Applying the correction term for the field strength along a direction to the original vector potential expression

$$\vec{A} = \frac{\mu_0 q \vec{v}}{4\pi r \left(1 + \frac{\dot{r}}{c}\right)}$$

, and this is the retarded vector potential originating from a particle, the Liénard-Wiechert vector potential. Since this relation should apply to the scalar potential as well, the expression for the retarded scalar potential is

$$\phi = \frac{q}{4\pi \varepsilon_0 r \left(1 + \frac{\dot{r}}{c}\right)}$$

. With this, the derivation of the retarded potential due to the moving source was done which Liénard and Wiechert first derived. Based on this, now is the turn to deal with the retarded field due to the moving source, which was originally intended to be dealt with.

3.4 Feynman's Formula

Let's go back to the starting point, Feynman's formula.

$$\vec{E} = \frac{q}{4\pi\varepsilon_0} \left[\frac{e_r'}{r'^2} + \frac{r'}{c} \frac{d}{dt} \left(\frac{e_r'}{r'^2} \right) + \frac{1}{c^2} \frac{d^2}{dt^2} e_r' \right]$$

This expression looks simple at first glance, but when we look into it, we realize that it is not as easy to recognize as we think. The term that differentiates once with respect to time is not a single term, but a term that is a combination of two terms by division, and then the term that differentiates twice with respect to time appears. If we try to interpret it, many hidden terms pop up. At first glance, this formula appears to be an expression in the form of a differential equation to express a certain principle, rather than a form that can be used immediately for practical purposes.

In the case of the formulas before Feynman, they are in the form of integral equations, but I consider that they are not formulas as answers, but rather as notations as the research plan of the people who wrote them, and have little value. In the case of differential equations, even if they are not solved, they often have useful physical meaning, but they are still mostly uncertain without a solution. Then again, the specific solution of a differential equation is often finding the integral of a certain function. After all, in the case of a differential or integral equation in physics, the meaning becomes clear when not only the physical equation but also the solution is clearly presented.

However, the peculiarity of this formula is that it does not appear to be trying to describe any particular physical principle. There is no description of any special principle, other than that it provides some support for what I described in the picture of the directionality of electromagnetic radiation. It means that Feynman saw it just as a consequence of ordinary electromagnetic laws. The only mention Feynman makes of the derivation is ”What is the formula for the electric and magnetic field produced by one individual charge? It turns out that this is very complicated, and it takes a great deal of study and sophistication to appreciate it.” When I first read this passage, I mistook Feynman's praise for someone else who derivated the formula, and, even looked for the original derivation. Since Feynman himself was the original derivation of the formula, it's a pretty self-congratulatory statement. But, as a result, I think it's a great achievement of Feynman's that deserves to be recognized.

This formula later reappears, in a different form, in Griffiths' book on electromagnetism, which is one of the main textbooks used in modern universities and will be dealt with again later. It does not appear in Purcell's book, published in the 1950s, which was the standard textbook on electromagnetism before Feynman. It is rumored that Feynman first succeeded in deriving the formula around 1950.

However, even though I don't know the exact meaning of the formula, at first glance it seems to be a clue to the Laplace problem I've been struggling with, so I decided to dig this formula as something meaningful.

First of all, I will get rid of the unfamiliar expression of the derivative of a unit vector. The definition of a unit vector is $e_r' = \frac{\vec{r}'}{r'}$, which can be used interchangeably. If apply this and decompose and write it, it becomes

$$\begin{aligned}
\vec{E} &= \frac{q}{4\pi\varepsilon_0}\left[\frac{e_r'}{r'^2} + \frac{r'}{c}\frac{d}{dt}\left(\frac{e_r'}{r'^2}\right) + \frac{1}{c^2}\frac{d^2}{dt^2}e_r'\right] \\
&= \frac{q}{4\pi\varepsilon_0}\left(\frac{\vec{r}'}{r'^3} + \frac{r'}{c}\frac{d}{dt}\left(\frac{\vec{r}'}{r'^3}\right) + \frac{1}{c^2}\frac{d^2}{dt^2}\left(\frac{\vec{r}'}{r'}\right)\right) \\
&= \frac{q}{4\pi\varepsilon_0}\left(\frac{\vec{r}'}{r'^3} - \frac{1}{c}\left(\frac{1}{r'^2}\vec{v}' + 3\vec{r}'\frac{\dot{r}'}{r'^3}\right) - \frac{1}{c^2}\frac{d}{dt}\left(\frac{\vec{v}'}{r'} + \frac{\vec{r}'\dot{r}'}{r'^2}\right)\right) \\
&= \frac{q}{4\pi\varepsilon_0}\left(\frac{\vec{r}'}{r'^3} - \frac{1}{c}\left(\frac{1}{r'^2}\vec{v}' + 3\vec{r}'\frac{\dot{r}'}{r'^3}\right) - \frac{1}{c^2}\left(\frac{\vec{a}'}{r'} - \vec{v}'\frac{\dot{r}'}{r'^2} - \vec{v}'\frac{\dot{r}'}{r'^2} + \ddot{r}'\frac{\vec{r}'}{r'^2} - 2\frac{\vec{r}'\dot{r}'^2}{r'^3}\right)\right) \\
&= \frac{q}{4\pi\varepsilon_0}\left(\left(\frac{1}{r'^3} - \frac{3}{c}\frac{\dot{r}'}{r'^3} - \frac{1}{c^2}\frac{\ddot{r}'}{r'^2} + \frac{2}{c^2}\frac{\dot{r}'^2}{r'^3}\right)\vec{r}' - \frac{1}{c^2}\left(\frac{c}{r'^2} - 2\frac{\dot{r}'}{r'^2}\right)\vec{v}' - \frac{1}{c^2}\frac{\vec{a}'}{r'}\right) \\
&= \frac{q}{4\pi\varepsilon_0 r'^2}\left(\left(\frac{1}{r'} - \frac{3}{c}\frac{\dot{r}'}{r'} - \frac{\ddot{r}'}{c^2} + \frac{2}{c^2}\frac{\dot{r}'^2}{r'}\right)\vec{r}' - \left(1 - \frac{2\dot{r}'}{c}\right)\frac{\vec{v}'}{c} - \frac{r'\vec{a}'}{c^2}\right)
\end{aligned}$$

Still, the implications are not easily reached. However, instead of unit vectors disappearing, familiar physical quantities such as position vectors, velocity vectors, and acceleration vectors appeared so that became a little more specific. The notation of the corresponding physical quantity with $'$ in the formula is a notation for the point of view, and it seems necessary to remove that part. In addition, the application of $\frac{d\vec{r}'}{dt} = -\vec{v}'$ during formula derivation may need to be explained to those who are not familiar with the change in expression depending on the coordinate system perspective. In fact, in traditional electromagnetism texts, explanations of retarded potentials start from this part, but I omitted it. Although it was said to be the retarded potential, it was actually a potential caused by a moving source, so it was not necessary to cover that part. Although it was a tradition of the past, from the point of view of the more refined knowledge of the present, I did not intend to follow the cumbersome explanation that contains the complex process of searching for the truth before the refinement of the past but also contains unnecessary superfluous concepts. The concept of negative numbers or zero is now learned in elementary school, but the process in which it was discovered in the past was not an elementary school process, and it is the same as seeing that it is not essential to start learning from the process of discovery. But now I've reached a point where I can't move on without dealing with that. There was a part of me that put it off because I had to draw a picture that required a complex explanation.

Physical quantities marked with $'$ in Feynman's formula mean, that although Feynman did not explain it directly, it is generally used in physics as a notation according to the subjective point of view of an observer on the other side. In other words, these expressions imply that there is an observer's subjective position and that it is not always a universally valid, objective, absolute space-time. In fact, since the appearance of the theory of relativity, it has been revealed that there is no universally valid, objective, absolute criterion of space and time. Therefore, in order to describe an event, it became one must describe how it is observed from at least two different perspectives in order to fully describe it. Thus, it is true that physics after the theory of relativity has become more difficult than before when only one description was required in the absolute coordinate system.

However, after the advent of relativity theory, even though there is no universally valid absolute space-time, it is still possible and must be possible, to have a third perspective in addition to the perspective of observer 1 and the perspective of observer 2. And, it is the position of most physics narratives that such a third perspective is appropriately selected and used as if it were an absolute coordinate system of the past.

For this problem, it might be a good idea to start by explaining that third point of view.

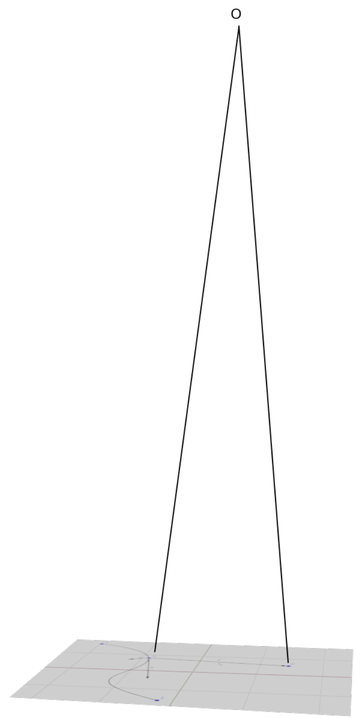

Figure 30: The third point of view

If a moving object and an observer exist on the same plane, at a position O that is between the two points and is perpendicularly far from the plane, the distance between the two positions is the same so that at point O, both positions can be observed simultaneously. It is similar to the concept of absolute time and space. This point of view is the original coordinate system without any mark, and the ′ ed coordinate system is defined based on this point of view.

However, this perspective is not always possible in practice. In this illustration, the O perspective is possible because the moving object is always moving perpendicular to the O perspective, which allows for simultaneous comparison of the moving object's perspective and ′'s perspective, but in reality, the moving object can move out of the plane of the illustration above, which breaks the property of the O perspective that allows for simultaneous viewing of the moving object and ′'s perspective. However, the fact that this perspective is not always possible in practice does not cause problems, as it is only intended to define the relationship between the ′ coordinate system and the moving object coordinate system, and is not used in actual computations.

Consider a moving object and the $'$ coordinate system that is observed from this perspective.

The events seen by observer O in 'The third point of view' figure are as follows.

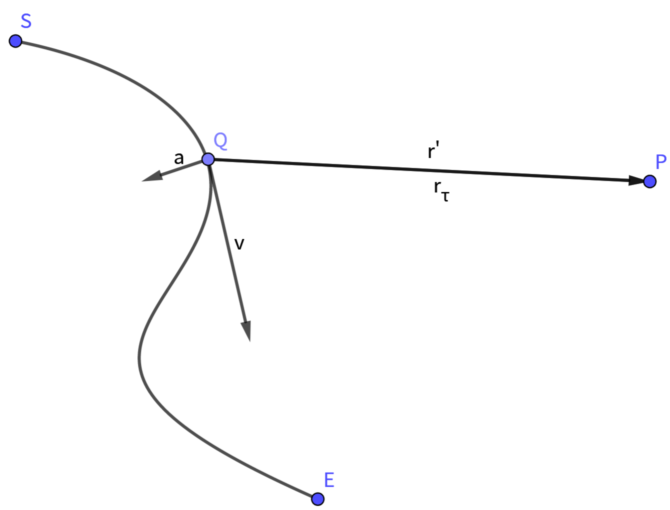

Figure 31: A moving charge Q and an observer at point P

I almost copied Feynman's drawing, but I removed the unmarked r vectors from his drawing because it will not be used at all anymore and I don't think it is helpful. Instead, I added the symbol $\vec{r}_\tau$.

Similar to Feynman's illustration, this is a picture of a charge Q traveling a random path from S to E, experiencing a velocity v and an acceleration "a" at a certain instant in time. The time at this moment is τ. If the distance between point Q and point P is r_τ, then the electromagnetic influence from Q takes $\frac{r_\tau}{c}$ to travel to point P, reaching and affecting it at time t'. At this time, the distance between Q and P as observed from point P is r'. So, it can be seen that it is the same $\vec{r}_\tau = \vec{r}'$, and there is no need to be distinguished.

In this case, when we observe various physical quantities and events in Q from a point P, the notation time can be called t'. And, if the time in Q when the event occurs is called τ, then the relationship between the two is $\tau = t' - \frac{r'_\tau}{c}$. The use of both superscript and subscript for r is a temporary way of saying that both are redundant at the same time since both are the same in either case.

Looking at other physical quantities based on these, first, in the case of v, it is $\vec{v}_\tau = -\frac{d\vec{r_\tau}}{d\tau}$ and $\vec{v}' = -\frac{d\vec{r}'}{dt'}$ from each point of view. The -sign is because the starting point of the r vector is the moving charge that is the source of the electric field, not the location where the electric field is measured. This use of the opposite definition from the usual case is also a convention originally used by Feynman to match the direction of the position vector r and the electric field vector E.

Among the physical quantities observed at point P, we will examine the case of v', whose characteristics can be most clearly known. We can see that nothing special happens when the v vector is perpendicular to the r vector and the distance between Q and P is constant, but this is not the case when the v vector has a component of the r vector and the distance between the two points changes.

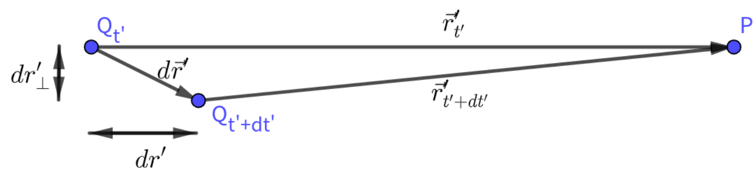

Figure 32: Microdisplacement of R

Note that in the figure above, since dr' is a small change $\vec{r}'_{t'}$ and $\vec{r}'_{t'+dt'}$ are actually parallel. In this case, the time it has taken when the distance from the source Q charge changes by dr is, of course, the time dτ at Q. τ is the time in Q, but not the time with relativistic correction in the Q inertial frame. It is simply time passing at the same rate from position Q as from position O, the 'third point of view'. The 'third point of view' illustration was necessary to explain that this is a real physical quantity that can be used in the calculation. The $d\vec{r}'$ event at time τ from the third point of view is transformed into an event during dt' based on time t' when observed from point P. The description at point P will deal with only the stationary state here, but relativistic corrections in the moving state will have to be made if necessary according to the physical quantity to be seen. However, in fact, even when point P moves, electromagnetic phenomena are already relativistic in nature, so there is no need for additional relativistic corrections to describe physical phenomena caused by the electromagnetic field felt by the object at point P observed from the point of view of O.

Comparing the time measured from the two perspectives under these conditions, it can be seen that it would be $dt' < d\tau$ in the case of the figure above. Observation at point P will show that the intervals of Q events from the τ perspective will be densely blue-shifted at the rate of the classical redshift equation, and accordingly, it can be predicted that all time-related physical quantities such as velocity and acceleration will look different. Since the inertial system which observes, belongs to the O point of view, r itself is measured the same regardless of the Q or P point of view.

I will now list the relationships between physical quantities in these two frameworks which I will call the $'$ and τ frameworks, respectively, but I will denote t' by t for compatibility with Feynman's formula. Since the difference between t' and time t is always constant and the absolute observer's view is not used anyway, removing $'$ is no problem. And, in the τ frame, τ notation will be omitted in all other physical quantities except time τ itself. Regarding physical quantities other than time, this is because the physical quantities in the τ framework are ordinary physical quantities that use the same time intervals as the physical quantities in the third objective observer's viewpoint, the temporal absolute coordinate framework, as described earlier. For example, for $\dot{r}$, the speed at which Q is getting closer or farther away, the range of possible values is $-c < \dot{r} < c$, which is the same as the normal concept, but $\dot{r}'$ the apparent speed from the $'$ perspective at point P, is not a real physical quantity but an apparent physical quantity with a range of $-\infty < \dot{r}' < \frac{c}{2}$. Furthermore, since the temporary absolute coordinate system described earlier is only introduced to illustrate the concept and will never be used directly in actual calculations or physical quantities, there is no need to leave a representation to distinguish between them. Therefore, in practice, note that $\vec{v} \rightarrow \vec{v}_\tau, \vec{a} \rightarrow \vec{a}_\tau, \dot{r} \rightarrow \dot{r}_\tau, \ddot{r} \rightarrow \ddot{r}_\tau$ and so on. Even if there is a correspondence relationship not marked here, all physical quantities are described in either the $'$ frame or the τ frame, and all physical quantities without $'$ notation except for t are described in the τ frame. However, the terms related to r as $r' = r_\tau, \vec{r}' = \vec{r}_\tau$ are indistinguishable in both frameworks, as depicted in the figure 'A moving charge Q and an observer at point P'.

First, the conversion relationship of physical quantities necessary for the analysis of Feynman's formula and its derivation are presented as follows. Since there are many necessary items, individual descriptions are omitted. Instead, the derivation process was described in detail as much as possible.

$$\tau = t - \tfrac{r}{c}, \quad r' = r, \quad \vec{r}' = \vec{r}$$

$$\begin{aligned}
\frac{d\tau}{dt} &= \frac{dt}{dt} - \frac{1}{c}\frac{dr}{dt} \\
&= 1 - \frac{\dot{r}'}{c}
\end{aligned}$$

$$\begin{aligned}
\dot{r}' &= \frac{dr}{dt} = \frac{dr}{d\tau}\frac{d\tau}{dt} \\
&= \dot{r}\left(1 - \frac{\dot{r}'}{c}\right) = \dot{r} - \frac{\dot{r}\dot{r}'}{c} \to \dot{r}' + \frac{\dot{r}\dot{r}'}{c} = \dot{r} \therefore \\
&= \frac{\dot{r}}{1+\frac{\dot{r}}{c}}
\end{aligned}$$

$$\begin{aligned}
\frac{d\tau}{dt} &= 1 - \frac{\dot{r}'}{c} = 1 - \frac{1}{c}\frac{\dot{r}}{1+\frac{\dot{r}}{c}} \\
&= \frac{1}{1+\frac{\dot{r}}{c}}
\end{aligned}$$

$$\begin{aligned}
\dot{r} &= \frac{dr}{d\tau} = \frac{dr}{dt}\frac{dt}{d\tau} \\
&= \frac{\dot{r}'}{1-\frac{\dot{r}'}{c}}
\end{aligned}$$

$$\begin{aligned}
\ddot{r}' &= \frac{d\dot{r}'}{dt} \\
&= \frac{d}{dt}\left(\frac{\dot{r}}{1+\frac{\dot{r}}{c}}\right) \\
&= \frac{1}{1+\frac{\dot{r}}{c}}\frac{d\dot{r}}{dt} - \frac{\dot{r}}{c\left(1+\frac{\dot{r}}{c}\right)^2}\frac{d\dot{r}}{dt} \\
&= \frac{1}{1+\frac{\dot{r}}{c}}\left(1 - \frac{\frac{\dot{r}}{c}}{1+\frac{\dot{r}}{c}}\right)\frac{d\dot{r}}{d\tau}\frac{d\tau}{dt} \\
&= \frac{1}{\left(1+\frac{\dot{r}}{c}\right)^2}\ddot{r}\frac{1}{1+\frac{\dot{r}}{c}} \\
&= \frac{\ddot{r}}{\left(1+\frac{\dot{r}}{c}\right)^3}
\end{aligned}$$

$$\begin{aligned}
\vec{v}' &= \frac{-d\vec{r}}{dt} = \frac{-d\vec{r}}{d\tau}\frac{d\tau}{dt} \\
&= \frac{\vec{v}}{1+\frac{\dot{r}}{c}}
\end{aligned}$$

$$\begin{aligned}
\vec{a}' &= \frac{d\vec{v}'}{dt} = \frac{d\vec{v}'}{d\tau}\frac{d\tau}{dt} \\
&= \frac{d}{d\tau}\left(\frac{\vec{v}}{1+\frac{\dot{r}}{c}}\right)\frac{1}{1+\frac{\dot{r}}{c}} \\
&= \frac{1}{\left(1+\frac{\dot{r}}{c}\right)^2}\frac{d\vec{v}}{d\tau} - \frac{\vec{v}}{\left(1+\frac{\dot{r}}{c}\right)^3}\frac{d\left(1+\frac{\dot{r}}{c}\right)}{d\tau} \\
&= \frac{\vec{a}}{\left(1+\frac{\dot{r}}{c}\right)^2} - \frac{\vec{v}}{\left(1+\frac{\dot{r}}{c}\right)^3}\frac{\ddot{r}}{c} \\
&= \frac{1}{\left(1+\frac{\dot{r}}{c}\right)^2}\left(\vec{a} - \frac{1}{1+\frac{\dot{r}}{c}}\frac{\ddot{r}}{c}\vec{v}\right)
\end{aligned}$$

By substituting these relations into the Feynman formula to eliminate the $'$ expressions, I have got the following.

$$
\begin{aligned}
\vec{E} &= \tfrac{q}{4\pi\varepsilon_0 r'^2}\left(\left(\tfrac{1}{r'} - \tfrac{3}{c}\tfrac{\dot{r}'}{r'} - \tfrac{\ddot{r}'}{c^2} + \tfrac{2}{c^2}\tfrac{\dot{r}'^2}{r'}\right)\vec{r}' - \left(1 - \tfrac{2\dot{r}'}{c}\right)\tfrac{\vec{v}'}{c} - \tfrac{r'\vec{a}'}{c^2}\right) \\
&= \tfrac{q}{4\pi\varepsilon_0 r^2}\left(\left(\tfrac{1}{r} - \tfrac{3}{cr}\tfrac{\dot{r}}{1+\frac{\dot{r}}{c}} - \tfrac{1}{c^2}\tfrac{\ddot{r}}{\left(1+\frac{\dot{r}}{c}\right)^3} + \tfrac{2}{c^2 r}\left(\tfrac{\dot{r}}{1+\frac{\dot{r}}{c}}\right)^2\right)\vec{r} - \left(1 - \tfrac{2}{c}\tfrac{\dot{r}}{1+\frac{\dot{r}}{c}}\right)\tfrac{1}{c}\tfrac{\vec{v}}{1+\frac{\dot{r}}{c}} - \tfrac{r}{c^2}\tfrac{1}{\left(1+\frac{\dot{r}}{c}\right)^2}\left(\vec{a} - \tfrac{1}{1+\frac{\dot{r}}{c}}\tfrac{\ddot{r}}{c}\vec{v}\right)\right) \\
&= \tfrac{q}{4\pi\varepsilon_0 r^2}\left(\left(\tfrac{1}{r}\left(1 - \tfrac{3}{c}\tfrac{\dot{r}}{1+\frac{\dot{r}}{c}} + \tfrac{2}{c^2}\left(\tfrac{\dot{r}}{1+\frac{\dot{r}}{c}}\right)^2\right) - \tfrac{1}{c^2}\tfrac{\ddot{r}}{\left(1+\frac{\dot{r}}{c}\right)^3}\right)\vec{r} - \tfrac{1-\frac{\dot{r}}{c}}{\left(1+\frac{\dot{r}}{c}\right)^2}\tfrac{\vec{v}}{c} - \tfrac{r}{c^2}\tfrac{1}{\left(1+\frac{\dot{r}}{c}\right)^2}\left(\vec{a} - \tfrac{1}{1+\frac{\dot{r}}{c}}\tfrac{\ddot{r}}{c}\vec{v}\right)\right) \\
&= \tfrac{q}{4\pi\varepsilon_0 r^2}\left(\left(\tfrac{1}{r}\left(1 - \tfrac{2}{c}\tfrac{\dot{r}}{1+\frac{\dot{r}}{c}}\right)\left(1 - \tfrac{1}{c}\tfrac{\dot{r}}{1+\frac{\dot{r}}{c}}\right) - \tfrac{1}{c^2}\tfrac{\ddot{r}}{\left(1+\frac{\dot{r}}{c}\right)^3}\right)\vec{r} - \tfrac{1-\frac{\dot{r}}{c}}{\left(1+\frac{\dot{r}}{c}\right)^2}\tfrac{\vec{v}}{c} + \tfrac{1}{\left(1+\frac{\dot{r}}{c}\right)^3}\tfrac{r\ddot{r}}{c^2}\tfrac{\vec{v}}{c} - \tfrac{1}{\left(1+\frac{\dot{r}}{c}\right)^2}\tfrac{r\vec{a}}{c^2}\right) \\
&= \tfrac{q}{4\pi\varepsilon_0 r^2}\left(\left(\tfrac{1}{r}\tfrac{1-\frac{\dot{r}}{c}}{1+\frac{\dot{r}}{c}}\tfrac{1}{1+\frac{\dot{r}}{c}} - \tfrac{1}{c^2}\tfrac{\ddot{r}}{\left(1+\frac{\dot{r}}{c}\right)^3}\right)\vec{r} - \tfrac{1}{\left(1+\frac{\dot{r}}{c}\right)^3}\left(1 - \tfrac{\dot{r}^2}{c^2} - \tfrac{r\ddot{r}}{c^2}\right)\tfrac{\vec{v}}{c} - \tfrac{1}{\left(1+\frac{\dot{r}}{c}\right)^2}\tfrac{r\vec{a}}{c^2}\right) \\
&= \tfrac{q}{4\pi\varepsilon_0 r^2\left(1+\frac{\dot{r}}{c}\right)^2}\left(\left(\tfrac{1}{r}\left(1 - \tfrac{\dot{r}}{c}\right) - \tfrac{\ddot{r}}{c^2}\tfrac{1}{1+\frac{\dot{r}}{c}}\right)\vec{r} - \tfrac{1}{1+\frac{\dot{r}}{c}}\left(1 - \tfrac{\dot{r}^2}{c^2} - \tfrac{r\ddot{r}}{c^2}\right)\tfrac{\vec{v}}{c} - \tfrac{r\vec{a}}{c^2}\right) \\
&= \tfrac{q}{4\pi\varepsilon_0 r^2\left(1+\frac{\dot{r}}{c}\right)^2}\left(\tfrac{1}{1+\frac{\dot{r}}{c}}\left(1 - \tfrac{\dot{r}^2}{c^2} - \tfrac{r\ddot{r}}{c^2}\right)\tfrac{\vec{r}}{r} - \tfrac{1}{1+\frac{\dot{r}}{c}}\left(1 - \tfrac{\dot{r}^2}{c^2} - \tfrac{r\ddot{r}}{c^2}\right)\tfrac{\vec{v}}{c} - \tfrac{r\vec{a}}{c^2}\right) \\
&= \tfrac{q}{4\pi\varepsilon_0 r^2\left(1+\frac{\dot{r}}{c}\right)^2}\left(\tfrac{1}{1+\frac{\dot{r}}{c}}\left(1 - \tfrac{v^2}{c^2} + \tfrac{\vec{a}\cdot\vec{r}}{c^2}\right)\tfrac{\vec{r}}{r} - \tfrac{1}{1+\frac{\dot{r}}{c}}\left(1 - \tfrac{v^2}{c^2} + \tfrac{\vec{a}\cdot\vec{r}}{c^2}\right)\tfrac{\vec{v}}{c} - \tfrac{r\vec{a}}{c^2}\right)
\end{aligned}
$$

The last line was expressed with the more familiar physical quantities of velocity and acceleration, using the following relational expression of the definition of centrifugal force.

$$
\begin{aligned}
\dot{r} &= -\vec{v}\cdot\tfrac{\vec{r}}{r} = -v_{\parallel} \\
\ddot{r} &= \tfrac{v_{\perp}^2}{r} - \vec{a}\cdot\tfrac{\vec{r}}{r} \\
\dot{r}^2 + r\ddot{r} &= v_{\parallel}^2 + v_{\perp}^2 - \vec{a}\cdot\vec{r} \\
&= v^2 - \vec{a}\cdot\vec{r}
\end{aligned}
$$

The meaning of Feynman's formula is now intuitively clear. The direction and magnitude of distance vectors, velocity vectors, and acceleration vectors are familiar and easy-to-use concepts. It is unknown why Feynman did not use this expression in the first place, but I will discuss the possible reason later.

Before that, I would like to look at Griffiths' book, which is often used as a textbook on electromagnetics, which is probably one of the most famous among many documents dealing with this similar formula after Feynman. In the book published in the 1990s, the following formula, which seems to be a different formula from the Feynman formula at first glance, appears. There is no mention of Feynman, and from the description of the derivation process, it seems that Griffiths himself derived the formula himself. The derivation process is also slightly different from Feynman's method, which I have investigated and reconstructed. Of course, they are essentially the same.

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{r}{(\vec{r}\cdot\vec{u})^3}((c^2-v^2)\vec{u}+\vec{r}\times(\vec{u}\times\vec{a}))$$

In fact, this is not as it is, and the r in the above formula must be replaced with the cursive r for which I could not find the font to display. It is presumed that the reason for the notation is that he wanted to introduce a different concept of r, which will be covered later. For now, I will transform this formula into a more readable form using the definition $\vec{u}\equiv c\hat{r}-\vec{v}$ provided by Griffith and the vector algebra formula $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$, which would be familiar now.

$$\begin{aligned}
\vec{E} &= \frac{q}{4\pi\varepsilon_0}\frac{r}{(\vec{r}\cdot\vec{u})^3}((c^2-v^2)\vec{u}+\vec{r}\times(\vec{u}\times\vec{a})) \\
&= \frac{q}{4\pi\varepsilon_0}\frac{r}{(cr-\vec{r}\cdot\vec{v})^3}((c^2-v^2)(c\hat{r}-\vec{v})+\vec{r}\times((c\hat{r}-\vec{v})\times\vec{a})) \\
&= \frac{q}{4\pi\varepsilon_0}\frac{r}{c^3r^3\left(1+\frac{\dot{r}}{c}\right)^3}((c^3-cv^2)\hat{r}-(c^2-v^2)\vec{v}+(\vec{r}\cdot\vec{a})(c\hat{r}-\vec{v})-(\vec{r}\cdot(c\hat{r}-\vec{v}))\vec{a}) \\
&= \frac{q}{4\pi\varepsilon_0 r^2}\frac{1}{c^3\left(1+\frac{\dot{r}}{c}\right)^3}((c^3-cv^2)\hat{r}-(c^2-v^2)\vec{v}+(\vec{r}\cdot\vec{a})(c\hat{r}-\vec{v})-(rc+r\dot{r})\vec{a}) \\
&= \frac{q}{4\pi\varepsilon_0 r^2\left(1+\frac{\dot{r}}{c}\right)^2c^3}\left(\frac{c^3-cv^2+c\vec{r}\cdot\vec{a}}{1+\frac{\dot{r}}{c}}\hat{r}-\frac{c^2-v^2+\vec{r}\cdot\vec{a}}{1+\frac{\dot{r}}{c}}\vec{v}-c\frac{1+\frac{\dot{r}}{c}}{1+\frac{\dot{r}}{c}}r\vec{a}\right) \\
&= \frac{q}{4\pi\varepsilon_0 r^2\left(1+\frac{\dot{r}}{c}\right)^2}\left(\frac{1-\frac{v^2}{c^2}+\frac{\vec{r}\cdot\vec{a}}{c^2}}{1+\frac{\dot{r}}{c}}\hat{r}-\frac{1-\frac{v^2}{c^2}+\frac{\vec{r}\cdot\vec{a}}{c^2}}{1+\frac{\dot{r}}{c}}\frac{\vec{v}}{c}-\frac{r\vec{a}}{c^2}\right)
\end{aligned}$$

As it turns out, as

$$\frac{q}{4\pi\varepsilon_0 r^2\left(1+\frac{\dot{r}}{c}\right)^2}\left(\frac{\left(\frac{1}{\gamma^2}+\frac{\vec{r}\cdot\vec{a}}{c^2}\right)(\hat{r}-\vec{\beta})}{1+\frac{\dot{r}}{c}}-\frac{r\vec{a}}{c^2}\right)$$

, it can be seen that it is identical to the Feynman formula. It is written as if the notion of R is slightly different, but the notion of R cannot be essentially different in formulas that are identically organized in this way. It seems that Griffiths thought that the r used in the formula was not an ordinary r, to avoid dealing with the troublesome concepts in the derivation method that will be discussed later.

Feynman made no mention in his book of his formula derivation methods. I don't know whether there are any other unknown documents besides Feynman's book, but even when searching for other materials such as Griffiths' book, only fragmentary materials were found, and no complete and specific method of deriving the formula was found. However, there were hints that appeared in common, and reconstructing Feynman's formula from them gave me some idea of why Feynman did not officially announce his derivation method. Now I will show the derivation of Feynman's formula.

First, to explain the hint that appears in common, it is a redefinition of the ∇ symbol. To introduce that redefinition, it is,

$$\begin{aligned}\nabla r &= \dot{r}'\nabla t = \tfrac{\dot{r}'}{\dot{r}}\tfrac{\vec{r}}{r} \\ &= \tfrac{\dot{r}}{1+\frac{\dot{r}}{c}}\tfrac{1}{\dot{r}}\tfrac{\vec{r}}{r} = \tfrac{\vec{r}}{r}\tfrac{1}{1+\frac{\dot{r}}{c}} \\ &= \tfrac{1-\frac{\dot{r}'}{c}}{\dot{r}'}\dot{r}'\tfrac{\vec{r}}{r} = \tfrac{\vec{r}}{r}\left(1-\tfrac{\dot{r}'}{c}\right) \\ &= \dot{r}\nabla\tau\end{aligned}$$

Derivation and explanation of the meaning of these conversion formulas will be postponed for a while, and for now, I will deal with it only as the deformation of the ∇ operation and a given rule for how to represent it in the τ framework and in the ′ framework.

The basic formula is

$$\vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t}$$

the electric field expression among the solutions of Maxwell's equations obtained above, and

$$\phi = \frac{q}{4\pi\varepsilon_0 r\left(1+\frac{\dot{r}}{c}\right)}$$

and

$$\vec{A} = \frac{\mu_0 q\vec{v}}{4\pi r\left(1+\frac{\dot{r}}{c}\right)}$$

the equations obtained for the retarded potential. Substituting the retarded potential equations into the electric field expressions using $\varepsilon_0\mu_0 = \frac{1}{c^2}$ results in

$$
\vec{E} = \frac{-q}{4\pi\epsilon_0}\left(\nabla\left(\frac{1}{r\left(1+\frac{\dot{r}}{c}\right)}\right)+\frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\vec{v}}{r\left(1+\frac{\dot{r}}{c}\right)}\right)\right)
$$

This can be computed as it is, but it gets a little complicated, and Feynman probably computed this equation in the $\prime$ frame. To do so, this equation must be converted into physical quantities of the $\prime$ frame, which has already been done once in the opposite direction. However, the following conversion equation must be added to the previous conversion equations.

$$
\begin{aligned}
\vec{v} &= \frac{-d\vec{r}}{d\tau} = \frac{-d\vec{r}}{dt}\frac{dt}{d\tau} \\
&= \frac{\vec{v}'}{1-\frac{\dot{r}'}{c}}
\end{aligned}
$$

Executing the conversion results in,

$$
\begin{aligned}
\vec{E} &= \frac{-q}{4\pi\epsilon_0}\left(\nabla\left(\frac{1}{r'\left(1+\frac{\frac{\dot{r}'}{1-\frac{\dot{r}'}{c}}}{c}\right)}\right)+\frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\frac{\vec{v}'}{1-\frac{\dot{r}'}{c}}}{r'\left(1+\frac{\frac{\dot{r}'}{1-\frac{\dot{r}'}{c}}}{c}\right)}\right)\right) \\
&= \frac{-q}{4\pi\epsilon_0}\left(\nabla\left(\frac{1}{r'}\left(1-\frac{\dot{r}'}{c}\right)\right)+\frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\vec{v}'}{r'}\right)\right)
\end{aligned}
$$

Based on this equation, if the modified definition of the ∇ operation introduced above is applied,

$$
\begin{aligned}
\vec{E} &= \frac{-q}{4\pi\epsilon_0}\left(\nabla\left(\frac{1}{r'}\left(1-\frac{\dot{r}'}{c}\right)\right)+\frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\vec{v}'}{r'}\right)\right) \\
&= \frac{-q}{4\pi\epsilon_0}\left(\nabla\left(\frac{1}{r'}-\frac{\dot{r}'}{r'c}\right)+\frac{1}{c^2}\left(\frac{\vec{a}'}{r'}-\frac{\dot{r}'\vec{v}'}{r'^2}\right)\right) \\
&= \frac{-q}{4\pi\epsilon_0}\left(-\frac{1}{r'^2}\nabla r-\frac{1}{r'c}\nabla\frac{dr'}{dt}+\frac{\dot{r}'}{cr'^2}\nabla r+\frac{1}{c^2}\left(\frac{\vec{a}'}{r'}-\frac{\dot{r}'\vec{v}'}{r'^2}\right)\right) \\
&= \frac{-q}{4\pi\epsilon_0}\left(-\frac{\vec{r}'}{r'^3}\left(1-\frac{\dot{r}'}{c}\right)-\frac{1}{r'c}\frac{d}{dt}\left(\frac{\vec{r}'}{r'}\left(1-\frac{\dot{r}'}{c}\right)\right)+\frac{\dot{r}'}{cr'^2}\frac{\vec{r}'}{r'}\left(1-\frac{\dot{r}'}{c}\right)+\frac{1}{c^2}\left(\frac{\vec{a}'}{r'}-\frac{\dot{r}'\vec{v}'}{r'^2}\right)\right) \\
&= \frac{-q}{4\pi\epsilon_0}\left(-\frac{\vec{r}'}{r'^3}+\frac{\dot{r}'\vec{r}'}{r'^3c}+\frac{\vec{v}'}{r'^2c}\left(1-\frac{\dot{r}'}{c}\right)+\frac{\vec{r}'}{r'c}\frac{\dot{r}'}{r'^2}\left(1-\frac{\dot{r}'}{c}\right)+\frac{\ddot{r}'\vec{r}'}{r'^2c^2}+\frac{\dot{r}'\vec{r}'}{cr'^3}-\frac{\dot{r}'^2\vec{r}'}{c^2r'^3}+\frac{1}{c^2}\left(\frac{\vec{a}'}{r'}-\frac{\dot{r}'\vec{v}'}{r'^2}\right)\right) \\
&= \frac{-q}{4\pi\epsilon_0}\left(-\frac{\vec{r}'}{r'^3}+\frac{\dot{r}'\vec{r}'}{r'^3c}+\frac{\vec{v}'}{r'^2c}-\frac{\dot{r}'\vec{v}'}{r'^2c^2}+\frac{\dot{r}'\vec{r}'}{r'^3c}-\frac{\dot{r}'^2\vec{r}'}{r'^3c^2}+\frac{\ddot{r}'\vec{r}'}{r'^2c^2}+\frac{\dot{r}'\vec{r}'}{r'^3c}-\frac{\dot{r}'^2\vec{r}'}{c^2r'^3}+\frac{\vec{a}'}{r'c^2}-\frac{\dot{r}'\vec{v}'}{r'^2c^2}\right) \\
&= \frac{-q}{4\pi\epsilon_0}\left(-\frac{\vec{r}'}{r'^3}+3\frac{\dot{r}'\vec{r}'}{r'^3c}+\frac{\ddot{r}'\vec{r}'}{r'^2c^2}-2\frac{\dot{r}'^2\vec{r}'}{r'^3c^2}+\frac{\vec{v}'}{r'^2c}-2\frac{\dot{r}'\vec{v}'}{r'^2c^2}+\frac{\vec{a}'}{r'c^2}\right) \\
&= \frac{q}{4\pi\epsilon_0 r'^2}\left(\left(\frac{1}{r'}-3\frac{\dot{r}'}{r'c}-\frac{\ddot{r}'}{c^2}+2\frac{\dot{r}'^2}{r'c^2}\right)\vec{r}'-\left(\frac{1}{c}-2\frac{\dot{r}'}{c^2}\right)\vec{v}'-\frac{r'\vec{a}'}{c^2}\right)
\end{aligned}
$$

It can be confirmed that the result is the same as the decomposition result of the Feynman formula.

In fact, it can be obtained directly from the τ frame without having to convert it to the t frame beforehand. However, the following terms must be precomputed.

$$
\begin{aligned}
\nabla \dot{r} &= \nabla \tfrac{dr}{d\tau} = \nabla \left(\tfrac{dr}{dt}\tfrac{dt}{d\tau}\right) = \nabla \left(\dot{r}' \frac{1}{1-\frac{\dot{r}'}{c}}\right) \\
&= \frac{1}{1-\frac{\dot{r}'}{c}} \nabla \dot{r}' + \frac{\dot{r}'}{c\left(1-\frac{\dot{r}'}{c}\right)^2} \nabla \dot{r}' = \left(\frac{1}{1-\frac{\dot{r}'}{c}} + \frac{\dot{r}'}{c\left(1-\frac{\dot{r}'}{c}\right)^2}\right) \nabla \dot{r}' \\
&= \left(\frac{1}{1-\frac{\dot{r}'}{c}} + \frac{\dot{r}'}{c\left(1-\frac{\dot{r}'}{c}\right)^2}\right) \frac{d}{dt} \nabla r = \left(\frac{1}{1-\frac{\dot{r}'}{c}} + \frac{\dot{r}'}{c\left(1-\frac{\dot{r}'}{c}\right)^2}\right) \frac{d}{dt}\left(\frac{\vec{r}}{r}\left(1 - \frac{\dot{r}'}{c}\right)\right) \\
&= \left(\frac{1}{1-\frac{\dot{r}'}{c}} + \frac{\dot{r}'}{c\left(1-\frac{\dot{r}'}{c}\right)^2}\right) \left(\frac{1}{r}\left(1 - \frac{\dot{r}'}{c}\right)\frac{d\vec{r}}{dt} - \frac{\vec{r}}{r^2}\left(1 - \frac{\dot{r}'}{c}\right)\frac{dr}{dt} - \frac{\vec{r}}{rc}\frac{d\dot{r}'}{dt}\right) \\
&= \left(\frac{1}{1-\frac{\dot{r}'}{c}} + \frac{\dot{r}'}{c\left(1-\frac{\dot{r}'}{c}\right)^2}\right) \left(\frac{-1}{r}\left(1 - \frac{\dot{r}'}{c}\right)\vec{v}' - \frac{\vec{r}}{r^2}\left(1 - \frac{\dot{r}'}{c}\right)\dot{r}' - \frac{\vec{r}}{rc}\ddot{r}'\right) \\
&= \left(1 + \frac{\dot{r}'}{c\left(1-\frac{\dot{r}'}{c}\right)}\right) \left(\frac{-1}{r}\vec{v}' - \frac{\vec{r}}{r^2}\dot{r}' - \frac{\vec{r}}{rc\left(1-\frac{\dot{r}'}{c}\right)}\ddot{r}'\right) \\
&= \frac{1}{1-\frac{\dot{r}'}{c}} \left(\frac{-1}{r}\vec{v}' - \frac{\vec{r}}{r^2}\dot{r}' - \frac{\vec{r}}{rc\left(1-\frac{\dot{r}'}{c}\right)}\ddot{r}'\right) \\
&= \left(1 + \tfrac{\dot{r}}{c}\right) \left(\frac{-1}{r}\vec{v}\left(\frac{1}{1+\frac{\dot{r}}{c}}\right) - \frac{\vec{r}}{r^2}\frac{\dot{r}}{1+\frac{\dot{r}}{c}} - \left(1 + \tfrac{\dot{r}}{c}\right)\frac{\vec{r}}{rc}\frac{\ddot{r}}{\left(1+\frac{\dot{r}}{c}\right)^2}\left(1 - \frac{\dot{r}}{c\left(1+\frac{\dot{r}}{c}\right)}\right)\right) \\
&= \frac{-\vec{v}}{r} - \frac{\vec{r}}{r}\frac{\dot{r}}{r} - \frac{\vec{r}}{r}\frac{\ddot{r}}{c}\left(1 - \frac{\dot{r}}{c\left(1+\frac{\dot{r}}{c}\right)}\right) \\
&= \frac{-\vec{v}}{r} - \left(\frac{\dot{r}}{r} + \frac{\ddot{r}}{c} - \frac{\dot{r}\ddot{r}}{c^2\left(1+\frac{\dot{r}}{c}\right)}\right)\frac{\vec{r}}{r}
\end{aligned}
$$

Applying this,

$$
\begin{aligned}
\vec{E} &= \frac{-q}{4\pi\epsilon_0}\left[\nabla\left(\frac{1}{r-\frac{\vec{v}\cdot\vec{r}}{c}}\right)+\frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\vec{v}}{r-\frac{\vec{v}\cdot\vec{r}}{c}}\right)\right]\\
&= \frac{-q}{4\pi\epsilon_0}\left[\nabla\left(\frac{1}{r+\frac{r\dot{r}}{c}}\right)+\frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\vec{v}}{r+\frac{r\dot{r}}{c}}\right)\right]\\
&= \frac{-q}{4\pi\epsilon_0}\left[\frac{-1}{\left(r+\frac{r\dot{r}}{c}\right)^2}\nabla\left(r+\frac{r\dot{r}}{c}\right)+\frac{1}{c^2}\left(\frac{1}{\left(r+\frac{r\dot{r}}{c}\right)}\frac{\partial\vec{v}}{\partial t}-\frac{\vec{v}}{\left(r+\frac{r\dot{r}}{c}\right)^2}\frac{\partial}{\partial t}\left(r+\frac{r\dot{r}}{c}\right)\right)\right]\\
&= \frac{-q}{4\pi\epsilon_0}\left[\frac{-1}{\left(r+\frac{r\dot{r}}{c}\right)^2}\left(\nabla r+\nabla\left(\frac{r\dot{r}}{c}\right)\right)+\frac{1}{c^2}\left(\frac{1}{\left(r+\frac{r\dot{r}}{c}\right)}\frac{\partial\vec{v}}{\partial\tau}\frac{d\tau}{dt}-\frac{\vec{v}}{\left(r+\frac{r\dot{r}}{c}\right)^2}\left(\frac{\partial r}{\partial t}+\frac{\dot{r}}{c}\frac{\partial r}{\partial t}+\frac{r}{c}\frac{\partial\dot{r}}{\partial t}\right)\right)\right]\\
&= \frac{-q}{4\pi\epsilon_0}\left[\frac{-1}{\left(r+\frac{r\dot{r}}{c}\right)^2}\left(\nabla r+\frac{\dot{r}}{c}\nabla r+\frac{r}{c}\nabla\dot{r}\right)+\frac{1}{c^2}\left(\frac{1}{\left(r+\frac{r\dot{r}}{c}\right)}\vec{a}-\frac{\vec{v}}{\left(r+\frac{r\dot{r}}{c}\right)^2}\left(\frac{\partial r}{\partial\tau}+\frac{\dot{r}}{c}\frac{\partial r}{\partial\tau}+\frac{r}{c}\frac{\partial\dot{r}}{\partial\tau}\right)\right)\frac{d\tau}{dt}\right]\\
&= \frac{-q}{4\pi\epsilon_0}\left[\frac{-1}{\left(r+\frac{r\dot{r}}{c}\right)^2}\left(\left(1+\frac{\dot{r}}{c}\right)\nabla r+\frac{r}{c}\nabla\dot{r}\right)+\frac{1}{c^2}\left(\frac{1}{\left(r+\frac{r\dot{r}}{c}\right)}\vec{a}-\frac{\vec{v}}{\left(r+\frac{r\dot{r}}{c}\right)^2}\left(\frac{\partial r}{\partial\tau}+\frac{\dot{r}}{c}\frac{\partial r}{\partial\tau}+\frac{r}{c}\frac{\partial\dot{r}}{\partial\tau}\right)\right)\frac{1}{1+\frac{\dot{r}}{c}}\right]\\
&= \frac{q}{4\pi\epsilon_0 r^2\left(1+\frac{\dot{r}}{c}\right)^2}\left[\left(\frac{1+\frac{\dot{r}}{c}}{1+\frac{\dot{r}}{c}}\frac{\vec{r}}{r}+\frac{r}{c}\nabla\dot{r}\right)-\frac{1}{c^2}\left(r\vec{a}-\frac{\vec{v}}{1+\frac{\dot{r}}{c}}\left(\dot{r}+\frac{r\ddot{r}}{c}+\frac{\dot{r}^2}{c}\right)\right)\right]\\
&= \frac{q}{4\pi\epsilon_0 r^2\left(1+\frac{\dot{r}}{c}\right)^2}\left[\left(\frac{\vec{r}}{r}+\frac{r}{c}\left(-\frac{\vec{v}}{r}-\left(\frac{\dot{r}}{r}+\frac{\ddot{r}}{c}-\frac{\ddot{r}\dot{r}}{c^2\left(1+\frac{\dot{r}}{c}\right)}\right)\frac{\vec{r}}{r}\right)\right)+\frac{\vec{v}}{c^2\left(1+\frac{\dot{r}}{c}\right)}\left(\dot{r}+\frac{r\ddot{r}}{c}+\frac{\dot{r}^2}{c}\right)-\frac{r\vec{a}}{c^2}\right]\\
&= \frac{q}{4\pi\epsilon_0 r^2\left(1+\frac{\dot{r}}{c}\right)^2}\left[\left(\frac{\vec{r}}{r}-\frac{r}{c}\left(\frac{\dot{r}}{r}+\frac{\ddot{r}}{c}-\frac{\ddot{r}\dot{r}}{c^2\left(1+\frac{\dot{r}}{c}\right)}\right)\frac{\vec{r}}{r}\right)-\frac{\vec{v}}{c}+\frac{\vec{v}}{c^2\left(1+\frac{\dot{r}}{c}\right)}\left(\dot{r}+\frac{r\ddot{r}}{c}+\frac{\dot{r}^2}{c}\right)-\frac{r\vec{a}}{c^2}\right]\\
&= \frac{q}{4\pi\epsilon_0 r^2\left(1+\frac{\dot{r}}{c}\right)^2}\left[\left(1-\frac{\dot{r}}{c}-\frac{r\ddot{r}}{c^2}+\frac{r\ddot{r}\dot{r}}{c^3\left(1+\frac{\dot{r}}{c}\right)}\right)\frac{\vec{r}}{r}-\left(1-\frac{1}{1+\frac{\dot{r}}{c}}\left(\frac{\dot{r}}{c}+\frac{r\ddot{r}}{c^2}+\frac{\dot{r}^2}{c^2}\right)\right)\frac{\vec{v}}{c}-\frac{r\vec{a}}{c^2}\right]\\
&= \frac{q}{4\pi\epsilon_0 r^2\left(1+\frac{\dot{r}}{c}\right)^2}\left[\frac{1}{1+\frac{\dot{r}}{c}}\left(\left(1-\frac{\dot{r}}{c}-\frac{r\ddot{r}}{c^2}\right)\left(1+\frac{\dot{r}}{c}\right)+\frac{r\ddot{r}\dot{r}}{c^3}\right)\frac{\vec{r}}{r}-\frac{1}{1+\frac{\dot{r}}{c}}\left(1+\frac{\dot{r}}{c}-\frac{\dot{r}}{c}-\frac{r\ddot{r}}{c^2}-\frac{\dot{r}^2}{c^2}\right)\frac{\vec{v}}{c}-\frac{r\vec{a}}{c^2}\right]\\
&= \frac{q}{4\pi\epsilon_0 r^2\left(1+\frac{\dot{r}}{c}\right)^2}\left[\frac{1}{1+\frac{\dot{r}}{c}}\left(1-\frac{\dot{r}}{c}-\frac{r\ddot{r}}{c^2}+\left(1-\frac{\dot{r}}{c}-\frac{r\ddot{r}}{c^2}\right)\frac{\dot{r}}{c}+\frac{r\ddot{r}\dot{r}}{c^3}\right)\frac{\vec{r}}{r}-\frac{1}{1+\frac{\dot{r}}{c}}\left(1-\frac{r\ddot{r}}{c^2}-\frac{\dot{r}^2}{c^2}\right)\frac{\vec{v}}{c}-\frac{r\vec{a}}{c^2}\right]\\
&= \frac{q}{4\pi\epsilon_0 r^2\left(1+\frac{\dot{r}}{c}\right)^2}\left[\frac{1}{1+\frac{\dot{r}}{c}}\left(1-\frac{\dot{r}}{c}-\frac{r\ddot{r}}{c^2}+\frac{\dot{r}}{c}-\frac{\dot{r}^2}{c^2}-\frac{r\ddot{r}\dot{r}}{c^3}+\frac{r\ddot{r}\dot{r}}{c^3}\right)\frac{\vec{r}}{r}-\frac{1}{1+\frac{\dot{r}}{c}}\left(1-\frac{r\ddot{r}}{c^2}-\frac{\dot{r}^2}{c^2}\right)\frac{\vec{v}}{c}-\frac{r\vec{a}}{c^2}\right]\\
&= \frac{q}{4\pi\epsilon_0 r^2\left(1+\frac{\dot{r}}{c}\right)^2}\left[\frac{1}{1+\frac{\dot{r}}{c}}\left(1-\frac{r\ddot{r}}{c^2}-\frac{\dot{r}^2}{c^2}\right)\frac{\vec{r}}{r}-\frac{1}{1+\frac{\dot{r}}{c}}\left(1-\frac{r\ddot{r}}{c^2}-\frac{\dot{r}^2}{c^2}\right)\frac{\vec{v}}{c}-\frac{r\vec{a}}{c^2}\right]
\end{aligned}
$$

We can verify that it is calculated directly.

However, there is a pitfall in the above computation.

Readers who have followed the discussion of the Feynman formula so far in this book may have noticed that these explanations are proceeding in reverse from the usual explanation order. It was a way of continuing to postpone the explanation of the most important point, doing the computations first, and then re-entering the derivation of the fundamental formula. Now it is the time to confess why I had to do that.

In the preparation computation of the last calculation, there is a problem with the part that started with $\nabla\dot{r} = \nabla\frac{dr}{d\tau} = \nabla\left(\frac{dr}{dt}\frac{dt}{d\tau}\right) = \nabla\left(\dot{r}'\frac{1}{1-\frac{\dot{r}'}{c}}\right)$ and was later treated as $\frac{d}{dt}\nabla r$. Originally, it would be more natural to derive as $\nabla\dot{r} = \nabla\frac{dr}{d\tau} = \frac{d}{d\tau}\nabla r$ from the beginning in the τ frame. However, the problem is that the results of the two paths are different. Originally, taking out the derivative with respect to the time from ∇ was a computational technique based on Clairaut's theorem that it did not matter if the order of the derivative was reversed. The fact that the result changes for the same calculation is equivalent to saying that the transformed ∇ is no longer an ordinary derivative. And because of this, it is not just the derivative with respect to τ that cannot be reordered and taken out of the ∇ operation, but also the derivative with respect to t, it became not clear whether it is justified to reorder it.

And also before, you may recall that the derivation of the definition of ∇r was the one that I put off. Looking back, I only presented the definition with

$$\begin{aligned}\nabla r &= \dot{r}'\nabla t = \frac{\dot{r}'}{\dot{r}}\frac{\vec{r}}{r}\\ &= \frac{\dot{r}}{1+\frac{\dot{r}}{c}}\frac{1}{\dot{r}}\frac{\vec{r}}{r} = \frac{\vec{r}}{r}\frac{1}{1+\frac{\dot{r}}{c}}\\ &= \frac{1-\frac{\dot{r}'}{c}}{\dot{r}'}\dot{r}'\frac{\vec{r}}{r} = \frac{\vec{r}}{r}\left(1-\frac{\dot{r}'}{c}\right)\\ &= \dot{r}\nabla\tau\end{aligned}$$

and passed by without explanation. Now, to confess about the derivation method, in fact, it is actually all there is. Precisely, the first line is the whole of the derivation process. The rest is just a list of various variant expressions for convenient use. However, the thought process that led to the first line would require explanation.

The definition of the ∇ operation is

$$\nabla f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

according to the definition introduced by William Rowan Hamilton, who was a mathematician but also contributed to physics. I'm going to assume that you've already learned about the history of this operation and its physical meaning in high school. Because thinking about complicated meanings makes only my head hurt, and mathematics, especially algebra, is a system of thought that is 'designed' to give correct answers if we follow the formal rules, even if we don't care about it. In science, these are basic concepts that are learned in high school, or at the latest in the first year of college. And the more elemental it is, the more difficult it is to discuss its meaning. Originally, at the beginning of this book, I decided to target the book's readership to science high school students or college beginners, but that is only for physics, and for mathematics, the explanation of these mathematical foundations is beyond my capabilities, so I will omit it. For now, I will just look at the form of these operations. In fact, that should be enough for most readers. The above operation can be transformed as follows according to the basic algebraic rules. I will just use r instead of f this time so that we don't have to hesitate for the next step.

$$\begin{aligned}
\nabla r &= \frac{\partial r}{\partial x}\hat{x} + \frac{\partial r}{\partial y}\hat{y} + \frac{\partial r}{\partial z}\hat{z} \\
&= \frac{dr}{dt}\frac{\partial t}{\partial x}\hat{x} + \frac{dr}{dt}\frac{\partial t}{\partial y}\hat{y} + \frac{dr}{dt}\frac{\partial t}{\partial z}\hat{z} \\
&= \frac{dr}{dt}\left(\frac{\partial t}{\partial x}\hat{x} + \frac{\partial t}{\partial y}\hat{y} + \frac{\partial t}{\partial z}\hat{z}\right) \\
&= \frac{dr}{dt}\nabla t \\
&= \dot{r}'\nabla t
\end{aligned}$$

The final ′ notation is a straightforward jump to the notation for our particular problem. Meanwhile, there is another well-known value for ∇r. This is the value $\nabla r = \frac{\vec{r}}{r}$, which is directly obtained from the mathematical definitions of ∇ and r. If we are going to apply this value to the present problem, we will have to consider the following two cases.

First of all, when Q is stationary or c is infinite, we can see that it converges to this value. In that case, it can be inferred that the value of ∇t is of course $\nabla t = \frac{1}{\dot{r}'}\frac{\vec{r}}{r}$. The next thing to consider now is if Q is moving and c is not infinite. In this case, in fact, the most common thought to consider is that ∇ is a differential operator, not a physical quantity such as time or distance, so it should remain unchanged without being affected by such things. However, the problem with such a method is that it fails to derive a physically consistent electromagnetic field corrected for special relativity based on Maxwell's equations.

If we look at Purcell's book, we shall see that there is no solution for charges in arbitrary motion that appeared after Feynman, but there is an analysis for charges in uniform linear motion already done. I will not deal with that directly. Purcell's formula used relativistic concepts of length contraction and time delay to estimate the properties of an electric field traveling at a constant velocity, but I will not attempt to verify Purcell's formula, as I have not done so. As for the question of how to trust Purcell's formula without testing it, Feynman, as an explanation of his formula, showed that it satisfies the conditions Purcell demanded. So did Griffiths. And I'm going to mention it briefly as well. In fact, even without computations like Purcell's, thought experiments alone can show that the important properties of Purcell's formula are relativistically indispensable. So, I don't think that Purcell's theory, which is incomplete compared to Feynman's, is indispensable knowledge just because it was the path of development. This is because the theories introduced later after Purcell convey knowledge in a more refined form, and by studying them, we can obtain more comprehensive and clearer knowledge without having to retrace the chaotic path of previous theories. Although it is important to look at the background of the development of theories because it gives us a sense of the direction in which physical theories have developed and an opportunity to think about the philosophical roots of physical theories, my limitations in this book will inevitably impose some restrictions on how far and in what detail I can look back into the surrounding knowledge. In the case of Purcell's formula, I will present only its results.

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r_p^2}\frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{3/2}}\hat{r}_p$$

This is the most commonly known formulation of the field due to a uniformly moving charge, analyzed based on special relativity. There are, of course, other ways to approximations or verifications with thought experiments that can be used for individual situations. Purcell's formula is just borrowed because it is a simple way to express the conclusion. The problem at the time was that it was derived from a special relativistic analysis of the electric field, and no specific method of deriving it in terms of Maxwell's equations was known, and it could only deal with the relatively simple situation of constant velocity linear motion, not arbitrary motion. I will not go into the derivation of this formula, but I will briefly discuss the minimum conditions required for an electric field due to a charge in a uniform linear motion to be consistent with relativity, which is implicit in this formula and which the electric field formulation must adhere to.

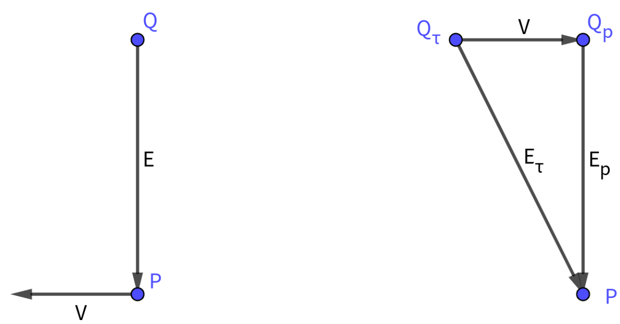

Figure 33: Relative perspective

First, let us consider the situation where the electric field E is observed while passing by the stationary charge Q at point P with a constant velocity.

It is a conclusion of classical electrodynamics that the electric field felt from the position passing through point P is directed in the Q direction. However, looking at this again from relativity, from that point of view, the observer at point P is stationary, and the electric charge Q moves at a constant velocity of v, feeling the electric field Eτ created at the position of Qτ in the past. However, if the electric field by the moving source is the same as the electric field by the stationary source, the position of Q obtained by feeling it becomes the past position, Qτ, and is different from the current position, Qp. In addition, if we look at it that way, the Qτ position is the position of Q that P sees through the phenomenon of velocity aberration in the light when P is moving, and P cannot see the actual position of Q through the light when it is moving, but when Q is moving, the position of Q seen through light and the position of Q felt by the electric field are the same, which is a phenomenon that negates relativity. Therefore, this cannot be correct and must be corrected. In order for relativity to be correct in this case, just as the case of P was moving, there also must be some process such that P sees the wrong position of Q through light and the electric field feels the true position of Q in the case that P is at rest and Q is moving at a constant velocity. Meanwhile, the position of Q as observed by light, whether P is moving or Q is moving, does not reflect the true current position of Q. This is due to the finite speed of light if Q is moving, and to the existence of the phenomenon of velocity aberration in the light if P is moving. Under the assumption that the electric field Eτ originating at Qτ should look like Ep reflecting the actual position of the charge Qp, Purcell's formula is the calculation of the strength in each direction, taking into account the effects of length contraction and time dilation.

In the previous sentence, there is a slightly complicated reason for expressing 'feel' rather than 'measurement'. It is expressed that the measurement or observation from the third point of view, which has been mentioned above, is felt from the point of view of P. Because, from the point of view of P, if the Q position is measured through the measurement of the electric field and light is emitted in that direction, the light also does not hit Qp, because of the Doppler beaming effect if P moves, and because the light is missed by moving Q before the light reaches Q if Q is moving. In fact, the distinction between the movement of P and the movement of Q in the first place is already an expression that has already introduced the third point of view. In the third point of view, it is expressed as 'feel' that the focus of the motion change of P by the field is directed toward Qp in any case. In this case, the proviso 'except for the influence of the magnetic field' is still attached.

If we check whether Feynman's formula satisfies the above condition through the case of uniform linear motion where no acceleration exists, it is

$$\begin{aligned}\vec{E} &= \frac{q}{4\pi\varepsilon_0 r^2\left(1+\frac{\dot{r}}{c}\right)^2}\left(\frac{1}{1+\frac{\dot{r}}{c}}\left(1-\frac{v^2}{c^2}+\frac{\vec{a}\cdot\vec{r}}{c^2}\right)\frac{\vec{r}}{r}-\frac{1}{1+\frac{\dot{r}}{c}}\left(1-\frac{v^2}{c^2}+\frac{\vec{a}\cdot\vec{r}}{c^2}\right)\frac{\vec{v}}{c}-\frac{r\vec{a}}{c^2}\right)\\ &= \frac{q\left(1-\frac{v^2}{c^2}\right)}{4\pi\varepsilon_0 r^2\left(1+\frac{\dot{r}}{c}\right)^3}\left(\frac{\vec{r}}{r}-\frac{\vec{v}}{c}\right)\end{aligned}$$

and it can be simply shown that it is consistent with Purcell's formula.

At first glance, the formulas look different, but if we note that r_p in the Purcell formula is a vector representing the current direction and distance of the charge, and r or r_τ in the Feynman formula is a vector representing the past direction and distance of the charge, fact that the direction of the electric field according to the two formulas is the same can be seen in the following figure.

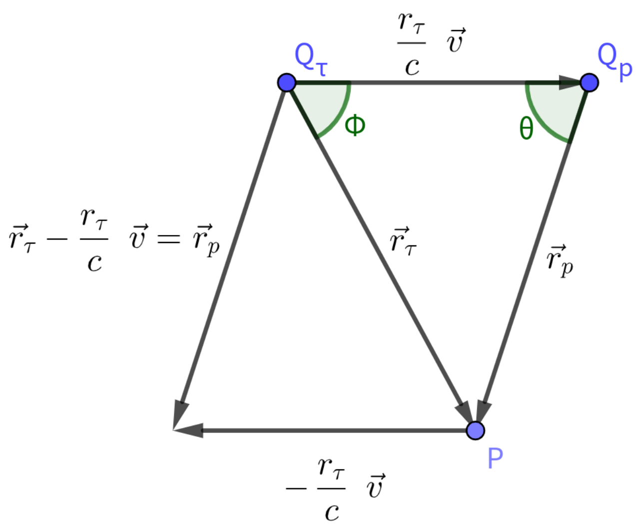

Figure 34: Comparison with the Purcell formula

It can be seen at a glance that $\vec{r}_p$ in the Purcell formula is the vector from the current position of the charge Q to the observation point P, which is in the same direction as $\vec{r}_\tau - \frac{r_\tau}{c}\vec{v}$ in the Feynman formula when a=0. However, the magnitude of the vectors needs to be verified by further calculation, but it can be guessed without further calculation that it would be very difficult for the overall magnitudes of vectors to be different with even the same direction obtained in conceptually independent calculations.

In order to compare whether Purcell's formula $\frac{q}{4\pi\varepsilon_0 r_p^2}\frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{3/2}}\hat{r}_p$ and Feynman's formula $\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2\left(1+\frac{\dot{r}}{c}\right)^2}\left(\frac{\left(1-\frac{v^2}{c^2}+\frac{\vec{a}\cdot\vec{r}}{c^2}\right)\left(\hat{r}-\frac{\vec{v}}{c}\right)}{1+\frac{\dot{r}}{c}} - \frac{r\vec{a}}{c^2}\right)$ are the same equation when acceleration a = 0, some preliminary preparation is required.

In the 'Comparison with Purcell Formula' figure, we can see $r_\tau \sin\phi = r_p \sin\theta \rightarrow \frac{r_p}{r} = \frac{\sin\phi}{\sin\theta}$. And, we can see the $r_\tau\cos\phi + r_p\cos\theta = r_\tau\frac{v}{c} \rightarrow \frac{r_p}{r} = \frac{\beta-\cos\phi}{\cos\theta}$ as well.

From these, equation $\frac{\sin\phi}{\sin\theta} = \frac{\beta-\cos\phi}{\cos\theta} \rightarrow \frac{\sin\phi}{\beta-\cos\phi} = \tan\theta \rightarrow \theta = \arctan\frac{\sin\phi}{\beta-\cos\phi}$ can be solved to obtain the expression of θ.

And, it would be a good idea to keep in mind the $\sin(\arctan a) = \frac{a}{\sqrt{a^2+1}}$ rule.

Apply these rules to the following equations to see if they are true.

$$
\begin{aligned}
\frac{q}{4\pi\varepsilon_0 r_p^2}\frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{3/2}}\hat{r}_p &= \frac{q}{4\pi\varepsilon_0 r^2\left(1+\frac{\dot{r}}{c}\right)^2}\left(\frac{1-\frac{v^2}{c^2}}{1+\frac{\dot{r}}{c}}\hat{r}-\frac{1-\frac{v^2}{c^2}}{1+\frac{\dot{r}}{c}}\frac{\vec{v}}{c}\right)\\
\frac{\hat{r}_p}{r_p^2(1-\beta^2\sin^2\theta)^{3/2}} &= \frac{\hat{r}-\frac{\vec{v}}{c}}{r^2\left(1+\frac{\dot{r}}{c}\right)^3}\\
\frac{\frac{r\left(\hat{r}-\frac{\vec{v}}{c}\right)}{r_p}}{r_p^2(1-\beta^2\sin^2\theta)^{3/2}} &= \frac{\hat{r}-\frac{\vec{v}}{c}}{r^2\left(1+\frac{\dot{r}}{c}\right)^3}\\
\frac{1}{r_p^3(1-\beta^2\sin^2\theta)^{3/2}} &= \frac{1}{r^3\left(1+\frac{\dot{r}}{c}\right)^3}\\
r\left(1+\frac{\dot{r}}{c}\right) &= r_p(1-\beta^2\sin^2\theta)^{1/2}\\
r\left(1-\frac{v}{c}\cos\phi\right) &= r_p\sqrt{1-\beta^2\sin^2\theta}\\
\frac{1-\frac{v}{c}\cos\phi}{\sqrt{1-\beta^2\sin^2\theta}} &= \frac{r_p}{r}\\
\frac{1-\frac{v}{c}\cos\phi}{\sqrt{1-\beta^2\sin^2\theta}} &= \frac{\sin\phi}{\sin\theta}\\
\frac{1+\beta^2\cos^2\phi-2\beta\cos\phi}{1-\beta^2\sin^2\theta} &= \frac{\sin^2\phi}{\sin^2\theta}\\
\frac{1+\beta^2\cos^2\phi-2\beta\cos\phi}{\sin^2\phi} &= \frac{1-\beta^2\sin^2\theta}{\sin^2\theta}\\
\frac{1+\beta^2\cos^2\phi-2\beta\cos\phi}{\sin^2\phi} &= \frac{1}{\sin^2\theta}-\beta^2\\
\frac{1+\beta^2\cos^2\phi-2\beta\cos\phi+\beta^2\sin^2\phi}{\sin^2\phi} &= \frac{\left(\frac{\sin\phi}{\beta-\cos\phi}\right)^2+1}{\left(\frac{\sin\phi}{\beta-\cos\phi}\right)^2}\\
\frac{1+\beta^2-2\beta\cos\phi}{\sin^2\phi} &= \frac{\frac{\sin^2\phi}{\beta^2+\cos^2\phi-2\beta\cos\phi}+1}{\frac{\sin^2\phi}{\beta^2+\cos^2\phi-2\beta\cos\phi}}\\
\frac{1+\beta^2-2\beta\cos\phi}{\sin^2\phi} &= \frac{\beta^2+1-2\beta\cos\phi}{\sin^2\phi}
\end{aligned}
$$

It was shown that Feynman's formula agrees with Purcell's formula in the absence of acceleration.

On the other hand, if the calculation is performed as strictly as possible without using the technique of transforming the differential operator used in the derivation of the Feynman formula, it is immediately apparent that

$$\vec{E}=\frac{q}{4\pi\epsilon_0 r^2\left(1+\frac{\dot{r}}{c}\right)^2}\left(\left(1-\frac{\dot{r}^2}{c^2}\right)\frac{\vec{r}}{r}-\left(1-\frac{\ddot{r}}{c^2}\right)\frac{\vec{v}}{c}-\frac{r\vec{a}}{c^2}\right)$$

is not consistent with the Purcell formula, which is a representative relativistic interpretation of the electric field. First of all, the direction of the electric field does not point to the current position of the charge moving at a constant velocity. However, since the advent of mathematics that twist spacetime, many people may try to attempt to distort spacetime with the method, rather than give up on the approach. On the other hand, whether Feynman's idea of abandoning this formula altogether and trying to touch the definition of differential operators is easy, or whether it is a brilliant idea that was difficult to come up with after many more attempts than I have briefly introduced here, I can't exactly feel the atmosphere of 1940s electromagnetism, so it is difficult to guess the difficulty of Feynman's idea. But it is not an easy idea by any means to me. However, while the idea may be difficult, its execution is not particularly difficult.

Earlier, it was shown that ∇r can be expressed as $\nabla r = \dot{r}'\nabla t = \frac{\dot{r}'}{\dot{r}'}\frac{\vec{r}}{r}$. If we try to see what transformations are possible that would cause this expression to change value due to the condition of a finite speed of light when moving while the value remains as it is in a state of rest, there is one candidate that immediately comes to mind. That is $\nabla r = \frac{\dot{r}'}{\dot{r}}\frac{\vec{r}}{r}$ that changed the expression in the ′ frame of the denominator in $\frac{\dot{r}'}{\dot{r}'}$, which is always 1 in the final expression, to the expression in τ. This is a method by directly substituting the characteristics of the ′ frame and the τ frame, which differ by the finite speed of light, and it can be seen at a glance that this is the only valid conversion that produces the required effect. There is no such thing as a formula derivation. If the definition of ∇ is something that can be influenced by the property of this universe, the finite speed of light, it is clear what the effect must be. And, simply by introducing this definition, an expression consistent with Purcell's formula is computed at once.

However, this method is not mathematically right. It was confirmed that the ∇ defined in this way loses a property of a differential operator that is independent of the order of differentiation, which means that ∇ used for physical phenomena is no longer a normal differential operator. Manipulating a differential operator to make it no longer a differential operator cannot be said mathematically valid. The first derivation that obtained a physically correct formula in the ′ frame was purely coincidental. Actually, it's not a coincidence, it's just that Feynman used the ′ template as a standard, so I just followed it without thinking. However, the method of calculation using the τ frame, which should be mathematically equivalent, produces different results. The method of direct calculation using the τ frame that I presented also involved temporarily transforming to the ′ frame and then reverting back during the intermediate stages of each transformation. What we can know from this is that the ∇ operator is a pseudo-differential operator that only operates as a real differential operator in the ′ frame. Additionally, it is recommended that the physical quantities be applied to r directly. This is undoubtedly a new property of mathematics. And this is not all.

So far, I have only discussed electric fields and have not discussed magnetic fields. Originally, I came to discuss electromagnetic fields in order to deal with Maxwellian gravity and Laplace's problem, so the magnetic field was not initially a priority. However, since I have discovered a mathematical anomaly in the derivation of electric fields, there arises a need to also investigate the magnetic field.

The magnetic field is defined by the equation using the vector potential.

$$\begin{aligned}\vec{B} &= \nabla \times \vec{A} \\ &= \tfrac{q}{4\pi\epsilon_0 c^2} \nabla \times \left(\tfrac{\vec{v}}{r\left(1+\frac{\dot{r}}{c}\right)} \right) \\ &= \tfrac{q}{4\pi\epsilon_0 c^2} \nabla \times \tfrac{\vec{v}'}{r}\end{aligned}$$

And, it is also

$$\vec{B} = \frac{1}{c^2}\vec{E} \times \vec{v}$$

for a moving observer P according to the law of electromagnetic induction. It is also

$$\vec{B} = \frac{q}{4\pi\varepsilon_0 c^2 r^2}\vec{v} \times \frac{\vec{r}}{r}$$

if the Biot-Savart law the first expression for a magnetic field, is rewritten, for a single charge. The opposite sign of the expressions in the Biot-Savart law and the law of electromagnetic induction is only the difference between whether the observer is moving or whether the charge, which is the source of the electric field, is moving. Here, of course, I am solving for the first approach, which is considered the most general. The other expressions are for the purpose of comparative verification. This is because the Biot-Savart law is an approximation at slow speeds that need to be corrected through relativity, and the electromagnetic induction law is a relativistically rigorous law that is not an approximation in terms of magnetic fields, but the E used in the electromagnetic induction law needs a relativistic correction, such as Purcell's formula. On the other hand, the first method, obtained by solving Maxwell's equations directly, does not require the complicated and obscure relativistic corrections that Purcell's method of physical intuition requires, and, as an added bonus, it yields expressions for general motion that are not restricted to constant velocity motion.

On the other hand, as a result of calculations using Purcell's formula and relativistic corrections, such as transforming a situation in which a magnetic field produced by a moving charge is measured by a stationary observer into a situation in which an electric field produced by a stationary charge is measured by a moving observer and a magnetic field is felt, it was known that the magnetic field can also be written as the following expression. This was the situation until about 1950 when the Feynman formula was calculated.

$$\vec{B} = \frac{\hat{r}}{c} \times \vec{E}$$

The r used here is the familiar r_τ and r' at the same time. And, E is the E that can be obtained by Purcell's formula or Feynman's formula, which are slightly different directions from r. This is already a relativistically correct expression. However, the calculation does not take into account the effect of the acceleration included in Feynman's formula, but to say the result in advance, the expression does not change when the effect of acceleration is included. I'll explain how to get this.

First, the following vector algebra basic formula is needed.

$$
\begin{aligned}
\nabla \times (u\vec{v}) &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ uv_x & uv_y & uv_z \end{vmatrix} \\
&= \left(\frac{\partial uv_z}{\partial y} - \frac{\partial uv_y}{\partial z}\right)\hat{x} + \left(\frac{\partial uv_x}{\partial z} - \frac{\partial uv_z}{\partial x}\right)\hat{y} + \left(\frac{\partial uv_y}{\partial x} - \frac{\partial uv_x}{\partial y}\right)\hat{z} \\
&= \left(\frac{duv_z}{dt}\frac{\partial t}{\partial y} - \frac{duv_y}{dt}\frac{\partial t}{\partial z}\right)\hat{x} + \left(\frac{duv_x}{dt}\frac{\partial t}{\partial z} - \frac{duv_z}{dt}\frac{\partial t}{\partial x}\right)\hat{y} + \left(\frac{duv_y}{dt}\frac{\partial t}{\partial x} - \frac{duv_x}{dt}\frac{\partial t}{\partial y}\right)\hat{z} \\
&= \left((ua_z + \dot{u}v_z)\frac{\partial t}{\partial y} - (ua_y + \dot{u}v_y)\frac{\partial t}{\partial z}\right)\hat{x} \\
&\quad + \left((ua_x + \dot{u}v_x)\frac{\partial t}{\partial z} - (ua_z + \dot{u}v_z)\frac{\partial t}{\partial x}\right)\hat{y} \\
&\quad + \left((ua_y + \dot{u}v_y)\frac{\partial t}{\partial x} - (ua_x + \dot{u}v_x)\frac{\partial t}{\partial y}\right)\hat{z} \\
&= \left(ua_z\frac{\partial t}{\partial y} - ua_z\frac{\partial t}{\partial z}\right)\hat{x} + \left(ua_x\frac{\partial t}{\partial z} - ua_z\frac{\partial t}{\partial x}\right)\hat{y} + \left(ua_y\frac{\partial t}{\partial x} - ua_x\frac{\partial t}{\partial y}\right)\hat{z} \\
&\quad + \left(\dot{u}v_z\frac{\partial t}{\partial y} - \dot{u}v_y\frac{\partial t}{\partial z}\right)\hat{x} + \left(\dot{u}v_x\frac{\partial t}{\partial z} - \dot{u}v_z\frac{\partial t}{\partial x}\right)\hat{y} + \left(\dot{u}v_y\frac{\partial t}{\partial x} - \dot{u}v_x\frac{\partial t}{\partial y}\right)\hat{z} \\
&= u\left(\left(a_z\frac{\partial t}{\partial y} - a_y\frac{\partial t}{\partial z}\right)\hat{x} + \left(a_x\frac{\partial t}{\partial z} - ua_z\frac{\partial t}{\partial x}\right)\hat{y} + \left(a_y\frac{\partial t}{\partial x} - a_x\frac{\partial t}{\partial y}\right)\hat{z}\right) \\
&\quad + \left(v_z\frac{\partial u}{\partial y} - v_y\frac{\partial u}{\partial z}\right)\hat{x} + \left(v_x\frac{\partial u}{\partial z} - v_z\frac{\partial u}{\partial x}\right)\hat{y} + \left(v_y\frac{\partial u}{\partial x} - v_x\frac{\partial u}{\partial y}\right)\hat{z} \\
&= u\nabla \times \vec{v} - \vec{v} \times \nabla u
\end{aligned}
$$

The reason for writing down the derivation process is that it's a technique that will be used frequently, so I want you to get used to it, and I want to show that it's mathematically justified up to this point.

Using this, if I introduce the calculation process first, it is as follows. If you see that introducing the result first is an unavoidable choice when the process is not completely justifiable, it is correct.

$$
\begin{aligned}
\vec{B} &= \tfrac{q}{4\pi\epsilon_0 c^2}\nabla\times\tfrac{\vec{v}'}{r} \\
&= \tfrac{q}{4\pi\epsilon_0 c^2}\left(\tfrac{1}{r}\nabla\times\vec{v}' - \vec{v}'\times\nabla\tfrac{1}{r}\right) \\
&= \tfrac{q}{4\pi\epsilon_0 c^2}\left(\tfrac{1}{r}\tfrac{d}{dt}(\nabla\times\vec{r}) + \vec{v}'\times\tfrac{\vec{r}}{r^3}\left(1-\tfrac{\dot{r}'}{c}\right)\right) \\
&= \tfrac{q}{4\pi\epsilon_0 c^2}\left(\tfrac{1}{r}\tfrac{d}{dt}\left(\tfrac{\vec{v}'}{c}\times\tfrac{\vec{r}}{r}\right) + \vec{v}'\times\tfrac{\vec{r}}{r^3}\left(1-\tfrac{\dot{r}'}{c}\right)\right) \\
&= \tfrac{q}{4\pi\epsilon_0 c^2}\left(\tfrac{1}{rc}\left(\vec{a}'\times\tfrac{\vec{r}}{r} - \vec{v}'\times\tfrac{\vec{v}'}{r} - \vec{v}'\times\tfrac{\dot{r}'\vec{r}}{r^2}\right) + \vec{v}'\times\tfrac{\vec{r}}{r^3}\left(1-\tfrac{\dot{r}'}{c}\right)\right) \\
&= \tfrac{q}{4\pi\epsilon_0 c^2}\left(\tfrac{1}{rc}\left(\vec{a}' - \vec{v}'\tfrac{\dot{r}'}{r}\right) + \tfrac{\vec{v}'}{r^2}\left(1-\tfrac{\dot{r}'}{c}\right)\right)\times\tfrac{\vec{r}}{r} \\
&= \tfrac{q}{4\pi\epsilon_0 c^2}\left(\tfrac{\vec{a}'}{rc} - \tfrac{\vec{v}'}{r^2}\tfrac{\dot{r}'}{c} + \tfrac{\vec{v}'}{r^2} - \tfrac{\vec{v}'}{r^2}\tfrac{\dot{r}'}{c}\right)\times\tfrac{\vec{r}}{r} \\
&= \tfrac{q}{4\pi\epsilon_0 r^2 c}\left(\tfrac{r\vec{a}'}{c^2} + \tfrac{\vec{v}'}{c}\left(1-\tfrac{2\dot{r}'}{c}\right)\right)\times\tfrac{\vec{r}}{r} \\
&= \tfrac{q}{4\pi\epsilon_0 r^2 c}\left(\left(1-2\tfrac{\dot{r}'}{c}\right)\tfrac{\vec{v}'}{c} + \tfrac{r'\vec{a}'}{c^2}\right)\times\tfrac{\vec{r}}{r}
\end{aligned}
$$

We can see that it is

$$\vec{B} = \frac{\hat{r}}{c}\times\vec{E}$$

in the ′ frame. Moving this to the τ frame, it is as follows

$$
\begin{aligned}
\vec{B} &= \tfrac{q}{4\pi\epsilon_0 r^2 c}\left(\left(1-2\tfrac{\dot{r}'}{c}\right)\tfrac{\vec{v}'}{c} + \tfrac{r'\vec{a}'}{c^2}\right)\times\tfrac{\vec{r}}{r} \\
&= \tfrac{q}{4\pi\epsilon_0 r^2 c}\left(\left(1-2\frac{\frac{\dot{r}}{1+\frac{\dot{r}}{c}}}{c}\right)\frac{\frac{\vec{v}}{1+\frac{\dot{r}}{c}}}{c} + \frac{r'\frac{1}{\left(1+\frac{\dot{r}}{c}\right)^2}\left(\vec{a}-\frac{1}{1+\frac{\dot{r}}{c}}\frac{\ddot{r}}{c}\vec{v}\right)}{c^2}\right)\times\tfrac{\vec{r}}{r} \\
&= \tfrac{q}{4\pi\epsilon_0 r^2 c}\left(\left(1-\frac{\frac{2\dot{r}}{c}}{1+\frac{\dot{r}}{c}}\right)\frac{\frac{\vec{v}}{c}}{1+\frac{\dot{r}}{c}} + r\frac{1}{c^2\left(1+\frac{\dot{r}}{c}\right)^2}\left(\vec{a}-\frac{1}{1+\frac{\dot{r}}{c}}\frac{\ddot{r}}{c}\vec{v}\right)\right)\times\tfrac{\vec{r}}{r} \\
&= \tfrac{q}{4\pi\epsilon_0 r^2 c}\left(\frac{1-\frac{\dot{r}}{c}}{1+\frac{\dot{r}}{c}}\frac{\frac{\vec{v}}{c}}{1+\frac{\dot{r}}{c}} + r\frac{1}{c^2\left(1+\frac{\dot{r}}{c}\right)^2}\left(\vec{a}-\frac{1}{1+\frac{\dot{r}}{c}}\frac{\ddot{r}}{c}\vec{v}\right)\right)\times\tfrac{\vec{r}}{r} \\
&= \tfrac{q}{4\pi\epsilon_0 r^2\left(1+\frac{\dot{r}}{c}\right)^2 c}\left(\left(1-\tfrac{\dot{r}}{c}\right)\tfrac{\vec{v}}{c} + \tfrac{r\vec{a}}{c^2} - \frac{\frac{r\ddot{r}}{c^2}}{1+\frac{\dot{r}}{c}}\tfrac{\vec{v}}{c}\right)\times\tfrac{\vec{r}}{r} \\
&= \tfrac{q}{4\pi\epsilon_0 r^2\left(1+\frac{\dot{r}}{c}\right)^2 c}\left(\tfrac{1}{1+\frac{\dot{r}}{c}}\left(1-\tfrac{\dot{r}^2}{c^2}-\tfrac{r\ddot{r}}{c^2}\right)\tfrac{\vec{v}}{c} + \tfrac{r\vec{a}}{c^2}\right)\times\tfrac{\vec{r}}{r} \\
&= \tfrac{q}{4\pi\varepsilon_0 r^2\left(1+\frac{\dot{r}}{c}\right)^2 c}\left(\tfrac{1}{1+\frac{\dot{r}}{c}}\left(1-\tfrac{v^2}{c^2}+\tfrac{\vec{a}\cdot\vec{r}}{c^2}\right)\tfrac{\vec{v}}{c} + \tfrac{r\vec{a}}{c^2}\right)\times\tfrac{\vec{r}}{r}
\end{aligned}
$$

Likewise, it can be seen that it is accurately converted to

$$\vec{B} = \frac{\hat{r}}{c}\times\vec{E}$$

in the τ frame.

The rule calculated in this way is as follows.

First, we saw that the ∇ operator in the previous electric field computation is no longer a true differential operator, and in the τ framework Clairaut's rule no longer applies, but in the ′ framework it still behaves as if it were a true differential operator. In addition, since the definition of r is the same in all perspectives, it was found that it is safe to always use r as the starting point from which other definitions such as velocity or acceleration are derived. And, if this guess is not correct, all these calculations are meaningless from the beginning.

So, the magnetic field was calculated based on r in the ′ frame, and the part substituted with $\nabla \times \vec{r} = \frac{\vec{v}'}{c} \times \frac{\vec{r}}{r}$ in the middle is based on the following speculation.

$$
\begin{aligned}
\nabla \times \vec{r} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r_x & r_y & r_z \end{vmatrix} \\
&= \left(\frac{\partial r_z}{\partial y} - \frac{\partial r_y}{\partial z}\right)\hat{x} + \left(\frac{\partial r_x}{\partial z} - \frac{\partial r_z}{\partial x}\right)\hat{y} + \left(\frac{\partial r_y}{\partial x} - \frac{\partial r_x}{\partial y}\right)\hat{z} \\
&= \left(\frac{dr_z}{dt}\frac{\partial t}{\partial y} - \frac{dr_y}{dt}\frac{\partial t}{\partial z}\right)\hat{x} + \left(\frac{dr_x}{dt}\frac{\partial t}{\partial z} - \frac{dr_z}{dt}\frac{\partial t}{\partial x}\right)\hat{y} + \left(\frac{dr_y}{dt}\frac{\partial t}{\partial x} - \frac{dr_x}{dt}\frac{\partial t}{\partial y}\right)\hat{z} \\
&= - \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{dr_x}{dt} & \frac{dr_y}{dt} & \frac{dr_z}{dt} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} & \frac{\partial t}{\partial z} \end{vmatrix} \\
&= \frac{-d\vec{r}}{dt} \times \nabla t \\
&= \vec{v}' \times \nabla t \\
&= \frac{\vec{v}'}{c} \times \frac{\vec{r}}{r}
\end{aligned}
$$

And, $\nabla t = \frac{1}{c}\frac{\vec{r}}{r}$. Here, strange speculation emerges. ∇ for time is not the same as the $\nabla t = \frac{1}{\dot{r}}\frac{\vec{r}}{r}$ in the previous ∇r. Of course, the previous case was also a conjecture based on physical intuition, and the only basis for doing so was that it was the simplest and most plausible candidate, but the result is that such a guess is not even allowed to be consistent. $\nabla t = \frac{1}{\dot{r}}\frac{\vec{r}}{r}$ cannot be used at all. If it is used, according to the above calculation, it shows a result of $\nabla \times r = \frac{\vec{v}'}{\dot{r}} \times \frac{\vec{r}}{r}$, which appears as infinity under certain conditions. If an infinity appears in a physical formula, it is not different from saying that it is a catastrophe unless the conditions under which the infinity appears are actually impossible conditions. As infinity appears within the range that is slower than the speed of light, this formula is not a consideration at all.

On the other hand, in its original vector algebraic definition, it is $\nabla \times r = 0$. However, substituting that definition into $\vec{B} = \nabla \times \vec{A}$ yields only the Biot-Savart formula. However, it is already known through the result of Purcell's relativistic analysis that the result is an approximation in the case of $v \ll c$ and cannot be exact. Therefore, it cannot be $\nabla \times r = 0$, and it is clear that it must have some value. By the way, the physical reason why the mathematical definition of the ∇ operation must be modified appears more clearly in the calculation of the magnetic field. It became clear that the modification had to be inevitable, and then considering what to do with ∇t, the condition was simple. The only other vector that can be used to calculate the vector product with v is the r vector itself, and it is 0 at slow speeds and becomes $\frac{\vec{r}}{r}$ at high speeds approaching the speed of light. This is a condition that can be easily created, it is $\nabla t = \frac{1}{c}\frac{\vec{r}}{r}$. If we test this, we can get the above result that is physically correct and mathematically impossible to explain. This is the gist of my process of reconstructing the Feynman formula.

Is this approach mathematically valid? There's no way it could be. Mathematically, this approach is not valid. The definition of ∇ as a differential operator has already been determined mathematically, but this approach treats ∇ as if it were an unknown function and redefines it. As a result, the characteristics of ∇ as a differential operator, which include the ability to change the order of differential operators, are maintained in some frames but not in others, and even in cases where they are maintained, it cannot be guaranteed that they will be. This approach suggests a mathematically invalid story that it produced the correct result when calculated under the belief that the characteristics are maintained.

But, I believe the result is correct even if it is not mathematically justified. The resulting Feynman formula has already been verified to be the answer when relativistically interpreting the electromagnetic field. Of course, Feynman's formula has not yet been experimentally verified. To determine the truth of such a formula experimentally, It is insufficient to accelerate an individual particle to a relativistic speed. One would need to accelerate an entire experimental setup to relativistic speeds, or at the very least, a substantial amount of material at relativistic speeds, which is not a technologically easy task. In the 21st century, papers about the experiment on the Feynman formula have recently been published, but it does not seem to be accepted as conclusive evidence yet. Unless it is a really ingenious experiment, it would be difficult to prove it experimentally with the current technology level. I predict that direct experiments will be possible only when technology is developed enough for humans to live in outer space. I do not consider this an experiment to be done on Earth because a mistake in accelerating a large amount of matter to near-light speeds required for the experiment would result in a nuclear explosion-like accident.

Nonetheless, I believe that the Feynman formula is correct, and I can only say that it is attributed purely to physical intuition aside from the agreement with other relativistic interpretations such as Purcell's formula.

The intuition to choose the latter between

$$\vec{E} = \frac{q}{4\pi\epsilon_0 r^2 \left(1 + \frac{\dot{r}}{c}\right)^2} \left(\left(1 - \frac{\dot{r}^2}{c^2}\right) \frac{\vec{r}}{r} - \left(1 - \frac{\ddot{r}}{c^2}\right) \frac{\vec{v}}{c} - \frac{r\vec{a}}{c^2} \right)$$

and

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2 \left(1 + \frac{\dot{r}}{c}\right)^2} \left(\frac{1}{1 + \frac{\dot{r}}{c}} \left(1 - \frac{v^2}{c^2} + \frac{\vec{a}\cdot\vec{r}}{c^2}\right) \frac{\vec{r}}{r} - \frac{1}{1 + \frac{\dot{r}}{c}} \left(1 - \frac{v^2}{c^2} + \frac{\vec{a}\cdot\vec{r}}{c^2}\right) \frac{\vec{v}}{c} - \frac{r\vec{a}}{c^2} \right)$$

is unexplainable, but it certainly exists. Physical intuition originally existed before mathematics. For example, the law of universal gravitation, published in 1687, Coulomb's law, published in 1785, and the Biot-Savart law, published in 1820, all have the inverse square law in common, but it was not until the introduction of the ∇ operator by Hamilton in 1837 that the inverse square law was mathematically explained by introducing the concept of potential. Of course, there was work by Gauss and Lagrange before that, but not yet the concept of a potential field or wave.

The inverse square law was not mathematically derived but was guessed intuitively and applied to actual phenomena in the beginnings of physics. In fact, a complete explanation still does not exist and never will, as the question 'why' can be infinitely repeated but the human ability to answer is finite. It is necessary to stop asking the question 'why' at some point, and it is important to stop at the right position. Determining where that position is a realm of physical intuition that cannot be fully explained. Anything under that limit, which can be explained perfectly, falls within the realm of mathematics.

I heard that it was mathematically proven that mathematics cannot be perfect by Russell and Gödel about a hundred years ago. It would be rather strange if mathematics, which is ultimately inherent in such imperfections, could explain nature perfectly. Whether nature is perfect or imperfect, an imperfect narrative system cannot fully describe an object. In the case of this problem, the concept ∇t, which is arguably physically ambiguous, and even its substantive meaning is questioned, adapts its form in mathematical calculations to meet the physical requirements within the bounds where its expression does not mathematically contradict. The result is a phenomenon in which, from one perspective, it still behaves as a differential operator, while from another perspective, it loses its defining characteristics. This, I believe, is a manifestation of mathematics' inherent imperfection when used to describe physics. Is it too hasty a judgment? Given that this pertains to the fundamental nature of the electromagnetic field, it is plausible that such mathematical limitations may indeed come into play.

Physics is obviously a discipline that leans very heavily on mathematics. Mathematics is the expression method of physics, and what cannot be expressed in mathematics is not a law of physics. However, in the process of searching for a truly new law in physics, physical intuition works more than mathematics, and although the expression is mathematical, the process of finding the expression is not always mathematical. Physics leans very heavily on mathematics, but it is not mathematics itself. Nature is certainly as mathematical as possible, but not perfectly mathematical. Since mathematics itself is not perfect, it is natural that nature is not perfectly mathematical.

However, this is not the end of this story. I see the essence of the Feynman formula as an example of physical laws confronting the limits of mathematics, but I think Feynman, the discoverer himself, might have thought differently. If he had thought like me, he would have had no reason to hesitate in revealing the derivation process, incomplete though it may be, and leaving that thought behind, but for some reason, he did not leave the record of the derivation process of this important formula. In fact, I think he may have been reluctant to cause a disturbance unnecessarily since the results are more important than the derivation process in this kind of basic law of physics. Or perhaps, rather than the incompleteness of mathematics, he saw it as the incompleteness of the derivation process and postponed the presentation to find a more complete method, which he ultimately could not find. Of course, I cannot know Feynman's thoughts for sure. From the insistence on the abbreviation of the formula in the ′ frame, it seems to be a similar view to mine, but it is impossible to know the thoughts of the passed predecessor without any words. However, in Griffiths' book, there is a dazzling mathematical skill that makes me wonder whether it is Griffiths' skill or Feynman's skill that has been transmitted.

I have no intention of explaining it the way Griffiths explains it. From my point of view, it is a technique that gathers mathematical imperfections in one place and slyly covers them up, so it is not possible to explain that aspect only by just repeating his words. Therefore, since this is to be summarized and conveyed from my point of view, if you want to make a clearer judgment on this matter, you will have to compare Griffiths' book with my next explanation.

Griffiths' book introduces t_r, which looks like it is derived from

$$\tau = t - \frac{r}{c} = t - t_r$$

and uses it to hide the problem of ∇t described above.

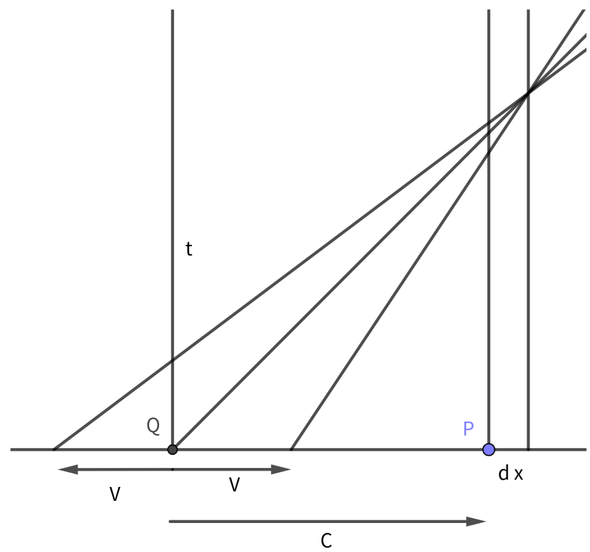

Figure 35: Definition of ∇r

Looking at the figure 'definition of ∇r', assuming that the information about the distance between Q and P is some physical quantity that originates from Q and travels to P at the speed of light, we can see that the slope of that quantity with respect to distance is $\frac{dy}{dx} = \frac{1}{1 \pm \frac{v}{c}}$. Replacing y with r to express this as a vector derivative in three dimensions, it is $\nabla r = \frac{1}{1+\frac{\dot{r}}{c}} \frac{\vec{r}}{r} = \frac{\dot{r}'}{\dot{r}} \frac{\vec{r}}{r}$, giving us a basis for the value of ∇r that we had previously guessed. So far, this seems like a pretty plausible solution.

And, the definition of $-c\nabla t_r = \nabla r$ was added to the previous definition, and then, using $\dot{r}' = \frac{\dot{r}}{1+\frac{\dot{r}}{c}}$, that resulted in

$$\nabla t_r = \frac{-\vec{r}}{rc - \vec{r} \cdot \vec{v}} = \frac{-1}{c} \frac{\dot{r}'}{\dot{r}} \frac{\vec{r}}{r}$$

, and it is $d\tau = dt - dt_r$ by definition. Since this is the case of addition between infinitesimals, so it is $dt = 0$, that it is $d\tau = -dt_r$, then the result of

$$
\begin{aligned}
\nabla \times \vec{r} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r_x & r_y & r_z \end{vmatrix} \\
&= \left(\frac{\partial r_z}{\partial y} - \frac{\partial r_y}{\partial z}\right)\hat{x} + \left(\frac{\partial r_x}{\partial z} - \frac{\partial r_z}{\partial x}\right)\hat{y} + \left(\frac{\partial r_y}{\partial x} - \frac{\partial r_x}{\partial y}\right)\hat{z} \\
&= \left(\frac{dr_z}{dt_r}\frac{\partial t_r}{\partial y} - \frac{dr_y}{dt_r}\frac{\partial t_r}{\partial z}\right)\hat{x} + \left(\frac{dr_x}{dt_r}\frac{\partial t_r}{\partial z} - \frac{dr_z}{dt_r}\frac{\partial t_r}{\partial x}\right)\hat{y} + \left(\frac{dr_y}{dt_r}\frac{\partial t_r}{\partial x} - \frac{dr_x}{dt_r}\frac{\partial t_r}{\partial y}\right)\hat{z} \\
&= -\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{dr_x}{dt_r} & \frac{dr_y}{dt_r} & \frac{dr_z}{dt_r} \\ \frac{\partial t_r}{\partial x} & \frac{\partial t_r}{\partial y} & \frac{\partial t_r}{\partial z} \end{vmatrix} \\
&= \frac{-d\vec{r}}{dt_r} \times \nabla t_r \\
&= \frac{d\vec{r}}{d\tau} \times \nabla t_r \\
&= -\vec{v} \times \frac{-1}{c}\frac{\dot{r}'}{r}\frac{\vec{r}}{r} \\
&= \vec{v} \times \frac{1}{c}\frac{1}{1+\frac{\dot{r}}{c}}\frac{\vec{r}}{r} \\
&= \frac{\vec{v}'}{c} \times \frac{\vec{r}}{r}
\end{aligned}
$$

is derived and used.

Has the previous lack of mathematical basis been completely resolved through this dazzling result? I look suspicious. If the ∇ operation still acts as a true differential operator after this transformation, then this method may be justified, but, since it is a real differential operator only in the ′ frame, the inconsistency that has been calculated as if nothing has happened after the transformation remains unresolved, even if it is disguised as a plausible calculation. It is mathematics that allows even the Ramanujan sum to be calculated using only plausible calculations. Nevertheless, the cleverness of combining the two different representations of ∇t and concealing the incompleteness in the derivation using it is undoubtedly impressive. Still, I prefer a crude way of revealing it rather than skillfully concealing it, if there is an imperfection. If we keep hiding it because it doesn't look good, later generations or even ourselves may forget that such incompleteness ever existed, and that's a regression. I think the fundamental problem of modern physics lies in the excessive use of overly sophisticated mathematics. As an aside, I don't like Griffiths' method of leaving most of the solutions in his book as practice problems without presenting the solution process directly to the reader. Would it be too much of a leap to say that not showing the derivation in such an important problem is a psychological trick that takes advantage of the fact that people who have succeeded in following the solution process based on weak evidence have shared the leap in the solution process and developed a bias toward the result? However, it is true that in a book written 30 years after Feynman, at least some explanation of the derivation and its basis is necessary. There is pressure to show the derivation process and the basis for anything. It would be the right of only Feynman, the first discoverer, to omit the derivation process and describe only the results.

Well, now that we have found how to calculate the electric field and even the magnetic field generated by moving charges through tedious and boring computations. Now it's time to go back to the original reason for this tedious computation, the solution to Laplace's problem.

3.5 The solution of the Laplace problem

I will compute the Laplace problem in a previous binary star system more specifically. For simplicity, I will use a binary star system with an exact circular orbit.

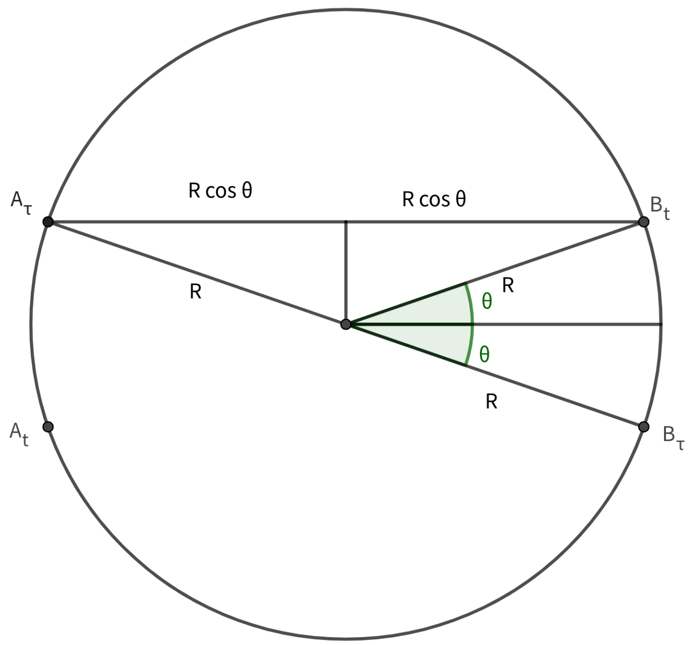

Figure 36: The Laplace problem in a binary star system

In a binary star system composed of stars of the same mass, A and B, even if there are other disturbance factors such as gravitational wave emission, it is safe to say that the two stars are orbiting in exactly opposite positions with the center of the orbit circle as the symmetry point. In this case, if the gravity of star A, which departed at time τ in the past, arrives at star B at time t, which is the present time, B moves from Bτ to Bt in the meantime. If $c \bigtriangleup t = 2R\cos\theta$ is the distance traveled the gravity at the speed of light, then $v \bigtriangleup t = 2R\theta$ is the distance traveled by

B. From this, it can be seen that

$$\frac{\cos\theta}{c} = \frac{\theta}{v} \to \frac{v}{c} = \frac{\theta}{\cos\theta}$$

Since c is a constant, it can be utilized as an equation to find the value of θ given a certain v.

Since this equation contains a function that includes the transcendental function cosine and cannot be solved algebraically, numerical calculation using a computer is required to obtain the value. In particular, in order to substitute the actual Earth's orbital speed or mass, the precision of numerical calculations is not enough with ordinary floating point numbers, and arbitrary precision calculations are required.

There are several programs available for arbitrary precision computation, but there is no public package that can calculate the numerical solution of this equation, so it is necessary to program it directly. The Python package 'mpmath' provides the functionality with the 'findroot' function, but as of 2022, there is a serious problem of inaccurate calculation values due to the floating-point input issue, so it is not recommended to use Python for verification. It is a problem common to all functions, and it is clear that other math packages such as friCAS have no problems with the corresponding input, so it is a problem of the package.

The easiest way to simply check the reliability of the calculations in the package for Python you are trying to use is to type a command such as 'mp.asin(mp.sin(0.1))'. You can see that 'mpmath' doesn't come out exactly 0.1 and the error starts to occur from the limit of floating point precision. On the other hand, in a proper package, such as friCAS, you can see that the answer of input 'asin(sin(0.1))' is exactly 0.1

The main package used in this book, friCAS, performs arbitrary-precision calculations by default, but does not have a root-finding function, and has its own programming function built in, but it is inconvenient for me. Therefore, I have written the calculation function using the computable-reals package of Common Lisp, which was first introduced in 1989 and has been verified reliable for a long time. In the computable-reals package, there is no arcsin function, but only the arctan function, so we can check that there is no problem with [atan-r (sin-r 1/10 /r cos-r 1/10)]. In computable-reals, only integer and rational inputs are accepted, and floating-point inputs are not accepted.

If you are using a computer that is following along with the calculations in this book, then you have already installed the necessary packages when you calculated the number of galaxies in the universe. So all you need to do is run the following commands. The first command extends the symbol table to redefine the operators of computable-reals so that they can be used in the SHN macro. The second command is Newton's method root-finding function that calculates the angle θ in radians from a given v/c. Finally, if you're curious about how many iterations Newton's method takes to find the root using the conventional root-finding function, you can use the last function instead of the second one.

A caution in use is that Newton's root-finding algorithm was used when writing this routine, but it is known that Newton's root-finding algorithm may fail to find roots in certain cases and fall into an infinite loop. In such a case, this is just a calculation routine made on the fly without any measures to escape from the infinite loop, so it should be used only when you can be sure that the root can be found with Newton's root-finding algorithm like this problem.

Of course, the following programs can also be downloaded and used from GitHub (https://github.com/kycgit/gsimm).

Program nroot.lisp

```
(defparameter *symbol-table*
(list
 '+ '(:s-type op :t-op + :arg-n 2)
 '- '(:s-type op :t-op - :arg-n 2)
 '* '(:s-type op :t-op * :arg-n 2)
 '+r '(:s-type op :t-op +r :arg-n 2)
 '-r '(:s-type op :t-op -r :arg-n 2)
 '*r '(:s-type op :t-op *r :arg-n 2)
 '× '(:s-type op :t-op * :arg-n 2)
 '/ '(:s-type op :t-op / :arg-n 2)
 '/r '(:s-type op :t-op /r :arg-n 2)
 '÷ '(:s-type op :t-op / :arg-n 2)
 '< '(:s-type op :t-op < :arg-n 2)
 '<= '(:s-type op :t-op <= :arg-n 2)
 '> '(:s-type op :t-op > :arg-n 2)
 '>= '(:s-type op :t-op >= :arg-n 2)
 '= '(:s-type op :t-op = :arg-n 2)
 '/= '(:s-type op :t-op /= :arg-n 2)
 '0- '(:s-type op :t-op 0- :arg-n 1)
 '1/ '(:s-type op :t-op 1/ :arg-n 1)
 '√ '(:s-type op :t-op sqrt :arg-n 1)
 'inDeg '(:s-type op :t-op deg :arg-n 1)
 'toDeg '(:s-type op :t-op radTodeg :arg-n 1)
 'e^ '(:s-type op :t-op exp :arg-n 1)
 '^ '(:s-type op :t-op expt :arg-n 2)
 'e^r '(:s-type op :t-op exp-r :arg-n 1)
 '^r '(:s-type op :t-op expt-r :arg-n 2)
 ':ra '(:s-type op :t-op rational-approx-r :arg-n 2)
 'ln '(:s-type op :t-op log :arg-n 1)
 'log '(:s-type op :t-op log :arg-n 2)
 'ln-r '(:s-type op :t-op log-r :arg-n 1)
 'log-r '(:s-type op :t-op log-r :arg-n 2)))

(defun nroot (F &optional dg)
(labels ((sign (x) (multiple-value-bind (a b c) (raw-approx-r x) a b c)))
  (let ((pct *CREAL-TOLERANCE*)
```

```
        (gx 0)
        eps)
    (if dg nil (setq dg *print-prec*))
    (setq *creal-tolerance* [ceiling (10 ^ dg log 2)])
    (setq eps [1/2 /r (10 ^ dg)])
    (loop while (not (= 0 (sign (funcall f gx))))
          do (setq gx [eps *r 2 /r (funcall(f gx -r eps) -r funcall(f gx +r eps)) *r
                   funcall(f gx) +r gx]))
    (setq *creal-tolerance* pct)
    gx)))

(defun nrootp (F &optional dg)
(labels ((sign (x) (multiple-value-bind (a b c) (raw-approx-r x) a b c)))
  (let ((pct *CREAL-TOLERANCE*)
        (gx 0)
        eps)
    (if dg nil (setq dg *print-prec*))
    (setq *creal-tolerance* [ceiling (10 ^ dg log 2)])
    (setq eps [1/2 /r (10 ^ dg)])
    (print (loop while (not (= 0 (sign (funcall f gx))))
                 doing (setq gx [eps *r 2 /r (funcall(f gx -r eps) -r funcall(f gx +r eps)) *r
                             funcall(f gx) +r gx])
                 collect gx))
    (setq *creal-tolerance* pct)
    gx)))
```

You can use this as follows. Execute common lisp (SBCL) in the directory where the downloaded program is located and enter the following commands.

```
CL-USER> (eval-when (:compile-toplevel :load-toplevel :execute)
  (ql:quickload :computable-reals)
  (use-package :computable-reals))
To load "computable-reals":
  Load 1 ASDF system:
    computable-reals
; Loading "computable-reals"

T
CL-USER> (load "shnv1-1.lisp")
To load "trivial-arguments":
  Load 1 ASDF system:
    trivial-arguments
; Loading "trivial-arguments"

T
CL-USER> (load "nroot.lisp")
T
CL-USER> (setq *print-prec* 100)
100 (7 bits, #x64, #o144, #b1100100)
CL-USER> (nroot (lambda (x) [x /r cos-r x -r 1/100000]))
+0.000009999999999500000000005416666665915277777895696924583245698306007967
64914388791406617643427328713...
CL-USER>
```

First, specify the total precision you want to get. The computable-reals package uses its own numerical specification called computable real, which allows infinite precision calculations, rather than the floating-point with fundamental errors. The value of the $*$print-prec$*$ variable represents the total number of digits, not just the number of significant digits. Also, when inputting, expressions using decimal points such as 0.23 cannot be used, and only integers or fractions can be entered. 0.23 should be entered as 23/100. This is to avoid the inherent error problem in the commonly used floating-point number representation specification. In the example above, 1/100000 was entered as v/c, which is equivalent to 0.00001. When using pi as an infinite precision value, the name provided by the package should be used. The computable-reals package, which implemented complete lazy evaluation in 1989 before Haskell, which is famous for its lazy evaluation method, even appeared, has been used for over 40 years without significant changes in usage and has been sufficiently verified for reliability, so it would be worth accepting some unfamiliarity.

The direction of the force felt by each star in a binary system can be considered symmetrical, so it is sufficient to calculate only one side. In reality, the velocities within a binary system are determined by the gravitational constant and mass in relation to the distance between the stars. However, this does not necessarily need to be accounted for in calculations The acceleration required for maintaining circular motion can be assumed to result from a cause other than gravity. Even in this scenario, the conservation of energy should not be violated by increasing kinetic energy when gravity is transmitted at a finite speed. This is essential for resolving the Laplace problem. If it can be shown that the Laplace problem is always solved for any arbitrary velocity and corresponding acceleration, then it has been solved for the specific case of gravity-based velocity and acceleration as well.

To perform specific numerical calculations, If represent the expanded form

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2 \left(1+\frac{\dot{r}}{c}\right)^2}\left(\frac{1}{1+\frac{\dot{r}}{c}}\left(1-\frac{v^2}{c^2}+\frac{\vec{a}\cdot\vec{r}}{c^2}\right)\frac{\vec{r}}{r}-\frac{1}{1+\frac{\dot{r}}{c}}\left(1-\frac{v^2}{c^2}+\frac{\vec{a}\cdot\vec{r}}{c^2}\right)\frac{\vec{v}}{c}-\frac{r\vec{a}}{c^2}\right)$$

of the previously calculated Feynman formula graphically, it is as shown below.

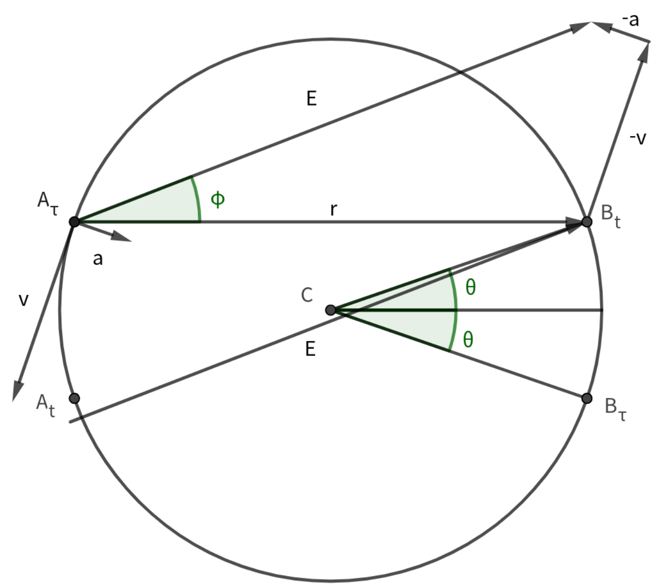

Figure 37: The direction of the force in a binary star system

Let me explain the direction of the force in the binary star system shown in the diagram and give the calculation result in advance. When one star is at Aτ and the other star is at Bτ in the binary star system, the field generated at Aτ affects the B star at Bt along the r path. The field is E, the sum of the r-direction component plus the component in the opposite direction of the velocity v at Aτ and the opposite direction of the acceleration 'a' at Aτ. When this field is felt at Bτ, it points slightly ahead of the center of the orbit or the current position of the A star which is the direction that slightly decelerates the B star.

If the direction of the force is the past of At, there is a component that accelerates star B even slightly, and energy conservation is not established, so the Laplace problem remains unsolved. And, the direction of the force cannot point exactly to the center of the orbit or the direction of At. As previously calculated, a trigonometric function, which is a transcendental function, appears in this angle calculation. If the direction of the center determined by the trigonometric function can be expressed as the r, v, and a of direction vectors of Feynman's formula, this is the result of expressing the transcendental function as an algebraic function. It is mathematically impossible.

The only remaining possibility is that if the direction of the force is slightly further into the future than At, star B will feel some resistance and its orbital energy will decrease. And then, there's another factor that can reduce orbital energy, namely, electromagnetic wave radiation or gravitational wave radiation. In this case, if the resistance component according to the direction change of the force is greater than the resistance due to the wave radiation, the Laplace problem still be unresolved. However, if the energy reduction due to the resistance component due to the change in the direction of the force is smaller than the amount lost to gravitational or electromagnetic wave radiation, the result will be one of two. Either the energy loss due to the force direction and the wave radiation will be combined, or the energy loss due to the force direction will be masked by the energy loss due to wave radiation. If the energy losses are combined, the law of conservation of energy will still be violated. Therefore, the only remaining possibility to maintain the conservation of energy law is if the wave radiation effect overshadows the friction effect.

The concept that a certain energy loss is masked by another larger loss has not yet appeared in physics. However, unless the result of the transcendental function can be expressed as an algebraic function, the force due to an electric field or gravity cannot precisely point toward the center of the circular orbit. Thus, it can only increase or decrease energy. If energy is increased, there is no way to establish the law of conservation of energy. Even when energy is decreased, if it exceeds the loss due to wave radiation resulting from accelerated motion, as calculated by the Larmor formula, it cannot be ignored. If energy simply disappears without any other effect, the law of energy conservation cannot be upheld. In the end, the only way to maintain the law of energy conservation, even if it may seem forced, is to assume that the energy consumed by resistance is less than the radiation energy according to the Larmor formula and that it is included in it. Therefore, a direct calculation is necessary to confirm whether the energy consumption due to the resistance component of the field is indeed less than the field's energy radiation due to acceleration.

In a circular orbit, v is the tangent vector of the circle at Aτ with an arbitrary velocity, and the acceleration a is a value dependent on v in the form of $\frac{v^2}{r}$, and its direction is from Aτ to the center of the circle. Therefore, knowing only v and r, we can calculate the force and its direction that arises from Aτ and affects Bt. To determine the direction, I will use the "The direction of the force in a binary star system" figure to find each component.

The magnetic field or the additional term of the previously discovered Lorentz force has a force that precisely points toward the center of the circle in a circular motion, so there is no resistance component. Therefore, it can be ignored. Besides, the force that maintains the binary system does not necessarily have to be the force between the two stars, it could be a physical string or something like a rail. So, there is no need to accurately describe the binary motion, and it should not be done. In reality, a perfect uniform circular motion cannot be sustained by itself due to energy radiation, such as electromagnetic or gravitational waves, and external energy must be added to maintain it. Therefore, it is entirely legitimate to assume an ideal situation and calculate only the necessary forces while omitting unnecessary details.

First of all, the following calculations are necessary in advance. It follows the figure of "The direction of the force in a binary star system", but the radius of the circular orbit will be called R to distinguish it from the r direction.

$$r = 2R\cos\theta$$

$$a = \frac{v^2}{R}$$

$$\dot{r} = -\vec{v}\cdot\frac{\vec{r}}{r} = v\sin\theta$$

$$\vec{a}\cdot\vec{r} = ar\cos\theta = \frac{v^2}{R}2R\cos\theta\cos\theta = 2v^2\cos^2\theta$$

The x and y-direction components are calculated through the force direction diagram in the binary system, respectively, as follows.

$$\begin{aligned}
E_x &= \tfrac{q}{4\pi\varepsilon_0 r^2\left(1+\frac{\dot{r}}{c}\right)^2}\left(\tfrac{1}{1+\frac{\dot{r}}{c}}\left(1-\tfrac{v^2}{c^2}+\tfrac{\vec{a}\cdot\vec{r}}{c^2}\right)\tfrac{r}{r}+\tfrac{1}{1+\frac{\dot{r}}{c}}\left(1-\tfrac{v^2}{c^2}+\tfrac{\vec{a}\cdot\vec{r}}{c^2}\right)\tfrac{v}{c}\sin\theta-\tfrac{ra}{c^2}\cos\theta\right)\\
&= \tfrac{q}{16\pi\varepsilon_0 R^2\cos^2\theta\left(1+\frac{v\sin\theta}{c}\right)^2}\left(\tfrac{1}{1+\frac{v\sin\theta}{c}}\left(1-\tfrac{v^2}{c^2}+\tfrac{2v^2\cos^2\theta}{c^2}\right)\left(1+\tfrac{v\sin\theta}{c}\right)-\tfrac{2R\cos\theta v^2}{c^2R}\cos\theta\right)\\
&= \tfrac{q}{16\pi\varepsilon_0 R^2\cos^2\theta\left(1+\frac{v\sin\theta}{c}\right)^2}\left(1-\tfrac{v^2}{c^2}\right)\\
E_y &= \tfrac{q}{4\pi\varepsilon_0 r^2\left(1+\frac{\dot{r}}{c}\right)^2}\left(\tfrac{1}{1+\frac{\dot{r}}{c}}\left(1-\tfrac{v^2}{c^2}+\tfrac{\vec{a}\cdot\vec{r}}{c^2}\right)\tfrac{v}{c}\cos\theta+\tfrac{ra}{c^2}\sin\theta\right)\\
&= \tfrac{q}{16\pi\varepsilon_0 R^2\cos^2\theta\left(1+\frac{v\sin\theta}{c}\right)^2}\left(\tfrac{1}{1+\frac{v\sin\theta}{c}}\left(1-\tfrac{v^2}{c^2}+\tfrac{2v^2\cos^2\theta}{c^2}\right)\tfrac{v}{c}\cos\theta+\tfrac{2R\cos\theta v^2}{c^2R}\sin\theta\right)\\
&= \tfrac{q}{16\pi\varepsilon_0 R^2\cos^2\theta\left(1+\frac{v\sin\theta}{c}\right)^3}\left(1-\tfrac{v^2}{c^2}+\tfrac{2v^2\cos^2\theta}{c^2}+\left(1+\tfrac{v\sin\theta}{c}\right)2\tfrac{v}{c}\sin\theta\right)\tfrac{v}{c}\cos\theta\\
&= \tfrac{q}{16\pi\varepsilon_0 R^2\cos^2\theta\left(1+\frac{v\sin\theta}{c}\right)^3}\left(1-\tfrac{v^2}{c^2}+\tfrac{2v^2\cos^2\theta}{c^2}+2\tfrac{v}{c}\sin\theta+\tfrac{2v^2\sin^2\theta}{c^2}\right)\tfrac{v}{c}\cos\theta\\
&= \tfrac{q}{16\pi\varepsilon_0 R^2\cos^2\theta\left(1+\frac{v\sin\theta}{c}\right)^3}\left(1+\tfrac{v^2}{c^2}+2\tfrac{v}{c}\sin\theta\right)\tfrac{v}{c}\cos\theta
\end{aligned}$$

Finally, the angle ϕ of the field vector at Bt can be obtained from

$$\tan\phi = \frac{E_y}{E_x} = \frac{\left(1+\frac{v^2}{c^2}+2\frac{v}{c}\sin\theta\right)\frac{v}{c}\cos\theta}{\left(1-\frac{v^2}{c^2}\right)\left(1+\frac{v\sin\theta}{c}\right)}$$

Now, i will find the angle of center, θ, and the angle of force, ϕ, of a binary system for v=9999c/10000, v=c/2, v=c/10, v=c/100, v=c/10000, and v=c/100000000, respectively.

First, if the angles θ from Bt to the center of the circle are obtained with the common-lisp program, it is

```
CL-USER> (setq *print-prec* 80)
80 (7 bits, #x50, #o120, #b1010000)
CL-USER> (loop for i in'(9999/10000 1/2 1/10 1/100 1/10000 1/100000000)
             collect (nroot (lambda (x) [x /r cos-r x -r i])))
(+0.73904096992784149842009277295660292864014156922029485014732271354005563251691328...
+0.45018361129487357303653869676268182732013650172305543401505849136364156698522338...
+0.09950534268738783481577354784178622555720388755521667951546674750794365368371520...
+0.00999950005415915395676999236483420598404766815205808975034755223694284492461535...
+0.00009999999950000000541666659152777895696922610284151349517116771524125383468111...
+0.00000000999999999999999995000000000000000541666666666666591527777777777895696925...)
CL-USER>
```

and the angles ϕ of the corresponding force are

$(67) - > digits(60)$

28

Type: PositiveInteger

(68) -> $\theta 1 := 0.739040969927841498420092772956602928640141569220294850147323;$

$$\operatorname{atan}\left(eval\left(\frac{\left(1+\frac{v^2}{c^2}+2\frac{v}{c}\,\mathsf{sin}(\theta)\right)\frac{v}{c}\,\mathsf{cos}(\theta)}{\left(1+\frac{v\,\mathsf{sin}(\theta)}{c}\right)\left(1-\frac{v^2}{c^2}\right)}, [c=1, v=9999/10000, \theta=\theta 1]\right)\right)$$

$1.5706610149\_8042923033\_7335987772\_8195857613\_5659890874\_027294838$

Type: Expression(Float)

(69) -> $\theta 2 := 0.450183611294873573036538696762681827320136501723055434015058;$

$$\operatorname{atan}\left(eval\left(\frac{\left(1+\frac{v^2}{c^2}+2\frac{v}{c}\,\mathsf{sin}(\theta)\right)\frac{v}{c}\,\mathsf{cos}(\theta)}{\left(1+\frac{v\,\mathsf{sin}(\theta)}{c}\right)\left(1-\frac{v^2}{c^2}\right)}, [c=1, v=1/2, \theta=\theta 2]\right)\right)$$

$0.6932110578\_6567610228\_0660508401\_7288222560\_3358879742\_4039714387$

Type: Expression(Float)

(70) -> $\theta 3 := 0.099505342687387834815773547841786225557203887555216679515466;$

$$\text{atan}\left(eval\left(\frac{\left(1+\frac{v^2}{c^2}+2\frac{v}{c}\,\text{sin}(\theta)\right)\frac{v}{c}\,\text{cos}(\theta)}{\left(1+\frac{v\,\text{sin}(\theta)}{c}\right)\left(1-\frac{v^2}{c^2}\right)}, [c=1, v=1/10, \theta=\theta 3]\right)\right)$$

$$0.1021376679\_7357788053\_3296234329\_8497973388\_4996878549\_2362462908$$

Type: Expression(Float)

(71) -> $\theta 4 := 0.009999500054159153956769992364834205984047668152058089750347;$

$$\text{atan}\left(eval\left(\frac{\left(1+\frac{v^2}{c^2}+2\frac{v}{c}\,\text{sin}(\theta)\right)\frac{v}{c}\,\text{cos}(\theta)}{\left(1+\frac{v\,\text{sin}(\theta)}{c}\right)\left(1-\frac{v^2}{c^2}\right)}, [c=1, v=1/100, \theta=\theta 4]\right)\right)$$

$$0.0100021663\_7419238476\_3529503499\_3910446092\_0418312358\_8443285399\_7$$

Type: Expression(Float)

(72) -> $\theta 5 := 0.000099999999950000000054166665915277789569692261028415134951 71;$

$$\text{atan}\left(eval\left(\frac{\left(1+\frac{v^2}{c^2}+2\frac{v}{c}\,\text{sin}(\theta)\right)\frac{v}{c}\,\text{cos}(\theta)}{\left(1+\frac{v\,\text{sin}(\theta)}{c}\right)\left(1-\frac{v^2}{c^2}\right)}, [c=1, v=1/10000, \theta=\theta 5]\right)\right)$$

$$0.0001000000\_0216666663\_7416666923\_9087240474\_4561638135\_6663689546\_388$$

Type: Expression(Float)

(73) -> $\theta 6 := 0.000000009999999999999999500000000000000054166666666666659153;$

$$\text{atan}\left(eval\left(\frac{\left(1+\frac{v^2}{c^2}+2\frac{v}{c}\,\text{sin}(\theta)\right)\frac{v}{c}\,\text{cos}(\theta)}{\left(1+\frac{v\,\text{sin}(\theta)}{c}\right)\left(1-\frac{v^2}{c^2}\right)}, [c=1, v=1/100000000, \theta=\theta 6]\right)\right)$$

$$0.1000000000\_0000002166\_6666666666\_6637416666\_6666666692\_3908730159E-7$$

Type: Expression(Float)

It can be seen that at all speeds v, the angle of the direction of the force is always greater than the angle θ at which the orbit passed the center, that is, the force felt by B is always in the direction of the resistance force consuming the energy of the orbit, not in the direction of increasing energy, which was first raised in the Laplace problem. Furthermore, it can be seen that it includes the resistive force increases when the speed is extremely close to the speed of light and becomes insignificant when it is close to the stationary state, showing the expected result that the total force is almost approximated to the central force. I will now compare the energy lost through this resistance with the energy lost through the radiation of waves.

3.6 Orbital energy loss and wave radiation

Previously, in order to find the direction of the force, the intensity of each xy component of the field with the moving Aτ as the source was obtained. This computes the full strength and direction of the field in the xy coordinate system.

$$\begin{aligned}\vec{E} &= E_x\hat{x} + E_y\hat{y} \\ &= \frac{q}{16\pi\varepsilon_0 R^2 \cos^2\theta\left(1+\frac{v\sin\theta}{c}\right)^2}\left(\left(1-\frac{v^2}{c^2}\right)\hat{x} + \left(\frac{1}{1+\frac{v\sin\theta}{c}}\left(1+\frac{v^2}{c^2}+2\frac{v}{c}\sin\theta\right)\frac{v}{c}\cos\theta\right)\hat{y}\right)\end{aligned}$$

This field can be represented by a component directed towards the center and the vertical component to it, as well. The direction of the vertical component is the resistive force in the direction opposite to the moving direction of B at Bτ.

$$\vec{E} = E_R\hat{R} + E_\theta\hat{\theta} = |\vec{E}|\cos(\varphi-\theta)\hat{R} + |\vec{E}|\sin(\varphi-\theta)\hat{\theta}$$

From this, the magnitude of the resistance component in the orbital motion of star B is

$$|\vec{E}|\sin(\varphi-\theta) = \frac{q\sin(\varphi-\theta)}{16\pi\varepsilon_0 R^2\cos^2\theta\left(1+\frac{v\sin\theta}{c}\right)^2}\sqrt{\left(1-\frac{v^2}{c^2}\right)^2 + \left(\frac{1}{1+\frac{v\sin\theta}{c}}\left(1+\frac{v^2}{c^2}+2\frac{v}{c}\sin\theta\right)\frac{v}{c}\cos\theta\right)^2}$$

As this will be applied to the binary star system problem, we can assume that the magnitudes of the velocities v of both stars A and B are equal, and the charges q are also identical. Therefore, it is sufficient to multiply this by q and v once more to calculate the energy loss at each moment.

$$\frac{q^2 v\sin(\varphi-\theta)}{16\pi\varepsilon_0 R^2\cos^2\theta\left(1+\frac{v\sin\theta}{c}\right)^2}\sqrt{\left(1-\frac{v^2}{c^2}\right)^2 + \left(\frac{1}{1+\frac{v\sin\theta}{c}}\left(1+\frac{v^2}{c^2}+2\frac{v}{c}\sin\theta\right)\frac{v}{c}\cos\theta\right)^2}$$

I will compare this with the Lamor formula

$$\frac{a^2q^2\gamma^6}{6c^3\pi\varepsilon_0}\left(1-\frac{v^2}{c^2}\sin^2\theta\right) = \frac{a^2q^2\gamma^6}{6c^3\pi\varepsilon_0}\left(1-\frac{v^2}{c^2}(\hat{a}\times\hat{v})^2\right) = \frac{a^2q^2\gamma^6}{6c^3\pi\varepsilon_0}\left(1-\frac{v^2}{c^2}\right) = \frac{a^2q^2\gamma^4}{6c^3\pi\varepsilon_0}$$

for circular motion. However, my goal is not to obtain exact values for the electromagnetic force; rather, I'm interested in comparing the magnitudes of the two values to determine whether the resistance due to the damping term in the electromagnetic or gravitational force, which is caused by the resistance term in the retarded potential theory, will be overshadowed by wave radiation even in the most extreme case of the binary star system. Therefore, I will eliminate as many common elements as possible from both equations and focus solely on comparing their magnitudes.

$$\frac{q^2 v \sin(\varphi-\theta)}{16\pi\varepsilon_0 R^2 \cos^2\theta\left(1+\frac{v\sin\theta}{c}\right)^2}\sqrt{\left(1-\frac{v^2}{c^2}\right)^2+\left(\frac{1}{1+\frac{v\sin\theta}{c}}\left(1+\frac{v^2}{c^2}+2\frac{v}{c}\sin\theta\right)\frac{v}{c}\cos\theta\right)^2} \quad : \quad \frac{a^2q^2\gamma^4}{6c^3\pi\varepsilon_0}$$

$$\frac{v \sin(\varphi-\theta)}{16 R^2 \cos^2\theta\left(1+\frac{v\sin\theta}{c}\right)^2}\sqrt{\left(1-\frac{v^2}{c^2}\right)^2+\left(\frac{1}{1+\frac{v\sin\theta}{c}}\left(1+\frac{v^2}{c^2}+2\frac{v}{c}\sin\theta\right)\frac{v}{c}\cos\theta\right)^2} \quad : \quad \frac{\gamma^4 v^4}{6c^3 R^2}$$

$$\frac{\sin(\varphi-\theta)}{8 \cos^2\theta\left(1+\frac{v\sin\theta}{c}\right)^2}\sqrt{\left(1-\frac{v^2}{c^2}\right)^2+\left(\frac{1}{1+\frac{v\sin\theta}{c}}\left(1+\frac{v^2}{c^2}+2\frac{v}{c}\sin\theta\right)\frac{v}{c}\cos\theta\right)^2} \quad : \quad \frac{v^3}{3c^3\left(1-\frac{v^2}{c^2}\right)^2}$$

I will compare the two values at different speeds using the above formula. First, I will enter the expressions into the variables.

$(70) - > digits(60);$

D $\theta := atan\left(\frac{\left(1+\frac{v^2}{c^2}+2\frac{v}{c}\sin(\theta)\right)\frac{v}{c}\cos(\theta)}{\left(1+\frac{v\sin(\theta)}{c}\right)\left(1-\frac{v^2}{c^2}\right)}\right)-\theta;$

Ex : = 1 - $\frac{v^2}{c^2}$;

Ey := $\frac{\left(1+\frac{v^2}{c^2}+2\frac{v}{c}\sin(\theta)\right)\frac{v}{c}\cos(\theta)}{1+\frac{v\sin(\theta)}{c}}$;

Type: Expression(Integer)

I will calculate the energy loss rate by substituting the actual values into the previously entered formulas.

- θi are the values for vi obtained with the lisp program above.
- fr is the orbital kinetic energy loss rate due to the resistive force, which is the force excluding the central force among the forces calculated by the equation of the retarded field.
- rr is the rate of energy loss due to electromagnetic/gravitational wave radiation calculated by the Larmor formula.

Finally, fr/rr is calculated to compare the magnitudes of the two loss rates. As this value does not exceed 1, it is shown that the wave radiation term covers the resistance term according to the retarded potential theory.

$(71) - > vi := 0.9999;$

$\theta i :=$ 0.739040969927841498420092772956602928640141569220294850147323;

fr := eval $\left(\frac{\sin(D\theta)}{8\cos(\theta)^2\left(1+\frac{v}{c}\sin(\theta)\right)^2}\sqrt{Ex^2+Ey^2}, [c=1, v=vi, \theta=\theta i] \right);$

rr := eval $\left(\frac{v^3}{3c^3\left(1-\frac{v^2}{c^2}\right)^2}, [c=1, v=vi] \right);$

print (fr), print (rr), print (fr / rr)

0.0892399649_9479155013_2068787228_8956260953_4052218224_9655564873_8
8331666.7291687500_5208333320_3111978190_0390584307_4542854622_508
0.1071093790_6621598234_4131331874_3801346668_3445450804_4894916308 E -7

LISP output:
( )

Type: Tuple(Void)

$(72) - > vi := 0.5;$

$\theta i :=$ 0.4501836112948735730365386967626818273201365017230554340150583;

fr := eval $\left(\frac{\sin(D\theta)}{8\cos(\theta)^2\left(1+\frac{v}{c}\sin(\theta)\right)^2}\sqrt{Ex^2+Ey^2}, [c=1, v=vi, \theta=\theta i] \right);$

rr := eval $\left(\frac{v^3}{3c^3\left(1-\frac{v^2}{c^2}\right)^2}, [c=1, v=vi] \right);$

print (fr), print (rr), print (fr / rr)

0.0244051180_7920067373_6001260480_2983665554_7662164751_6962118644_2
0.0740740740_7407407407_4074074074_0740740740_7407407407_4074074074
0.3294690940_6920909543_6017016484_0279484989_3439224147_8988601697

LISP output:
( )

Type: Tuple(Void)

$(73) - > vi := 0.1;$

$\theta i :=$ 0.099505342687387834815773547841786225557203887555216679515466;

fr := eval $\left(\frac{\sin(D\theta)}{8\cos(\theta)^2\left(1+\frac{v}{c}\sin(\theta)\right)^2}\sqrt{Ex^2+Ey^2}, [c=1, v=vi, \theta=\theta i] \right);$

rr := eval $\left(\frac{v^3}{3c^3\left(1-\frac{v^2}{c^2}\right)^2}, [c=1, v=vi] \right);$

print (fr), print (rr), print (fr / rr)

0.0003242459_9695416441_8468171497_7894890778_8117225895_4766710373_687
0.0003401013_5020236030_3370404380_5053906064_0070741080_8420909431_011
0.9533805048_4432963962_1964654950_4347356940_1079300470_0558511753

LISP output:
( )

Type: Tuple(Void)

$(74) - > vi := 0.01;$

$\theta i := 0.009999500054159153956769992364834205984047668152058089750347;$

fr := eval $\left(\frac{\sin(D\theta)}{8\cos(\theta)^2 \left(1+\frac{v}{c}\sin(\theta)\right)^2} \sqrt{Ex^2+Ey^2}, [c=1, v=vi, \theta=\theta i] \right);$

rr := eval $\left(\frac{v^3}{3c^3\left(1-\frac{v^2}{c^2}\right)^2}, [c=1, v=vi] \right);$

print (fr), print (rr), print (fr / rr)

0.3332400252_3155156473_6493549399_3941077973_1737035689_8370085164 E -6
0.3334000100_0133350002_0002333600_0300033337_0004000433_3800050005 E -6
0.9995201416_7671652021_7185694163_3491689090_9695456800_2081940393

LISP output:
( )

Type: Tuple(Void)

$(75) - > vi := 0.0001;$

$\theta i := 0.000099999999500000000541666659152777895696922610284151349517 1;$

fr := eval $\left(\frac{\sin(D\theta)}{8\cos(\theta)^2 \left(1+\frac{v}{c}\sin(\theta)\right)^2} \sqrt{Ex^2+Ey^2}, [c=1, v=vi, \theta=\theta i] \right);$

rr := eval $\left(\frac{v^3}{3c^3\left(1-\frac{v^2}{c^2}\right)^2}, [c=1, v=vi] \right);$

print (fr), print (rr), print (fr / rr)

0.3333333240_0000025238_0945835626_2676257775_2266045256_9138108882 E -12
0.3333333400_0000010000_0001333333_3500000002_0000000233_333336 E -12
0.9999999520_0000141714_2819564022_1284540402_6112258663_5520429297

LISP output:
( )

Type: Tuple(Void)

$(76) - > vi := 0.00000001;$

$\theta i := 0.000000009999999999999999500000000000000054166666666666659153;$

fr := eval $\left(\frac{\sin(D\theta)}{8\cos(\theta)^2 \left(1+\frac{v}{c}\sin(\theta)\right)^2} \sqrt{Ex^2+Ey^2}, [c=1, v=vi, \theta=\theta i] \right);$

rr := eval $\left(\frac{v^3}{3c^3\left(1-\frac{v^2}{c^2}\right)^2}, [c=1, v=vi] \right);$

print (fr), print (rr), print (fr / rr)

0.3333333333_3333324000_0000000000_0252380674_6031781694_164308759 E -24
0.3333333333_3333340000_0000000000_0100000000_0000000133_3333333333 E -24
0.9999999999_9999952000_0000000000_1417142023_8095343288_208878658

LISP output:
( )

Type: Tuple(Void)

When the speed is high enough to be comparable to the speed of light, the loss due to wave radiation is overwhelmingly large. However, the difference starts to decrease rapidly, and the difference decreases to about the same unit around half the speed of light, and as the speed slows down, it can be seen that the two values become very similar from around 0.1c. However, the energy loss due to wave radiation is always slightly large.

This is a surprising result. It coincides too well to be an unrelated coincidence. To claim there is a connection, this is a comparison of the computational result of a transcendental function with that of an algebraic function, and the relation cannot be expressed by a finite polynomial. The calculation started with the expectation that the loss due to wave radiation would cover the loss term due to resistivity, but I couldn't expect that it would cover this very slightly. When the speed is very slow, it is practically the same. I think this is an example that shows that nature is never completely mathematical, but it is as mathematical as possible.

According to this result, the energy emission rate of gravitational waves based on general relativity, which was used to calculate the gravitational wave emission rate in the binary neutron star system PSR B1913+16 introduced earlier, is $\frac{64G^4m^5}{5c^5r^5}$. It is excessively small compared to the energy emitted based on Maxwell's gravity, making it impossible to overshadow the losses calculated based on the concept of the retarded potential. Consequently, the Laplace problem cannot be resolved, and this is one of the important pieces of evidence that general relativity cannot be correct. If I consider whether there is any other way to solve the Laplace problem that satisfies special relativity, which is completely different from the method presented in this book derived from the concept of retarded potential, I cannot even imagine it. I cannot think of any reason why nature should prepare different cumbersome mathematics for each force. Forces should arise from the nature of spacetime, and therefore it is logical that similar mathematical methods should be shared for similar problems, even if they are different types of forces.

Even if each force has its own mathematics, and we make the unreasonable assumption that general relativity has a small resistance component of the force that can be masked by the small gravitational wave emissivity in a different mathematical way, the difference between the angles θ and ϕ becomes slightly smaller than that of the electromagnetic force. This means that the direction of gravity is more accurately directed toward At, leading to the absurd conclusion that the position of an object with an accelerating charge and mass, measured by electrostatic force and by gravity, is different. There is no reason for such an absurd phenomenon to exist. After all, in order for general relativity to be correct, at least two levels of unnecessary complexity beyond imagination are required. I don't believe nature provides for such superfluity. Since general relativity does not fit the concept of retarded potential, one of the two must be wrong, and there is no possibility that the theory of retarded potential is wrong. Therefore, I consider general relativity to be wrong.

Conversely, it is possible to question whether the gravitational wave emission rate is too large. If the emission rate is large, the question may be why detecting gravitational waves is so difficult. Actually, I'm not sure about the cause. I haven't designed or attempted a gravitational wave detection experiment myself or thoroughly reviewed the work of others. However, I can suggest a few basic factors that make detecting gravitational waves difficult.

The first challenge arises from the fundamental principle that gravitational waves affect all matter equally, rendering single-location detection impossible. To measure gravitational waves, enormous structures tuned to their wavelengths are needed, capable of detecting subtle changes in acceleration relative to distance. Furthermore, gravitational waves constitute signals that are orders of magnitude smaller, typically around 10^{-30} to 10^{-40} times weaker, than electromagnetic waves, making their detection exceptionally challenging.

The second is that if we consider an event such as a neutron star collision as a gravitational wave source that can be detected in cosmological phenomena, the difference between the Schwarzschild radius and the radius of a neutron star near the mass of the Sun is only a few tens of magnitude, so at that scale, the difference in strength between general relativity-based gravitational waves and Maxwellian gravity-based gravitational waves is only a few tens of magnitude. At that level, it is difficult to say that gravitational waves based on Maxwellian gravity are clearly easier to detect.

Third, the universe is not transparent to gravitational waves because it is composed of matter with a single polarity that is not neutral with respect to gravity. All stars, planets, and interstellar matter will absorb and scatter gravitational waves, so a gravitational wave signal will quickly become noisy and difficult to interpret as evidence of gravitational waves, even if it is slightly away from its source.

Considering these things, I suspect that detecting gravitational waves will become more difficult than general relativity-based theories, but I'll leave the exact details to future research.

The idea that I am now proposing that the loss of orbital energy due to resistance due to the finite speed of propagation of the field is masked by another, larger term, wave radiation, is hardly a mathematical idea. It is just a realistic, substantive physical idea. And the calculations show a strange, slightly off, but realistic agreement that is sufficient to prove it. This kind of idea is not actually new to physics. I see quantum theory as a product of a similar way of thinking. Quantum theory is a theory that can never come out simply by mathematically extending the concept of classical physics. I think there is already a precedent for a theory that complements the limitations of classical theory and mathematics with realistic thinking.

And of course, this strange coincidence and slight discrepancy may essentially indicate the imperfections of mathematics, but it may also be telling us about the incompleteness of physical theories. It may be possible that there exists a theory that could fill in this tiny gap. However, there will always be even finer gaps, and ultimately they will never be completely filled.

The case of a circular orbital binary system of the same mass is the largest conceivable case of the resistance component of the field according to the retarded potential theory, and in other cases, the difference between the loss due to wave radiation and the resistance component of the field will be larger. In cases where the mass difference is similar to that of the Earth and the Sun, the effect of the Laplace problem is so negligible that Laplace gave up trying to handle it directly and instead focused on the orbit of the Moon. However, the Earth is also in an accelerated motion and is emitting gravitational waves accordingly. By calculating and appreciating it, I try to conclude the Laplace problem.

Since the Earth is also close to a circular orbit, it is possible to use the Larmor formula assuming a constant acceleration in a circular orbit. But, we have to use η instead of ε. Also, since the output is always a positive number, we have to change the sign as well. Then the formula is

$$\frac{-a^2m^2\gamma^4}{6c^3\pi\eta} = \frac{a^2m^2\gamma^4}{6c^3\pi\frac{1}{4\pi G}} = \frac{2a^2m^2G\gamma^4}{3c^3} = \frac{2v^4m^2G}{3c^3r^2\left(1-\frac{v^2}{c^2}\right)^2}$$

Substituting the speed of light $c = 299792458m/s$, the gravitational constant $G = 6.67430e-11m^3Kg^{-1}s^{-2}$, the Earth's average orbital speed 29780m/s, the Earth's average orbital radius 149597870700m, and the Earth's mass $5.9736e24Kg$,

(77) -> `PL:=[c=299792458,G=6.67430e-11,v=29780,r=149597870700,m=5.9736e24]`

$[c = 2\_99792458.0, G = 0.66743E-10, v = 29780.0, r = 1495\_97870700.0,$
$m = 5973600\_0000000000\_00000000.0]$

Type: List(Equation(Polynomial(Float)))

(78) -> eval $\left(\frac{2v^4m^2G}{3c^3r^2\left(1-\frac{v^2}{c^2}\right)^2}, PL\right)$

$2070964195.8712177347\_5882254230\_8379199191\_3710703980\_8297222483$

Type: Fraction(Polynomial(Float))

It has a large power output of about 2 gigawatts. However, on a global scale, this is much smaller than the power consumption of a single metropolitan city, Seoul.

(79) -> Te := eval $\left(\frac{1}{2}mv^2, [m = \texttt{5.9736e24}, \texttt{v=29780}]\right)$

$2648\_8388011200\_0000000000\_0000000000.0$

Type: Polynomial(Float)

(80) -> Te / 2070964195.87 / 60 / 60 / 24 / 365 / 1.0 e 12

$40557.9814221530\_4209212684\_7023449879\_1656668045\_4927328379\_01691$

Type: Polynomial(Float)

At this emission rate, it takes about 40 quadrillion years for the Earth to lose all of its kinetic energy. So we don't have to worry about the Earth falling into the Sun due to orbital energy loss due to gravitational wave emission. In reality, there are factors that lose and gain much larger orbital energy, and irrespective of whether or not orbital energy is lost, after several billion years, which is only an instant compared to a few tens quadrillion years, it is said that the Sun will become a red giant in several billion years, expanding beyond Earth's orbit and eventually engulfing it.

3.7 Summary

Having shown that Maxwellian gravitation with some modifications to the Lorentz force, can solve a number of problems, I checked whether Maxwellian gravity solves the Laplace problem, the problem of orbital stabilization by a force transmitted at the speed of light, which has long been a subject of my curiosity in addition to the particle density function, the structure of the universe.

In this process, it was confirmed that the Heaviside-Feynman formula

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\left[\frac{e_{r'}}{r'^2} + \frac{r^{'}}{c}\frac{d}{dt}\left(\frac{e_{r'}}{r'^2}\right) + \frac{1}{c^2}\frac{d^2}{dt^2}e_{r'}\right]$$

for the field due to moving point charges and the Griffiths formula

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{r}{(\vec{r}\cdot\vec{u})^3}((c^2 - v^2)\vec{u} + \vec{r}\times(\vec{u}\times\vec{a}))$$

for the field due to moving point charges are the same formulas that can be easily converted into

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2 \left(1+\frac{\dot{r}}{c}\right)^3}\left(\left(1-\frac{v^2}{c^2}+\frac{\vec{a}\cdot\vec{r}}{c^2}\right)\frac{\vec{r}}{r}-\left(1-\frac{v^2}{c^2}+\frac{\vec{a}\cdot\vec{r}}{c^2}\right)\frac{\vec{v}}{c}-\left(1+\frac{\dot{r}}{c}\right)\frac{r\vec{a}}{c^2}\right)$$

form, and it was confirmed that all these formulas are the same as Purcell's relativistic electric field formula

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r_p^2}\frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{3/2}}\hat{r}_p$$

due to point charges moving at a constant velocity.

Based on this, a solution that satisfies the condition that a force transmitted at a finite speed of light can maintain an energy-conserving orbital motion is that, in the case of a binary system, the direction in which the field from the moving source of one star acts on the opposite star is slightly future than the current position of the source star, and the direction of the force is slightly resistive as seen from the side receiving the force, but the magnitude of the energy loss due to the resistive force is less than the energy loss due to the gravitational wave emission, I conjectured that the Laplace problem could be solved by masking and integrating the effect of the resistive force into the effect of the gravitational wave emission. The case of zero resistive force, the case where the force is directed exactly at the opposite star, as in the case of linear motion, was ruled out at the conjecture stage. Since the function that determines the position of the opposite star in the motion of a binary system involves a trigonometric function, a transcendental function, it was clear from the beginning that it could not be expressed exactly as an algebraic function that determines the direction of the field. I tried to verify this conjecture through specific calculations.

The result was as expected for the direction speculation and the assumption that the loss due to radiation of the wave would be greater than the loss due to resistance, but for the magnitude itself, quite surprisingly, in the case of a binary system with the same mass and orbiting a circle, it was confirmed that the loss due to wave radiation and the loss due to resistance due to the finite speed of the field converged as the orbital speed slowed down. This means that general relativity, which emits less gravitational wave energy compared to Maxwellian gravity, cannot coexist with the theory of retarded potential. This unexpected result serves as additional evidence against general relativity, adding to the various refutations of general relativity discussed in this book. Of course, equal-mass circular orbiting binary systems would have the smallest gravitational wave emission and the largest drag due to force deflection. In cases with different mass ratios or non-circular orbits, as well as orbits deviating from perfect circles, the difference between gravitational wave emissions and resistance force losses will increase. But a correct theory should not break down in the worst case. Maxwellian gravity is progressively consistent with its limits, but general relativity is not consistent with its limits but breaks down. This seems to be the most conclusive evidence against general relativity of the various refutations of general relativity discussed in this book.

During this process, I examined intriguing hints about the connection between mathematics and physics found in the derivation process of Feynman's formula. Additionally, as the preparation process, I attempted to provide explanations for the Liénard-Wiechert potential and the Lamor formula that I believe are clearer and more up-to-date than the old explanations that predate the theory of relativity, which are still included in electromagnetics textbooks.

The derivation of expression

$$\vec{B} = \frac{\hat{r}}{c} \times \vec{E}$$

for the magnetic field due to a moving point charge was not directly related to the Laplace problem of planetary motion and its conservation of energy, but it was done in addition to deal with an oddity that I discovered in the derivation of Feynman's formula that might have something to do with the limits of mathematics as applied to physics.

The discovery that at the very bottom of the process of formulating and describing force, there was a mathematical incompleteness that Feynman missed or passed undisclosed was a meaningful experience in many ways, and I think it gives us something to think about in many ways about the relationship between physics and mathematics.

I think this part is an important one that needs to be revisited from time to time in the process of discovering future physics theories.