1、$\frac{1}{2}\sin 1$.

由于 $y(0)=0$，由牛顿-莱布尼茨公式：

$$y(x)=y(x)-y(0)=\int_0^x y'(t)dt=\int_0^x \cos(1-t)^2dt.$$

因此，

$$\int_0^1 y(x)dx=\cancel{\int_0^1 dx\int_0^x \cos(1-t)^2dt=\int_0^1 dt\int_t^1 \cos(1-t)^2 dx.}$$

$$=x\,y(x)\Big|_0^1-\int_0^1 x\,y'(x)dx=y(1)-\int_0^1 x\cos(1-x)^2dx.$$

$$=\int_0^1\cos(1-x)^2dx-\int_0^1 x\cos(1-x)^2dx.=\int_0^1(1-x)\cos(1-x)^2dx.$$

$$=-\frac{1}{2}\int_0^1\cos(1-x)^2\,d[(1-x)^2]=-\frac{1}{2}\sin(1-x)^2\Big|_0^1$$

$$=\frac{1}{2}\sin 1.$$

2、对 $x(x+1)f'(x)-(x+1)f(x)+\int_1^x f(t)dt=x-1$ 两端求导并整理得：

$$(x^2+x)f''(x)+xf'(x)=1,\ 即\ f''(x)+\frac{1}{x+1}f'(x)=\frac{1}{x(x+1)}.$$

$\Rightarrow$ 一个关于 $f'(x)$ 的一阶非齐次线性微分方程.

$$\Rightarrow\quad f'(x)=e^{-\int\frac{1}{x+1}dx}\left[\int e^{\int\frac{1}{x+1}dx}\frac{1}{x(x+1)}dx+C\right]=\frac{1}{x+1}(\ln x+C).$$

在原方程中令 $x=1$，得 $f'(1)=0$. 从而 $C=0$. $f'(x)=\frac{\ln x}{x+1}$. 则 $f'(2)=\frac{\ln 2}{3}$.

- - - - 令 $x=2$，得 $6f'(2)-3f(2)+\int_1^2 f(x)dx=1$. 代入 $f'(2)=\frac{\ln 2}{3}$.

$$\Rightarrow\quad \int_1^2 f(x)dx-3f(2)=1-2\ln 2.$$

另一方面，

$$\lim_{x\to 1}\frac{\int_1^x\frac{\sin(t-1)^2}{t-1}dt}{f(x)}\ \underline{\underline{洛必达}}\ \lim_{x\to 1}\frac{\frac{\sin(x-1)^2}{x-1}}{f'(x)}=\lim_{x\to 1}\frac{\sin(x-1)^2}{x-1}\cdot\frac{x+1}{\ln x}.$$

$$=\lim_{x\to 1}\frac{2\sin(x-1)^2}{(x-1)\ln(1+x-1)}=2\lim_{x\to 1}\frac{(x-1)^2}{(x-1)^2}=2.$$

因此，原式 $=1-2\ln 2+2=3-2\ln 2$.



3、令 $t-s=u$，则 $y=\int_t^0\sin u^2(-du)=\int_0^t\sin u^2du$.

$$\Rightarrow\ \frac{dy}{dx}=\frac{\frac{d}{dt}\left(\int_0^t\sin u^2du\right)}{\frac{d}{dt}\left(2\int_0^t e^{-s^2}ds\right)}=\frac{\sin t^2}{2e^{-t^2}}=\frac{1}{2}e^{t^2}\sin t^2.$$

$$\frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}=\frac{d}{dt}\left(\frac{1}{2}e^{t^2}\sin t^2\right)\frac{dt}{dx}=(te^{t^2}\sin t^2+te^{t^2}\cos t^2)\frac{1}{\frac{d}{dt}\left(2\int_0^t e^{-s^2}ds\right)}$$

$$=\frac{t}{2}e^{2t^2}(\sin t^2+\cos t^2).$$

$\therefore$ 原式 $=-\frac{\sqrt{\pi}e^{2\pi}}{2}$.

4、令 $x=-t$，则.

$$I=\int_{-1}^1\frac{dx}{(1+e^x)(1+x^2)}=\int_{-1}^1\frac{dt}{(1+e^{-t})(1+t^2)}=\int_{-1}^1\frac{e^tdt}{(1+e^t)(1+t^2)}.$$

$$\therefore I=\frac{1}{2}\left[\int_{-1}^1\frac{dx}{(1+e^x)(1+x^2)}+\int_{-1}^1\frac{e^xdx}{(1+e^x)(1+x^2)}\right].$$

$$=\frac{1}{2}\int_{-1}^1\frac{dx}{1+x^2}=\int_0^1\frac{dx}{1+x^2}=\frac{\pi}{4}.$$

5. 由已知，$f'(x)=\frac{8}{\int_0^2 f(x)dx}$. $\Rightarrow$ $f(x)=\frac{8}{\int_0^2 f(x)dx}\cdot x+C$.

由 $f(0)=0$ 知 $C=0$，等式两端在 $[0,2]$ 上积分.

$$\Rightarrow\ \int_0^2 f(x)dx=\frac{8}{\int_0^2 f(x)dx}\cdot\int_0^2 x\,dx.$$

$$\Rightarrow\ \int_0^2 f(x)dx=4.\ \Rightarrow\ f(x)=2x\ (x\geq 0).$$

6、$xy' = y(\ln y - \ln x) \Rightarrow y' = \frac{y}{x}\ln\frac{y}{x}$.

令$\frac{y}{x}=u$，则$\frac{dy}{dx} = x\frac{du}{dx}+u$，因此 $x\frac{du}{dx}+u=u\ln u$.

$\Rightarrow \frac{du}{u\ln u-u} = \frac{dx}{x} \Rightarrow \int\frac{du}{u\ln u-u} = \int\frac{dx}{x} \Rightarrow \ln|\ln u-1| = \ln|Cx|$.

故$\ln u-1 = Cx$，$\ln\frac{y}{x}-1 = Cx \Rightarrow y = xe^{Cx+1}$.

7、将x视为y的函数，原式化为$\frac{dx}{dy} = \frac{x+y^2}{y} = \frac{x}{y}+y$.

令$\frac{x}{y}=u$. 则$u+y\frac{du}{dy} = u+y \Rightarrow \frac{du}{dy}=1. \Rightarrow u=y+C$.

故$\frac{x}{y}=y+C$. $\therefore x=y^2+Cy$.

8. 由奇偶函数的性质. ($\frac{\tan x}{1+x^4}$, $\ln(x+\sqrt{x^2+1})$, $(e^x-e^{-x})\cos x$ 均为奇函数).

$M=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}x^8dx$. $N=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sin^8 x\,dx$. $P=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\tan^4 x\,dx$.

易知 $P>M>N$.

9、原式$= \int_0^{\sqrt{3}}\frac{(x^4-1+1)\arctan x}{x^2+1}dx = \int_0^{\sqrt{3}}(x^2-1)\arctan x\,dx + \int_0^{\sqrt{3}}\frac{\arctan x}{x^2+1}dx$.

$= (\frac{1}{3}x^3-x)\arctan x\Big|_0^{\sqrt{3}} - \frac{1}{3}\int_0^{\sqrt{3}}\frac{(x^2-3)x}{1+x^2}dx. + \frac{1}{2}\arctan^2 x\Big|_0^{\sqrt{3}}$.

$= -\frac{1}{3}\int_0^{\sqrt{3}}(x-\frac{4x}{1+x^2})dx + \frac{\pi^2}{18} = -\frac{1}{3}\int_0^{\sqrt{3}}x\,dx + \frac{2}{3}\int_0^{\sqrt{3}}\frac{d(1+x^2)}{1+x^2} + \frac{\pi^2}{18}$.

$= -\frac{1}{2}+\frac{4}{3}\ln 2+\frac{\pi^2}{18}$.



10、设$F(x)=(1-x)\int_0^x f(t)dt$，$x\in[0,1]$.

因$f(x)$在$[0,1]$连续，则$F(x)$在$[0,1]$上连续，$(0,1)$内可导，且$F(0)=F(1)=0$.

满足中值定理条件（罗尔）.

$\therefore \exists\, \xi\in(0,1)$，使$F'(\xi)=0$. 且$F'(\xi) = -\int_0^{\xi}f(t)dt + (1-\xi)f(\xi)=0$.

故 $\int_0^{\xi}f(t)dt = (1-\xi)f(\xi)$.

下用反证法证明：$f(x)>0$，且单减时，$\xi$唯一.

若同时$\exists\, \xi_1,\xi_2\in(0,1)$，$\xi_1<\xi_2$. 满足.

$\int_0^{\xi_1}f(t)dt = (1-\xi_1)f(\xi_1)$，$\int_0^{\xi_2}f(t)dt = (1-\xi_2)f(\xi_2)$.

相减得，$\int_{\xi_1}^{\xi_2}f(t)dt = f(\xi_2)-f(\xi_1)-\xi_2 f(\xi_2)+\xi_1 f(\xi_1)$.

$= (1-\xi_2)[f(\xi_2)-f(\xi_1)] - (\xi_2-\xi_1)f(\xi_1)$.

由所给条件. $\int_{\xi_1}^{\xi_2}f(t)dt>0$.

$(1-\xi_1)[f(\xi_2)-f(\xi_1)] - (\xi_2-\xi_1)f(\xi_1)<0$. 矛盾.

$\therefore \xi$唯一.