1. a). soln: $f(x)=\frac{1}{2}x^TAx+b^Tx \quad x\in R^n \quad A\in R^{n\times n} \quad b\in R^n$

$\therefore f(x)=\frac{1}{2}x^T\cdot x\cdot A+b^Tx$

$\because f(x)=\frac{1}{2}[x_1,x_2\cdots x_n]\cdot\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}\cdot\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \ddots & \vdots\\ \vdots & \vdots & & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{bmatrix}+b^Tx$

$\therefore f(x)=\frac{1}{2}Ax^2+bx$

$\nabla_x f(x)=\frac{1}{2}\cdot 2Ax+b$

$\nabla_x f(x)=Ax+b$

b). soln: $f(x)=g(h(x))$

$\because$ the chain rule

$\therefore \nabla_x f(x)=g'(h(x))\nabla_x f(x)$

c). soln: According a). we know $\nabla_x f(x)=Ax+b$

$\therefore \nabla_x^2 f(x)=A$

d). soln: $f(x)=g(a^Tx)$

$\because$ the chain rule

$\therefore \nabla_x f(x)=g'(a^Tx)\cdot a$

$\therefore \nabla_x^2 f(x)=g''(a^Tx)\cdot a\cdot a^T$

problem 2

a). soln : $A = T\Lambda T^{-1} \Rightarrow AT = T\Lambda \quad T = [t^{(1)}, t^{(2)}, \cdots t^{(n)}] \quad \Lambda = diag(\lambda_1 \cdots \lambda_n)$

$$AT = T\Lambda \Rightarrow A \cdot [t^{(1)}, t^{(2)}, \cdots t^{(n)}] = [t^{(1)}, t^{(2)}, \cdots t^{(n)}] \cdot \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$\Rightarrow [At^{(1)} \; At^{(2)} \cdots At^{(n)}] = [\lambda_1 t^{(1)} \; \lambda_2 t^{(2)} \cdots \lambda_n t^{(n)}]$$

$$\Rightarrow At^{(i)} = \lambda_i t^{(i)} \quad i \in \{1, 2, \cdots n\}$$

b). soln : $A = U\Lambda U^T \Rightarrow AU = U\Lambda \quad U = [u^{(1)} u^{(2)} \cdots u^{(n)}] \quad \Lambda = diag(\lambda_1 \cdots \lambda_n)$

$$AU = U\Lambda \Rightarrow A [u^{(1)} \; u^{(2)} \cdots u^{(n)}] = [u^{(1)} \; u^{(2)} \cdots u^{(n)}] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$\Rightarrow [Au^{(1)} \; Au^{(2)} \cdots Au^{(n)}] = [\lambda_1 u^{(1)} \; \lambda_2 u^{(2)} \cdots \lambda_n u^{(n)}]$$

$$\Rightarrow Au^{(i)} = \lambda_i u^{(i)} \quad i \in \{1, 2, \cdots, n\}$$

c). soln : if A is PSD, denote $A \succeq 0$

according to a). and b). we know $At^{(i)} = \lambda_i t^{(i)} \quad i \in \{1, 2, \cdots n\}$

$Au^{(i)} = \lambda_i u^{(i)} \quad i \in \{1, 2, \cdots n\}$

$\because A \succeq 0 \quad \therefore \lambda_i \geq 0 \quad \because \lambda_i = \lambda_i(A)$ denote the ith eigenvalue of A

$\therefore \lambda_i(A) \geq 0$

problem 3:

a). soln: ∵ I_n is a matrix of size $n \times n$ with ones along the diagonal.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore |I_n| = |E|$

If I_n is positive definite $\Rightarrow |I_n - \lambda E| = 0$, all $\lambda > 0$

$\therefore \lambda = 1 \Rightarrow \lambda > 0$

$\therefore I_n$ is positive definite

b). soln: ∵ $z \in \mathbb{R}^n$ $A = zz^T$

$\therefore A \in \mathbb{R}^{n \times n} \Rightarrow A = A^T$

$$x^T A x = x^T z z^T x$$
$$= (x^T z)^2 \geq 0$$

$\therefore A = zz^T$ is positive semidefinite

c). soln: $Ax = 0$ $A = zz^T \Rightarrow zz^T x = 0$ $z^T x = 0$

if x is orthogonal to z, x is in the null-space of A.

So the dimension of the nullspace is $n-1$.

$$\therefore \text{rank}(A) + \text{null}(A) = n$$
$$\text{rank}(A) + (n-1) = n$$
$$\text{rank}(A) = 1$$

$\therefore$ the rank of A is 1

problem 3

d.). soln: if BAB^T is PSD, then $x^TBAB^Tx \geq 0 \ \forall x \in R^m$

$$x^TBAB^Tx = (x^TB)A(B^Tx) = (B^Tx)^TA(B^Tx)$$

where $z = B^Tx$ $z \in R^n$, and since A is PSD

$\therefore (B^Tx)^TA(B^Tx) = z^TAz \geq 0$

$\therefore BAB^T$ is PSD