}{\cancel{\frac{0.06}{4}}} = 1.985$

Differentials



$df = f'(x)dx$

$y = f(x)$

uncertainty in $x \to dx$

uncertainty in $y \to |dy| = |f'(x)dx|$

relative error:

$\left|\frac{dx}{x}\right|$ and $\left|\frac{\Delta y}{y}\right| \approx \left|\frac{dy}{y}\right|$

relative error -> 3% = $\left|\frac{dr}{r}\right|$ $\qquad \frac{dV}{V} = \frac{4\pi r^2 dr}{\frac{4}{3}\pi r^3} = 3\frac{dr}{r} = 9\%$

$V = \frac{4}{3}\pi r^3 \Rightarrow dV = f'(r)dr = 4\pi r^2 dr$

$y = \cos(x) \qquad dy = f'(x)dx = \underbrace{-\sin(x)}_{x=\frac{1}{2}\pi}dx$

fixed absolute error

$\sqrt[3]{1001}$

$f(x) = \sqrt[3]{1000+x} \quad f'(x) = \frac{1}{3\sqrt[3]{(1000+x)^2}}$

$a=0 \quad f(a)=10 \quad f'(a) = \frac{1}{300}$

$L(x) = 10 + \frac{x}{300}$

$\sqrt[3]{1001} = L(1) = 10\frac{1}{300}$

$e^{0.1}$

$f(x) = e^{1-x} \quad f'(x) = -e^{1-x}$

$a=1 \quad f(a)=1 \quad f'(a) = -1$

$L(x) = 1-(x-1) = 2-x$

$e^{0.1} = L(0.9) = 1.1$

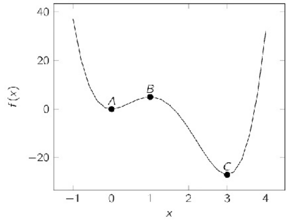
$y = e^{\frac{x}{10}} \quad |dy| = |f'(x)dx| \quad |dy| = \left|\frac{e^{\frac{x}{10}}}{10}dx\right|$

$x=0 \quad dx=0.1 \quad |dy| = 0.01$

Extreme Values

$c \in D$, $f(c)$:

- absolute maximum of f on D if $\forall x \in D$, $f(c) \geq f(x)$
- absolute minimum of f on D if $\forall x \in D$, $f(c) \leq f(x)$
- local maximum of f if $f(c) \geq f(x)$ when x is near c.
- local minimum of f if $f(c) \leq f(x)$ when x is near c.



Point A is a local minimum and point B is a local maximum. Point C is both a local and absolute minimum.

Fermat's Theorem (Critical points):
f - local extremum at c and $\exists f'(c) \to f'(c)=0$
critical point: $f'(c)=0$ or DNE

Extreme Value Theorem:
f - continuous on $[a,b]$
$\exists c$, $f(c)$ - absolute maximum value
$\exists d$, $f(d)$ - absolute minimum value

Mean Value

f - continuous on $[a,b]$; differentiable on (a,b)
$\to \exists c \in (a,b)$ s.t. $f'(c)=\dfrac{f(b)-f(a)}{b-a}$

$f'(x)=0$ for $\forall x \in (a,b) \Rightarrow f$ - constant on (a,b)
$f'(x)=g'(x)$ for $\forall x \in (a,b) \Rightarrow f-g$ - constant on (a,b)

$f(x)=2x^3-3x^2-12x+1$ $[-2,3]$ $\underset{\text{abs. min.}}{f(-2)=-3}$ $\underset{\text{abs. max.}}{f(3)=-8}$
$f'(x)=6x^2-6x-12=6(x-2)(x+1)$ $f(2)=-19$ $f(-1)=8$

$g(x)=x+\dfrac{1}{x}$ $[0.2,4]$ $\overset{\text{abs. max.}}{g(0.2)=5.2}$ $g(4)=4.25$
$g'(x)=1-\dfrac{1}{x^2}=\dfrac{x^2-1}{x^2}=\dfrac{(x-1)(x+1)}{x^2}$ $\underset{\text{abs. min.}}{g(1)=2}$ $x=0 \to$ vertical asymptote
$g(-1)=-2$

$x^3+e^x=0 \to$ no asymptotes
$\dfrac{d}{dx}(x^3+e^x)=3x^2+e^x>0 \Rightarrow$ always increasing
$\Rightarrow$ pass through y-axis just once

22/11

Implicit differentiation and Newton's method [3.5, 4.8]

$f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)} = \sin(x)(\cos(x))^{-1}$

$f'(x) = (\sin(x))'(\cos(x))^{-1} + \sin(x)(\cos(x))^{-1'}$

$= \frac{\cancel{\cos(x)}}{\cancel{\cos(x)}}^{1} - \frac{\sin(x)}{\cos^2(x)} \cdot (-\sin(x)) = 1 + \frac{\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$

$(\tan(x))' = \frac{1}{\cos^2(x)}$

$f(x) = 3^x = e^{\ln(3)x}$

$f'(x) = \ln(3)e^{\ln(3)x} = \ln(3) \cdot 3^x$

$(a^x)' = \ln(a)\, a^x$

$(x^x)' = (1+\ln(x))x^x$

$f(x) = x^x = e^{\ln(x)x}$

$f'(x) = e^{\ln(x)x} \cdot (\ln(x)x)' = e^{\ln(x)x}\left(\frac{1}{\cancel{x}} \cdot \cancel{x} + \ln(x) \cdot 1\right) = x^x(1+\ln(x))$

Implicit differentiation

$x^2 + y^2 = 25 \qquad \frac{d}{dx}(x^2+y^2) = 0 \qquad 2x + 2y\frac{dy}{dx} = 0 \qquad \frac{dy}{dx} = -\frac{x}{y}$

$(3,-4)\ \frac{dy}{dx} = \frac{3}{4} \qquad (-4,3)\ \frac{dy}{dx} = \frac{4}{3} \qquad (1,2)\ \frac{dy}{dx} = \frac{4}{5}$

$x^3 + y^3 = \frac{9}{2}xy \qquad 3x^2 + 3y^2\frac{dy}{dx} - \frac{9}{2}y - \frac{9}{2}x\frac{dy}{dx} = 0 \qquad \frac{dy}{dx} = \frac{\frac{9}{2}y - 3x^2}{3y^2 - \frac{9}{2}x}$

$\frac{dy}{dx} = 0 \qquad \frac{9}{2}y = 3x^2$

$y = \frac{2}{3}x^2$

$x^3 + y^3 = \frac{9}{2}xy \to x^3 + \frac{8}{27}x^6 = 3x^3$

$\frac{8}{27}x^6 - 2x^3 = 0 \qquad 2x^3\left(\frac{4}{27}x^3 - 1\right) = 0$

$y_2 = \frac{2}{3}\left(\frac{3}{\sqrt[3]{4}}\right)^2$

$= \frac{3}{\sqrt[3]{2}}$

$x_1 = 0 \qquad x_2 = \frac{3}{\sqrt[3]{4}} \approx 1.89 \qquad y_2 = \frac{3}{\sqrt[3]{2}} \approx 2.38$

$y_1 = 0$

Derivatives of inverse trigonometric functions

$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$

$\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}}$

$f(x) = \arctan(x)$ $a=1$ $f(a) = \frac{\pi}{4}$

$f'(x) = \frac{1}{1+x^2}$ $f'(a) = \frac{1}{2}$ $\Big\} L(x) = \frac{\pi}{4} + \frac{1}{2}(x-1)$

$x^2 - xy - y^2 = 1$ $2x - y - x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$ $\frac{dy}{dx} = \frac{2x-y}{x+2y} = \frac{3}{4}$

$(2,1)$

$y = \arctan(\sqrt{x})$ $y' = \frac{1}{2(1+x)\sqrt{x}}$

Newton's method

$f(x) = x^3 - 2x - 5$ $f(x) = 0$ $f(2.1) = 0.061$

$f'(x) = 3x^2 - 2$ $f'(2.1) = 11.23$

$L(x) = 0.061 + 11.23(x-2.1) = 0$ $x = 2.0945681...$ $f(x) = 0.0000\,1857...$

Newton-Raphson method

$f(x) = 0$ $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ $f(\alpha) = 0$ $\lim_{n\to\infty} x_n = \alpha$

$\sqrt[3]{2}$

$f(x) = 0$ $f(x) = x^3 - 2$ $f'(x) = 2x^3$

$x_0 = 1$ $f(x_0) = -1$ $f'(x_0) = 3$

$x_1 = \frac{4}{3}$ $f(x_1) = 0.\overline{370}$ $f'(x_1) = 5.\overline{33}$

$x_2 = 1.26\overline{388}$ $f(x_2) = 0.018955$ $f'(x_2) = 4.7922\overline{45370}$

$x_3 = 1.259933$ $f(x_3) = 0.0000592593$

$f(x), f'(x), f''(x)$ – continuous for x around α
$f(\alpha)=0, \; f'(\alpha)\neq 0$
$\Rightarrow$ x_0 – close to α $\rightarrow \lim_{n\to\infty} x_n = \alpha$

$\frac{2}{x} - x^2 + 1 = 0$

$x_n - \frac{f(x_n)}{f'(x_n)}$ $\qquad f(x) = \frac{2}{x} - x^2 + 1$ $\qquad f'(x) = -\frac{2}{x^2} - 2x$

$x_1 = 2$ $\qquad f(x_1) = -2$ $\qquad f'(x_1) = -\frac{9}{2} = -4.5$

$x_2 = 2 - \frac{-2}{-\frac{9}{2}} = \frac{14}{9}$ $\qquad f(x_2) = -0.134$ $\qquad f'(x_2) = -3.94$

$x_3 = \frac{14}{9} - \frac{-0.134}{-3.94} = 1.522$

$\sqrt[4]{75} = 2.94283096$

$x_n - \frac{f(x_n)}{f'(x_n)}$ $\qquad f(x) = x^4 - 75$ $\qquad f'(x) = 4x^3$

$x_0 = 3$ $\qquad f(x_0) = 6$ $\qquad f'(x_0) = 108$

$x_1 = 3 - \frac{6}{108} = 2.9\overline{4}$ $\qquad f(x_1) = 0.165$ $\qquad f'(x_1) = 102.110$

$f(x_2) = -0.0002463$

$x_2 = 2.9\overline{4} - \frac{0.165}{102.110} = 2.94282854$ $\qquad f'(x_2) = 101.9424$

$x_3 = 2.94282854 + \frac{0.0002463}{101.9424} = 2.94283096$ ✓

23.11

L'Hospital's rule [4.4]

$$\lim_{x\to 1} \frac{\ln(x)}{1-\sqrt{x}} = \lim_{x\to 1} \frac{\frac{1}{x}}{\frac{-1}{2\sqrt{x}}} = \lim_{x\to 1} \frac{-2}{\sqrt{x}} = -2$$

$$\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \lim_{x\to 0} \frac{\sin(x)}{2x} = \frac{1}{2} \lim_{x\to 0} \cancel{\frac{\sin(x)}{x}}^{\to 1} = \frac{1}{2}$$

L'Hopital's rule

f, g - differentiable

$g'(x) \neq 0$

$$\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x) \to \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

holds for: $\lim_{x\to a} f(x) = \pm\infty$ or $\lim_{x\to a} g(x) = \pm\infty$

$\lim_{x\to a^-}$ or $\lim_{x\to a^+}$ instead od $\lim_{x\to a}$

$a = \pm\infty$

Indeterminate forms

$$\lim_{x\to 2} \frac{x^2-4x+4}{x^2-x-2} = \lim_{x\to 2} \frac{2x-4}{2x-1} = \frac{0}{3} = 0$$

$$= \lim_{x\to 2} \frac{(x-2)^{\cancel{2}}}{\cancel{(x-2)}(x+1)} = \frac{0}{3} = 0$$

$$\lim_{x\to\infty} \frac{2x-\sqrt{x}+x^3}{x^3-x-2} = \lim_{x\to\infty} \frac{3x^2+2-\cancel{\frac{1}{2\sqrt{x}}}^{\to 0}}{3x^2-1} = \lim_{x\to\infty} \frac{\cancel{6x}}{\cancel{6x}} = 1$$

$$= \lim_{x\to\infty} \frac{\cancel{\frac{2}{x^2}}^{\to 0} - \cancel{\frac{\sqrt{x}}{x^3}}^{\to 0} + 1}{1 - \cancel{\frac{1}{x^2}}^{\to 0} - \cancel{\frac{2}{x^3}}^{\to 0}} = 1$$

$$\lim_{x\to 0} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to 0} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \qquad y=\frac{1}{x}$$

$$\Rightarrow \lim_{y\to\infty} \frac{\ln(1+y)}{y} = \frac{1}{\cancel{1+y}}^{\to 0} = 0$$

$$\lim_{x\to\infty} x-\sqrt{x^2+1} = \lim_{x\to\infty} 1-\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}} = \lim_{x\to\infty} 1-\frac{1}{\sqrt{1+\cancel{\frac{1}{x^2}}^{\to 0}}} = 0$$

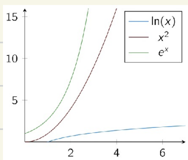
$$\lim_{x\to 1}\left(\frac{1}{x}\right)^{\frac{1}{x-1}} = \lim_{x\to 1} e^{\ln\left(\frac{1}{x}\right)\left(\frac{1}{x-1}\right)} = e^{\lim_{x\to 1}\frac{\ln\left(\frac{1}{x}\right)}{x-1}} = e^{\lim_{x\to 1}\frac{-\frac{1}{x}}{1}} = e^{-1} = \frac{1}{e}$$

$$\lim_{x\to 0}\left(1+\frac{1}{x}\right)^x = \lim_{x\to 0} e^{x\ln\left(1+\frac{1}{x}\right)} = e^{\lim_{x\to 0} x\ln\left(1+\frac{1}{x}\right)} = e^{\lim_{x\to 0}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}} = e^{\lim_{x\to 0}\frac{1}{1+\frac{1}{x}}} = e^0 = 1$$

$$\lim_{x\to 0} x^x = \lim_{x\to 0} e^{x\ln(x)} = e^{\lim_{x\to 0}\frac{\ln(x)}{\frac{1}{x}}} = e^{\lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{x^2}}} = e^{\lim_{x\to 0} -x} = e^0 = 1$$

$$\lim_{x\to\infty}\frac{x^r}{e^x} = 0$$

$$\lim_{x\to\infty}\frac{\ln(x)}{x^r} = 0$$



$$\lim_{x\to\infty}\frac{x}{\sqrt{1+x^2}} = 1$$

$$\lim_{x\to 0}(1-x)^{\frac{1}{x}} = \lim_{x\to 0} e^{\frac{\ln(1-x)}{x}} = e^{\lim_{x\to 0} -\frac{1}{1-x}} = e^{-1} = \frac{1}{e}$$

$$\lim_{x\to 0^+} x^{\sqrt{x}} = \lim_{x\to 0^+} e^{\ln(x)\sqrt{x}} = e^{\lim_{x\to 0}\frac{\ln(x)}{\frac{1}{\sqrt{x}}}} = e^{\lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{2x\sqrt{x}}}} = e^0 = 1$$

$$\lim_{x\to 4}\frac{x^2-2x-8}{x-4} = \lim_{x\to 4} 2x-2 = 6 \qquad \lim_{x\to 0}\frac{\ln(1+x)}{\cos(x)+e^x-1} = \frac{\cancel{\ln(1)}^{\to 0}}{\cancel{x}+1\cancel{x}} = 0$$

$$\lim_{x\to\infty}\frac{\ln\sqrt{x}}{x^2} = \frac{1}{4x^2} = 0 \qquad \left(\ln\sqrt{x}\right)' = \frac{1}{\sqrt{x}}\cdot\left(\sqrt{x}\right)' = \frac{1}{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{2x}$$

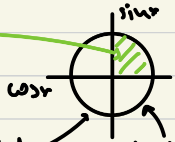
$$\lim_{x\to 1}\frac{x^a-ax+a-1}{(x-1)^2} = \lim_{x\to 1}\frac{ax^{a-1}-a}{2x-2} = \lim_{x\to 1}\frac{a(a-1)}{2}$$

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right) = \lim_{x\to 0^+}\frac{e^x-1-x}{x(e^x-1)} = \lim_{x\to 0^+}\frac{e^x-1}{xe^x+e^x-1} = \lim_{x\to 0^+}\frac{e^x}{(x+2)e^x} = \frac{1}{2}$$

28.11

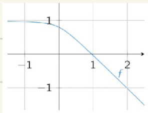
Integrals & Substitution rule [5.1 – 5.5]

$$\int_0^1 \sqrt{1-x^2}\, dx = \frac{\pi}{4}$$



circle: $x^2+y^2=1$

right semicircle: $y=\sqrt{1-x^2}$



$$\int_{-1}^{2} f(x)\,dx \approx 0.8026$$

Computation rules

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$\Delta x = x_i - x_{i-1} = \frac{b-a}{n}$$

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

$$\int_a^b c\,dx = c(b-a)$$

$$\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Mean value theorem

$f(x)$ – continuous on $[a,b]$

$\exists c \in [a,b] \quad f(c) = \dfrac{\int_a^b f(x)\,dx}{b-a}$

Fundamental theorem

f - continuous on $[a,b]$

$F(x) = \int_a^x f(t)\,dt$, $a \le x \le b$, continuous on (a,b)

$\Rightarrow F'(x) = f(x)$ → antiderivative function

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$\int \frac{1}{5x}\,dx = \frac{\ln|5x|}{5} + C \quad \text{or} \quad \frac{\ln|x|}{5} + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C \quad \text{or} \quad -\arccos(x) + C$$

$$\int \cos(2x)\,dx = \frac{1}{2}\sin(2x) + C$$

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C \qquad \int e^x\,dx = e^x + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$$

$$\int \cos(x)\,dx = \sin(x) + C \qquad \int -\frac{1}{\sqrt{1-x^2}}\,dx = \arccos(x) + C$$

$$\int \sin(x)\,dx = -\cos(x) + C \qquad \int \frac{1}{1+x^2}\,dx = \arctan(x) + C$$

$$\int_0^4 x^2-4x+4 \overset{?}{\geq} 0 \qquad x^2-4x+4 = (x-2)^2 \geq 0 \checkmark$$

$$\int_0^1 \sqrt{1+x^2}\,dx \overset{?}{\leq} \int_0^1 \sqrt{1+x}\,dx \quad \text{on } [0,1]: \; x^2 < x \quad x^2+1 < x+1 \quad \sqrt{x^2+1} < \sqrt{x+1}$$

$$\int_0^1 x^3\,dx = \frac{x^4}{4}\Big|_0^1 = \frac{1}{4}$$

$$\int_0^2 xe^{-x}\,dx = -(x+1)e^{-x}\Big|_0^2 = 1 - 3e^{-2}$$

$$g(x) = \int_{2x}^{3x} \frac{u^2-1}{u^2+1}\,du \qquad g'(x) = 3\left(\frac{9x^2-1}{9x^2+1}\right) - 2\left(\frac{4x^2-1}{4x^2+1}\right)$$

$$F(x) = \int_x^{x^2} e^{t^2}\,dt \qquad F'(x) = 2x\,e^{x^4} - e^{x^2}$$

Even and odd functions

even: $f(-x) = f(x)$ y-axis ($x=0$) symmetric   even $\cdot$ odd = odd

odd: $f(-x) = -f(x)$ $x=y$ symmetric   odd $\cdot$ odd = even

odd + odd = odd   even + even = even   even $\cdot$ even = even odd: $\int_{-a}^{a} f(x)dx = 0$

even: $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)\,dx$

Substitution Rule

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(u)du \qquad u = g(x)$$

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

$\int e^x\sqrt{1+e^x}\,dx$

$u = e^x \rightarrow \int \sqrt{1+u}\,du$

$u = 1+e^x \rightarrow \int \sqrt{u}\,du$

$\int e^{x^2} dx \qquad u = x^2$

$\int x e^{x^2} dx = \int \frac{e^u}{2}\,du$

$\int \sqrt{x}\,\sin(1+\sin x^{\frac{3}{2}})\,dx = -\frac{3}{2}\cos(1+x^{\frac{3}{2}}) + C$

$\int x\sqrt{1-x^2}\,dx = \int \frac{\sqrt{u}}{2}\,du = \frac{\cancel{2}u^{\frac{3}{2}}}{3\cdot\cancel{2}} = \frac{(1-x^2)^{\frac{3}{2}}}{3} + C \qquad u = 1-x^2, \; du = 2x\,dx$

$\int \frac{dx}{\arcsin(x)\sqrt{1-x^2}} = \int \frac{1}{u}\,du = \ln(u) = \ln(\arcsin(x)) + C \qquad u = \arcsin(x), \; du = \sqrt{1-x^2}\,dx$

$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} x^4\sin(x)\,dx = 0 \qquad \left.\begin{matrix} x^4 - \text{even} \\ \sin(x) - \text{odd} \end{matrix}\right\}\text{odd}$

$\int \frac{1+x}{1+x^2}dx = \int \frac{1}{2u}du + \int \frac{1}{x^2+1}\,dx = \frac{\ln(x^2+1)}{2} + \arctan(x) + C \qquad u = x^2+1, \; du = 2x\,dx$

$\int \frac{e^u}{(1-e^u)^2}du = -\int \frac{1}{x^2}dx = \frac{1}{x} + C = \frac{1}{1-e^u} + C \qquad x = 1-e^u, \; dx = -e^u du$

$\int_1^2 \frac{e^{\frac{1}{x}}}{x^2}dx = \int_1^{\frac{1}{2}} e^u du = e - \sqrt{e} \qquad u = \frac{1}{x} \quad du = -x^{-2}dx$

29.11

Integration by parts [7.1]

$$\int (3x+4)\cos(x)\,dx = (3x+4)\sin(x) + 3\cos(x) + C$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

$g'(x)\,dx = du$, $f'(x)\,dx = dv$ } $\int u\,dv = uv - \int v\,du$

$$\int \ln(x)\,dx = x\ln(x) - x + C$$

$$\int_a^b f(x)g'(x)\,dx = \left[f(x)g(x)\right]_a^b - \int_a^b f'(x)g(x)\,dx$$

$$\int x^2e^x\,dx = x^2e^x - \int 2xe^x\,dx = x^2e^x - \left(2xe^x - \int 2e^x\,dx\right) = e^x(x^2-2x+2)$$

$$\int \frac{e^x}{x}\,dx = \frac{e^x}{x} + \int \frac{e^x}{x^2}\,dx$$

$u = x$ $\quad dv = \sin(x)$

$du = 1$ $\quad v = -\cos(x)$

$$\int_0^{\frac{\pi}{2}} x\sin(x)\,dx = -x\cos(x)\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}\cos(x)\,dx = -\frac{\pi}{2} - \cancel{\sin\left(\frac{\pi}{2}\right)}^{1} + \cancel{\sin(0)}^{0} = 1 - \frac{\pi}{2}$$

choose u: LIATE

Logarithms Inverse trigonometric functions
Algebraic functions Trigonometric functions Exponentials

$$\int x^2\ln(x)\,dx = \frac{x^3}{3}\ln(x) - \int \frac{x^2}{3}\,dx = \frac{x^3\ln(x)}{3} - \frac{x^3}{9} + C$$

$u = \ln(x)$ $\quad dv = x^2$

$du = \frac{1}{x}$ $\quad v = \frac{x^3}{3}$

$$\int \arctan(x)\,dx = x.\arctan(x) - \int \frac{x}{x^2+1}\,dx = x.\arctan(x) - \frac{\ln(x^2+1)}{2} + C$$

$u = \arctan(x)$ $\quad du = \frac{1}{x^2+1}$ $\quad dv = 1$ $\quad v = x$

$$\int \underline{e^{2x}\cos(x)\,dx} = \frac{e^{2x}\cos(x)}{2} + \int \frac{e^{2x}\sin(x)\,dx}{2} = \frac{e^{2x}\cos(x)}{2} + \frac{\sin(x)e^{2x}}{4}$$

$u = \cos(x)$ $\quad dv = e^{2x}$

$du = -\sin(x)$ $\quad v = \frac{e^{2x}}{2}$

$u = \sin(x)$ $\quad dv = \frac{e^{2x}}{2}$

$du = \cos(x)$ $\quad v = \frac{e^{2x}}{4}$

$$-\frac{1}{4}\int \underline{e^{2x}\cos(x)\,dx}$$

$$\frac{5}{4}\int e^{2x}\cos(x)\,dx = \frac{e^{2x}\cos(x)}{2} + \frac{e^{2x}\sin(x)}{4} \;\Big|\cdot\frac{4}{5} \quad + C$$

Reduction formulas

$$\int x^4 e^{-x}\,dx = -x^4e^{-x} + \int 4x^3 e^{-x}\,dx = -x^4e^{-x} - 4x^3e^{-x} - 12x^2e^{-x} - 24xe^{-x} - 24e^{-x} + C$$

$u = x^4$ $\quad du = 4x^3$ $\quad I_n = \int x^n e^{-x}\,dx$

$dv = e^{-x}$ $\quad v = -e^{-x}$ $\quad I_0 = -e^{-x} + C$

reduction formula $\to$ $I_n = -x^n e^{-x} + nI_{n-1}$

$$\int x^n e^x\,dx = x^n e^x - \int n e^x x^{n-1}\,dx \quad u = x^n \quad du = nx^{n-1} \quad dv = e^x \quad v = e^x$$

$$\int \sin^n x\,dx = \int \sin^2 x \sin^{n-2} x\,dx = \sin^2 x\, I_{n-2} - \int I_{n-2}\sin 2x\,dx$$

$\hookrightarrow I_n$ $\quad u = \sin^2 x$ $\quad du = 2\sin x\cos x = \sin 2x$

$dv = \sin^{n-2} x$ $\quad v = I_{n-2}$

$$\int (x^2+2x)\cos(x)\,dx = (x^2+2x)\sin(x) - \int (2x+2)\sin(x)\,dx = (x^2+2x)\sin(x) + (2x+2)\cos x - \cancel{\int 2\cos(x)\,dx}\ 2\sin(x) + C$$

$u = x^2+2x$ $\quad dv = \cos(x)$ $\quad u = 2x+2$ $\quad dv = \sin(x)$

$du = 2x+2$ $\quad v = \sin(x)$ $\quad du = 2$ $\quad v = -\cos(x)$

$$\int_1^2 \frac{(\ln(x))^2}{x^3}\,dx = -\frac{\ln^2(x)}{2x^2} + \int \frac{\ln(x)}{x^3}\,dx = -\frac{\ln^2(x)}{2x^2} - \frac{\ln(x)}{2x^2} + \cancel{\int \frac{1}{2x^3}\,dx} - \frac{1}{4x^2} =$$

$u = \ln^2(x)$ $\quad v' = x^{-3}$ $\quad u = \ln(x)$ $\quad dv = x^{-3}$ $\quad = \dfrac{2\ln^2(2) + 2\ln(2) - 3}{16}$

$du = \dfrac{2\ln(x)}{x}$ $\quad v = -\dfrac{x^{-2}}{2}$ $\quad du = \dfrac{1}{x}$ $\quad v = -\dfrac{x^{-2}}{2}$

$$\int t^4 \ln(t)\,dt = \frac{t^5\ln(t)}{5} - \int \frac{t^4}{5}\,dt = \frac{t^5\ln(t)}{5} - \frac{t^5}{25} + C \quad u = \ln(t) \quad dv = t^4$$

$du = \dfrac{1}{t}$ $\quad v = \dfrac{t^5}{5}$

$$\int_0^{\frac{\pi}{3}} \sin(x)\ln(\cos x)\,dx = -\cos(x)\ln(\cos x) - \int \sin x\,dx = -\cos(x)\ln(\cos x) + \cos(x)$$

$u = \ln(\cos x)$ $\quad du = \dfrac{-\sin x}{\cos x} = -\tan x$ $\quad dv = \sin(x)$ $\quad v = -\cos(x)$

$$\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right)dx = \int_1^{\sqrt{3}} \operatorname{arccot}(x)\,dx = x\operatorname{arccot}(x) + \int \frac{x}{x^2+1}\,dx = x\operatorname{arccot}(x) + \frac{\ln(x^2+1)}{2}\Bigg|_1^{\sqrt{3}}$$

$u = \operatorname{arccot}(x)$ $\quad dv = 1$ $\quad u = x^2+1$

$du = -\dfrac{1}{x^2+1}$ $\quad v = x$ $\quad du = 2x\,dx$ $\quad \displaystyle\int \frac{1}{2u}\,du = \frac{\ln(u)}{2}$

$$\int_{\sqrt{\frac{\pi}{2}}}^{\sqrt{\pi}} \theta^3\cos(\theta^2)\,d\theta = \int_{\frac{\pi}{2}}^{\pi} \frac{x\cos(x)}{2}\,dx = \frac{x\sin(x)}{2} - \int \frac{\sin(x)}{2}\,dx = \frac{x\sin(x) + \cos(x)}{2}\Bigg|_{\frac{\pi}{2}}^{\pi} =$$

$x = \theta^2$ $\quad u = x$ $\quad dv = \cos(x)$ $\quad = \dfrac{\pi\sin\pi + \cos\pi}{2} - \dfrac{\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2}}{2} = -\dfrac{\pi}{4} - \dfrac{1}{2}$

$dx = 2\theta\,d\theta$ $\quad du = 1$ $\quad v = \sin(x)$

30.11

Improper integrals [7.8]

$$\int_1^{\infty} \frac{1}{x^4}\,dx = -\frac{1}{3x^3}\Big|_1^{\infty} = \cancel{\frac{1}{-3\infty^3}}^{\,0} + \frac{1}{3} = \frac{1}{3}$$

$$\int_{-1}^{1} \frac{1}{x^4}\,dx = 2\int_0^1 \frac{1}{x^4}\,dx = -\frac{2}{3x^3}\Big|_0^1 = -\frac{2}{3} + \frac{2}{3\cdot 0^2} = \text{DNE}$$

Improper integrals: type I

infinite interval of integration

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx$$

improper integral:
- **convergent** if limit exists
- **divergent** otherwise

$f(x) = x$ - odd

$$\int_{-\infty}^{\infty} x\,dx = \int_{-\infty}^{0} x\,dx + \int_0^{+\infty} x\,dx$$ both limits must exist

divergent <= divergent

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \arctan(x)\Big|_{-\infty}^{+\infty} = \pi$$ convergent

$$\int_1^{\infty} \frac{x}{1+x^2}\,dx = \int_2^{\infty} \frac{1}{2u}\,du = \frac{\ln(u)}{2}\Big|_2^{\infty}$$ divergent

$u = x^2+1$

$du = 2x\,dx$

$$\int_1^{\infty} -x^{-\frac{2}{3}}\,dx = -3x^{\frac{1}{3}}\Big|_1^{\infty}$$ divergent

$$\int_a^{\infty} x^r\,dr = \begin{cases} -\frac{a^{r+1}}{r+1} & r < -1 \\ \text{divergent} & r \geq -1 \end{cases}$$

$$\int_a^{\infty} e^{rx}\,dx = \begin{cases} -\frac{e^{ra}}{r} & r < 0 \\ \text{divergent} & r \geq 0 \end{cases}$$

Improper integrals: type II

discontinuity in the integrand

discontinuity at b: $\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$

discontinuity at a: $\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$

discontinuity at $c \in (a,b)$: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

improper integral:
- convergent if limit exists
- divergent otherwise

$$\int_0^b x^r\,dx = \begin{cases} \text{divergent} & r \leq -1 \\ \frac{b^{r+1}}{r+1} & r > -1 \end{cases}$$

$$\int_0^b \ln(rx)\,dx = b\ln(rb) - b \quad r > 0$$

Comparison test

f, g - continuous functions $0 \leq g(x) \leq f(x)$, $x \geq a$

$\int_a^\infty f(x)\,dx$ convergent $\Rightarrow \int_a^\infty g(x)\,dx$ convergent

$\int_a^\infty g(x)\,dx$ divergent $\Rightarrow \int_a^\infty f(x)\,dx$ divergent

$$\int_0^1 \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = \int_0^1 2\cos(u)\,du = 2\sin(u)\Big|_0^1 \text{ convergent} \quad u = \sqrt{x} \quad du = \frac{1}{2\sqrt{x}}\,dx$$

$$\int_0^1 x\ln(x)\,dx = \frac{\ln(x)x^2}{2} - \int \frac{x}{2}\,dx = \frac{\ln(x)x^2}{2} - \frac{x^2}{4}\Big|_0^1 = -\frac{1}{4} \text{ convergent} \quad du = \frac{1}{x} \quad dv = x \quad u = \ln(x) \quad v = \frac{x^2}{2}$$

$$\int_0^\infty \frac{1}{\sqrt{x} + x\sqrt{x}}\,dx = \int_0^\infty \frac{2}{u^2+1}\,du = 2\arctan(u)\Big|_0^\infty = \pi \text{ convergent} \quad du = \frac{1}{2\sqrt{x}}\,dx \quad u = \sqrt{x}$$

$$\int_1^\infty \frac{x}{1+x^3}\,dx = -\frac{\ln|x+1|}{3} + \frac{\ln(x^2-x+1)}{6} + \frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)}{\sqrt{3}}\Bigg|_1^\infty = \frac{\ln 2}{3} + \frac{\pi}{3^{\frac{3}{2}}} \text{ convergent}$$

05.12

Sequences [11.1]

sequence - function with domain $\mathbb{N} = \{1, 2, 3, \dots\}$

$\{a_1, a_2, \dots\}$ $\{a_n\}_{n=1}^{\infty}$ (a_n) $(a_n)_{n=k}^{\infty}$ ← notations

defining: explicitly $a_n = f(n)$

recursively $a_{n+1} = f(a_n, a_{n-1}, \dots)$

Limit

$\lim_{n\to\infty} a_n = L \rightarrow$ sequence convergent $\quad \lim_{n\to\infty} a_n = DNE$ → limit divergent

$\forall \varepsilon > 0 \rightarrow \exists N \geq k$ s.t. $n \geq N \rightarrow |a_n - L| < \varepsilon$

$a_n = f(n)$ and $\lim_{x\to\infty} f(x) = L \Rightarrow \lim_{n\to\infty} a_n = L$

$\begin{cases} a_0 = 3 \\ a_{n+1} = -\frac{1}{3} a_n \end{cases} \qquad a_n = \frac{a_0}{(-3)^n} = \frac{3}{(-3)^n} = \frac{(-1)^n}{3^{n-1}}$

$\lim_{n\to\infty} \frac{\ln(n)}{n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \lim_{n\to\infty} \cancel{\frac{1}{n}}^{0} = 0$

$\lim_{n\to\infty} n^2 \sin(n\pi) = DNE$ ← function $\quad \sin(n\pi) = 0 \quad a_n = n^2 \sin(\pi n) \quad \lim_{n\to\infty} a_n = 0$

Computation rules and techniques

$\lim_{n\to\infty} a_n = L \quad \lim_{n\to\infty} b_n = M$

$\lim_{n\to\infty} c a_n = c \cdot L \qquad \lim_{n\to\infty} a_n b_n = LM$

$\lim_{n\to\infty} a_n + b_n = L + M \qquad \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M},\ M \neq 0$

$\lim_{n\to\infty} a_n = L$

f - continuous at L

$\lim_{n\to\infty} f(a_n) = f(L)$

$a_n \leq b_n \leq c_n$

$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n \Rightarrow \lim_{n\to\infty} b_n = L$

$\lim_{n\to\infty} (-1)^n \frac{n}{n+1} = DNE$

$\lim_{n\to\infty} \frac{2^n}{n!} = 0$

$\left\{\frac{1}{2}, -\frac{4}{3}, \frac{9}{4}, -\frac{16}{5}, \frac{25}{6}, \dots\right\} \quad a_n = \frac{(-1)^n (n+1)^2}{n+2}$

$a_n = \frac{4n^2-3n}{2n^2+1}$ $\lim\limits_{n\to\infty} a_n = 2$ convergent

$a_n = \frac{\cos^2(n)}{2^n}$ $\lim\limits_{n\to\infty} a_n = 0$ convergent

$a_n = \frac{(-1)^{n-1} n}{n^2+1}$ $\lim\limits_{n\to\infty} a_n = 0$ convergent

$f(c)=c \to$ fixed point

$a_{n+1} = f(a_n)$ f-continuous $\lim\limits_{n\to\infty} a_n = L \to$ fixed point of f

$L = \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} f(a_n) = f(\lim\limits_{n\to\infty} a_n) = f(L)$

$x=f(x)$ $x_0 = \frac{\pi}{3}$ $x_1 = \cos(\frac{\pi}{3}) = \frac{1}{2}$ $x_2 = \cos(\frac{1}{2}) = 0.87758$ $x_3 = \cos(x_2) = 0.63901$

$x = \cos(x)$ $x \in [0, \frac{\pi}{3}]$ $x_4 = \cos(x_3) = 0.80268$ $x_{10} = \cos(x_9) = 0.73501$

simple iteration $0.73908513 = \cos(0.73908513)$

Newton's method $f(x) = \cos(x) - x$

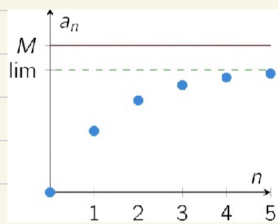
$x_0 = \frac{\pi}{3}$ $x_1 = 0.753951$ $x_2 = 0.739133$ $x_3 = 0.7390851$

$f: [a,b] \to [a,b]$ - continuous

$\exists k<1$ s.t. $|f'(x)| < k$ for $\forall x \in [a,b]$

$x_{n+1} = f(x_n)$ sequence converges for $\forall x_0 \in [a,b]$

Monotone convergence



(a_n), 1) $\exists M$ s.t. $a_n \le M$ for $\forall n$ (bounded above)

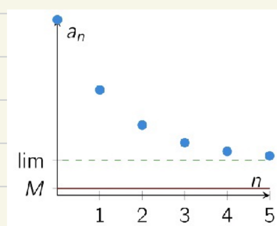
2) $a_{n+1} \ge a_n$ for $\forall n$ (increasing)

$\Rightarrow (a_n)$ - convergent

(a_n), 1) $\exists M$ s.t. $a_n \ge M$ for $\forall n$ (bounded below)

2) $a_{n+1} \le a_n$ for $\forall n$ (decreasing)

$\Rightarrow (a_n)$ - convergent



$a_0 = 0$ $a_{n+1} = 1 + \frac{a_n}{3}$ $a_{n+1} > a_n$ $a_n < 2$

$a_0 = 5$ $a_{n+1} = \sqrt{2 + a_n}$ $a_n < a_{n+1}$ $a_n > 1$

$\begin{cases} a_0 = 2 \\ a_{n+1} = 6 - \frac{5}{a_n} \end{cases}$ $\quad a_{n+1} > a_n \quad 6 - \frac{5}{a_n} > a_n \;\Big|\cdot a_n \quad a_n^2 - 6a_n + 5 < 0 \quad (a_n - 5)(a_n - 1) < 0$

$+ \quad 1 \quad - \quad 5 \quad +$

decr. fix incr. fix decr.

$1 \qquad 5 \quad a_0 \in (1, 5)$ $\Rightarrow a_{n+1} \geq a_n \quad a_n \leq 5 \Rightarrow a_n$ - increasing

$a_0 = \sqrt{2}$ $\quad a_{n+1} > a_n \quad \sqrt{2}\sqrt{a_n} > a_n \Big|^2 \quad 2a_n > a_n^2$

$a_{n+1} = \sqrt{2a_n}$ $\quad 0 > a_n(a_n - 2)$ $\quad + \quad 0 \quad - \quad 2 \quad +$

decr. fix incr fix decr.

$0 \qquad 2$

$a_0 \in (0, 2) \Rightarrow a_{n+1} \geq a_n \quad a_n \leq 2 \Rightarrow a_n$ - increasing

$a_0 = \sqrt{2}$ $\quad a_{n+1} > a_n \quad \sqrt{2 + a_n} > a_n \Big|^2 \quad 2 + a_n > a_n^2$

$a_{n+1} = \sqrt{2 + a_n}$ $\quad a_n^2 - a_n - 2 < 0 \quad (a_n - 2)(a_n + 1) < 0$ $\quad + \quad -1 \quad - \quad 2 \quad +$

decr. fix incr. fix decr. $\Rightarrow a_{n+1} \geq a_n \quad a_n \leq 2 \Rightarrow a_n$ - increasing

$-1 \qquad 2 \quad a_0 \in (-1, 2)$

06.12

Series [11.2]

$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots=1$

$1\underbrace{-1+2}_{+1}\underbrace{-2+3}_{+1}-3\ldots=DNE$

Series

infinite sum: $\sum_{n=0}^{\infty} a_n = a_1+a_2+a_3+\ldots$

partial sum: $S_N=\sum_{n=0}^{N} a_n$

$\lim_{N\to\infty} S_N = L \to$ convergent

$\sum_{n=0}^{\infty} a_n = L$ otherwise divergent

Geometric series

$r=\frac{a_{n+1}}{a_n} \to$ common ratio

$r\in(-1,1)$

$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a_0 r^n = \frac{a_0}{1-r}$ > convergent

$r\le -1$ or $r\ge 1 \to$ divergent

$\frac{3}{4}-1+\frac{4}{3}-\frac{16}{9}+\ldots \quad a_0=\frac{3}{4} \quad r=-\frac{4}{3} \to$ div.

$\sum_{n=0}^{\infty}\frac{2}{3}\left(-\frac{1}{3}\right)^n=\frac{2}{3}/\left(1-\left(-\frac{1}{3}\right)\right)=\frac{1}{2}$

$3-4+\frac{16}{3}-\frac{64}{9}+\ldots \quad a_0=3 \quad r=-\frac{4}{3} \to$ diverges

$\sum_{n=1}^{\infty}-\frac{1}{3}\left(-\frac{3}{4}\right)^n=-\frac{1}{3}/\left(1-\left(-\frac{3}{4}\right)\right)=-\frac{4}{21} \qquad a_0=-\frac{1}{3} \quad r=-\frac{3}{4}$

$\sum_{n=1}^{\infty}(x+2)^n \qquad -1<x+2<1 \mid -2 \qquad -3<x<-1 \qquad \sum_{n=1}^{\infty}(x+2)^n=\frac{1}{1-(x+2)}=-\frac{1}{x+1}$

Computation rules

$\sum_{n=1}^{\infty} a_n+b_n=\sum_{n=1}^{\infty} a_n+\sum_{n=1}^{\infty} b_n$

$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

Divergence test

$\sum_{n=1}^{\infty} a_n$ - convergent $\Rightarrow \lim_{n\to\infty} a_n=0$

$\lim_{n\to\infty} a_n \ne 0$ or DNE $\Rightarrow \sum_{n=1}^{\infty} a_n$ - divergent

Integral test and p-series

$f(x)$ - positive, continuous, decreasing on $[1,\infty) \Rightarrow a_n=f(n)$

$\sum_{n=1}^{\infty} a_n$ - convergent if $\int_1^{\infty} f(x)\,dx$ - convergent

$\sum_{n=1}^{\infty} a_n$ - divergent if $\int_1^{\infty} f(x)\,dx$ - divergent

$\sum_{n=1}^{\infty}\frac{1}{n^p}$ - p-series; $p>1$ convergent; $p\le 1$ divergent

$\sum_{n=1}^{\infty}\frac{3+n}{n^2}=3\underbrace{\sum_{n=1}^{\infty}\frac{1}{n^2}}_{p=2\text{ con.}}+\underbrace{\sum_{n=1}^{\infty}\frac{1}{n}}_{p=1}\to$ div. $\to$ divergent

$\sum_{n=0}^{\infty}\frac{1+2^n}{3^n}=\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n+\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n=1/1-\left(\frac{1}{3}\right)+1/1-\left(\frac{2}{3}\right)=\frac{3}{2}+3=4.5\to$ convergent

$\sum_{n=1}^{\infty}\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \qquad \lim_{n\to\infty}\frac{\sin\left(\cancel{\frac{1}{n}}\to 0\right)}{\cancel{\frac{1}{n}}\to 0}=\frac{\sin(0)}{0}=1 \to$ divergent

$$\sum_{n=1}^{\infty} \frac{2n}{3n+1} \quad \lim_{n\to\infty} \frac{2n}{3n+1} = \frac{2}{3} \Rightarrow \text{divergent}$$

$$\sum_{n=1}^{\infty} \arctan(n) \quad \lim_{n\to\infty} \arctan(n) = \frac{\pi}{2} \Rightarrow \text{divergent}$$

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1} \quad \int_1^{\infty} \frac{x}{x^2+1}\,dx = \int_2^{\infty} \frac{1}{2u}\,du = \left.\frac{\ln(u)}{2}\right|_2^{\infty} \Rightarrow \text{divergent}$$

$$u = x^2+1 \quad du = 2x\,dx$$

$$\sum_{n=1}^{\infty} \frac{2}{n^{0.85}} \quad p \leq 1 \Rightarrow \text{divergent}$$

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}+4}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^{1.5}} + \sum_{n=1}^{\infty} \frac{4}{n^2} \quad p_1 > 1,\ p_2 > 1 \Rightarrow \text{convergent}$$

$$5.\overline{71358} = 5 + \frac{71358}{\cancel{10^5}} \cdot \frac{\cancel{10^5}}{99999} = 5\,\frac{23786}{33333} \quad a = 0.71358 = \frac{71358}{10^5} \quad r = 10^{-5}$$

07.12

Alternating series and comparison test [11.3 – 11.5]

Alternating series

$r<0 \to$ alternating series

terms are alternatively positive and negative

alternating series $\sum_{n=1}^{\infty} a_n$ is convergent iff

$|a_{n+1}| \leq |a_n|$ for $\forall n$ and $\lim_{n\to\infty} |b_n| = 0$

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \to$ alternating and convergent

$\sum_{n=1}^{\infty} \cos(\pi n) \to$ alternating and divergent

Comparison test

$\sum a_n$ and $\sum b_n \to$ series with positive terms

if $\sum b_n$ - convergent, $a_n \leq b_n$ for $\forall n \to \sum a_n$ - convergent

if $\sum b_n$ - divergent, $a_n \geq b_n$ for $\forall n \to \sum a_n$ - divergent

if $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, c - finite number $>0 \Rightarrow$ both series converge or diverge

$\sum_{n=0}^{\infty} \frac{1}{1+n^2}$ $\quad \int_0^{\infty} (x^2+1)^{-1}\,dx = \arctan(x)\Big|_0^{\infty} = \frac{\pi}{2}$ convergent

$\sum_{n=1}^{\infty} \frac{n}{\sqrt{1+3n^4}}$ $\quad \lim_{n\to\infty} \frac{\frac{n}{\sqrt{1+3n^4}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{\sqrt{1+3n^4}} = \lim_{n\to\infty} \frac{1}{\sqrt{\frac{1}{n^4}+3}} = \frac{1}{\sqrt{3}} > 0 \Rightarrow$ divergent

$-\frac{2}{5} + \frac{4}{6} - \frac{6}{7} + \frac{8}{8} - \frac{10}{9} + \dots$ divergent

$\sum_{n=1}^{\infty} \frac{1}{n!} \leq \sum_{n=1}^{\infty} \frac{1}{n^n}$ $\quad n=1 \quad a_n = b_n = 1 \Rightarrow \sum_{n=2}^{\infty} \frac{1}{n!} < \sum_{n=2}^{\infty} \frac{1}{n^n} \to p>1 \Rightarrow$ both convergent

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}$ $\quad \lim_{n\to\infty} \frac{n^2}{n^3+4} = 0 \quad |a_{n+1}| \leq |a_n| \Rightarrow$ convergent

$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ $\quad \lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} = 1 \Rightarrow$ divergent

Error bounds on partial sums

$f(k) = a_k$ f - continuous, positive, decreasing on $[n, \infty)$

$\sum a_n$ - convergent

$s = \sum_{k=1}^{\infty} a_k$

$s_n = \sum_{k=1}^{n} a_k$

$$\int_{n+1}^{\infty} f(x)\,dx \le s - s_n \le \int_{n}^{\infty} f(x)\,dx$$

$s_N = \sum_{n=1}^{N} a_n$ $\quad |s - s_N| \le |a_{N+1}|$

$s = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ $\quad |s - s_{100}| \le |a_{101}| = \frac{1}{101}$

$\sum a_n$ converges by comparison with $\sum b_n$

$s - s_n = \sum_{k=n+1}^{\infty} a_k \le \sum_{k=n+1}^{\infty} b_k = t - t_n$

$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^6}$ $\quad |s - s_N| \le |a_{N+1}| \le 10^{-6}$ $\quad (2N+3)^6 \ge 10^6$ $\quad N \ge 3.5$ $\quad N = 4$

$$s_N = s_4 = \int_1^4 (2n+1)^{-6}\,dn = -\frac{(2n+1)^{-5}}{10}\Bigg|_1^4 = \frac{1}{3^5 \times 10} - \frac{1}{9^5 \times 10}$$

$\sum_{n=1}^{\infty} 5^{-n} \cos^2(n)$ $\quad |s - s_N| \le |a_{N+1}|$ $\quad a_{11} = 5^{-11} \cos^2(11)$

$$s_{10} = \int_1^{11} 5^{-n} \cos^2(n)\,dn =$$

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$ $\rightarrow$ converges

$(N+1)^6 \ge 2 \times 10^4$

$|s - s_N| \le |a_{N+1}| \le \frac{1}{2 \times 10^4}$ $\quad N = 5$