Numerical Stochastic Differential Equations

Kaloyan Yanchev

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1. A stochastic process $\{W(t),\, t \geq 0\}$ is a Wiener process if
   1) $W(0) = 0$, w.p. 1
   2) $\{W(t),\, t \geq 0$ has independent and stationary increments.
   3) $W(t) \sim N(0, \sigma^2 t)$, $\forall t > 0$ (usually $\sigma = 1$)
   Show that the process $X(t) = tW(\frac{1}{t})$ for $t > 0$ and $X(0) = 0$ is a Wiener process.

   Since $X(0) = 0$, condition 1 is satisfied.

   To investigate condition 2, take $t_1 < t_2$:

   $$X(t_2) - X(t_1) = t_2 W(\frac{1}{t_2}) - t_1 W(\frac{1}{t_1})$$

   Since $W(t)$ is a Wiener process, so is $W(\frac{1}{t})$ and it inherits the independent and stationary increments properties. Therefore $X(t)$ has increments that do not depend on the past, but only on the difference $t_2 - t_1$, thus condition 2 is satisfied.

   In order to satisfy condition 3, two statements need to be proved:
   1) $\mathbb{E}[X(t)] = 0$
   2) $Var[X(t)] = \sigma^2 t$
   Since $W(\frac{1}{t})$ is Wiener process, $W(\frac{1}{t}) \sim N(0, \frac{\sigma^2}{t}) => \mathbb{E}[W(\frac{1}{t})] = 0, Var[W(\frac{1}{t})] = \frac{\sigma^2}{t}$.

   $$\mathbb{E}[X(t)] = \mathbb{E}[tW(\frac{1}{t})] = t\mathbb{E}[W(\frac{1}{t})] = 0$$

   $$Var[X_t] = Var[tW(\frac{1}{t})] = t^2 Var[W(\frac{1}{t})] = t^2 \frac{\sigma^2}{t} = \sigma^2 t$$

   Thus condition 3 is also satisfied, therefore $X(t)$ is indeed a Wiener process.

2. Show that

   $$\Phi_t = cos(aW_t + b)$$

   is a solution of the Itô SDE

   $$d\Phi_t = -\frac{1}{2}a^2 \Phi_t dt - a\sqrt{1 - \Phi_t^2} dW_t, \Phi_0 = cos(b)$$

   Itô's Differential Rule:

   $$X_t = W_t => dX_t = dW_t, g = 1$$

   Let $\phi(x) = cos(ax + b)$:

   $$\frac{\partial \phi}{\partial t} = 0, \frac{\partial \phi}{\partial x} = -asin(ax + b), \frac{\partial^2 \phi}{\partial x^2} = -a^2 cos(ax + b)$$

   Since $d\Phi_t \overset{\text{Itô}}{=} \frac{\partial \phi}{\partial t} dt + \frac{\partial \phi}{\partial x} dX_t + \frac{1}{2} g^2 \frac{\partial^2 \phi}{\partial x^2} dt$:

   $$d\Phi_t \overset{\text{Itô}}{=} -asin(aX_t + b)dX_t - \frac{1}{2}a^2 cos(aX_t + b)dt$$

   Substitute back $X_t = W_t$:

   $$d\Phi_t \overset{\text{Itô}}{=} -asin(aW_t + b)sW_t - \frac{1}{2}a^2 cos(aW_t + b)dt$$

Since $sinx = \sqrt{1 - cos^2 x}$:

$$d\Phi_t \overset{\text{Itô}}{=} -a\sqrt{1 - cos^2(aW_t + b)}dW_t - \frac{1}{2}a^2 cos(aW_t + b)dt$$

Since $\Phi_t = cos(aW_t + b)$:

$$d\Phi_t \overset{\text{Itô}}{=} -\frac{1}{2}a^2\Phi_t dt - a\sqrt{1 - \Phi_t^2}dW_t$$

Since W_t - Wiener process => $W_0 = 0$:

$$\Phi_0 = cos(aW_0 + b) = cos(b)$$

3. Derive the Itô SDE for the process $\Phi_t = W_t^4$.

Itô's Differential Rule:

$$X_t = W_t => dX_t = dW_t, g = 1$$

Let $\phi(x) = x^4$:

$$\frac{\partial \phi}{\partial t} = 0, \frac{\partial \phi}{\partial x} = 4x^3, \frac{\partial^2 \phi}{\partial x^2} = 12x^2$$

Since $d\Phi_t \overset{\text{Itô}}{=} \frac{\partial \phi}{\partial t}dt + \frac{\partial \phi}{\partial x}dX_t + \frac{1}{2}g^2\frac{\partial^2 \phi}{\partial x^2}dt$:

$$d\Phi_t \overset{\text{Itô}}{=} 4X_t^3 dX_t + 6X_t^2 dt$$

Substitute back $X_t = W_t$:

$$d\Phi_t \overset{\text{Itô}}{=} 6W_t^2 dt + 4W_t^3 dW_t$$

Since $\Phi_t = W_t^4 => W_t = \Phi_t^{\frac{1}{4}}$:

$$d\Phi_t \overset{\text{Itô}}{=} 6\Phi_t^{\frac{1}{2}} dt + 4\Phi_t^{\frac{3}{4}} dW_t$$

Since W_t - Wiener process => $W_0 = 0$:

$$\Phi_0 = W_0^4 = 0$$

4. The stochastic process X_t satisfies the stochastic differential equation

$$dX_t = -\gamma X_t dt + X_t^3 dW_t$$

1) Calculate the differential equations for $\mathbb{E}[X_t]$ and $\mathbb{E}[X_t^2]$.
2) Derive the differential equation for the variance $V(X_t)$ and solve it.

1.1) Calculate the differential equation for $\mathbb{E}[X_t]$.

$$d\mathbb{E}[X_t] = \mathbb{E}[dX_t] = \mathbb{E}[-\gamma X_t dt + X_t^3 dW_t]$$

Since $\mathbb{E}[dW_t] = 0$:

$$d\mathbb{E}[X_t] = \mathbb{E}[-\gamma X_t dt] = -\gamma\mathbb{E}[X_t]dt$$

Differentiating with respect to t:

$$\frac{\partial}{\partial t}\mathbb{E}[X_t] = -\gamma\mathbb{E}[X_t]$$

1.2) Calculate the differential equation for $\mathbb{E}[X_t^2]$.
Itô's Differential Rule:

$$dX_t = -\gamma X_t dt + X_t^3 dW_t, g = X_t^3$$

Let $\phi(x) = x^2$:

$$\frac{\partial \phi}{\partial t} = 0, \frac{\partial \phi}{\partial x} = 2x, \frac{\partial^2 \phi}{\partial x^2} = 2$$

Since $d\Phi_t \overset{\text{Itô}}{=} \frac{\partial \phi}{\partial t}dt + \frac{\partial \phi}{\partial x}dX_t + \frac{1}{2}g^2\frac{\partial^2 \phi}{\partial x^2}dt$:

$$d\Phi_t \overset{\text{Itô}}{=} 2dX_t + X_t^6 dt$$

Substituting back $\Phi_t = X_t^2$ and $dX_t = -\gamma X_t dt + X_t^3 dW_t$

$$dX_t^2 \overset{\text{Itô}}{=} -2\gamma X_t^2 dt - 2X_t^4 dW_t + X_t^6 dt$$

$$d\mathbb{E}[X_t^2] = \mathbb{E}[dX_t^2] = \mathbb{E}[-2\gamma X_t^2 dt - 2X_t^4 dW_t + X_t^6 dt]$$

Since $\mathbb{E}[dW_t] = 0$:

$$d\mathbb{E}[X_t^2] = \mathbb{E}[-2\gamma X_t^2 dt + X_t^6] = -2\gamma \mathbb{E}[X_t^2]dt + \mathbb{E}[X_t^6]dt$$

Differentiating with respect to t:

$$\frac{\partial}{\partial t}\mathbb{E}[X_t^2] = -2\gamma \mathbb{E}[X_t^2] + \mathbb{E}[X_t^6]$$

2.1) Derive the differential equation for the variance $V(X_t)$.

$$V(X_t) = \mathbb{E}[X_t^2] - (\mathbb{E}[X_t])^2$$

Differentiating with respect to t:

$$\frac{\partial}{\partial t}V(X_t) = \frac{\partial}{\partial t}(\mathbb{E}[X_t^2] - (\mathbb{E}[X_t])^2)$$

Applying chain rule:

$$\frac{\partial}{\partial t}V(X_t) = \frac{\partial}{\partial t}\mathbb{E}[X_t^2] - 2\mathbb{E}[X_t]\frac{\partial}{\partial t}\mathbb{E}[X_t]$$

Substituting with previously derived $\frac{\partial}{\partial t}\mathbb{E}[X_t^2] = -2\gamma\mathbb{E}[X_t^2] + \mathbb{E}[X_t^6]$ and $\frac{\partial}{\partial t}\mathbb{E}[X_t] = -\gamma\mathbb{E}[X_t]$:

$$\frac{\partial}{\partial t}V(X_t) = -2\gamma\mathbb{E}[X_t^2] + \mathbb{E}[X_t^6] + 2\gamma(\mathbb{E}[X_t])^2$$

Substituting back $V(X_t) = \mathbb{E}[X_t^2] - (\mathbb{E}[X_t])^2$:

$$\frac{\partial}{\partial t}V(X_t) = -2\gamma V(X_t) + \mathbb{E}[X_t^6]$$

2.2) Solve the differential equation for the variance $V(X_t)$.

Let $k = \mathbb{E}[X_t^6]$:

$$\frac{\partial}{\partial t}V(X_t) = -2\gamma V(X_t) + k$$

Rearrange the terms:

$$\frac{\partial}{\partial t}V(X_t) + 2\gamma V(X_t) = k$$

Multiplying by $e^{2\gamma t}$:

$$e^{2\gamma t}\frac{\partial}{\partial t}V(X_t) + 2\gamma V(X_t)e^{2\gamma t} = ke^{2\gamma t}$$

Since $\frac{\partial}{\partial t}(e^{2\gamma t}V(X_t)) = e^{2\gamma t}\frac{\partial}{\partial t}V(X_t) + 2\gamma V(X_t)e^{2\gamma t}$:

$$\frac{\partial}{\partial t}(e^{2\gamma t}V(X_t)) = ke^{2\gamma t}$$

Integrating with respect to t:

$$e^{2\gamma t}V(X_t) = \frac{k}{2\gamma}e^{2\gamma t} + C$$

Dividing by $e^{2\gamma t}$:

$$V(X_t) = \frac{k}{2\gamma} + Ce^{-2\gamma t}$$

Since $V(X_0) = \frac{k}{2\gamma} + C \Rightarrow C = V(X_0) - \frac{k}{2\gamma}$:

$$V(X_t) = \frac{k}{2\gamma} + (V(X_0) - \frac{k}{2\gamma})e^{-2\gamma t}$$

5. Consider the deterministic differential equation for the process $x(t)$:

$$\frac{\partial x}{\partial t} = ax^2 - bx^3 + csinx, x(0) = 1, t \geq 0$$

where a,b,c are constants. Now assume that the parameter a is not fixed, but subject to fluctuations in time. So we replace a by the stochastic quantity A_t that can be described as stochastic process:

$$A_t = a + \xi(t),$$

where $\xi(t)$ is a zero mean Gaussian white noise process. The fluctuations in A_t will make the trajectory of $x(t)$ also noisy.
Derive the Itô SDE for the stochastic process X_t.

White noise process N_t in ODE = Stratonovitz SDE:

$$\frac{\partial x}{\partial t} \stackrel{\text{Str}}{=} (a + N_t)X_t^2 - bX_t^3 + csin(X_t) = aX_t^2 - bX_t^3 + csin(X_t) + X_t^2 N_t, X_0 = 1$$

Multiplying both sides by dt ($N_t dt = W_t$):

$$dX_t \stackrel{\text{Str}}{=} (aX_t^2 - bX_t^3 + csin(X_t))dt + X_t^2 dW_t, X_0 = 1$$

Since $g = X_t^2$:

$$\frac{\partial g}{\partial x} = 2X_t$$

Itô correction term:

$$\frac{1}{2} g \frac{\partial g}{\partial x} dt = X_t^3 dt$$

Itô SDE = Stratonovitz SDE + Itô correction term:

$$dX_t \stackrel{\text{Itô}}{=} (aX_t^2 - bX_t^3 + csin(X_t) + X_t^3)dt + X_t^2 dW_t, X_0 = 1$$

6. Show by means of a counter example that $\mathbb{E}[\int_{t_0}^t G_s\, dW_s]$ in the notes does not always hold for Stratonovich integrals.

Take Stratonovitz integral:

$$\int_{t_0}^t W_s\, dW_s \stackrel{\text{Str}}{=} \frac{W_s^2}{2}|_{t_0}^t = \frac{W_t^2}{2} - \frac{W_{t_0}^2}{2}$$

Since $\mathbb{E}[W_t^2] = t$:

$$\mathbb{E}[\int_{t_0}^t W_s\, dW_s] \stackrel{\text{Str}}{=} \mathbb{E}[\frac{W_t^2}{2} - \frac{W_{t_0}^2}{2}] = \frac{1}{2}(t - t_0)$$

7. Derive the Stratonotivz SDE for the process $\Phi_t = W_t^4$.

Itô SDE:

$$d\Phi_t \stackrel{\text{Itô}}{=} 6\Phi_t^{\frac{1}{2}} dt + 4\Phi_t^{\frac{3}{4}} dW_t$$

Since $g = 4\Phi_t^{\frac{3}{4}}$:

$$\frac{\partial g}{\partial \phi} = 3\Phi_t^{-\frac{1}{4}}$$

Itô correction term:

$$\frac{1}{2} g \frac{\partial g}{\partial \phi} dt = 6\Phi_t^{\frac{1}{2}} dt$$

Stratonovitz SDE = Itô SDE - Itô correction term:

$$d\Phi_t \stackrel{\text{Str}}{=} 4\Phi_t^{\frac{3}{4}} dW_t$$

8. Consider the Itô SDE as a particle model for dissolved matter in a river:

$$dX_t = Udt + gdW_t, X_0 = 0, t \geq 0$$

$$dM_t = -aM_t dt, M_0 = 1, t \geq 0,$$

where X_t is the position of a particle and M_t its mass. U is the velocity of the water, and g and a are positive constants. Because of a chemical reaction, the dissolved matter slowly decays, and the mass of the particles decreases in time. Derive the partial differential equation for the probability $p(x, m, t)$ at time t to find a particle at position x, with mass m.

$$d\begin{pmatrix} X_t \\ M_t \end{pmatrix} \overset{\text{Itô}}{=} \begin{pmatrix} U \\ -aM_t \end{pmatrix} dt + \begin{pmatrix} g \\ 0 \end{pmatrix} dW_t, \begin{pmatrix} X_0 \\ M_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$=> \vec{f}(X_t, M_t, t) = \begin{pmatrix} U \\ -aM_t \end{pmatrix}, \vec{g}(X_t, M_t, t) = \begin{pmatrix} g \\ 0 \end{pmatrix}$$

Fokker-Planck equation $\frac{\partial p}{\partial t} = -\sum \frac{\partial (f_i p)}{\partial x_i} + \frac{1}{2}\sum\sum \frac{\partial^2((gg^T)_{ij}p)}{\partial x_i \partial x_j}, t \geq t_0$:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}(Up) - \frac{\partial}{\partial m}(-amp) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(g^2 p) = -U\frac{\partial p}{\partial x} + a\frac{\partial mp}{\partial m} + \frac{g^2}{2}\frac{\partial^2 p}{\partial x^2}$$

$$p(x, m, 0) = \delta(x - x_0)\delta(m - m_0) = \delta(x)\delta(m - 1)$$