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Lecture 4 - Linear transformations

Matrix transformations

$T(x)=\begin{bmatrix}3&5\\0&4\\0&0\end{bmatrix}x$

domain: $\mathbb{R}^2$ (columns)
codomain: $\mathbb{R}^3$ (rows)
range: span of columns

$x=\begin{bmatrix}1\\2\\3\end{bmatrix}$ $x \notin T(y)$ $T(y)=\begin{bmatrix}3y_1+5y_2\\4y_2\\0\end{bmatrix}$

$A=\begin{bmatrix}-2&5&1\\3&-1&4\end{bmatrix}$ $Ax=\begin{bmatrix}-2x_1+5x_2+x_3\\3x_1-x_2+4x_3\end{bmatrix}$

Def. matrix $A(m\times n)$ defines tranformation $T: \mathbb{R}^n \to \mathbb{R}^m$ $T(x)=Ax$
$b \in Range(T) \longleftrightarrow Ax=b$ - consistent

Linear transformations

Def. $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear if:
- $T(u+v)=T(u)+T(v)$ for $\forall u,v \in \mathbb{R}^n$
- $T(cu)=cT(u)$ for $\forall c \in \mathbb{R}$, $\forall u \in \mathbb{R}^n$

Th $\forall$ matrix A $(m\times n)$, $T(x)=Ax$ is linear transformation $\mathbb{R}^n \to \mathbb{R}^m$
$T(0)\neq 0 \to T$ is nonlinear
$T(0)=0 \to$ property of linear transformations, not a condition, $\sin(0)=0$

$T: \mathbb{R}^2 \to \mathbb{R}^3 \Rightarrow A$ is 3×2
$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}4\\2\\1\end{bmatrix}$ $T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)=\begin{bmatrix}2\\0\\3\end{bmatrix}$
$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=\begin{bmatrix}3x_1+x_2\\x_1+x_2\\2x_1-x_2\end{bmatrix} \Rightarrow A=\begin{bmatrix}3&1\\1&1\\2&-1\end{bmatrix}$ $T\left(\begin{bmatrix}0\\2\end{bmatrix}\right)=\begin{bmatrix}2\\2\\-2\end{bmatrix}$

$\left.\begin{matrix}T: \mathbb{R}^3 \to \mathbb{R}^2\\ S: \mathbb{R}^3 \to \mathbb{R}^2\end{matrix}\right\}$ linear
$\Rightarrow R(x)=T(x)+S(x)$
- linear $\mathbb{R}^3 \to \mathbb{R}^2$

1.8./ 9. $A=\begin{bmatrix}1 & -4 & 7 & -5\\ 0 & 1 & -4 & 3\\ 2 & -6 & 6 & -4\end{bmatrix}$ $\begin{matrix} \\ -2 \\ 1/2 \end{matrix}$ -1 $\xrightarrow{\text{reduced echelon}}$ $A=\begin{bmatrix}1 & -4 & 7 & -5\\ 0 & 1 & -4 & 3\\ 0 & 0 & 0 & 0\end{bmatrix}$ $T(x)=Ax=\begin{bmatrix}x_1-4x_2+7x_3-5x_4\\ x_2-4x_3+3x_4\\ 0\end{bmatrix}$

$T(x)=0 \Rightarrow \left|\begin{array}{l}x_1-4x_2+7x_3-5x_4=0\\ x_2-4x_3+3x_4=0\end{array}\right.$ $\left|\begin{array}{l}x_1=9x_3-7x_4\\ x_2=4x_3-3x_4\end{array}\right. \Rightarrow x=x_3\begin{bmatrix}9\\4\\1\\0\end{bmatrix}+x_4\begin{bmatrix}-7\\-3\\0\\1\end{bmatrix}$

16. $T(x)=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ $u=\begin{bmatrix}5\\2\end{bmatrix}$ $T(u)=\begin{bmatrix}2\\5\end{bmatrix}$ $v=\begin{bmatrix}-2\\4\end{bmatrix}$ $T(v)=\begin{bmatrix}4\\-2\end{bmatrix}$ $x=y$ inversion

22. $x=\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ $v_1=\begin{bmatrix}-2\\5\end{bmatrix}$ $v_2=\begin{bmatrix}7\\-3\end{bmatrix}$ $T:\mathbb{R}^2\to\mathbb{R}^2$ $T(x)=x_1v_1+x_2v_2$ $T(x)=Ax$ $A=\begin{bmatrix}-2 & 7\\ 5 & -3\end{bmatrix}$

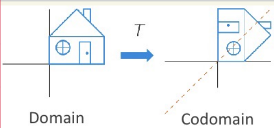
31. $T:\mathbb{R}^n\to\mathbb{R}^m$ (linear) $\{v_1,v_2,v_3\}$ - linearly dependent, $\mathbb{R}^n \Rightarrow \{T(v_1),T(v_2),T(v_3)\}$ - linearly dependent

Standard matrices

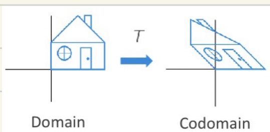
Th $T:\mathbb{R}^n\to\mathbb{R}^m$ (linear)

$\exists A$ $(m\times n)$ s.t. $T(x)=Ax$ standard matrix

columns of A → images under T od unit vectors: $A=\begin{bmatrix}T(e_1)\dots T(e_n)\end{bmatrix}$



$A=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ $u=\begin{bmatrix}x\\y\end{bmatrix}$ $T(u)=Au=\begin{bmatrix}y\\x\end{bmatrix}$

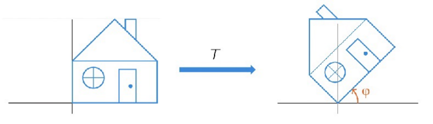


$A=\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}$ $u=\begin{bmatrix}x\\y\end{bmatrix}$ $T(u)=Au=\begin{bmatrix}x-y\\y\end{bmatrix}$

$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=\begin{bmatrix}2x_1+3x_2\\ x_1+4x_2\\ -x_1\end{bmatrix}$ $A=\begin{bmatrix}2 & 3\\ 1 & 4\\ 0 & -1\end{bmatrix}$

$T:\mathbb{R}^2\to\mathbb{R}^3$ (linear) $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}2\\0\\3\end{bmatrix}$ $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)=\begin{bmatrix}1\\1\\1\end{bmatrix}$ $\Rightarrow Ax=\begin{bmatrix}3x_1-x_2\\ -x_1+x_2\\ 5x_1-2x_2\end{bmatrix}$ $A=\begin{bmatrix}3 & -1\\ -1 & 1\\ 5 & -2\end{bmatrix}$

Th T - rotation about the origin, φ angle



$T(x) = Ax \Rightarrow A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$

linear

1.9./ 7. $T: \mathbb{R}^2 \to \mathbb{R}^2$ rotate $\frac{3\pi}{4}$, reflect on x_1

$\Rightarrow A = \begin{bmatrix} -\cos(\frac{3\pi}{4}) & -\sin(\frac{3\pi}{4}) \\ -\sin(\frac{3\pi}{4}) & \cos(\frac{3\pi}{4}) \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

9. $T: \mathbb{R}^2 \to \mathbb{R}^2$ shear e_2 into $e_2 - 2e_1 \to A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ reflect on $x_2 = -x_1 \to A = \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$

16. $\begin{bmatrix} 3 & 0 & -2 \\ 4 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 - 2x_3 \\ 4x_1 \\ x_1 - x_2 + x_3 \end{bmatrix}$

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Lecture 5 – Matrix operations

Matrix addition and scalar multiplication

$A = \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -2 \\ 4 & -1 \\ -2 & 3 \end{bmatrix}$

$A + B =$ undefined

$A = \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -2 \\ 4 & -1 \end{bmatrix}$ $2A - B = \begin{bmatrix} 0 & 4 \\ 2 & 11 \end{bmatrix}$

Def. A - $m \times n$ matrix:

- zero - all entries are 0
- square - $m = n$

main diagonal of a matrix - a_{ii} entries

Square matrix:

- diagonal matrix - all off-diagonal entries are 0
- identity matrix - diagonal matrix with 1's on the main diagonal
- Lower/upper triangular matrix - all entries above/below main diagonal are 0

Th A, B, C - same size matrices r, s - scalars

$A+B=B+A$

$(A+B)+C=A+(B+C)$

$A+0=A$

$r(A+B)=rA+rB$

$(r+s)A=rA+sA$

$r(sA)=(rs)A$

Matrix multiplication

Th $T: \mathbb{R}^p \to \mathbb{R}^n \quad S: \mathbb{R}^n \to \mathbb{R}^m$

$(S \circ T)(x) = S(T(x))$ all linear

Def. A - $m \times n$ matrix $S(x)=Ax$

B - $n \times p$ matrix $T(x)=Bx$

$AB = S \circ T$ — $m \times p$ matrix

$$A=\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \quad B=\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

AB — defined

Th A - $m \times n$ matrix

B — $n \times p$ matrix, columns: $b_1 \dots b_p$

$AB = A[b_1 \; b_2 \; \dots \; b_p] = [Ab_1 \; Ab_2 \; \dots \; Ab_p]$ - $m \times p$ matrix

$$A=\begin{bmatrix} -5 & 3 \\ -4 & -1 \end{bmatrix} \quad B=\begin{bmatrix} 1 & -1 & 2 \\ 2 & 6 & 0 \end{bmatrix} \quad AB=\begin{bmatrix} 1 & 23 & -10 \\ -6 & -2 & -8 \end{bmatrix}$$

$$\left.\begin{array}{l} A - 6 \times 5 \text{ matrix} \\ B - 5 \times 2 \text{ matrix} \end{array}\right\} AB - 6 \times 2 \text{ matrix}$$

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ \boxed{-1 \quad -2 \quad -3 \quad -4} \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} \qquad (AB)_{3,2} = -1.0 - 2.1 - 3.0 - 4.(-1) = 2$$

$$A = \begin{bmatrix} -1 & 7 \\ 0 & 4 \\ 3 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \qquad AB = \begin{bmatrix} -7 & 13 \\ -4 & 8 \\ 2 & -1 \end{bmatrix}$$

Th | A – $m \times n$ matrix, B, C – appropriate size matrices

$A(BC) = (AB)C$

$A(B+C) = AB + AC$

$(B+C)A = BA + CA$

$r(AB) = (rA)B = A(rB)$ for scalar r

$I_m A = A = A I_n$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad AB = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad BA = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow AB = -BA$$

2.1./ 5. $A = \begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} \qquad AB = \begin{bmatrix} -7 & 4 \\ 7 & -6 \\ 12 & -7 \end{bmatrix}$

7. $\left.\begin{array}{l} A - 5 \times 3 \text{ matrix} \\ AB - 5 \times 7 \text{ matrix} \end{array}\right\} \Rightarrow B - 3 \times 7$ matrix

8. $BC - 3 \times 4$ matrix $\Rightarrow B - 3$ rows

9. $A = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix} \qquad AB = \begin{bmatrix} 23 & 5(k-2) \\ -9 & 15+k \end{bmatrix} \qquad BA = \begin{bmatrix} 23 & 15 \\ 3(2-k) & 15+k \end{bmatrix} \qquad AB = BA \Rightarrow k = 5$

Def. A – $n\times n$ matrix, $k\in\mathbb{N}$, $k\geq 0$

$A^k = \underbrace{AA\dots A}_{k \text{ factors}}$, $k=0$, $A^0 = I$

$$A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \quad A^2 = \begin{bmatrix} \cos^2\varphi - \sin^2\varphi & -2\sin\varphi\cos\varphi \\ 2\sin\varphi\cos\varphi & \cos^2\varphi - \sin^2\varphi \end{bmatrix} = \begin{bmatrix} \cos 2\varphi & -\sin 2\varphi \\ \sin 2\varphi & \cos 2\varphi \end{bmatrix}$$

$$K = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad M = \begin{bmatrix} 3 & 21 & 0 & 5 \\ 15 & 20 & 19 & 14 \\ 13 & 5 & 3 & 3 \\ 16 & 18 & 9 & 5 \end{bmatrix} \quad KM = \begin{bmatrix} 47 & 64 & 31 & 27 \\ 44 & 43 & 31 & 22 \\ 29 & 23 & 12 & 8 \\ 16 & 18 & 9 & 5 \end{bmatrix}$$

$$KM = \begin{bmatrix} 16 & 30 & 43 & 50 & 15 & 22 & 29 & 46 \\ 15 & 14 & 30 & 30 & 15 & 21 & 20 & 30 \\ 1 & 9 & 21 & 21 & 14 & 5 & 1 & 21 \\ 1 & 9 & 1 & 7 & 0 & 0 & 1 & 7 \end{bmatrix} \quad M = \begin{bmatrix} 1 & 16 & 13 & 20 & 0 & 1 & 9 & 16 \\ 14 & 5 & 9 & 9 & 1 & 16 & 19 & 9 \\ 0 & 0 & 20 & 14 & 14 & 5 & 0 & 14 \\ 1 & 9 & 1 & 7 & 0 & 0 & 1 & 7 \end{bmatrix}$$

An ape imitating an ape is aping.

Transpose of a matrix

Def. A – $m\times n$ matrix

A^T – $n\times m$ form of the same matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad A^2 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} \quad AB = \begin{bmatrix} -1 & 4 \\ -1 & 10 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \quad B^T = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$(A^T)^2 = \begin{bmatrix} 7 & 15 \\ 10 & 22 \end{bmatrix} \quad A^T B^T = \begin{bmatrix} 7 & 2 \\ 10 & 2 \end{bmatrix}$$

$$(A^2)^T = (A^T)^2 \qquad (AB)^T \neq A^T B^T$$

Th A, B - appropriate size matrices

$(A^T)^T = A$

$(A+B)^T = A^T + B^T$

$(rA)^T = rA^T$, r - scalar

$(AB)^T = B^T A^T$

15. a) F b) F c) T d) T e) F

19. $B = \begin{bmatrix} a & b & a+b \end{bmatrix}$

$A = \begin{bmatrix} x \end{bmatrix}$

$AB = \begin{bmatrix} ax & bx & (a+b)x \end{bmatrix}$

21. Columns of A are linearly dependent.

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Lecture 6 - Invertibility of matrices

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\det(A) = ad - bc = 0$

$\Rightarrow A$ - singular

$A = \begin{bmatrix} 3 & -2 \\ 1 & x \end{bmatrix}$ $\det(A) = 3x + 2 = 0$

$\Rightarrow x = -\frac{2}{3}$, A - singular

$A = \begin{bmatrix} 4 & -2 \\ -3 & 5 \end{bmatrix}$ $\det(A) = 16$ $\Rightarrow \exists A^{-1}$ $A^{-1} = \frac{1}{14} \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$

Inverse of a matrix

Th A - invertible $n \times n$ matrix

$\forall b \in \mathbb{R}^n$, $Ax = b$ has unique solution $x = A^{-1} b$

Th A, B – $n \times n$ invertible matrices
A^{-1} – invertible, $(A^{-1})^{-1} = A$
AB – invertible, $(AB)^{-1} = B^{-1}A^{-1}$
A^T – invertible, $(A^T)^{-1} = (A^{-1})^T$

$$\text{Key} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{Key}^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Computing the inverse of a matrix

Th A – $n \times n$ matrix invertible $\iff$ A – row equivalent to I_n

2.2.1

1. $A = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$ $\det(A) = 2$ $A^{-1} = \frac{1}{2} \begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -\frac{5}{2} & 4 \end{bmatrix}$

5. $8x_1 + 6x_2 = 2$
$5x_1 + 4x_2 = -1$
$\begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix} x = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $x = A^{-1}b = \frac{1}{2} \begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 14 \\ -18 \end{bmatrix} = \begin{bmatrix} 7 \\ -9 \end{bmatrix}$

19. A, B, C – $n \times n$ invertible matrices
$B^{-1}(A+X)C^{-1} = I_n$ $X = CB - A$

31. $A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$ $A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix}$ $\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{array}\right]$

Invertible matrix theorem

Th A – $n \times n$ matrix

$$\left.\begin{array}{ll} A\text{-invertible} & A\text{-row equivalent to } I_n \\ A\text{-}n \text{ pivots} & Ax = 0 \text{ - only trivial solution} \\ A\text{-independent columns} & \text{columns of } A \text{ - span } \mathbb{R}^n \\ A^T\text{-invertible} & Ax = b \quad \exists x \text{ for } \forall b \in \mathbb{R}^n \\ \exists C\text{-}n \times n \text{ s.t. } CA = I & \exists D\text{-}n \times n \text{ s.t. } AD = I \end{array}\right\} \text{equivalent}$$

$Ax=b$, A – invertible $n\times n$ matrix

$[A \mid b] \to [I_n \mid A^{-1}b]$

A – $n\times n$ matrix, n linearly independent columns

$\Rightarrow A^2$ – invertible

Def. $T: \mathbb{R}^n \to \mathbb{R}^n$ (linear) – invertible if $\exists S$, $S: \mathbb{R}^n \to \mathbb{R}^n$
s.t. $\forall x \in \mathbb{R}^n$, $S(T(x)) = x$ and $T(S(x)) = x$, $S = T^{-1}$ – inverse of T

Th $T: \mathbb{R}^n \to \mathbb{R}^n$ – invertible if A – invertible $T(x) = Ax$
$\Rightarrow T^{-1}(x) = A^{-1}x$

2.3. / 3.

$$A = \begin{bmatrix} 5 & 0 & 0 \\ -3 & -7 & 0 \\ 8 & 5 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 5 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$ invertible

15. not invertible

27. AB – invertible
$\Rightarrow \exists W$ s.t. $ABW = I$
$\Big\}$ $\Rightarrow W = (AB)^{-1} = B^{-1}A^{-1}$
$\Rightarrow A, B$ – invertible