Maximal Structural Balanced k-plex Enumeration in Signed Graphs

Kai Yao*†, Xinchen Zhang‡, Lijun Chang‡
* College of Computer Science, Sichuan Normal University, Chengdu, China
† Visual Computing and Virtual Reality Key Lab, Sichuan Normal University, Chengdu, China
‡ School of Computer Science, The University of Sydney, Sydney, Australia
kaiyao@sicnu.edu.cn, xczhang0508@outlook.com, lijun.chang@sydney.edu.au

Abstract*—Signed graphs, which represent interactions as friendly (positive edges) or antagonistic (negative edges), offer a powerful model to capture interesting structural properties of real-world phenomena. Recently, the structural balanced clique model has been proposed to identify polarized structures in signed graphs. A signed graph is a structural balanced clique if it is a clique (i.e., every pair of its vertices is connected by an edge) and is structural balanced (i.e., its vertices can be partitioned into two sets such that all intra-partition edges are positive and all cross-partition edges are negative). However, this model's rigidity restricts its applicability in practice. In this paper, we formulate the structural balanced k-plex, which relaxes structural balanced clique by allowing each vertex to miss a few connections or slightly violate structural balance theory. We prove that the problem of enumerating all maximal balanced k-plexes is #P-hard. To solve this problem, we first propose a backtracking algorithm MBPE-BK as well as optimization techniques to enhance its practical performance. However, MBPE-BK's efficiency is still hindered by the issue of overlapping candidate sets. To tackle this challenge, we then propose a novel algorithm MBPE with a better time complexity than MBPE-BK (i.e., $\mathcal{O}^*(2^\delta)$ v.s. $\mathcal{O}^*(3^{\delta D})$). Furthermore, we adopt the minimum-degree branching strategy to improve the worst-case time complexity of MBPE to $\mathcal{O}^*(\alpha_k^\delta)$, where $1 < \alpha_k < 2$ is a constant that depends only on k. Extensive experiments on real-world and synthetic datasets are conducted to demonstrate the efficiency of our algorithms and the effectiveness of our model.

I. Introduction

Signed graphs provide an enriched representation of conventional graphs by incorporating the concept of *polarity* to signify relationships between entities/vertices using *positive* and *negative* edge signs [1]. This enhancement enables the capture of various types of relationships in different domains. For example, in social networks, signed graphs model friend-foe relationships [2]. In opinion networks, they represent support-dissent opinions [3]. In protein-protein interaction networks, they model activation-inhibition interactions [4].

In the literature, many traditional graph analysis tasks have been extended to signed graphs by treating positive and negative edges differently, such as community detection [5], [6], link prediction [7], [8] and recommendation systems [9], [10]. Among them, the problem of structural balanced clique computation receives an increasing attention recently [11]–[15]. A graph is a structural balanced clique if (1) it is a clique (i.e., every pair of its vertices is connected by an edge), and (2) it obeys the structural balance theory (i.e., its vertices can be partitioned into two sets such that all intra-partition edges are positive and all inter-partition edges are negative). Identifying structural balanced cliques holds significant relevance in various practical domains, such as polarized community discovery in social networks [6], [16], [17], synonym-antonym groups detection in word networks [5], and protein complexes identification in protein-protein interaction networks [18]. However, the structural balanced clique concept is often too restrictive for practical applications. Firstly, not every user is connected to all other users in a polarized structure. Secondly, not every user strictly agrees with all members of the same group or disagrees with all members of the opposite group. Figure 1 shows a real-world polarized group (i.e., red vertices v.s. blue vertices) in Tribes [19], which depicts the friendship (solid lines) and enmity (dashed lines) among 8 tribes situated in the Eastern Central Highlands. Although this is a clearly defined polarized structure, not every user is connected to all others (e.g., "Gahuku" and "Gehamo"), and it does not fully obey the structural balance theory (e.g., "Asarodzuha" and "Kotuni" are friends).

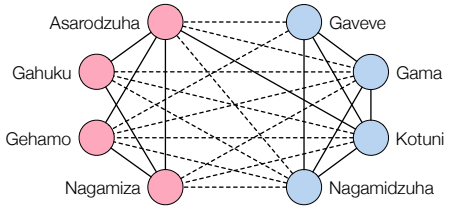


Fig. 1. Polarized groups in Tribes

In this paper, we formulate the structural balanced k-plex, which relaxes structural balanced clique by allowing each vertex to miss a few connections or slightly violate the structural balance theory. Specifically, given a signed graph G, two disjoint vertex subsets P_L and P_R induce a structural balanced k-plex if for each vertex $u \in P_L \cup P_R$, there are no more than k vertices that are either (1) non-neighbors of u, or (2) negative neighbors of u but in the same set as u, or (3) positive neighbors of u but in the opposite set of u. Equivalently, for each vertex $u \in P_L \cup P_R$, the total number of u's same-set positive neighbors and opposite-set negative neighbors is at least $|P_L \cup P_R| - k$. This ensures that each member holds a clear standpoint within a polarized structure while still allowing a limited number of missing or

personal relationships through the parameter k. Referring back to Figure 1, the sets of red vertices and blue vertices induce a structural balanced 3-plex, as each vertex has at least 5 same-set positive or opposite-set negative neighbors. For simplicity, *we refer to structural balanced as balanced and denote a balanced k-plex as (P_L, P_R) in the following.*

We prove that the problem of enumerating all maximal balanced k-plexes is #P-hard. To solve this problem, we first propose a backtracking algorithm MBPE-BK, by drawing inspiration from the well-known Bron-Kerbosch algorithm [20], which is designed for maximal clique enumeration in unsigned graphs. The main idea of MBPE-BK is that, for each vertex v_i in the graph, we iteratively build up a partial balanced k-plex (P_L, P_R), where $P_L = \{v_i\}$ and $P_R = \emptyset$ initially. In addition, we maintain two candidate vertex sets C_L and C_R which are used for growing P_L and P_R, respectively. Then, we iteratively and recursively try each vertex of C_L to be added to P_L and each vertex of C_R to be added to P_R, to grow the solution until maximal balanced k-plexes are reached.

The performance of MBPE-BK is however unsatisfactory due to the heavy overlap between C_L and C_R. To overcome this issue, we propose the algorithm MBPE that adopts a different strategy at the root level of the backtracking search tree. Specifically, for each vertex v_i, we enumerate all possible initial balanced k-plexes (P_L, P_R) such that $v_i \in P_L$ and (P_L, P_R) contains at most $k-1$ other vertices that are either (1) non-neighbors of v_i, or (2) negative neighbors of v_i within P_L, or (3) positive neighbors of v_i within P_R. For each initial balanced k-plex, we set C_L and C_R as v_i's remaining positive neighbors and negative neighbors, respectively. In this way, C_L and C_R become disjoint automatically. By utilizing this fact, we propose subgraph sparsification and partition-based vertex reduction techniques to further improve the efficiency of MBPE. We prove that MBPE achieves a better theoretical time complexity than MBPE-BK (i.e., $\mathcal{O}^*(2^\delta)$ v.s. $\mathcal{O}^*(3^{\delta D})$). [1] Furthermore, we speed up the enumeration procedure by adopting the minimum-degree pivoting strategy [21], i.e., the minimum-degree vertex of $P_L \cup P_R$ is chosen as the branching vertex throughout the enumeration process. This also improves the worst-case time complexity of MBPE to $\mathcal{O}^*(\alpha_k^\delta)$, where $1 < \alpha_k < 2$ is a constant depending only on k, e.g., $\alpha_2 = 1.839$, $\alpha_3 = 1.928$, and $\alpha_4 = 1.966$.

Our main contributions are summarized as follows.

- We formalize the structural balanced k-plex model in signed graphs, analyze its properties, and prove that enumerating all maximal balanced k-plexes is #P-hard.
- We propose a backtracking algorithm MBPE-BK as well as optimization techniques to improve its performance.
- We propose a novel algorithm MBPE with better time complexity than MBPE-BK (i.e., $\mathcal{O}^*(2^\delta)$ v.s. $\mathcal{O}^*(3^{\delta D})$), and also subgraph sparsification and partition-based vertex reduction to improve the efficiency of MBPE.
- We adopt the minimum-degree branching strategy to accelerate the enumeration procedure, improving the worst-case time complexity of MBPE to $\mathcal{O}^*(\alpha_k^\delta)$, where $1 < \alpha_k < 2$ is a constant that depends only on k.

We conduct extensive experiments on both real-world and synthetic signed graphs. The results demonstrate that our algorithm MBPE not only has a good theoretical time complexity but also runs efficiently in practice. Furthermore, case studies on real-world datasets show that our model can discover more interesting results compared to existing models.

Organization. The rest of the paper is organized as follows. Section II provides preliminaries and problem definition. Section III presents our MBPE-BK algorithm and its optimizations. Section IV presents our MBPE algorithm and a novel branching strategy. Experimental results are reported in Section V. Related works are discussed in Section VI. Finally, Section VII concludes the paper. All proofs for lemmas and theorems can be found in Appendix A.

II. Preliminaries

In the paper, we consider undirected signed graphs $G = (V, E^+, E^-)$, where V is the set of vertices, and E^+ and E^- are the sets of positive and negative edges, respectively. The numbers of vertices and edges in G are denoted by $n = |V|$ and $m = |E^+| + |E^-|$. For each vertex $v \in V$, $N_G^+(v)$ denotes the positive neighbors of v, i.e., $N_G^+(v) = \{u \in V \mid (v,u) \in E^+\}$, and $N_G^-(v)$ denotes the negative neighbors of v, i.e, $N_G^-(v) = \{u \in V \mid (v,u) \in E^-\}$. $N_G(v) = N_G^-(v) \cup N_G^+(v)$ denotes all neighbors of v. We use $d_G^+(v) = |N_G^+(v)|$, $d_G^-(v) = |N_G^-(v)|$, and $d_G(v) = |N_G(v)|$ to respectively denote the positive degree, negative degree, and (total) degree of v. Given a vertex subset $S \subseteq V$, we use $G[S]$ to denote the induced subgraph of G that consists of all edges between vertices of S, i.e., $G[S] = (S, \{(u,v) \in E^+ \cup E^- \mid u, v \in S\})$. We use $N_G^2(v)$ to denote 2-hop neighbors of v, i.e., the set of vertices with distance exactly 2 to v in G. When the context is clear, we omit the subscript G.

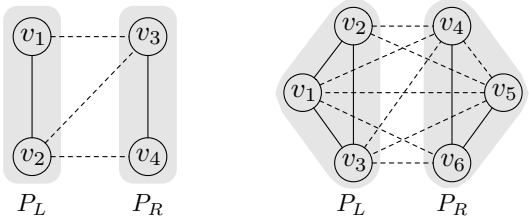
Definition 1 (Structural Balanced Graph [22])**.** A signed graph $G = (V, E^+, E^-)$ is *structural balanced* if its vertices can be partitioned into two groups V_L and V_R such that all edges between vertices in the same group are positive and all edges between vertices from different groups are negative.

Definition 2 (k-Plex [23])**.** A graph $G = (V, E)$ is a *k-plex* if for each $u \in V$, its degree satisfies $d_G(u) \geq |V| - k$, i.e., u misses connections to at most k vertices (including u itself).

Definition 3 (Antagonistic Structural Balanced k-Plex)**.** Given a signed graph G, two disjoint vertex subsets P_L and P_R induce an *antagonistic balanced k-plex* if the subgraph of G induced by $P_L \cup P_R$ is structural balanced and is also a k-plex.

For the signed graph in Figure 2(a), $P_L = \{v_1, v_2\}$ and $P_R = \{v_3, v_4\}$ form an antagonistic structural balanced 2-plex. However, this model is restrictive because it demands that all members within the same group are completely friendly with one another, while all members of one group are entirely

[1] For presentation simplicity, we also use the $\mathcal{O}^*$ notation, in addition to the usual $\mathcal{O}$ notation, for time complexity analysis. $\mathcal{O}^*$ notation focuses on the exponential part of the time complexity by hiding polynomial factors.



(a) Antagonistic balanced 2-plex (b) Structural balanced 2-plex

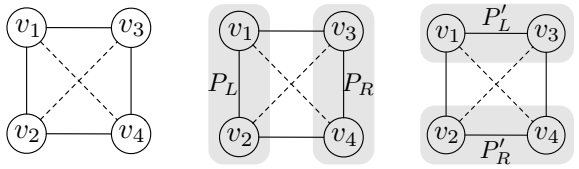
Fig. 2. Illustration of (antagonistic) structural balanced k-plex

antagonistic towards those in the other group. In practice, a cohesive group might include some members who don't get along with each other and may also have a small number of friends in the antagonistic group. For example, Figure 2(b) shows such a situation. Motivated by this, we propose a relaxed model that is more suitable for practical applications.

Definition 4 (Structural Balanced k-Plex)**.** Given a signed graph G, two disjoint vertex subsets P_L and P_R induce a *structural balanced k-plex* if the maximal structural balanced subgraph of G induced by P_L and P_R is a k-plex.

Specifically, the *maximal structural balanced subgraph of G induced by* (P_L, P_R), denoted $g(P_L, P_R)$, is the subgraph of G whose vertices are $P_L \cup P_R$ and whose edges consist of (1) all positive edges of G between vertices of P_L, (2) all positive edges between vertices of P_R, and (3) all negative edges between vertices of P_L and vertices of P_R. That is, $g(P_L, P_R)$ is obtained from $G[P_L, P_R]$ by removing all edges that violate the structural balance theory. For example, consider the graph G in Figure 2(b), the maximal structural balanced subgraph of G induced by $P_L = \{v_1, v_2, v_3\}$ and $P_R = \{v_4, v_5, v_6\}$ contains all edges except (v_4, v_5); it is easy to see that this subgraph is a 2-plex. Therefore, ($P_L = \{v_1, v_2, v_3\}$,$P_R = \{v_4, v_5, v_6\}$) can be identified as a structural balanced 2-plex.

In this paper, we focus our discussions on balanced k-plex (Definition 4), while our techniques can be easily adapted to antagonistic balanced k-plex (Definition 3); we will empirical compare the two models in Section V. For simplicity, we refer to (P_L, P_R) as a *balanced k-plex* if P_L and P_R induce a structural balanced k-plex. For a balanced k-plex, the role of P_L and P_R can be swapped, i.e., (P_L, P_R) is the same as (P_R, P_L). We measure the size of a balanced k-plex by its number of vertices, i.e., $|P_L \cup P_R|$. The property of balanced k-plex is *hereditary*; that is, given a balanced k-plex (P_L, P_R), (P'_L, P'_R) is also a balanced k-plex for any $P'_L \subseteq P_L$ and $P'_R \subseteq P_R$. A balanced k-plex (P_L, P_R) is *maximal* if adding any vertex to either P_L or P_R would violate the balanced k-plex property. We aim to enumerate all maximal balanced k-plexes (P_L, P_R) in a signed graph. To avoid generating trivial solutions, we require the numbers of vertices in P_L and in P_R to be no less than a user-given threshold τ, i.e., $|P_L| \geq \tau$ and $|P_R| \geq \tau$. We also require $\tau \geq k$ such that each vertex v in the balanced k-plex possesses a sufficient number of neighbors (i.e., positive neighbors in the same set with v and negative neighbors in the opposite set with v).



(a) A signed graph (b) Case I (c) Case II

Fig. 3. Uniqueness of the balanced k-plex

Problem Statement (Maximal Balanced k-Plex Enumeration)**.** Given a signed graph G and two positive integers k and τ such that $\tau \geq k$, the maximal balanced k-plex enumeration problem aims to identify all maximal balanced k-plexes (P_L, P_R) in G s.t. $|P_L| \geq \tau$ and $|P_R| \geq \tau$.

A. Properties of Maximal Balanced k-Plexes

Theorem 1. *The maximal balanced k-plex enumeration problem is #P-hard.*

Note that two distinct balanced k-plexes, e.g., (P_L, P_R) and (P'_L, P'_R), may share the same set of vertices, i.e., $P_L \cup P_R = P'_L \cup P'_R$. For instance, for the signed graph in Figure 3(a), both ($P_L = \{v_1, v_2\}, P_R = \{v_3, v_4\}$) and ($P'_L = \{v_1, v_3\}, P'_R = \{v_2, v_4\}$) are balanced 2-plexes, yet they consist of the same vertices. In this study, we regard them as different balanced k-plexes since different partitions imply different perspectives of the vertices. Nevertheless, we prove in the lemma below that for sufficiently large balanced k-plexes, different balanced k-plexes will not have exactly the same set of vertices.

Lemma 1. *Given two different balanced k-plexes (P_L, P_R) and (P'_L, P'_R), if $|P_L \cup P_R| = |P'_L \cup P'_R| > 4k - 4$, then $P_L \cup P_R \neq P'_L \cup P'_R$.*

III. A Bron-Kerbosch-based Algorithm

The classic Bron-Kerbosch algorithm is proposed in [20] for enumerating maximal cliques in unsigned graphs. In this section, we adopt the Bron-Kerbosch framework to solve the maximal balanced k-plex enumeration problem. The main idea is to iteratively build up two vertex sets P_L and P_R such that (P_L, P_R) is a balanced k-plex. In addition, we maintain two candidate vertex sets C_L and C_R which are used for growing P_L and P_R, respectively. Then, we iteratively try each vertex of C_L to be added to P_L and each vertex of C_R to be added to P_R, to grow the partial balanced k-plex and conduct the recursion. We also maintain exclusive sets X_L and X_R to record the vertices that have been processed to avoid generating duplicate results. Before introducing the details of our algorithm, we analyze two properties of balanced k-plex that are important for our algorithm design.

Lemma 2. *Given a signed graph G, if the size of a balanced k-plex (P_L, P_R) is at least $2k-1$, the diameter of the subgraph of G induced by $P_L \cup P_R$ is bounded by 2.*

We categorize the 2-hop neighbors of v into four distinct types: positive-positive, positive-negative, negative-positive,

and negative-negative neighbors. These are denoted as $N_G^{++}(v)$, $N_G^{+-}(v)$, $N_G^{-+}(v)$ and $N_G^{--}(v)$, respectively; note that, different types may overlap.

Lemma 3. *Suppose v is an arbitrary vertex in a signed graph G. For any balanced k-plex (P_L, P_R) with size at least $2k-1$ and $v \in P_L$, we have that $P_L \subseteq \{v\} \cup N_G(v) \cup N_G^{++}(v) \cup N_G^{--}(v)$ and $P_R \subseteq N_G(v) \cup N_G^{+-}(v) \cup N_G^{-+}(v)$.*

Algorithm 1: MBPE-BK

Input: A signed graph $G = (V, E^+, E^-)$ and integers k, τ
Output: All maximal balanced k-plexes

1 Sort V by degeneracy order as $\{v_1, \cdots, v_n\}$;
2 **for each** v_i *in degeneracy order* **do**
3 $P_L \leftarrow \{v_i\}$, $P_R \leftarrow \emptyset$;
4 $C_L \leftarrow \{v_{i+1}, \cdots, v_n\} \cap (N_G(v_i) \cup N_G^{++}(v_i) \cup N_G^{--}(v_i))$;
5 $C_R \leftarrow \{v_{i+1}, \cdots, v_n\} \cap (N_G(v_i) \cup N_G^{+-}(v_i) \cup N_G^{-+}(v_i))$;
6 $X_L \leftarrow \{v_1, \cdots, v_{i-1}\} \cap (N_G(v_i) \cup N_G^{++}(v_i) \cup N_G^{--}(v_i))$;
7 $X_R \leftarrow \{v_1, \cdots, v_{i-1}\} \cap (N_G(v_i) \cup N_G^{+-}(v_i) \cup N_G^{-+}(v_i))$;
8 Enum$(P_L, P_R, C_L, C_R, X_L, X_R)$;

Procedure Enum$(P_L, P_R, C_L, C_R, X_L, X_R)$
9 $C_L \leftarrow \{v \in C_L : (P_L \cup \{v\}, P_R)$ is a balanced k-plex$\}$;
10 $C_R \leftarrow \{v \in C_R : (P_L, P_R \cup \{v\})$ is a balanced k-plex$\}$;
11 $X_L \leftarrow \{v \in X_L : (P_L \cup \{v\}, P_R)$ is a balanced k-plex$\}$;
12 $X_R \leftarrow \{v \in X_R : (P_L, P_R \cup \{v\})$ is a balanced k-plex$\}$;
13 **if** $C_L \cup C_R = \emptyset$ **and** $X_L \cup X_R = \emptyset$ **then**
14 **if** $|P_L| \geq \tau$ **and** $|P_R| \geq \tau$ **then** emit (P_L, P_R);
15 **for each** $v \in C_L$ **do**
16 $C_L \leftarrow C_L \setminus \{v\}$;
17 Enum$(P_R, P_L \cup \{v\}, C_R \setminus \{v\}, C_L, X_R \setminus \{v\}, X_L)$;
18 $X_L \leftarrow X_L \cup \{v\}$;
19 **for each** $v \in C_R$ **do**
20 $C_R \leftarrow C_R \setminus \{v\}$;
21 Enum$(P_R \cup \{v\}, P_L, C_R, C_L \setminus \{v\}, X_R, X_L \setminus \{v\})$;
22 $X_R \leftarrow X_R \cup \{v\}$;

The pseudo-code of our algorithm is presented in Algorithm 1, denoted MBPE-BK. The major framework is firstly sorting vertices in degeneracy order $\eta = \{v_1, \cdots, v_n\}$ (Line 1), and then iteratively searching for all maximal v_i-lead balanced k-plex, i.e., those balanced k-plexes in which v_i precedes all other vertices according to η (Lines 2-8). The degeneracy order $\eta = \{v_1, \cdots, v_n\}$ is defined such that for each $1 \leq i \leq n$, v_i is the minimum-degree vertex in $G[\{v_i, \ldots, v_n\}]$ [24]. This ordering minimizes the maximum number of neighbors that succeed v_i for each $v_i \in V$, potentially reducing the size of initial candidate sets. It uses P_L and P_R to maintain the balanced k-plex, which are initialized with v_i and $\emptyset$. It also uses C_L and C_R to maintain the set of candidates that are used to grow the partial balanced k-plex. Since we require $\tau \geq k$, the target balanced k-plex must have size of at least $2k-1$. Thus Lemma 3 can be employed to optimize the initialization of C_L and C_R (Lines 4-5). Note that here we only need to consider the vertices that succeed v_i in the ordering η. To avoid outputting duplicate results, X_L and X_R are used to record the vertices that have been processed (Lines 6-7). After initializing these six sets, it invokes the enumeration procedure Enum to find maximal v_i-lead balanced k-plexes (Line 8).

Procedure Enum performs the maximal balanced k-plex enumeration based on the given six sets. Firstly, C_L, C_R, X_L and X_R are refined by removing vertices that cannot form a larger balanced k-plex with the partial solution (P_L, P_R) (Lines 9-12). If these four sets are empty and (P_L, P_R) satisfies the size constraint, (P_L, P_R) is selected as the result (Line 14). Otherwise, it expands (P_L, P_R) by adding each vertex $v \in C_L$ to P_L and recursively invokes itself to further enlarge the balanced k-plex (Line 17). Note that in the new recursion, v's copies in C_R and X_R (if there are) should be removed. After v is processed, it is added to X_L (Line 18). Then, the algorithm continues to expand (P_L, P_R) by adding each vertex $v \in C_R$ to P_R in a similar way (Lines 19-22). Note that at Lines 17 and 21, we swap the roles of P_L and P_R, C_L and C_R, X_L and X_R, such that we are adding vertices to the two sides of the growing balanced k-plex in alternating order; this is to avoid generating too skewed intermediate results.

Theorem 2. *The time complexity of* MBPE-BK *(Algorithm 1) is $\mathcal{O}(n(\delta D)^2 3^{\delta D})$, where δ and D are the degeneracy and maximum degree of G, respectively.*

A. Optimization Techniques

In this subsection, we present optimization techniques to improve the practical efficiency of MBPE-BK.

Vertex reduction. By utilizing the size constraint τ, we can remove unpromising vertices that are not contained in any maximal balanced k-plex. We have the following lemma.

Lemma 4. *Given a signed graph G, and integers k and τ, each vertex v in a maximal balanced k-plex must satisfy:*

- $d_G^+(v) \geq \tau - k$;
- $d_G^-(v) \geq \tau - k + 1$;
- $d_G(v) \geq 2\tau - k$.

The pseudo-code of vertex reduction is shown in Algorithm 2. Firstly, it identifies the unqualified vertex v (Line 1). Since v will be removed from the graph, it decreases the positive (resp. negative) degree and degree of its positive (resp. negative) neighbors by 1 (Lines 2-5). Lastly, the algorithm removes v and its associated edges from the graph (Line 6). The algorithm repeats the above steps until all the remaining vertices satisfy the degree constraints. The time complexity of VertexReduction is $\mathcal{O}(n+m)$.

Algorithm 2: VertexReduction

Input: A signed graph $G = (V, E^+, E^-)$ and integers k, τ
Output: A reduced signed graph

1 **while** $\exists v \in V$ *s.t.* $d_G^+(v) < \tau - k$ **or** $d_G^-(v) < \tau - k + 1$ **or** $d_G(v) < 2\tau - k$ **do**
2 **for each** $u \in N_G^+(v)$ **do**
3 $d_G^+(u) \leftarrow d_G^+(u) - 1$; $d_G(u) \leftarrow d_G(u) - 1$;
4 **for each** $u \in N_G^-(v)$ **do**
5 $d_G^-(u) \leftarrow d_G^-(u) - 1$; $d_G(u) \leftarrow d_G(u) - 1$;
6 Remove v from G;

Edge reduction. We can also prune unpromising edges based on the common neighbors between two vertices. For each edge (u, v) in G, we define its positive-positive support as the number of vertices that positively link to both u and v (i.e., $\Delta_G^{++}(u,v) = |\{w \mid (u,w) \in E^+ \wedge (v,w) \in E^+\}|$), its negative-negative support as the number of vertices that negatively link to both u and v (i.e., $\Delta_G^{--}(u,v) = |\{w \mid (u,w) \in E^- \wedge (v,w) \in E^-\}|$), its positive-negative support as the number of vertices that positively link to u and negatively link to v (i.e., $\Delta_G^{+-}(u,v) = |\{w \mid (u,w) \in E^+ \wedge (v,w) \in E^-\}|$), and its negative-positive support as the number of vertices that negatively link to u and positively link to v (i.e., $\Delta_G^{-+}(u,v) = |\{w \mid (u,w) \in E^- \wedge (v,w) \in E^+\}|$). Then, we have the following lemma.

Lemma 5. *Given a signed graph G, and integers k and τ. Suppose (P_L, P_R) is a valid balanced k-plex, any positive edge (u, v) in $g(P_L, P_R)$ must satisfy:*

- $\Delta_G^{++}(u,v) \geq \tau - 2k$;
- $\Delta_G^{--}(u,v) \geq \tau - 2k + 2$;
- $\Delta_G^{++}(u,v) + \Delta_G^{--}(u,v) \geq 2\tau - 2k$.

Similarly, any negative edge (u, v) in $g(P_L, P_R)$ must satisfy:

- $\Delta_G^{+-}(u,v) \geq \tau - 2k + 1$;
- $\Delta_G^{-+}(u,v) \geq \tau - 2k + 1$;
- $\Delta_G^{+-}(u,v) + \Delta_G^{-+}(u,v) \geq 2\tau - 2k$.

The pseudo-code of edge reduction is shown in Algorithm 3. Firstly, it computes $\Delta_G^{++}(u,v)$ and $\Delta_G^{--}(u,v)$ for each positive edge, and $\Delta_G^{+-}(u,v)$ and $\Delta_G^{-+}(u,v)$ for each negative edge (Lines 1-2). Then, all positive edges that violate Lemma 5 are removed (Line 5); also, the algorithm updates the common neighbor supports for those affected edges (Lines 6-11). Negative edges are handled similarly at Lines 13-19. The time complexity of EdgeReduction is $\mathcal{O}(m^{1.5})$.

In our implementation, we first apply vertex and edge reductions on the original graph G at the beginning of MBPE-BK. Note that vertex and edge reductions may interact with each other. That is, removing unqualified vertices can cause some edges to become invalid, and removing unqualified edges can cause some vertices to become invalid. Therefore, we repeatedly apply vertex and edge reductions until no more vertices or edges can be pruned. Additionally, during the enumeration process, we perform reductions on the subgraph that the current search instance is focusing on. Specifically, at the beginning of Enum, we apply vertex reduction on the subgraph $G[P_L \cup P_R \cup C_L \cup C_R]$; note that we do not apply edge reduction in Enum due to its high computational cost.

Pivoting technique. The Bron-Kerbosch algorithm designed for maximal clique enumeration uses an effective technique called pivoting to reduce the number of branches in the search tree. The main idea of pivoting is that each maximal clique must include either the vertex u (i.e., the pivot) or a non-neighbor of u. Otherwise, a clique only contains neighbors of u and we can enlarge it by adding u to it, which makes it non-maximal. Motivated by this, we present a generalized version of pivoting that is applicable to the problem of maximal balanced k-plex enumeration. For clarity, we use P to denote (P_L, P_R) and C to denote (C_L, C_R). We define $u \in C$ if $u \in C_L \cup C_R$. For a vertex $u \in C$, $P \cup \{u\}$ is defined as $(P_L \cup \{u\}, P_R)$ if $u \in C_L$ and as $(P_L, P_R \cup \{u\})$ if $u \in C_R$.

Algorithm 3: EdgeReduction

Input: A signed graph $G = (V, E^+, E^-)$ and integers k, τ
Output: A reduced signed graph

1 **for each** $(u,v) \in E^+$ **do** Obtain $\Delta_G^{++}(u,v)$ and $\Delta_G^{--}(u,v)$;
2 **for each** $(u,v) \in E^-$ **do** Obtain $\Delta_G^{+-}(u,v)$ and $\Delta_G^{-+}(u,v)$;
3 **while** $\exists (u,v) \in E^+ \cup E^-$ *violates Lemma 5* **do**
4 **if** $(u,v) \in E^+$ **then**
5 Remove (u,v) from G;
6 **for each** w *s.t.* $(u,w) \in E^+$ **and** $(v,w) \in E^+$ **do**
7 $\Delta_G^{++}(u,w) \leftarrow \Delta_G^{++}(u,w) - 1$;
8 $\Delta_G^{++}(v,w) \leftarrow \Delta_G^{++}(v,w) - 1$;
9 **for each** w *s.t.* $(u,w) \in E^-$ **and** $(v,w) \in E^-$ **do**
10 $\Delta_G^{+-}(u,w) \leftarrow \Delta_G^{+-}(u,w) - 1$;
11 $\Delta_G^{+-}(v,w) \leftarrow \Delta_G^{+-}(v,w) - 1$;
12 **else**
13 Remove (u,v) from G;
14 **for each** w *s.t.* $(u,w) \in E^+$ **and** $(v,w) \in E^-$ **do**
15 $\Delta_G^{--}(u,w) \leftarrow \Delta_G^{--}(u,w) - 1$;
16 $\Delta_G^{-+}(v,w) \leftarrow \Delta_G^{-+}(v,w) - 1$;
17 **for each** w *s.t.* $(u,w) \in E^-$ **and** $(v,w) \in E^-$ **do**
18 $\Delta_G^{-+}(u,w) \leftarrow \Delta_G^{-+}(u,w) - 1$;
19 $\Delta_G^{--}(v,w) \leftarrow \Delta_G^{--}(v,w) - 1$;

Lemma 6. *Let $P = (P_L, P_R)$ be a partial balanced k-plex, and $C = (C_L, C_R)$ be the corresponding candidate set. Given a vertex $u \in C$, any expanded balanced k-plex must contain either u or the vertex v that $v \neq u$ and satisfy one of the following conditions:*

(1) *$v \in C$ and u, v are non-neighbors in $g(P \cup \{u, v\})$;*
(2) *$v \in C$ and u, v are neighbors in $g(P \cup \{u, v\})$ and u, v have common non-neighbors in $g(P \cup \{u, v\})$.*

Lemma 6 suggests that if a candidate vertex v is a neighbor of candidate vertex u and does not share any common non-neighbor with u in $g(P \cup \{u, v\})$, then we do not need to branch on v without affecting the correctness of the algorithm. Suppose u is the pivot from C_L, we have the following branching rule. At Line 15 of Algorithm 1, we only generate branches for $v \in C_L$ such that $v \notin N^+(u)$ or $P_L \setminus (N^+(u) \cup N^+(v)) \neq \emptyset$ or $P_R \setminus (N^-(u) \cup N^-(v)) \neq \emptyset$. At Line 19 of Algorithm 1, we only generate branches for $v \in C_R$ such that $v \notin N^-(u)$ or $P_L \setminus (N^+(u) \cup N^-(v)) \neq \emptyset$ or $P_R \setminus (N^-(u) \cup N^+(v)) \neq \emptyset$. The branching rule is derived similarly when the pivot is from C_R. To reduce the number of recursive calls as many as possible, we pick the vertex u that cuts the most branches from the search tree as the pivot.

B. Drawback of MBPE-BK

Despite incorporating all optimizations above, the performance of MBPE-BK is still unsatisfactory. The primary reason is that the candidate sets C_L and C_R often overlap significantly. This overlap arises from the fact that balanced k-plex

model allows each member u to have a few non-neighbors or neighbors that violate the structural balance theory. Thus, at the early stage of the enumeration procedure, C_L and C_R may overlap heavily with each other. Such overlap can further weaken the effectiveness of vertex reduction during the enumeration process. To address this issue, we propose a new enumeration framework to ensure that C_L and C_R are disjoint and contain only neighbors of the lead vertex v_i. Consequently, the candidate set will be of size at most the degeneracy δ that is typically a small value in real-world graphs.

IV. Our Advanced Algorithm MBPE

In this section, we first propose a new search framework in Section IV-A. Then, we optimize the enumeration procedure in Section IV-B by proposing a minimum degree pivoting technique. We discuss the extension of our algorithms to the antagonistic balanced k-plex enumeration in Section IV-C.

A. A New Enumeration Framework

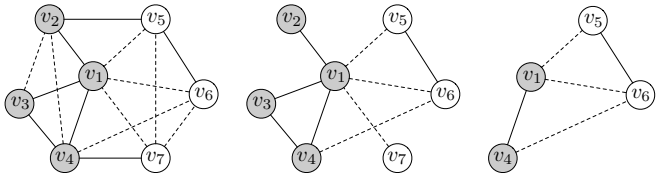
In MBPE-BK, the overlap between C_L and C_R arises from the fact that each vertex v_i in the balanced k-plex (P_L, P_R) is allowed to have at most k non-neighbors (including itself) in $g(P_L, P_R)$. In an ideal scenario, if we can determine in advance the set of v_i's non-neighbors in $g(P_L, P_R)$, then we only need to focus on its neighbors, i.e., C_L only contains v_i's positive neighbors, and C_R only contains v_i's negative neighbors. Thus C_L and C_R automatically become disjoint. To achieve this, for each vertex v_i, we enumerate all possible initial balanced k-plexes (P_L, P_R) such that $v_i \in P_L$ and (P_L, P_R) contains at most k vertices that are either (1) non-neighbors of v_i, (2) negative neighbors of v_i within P_L, or (3) positive neighbors of v_i within P_R. For each initial balanced k-plex, we set C_L and C_R as v_i's remaining positive neighbors and negative neighbors, respectively, and identify all maximal v_i-lead balanced k-plexes under this search instance.

Algorithm 4: MBPE

Input: A signed graph $G = (V, E^+, E^-)$ and integers k, τ
Output: All maximal balanced k-plexes

1 Sort V by degeneracy order: $\eta = \{v_1, \cdots, v_n\}$;
2 **for each** v_i *in degeneracy order* **do**
3 **for each** $S \subseteq N_{\succ\eta}(v_i) \cup N^2_{\succ\eta}(v_i)$ *s.t.* $|S| \leq k-1$ **do**
4 $S_L \leftarrow \{u \in S \colon u \in N^-_{\succ\eta}(v_i)\}$;
5 $S_R \leftarrow \{v \in S \colon v \in N^+_{\succ\eta}(v_i)\}$;
6 $S_C \leftarrow \{v \in S \colon v \in N^2_{\succ\eta}(v_i)\}$;
7 **for each** *partition* (T_L, T_R) *of* S_C **do**
8 $P_L \leftarrow \{v_i\} \cup S_L \cup T_L$;
9 $P_R \leftarrow S_R \cup T_R$;
10 $C_L \leftarrow N^+_{\succ\eta}(v_i) \setminus S_R$;
11 $C_R \leftarrow N^-_{\succ\eta}(v_i) \setminus S_L$;
12 $X_L \leftarrow N_{\prec\eta}(v_i) \cup N^2_{\prec\eta}(v_i) \cup (N^-_{\succ\eta}(v_i) \cup N^2_{\succ\eta}(v_i)) \setminus (P_L \cup P_R)$;
13 $X_R \leftarrow N_{\prec\eta}(v_i) \cup N^2_{\prec\eta}(v_i) \cup (N^+_{\succ\eta}(v_i) \cup N^2_{\succ\eta}(v_i)) \setminus (P_L \cup P_R)$;
14 Enum$(P_L, P_R, C_L, C_R, X_L, X_R)$;

The pseudo-code of our algorithm MBPE is shown in Algorithm 4. It first sorts the vertices in the degeneracy order $\eta = \{v_1, \cdots, v_n\}$ (Line 1). Then, from v_1 to v_n, the algorithm iteratively finds maximal v_i-lead balanced k-plexes (Lines 2-14). Let $N_{\prec_\eta}(v_i) = N(v_i) \cap \{v_1, \ldots, v_{i-1}\}$ and $N_{\succ_\eta}(v_i) = N(v_i) \cap \{v_{i+1}, \ldots, v_n\}$ be backward and forward neighbors of v_i, respectively; $N^2_{\prec_\eta}(v_i) = N^2(v_i) \cap \{v_1, \ldots, v_{i-1}\}$, $N^2_{\succ_\eta}(v_i) = N^2(v_i) \cap \{v_{i+1}, \ldots, v_n\}$ be the backward and forward 2-hop neighbors of v_i, respectively. For each vertex $v_i \in V$, it obtains the candidates that are at most 2-hop neighbors succeeding v_i, i.e., $N_{\succ\eta}(v_i) \cup N^2_{\succ\eta}(v_i)$. Then, it exhaustively explores all possible vertex subsets $S \subseteq N_{\succ\eta}(v_i) \cup N^2_{\succ\eta}(v_i)$ of size at most $k-1$, which contain v_i's potentially non-neighbors in $g(P_L, P_R)$ (Line 3). Next, the vertices in S are allocated to sets S_L, S_R, and S_C as follows: S_L (resp. S_R) is the set of negative (resp. positive) neighbors of v_i, which will be placed in the same (resp. opposite) set with v_i to become v_i's non-neighbors in $g(P_L, P_R)$ (Lines 4-6). S_C contains v_i's 2-hop neighbors and a further partition of S_C ensures all possibilities are exhausted (Line 7), since any vertex $u \in S_C$ will be v_i's non-neighbor in $g(P_L, P_R)$ regardless of which set u is placed in. For each instance, the initial balanced k-plex (P_L, P_R) is established (Lines 8-9). Sets C_L and C_R are initialized to contain only the positive and negative neighbors of v_i by excluding vertices already in the initial balanced k-plex (Lines 10-11). The exclusive sets X_L and X_R include any other vertices that can be added to the initial balanced k-plex (Lines 12-13). After initializing these six sets, the algorithm invokes Enum to search for all maximal v_i-lead balanced k-plexes under this instance (Line 14).



(a) A signed graph G (b) Conflict edge removal (c) Vertex reduction

Fig. 4. Conflict edge removal and vertex reduction

Subgraph sparsification. In MBPE, it is guaranteed that C_L and C_R are disjoint (i.e., C_L contains only v_i's positive neighbors and C_R contains only v_i's negative neighbors). By utilizing this fact, we can reduce the subgraph $G[P_L \cup P_R \cup C_L \cup C_R]$ in the Enum procedure, by removing *conflicting* edges that violate the structural balance theory, i.e.,

- negative edges between vertices of $P_L \cup C_L$,
- negative edges between vertices of $P_R \cup C_R$, and
- positive edges between a vertex of $P_L \cup C_L$ and a vertex of $P_R \cup C_R$.

This approach is justified as vertices from $P_L \cup C_L$ will belong to the same set of any derived balanced k-plex, and only positive edges among them contribute to forming a balanced k-plex. Thus, removing negative edges between vertices of $P_L \cup C_L$ does not compromise the algorithm's correctness.

Similarly, negative edges between vertices of $P_R \cup C_R$ can also be removed. Additionally, since any vertex $v \in P_L \cup C_L$ and any vertex $v' \in P_R \cup C_R$ will be on opposite sets of any derived balanced k-plex, only the negative edges between them will be counted for the formation of balanced k-plex. Therefore, we can safely remove positive edges between vertices of $P_L \cup C_L$ and $P_R \cup C_R$. Removing conflicting edges offers two significant advantages. Firstly, it leads to the sparsification of the subgraph $G[P_L \cup P_R \cup C_L \cup C_R]$ that the current search instance is focusing on. Secondly, it enhances the efficiency of the vertex reduction methods proposed in Section III-A, resulting in further reductions in the computational costs. The following example illustrates this benefit.

Example 1. *Consider the signed graph G in Figure 4(a) with $k = 2$ and $\tau = 2$. Suppose v_1 precedes all other vertices in the ordering η. When processing v_1 with $S = \emptyset$, we have $P_L = \{v_1\}$, $P_R = \emptyset$, $C_L = \{v_2, v_3, v_4\}$, and $C_R = \{v_5, v_6, v_7\}$. Conflicting edges to be removed include negative edges (v_2, v_3), (v_2, v_4), (v_5, v_7), and (v_6, v_7), as well as positive edges (v_2, v_5) and (v_4, v_7). The subgraph after removing these conflicting edges is shown in Figure 4(b). We can then apply vertex reduction to further prune unpromising vertices, resulting in the final reduced subgraph shown in Figure 4(c). In contrast, apply vertex reduction without removing conflicting edges will not have any pruning effect.*

Reduce the number of generated subsets S. Our new algorithm MBPE makes candidate sets disjoint by considering all possible subsets S of size at most $k-1$ from $N_{\succ_\eta}(v_i) \cup N^2_{\succ_\eta}(v_i)$. In the worst case, the size of $N_{\succ_\eta}(v_i) \cup N^2_{\succ_\eta}(v_i)$ could be as large as δD, leading to a substantial number of generated instances. We aim to prune unpromising vertices from $N_{\succ_\eta}(v_i) \cup N^2_{\succ_\eta}(v_i)$. This is challenging since the vertices in $N_{\succ_\eta}(v_i) \cup N^2_{\succ_\eta}(v_i)$ can be placed in either P_L or P_R of the initial balanced k-plex, making it difficult to derive the pruning criteria. The vertex reduction proposed in Section III-A is ineffective here, since it does not consider the possibility that vertices in $N_{\succ_\eta}(v_i) \cup N^2_{\succ_\eta}(v_i)$ can be placed into different sets of a balanced k-plex. To address this, we propose a partition-based vertex reduction technique. The main idea is that, for each vertex $u \in N_{\succ_\eta}(v_i) \cup N^2_{\succ_\eta}(v_i)$, we explore the cases where it is placed in P_L or P_R. In each case, we derive the minimum criteria that u must meet to be part of a valid balanced k-plex with v_i. If u fails to satisfy this criterion, it can be safely removed from $N_{\succ_\eta}(v_i) \cup N^2_{\succ_\eta}(v_i)$.

Specifically, we partition $N_{\succ_\eta}(v_i)$ into $N^+_{\succ_\eta}(v_i) \cup N^-_{\succ_\eta}(v_i)$ and denote $L = \{v_i\} \cup N^+_{\succ_\eta}(v_i)$, $R = N^-_{\succ_\eta}(v_i)$, and $T = N^2_{\succ_\eta}(v_i)$. For each vertex $u \in L \cup R \cup T$, we define its L-part positive (or negative) neighbors as $N^+_L(u) = \{v \in L \mid (u, v) \in E^+\}$ (or $N^-_L(u) = \{v \in L \mid (u, v) \in E^-\}$). We also define its L-part positive (or negative) degree as $d^+_L(u) = |N^+_L(u)|$ (or $d^-_L(u) = |N^-_L(u)|$). Similarly, R-part positive and negative neighbors/degrees are defined by substituting L with R. In MBPE, for each vertex v_i, before generating all possible subsets S from $N_{\succ_\eta}(v_i) \cup N^2_{\succ_\eta}(v_i)$, we can reduce the size of $N_{\succ_\eta}(v_i) \cup N^2_{\succ_\eta}(v_i)$ using the following lemma.

Lemma 7. *Consider a signed graph $G = (V, E^+, E^-)$ and an arbitrary order of vertices $\eta = \{v_1, \cdots, v_n\}$. For a vertex $u \in N_{\succ_\eta}(v_i) \cup N^2_{\succ_\eta}(v_i)$ to be part of a v_i-lead balanced k-plex, it must meet the following conditions:*

- *If $u \in N^+_{\succ_\eta}(v_i)$, it satisfies at least one of the following:*
  - $d^+_L(u) \geq \tau - k - |S| \wedge d^-_R(u) \geq \tau - k - |S| + 1$
  - $d^+_R(u) \geq \tau - k - |S| + 2 \wedge d^-_L(u) \geq \tau - k - |S| + 2$
- *If $u \in N^-_{\succ_\eta}(v_i)$, it satisfies at least one of the following:*
  - $d^+_L(u) \geq \tau - k - |S| + 1 \wedge d^-_R(u) \geq \tau - k - |S| + 3$
  - $d^+_R(u) \geq \tau - k - |S| \wedge d^-_L(u) \geq \tau - k - |S| + 1$
- *If $u \in N^2_{\succ_\eta}(v_i)$, it satisfies at least one of the following:*
  - $d^+_L(u) \geq \tau - k - |S| + 1 \wedge d^-_R(u) \geq \tau - k - |S| + 3$
  - $d^+_R(u) \geq \tau - k - |S| + 2 \wedge d^-_L(u) \geq \tau - k - |S| + 2$

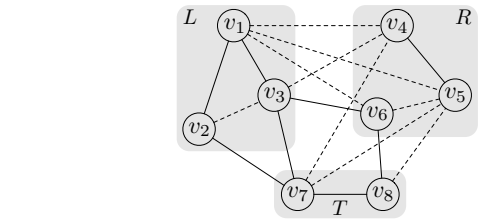
The pseudo-code of PartitionVertexReduction is shown in Algorithm 5. Firstly, it obtains the three vertex sets L, R, and T, and computes the L-part and R-part positive/negative degrees for each vertex (Lines 1-2). Then it identifies the vertex u that violates the degree constraints in Lemma 7. If u is from L, it decreases the L-part positive (resp. negative) degree by one for each of u's positive (resp. negative) neighbors in $L \cup R \cup T$ (Lines 4-6). If u is from R, it decreases the R-part positive (resp. negative) degree by one for each of u's positive (resp. negative) neighbors in $L \cup R \cup T$ (Lines 7-9). Then, it removes u from $N_{\succ\eta}(v_i) \cup N^2_{\succ\eta}(v_i)$ (Line 10). Note that the removal of a vertex in T will not cause the update of L-part and R-part degrees of other vertices. The algorithm terminates when no vertex violates Lemma 7 (Line 3).

Algorithm 5: PartitionVertexReduction

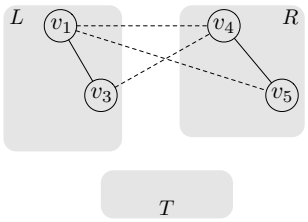
Input: The candidate set $N_{\succ\eta}(v_i) \cup N^2_{\succ\eta}(v_i)$
Output: Reduced candidate set

1 $L \leftarrow \{v_i\} \cup N^+_{\succ_\eta}(v_i)$, $R \leftarrow N^-_{\succ_\eta}(v_i)$, $T \leftarrow N^2_{\succ_\eta}(v_i)$;
2 Compute $d^+_L(u)$, $d^-_L(u)$, $d^+_R(u)$, $d^-_R(u)$ for each $u \in L \cup R \cup T$;
3 **while** $\exists u \in N_{\succ\eta}(v_i) \cup N^2_{\succ\eta}(v_i)$ *s.t.* $u \neq v_i$ *violates Lemma 7* **do**
4 **if** $u \in L$ **then**
5 **for each** $v \in N^+_{L \cup R \cup P}(u)$ **do** $d^+_L(v) \leftarrow d^+_L(v) - 1$;
6 **for each** $v \in N^-_{L \cup R \cup P}(u)$ **do** $d^-_L(v) \leftarrow d^-_L(v) - 1$;
7 **else if** $u \in R$ **then**
8 **for each** $v \in N^+_{L \cup R \cup P}(u)$ **do** $d^+_R(v) \leftarrow d^+_R(v) - 1$;
9 **for each** $v \in N^-_{L \cup R \cup P}(u)$ **do** $d^-_R(v) \leftarrow d^-_R(v) - 1$;
10 Remove u from $N_{\succ\eta}(v_i) \cup N^2_{\succ\eta}(v_i)$;

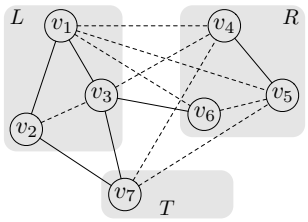
Example 2. *Consider the signed graph G in Figure 5(a) and $k = 2, \tau = 2$. Suppose v_1 precedes all other vertices in the ordering η. We have $L = \{v_1, v_2, v_3\}$, $R = \{v_4, v_5, v_6\}$, and $T = \{v_7, v_8\}$. First let's see how to prune unpromising vertices when $|S| = 0$. In this case, v_2 is unqualified since $d^-_R(v_2) = 0 < \tau - k - |S| + 1 = 1$ and $d^+_R(v_2) = 0 < \tau - k - |S| + 2 = 2$. Besides, v_6 is unqualified since $d^-_R(v_6) = 1 < \tau - k - |S| + 3 = 3$ and $d^+_R(v_6) + d^-_L(v_6) = 1 < 2\tau - k - |S| = 2$, and v_7 is unqualified since $d^-_R(v_7) = 2 < \tau - k - |S| + 3 = 3$ and $d^-_L(v_7) = 0 < \tau - k - |S| + 2 = 2$. v_8 is also*



(a) A signed graph G



(b) The case of $|S| = 0$



(c) The case of $|S| = 1$

Fig. 5. Partition-based vertex reduction

unqualified due to the same reason as v_7. After removing all unpromising vertices, $N_{\succ_\eta}(v_i) \cup N^2_{\succ_\eta}(v_i)$ is reduced to $\{v_3, v_4, v_5\}$, as shown in Figure 5(b). For the case of $|S| = 1$, v_8 is unqualified since $d^-_R(v_8) = 1 < \tau - k - |S| + 3 = 2$ and $d^-_L(v_8) = 0 < \tau - k - |S| + 2 = 1$. In this case, $N_{\succ_\eta}(v_i) \cup N^2_{\succ_\eta}(v_i)$ is reduced to $\{v_2, v_3, v_4, v_5, v_6, v_7\}$, as shown in Figure 5(c).

Theorem 3. *The time complexity of* MBPE *(Algorithm 4) is $\mathcal{O}(n2^{k-1}(\delta D)^{k+1}2^\delta)$, where δ and D are the degeneracy and maximum degree of G, respectively.*

B. Minimum Degree Pivot

In our algorithm MBPE, Enum is the most critical component and has exponential time complexity. To improve practical performance of our algorithm, we adopt a novel branching technique to speed up the enumeration procedure, which can reduce the generated branches in the search tree. In addition, we prove that this branching technique leads to a better time complexity of MBPE algorithm. The main idea of this branching strategy is to select the vertex $u \in P_L \cup P_R$ with the minimum degree in $g(P_L \cup C_L, P_R \cup C_R)$. Then, based on u, we expand the partial balanced k-plex (P_L, P_R) by prioritizing the candidate vertices in $C_L \cup C_R$ that are u's non-neighbors in $g(P_L \cup C_L, P_R \cup C_R)$. When the number of u's non-neighbors in $g(P_L, P_R)$ reaches k, we no longer generate branches for u's remaining non-neighbors in $C_L \cup C_R$. In this way, the number of generated branches in the enumeration process can be reduced significantly. For simplicity, we will refer to $g(P_L \cup C_L, P_R \cup C_R)$ as g in the following sections.

The pseudo-code of Enum-MDP is shown in Algorithm 6. First, C_L, C_R, X_L and X_R are refined by removing vertices that cannot form a larger balanced k-plex with the current balanced k-plex (P_L, P_R) (Lines 1-4). If these four sets are empty and (P_L, P_R) satisfies the size constraint, it emits this maximal balanced k-plex (Lines 5-7). Next, the algorithm selects a vertex u with the minimum degree in g (Line 8)

Algorithm 6: Enum-MDP

Input: P_L, P_R, C_L, C_R, X_L, X_R
Output: Maximal balanced k-plexes

1 $C_L \leftarrow \{v \in C_L : (P_L \cup \{v\}, P_R)$ is a balanced k-plex$\}$;
2 $C_R \leftarrow \{v \in C_R : (P_L, P_R \cup \{v\})$ is a balanced k-plex$\}$;
3 $X_L \leftarrow \{v \in X_L : (P_L \cup \{v\}, P_R)$ is a balanced k-plex$\}$;
4 $X_R \leftarrow \{v \in X_R : (P_L, P_R \cup \{v\})$ is a balanced k-plex$\}$;
5 **if** $C_L \cup C_R = \emptyset$ **and** $X_L \cup X_R = \emptyset$ **then**
6 **if** $|P_L| \geq \tau$ **and** $|P_R| \geq \tau$ **then** emit (P_L, P_R);
7 **return**
/* denote $g = g(P_L \cup C_L, P_R \cup C_R)$ */
8 $u \leftarrow$ the vertex of minimum degree in g;
9 **if** $d_g(u) \geq V(g) - k$ **then**
10 **if** *g is maximal and satisfies the size constraint* **then**
11 emit $(P_L \cup C_L, P_R \cup C_R)$;
12 **else if** $u \in P_L \cup P_R$ **then**
13 Let $S = \{v_1, \cdots, v_{q_2}\}$ be the set of vertices in $(C_L \cup C_R) \setminus N_g(u)$;
14 $k' \leftarrow k - |(P_L \cup P_R) \setminus N_g(u)|$;
15 **for each** $i \in \{1, \cdots, k'\}$ **do**
16 **if** $v_i \in C_L$ **then**
17 $C_L \leftarrow C_L \setminus \{v_i\}$;
18 Enum-MDP$(P_L, P_R, C_L, C_R, X_L \cup \{v_i\}, X_R)$;
19 $P_L \leftarrow P_L \cup \{v_i\}$; $X_R \leftarrow X_R \setminus \{v_i\}$;
20 **else**
21 $C_R \leftarrow C_R \setminus \{v_i\}$;
22 Enum-MDP$(P_L, P_R, C_L, C_R, X_L, X_R \cup \{v_i\})$;
23 $P_R \leftarrow P_R \cup \{v_i\}$; $X_L \leftarrow X_L \setminus \{v_i\}$;
24 Enum-MDP$(P_L, P_R, C_L \setminus S, C_R \setminus S, X_L, X_R)$;
25 **else if** $u \in C_L$ **then**
26 Enum-MDP$(P_L, P_R, C_L \setminus \{u\}, C_R, X_L \cup \{u\}, X_R)$;
27 Enum-MDP$(P_L \cup \{u\}, P_R, C_L \setminus \{u\}, C_R, X_L, X_R \setminus \{u\})$;
28 **else**
29 Enum-MDP$(P_L, P_R, C_L, C_R \setminus \{u\}, X_L, X_R \cup \{u\})$;
30 Enum-MDP$(P_L, P_R \cup \{u\}, C_L, C_R \setminus \{u\}, X_L \setminus \{u\}, X_R)$;

and checks if g is maximal and satisfies the size constraint. If so, it emits $(P_L \cup C_L, P_R \cup C_R)$ and terminates the current branch (Lines 9-11). Otherwise, we know that u has more than k non-neighbors in g and branches are generated as follows:

- If $u \in P_L \cup P_R$, let $q_1 = |(P_L \cup P_R) \setminus N_g(u)|$, representing the number of u's non-neighbors in $P_L \cup P_R$, and $q_2 = |(C_L \cup C_R) \setminus N_g(u)|$, representing the number of u's non-neighbors in $C_L \cup C_R$. Since u has more than k non-neighbors in g, we have $q_1 < k$ (otherwise, u cannot have more than k non-neighbors in g due to the refinement of C_L and C_R at Lines 1-2). Let $k' = k - q_1$. At most k' vertices in $(C_L \cup C_R) \setminus N_g(u)$ can be included in the current balanced k-plex. If we use $\{v_1, \ldots, v_{q_2}\}$ to denote u's non-neighbors in the candidate set $C_L \cup C_R$, $k' + 1$ branches will be generated (Lines 12-24):
  1) The first branch moves v_1 from C_L (resp. C_R) to X_L (resp. X_R);
  2) The second branch moves v_1 from C_L (resp. C_R) to P_L (resp. P_R) and v_2 from C_L (resp. C_R) to X_L (resp. X_R);
  3) The i-th ($3 \leq i \leq k'$) branch moves $\{v_1, \cdots, v_{i-1}\}$ from C_L (resp. C_R) to P_L (resp. P_R) and v_i from C_L (resp. C_R) to X_L (resp. X_R);
  4) The last branch moves $\{v_1, \cdots, v_{k'}\}$ from C_L (resp.

C_R) to P_L (resp. P_R) and the rest of u's non-neighbors $\{v_{k'+1}, \cdots, v_{q_2}\}$ are removed from C_L (resp. C_R) to X_L (resp. X_R).

- If $u \notin P_L \cup P_R$, two branches are generated by moving u from C_L (resp. C_R) to X_L (resp. X_R) and by moving u from C_L (resp. C_R) to P_L (resp. P_R) for the case of $u \in C_L$ (resp. $u \in C_R$) (Lines 25-30). The second branch will fall into the first case.

The maximality check at Line 10 can be efficiently performed by verifying if there is any vertex in $X_L \cup X_R$ that can be added to g to form a larger balanced k-plex. This process can be efficient since if there is a vertex v in g that has already missed k vertices, we only need to consider v's neighbors in $X_L \cup X_R$. By substituting the Enum procedure with Enum-MDP in MBPE (Algorithm 4), we obtain a new algorithm, MBPE*. The time complexity of MBPE* is analyzed in the theorem below.

Theorem 4. *The time complexity of* MBPE* *is* $\mathcal{O}(n2^{k-1}(\delta D)^{k+1}\alpha_k^{\delta})$*, where* δ *and* D *are the degeneracy order and the maximum degree of the graph, respectively, and* α_k *is strictly smaller than* 2 *and depends only on* k*, e.g.,* $\alpha_2 = 1.839$*,* $\alpha_3 = 1.928$*, and* $\alpha_4 = 1.966$*.*

C. Extension to Antagonistic Balanced k-plex Computation

Our algorithms can be easily adapted for enumerating the maximal antagonistic balanced k-plexes (Definition 3). Specifically, at Line 3 of Algorithm 4, we replace $S \subseteq N_{\succ\eta}(v_i) \cup N^2_{\succ\eta}(v_i)$ by $S \subseteq N^2_{\succ\eta}(v_i)$. This is because an antagonistic balanced k-plex requires each vertex to be positively connected to same-set vertices and negatively connected to opposite-set vertices. Therefore, S only contains v_i's 2-hop neighbors, and S_L and S_R will always be empty. The subsequent initialization of six vertex sets and the invocation of the Enum procedure follow the same way. We refer to this adapted algorithm for maximal antagonistic balanced k-plex enumeration as MBPE*-Atg.

V. Experiments

In this section, we evaluate the efficiency and effectiveness of our techniques by comparing the following algorithms:

- MBPE-BK: The baseline algorithm (Algorithm 1) equipped with all optimizations proposed in Section III-A except the pivoting technique.
- MBPE-BK*: MBPE-BK equipped with the pivoting technique.
- MBPE: The advanced algorithm (Algorithm 4) proposed in Section IV-A, equipped with subgraph sparsification and partition-based vertex reduction.
- MBPE*: MBPE equipped with the minimum degree pivot technique, i.e., by replacing Enum with Enum-MDP (Algorithm 6).

All the algorithms are implemented in C++, compiled with g++ 7.5.0 with the -O3 flag.[2] All the experiments are conducted on a machine with an Intel Core-i7 3.20GHz CPU and 64GB RAM running Ubuntu 18.04. The time cost is measured as the amount of wall-clock time elapsed during the program's execution. The time limit is set as 20 hours.

[2]Source code is available at https://github.com/kyaocs/MBPE

TABLE I
Statistics of datasets

Dataset	$\|V\|$	$\|E\|$	$\frac{\|E^-\|}{\|E\|}$	D	δ	Category
Bitcoin	5,881	21,492	15%	795	21	Trade
AdjWordNet	16,259	76,845	32%	477	63	Language
WikiElection	7,118	100,693	22%	1065	53	Editing
Reddit	54,075	220,151	8%	4,164	76	Social
Referendum	10,884	251,406	5%	2,784	104	Political
Slashdot	82,140	500,481	24%	2,548	54	Social
Epinions	131,828	841,372	17%	3,558	121	Social
WikiConflict	116,836	2,027,871	62%	20,153	145	Editing
DBLP	2,387,365	11,915,023	72%	3,282	448	Cooperation
SN1	2,000,000	83,326,093	29%	1,301	489	Synthetic
SN2	3,000,000	124,990,528	31%	1,448	538	Synthetic

Datasets. We evaluate our algorithms on nine publicly-available real-world signed graphs, whose main characteristics are summarized in Table I. Bitcoin[3] and Epinions[3] are who-trusts-whom graphs for users on the Bitcoin OTC and Epinions platforms, respectively. AdjWordNet[4] captures the synonym-and-antonym relation among adjectives in the English language. WikiElection[5] is the graph of users from the English Wikipedia that voted for and against each other during administration elections. Reddit[3] captures the sentiment among different subreddits on Reddit. Referendum [25] records Twitter data on the 2016 Italian Referendum, where interactions are negative for users with conflicting stances and positive otherwise. Slashdot[3] contains friend/foe links among Slashdot users. WikiConflict[5] represents positive and negative edit conflicts among the users of the English Wikipedia. DBLP is the signed network used in [26], which is downloaded from the *dblp database* [6] and processed in the same way as [26]. In addition, we also evaluate the algorithms on two synthetic datasets, SN1 and SN2, which are generated by the synthetic signed network generator SRN with default settings [27].

Exp-1: Efficiency when varying τ. In this experiment, we evaluate the efficiency of the algorithms by varying τ from 3 to 5 (with k fixed at 2). The running time for each algorithm is presented in Table II. The fastest running time for each instance is highlighted in bold. As τ increases, the running time decreases. This is because a larger size constraint τ enables the algorithm to prune more unpromising vertices early in the process. Notably, the pivot technique employed by MBPE-BK* significantly accelerates its performance, making it approximately two orders of magnitude faster than MBPE-BK. For instance, on AdjWordNet with $\tau = 4$, MBPE-BK* completes in 9.57 seconds, while MBPE-BK takes 7923.66 seconds. This highlights the effectiveness of the pivot technique. However, despite the gains made with the

[3]https://snap.stanford.edu/
[4]https://wordnet.princeton.edu/
[5]https://konect.cc
[6]https://dblp.uni-trier.de/

TABLE II
THE RUNNING TIME OF ENUMERATING MAXIMAL BALANCED k-PLEXES BY VARYING τ ($k = 2$)

Graphs	τ	Running time (sec)			
		MBPE-BK	MBPE-BK*	MBPE	MBPE*
Bitcoin	3	0.19	0.077	0.028	**0.021**
	4	0.06	0.015	0.013	**0.008**
	5	0.05	0.006	0.004	**0.002**
AdjWordNet	3	8930.63	15.15	8.39	**7.01**
	4	7923.66	9.57	4.26	**3.26**
	5	1929.86	4.69	2.30	**1.60**
WikiElection	3	15.38	14.31	1.21	**1.17**
	4	0.92	0.92	0.23	**0.23**
	5	0.87	0.87	0.16	**0.16**
Reddit	3	64.94	58.85	4.57	**4.45**
	4	7.77	7.67	0.69	**0.66**
	5	5.90	5.90	0.29	**0.29**
Referendum	3	1341.34	34.75	12.21	**10.36**
	4	114.98	6.91	2.74	**1.85**
	5	10.94	1.26	0.63	**0.39**
Slashdot	3	9.94	8.86	2.41	**2.33**
	4	0.68	0.67	0.64	**0.62**
	5	0.52	0.56	0.52	**0.50**
Epinions	3	594.79	483.13	42.82	**40.69**
	4	92.53	73.86	8.53	**7.32**
	5	27.83	19.60	2.69	**2.31**
WikiConflict	3	1143.19	1055.63	**141.43**	142.22
	4	4.42	4.43	3.10	**3.10**
	5	2.56	2.44	2.39	**2.26**
DBLP	3	17394.57	4620.07	381.52	**320.31**
	4	7305.79	2746.66	53.63	**46.87**
	5	46.21	46.50	32.01	**28.50**
SN1	3	-	19486.49	2249.52	**2136.59**
	4	2139.66	2136.74	428.77	**406.65**
	5	502.34	501.92	392.01	**388.82**
SN2	3	-	33747.44	3495.38	**3371.46**
	4	3130.09	3128.66	643.63	**635.24**
	5	735.60	734.53	585.92	**583.66**

TABLE III
THE RUNNING TIME OF ENUMERATING MAXIMAL BALANCED k-PLEXES BY VARYING k ($\tau = k + 2$)

Graphs	k	Running time (sec)			
		MBPE-BK	MBPE-BK*	MBPE	MBPE*
Bitcoin	2	0.06	0.015	0.013	**0.008**
	3	0.35	0.079	0.097	**0.024**
	4	1.03	0.29	0.34	**0.081**
AdjWordNet	2	7923.66	9.57	4.26	**3.26**
	3	13492.20	564.26	50.34	**21.22**
	4	-	14356.45	920.71	**792.94**
WikiElection	2	0.92	0.92	0.23	**0.23**
	3	438.69	499.22	1.36	**1.31**
	4	-	-	97.51	**94.23**
Reddit	2	7.77	7.67	0.69	**0.66**
	3	5827.53	6667.67	84.66	**70.82**
	4	-	-	10403.30	**8759.93**
Referendum	2	114.98	6.91	2.74	**1.85**
	3	3818.18	670.82	122.70	**62.51**
	4	-	-	3213.36	**1676.66**
Slashdot	2	0.68	0.67	0.64	**0.62**
	3	1.75	1.45	0.68	**0.67**
	4	85.84	89.61	1.39	**1.34**
Epinions	2	92.53	73.86	8.53	**7.32**
	3	-	-	568.62	**465.23**
	4	-	-	53502.45	**46098.02**
WikiConflict	2	4.42	4.43	3.10	**3.10**
	3	2569.45	2613.50	6.50	**6.40**
	4	-	-	1413.70	**1389.39**
DBLP	2	7305.79	2746.66	53.63	**46.87**
	3	-	-	1614.75	**1487.92**
	4	-	-	37495.30	**29516.18**
SN1	2	2139.66	2136.74	428.77	**406.65**
	3	4921.64	4926.66	988.77	**936.65**
	4	-	-	18793.56	**18108.82**
SN2	2	3130.09	3128.66	643.63	**635.24**
	3	7470.37	7469.89	1560.08	**1548.87**
	4	-	-	27765.03	**25829.51**

pivot technique, MBPE-BK* remains slower than MBPE in all cases, with a gap of approximately one order of magnitude. For instance, on Reddit with $\tau = 3$, MBPE takes 4.57 seconds, whereas MBPE-BK* takes 58.85 seconds. This discrepancy does demonstrate the effectiveness of the new search framework, thereby benefiting practical performance. The running time of MBPE and MBPE* are comparable, with MBPE* being slightly faster in most cases.

Exp-2: Efficiency when varying k. In this experiment, we evaluate the efficiency of the algorithms by varying k from 2 to 4 (setting $\tau = k + 2$). The results, shown in Table III, reveal that the running time of all algorithms increases with larger values of k. This is attributed to the fact that a larger k allows each member in a balanced k-plex to have more non-neighbors, significantly expanding the search space. In this case, the superiority of our minimum degree pivot technique becomes evident. MBPE* consistently outperforms MBPE, especially when k becomes large. For instance, on Referendum with $k = 4$, MBPE* is around 2 times faster than MBPE (i.e., 1676.66 seconds v.s. 3213.36 seconds), illustrating the improved efficiency by the minimum degree pivot technique.

Exp-3: Evaluation of each technique. In this experiment, we evaluate the effectiveness of each optimization technique by varying k from 2 to 4 (setting $\tau = k + 2$). We implement different variants of MBPE* as follows. MBPE*-A does not apply vertex reduction and edge reduction on the original graph (proposed in Section III). MBPE*-B does not apply the subgraph sparsification technique (proposed in Section IV). MBPE*-C does not apply the partition-based vertex reduction technique (proposed in Section IV). As shown in Table IV, MBPE* consistently outperforms all other algorithms. For example, on WikiConflict, when $k = 3$, MBPE* is three orders of magnitude faster than MBPE*-A, which demonstrates the effectiveness of our vertex and edge reduction techniques. On Reddit, when $k = 3$, MBPE* is around 6 times faster than MBPE*-B, showcasing the effectiveness of our subgraph sparsification technique. On WikiElection, when $k = 4$, MBPE* is around one order of magnitude faster than MBPE*-C, validating the effectiveness of our partition based vertex reduction. This experiment demonstrates that each of our optimization techniques contributes to the algorithm efficiency.

Exp-4: Evaluation of different balanced k-plex models. In this experiment, we evaluate the two alternative definitions: antagonistic balanced k-plex (Definition 3) and balanced k-plex (Definition 4), by varying k from 2 to 4. The algorithm MBPE* aims to detect maximal balanced k-plexes, while MBPE*-Atg focuses on detecting maximal antagonistic balanced k-plexes. As shown in Table V, MBPE* can detect more results than MBPE*-Atg in most cases, albeit with

TABLE IV
EVALUATION OF EACH TECHNIQUE BY VARYING k ($\tau = k + 2$)

Graphs	k	Running time (sec)			
		MBPE*-A	MBPE*-B	MBPE*-C	MBPE*
Bitcoin	2	0.27	0.009	0.009	**0.008**
	3	0.29	0.025	0.026	**0.024**
	4	0.36	0.090	0.081	**0.081**
AdjWordNet	2	62.82	3.38	3.80	**3.26**
	3	79.60	31.11	24.46	**21.22**
	4	865.18	1821.60	802.84	**792.94**
WikiElection	2	3.18	0.26	0.30	**0.23**
	3	8.89	4.97	3.92	**1.31**
	4	182.77	299.75	839.88	**94.23**
Reddit	2	199.03	1.26	0.83	**0.66**
	3	467.32	422.72	79.52	**70.82**
	4	-	-	36455.61	**8759.93**
Referendum	2	12.32	2.06	2.10	**1.85**
	3	94.94	77.46	66.61	**62.51**
	4	2477.84	2768.67	1825.48	**1676.66**
Slashdot	2	124.24	0.65	0.65	**0.62**
	3	235.22	0.71	0.70	**0.67**
	4	373.80	1.85	2.60	**1.34**
Epinions	2	2384.19	10.94	9.50	**7.32**
	3	4540.34	1775.48	531.59	**465.23**
	4	-	-	-	**46098.02**
WikiConflict	2	1498.75	3.14	3.24	**3.10**
	3	6129.94	12.72	19.10	**6.40**
	4	-	2383.00	11029.96	**1389.39**
DBLP	2	3394.64	53.14	159.25	**46.87**
	3	-	1957.65	9566.00	**1487.92**
	4	-	-	-	**29516.18**
SN1	2	36546.30	460.06	1513.33	**406.65**
	3	-	1112.40	8639.49	**936.65**
	4	-	-	-	**18108.82**
SN2	2	57143.52	696.56	2194.52	**635.24**
	3	-	1676.66	13239.49	**1548.87**
	4	-	-	-	**25829.51**

TABLE V
EVALUATION OF DIFFERENT MODELS BY VARYING k ($\tau = k + 2$)

Graphs	k	MBPE*		MBPE*-Atg	
		# results	Time (sec)	# results	Time (sec)
Bitcoin	2	1026	0.008	1026	0.007
	3	7583	0.024	7583	0.023
	4	23739	0.081	23739	0.072
AdjWordNet	2	52374	3.26	52209	3.26
	3	3213645	21.22	3201555	20.98
	4	145899163	792.94	145455233	800.39
WikiElection	2	13	0.23	12	0.22
	3	7	1.31	5	1.00
	4	1	94.23	1	77.51
Reddit	2	752	0.66	520	0.64
	3	3646	70.82	1361	43.66
	4	11873	8759.93	2136	4897.61
Referendum	2	138924	1.85	138924	1.81
	3	3378920	62.51	3378920	62.11
	4	47040124	1676.66	47040124	1659.07
Slashdot	2	100	0.62	100	0.64
	3	40	0.67	40	0.60
	4	14	1.34	14	1.11
Epinions	2	319386	7.32	294265	7.26
	3	7961979	465.23	6702015	371.08
	4	146189731	46098.02	116817331	33502.45
WikiConflict	2	623	3.10	506	3.03
	3	762	6.40	528	5.93
	4	464	1389.39	291	1250.03
DBLP	2	612062	46.87	523948	43.54
	3	75018028	1487.92	63294958	1406.33
	4	820939485	29516.18	793049854	29176.66
SN1	2	102	406.65	97	423.59
	3	13	936.65	11	976.65
	4	2	18108.82	2	18108.82
SN2	2	120	635.24	114	650.01
	3	19	1548.87	16	1565.24
	4	1	25829.51	1	27462.79

a slightly increased running time. For instance, on Reddit, when $k = 3$ and $\tau = 5$, there are 3646 maximal balanced 3-plexes compared to 1361 maximal antagonistic balanced 3-plexes, and MBPE* and MBPE*-Atg use 70.82 seconds and 43.66 seconds, respectively. This outcome aligns with our expectations, as the balanced k-plex model generalizes the antagonistic balanced k-plex model, allowing vertices to slightly violate the structural balance theory.

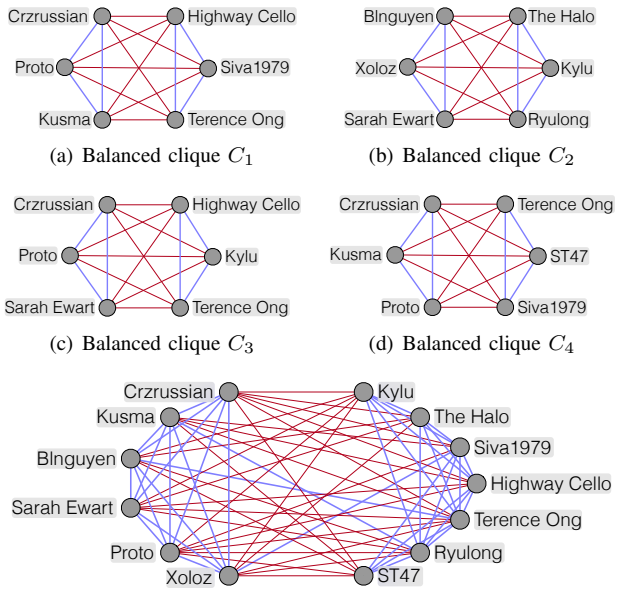
Exp-5: Synonym and antonym group detection on AdjWordNet. In this experiment, we evaluate the effectiveness of the balanced k-plex by detecting synonym and antonym groups on AdjWordNet. This dataset is the same as the one used for efficiency evaluation (see Table I), where each vertex represents an adjective and the edge between a pair of synonyms (resp. antonyms) is positive (resp. negaitve). Using our algorithm MBPE*, we identify maximal balanced k-plexes with $\tau = 6$ and $k = 3$. For comparison, we employ MBCEnum [11] to detect maximal balanced cliques under the same parameter $\tau = 6$. Our results reveal $8,857$ maximal balanced k-plexes (with sizes ranging from 12 to 64) and 183 maximal balanced cliques (with sizes ranging from 12 to 60). Notably, some synonym and antonym groups are uniquely detected by MBPE*, as k-plexes allow for missing connections. Representative results are shown in Table VI. For instance, the second row shows a balanced 3-plex with

TABLE VI
CASE STUDY ON ADJWORDNET

P_L	P_R
big, high, immodest, **up**, large, proud, tall	humble, modest, **pocket-size**, lowly, miserable, low, small
dirty, black, cloudy, ill-defined, dark, **heavy**, opaque, preserved, stale, unclean, unclear, unfair....	clean, blank, clear, fair, fresh, **light**, neat, unclouded, uncontaminated, uninfected, white....
...	...

$|P_L| = 7$ and $|P_R| = 7$. Compared to the balanced clique detected by MBCEnum, the adjectives "up" and "pocket-size" (highlighted in gray) newly appear, since they do not connect to all other words. However, these two words fit well within the context of this synonym and antonym group. A similar phenomenon is observed in the third row, which is a balanced 3-plex with $|P_L| = 14$ and $|P_R| = 16$. In this case, "heavy" and "light" newly appear compared to the balanced clique detected by MBCEnum. From this experiment, we conclude that balanced k-plexes can identify more interesting synonym and antonym groups in AdjWordNet than balanced cliques.

Exp-6: Conflicts discovery on wiki-RfA. In this experiment, we evaluate the effectiveness of the balanced k-plex by detecting conflicts on wiki-RfA[3]. wiki-RfA records the mutual voting results of Wikipedia editors running for administrator, which contains $11,381$ Wikipedia members (voters and



(a) Balanced clique C_1 (b) Balanced clique C_2

(c) Balanced clique C_3 (d) Balanced clique C_4

(e) Balanced 4-plex

Fig. 6. Polarized structures detected in wiki-RfA

votees) forming $189,004$ distinct voter/votee pairs. We build an undirected signed graph by regarding each user as a node and create a positive (or negative) edge between two nodes if there is an supporting (or opposing) vote between them. Our goal is to find polarized structures in wiki-RfA. First, we apply the MBCEnum algorithm [11], which identifies only maximal balanced cliques of size 6. Figures 6(a) to 6(d) display four of the detected maximal balanced cliques, with blue edges indicating positive connections and red edges indicating negative connections. Next, using our MBPE* algorithm with $k = 4$, we discover a balanced 4-plex of size 13 with $P_L = \{$"Crzrussian", "Kusma", "Blnguyen", $\cdots\}$ and $P_R = \{$"Kylu", "The Halo", "Siva1979", $\cdots\}$ (Figure 6(e)). This balanced 4-plex is a more significant polarized structure and captures all four maximal balanced cliques shown in Figure 6. Interestingly, we observe three positive edges connecting vertices from opposite sets, such as the positive edge between "Kusma" and "Terence Ong". This is allowed by our balanced k-plex model, and a limited number of such "conflicting edges" does not compromise the overall polarized structure. Note that, the result in Figure 6(e) also cannot be found by the antagonistic balanced k-plex model. This highlights the advantages of our balanced k-plex model; by allowing slight deviations from structural balance theory, our model can uncover larger and more intriguing patterns.

VI. Related Works

Signed graph was firstly studied by Harary et al. [22], where the notion of structural balance theory is introduced. The problem of finding the largest (in terms of vertex number) vertex-induced subgraph that is structural balanced, known as the *maximum balanced subgraph problem*, is studied by Figueiredo et al. [28] and Ordozgoiti et al. [6]. The problem is NP-hard. A branch-and-cut exact algorithm, which only works for graphs with up-to a few thousand vertices, is proposed by Figueiredo et al. [28]. Heuristic algorithms without any guarantee on the solution optimality are investigated in [6], [28]. As the maximum balanced subgraph problem does not limit the non-neighbors of each vertex in the identified subgraph, these techniques cannot be applied to our problem.

The notion of structural balanced clique is formulated by Chen et al. [11] and the detection of balanced cliques receives an increasing attention recently [11], [12], [14], [15]. However, balanced clique requires each pair of vertices to be linked by an edge. Thus, their algorithms cannot be applied to our problem. Hao et al. [29] introduced the notion of k-*balanced trusted clique* for signed graphs, which is essentially the same problem as the traditional k-clique problem over unsigned graphs. Thus, the techniques proposed by Hao et al. [29] cannot be applied to solve our problem. The notion of (α, k)-clique is defined by Li et al. [26] for signed graphs, which is a clique such that each vertex has at most k negative neighbors and at least αk positive neighbors in the clique. As the structural balanced constraint is ignored, the techniques proposed by Li et al. [26] cannot be applied to our problem.

The problem of enumerating maximal k-plexes in unsigned graph has received extensive attention [21], [30]–[36]. Most of the maximal k-plex enumeration algorithms follow the framework of the Bron-Kerbosch algorithm, such as [30], [33], [34]. Aside from the Bron-Kerbosch variants, a polynomial delay algorithm is proposed by Berlowitz et al. [37]. Zhou et al. [35] proposed a novel branching scheme with a worst-case time complexity analysis. With their branching scheme, the running time of the algorithm is improved from $\mathcal{O}^*(2^n)$ to $\mathcal{O}^*(\gamma_k^n)$ in a graph with n vertices, where γ_k is related to k but strictly smaller than 2. Recently, Dai et al. [36] introduced a different branching strategy, which shares the same time complexity as the algorithm presented in [35]. Wang et al. [21] further improved the worst-case time complexity to $\mathcal{O}^*(\gamma_k^\delta)$, where δ is the degeneracy of the graph. However, these techniques cannot be directly applied to our problem due to the existence of edge signs and the enforcement of structural balance constraint.

VII. Conclusion

In this paper, we formulated the balanced k-plex model in signed graphs and proved that the problem of enumerating all maximal balanced k-plexes is #P-hard. To solve this problem, we first introduced a basic backtracking algorithm MBPE-BK and proposed several optimization techniques to enhance its practical performance. Then, we proposed a novel algorithm MBPE with better time complexity than MBPE-BK. We also proposed subgraph sparsification and partition-based vertex reduction to improve its efficiency. Finally, we adopted the minimum-degree branching strategy to speed up the enumeration procedure, which further improved the worst-case time complexity of MBPE. Experimental results on real-world and synthetic signed graphs demonstrated the efficiency of our algorithms and the effectiveness of our model.

REFERENCES

[1] Jiliang Tang, Yi Chang, Charu Aggarwal, and Huan Liu. A survey of signed network mining in social media. *ACM Computing Surveys (CSUR)*, 49(3):1–37, 2016.

[2] David A. Easley and Jon M. Kleinberg. *Networks, Crowds, and Markets - Reasoning About a Highly Connected World*. Cambridge University Press, 2010.

[3] Jérôme Kunegis, Andreas Lommatzsch, and Christian Bauckhage. The slashdot zoo: mining a social network with negative edges. In *Proc. of WWW'09*, pages 741–750, 2009.

[4] Le Ou-Yang, Dao-Qing Dai, and Xiao-Fei Zhang. Detecting protein complexes from signed protein-protein interaction networks. *IEEE/ACM transactions on computational biology and bioinformatics*, 12(6):1333–1344, 2015.

[5] Lingyang Chu, Zhefeng Wang, Jian Pei, Jiannan Wang, Zijin Zhao, and Enhong Chen. Finding gangs in war from signed networks. In *Proc. of SIGKDD'16*, pages 1505–1514, 2016.

[6] Bruno Ordozgoiti, Antonis Matakos, and Aristides Gionis. Finding large balanced subgraphs in signed networks. In *Proceedings of The Web Conference 2020*, pages 1378–1388, 2020.

[7] Jure Leskovec, Daniel Huttenlocher, and Jon Kleinberg. Predicting positive and negative links in online social networks. In *Proc. of WWW'10*, pages 641–650, 2010.

[8] Jihang Ye, Hong Cheng, Zhe Zhu, and Minghua Chen. Predicting positive and negative links in signed social networks by transfer learning. In *Proc. of WWW'13*, pages 1477–1488, 2013.

[9] Chien Chin Chen, Yu-Hao Wan, Meng-Chieh Chung, and Yu-Chun Sun. An effective recommendation method for cold start new users using trust and distrust networks. *Information Sciences*, 224:19–36, 2013.

[10] Jiliang Tang, Charu Aggarwal, and Huan Liu. Recommendations in signed social networks. In *Proceedings of the 25th International Conference on World Wide Web*, pages 31–40, 2016.

[11] Zi Chen, Long Yuan, Xuemin Lin, Lu Qin, and Jianye Yang. Efficient maximal balanced clique enumeration in signed networks. In *Proceedings of The Web Conference 2020*, pages 339–349, 2020.

[12] Zi Chen, Long Yuan, Xuemin Lin, Lu Qin, and Wenjie Zhang. Balanced clique computation in signed networks: Concepts and algorithms. *IEEE Transactions on Knowledge and Data Engineering*, 2022.

[13] Kai Yao, Lijun Chang, and Lu Qin. Computing maximum structural balanced cliques in signed graphs. In *2022 IEEE 38th International Conference on Data Engineering (ICDE)*, pages 1004–1016. IEEE, 2022.

[14] Renjie Sun, Yanping Wu, Xiaoyang Wang, Chen Chen, Wenjie Zhang, and Xuemin Lin. Clique identification in signed graphs: A balance theory based model. *IEEE Transactions on Knowledge and Data Engineering*, 2023.

[15] Kai Yao, Lijun Chang, and Lu Qin. Identifying large structural balanced cliques in signed graphs. *IEEE Transactions on Knowledge and Data Engineering*, 2023.

[16] Francesco Bonchi, Edoardo Galimberti, Aristides Gionis, Bruno Ordozgoiti, and Giancarlo Ruffo. Discovering polarized communities in signed networks. In *Proc. of CIKM'19*, pages 961–970, 2019.

[17] Han Xiao, Bruno Ordozgoiti, and Aristides Gionis. Searching for polarization in signed graphs: a local spectral approach. In *Proceedings of The Web Conference 2020*, pages 362–372, 2020.

[18] Apichat Suratanee, Martin H Schaefer, Matthew J Betts, Zita Soons, Heiko Mannsperger, Nathalie Harder, Marcus Oswald, Markus Gipp, Ellen Ramminger, Guillermo Marcus, et al. Characterizing protein interactions employing a genome-wide sirna cellular phenotyping screen. *PLoS Comput Biol*, 10(9):e1003814, 2014.

[19] Kenneth E Read. Cultures of the central highlands, new guinea. *Southwestern Journal of Anthropology*, 10(1):1–43, 1954.

[20] Coenraad Bron and Joep Kerbosch. Finding all cliques of an undirected graph (algorithm 457). *Commun. ACM*, 16(9):575–576, 1973.

[21] Zhengren Wang, Yi Zhou, Mingyu Xiao, and Bakhadyr Khoussainov. Listing maximal k-plexes in large real-world graphs. In *Proceedings of the ACM Web Conference 2022*, pages 1517–1527, 2022.

[22] Frank Harary et al. On the notion of balance of a signed graph. *The Michigan Mathematical Journal*, 2(2):143–146, 1953.

[23] Stephen B Seidman and Brian L Foster. A graph-theoretic generalization of the clique concept. *Journal of Mathematical sociology*, 6(1):139–154, 1978.

[24] David W. Matula and Leland L. Beck. Smallest-last ordering and clustering and graph coloring algorithms. *J. ACM*, 30(3):417–427, 1983.

[25] Mirko Lai, Viviana Patti, Giancarlo Ruffo, and Paolo Rosso. Stance evolution and twitter interactions in an italian political debate. In *Natural Language Processing and Information Systems: 23rd International Conference on Applications of Natural Language to Information Systems, NLDB 2018, Paris, France, June 13-15, 2018, Proceedings 23*, pages 15–27. Springer, 2018.

[26] Rong-Hua Li, Qiangqiang Dai, Lu Qin, Guoren Wang, Xiaokui Xiao, Jeffrey Xu Yu, and Shaojie Qiao. Efficient signed clique search in signed networks. In *2018 IEEE 34th International Conference on Data Engineering (ICDE)*, pages 245–256. IEEE, 2018.

[27] Yansen Su, Bangju Wang, Fan Cheng, Lei Zhang, Xingyi Zhang, and Linqiang Pan. An algorithm based on positive and negative links for community detection in signed networks. *Scientific reports*, 7(1):10874, 2017.

[28] Rosa Maria Videira de Figueiredo and Yuri Frota. The maximum balanced subgraph of a signed graph: Applications and solution approaches. *Eur. J. Oper. Res.*, 236(2):473–487, 2014.

[29] Fei Hao, Stephen S Yau, Geyong Min, and Laurence T Yang. Detecting k-balanced trusted cliques in signed social networks. *IEEE internet computing*, 18(2):24–31, 2014.

[30] Bin Wu and Xin Pei. A parallel algorithm for enumerating all the maximal k-plexes. In *Pacific-Asia conference on knowledge discovery and data mining*, pages 476–483. Springer, 2007.

[31] Sara Cohen, Benny Kimelfeld, and Yehoshua Sagiv. Generating all maximal induced subgraphs for hereditary and connected-hereditary graph properties. *Journal of Computer and System Sciences*, 74(7):1147–1159, 2008.

[32] Alessio Conte, Donatella Firmani, Caterina Mordente, Maurizio Patrignani, and Riccardo Torlone. Fast enumeration of large k-plexes. In *Proceedings of the 23rd ACM SIGKDD international conference on knowledge discovery and data mining*, pages 115–124, 2017.

[33] Zhuo Wang, Qun Chen, Boyi Hou, Bo Suo, Zhanhuai Li, Wei Pan, and Zachary G Ives. Parallelizing maximal clique and k-plex enumeration over graph data. *Journal of Parallel and Distributed Computing*, 106:79–91, 2017.

[34] Alessio Conte, Tiziano De Matteis, Daniele De Sensi, Roberto Grossi, Andrea Marino, and Luca Versari. D2k: scalable community detection in massive networks via small-diameter k-plexes. In *Proceedings of the 24th ACM SIGKDD International Conference on Knowledge Discovery & Data Mining*, pages 1272–1281, 2018.

[35] Yi Zhou, Jingwei Xu, Zhenyu Guo, Mingyu Xiao, and Yan Jin. Enumerating maximal k-plexes with worst-case time guarantee. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 34, pages 2442–2449, 2020.

[36] Qiangqiang Dai, Rong-Hua Li, Hongchao Qin, Meihao Liao, and Guoren Wang. Scaling up maximal k-plex enumeration. In *Proceedings of the 31st ACM International Conference on Information & Knowledge Management*, pages 345–354, 2022.

[37] Devora Berlowitz, Sara Cohen, and Benny Kimelfeld. Efficient enumeration of maximal k-plexes. In *Proceedings of the 2015 ACM SIGMOD International conference on management of data*, pages 431–444, 2015.

[38] Leslie G Valiant. The complexity of enumeration and reliability problems. *siam Journal on Computing*, 8(3):410–421, 1979.

[39] Vladimir Batagelj and Matjaz Zaversnik. An o (m) algorithm for cores decomposition of networks. *arXiv preprint cs/0310049*, 2003.

[40] Fedor V Fomin and Petteri Kaski. Exact exponential algorithms. *Communications of the ACM*, 56(3):80–88, 2013.

APPENDIX A. PROOFS OF LEMMAS AND THEOREMS

Proof of Theorem 1. We prove the theorem by reducing from the problem of enumerating all maximal k-plexes that are of size at least τ in an unsigned graph, which is #P-complete [38]. Given an unsigned input graph $G^u = (V^u, E^u)$ and an integer τ of the maximal k-plex enumeration problem, we construct a signed graph $G = (V, E^+, E^-)$ as follows. Firstly, $V = V^u \cup \{v_1, \ldots, v_\tau\}$ where $v_i \notin V^u, \forall 1 \leq i \leq \tau$. Denote $\{v_1, \ldots, v_\tau\}$ by V^a. Then, $E^+ = E^u \cup \{(v_i, v_j) \mid v_i \in V^a, v_j \in V^a, v_i \neq v_j\}$ and $E^- = \{(u, v) \mid u \in V^u, v \in V^a\}$. We first prove by contradiction that every balanced k-plex

(P_L, P_R) in G with $\min\{|P_L|, |P_R|\} \geq \tau \geq k$ must satisfy either $P_L \subseteq V^u \wedge P_R = V^a$ or $P_L = V^a \wedge P_R \subseteq V^u$.

1) Suppose $P_L \subseteq V^u$ and $P_R \subseteq V^u$. Consider any vertex $u \in P_L$, we have $d^-_{P_R}(u) = 0 \leq |P_R| - k$, contradicting that (P_L, P_R) is a balanced k-plex.
2) Suppose $P_L \cap V^a \neq \emptyset$ and $P_R \cap V^a \neq \emptyset$. Let $X_1 = P_L \cap V^a$, $X_2 = P_L \cap V^u$, $Y_1 = P_R \cap V^a$ and $Y_2 = P_R \cap V^u$. Then, none of X_1, X_2, Y_1, Y_2 is empty. Let's consider any vertex $u \in X_2$, then $d^+_{P_L}(u) + d^-_{P_R}(u) \geq |P_L| + |P_R| - k$ and thus $|X_1| + |Y_2| \leq k-1$. Similarly we have $|X_2| + |Y_1| \leq k-1$, contradicting that $|P_L| \geq \tau$ and $|P_R| \geq \tau$.
3) Suppose $P_L \cap V^a \neq \emptyset$, $P_L \cap V^u \neq \emptyset$ and $P_R \subseteq V^u$. Consider any vertex $u \in P_L$, we have $d^-_{P_R}(u) = 0 \leq |P_R| - k$, contradiction (similar to the first case).
4) Suppose $P_R \cap V^a \neq \emptyset$, $P_R \cap V^u \neq \emptyset$ and $P_L \subseteq V^u$. Similar to the third case above.

Without loss of generality, assume $P_R = V^a$. Then, it is easy to see that (P_L, P_R) is a maximal balanced k-plex in G with $\min\{|P_L|, |P_R|\} \geq \tau$ if and only if P_L is a maximal k-plex in G^u with $|P_L| \geq \tau$. □

Proof of Lemma 1. Suppose that the partitioning is not unique, and let (P_L, P_R) and (P'_L, P'_R) be two possible ways to partition it. Denote $X_1 = P_L \cap P'_L$, $X_2 = P_L \cap P'_R$, $Y_1 = P_R \cap P'_L$ and $Y_2 = P_R \cap P'_R$. Without loss of generality, assume $X_1 \neq \emptyset$ and let u be an arbitrary vertex in X_1. Then, by the definition of balanced k-plex, it must be the case that:

- $d^+_{X_1 \cup X_2}(u) + d^-_{Y_1 \cup Y_2}(u) \geq |X_1| + |X_2| + |Y_1| + |Y_2| - k$
- $d^+_{X_1 \cup Y_1}(u) + d^-_{X_2 \cup Y_2}(u) \geq |X_1| + |X_2| + |Y_1| + |Y_2| - k$

By adding up both inequalities, we have

$$\begin{aligned} &2d^+_{X_1}(u) + 2d^-_{Y_2}(u) + d^+_{X_2}(u) + d^+_{Y_1}(u) + d^-_{Y_1}(u) + d^-_{X_2}(u) \\ \geq &2|X_1| + 2|X_2| + 2|Y_1| + 2|Y_2| - 2k \end{aligned} \tag{1}$$

This is because $d^+_{X_1 \cup X_2}(u) = d^+_{X_1}(u) + d^+_{X_2}(u)$, $d^-_{Y_1 \cup Y_2}(u) = d^-_{Y_1}(u) + d^-_{Y_2}(u)$, $d^+_{X_1 \cup Y_1}(u) = d^+_{X_1}(u) + d^+_{Y_1}(u)$, $d^-_{X_2 \cup Y_2}(u) = d^-_{X_2}(u) + d^-_{Y_2}(u)$ since all sets are disjoint. Moreover, the LHS of Equation 1 satisfies:

$$LHS \leq 2(|X_1| - 1) + 2|Y_2| + |X_2| + |Y_1| \tag{2}$$

Combining Equations 1 and 2, we have

$$|X_2| + |Y_1| \leq 2k - 2. \tag{3}$$

Note that X_2 and Y_1 cannot be empty at the same time; otherwise the two ways of partitioning are the same. Without loss of generality, we pick a vertex $v \in X_2$, following the same logic, we derive

$$|X_1| + |Y_2| \leq 2k - 2 \tag{4}$$

Combining Equations 3 and 4, we have

$$|X_1| + |X_2| + |Y_1| + |Y_2| \leq 4k - 4.$$

Hence we conclude the proof. □

Proof of Lemma 2. Since (P_L, P_R) is a balanced k-plex, we know that the maximal balanced subgraph of G induced by (P_L, P_R), i.e., $g(P_L, P_R)$, must be a k-plex (after removing edges that violate structural balance), whose diameter is bounded by 2 [34]. Thus, we can easily know that the diameter of the subgraph $G[P_L \cup P_R]$ must also be bounded by 2. □

Proof of Lemma 3. We first prove that $P_L \subseteq \{v\} \cup N_G(v) \cup N^{++}_G(v) \cup N^{--}_G(v)$. For any other vertex $u \in P_L$ such that $u \neq v$ and $u \notin N_G(v)$, u and v must have a common neighbor w in $g(P_L, P_R)$. This is because the diameter of this maximal structural balanced subgraph is bounded by 2 (Lemma 2). Moreover, w's links to u and v must be of the same sign (i.e., $(u,w) \in E^+ \wedge (v,w) \in E^+$ or $(u,w) \in E^- \wedge (v,w) \in E^-$) since $g(P_L, P_R)$ is structural balanced. Thus, we derive that $u \in N^{++}_G(v) \cup N^{--}_G(v)$. $P_R \subseteq N_G(v) \cup N^{+-}_G(v) \cup N^{-+}_G(v)$ can be proved in a similar way. We omit the details. □

Proof of Theorem 2. Firstly, the time cost of computing degeneracy order of V is bounded by $\mathcal{O}(m)$ [39]. Secondly, the for loop at Line 2 processes each vertex v_i by constructing the partial solution, candidate sets and exclusive sets, which can be done in $\mathcal{O}(\delta D)$. Then the set of v_i-lead maximal balanced k-plexes are computed by invoking Enum, which then recursively invokes itself (Lines 18 and 22). Note that the number of vertices in C_L, C_R, X_L and X_R is bounded by δD since we only consider the forward two-hop neighbors of each vertex v_i. For each v_i, the total number of invocations to Enum is at most $3^{\delta D}$, since in each Enum we reduce the size of C_L or C_R by at least 1 (Lines 17 and 21). The running time of Enum without considering Lines 18 and 22 is dominated by the refinement of C_L, C_R, X_L and X_R at Lines 9-12, whose time complexity is bounded by $\mathcal{O}((\delta D)^2)$. Thus, the total time complexity of Algorithm 1 is $\mathcal{O}(m + \sum_{v_i \in V}(\delta D + (\delta D)^2 3^{\delta D})) = \mathcal{O}(n(\delta D)^2 3^{\delta D})$. □

Proof of Lemma 4. For each vertex v in a balanced k-plex (P_L, P_R), its positive degree and negative degree in $g(P_L, P_R)$ should be at least $\tau - k$ and $\tau - k + 1$, according to the definition of balanced k-plex. Thus, v's positive degree and negative degree should also satisfy these conditions in G. In a similar way, we can also prove the degree of each vertex v in a balanced k-plex (P_L, P_R) should be at least $2\tau - k$ in G. We complete the proof. □

Proof of Lemma 5. Suppose (u, v) is a positive edge in $g(P_L, P_R)$. Without loss of generality, we assume $u, v \in P_L$. In this case, u and v has at most $k-1$ non-neighbors from P_L in $g(P_L, P_R)$, respectively. Since $|P_L| \geq \tau$, we know that the number of vertices in P_L that positively link to both u and v is at least $\tau - 2k$. Thus, in G, which is a super graph of $g(P_L, P_R)$, the number of vertices that positively link to both u and v is also at least $\tau - 2k$. We have $\Delta^{++}_G(u, v) \geq \tau - 2k$. Also, u and v has at most $k-1$ non-neighbors from P_R in $g(P_L, P_R)$, respectively. Since $|P_R| \geq \tau$, we know that the number of vertices in P_R that negatively link to both u and v is at least $\tau - 2k + 2$. Thus, we have $\Delta^{--}_G(u, v) \geq \tau - 2k + 2$. Moreover, u and v has at most $k-1$ non-neighbors in $g(P_L, P_R)$, respectively, and the size of $g(P_L, P_R)$ is at least

2τ. Thus we have $\Delta_G^{++}(u,v) + \Delta_G^{--}(u,v) \geq 2\tau - 2k$. In a similar way, we can prove the conditions for the negative edge (u,v). We omit the details. □

Proof of Lemma 6. We prove the lemma by contradiction. Suppose there exists a maximal balanced k-plex P' that does not satisfy the given condition: P' contains P but

(1) P' does not contain u,
(2) $P' \setminus P$ does not contain any $v \in C$ such that $v \neq u$ and u and v are non-neighbors in $g(P \cup \{u,v\})$,
(3) $P' \setminus P$ does not contain any $v \in C$ such that u and v are neighbors in $g(P \cup \{u,v\})$ and they have common non-neighbors in $g(P \cup \{u,v\})$.

We will show that u can be added to P' to form a larger balanced k-plex. From (2), we know that $P' \setminus P$ contains only neighbors of u in $g(P' \cup \{u\})$. Given that u is a valid candidate vertex and $P \cup \{u\}$ is a balanced k-plex, the only thing left to prove is that u's non-neighbors in $g(P \cup \{u\})$ have at most $k-1$ non-neighbors in $g(P' \cup \{u\})$. This can be derived from (3), since it states that any vertex $v \in P' \setminus P$ does not have common non-neighbors with u in $g(P \cup \{u,v\})$. Thus, the presence of u in P' would not cause its non-neighbors in $g(P \cup \{u\})$ to have more than k non-neighbors. Hence, $P' \cup \{u\}$ is also a balanced k-plex, which contradicts the claim that P' is maximal. □

Proof of Lemma 7. Firstly, let us consider that the vertex $u \in N^+_{\succ_\eta}(v_i)$. Suppose $P' = (P'_L, P'_R)$ is a maximal v_i-lead balanced k-plex derived from this instance and $u \in P'$. There are two cases (1) $u \in P'_L$ and (2) $u \in P'_R$. For the first case, the size of P'_L is at most $d^+_L(u) + k + |S|$ since u has at most k non-neighbors or wrong-signed neighbors in P'_L. Also, at most $|S|$ vertices in P'_L may come from R and T, and most ideally all of them are positive neighbors of u. Combining with the size constraint τ, we have $d^+_L(u) \geq \tau - k - |S|$. Similarly, we can derive that $d^-_R(u) \geq \tau - k - |S| + 1$. For the second case that u is in the opposite side with v_i, i.e., $u \in P'_R$. u and v_i are wrong-signed neighbors with each other. In this case, the size of P'_L is at most $d^-_L(u) + k - 1 + |S| - 1$ since u has at most $k-1$ non-neighbors or wrong-signed neighbors in P'_L. Also, at most $|S| - 1$ vertices in P'_L may come from R and T, and most ideally all of them are negative neighbors of u. Thus, we can derive that $d^-_L(u) \geq \tau - k - |S| + 2$. Similarly, we can also derive that $d^+_R(u) \geq \tau - k - |S| + 2$. The degree constraints of the vertex $u \in N^-_{\succ_\eta}(v_i)$ and $u \in N^2_{\succ_\eta}(v_i)$ can be derived similarly. We omit the details. □

Proof of Theorem 3. Firstly, the time cost of computing degeneracy order of V is bounded by $\mathcal{O}(m)$ [39]. For each v_i in the degeneracy order, we reduce the search space to its 2-hop neighbors $N_{\succ_\eta}(v_i) \cup N^2_{\succ_\eta}(v_i)$. Then we enumerate all subsets $S \subseteq N_{\succ_\eta}(v_i) \cup N^2_{\succ_\eta}(v_i)$ with size at most $k-1$. S is then added to P_L and P_R to generate a partial solution. This will give in total $|N_{\succ_\eta}(v_i) \cup N^2_{\succ_\eta}(v_i)|^{k-1} = O((\delta D)^{k-1})$ instances. For each instance, we will partition it at most another $2^{|S|} = O(2^{k-1})$ times and invoke Enum. Here, the size of C_L and C_R is bounded by δ and the size of X_L and X_R is bounded by δD. As the analysis in Theorem 2, for each instance the time complexity of Enum is bounded by $\mathcal{O}((\delta D)^2 2^\delta)$. Thus, the total time complexity is $\mathcal{O}(m + \sum_{v_i \in V}((\delta D)^{k-1} 2^{k-1} (\delta D)^2 2^\delta)) = \mathcal{O}(n 2^{k-1} (\delta D)^{k+1} 2^\delta)$. □

Proof of Theorem 4. We only need to prove the time complexity of Enum-MDP. When $C_L \cup C_R = \emptyset$, it means the candidate sets have been exhausted and no branching is needed. Then we can analyze from the perspective of the number of vertices removed from $C_L \cup C_R$. When the minimum degree vertex u is from $P_L \cup P_R$, the algorithm will generate $k' + 1$ branches. In the first branch, one vertex (i.e., $\{v_1\}$) is removed from $C_L \cup C_R$. In the second branch, two vertices (i.e., $\{v_1, v_2\}$) are removed from $C_L \cup C_R$. In the i-th branch ($3 \leq i \leq k'$), i vertices (i.e., $\{v_1, \cdots, v_i\}$) are removed from $C_L \cup C_R$. Finally, in the last branch, q_2 vertices (i.e., $\{v_1, \cdots, v_{q_2}\}$) are removed from $C_L \cup C_R$. Note that we have $q_2 > k'$ since $q_1 + q_2 > k$. If we use $T(c)$ to represent the number of branches generated by Enum-MDP on the instance $C_L \cup C_R$ of size c, then we can obtain the following recurrence:

$$T(c) \leq T(c-1) + \cdots + T(c-k') + T(c-q_2).$$

When $u \in C_L \cup C_R$, the algorithm will generate two branches. The first branch removes u from $C_L \cup C_R$ to $X_L \cup X_R$. The second branch removes u from $C_L \cup C_R$ to $P_L \cup P_R$, which follows with the above recurrence. Combining them together, we have:

$$T(c) \leq T(c-1) + \cdots + T(c-k'-1) + T(c-q_2-1).$$

Given $k' \leq k-1$ and $q_2 \geq k'+1$, the worst case is when $k' = k-1$ and $q_2 = k'+1 = k$. After substituting into the above recurrence, we obtain:

$$T(c) \leq T(c-1) + \cdots + T(c-k) + T(c-k-1).$$

Based on the theoretical result in [40], the branching factor α_k of the above recurrence is the largest real root of function $x^{k+2} - 2x^{k+1} + 1 = 0$. Note that α_k is strictly smaller than 2 and only depends on k. For example, $\alpha_2 = 1.839$, $\alpha_3 = 1.928$, and $\alpha_4 = 1.966$. We also have $c \leq \delta$ since Enum-MDP starts with $C_L \cup C_R$ being $N^+_{\succ_\eta}(v_i) \cup N^-_{\succ_\eta}(v_i)$. In addition, the running time of Enum-MDP without considering the recursive invocations of itself is dominated by the refinement of C_L, C_R, X_L and X_R and the minimum-degree vertex selection, whose time complexity can be bounded by $\mathcal{O}((\delta D)^2)$. Thus, the running time of Enum-MDP is $\mathcal{O}((\delta D)^2 \alpha_k^\delta)$. As the analysis in Theorem 3, at most $(2\delta D)^{k-1}$ instances will be generated for each vertex v_i. Thus, the time complexity of MBPE* is $\mathcal{O}(m + \sum_{v_i \in V}((2\delta D)^{k-1} (\delta D)^2 \alpha_k^\delta)) = \mathcal{O}(n 2^{k-1} (\delta D)^{k+1} \alpha_k^\delta)$. □