Assignment 1
Derivation and analysis of PDE's

Numerical Analysis For PDE's (WI3730TU)
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Finite, simply-connected open domain: $\Omega \subset \mathbb{R}^2$
Smooth boundary: $\partial\Omega$
Local rate of production: $f(x, y, t) > 0$
Diffusion coefficient: $k(x, y) > 0$

1. Concentration of substance: $c(x, y, t)$ $[m^{-2}]$
   Flux density: $\mathbf{J}(x, y, t)$

   Global conservation law: $\frac{\partial Q}{\partial t} = -\phi + R$

   $Q = \iint_\Omega c \, d\Omega$

   $R = \iint_\Omega f \, d\Omega$

   $\phi = \int_{\partial\Omega} \mathbf{J} \cdot \, d\boldsymbol{\partial\Omega}$

   Global conservation law: $\frac{\partial}{\partial t} \iint_\Omega c \, d\Omega = -\int_{\partial\Omega} \mathbf{J} \cdot \, d\boldsymbol{\partial\Omega} + \iint_\Omega f \, d\Omega$

   $\frac{\partial}{\partial t} \iint_\Omega [m^{-2}] \, d\Omega = -\int_{\partial\Omega} [\mathbf{J}] \cdot \, d\boldsymbol{\partial\Omega} + \iint_\Omega [f] \, d\Omega$

   $\frac{\partial}{\partial t}[1] = -[\mathbf{J}][m] + [f][m^2]$

   $[s^{-1}] = -[\mathbf{J}][m] + [f][m^2]$

   $[\mathbf{J}] = [m^{-1}s^{-1}]$

   $[f] = [m^{-2}s^{-1}]$

2. $\phi = \int_{\partial\Omega} \mathbf{J} \cdot \, d\boldsymbol{\partial\Omega} = \iint_\Omega \nabla \cdot \mathbf{J} \, d\Omega$ Divergence Theorem

   $\frac{\partial}{\partial t} \iint_\Omega c \, d\Omega = -\iint_\Omega \nabla \cdot \mathbf{J} \, d\Omega + \iint_\Omega f \, d\Omega$

   If this holds for any Ω, then local conservation law: $\frac{\partial c}{\partial t} = -\nabla \cdot \mathbf{J} + f$

3. $\mathbf{J}_{\mathbf{diffusion}} = -k\nabla c$ Fick's Law

   $[\mathbf{J}] = [k]\nabla[c]$

   $[m^{-1}s^{-1}] = [k]\nabla[m^{-2}]$

   $[m^{-1}s^{-1}] = [k][m^{-3}]$

   $[k] = [m^2 s^{-1}]$

   $\frac{\partial c}{\partial t} - \nabla \cdot (k\nabla c) = f$

4. Initial condition: $c(x, y, 0) = 0, \forall (x, y) \in \overline{\Omega}$

5. $\alpha = \frac{\mathbf{J}\cdot\mathbf{n}}{c}$

   $[\alpha] = \frac{[\mathbf{J}]}{[c]}$

   $[\alpha] = \frac{[m^{-1}s^{-1}]}{[m^{-2}]}$

   $[\alpha] = [ms^{-1}]$

   $\alpha = \frac{-k\nabla c\cdot\mathbf{n}}{c}$

   $\alpha c + k\nabla c \cdot \mathbf{n} = 0$

   $\alpha c + kD_{\mathbf{n}}c = 0$ Robin boundary condition

6. (a) Steady-state => $\frac{\partial c}{\partial t} = 0$

   $-\nabla \cdot (k\nabla c) = f$

   $-k\nabla \cdot \nabla c - (\nabla k)\cdot(\nabla c) = f$

   $-k\triangle c - (\nabla k)\cdot(\nabla c) = f$

   $-k\frac{\partial^2 c}{\partial x^2} - k\frac{\partial^2 c}{\partial y^2} - \frac{\partial k}{\partial x}\frac{\partial c}{\partial x} - \frac{\partial k}{\partial y}\frac{\partial c}{\partial y} = f$

   Variables: x, y
   Coefficients of second-order derivatives of c with respect to x and y are of the same sign. => elliptic

   (b) $-k\triangle c - (\nabla k)\cdot(\nabla c) = f$

   $k = 1 \Rightarrow -\triangle c = f, \mathbf{x} \in \Omega$ Poisson's equation

   Robin boundary condition: $\alpha c + D_{\mathbf{n}}c = 0, \mathbf{x} \in \partial\Omega$

   Claim: There exists at most one solution with Robin boundary condition.
   <u>Proof</u>:
   Assume there exist two solutions c_1 and c_2 such that $c_1 - c_2 \neq 0$, $c_1, c_2 \in C^2(\Omega)\bigcap C(\overline{\Omega})$.

   $-\triangle c_1 = f, \mathbf{x} \in \Omega$ (1)
   $\alpha c_1 + D_{\mathbf{n}}c_1 = 0, \mathbf{x} \in \partial\Omega$ (2)
   $-\triangle c_2 = f, \mathbf{x} \in \Omega$ (3)
   $\alpha c_2 + D_{\mathbf{n}}c_2 = 0, \mathbf{x} \in \partial\Omega$ (4)

   $(1) - (3) \Rightarrow -\triangle(c_1 - c_2) = f - f, \mathbf{x} \in \Omega$
   $(2) - (4) \Rightarrow \alpha(c_1 - c_2) + D_{\mathbf{n}}(c_1 - c_2) = 0, \mathbf{x} \in \partial\Omega$

   Let $v = c_1 - c_2 \neq 0$, $\mathbf{x} \in C^2(\Omega)\bigcap C(\overline{\Omega})$.
   $\triangle v = 0, \mathbf{x} \in \Omega$
   $\alpha v + D_{\mathbf{n}}v = 0 \Rightarrow \alpha v = -D_{\mathbf{n}}v = -\nabla v \cdot \mathbf{n}, \mathbf{x} \in \partial\Omega$

   $\iint_\Omega \nabla\cdot(v\nabla v)\, d\Omega = \iint_\Omega v\triangle v\, d\Omega + \iint_\Omega (\nabla v)^2\, \Omega = \iint_\Omega (\nabla v)^2\, d\Omega \geq 0$

   $\iint_\Omega \nabla\cdot(v\nabla v)\, d\Omega = \int_{\partial\Omega} v\nabla v \cdot\, d\partial\boldsymbol{\Omega} = \int_{\partial\Omega} v\nabla v\cdot\mathbf{n}\, d\partial\Omega = \int_{\partial\Omega} -\alpha v^2\, d\partial\Omega \leq 0$ Divergence Theorem

   $\Rightarrow \iint_\Omega \nabla\cdot(v\nabla v)\, d\Omega = 0$

   $\Rightarrow \iint_\Omega (\nabla v)^2\, d\Omega = 0 \Rightarrow \nabla v = 0, \mathbf{x} \in \Omega$

   $\Rightarrow \int_{\partial\Omega} -\alpha v^2\, d\partial\Omega = 0, \alpha > 0 \Rightarrow v = 0, \mathbf{x} \in \partial\Omega$

   $\Rightarrow v = 0, \mathbf{x} \in \overline{\Omega}$ Contradiction!

   $\therefore$ Solution is unique.