27^{th} TJIMO

Alexandria, Virginia

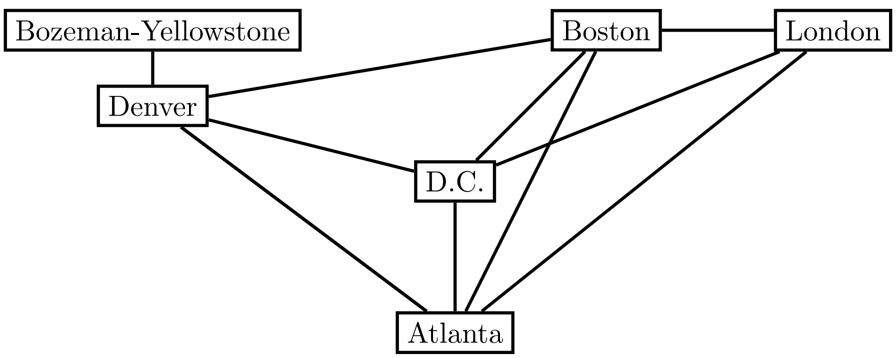
Round: **Power**

1 What is a Graph?

At its core, a graph consists of two sets: a group of objects, and a group of connectors that each join two objects.

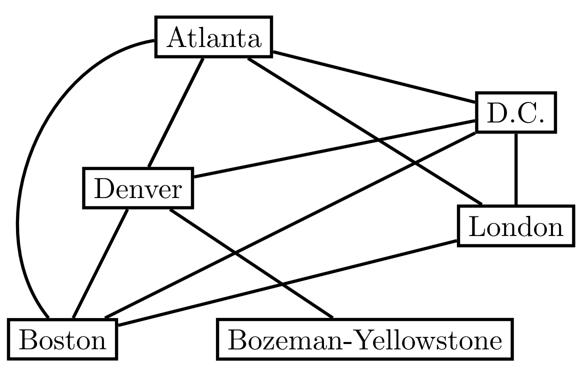
To gain insight into what a graph is, let's first consider flights that connect airports around the world. We depend on airplanes to travel quickly and efficiently. When we want to travel between major cities, like between D.C. and Boston, we can often take a single flight to make the trip. Enough people want to travel between these two cities for it to be economical to offer a single flight between them.

However, it is not always the case that single-flight trips are economically feasible, either due to lack of demand or due to needing to refuel. To travel from London to Yellowstone, for example, one must first stop in Atlanta, hop on a connecting flight to Denver, and then finally travel to Yellowstone. Denver and Atlanta are what we call hubs, since they are very large airports that accomodate heavy travel loads. To travel between smaller airports, we often need to stop at one of these larger hubs.



Our connections between our cities in the graph represent only direct connections between cities, but from our diagram, we can still see it is possible to travel between any two cities in our graph.

Neither the exact position of each city nor how connections might intersect matters. In other words, the way we draw a graph on paper does not change its properties, and for the most part are not useful to us. The following is an equivalent graph:

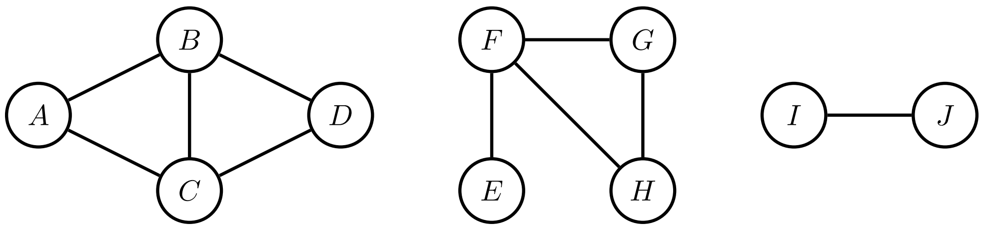


The only characteristics of a graph we care about are the objects (in this case, cities) themselves, which we'll call *vertices*, and the connections joining the objects, which we'll call *edges*.

Time limit: 45 minutes.

2 Definition of a Graph

We now have the background necessary to present the mathematical definition of a graph:
Definition 1. A *graph* $G(V, E)$ consists of a set of vertices V and a set of edges E.



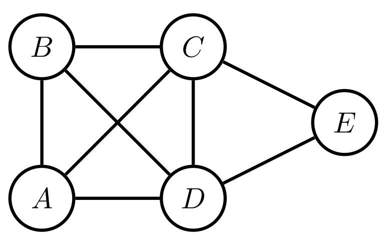
Definition 2. An *edge* is a collection of exactly two vertices. In the above graph, there is an edge between A and B, so $\{A, B\}$ is an edge. Note that $\{A, B\}$ is the same edge as $\{B, A\}$.
Definition 3. A *set* is a collection of any number of objects. In the above graph, we have the set of vertices

$$V = \{A, B, C, D, E, F, G, H, I, J\}.$$

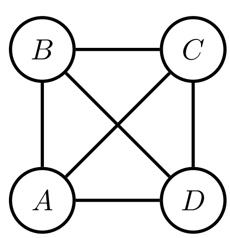
We also have the set of edges

$$E = \{\{A, B\}, \{B, C\}, \{C, D\}, \{B, D\}, \{A, C\}, \{E, F\}, \{F, G\}, \{G, H\}, \{H, F\}, \{I, J\}\}.$$

Definition 4. Given a set A, the cardinality $|A|$ denotes the number of elements in A.
Problem 1. Evaluate $|V|$ and $|E|$ for the sets V and E in Definition 3, the definition of the set. Note that $\{A, B\}$ counts as one single edge.
Solution. Answer: $|V| = |E| = \boxed{10}$.
There are 10 vertices and 10 edges in the graph.
Problem 2. Draw any graph with 5 vertices and 8 edges.
Solution. The following graph has 5 vertices and 8 edges.



Definition 5. A *clique* is a graph such that there is exaclty one edge covering every pair of distince vertices in the graph. We denote a clique of n vertices as K_n. Pictured is a clique of 4 vertices, or K_4.



Time limit: 45 minutes.

Problem 3. Solve each of the following:

(a) How many edges are in the clique of 4 vertices, K_4?

(b) How many edges are in the clique of 5 vertices, K_5?

(c) In terms of n, how many edges are in the clique of n vertices, K_n, where n is any positive integer?

Solution. The number of edges in a clique are as follows:

(a) K_4 has $\boxed{6}$ edges: $\{A, B\}$, $\{A, C\}$, $\{B, C\}$, $\{A, D\}$, $\{B, D\}$, $\{C, D\}$.

(b) K_5 has $\boxed{10}$ edges: $\{A, B\}$, $\{A, C\}$, $\{B, C\}$, $\{A, D\}$, $\{B, D\}$, $\{C, D\}$, $\{A, E\}$, $\{B, E\}$, $\{C, E\}$, $\{D, E\}$.

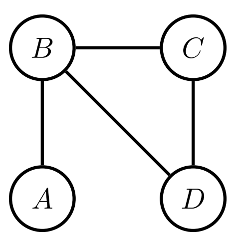
(c) In general, K_n has

$$\binom{n}{2} = \boxed{\frac{n \cdot (n-1)}{2}}$$

edges.
We need to count the total number of possible ways to choose two different vertices from n total vertices. We have n ways to choose the first vertex and $n - 1$ ways to choose the second vertex, but the order in which we choose the vertices doesn't matter (that is, $\{A, B\}$ represents the same edge as $\{B, A\}$), so we must divide by 2 to get the correct answer of $\frac{n \cdot (n-1)}{2}$.

3 Paths

Definition 6. A *path* is a sequence of vertices, such that every two consecutive vertices in the graph are connected by an edge. All edges in the path must be distinct, and all the vertices must be distinct.
Problem 4. In the following graph, answer the questions below.



(a) Is B, C a path?

(b) Is B, C, D a path?

(c) Is A, C, B a path?

(d) Is B, C, B a path?

(e) Is B, C, D, B a path?

(f) Is B a path?

If your answer for any of these was "no," explain why.
Solution. From Definition 6 of a path,

(a) B, C is a path.

(b) B, C, D is a path.

(c) A, C, B is not a path, since A and C are not connected.

(d) B, C, B is not a path, since B is repeated.

(e) B, C, D, B is not a path, since B is repeated.

(f) B is a path.

Problem 5. In the clique of n vertices K_n, where $n \geq 2$, how many paths are there from A to B, were A and B are different vertices in the clique?