27^{th} TJIMO

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Round: **Practice Guts**

Set 1

Problem 1. Alfredo rolls a fair, six-sided die. What is the probability that he rolls an odd number?

Answer. $\boxed{\frac{1}{2}}$.

Solution. There are six equally likely outcomes: Alfredo can roll a 1, 2, 3, 4, 5, or 6. The odd numbers Alfredo could roll are 1, 3, and 5. So, the probability that he rolls an odd number is $\frac{3}{6} = \boxed{\frac{1}{2}}$.

Problem 2. The expression $(2x-3) \times (5x+2)$ can be written as $ax^2 + bx + c$. Find $a + b + c$.

Answer. $\boxed{-7}$.

Solution. We multiply the given expression using the distributive property (or FOIL):

$$\begin{aligned}(2x-3)(5x+2) &= 5x(2x-3) + 2(2x-3) \\ &= 10x^2 - 15x + 4x - 6 \\ &= 10x^2 - 11x - 6.\end{aligned}$$

So, $a = 10$, $b = -11$, and $c = -6$, and $a + b + c = \boxed{-7}$.

Problem 3. Simplify the fraction $\frac{\frac{12}{33}}{\frac{24}{35}}$.

Answer. $\boxed{\frac{35}{66}}$.

Solution. Invert the denominator, prime factor, then cancel:

$$\begin{aligned}\frac{\frac{12}{33}}{\frac{24}{35}} &= \frac{12}{33} \times \frac{35}{24} \\ &= \frac{2^2 \cdot 3}{3 \cdot 11} \frac{5 \cdot 7}{2^3 \cdot 3} \\ &= \frac{5 \cdot 7}{2 \cdot 3 \cdot 11} \\ &= \boxed{\frac{35}{66}}.\end{aligned}$$

Problem 4. How many factors does $2^4 \cdot 3^2 \cdot 4^3$ have?

Answer. $\boxed{33}$.

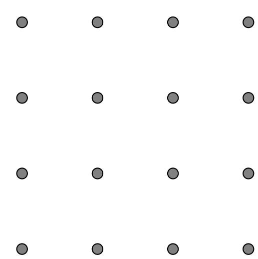
Solution. We have to be careful on this problem because we are *not* given the prime factorization of the number. So we rewrite the factorization as

$$2^4 \cdot 3^2 \cdot 4^3 = 2^4 \cdot 3^2 \cdot 2^6 = 2^{10} \cdot 3^2.$$

And now we can proceed normally by finding the number of factors as $(10+1)(2+1) = 11 \cdot 3 = \boxed{33}$.

Set 2

Problem 5. Triangle ABC has $AB = 25$ and $BC = 7$. If $\angle C = 90°$, find the length of AC.
Answer. $\boxed{24}$.
Solution. Using the Pythagorean Theorem on right triangle ABC, $(AC)^2 + (BC)^2 = (AB)^2$. Solving for AC, $AC = \sqrt{(AB)^2 - (BC)^2} = \sqrt{625 - 49} = \sqrt{576} = \boxed{24}$.
Problem 6. Let $f(x) = 7x - \sqrt{x} + 3$. Compute $f(4)$.
Answer. $\boxed{29}$.
Solution. Substituting $x = 4$, $f(4) = 7 \cdot 4 - \sqrt{4} + 3 = 28 - 2 + 3 = \boxed{29}$.
Problem 7. How many squares have all 4 vertices in the array of 16 points below?



Answer. $\boxed{20}$.
Solution. We break down cases based on the slopes of the sides:

1. *Sides are horizontal/vertical.* There are 9 squares of side length 1, 4 of side length 2, and 1 of side length 3, for $9 + 4 + 1 = 14$ total.

2. *Sides have slope* ± 1. There are 4 such squares.

3. *Sides have slope* $\pm 2, \pm \frac{1}{2}$. There 2 such squares.

Note that no other cases fit in the array of points. Adding these up, there are $14 + 4 + 2 = \boxed{20}$ rectangles.
Problem 8. Express $0.\overline{47} = 0.47474747\ldots$ as a simplified fraction.
Answer. $\boxed{\dfrac{47}{99}}$.
Solution. $0.47474747\ldots = 47(0.01 + 0.0001 + 0.000001 + \ldots)$. Using the expression for an infinite sum, this equals $\dfrac{0.47}{1 - 0.01} = \boxed{\dfrac{47}{99}}$.

Set 3

Problem 9. Compute 25×316484.
Answer. $\boxed{7912100}$.
Solution. We use

$$25 = \frac{100}{4}$$

so that

$$\begin{aligned} 25 \times 316484 &= \frac{100}{4} \times 316484 \\ &= 100 \times \frac{316484}{4} \\ &= 100 \times 79121 \\ &= \boxed{7912100}. \end{aligned}$$

Problem 10. What is the largest prime factor of $25^2 - 14^2$?
Answer. $\boxed{13}$.
Solution. We use the property

$$a^2 - b^2 = (a+b)(a-b).$$

So,

$$25^2 - 14^2 = (25+14)(25-14) = 39 \cdot 11 = 3 \cdot 11 \cdot 13,$$

and we see our largest prime factor is $\boxed{13}$.
Problem 11. Three consecutive odd integers add up to 27. If I subtract 1 from each of these numbers and multiply them all by 6, what is their new sum?
Answer. $\boxed{144}$.
Solution. We can find the three odd integers and perform the operations described by the problem, but there is a faster way.

Instead, why can call the three consecutive odd integers a, b, c, so

$$a + b + c = 27.$$

We then subtract one from each

$$(a-1) + (b-1) + (c-1)$$

, and then multiply them all by 6,

$$6(a-1) + 6(b-1) + 6(c-1)$$

, we can simplify them

$$\begin{aligned} 6(a-1) + 6(b-1) + 6(c-1) &= 6a - 6 + 6b - 6 + 6c - 6 \\ &= 6(a+b+c) - 18 \end{aligned}$$

and since we know $a + b + c = 27$ from the problem statement, we can substitute that in:

$$6(27) - 18 = \boxed{144}.$$

Problem 12. Rectangle A has side lengths 5 and 4, and Rectangle B has side lengths 7 and 2. What percentage of Rectangle A's area is Rectangle B's area?

Answer. $\boxed{70\%}$.

Solution. In order to calculate how much of Rectangle A's area Rectangle B makes up, we do

$$\begin{aligned}\text{Percent B of A} &= \frac{\text{Area of Rectangle B}}{\text{Area of Rectangle A}} \times 100 \\ &= \frac{7 \cdot 2}{5 \cdot 4} \times 100 \\ &= \frac{14}{20} \times 100 \\ &= 14 \times 5 \\ &= \boxed{70\%}.\end{aligned}$$