Population Dynamics
SDE Project

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Consider the following population model for a population y:

$$\frac{\partial y}{\partial t} = b(t)y - d(t)y$$

In a varying environment, the per capita birth $b(t)$ and death rates $d(t)$, would be functions of additional environmental variables and would have the forms $b(t, v_1, v_2, \cdots, v_n)$ and $d(t, v_1, v_2, \cdots, v_n)$, respectively, where $v_1, v_2, \cdots, v_n$ represent n different environmental variables. Thus, as $v_1, v_2, \cdots, v_n$ vary, the per capita birth and death rates also vary. This suggests that an approximate way to include environmental variability, without modeling additional environmental factors, would be to vary the per capita birth and death rates in a random manner.
The new SDEs that can be derived satisfy:

$$dy(t) = (b(t) - d(t))y(t)dt + \sqrt{b(t) + d(t)}\sqrt{y(t)}dW_1(t)$$

$$db(t) = \beta_1(b_e - b(t))dt + \alpha_1 dW_2(t)$$

$$dd(t) = \beta_2(d_e - d(t))dt + \alpha_2 dW_3(t),$$

where $\beta_1, \beta_2, \alpha_1, \alpha_2, b_e, d_e$ are constants and $W_i(t)$ are independent Wiener processes for $i = 1, 2, 3$.

1. Apply the Euler-Maruyama scheme to approximate the model numerically. Analyze by means of experiments the strong and weak order of convergence of this scheme. Repeat the experiments for different simulation times, and for various values of $\alpha_{1,2}, \beta_{1,2}$. A good starting point would be $\beta_1 = \beta_2 = 1, \alpha_1 = \alpha_2 = 0.5, b_e = 1, d_e = 1.4$. Suppose that $y(0) = 30$, $b(0) = b_e$, and $d(0) = d_e$.

   Euler-Maruyama scheme:

   $$y_{n+1} = y_n + (b(t_n) - d(t_n))y_n\Delta t + \sqrt{b(t_n) + d(t_n)}\sqrt{y_n}\Delta W_1(t_n)$$

   $$b_{n+1} = b_n + \beta_1(b_e - b_n)\Delta t + \alpha_1\Delta W_2(t_n)$$

   $$d_{n+1} = d_n + \beta_2(d_e - d_n)\Delta t + \alpha_2\Delta W_3(t_n)$$

   In order to evaluate the approximation scheme, first, a Wiener process should be derived. For this purpose, a method is designed such that the values of the three Wiener processes are derived based on a reference time step (in this case 0.0001). This way for time steps bigger than the reference one, the relevant values can be selected and the approximation is still correctly evaluated. Furthermore, the randomness of the Wiener process can be seeded such that results are reproducible as well as the fact that for the convergence tests it is important to compare results based on the same random sequences (or generally said for the same Wiener process values).

   Also, it should be noted that for the whole report, the following parameters are always the same: $y(0) = 30, b_e = 1, d_e = 1.4$. Another important remark concerns the other formula constants in order to perform reasonable and sensible tests for convergence. There are three sources of randomness (namely W_1, W_2 and W_3) in each of the three simulated parameters (respectively y, b, and d), however, in order to properly evaluate the approximations abilities of the different schemes, only a single isolated randomness should be considered which is why for all convergence tests the following parameters are always the same $\alpha_1 = \alpha_2 = 0$, since they regulate the randomness respectively in b and d. Thus the convergence tests examine the noise in the computation of y and are independent from the values of β_1 and β_2. Lastly, it should be noted that since the strong and weak errors are analyzed in terms of upper-bounds, only the maximal error for each test case is noted and represented in the plots.

Now that all the preliminaries have been explained, the actual convergence tests should be examined. First of all, it should be noted that all the tests are run for a number of samples (always 100) and the sample index is used as a seed for the randomness such that for the same sample, but for different time steps, the same Wiener process values are used. Since there is no analytical solution to the SDE, a reference solution is used to evaluate the results that is derived with a significantly small time step (namely 0.0001), assuming it will be closer to the real solution than using bigger time step. For the test for the weak convergence, first the average population value at each time frame of the reference solution is computed over 100 samples, and then the same is done for the solution at hand, derived with the test specified time step. This way the expected population value at this time frame is derived and then the maximal absolute difference between the reference and test solutions for all overlapping time frames is considered the weak error. For the test for strong convergence, the procedure is similar, however, all the values for all time steps for each sample is saved, rather than the average. Then the difference between the population values at the overlapping time frames for the same sample index (thus same Wiener process) are computed and finally for each time frame, the average difference is evaluated, effectively representing the expected difference between the population values for the reference and test solutions. Lastly, the strong error is considered to be the maximal error found for all the time frames.

Now that the convergence testing mechanism has been covered, the actual results can be analyzed. Note that in all test cases, the reference solution is obtained with time step of 0.0001 and the test case consider time steps 0.001, 0.005, 0.01 and 0.02. Figure 1 represents the strong and weak errors for Euler-Maruyama. As can be observed, the strong error for Euler-Maruyama does indeed follow a pattern close to a square root function based on the time step, while the weak error is closer to a linear function. This is consistent with the theoretically derived upper bounds that state that Euler-Maruyama is $O(\sqrt{\Delta t})$ in the strong sense and $O(\Delta t)$ in the weak sense.

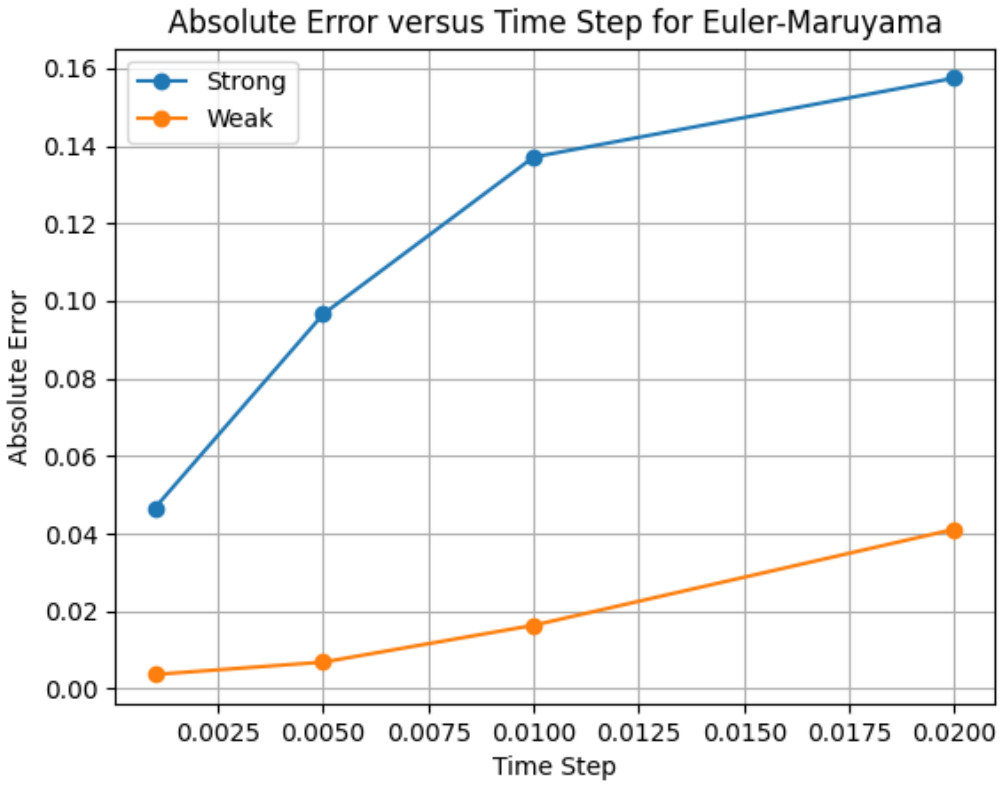

Figure 1: Absolute strong and weak errors for Euler-Maruyama approximations over 100 samples with $\alpha_1 = \alpha_2 = 0$ and default $b_e = 1, d_e = 1.4, \beta_1 = \beta_2 = 1$ for time steps 0.001, 0.005, 0.01 and 0.02, compared to the reference solution, derived with time step 0.0001.

2. Make pictures of the population distribution at three different times and compare the cases with $\alpha_i = 0.5$ and $\beta_i = 1$ and with $\alpha_i = \beta_i = 0$ for $i = 1, 2$.

   Figure 2 represents the population distribution for the Euler-Maruyama approximation at three time frames, for two executions – with noise (meaning $\alpha_1 = \alpha_2 = 0.5, \beta_1 = \beta_2 = 1$) and without noise (meaning $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$), obtained over 100 samples. As can be observed the noise approximation has a wider range of values than the one without noise, however, the frequency peaks are generally consistent.

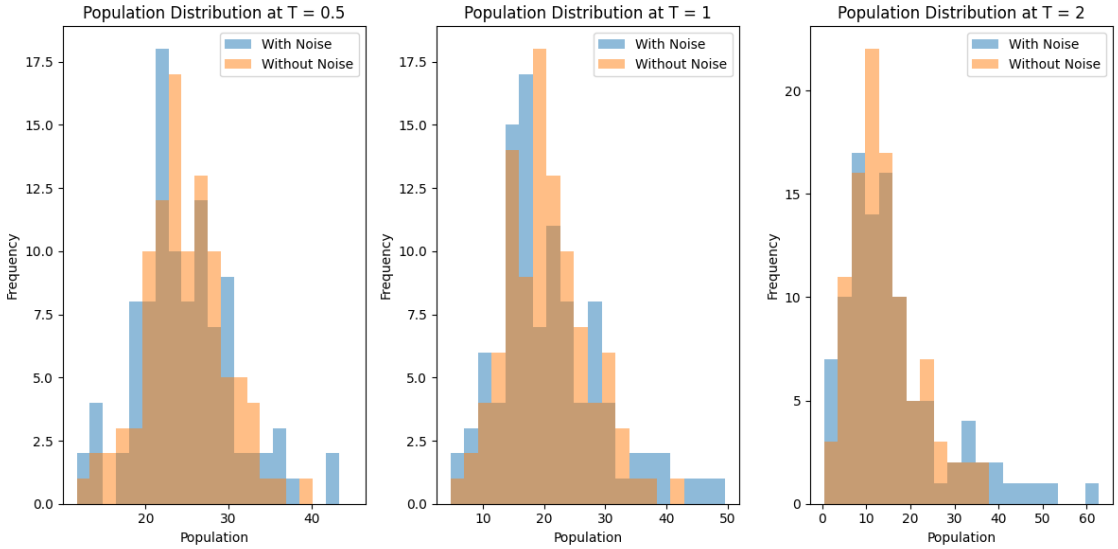

Figure 2: Distributions of population dynamics with $(\alpha_1 = \alpha_2 = 0.5, \beta_1 = \beta_2 = 1)$ and without $(\alpha_1 = \alpha_2 = 0, \beta_1 = \beta_2 = 0)$ noise, with timestep $dt = 0.001$, obtained over 100 samples at times $t_1 = 0.5, t_2 = 1, t_3 = 2$.

3. Use a Milstein scheme to numerically simulate the equations in (a) and investigate the behavior of the error as a function of the size of the time step and explain your results.

   Milstein term is $\frac{1}{2}g(x)\frac{\partial g}{\partial x}(\Delta W_n^2 - \Delta t)$.

$$g_y = \sqrt{b(t_n) + d(t_n)}\sqrt{y_n} \Rightarrow \frac{\partial g_y}{\partial y} = \frac{\sqrt{b(t_n) + d(t_n)}}{2\sqrt{y_n}} \Rightarrow \text{Milstein term} = \frac{b(t_n) + d(t_n)}{4}(\Delta W_1^2(t_n) - \Delta t)$$

$$g_b = \alpha_1 \Rightarrow \frac{\partial g_b}{\partial b} = 0 \Rightarrow \text{Milstein term} = 0$$

$$g_d = \alpha_2 \Rightarrow \frac{\partial g_d}{\partial d} = 0 \Rightarrow \text{Milstein term} = 0$$

   Milstein scheme:

$$y_{n+1} = y_n + (b(t_n) - d(t_n))y_n\Delta t + \sqrt{b(t_n) + d(t_n)}\sqrt{y_n}\Delta W_1(t_n) + \frac{b(t_n) + d(t_n)}{4}(\Delta W_1^2(t_n) - \Delta t)$$

$$b_{n+1} = b_n + \beta_1(b_e - b_n)\Delta t + \alpha_1\Delta W_2(t_n)$$

$$d_{n+1} = d_n + \beta_2(d_e - d_n)\Delta t + \alpha_2\Delta W_3(t_n)$$

   The same tests for the convergence as the beforehand explained ones have been applied to the Milstein approximation scheme. Figure 3 represents the strong and weak errors for Milstein scheme. As can be observed, both errors are almost linear to the time step, which is consistent with the theoretically derived upper bounds that states that both the strong and weak errors of the Milstein scheme are $O(\Delta t)$. Furthermore, the difference in weak and strong errors between Euler-Maruayama and Milstein scheme can be analyzed in order to observe the contribution of the Milstein term as an approximation correction. Figure 4 compares the strong and weak errors of the two approximations. As can be observed, the weak errors of the two methods are comparable, however, the Milstein scheme is generally better. For the strong error, on the other hand, there is an evident difference that suggests that the Misltein scheme is globally noticeably more accurate. This is consistent with the theorical idea of the Milstein term which is not significant locally (weak error), however, is noticeable globally (strong error).

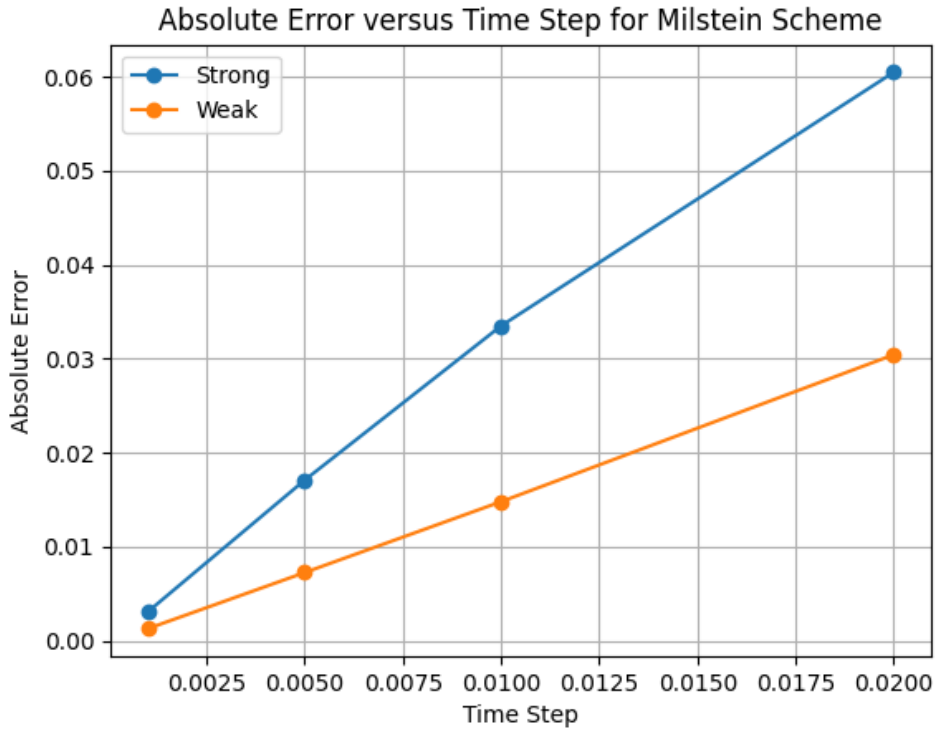

Figure 3: Absolute strong and weak errors for Milstein scheme approximations over 100 samples with $\alpha_1 = \alpha_2 = 0$ and default $b_e = 1, d_e = 1.4, \beta_1 = \beta_2 = 1$ for time steps 0.001, 0.005, 0.01 and 0.02, compared to the reference solution, derived with time step 0.0001.

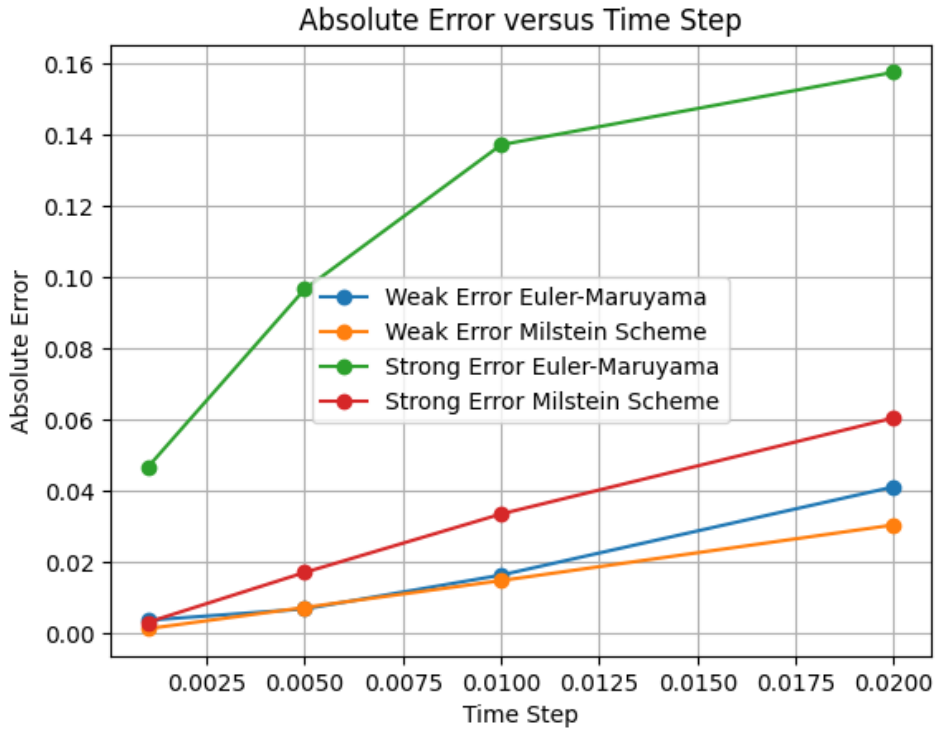

Figure 4: Absolute strong and weak errors for Euler-Maruyama and Milstein scheme approximations over 100 samples with $\alpha_1 = \alpha_2 = 0$ and default $b_e = 1, d_e = 1.4, \beta_1 = \beta_2 = 1$ for time steps 0.001, 0.005, 0.01 and 0.02, compared to the reference solution, derived with time step 0.0001.