25/10/23 # Assignment 4: Transformations and Barycentric Coordinates

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Exercise 1

$\mathbb{R} \in 2D$

$u = p - p_1$

$p_1 = (1, 1)^T$

$p = (1.5, 2.5)^T$

Task 1

$R_{90} \mid R \in \mathbb{R}^{3\times3}$

Counter-clockwise rotation around the center by $90°$:

$$R_{90} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(90) & -\sin(90) & 0 \\ \sin(90) & \cos(90) & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Translation by $t = (1, -2)^T$:

$$T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Task 2

$p_1 = (1, 1)^T$

$p = (1.5, 2.5)^T$

$u = p - p_1$

$\Rightarrow$ Represented in homogeneous coordinates:

$p_1 = (1, 1, 1)^T$

$p = (1{,}5, 2{,}5, 1)^T$

$u = p - p_1 = (0{,}5, 1{,}5, 0)^T$

Rotate by R_{90}:

$$R_{90} \cdot p_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow p_1' = (-1, 1)^T$$

$$R_{90} \cdot p = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1{,}5 \\ 2{,}5 \\ 1 \end{bmatrix} = \begin{bmatrix} -2{,}5 \\ 1{,}5 \\ 1 \end{bmatrix} \Rightarrow p' = (-2{,}5, 1{,}5)^T$$

$$R_{90} \cdot u = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0{,}5 \\ 1{,}5 \\ 0 \end{bmatrix} = \begin{bmatrix} -1{,}5 \\ 0{,}5 \\ 0 \end{bmatrix} > u' = (-1{,}5, 0{,}5)^T$$

Translate:

$$T \cdot p_1' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \Rightarrow p_1'' = (0, -1)^T$$

$$T \cdot p' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2{,}5 \\ 1{,}5 \\ 1 \end{bmatrix} = \begin{bmatrix} -1{,}5 \\ -0{,}5 \\ 1 \end{bmatrix} \Rightarrow p'' = (-1{,}5, -0{,}5)^T$$

$$T \cdot u' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1{,}5 \\ 0{,}5 \\ 0 \end{bmatrix} = \begin{bmatrix} -1{,}5 \\ 0{,}5 \\ 0 \end{bmatrix} \Rightarrow u'' = (-1{,}5, 0{,}5)^T$$

Verify

$u' = p' - p_1'$

$$= \begin{bmatrix} -1{,}5 \\ -0{,}5 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

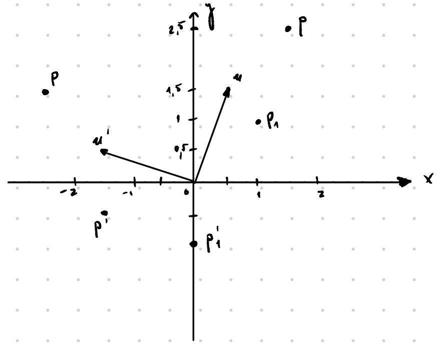
$$= \begin{bmatrix} -1{,}5 \\ 0{,}5 \end{bmatrix}$$

$= u'$ $\boxed{V}$

$\Rightarrow u' = u$

$\therefore$ u is not affected by T since it is a vector.

(cont.)



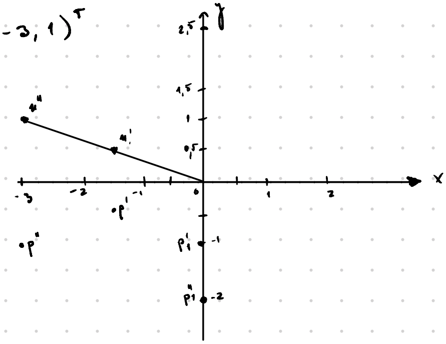
Task 3 Scaling matrix $S \in \mathbb{R}^{3,3}$, factor of 2.

$$\Rightarrow S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S \cdot p_1' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \Rightarrow p_1'' = (0, -2)^T$$

$$S \cdot p' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1,5 \\ -0,5 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} \Rightarrow p'' = (-3, -1)^T$$

$$S \cdot u' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1,5 \\ 0,5 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \Rightarrow u'' = (-3, 1)^T$$



Task 4 $R_{90}^{-1}, T^{-1}, S^{-1} \in \mathbb{R}^{3,3}$

$$\Rightarrow R_{90}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \Rightarrow T^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad \Rightarrow S^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M = R_{90}^{-1} T^{-1} S^{-1} \Rightarrow \dots \Rightarrow \begin{bmatrix} 0 & 1/2 & 2 \\ -1/2 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$p_1 = M p_1'' = \begin{bmatrix} 0 & 1/2 & 2 \\ -1/2 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = p_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \blacksquare$$

$$p = M p'' = \begin{bmatrix} 0 & 1/2 & 2 \\ -1/2 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1,5 \\ 2,5 \\ 1 \end{bmatrix} = p = \begin{bmatrix} 1,5 \\ 2,5 \end{bmatrix} \blacksquare$$

$$u = M u'' = \begin{bmatrix} 0 & 1/2 & 2 \\ -1/2 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0,5 \\ 1,5 \\ 0 \end{bmatrix} = u = \begin{bmatrix} 0,5 \\ 1,5 \end{bmatrix} \blacksquare$$

Exercise 2

$p_1 = (6, 0, 4)$ $\qquad$ $p = (4, 1, 3)$

$p_2 = (2, 0, 0)$

$p_3 = (2, 4, 4)$

area of triangles p_1, p_2, p_3 $= (p_2 - p_1) \times (p_3 - p_1) = \left(\begin{bmatrix}2\\0\\0\end{bmatrix} - \begin{bmatrix}6\\0\\4\end{bmatrix}\right) \times \left(\begin{bmatrix}2\\4\\4\end{bmatrix} - \begin{bmatrix}6\\0\\4\end{bmatrix}\right) = \begin{bmatrix}-4\\0\\-4\end{bmatrix} \times \begin{bmatrix}-4\\4\\0\end{bmatrix} = \begin{bmatrix}16\\16\\-16\end{bmatrix}$

Permuting normals:

resulting normals of $\underline{p, p_1, p_2}$ $= \begin{pmatrix}6\\0\\4\end{pmatrix} - \begin{pmatrix}4\\1\\3\end{pmatrix} \times \begin{pmatrix}2\\0\\0\end{pmatrix} - \begin{pmatrix}4\\1\\3\end{pmatrix} = \begin{pmatrix}2\\-1\\1\end{pmatrix} \times \begin{pmatrix}-2\\-1\\-3\end{pmatrix} = \begin{pmatrix}4\\4\\-4\end{pmatrix}$

resulting normals of $\underline{p, p_1, p_3}$ $= \begin{pmatrix}2\\4\\4\end{pmatrix} - \begin{pmatrix}4\\1\\3\end{pmatrix} \cdot \begin{pmatrix}6\\0\\4\end{pmatrix} - \begin{pmatrix}4\\1\\3\end{pmatrix} = \begin{pmatrix}-2\\3\\1\end{pmatrix} \times \begin{pmatrix}2\\-1\\1\end{pmatrix} = \begin{pmatrix}4\\4\\-4\end{pmatrix}$

resulting normals of $\underline{p, p_2, p_3}$ $= \begin{pmatrix}2\\0\\0\end{pmatrix} - \begin{pmatrix}4\\1\\3\end{pmatrix} \times \begin{pmatrix}2\\4\\4\end{pmatrix} - \begin{pmatrix}4\\1\\3\end{pmatrix} = \begin{pmatrix}-2\\-1\\-3\end{pmatrix} \times \begin{pmatrix}-2\\3\\1\end{pmatrix} = \begin{pmatrix}8\\8\\-8\end{pmatrix}$

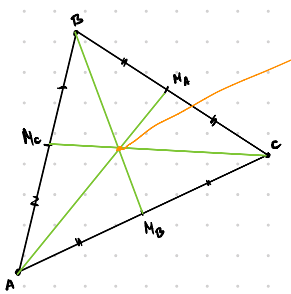
By checking the dot product between the normals of the p with the full area, we can tell from the +/- of the result, whether p lies within the original area:

1. $\left\langle \begin{pmatrix}4\\4\\-4\end{pmatrix}, \begin{pmatrix}16\\16\\-16\end{pmatrix} \right\rangle = 64 + 64 + 64 = 192 \rightsquigarrow +$
2. $\left\langle \begin{pmatrix}4\\4\\-4\end{pmatrix}, \begin{pmatrix}16\\16\\-16\end{pmatrix} \right\rangle = 64 + 64 + 64 = 192 \rightsquigarrow +$
3. $\left\langle \begin{pmatrix}8\\8\\-8\end{pmatrix}, \begin{pmatrix}16\\16\\-16\end{pmatrix} \right\rangle = 128 + 128 + 128 = 384 \rightsquigarrow +$

therefore, the point lies inside the triangle ■

Exercise 3

Using barycentric coordinates, prove the centroid of a triangle divides its medians in 2:1 ratio.



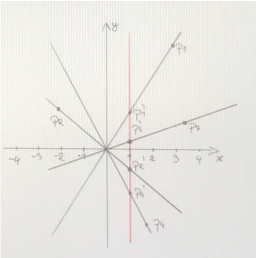
Since every point point on the triangle can be expressed in barycentric coordinates as α, β, γ respectively, then, $\alpha + \beta + \gamma = 1$.
Therefore, the medians M_A, M_B, M_C can be expressed similarly:

$M_A = (\alpha, \beta, \gamma)$, where $\beta = \gamma$
$M_B =$ " , where $\alpha = \gamma$
$M_C =$ " , where $\alpha = \beta$

$\Rightarrow \alpha + \beta + \gamma = 1, \; \alpha = \beta = \gamma = \frac{1}{3}$
$\Rightarrow$ centroid $\Rightarrow \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

Barycentric coordinates of points inside triangle are expressed $[0, 1]$, therefore the centroid representation divides the medians accordingly $\frac{2}{3} : \frac{1}{3}$

Exercise 4



- → Transformation that projects $p_i \in \mathbb{R}^2$ onto $x = 1$. ⟹ p_i'
- → Create $M \in \mathbb{R}^{3\times3}$ that does that (homogeneous).
- → Then, convert back to Cartesian, and result in p_i'
- → How does M affects the points on y-axis?

Abstract homogeneous coordinate $= \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $\qquad P \to P''$

Conversion to cartesian ⟹ $C = \begin{bmatrix} \frac{a}{c} \\ \frac{b}{c} \end{bmatrix}$ $\Rightarrow$ $P'' = \begin{bmatrix} 1 \\ \frac{y}{x} \end{bmatrix}$

Rewritten as M (homogeneous):

$$p'' = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \cdot \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{p_y}{p_x} \\ 1 \end{bmatrix} \xRightarrow[\cdot\ \text{set } p_z \text{ as } p_x]{\cdot\ \text{for } x \text{ to equal } 1 \text{ in cartesian}} \begin{bmatrix} \frac{p_x}{p_x} \\ \frac{\frac{1}{p_x}}{\frac{p_x}{p_x}} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{p_x} \\ 1 \end{bmatrix}$$

<u>But</u> lost p_y and matrix M unpreserved, so we rewrite M:

$$p'' = \underbrace{\begin{bmatrix} 1 & b & c \\ d & 1 & f \\ 1 & h & i \end{bmatrix}}_{M} \cdot \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{p_y}{p_x} \\ 1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

∴ If $p_x = 0$, then p_y'' would not exist since $p_y'' = \frac{p_y}{p_x}$ (dividing by zero).

Bonus:

$y = ax + b \mid a, b \in \mathbb{R}$. M?

from ex. 4: $y = \frac{p_y}{p_x} x$

$$\Rightarrow \begin{cases} y = ax + b \\ y = \frac{p_y}{p_x} x \end{cases} \Rightarrow \begin{cases} x = \frac{bp_x}{p_y - ap_x} \\ y = \frac{bp_y}{p_y - ap_x} \end{cases}$$

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \qquad \frac{\sum_i R_{1i}}{\sum_i R_{3i}} = \frac{bp_x}{p_y - ap_x} \qquad \frac{\sum_i R_{2i}}{\sum_i R_{3i}} = \frac{bp_y}{p_y - ap_x}$$

Thus, the matrix M must be of the form:

$$\begin{bmatrix} b & 0 & 0 \\ 0 & b & 0 \\ -a & 1 & 0 \end{bmatrix}$$

in order to preserve the transformation and its constraints.