Relativistic Consistency of Electromagnetic force

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Abstract

According to special relativity and electromagnetism, the electromagnetic force, also known as the Lorentz force $\vec{F} = q\,(\vec{E} + \vec{v} \times \vec{B})$, was presumed to take the same form in every inertial frame. However, actual theoretical validation has not been conducted until now. Hence, I undertook this task independently, introducing a new form of the Heaviside-Feynman formula.

$$\vec{E} = \frac{q}{4\pi\varepsilon_0\,r^2\,(1+\frac{\dot{r}}{c})^3}\left(\left(1 - \frac{v^2}{c^2} + \frac{\vec{a}\cdot\vec{r}}{c^2}\right)\left(\hat{r} - \frac{\vec{v}}{c}\right) - \left(1 + \frac{\dot{r}}{c}\right)\frac{r\,\vec{a}}{c^2}\right)$$

1 Introduction

According to special relativity and electromagnetism, the electromagnetic force, also known as the Lorentz force $\vec{F} = q\,(\vec{E} + \vec{v} \times \vec{B})$, was presumed to take the same form in every inertial frame. However, actual theoretical validation has not been conducted until now. Hence, I undertook this task independently.

2 Numerical computation

This calculation aims to demonstrate the consistent expression of the electromagnetic field's effect from one particle, Qa, on another particle, Qb, in all inertial systems. Three inertial systems are selected for this purpose. Firstly, there is the stationary inertial frame, representing the observer's inertial frame and expressed by physical quantities with the subscript r. Next is the v inertial system, commonly denoted by physical quantities with the subscript v. The description of this inertial system is a bit intricate. Initially, v is the velocity of Qa relative to the stationary inertial system at the moment the electromagnetic field affecting Qb leaves Qa. When Qa proceeds without any change in velocity, the position of Qa becomes the origin of the v inertial coordinate system, and the origin of the stationary inertial coordinate system is also chosen to match there. Finally, there's the u inertial system with the subscript u, representing Qb's inertial system. In each of these three inertial systems, the velocity of Qa is expressed differently, and the resulting electromagnetic field is different, leading to different accelerations received by Qb. The acceleration of Qb in each inertial frame can be converted to the acceleration observed in other inertial frames through acceleration conversion formulas. Ultimately, this calculation proves the relativistic consistency of the electromagnetic force by showing that the acceleration converted from other inertial systems always matches the acceleration due to the electromagnetic field observed in that inertial system.

Before delving into the computation, first, it needs the basic knowledge of reletividtic force-acceleration relationship.

$$\begin{aligned}
\frac{d\vec{p}}{dt} &= \gamma^3\,m\,\frac{\vec{a}\cdot\vec{v}}{c^2}\,\vec{v} + \gamma\,m\,\vec{a} \\
&= q\,(\vec{E} + \vec{v} \times \vec{B}) \\
\vec{a} &= \frac{q}{\gamma\,m}\left(\vec{E} + \vec{v} \times \vec{B} - \frac{1}{c^2}\,\vec{v}\,(\vec{v}\cdot\vec{E})\right)
\end{aligned}$$

And, it requires a new practical form of the Heaviside-Feynman formula that I derived. The equation $\vec{E}=\frac{q}{4\pi\varepsilon_0}\left[\frac{e_r'}{r'^2}+\frac{r'}{c}\frac{d}{\mathrm{dt}}\left(\frac{e_r'}{r'^2}\right)+\frac{1}{c^2}\frac{d^2}{\mathrm{dt}^2}e_r'\right]$ is from Feynman's lecture book. And, I derived

$$\vec{E}=\frac{q}{4\pi\varepsilon_0 r^2\left(1+\frac{\dot{r}}{c}\right)^3}\left(\left(1-\frac{v^2}{c^2}+\frac{\vec{a}\cdot\vec{r}}{c^2}\right)\left(\frac{\vec{r}}{r}-\frac{\vec{v}}{c}\right)-\left(1+\frac{\dot{r}}{c}\right)\frac{r\,\vec{a}}{c^2}\right)$$

with the following rules. $\tau=t-\frac{r}{c}$, $\vec{r}'=\vec{r}$, $\dot{r}'=\frac{dr}{dt}$, $\dot{r}=\frac{dr}{d\tau}$, $\vec{v}'=\frac{-d\vec{r}}{dt}$, and $\vec{v}=\frac{-d\vec{r}}{d\tau}$. The negative sign when defining $\vec{v}$ is Feynman's convention for matching the directions of the distance vector and the electric field vector. And, of course, $\vec{B}=\frac{\hat{r}}{c}\times\vec{E}$. I employed τ for non-primed physical quantities because the t is already allocated for primed physical quantities by Feynman. I found that non-primed physical quantities share the same properties as ordinary physical quantities.

I performed the calculation using a Computer Algebra System (CAS) called friCAS. Symbolic calculations were so extensive that they were impractical even with my computer, so numerical computations were employed. The following are inputs and comments for friCAS.

(1) -> $\mathrm{cX}(a,b)==\mathrm{vector}[a.2\,b.3-a.3\,b.2,\,a.3\,b.1-a.1\,b.3,\,a.1\,b.2-a.2\,b.1]$

Type: Void

(2) -> $\mathrm{dX}(a,b)==a.1\,b.1+a.2\,b.2+a.3\,b.3$

Type: Void

(3) -> $\mathrm{sq}(v)==\mathrm{dX}(v,v)$

Type: Void

(4) -> $\mathrm{Gm}(v,c)==\frac{1}{\sqrt{1-\frac{\mathrm{dX}(v,v)}{c^2}}}$

Type: Void

$\mathrm{cX}(\vec{a},\vec{b})=\vec{a}\times\vec{b}$, $\mathrm{dX}(\vec{a},\vec{b})=\vec{a}\cdot\vec{b}$, $\mathrm{Gm}(\vec{v},c)=\gamma_v$

(5) -> $E(r,v,a,c)==$

$$\frac{1-\frac{\mathrm{sq}(v)}{c^2}+\frac{\mathrm{dX}(a,r)}{c^2}}{\mathrm{sq}(r)\left(1-\frac{\mathrm{dX}(v,r)}{c\sqrt{\mathrm{sq}(r)}}\right)^3}\left(\frac{1}{\sqrt{\mathrm{sq}(r)}}r-\frac{1}{c}v\right)-\frac{1}{\mathrm{sq}(r)\left(1-\frac{\mathrm{dX}(v,r)}{c\sqrt{\mathrm{sq}(r)}}\right)^2}\frac{\sqrt{\mathrm{sq}(r)}}{c^2}a$$

Type: Void

This is the practical form of the Heaviside-Feynman formula for the electric field. $\vec{E}=\frac{1}{r^2\left(1+\frac{\dot{r}}{c}\right)^3}\left(\left(1-\frac{v^2}{c^2}+\frac{\vec{a}\cdot\vec{r}}{c^2}\right)\left(\frac{\vec{r}}{r}-\frac{\vec{v}}{c}\right)-\left(1+\frac{\dot{r}}{c}\right)\frac{r\,\vec{a}}{c^2}\right)$, $\dot{r}=-\hat{r}\cdot\vec{v}$

(6) -> $U(v,d,c)==\mathrm{if}\,(v=[0,0,0])\,\mathrm{then}\,d\,\mathrm{else}$

$$\frac{1}{1+\frac{\mathrm{dX}(d,v)}{c^2}}\left(\frac{1}{\mathrm{Gm}(v,c)}d+v+\left(1-\frac{1}{\mathrm{Gm}(v,c)}\right)\frac{\mathrm{dX}(d,v)}{\mathrm{sq}(v)}v\right)$$

Type: Void

This is the relativistic velocity sum function. $\vec{u}=\vec{v}\oplus\vec{d}=\frac{1}{1+\frac{\vec{v}\cdot\vec{d}}{c^2}}\left(\frac{\vec{d}}{\gamma}+\vec{v}+\frac{\gamma-1}{\gamma}(\vec{d}\cdot\hat{v})\,\hat{v}\right)$

(7) -> $\mathrm{rr}(d,r,c)==\mathrm{if}\,(d=[0,0,0])\,\mathrm{then}\,r$

$$\mathrm{else}\,r-\left(1-\frac{1}{\mathrm{Gm}(d,c)}\right)\frac{\mathrm{dX}(d,r)}{\mathrm{dX}(d,d)}d$$

Type: Void

This is the relativistic length contraction formula in three dimensions. $\vec{r}=\vec{r}'-\frac{\gamma-1}{\gamma}(\vec{r}'\cdot\hat{v})\,\hat{v}$

(8) -> $A(a,u,c)==\mathrm{if}\,(u=[0,0,0])\,\mathrm{then}\,a$

$$\mathrm{else}\,\frac{1}{\mathrm{Gm}(u,c)^2}\left(a+\left(\frac{1}{\mathrm{Gm}(u,c)}-1\right)\frac{\mathrm{dX}(a,u)}{\mathrm{sq}(u)}u\right)$$

Type: Void

(9) -> $\mathrm{rA}(a,v,c) == \mathrm{if}\,(v=[0,0,0])\ \mathrm{then}\ a\ \mathrm{else}\ \mathrm{Gm}(v,c)^2\left(a+(\mathrm{Gm}(v,c)-1)\frac{\mathrm{dX}(a,v)}{\mathrm{sq}(v)}v\right)$

Type: Void

These are well-known relativistic acceleration conversion formulas. When $\vec{u}=\vec{v}\oplus\vec{u}'$, $\vec{a}=\frac{1}{\gamma^2}\left(\vec{a}'+\frac{1-\gamma}{\gamma}\frac{\vec{a}'\cdot\vec{v}}{v^2}\vec{v}\right)$, $\vec{a}'=\gamma^2\left(\vec{a}+(\gamma-1)\frac{\vec{a}\cdot\vec{v}}{v^2}\vec{v}\right)$.

(10) -> $\mathrm{dA}(a,u,d,c) == \mathrm{if}\,(u=[0,0,0])\ \mathrm{then}\ a\ \mathrm{else}\ \frac{1}{\mathrm{Gm}(u,c)^2\left(1+\frac{\mathrm{dX}(u,d)}{c^2}\right)^3}\left(\left(1+\frac{\mathrm{dX}(u,d)}{c^2}\right)a+\frac{1-\mathrm{Gm}(u,c)}{\mathrm{Gm}(u,c)}\frac{\mathrm{dX}(a,u)}{\mathrm{sq}(u)}u-\frac{\mathrm{dX}(a,u)}{c^2}d\right)$

Type: Void

This is more general relativistic acceleration conversion formula. Both directions can share a function. $\vec{a}=\frac{1}{\gamma^2\left(1+\frac{\vec{v}\cdot\vec{u}'}{c^2}\right)^3}\left(\left(1+\frac{\vec{v}\cdot\vec{u}'}{c^2}\right)\vec{a}'-\frac{\vec{v}\cdot\vec{a}'}{c^2}\vec{u}'-\frac{\gamma-1}{\gamma}\frac{\vec{a}'\cdot\vec{v}}{v^2}\vec{v}\right)$, and $\vec{a}'=\frac{1}{\gamma^2\left(1-\frac{\vec{v}\cdot\vec{u}}{c^2}\right)^3}\left(\left(1-\frac{\vec{u}\cdot\vec{v}}{c^2}\right)\vec{a}+\frac{\vec{v}\cdot\vec{a}}{c^2}\vec{u}-\frac{\gamma-1}{\gamma}\frac{\vec{a}\cdot\vec{v}}{v^2}\vec{v}\right)$

(11) -> $\Lambda\mathrm{matrix} := \begin{pmatrix}\gamma & -\beta_x\gamma & -\beta_y\gamma & -\beta_z\gamma\\ -\beta_x\gamma & 1+\frac{(\gamma-1)\beta_x^2}{\beta^2} & \frac{(\gamma-1)\beta_x\beta_y}{\beta^2} & \frac{(\gamma-1)\beta_x\beta_z}{\beta^2}\\ -\beta_y\gamma & \frac{(\gamma-1)\beta_x\beta_y}{\beta^2} & 1+\frac{(\gamma-1)\beta_y^2}{\beta^2} & \frac{(\gamma-1)\beta_y\beta_z}{\beta^2}\\ -\beta_z\gamma & \frac{(\gamma-1)\beta_x\beta_z}{\beta^2} & \frac{(\gamma-1)\beta_y\beta_z}{\beta^2} & 1+\frac{(\gamma-1)\beta_z^2}{\beta^2}\end{pmatrix};$

Type: Matrix(Fraction(Polynomial(Integer)))

(12) -> $\Lambda M := \mathrm{eval}\left(\Lambda\mathrm{matrix},\left[\gamma=\frac{1}{\sqrt{1-\beta_x^2-\beta_y^2-\beta_z^2}},\beta=\sqrt{\beta_x^2+\beta_y^2+\beta_z^2}\right]\right);$

Type: SquareMatrix(4,Expression(Integer))

ΛM is the Lorentz transformation matrix in three dimensions.

(14) -> $\mathrm{digits}(200);$
$c := 2;$
$r := \mathrm{vector}[0.6, 1.0, -0.7];$
$d := \mathrm{vector}[0.2, 0.1, 0.4];$
$v := \mathrm{vector}[-0.5, 0.2, -0.1];$
$\mathrm{aqv} := \mathrm{vector}[0.2, 0.1, -2.3];$
$u := U(v,d,c);$

Type: Vector(Float)

The calculation precision was set to 200 digits. c is the speed of light, r and d are the position vector and velocity of Qb in the v inertial frame, v is the velocity of the v inertial frame based on the stationary inertial frame, and aqv is the acceleration of Qa when emitting the electromagnetic field currently acting on Qb based on the v inertial frame. All these parameters were assigned random values. $\vec{u}=\vec{v}\oplus\vec{d}$ is the calculated velocity of Qb based on the stationary inertial frame.

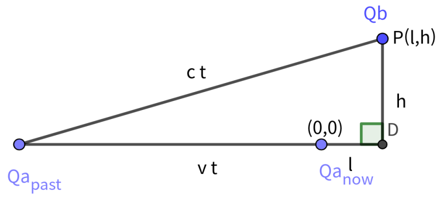
(15) -> $\Lambda v := \mathrm{eval}\left(\Lambda M,\left[\beta_x=\frac{v.1}{c},\beta_y=\frac{v.2}{c},\beta_z=\frac{v.3}{c}\right]\right);$
$\Lambda d := \mathrm{eval}\left(\Lambda M,\left[\beta_x=\frac{d.1}{c},\beta_y=\frac{d.2}{c},\beta_z=\frac{d.3}{c}\right]\right);$
$\Lambda u := \mathrm{eval}\left(\Lambda M,\left[\beta_x=\frac{u.1}{c},\beta_y=\frac{u.2}{c},\beta_z=\frac{u.3}{c}\right]\right);$
$i\Lambda u := \mathrm{eval}\left(\Lambda M,\left[\beta_x=-1\frac{u.1}{c},\beta_y=-1\frac{u.2}{c},\beta_z=-1\frac{u.3}{c}\right]\right);$
$\Lambda R := i\Lambda u\,\Lambda v\,\Lambda d;$
$r\Lambda R := \Lambda d\,\Lambda v\,i\Lambda u;$
$\mathrm{rM} := \begin{pmatrix}\Lambda R(2,2) & \Lambda R(2,3) & \Lambda R(2,4)\\ \Lambda R(3,2) & \Lambda R(3,3) & \Lambda R(3,4)\\ \Lambda R(4,2) & \Lambda R(4,3) & \Lambda R(4,4)\end{pmatrix};$
$\mathrm{rrM} := \begin{pmatrix}r\Lambda R(2,2) & r\Lambda R(2,3) & r\Lambda R(2,4)\\ r\Lambda R(3,2) & r\Lambda R(3,3) & r\Lambda R(3,4)\\ r\Lambda R(4,2) & r\Lambda R(4,3) & r\Lambda R(4,4)\end{pmatrix};$

Type: Matrix(Expression(Float))

The Wigner rotation must be considered. When $\vec{u}=\vec{v}\oplus\vec{d}$, the Lorentz transform matrix for $\vec{u}$ is Λu , for $\vec{v}$ is Λv , and for $\vec{d}$ is Λd. Thus the ratation matrix is $\Lambda R=\Lambda u^{-1}\Lambda v\,\Lambda d$, and the 3D rotation matrix rM was extracted from ΛR. The reverse rotation matrix is, $r\Lambda R=\Lambda d\,\Lambda v\,\Lambda u^{-1}$, and rrM was extracted from $r\Lambda R$.

(17) -> $p := \mathrm{rr}(v, r, c);$
$l := \frac{\mathrm{dX}(p, v)}{\mathrm{sq}(v)} v;$
$t := \frac{\mathrm{Gm}(v, c)^2}{c^2}\left(\mathrm{dX}(l, v) + \sqrt{\mathrm{dX}(l, v)^2 + c^2 \frac{\mathrm{sq}(p)}{\mathrm{Gm}(v, c)^2}}\right);$
$o := p + t\, v;$
$\mathrm{ro} := \frac{1}{\sqrt{\mathrm{sq}(o)}} o;$

Type: Vector(Float)



p is the position vector of Qb based on the stationary inertial frame. l is a vertical line from p to the v vector and is a value used to calculate the time t for the electromagnetic field to depart from Qa and arrive at Qb. o is the path vector through which the electromagnetic field departs from Qa and arrives at Qb. ro is the direction vector of o.

(18) -> $\mathrm{er} := E(o, v, A(\mathrm{aqv}, v, c), c);$
$\mathrm{ev} := E(r, \mathrm{vector}[0, 0, 0], \mathrm{aqv}, c);$
$\mathrm{tmp} := \Lambda u\, \mathrm{transpose}([\mathrm{cons}(c\, t, o)]);$
$\mathrm{eu} := E([\mathrm{tmp}(2, 1), \mathrm{tmp}(3, 1), \mathrm{tmp}(4, 1)], -\mathrm{rM}\, d, A(\mathrm{rM}\, \mathrm{aqv}, -\mathrm{rM}\, d, c), c);$

Type: Vector(Expression(Float))

er is the electric field in the stationary inertial frame, and ev is the electric field in the v inertial frame. The path vector of the electric field to obtain the electric field in the u inertial frame is obtained by Lorentz transforming the electric field path vector in the stationary inertial frame into the u inertial frame. The motion of Qa from the perspective of the u inertial frame is obtained by considering the Wigner rotation to the motion of Qb in the v inertial frame. While there is no rotation between the stationary inertial frame and the v and u inertial frames, there is the Wigner rotation between the v and u inertial frames.

(21) -> $\mathrm{ar} := \frac{1}{\mathrm{Gm}(u, c)}\left(\mathrm{er} + \frac{1}{c}\mathrm{cX}(u, \mathrm{cX}(\mathrm{ro}, \mathrm{er})) - \frac{1}{c^2} u\, \mathrm{dX}(\mathrm{er}, u)\right);$
$\mathrm{av} :=$
$\frac{1}{\mathrm{Gm}(d, c)}\left(\mathrm{ev} + \frac{1}{c}\mathrm{cX}\left(d, \mathrm{cX}\left(\frac{1}{\sqrt{\mathrm{sq}(r)}} r, \mathrm{ev}\right)\right) - \frac{1}{c^2} d\, \mathrm{dX}(\mathrm{ev}, d)\right);$
$\mathrm{au} := \mathrm{eu};$

Type: Vector(Expression(Float))

$$\vec{a} = \frac{1}{\gamma}\left(\vec{E} + \vec{v} \times \left(\frac{\hat{r}}{c} \times \vec{E}\right) - \frac{\vec{v}}{c^2}(\vec{v} \cdot \vec{E})\right)$$

ar is the acceleration of Qb due to the electromagnetic field in the stationary inertial frame.
av is the acceleration due to the electromagnetic field in the v inertial frame. In the v inertial frame, the velocity of Qa, the source of the field, is 0. However, when Qa accelerates, a magnetic field results from it, so the magnetic field term must also be calculated. au is the acceleration due to the electromagnetic field in the u inertial frame. Here, since Qb is stationary, the electric field term alone is sufficient. The mass of Qb is 1.

(22) -> $A(\mathrm{au}, u, c) - \mathrm{ar} = \mathrm{dA}(\mathrm{au}, u, \mathrm{vector}[0, 0, 0], c) - \mathrm{ar},$
$A(\mathrm{rM}\, \mathrm{rA}(\mathrm{av}, d, c), u, c) - \mathrm{ar} = \mathrm{dA}(\mathrm{av}, v, d, c) - \mathrm{ar},$
$\mathrm{rA}(\mathrm{ar}, u, c) - \mathrm{au} = \mathrm{dA}(\mathrm{ar}, -u, u, c) - \mathrm{au},$
$\mathrm{rM}\, \mathrm{rA}(\mathrm{av}, d, c) - \mathrm{au} = \mathrm{rM}\, \mathrm{dA}(\mathrm{av}, -d, d, c) - \mathrm{au},$
$\mathrm{dA}(\mathrm{dA}(\mathrm{av}, v, d, c), -u, u, c) - \mathrm{au},$
$\mathrm{rrM}\, A(\mathrm{au}, -\mathrm{rM}\, d, c) - \mathrm{av} = \mathrm{rrM}\, \mathrm{dA}(\mathrm{au}, -\mathrm{rM}\, d, \mathrm{vector}[0, 0, 0], c) - \mathrm{av},$
$\mathrm{dA}(\mathrm{dA}(\mathrm{au}, u, \mathrm{vector}[0, 0, 0], c), -v, u, c) - \mathrm{av},$
$\mathrm{rrM}\, A(\mathrm{rA}(\mathrm{ar}, u, c), -\mathrm{rM}\, d, c) - \mathrm{av} = \mathrm{dA}(\mathrm{ar}, -v, u, c) - \mathrm{av}$

$[[0.3\,E-200, 0.0, 0.7\,E-200] = [0.3\,E-200, 0.0, 0.7\,E-200], [0.2\,E-200, 0.1\,E-199, -0.1\,E-200] = [0.2\,E-200, 0.3\,E-200, -0.3\,E-201], [-0.2\,E-200, 0.3\,E-200, -0.7\,E-200] = [0.2\,E-200, 0.7\,E-200, -0.7\,E-200], [-0.2\,E-200, 0.1\,E-199, -0.8\,E-200] = [0.5\,E-200, 0.3\,E-199, -0.6\,E-200], [0.2\,E-200, 0.1\,E-199, -0.7\,E-200], [0.7\,E-200, 0.1\,E-199, 0.8\,E-200] = [0.7\,E-200, 0.1\,E-199, 0.8\,E-200], [0.2\,E-200, -0.3\,E-200, 0.7\,E-200], [0.5\,E-200, 0.1\,E-199, 0.1\,E-200] = [0.0, -0.3\,E-200, 0.2\,E-201]]$

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Type: Tuple(Any)
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This calculation is the test of the following identities:

$$\begin{aligned}
\vec{a}_r &= A(\vec{a}_u, \vec{u}) = \mathrm{dA}(\vec{a}_u, \vec{u}, 0)\\
\vec{a}_r &= A(\mathrm{rM}\,\mathrm{rA}(\vec{a}_v, \vec{d}), \vec{u}) = \mathrm{dA}(\vec{a}_v, \vec{v}, \vec{d})\\
\vec{a}_u &= \mathrm{rA}(\vec{a}_r, \vec{u}) = \mathrm{dA}(\vec{a}_r, -\vec{u}, \vec{u})\\
\vec{a}_u &= \mathrm{rM}\,\mathrm{rA}(\vec{a}_v, \vec{d}) = \mathrm{rM}\,\mathrm{dA}(\vec{a}_v, -\vec{d}, \vec{d})\\
&= \mathrm{dA}(\mathrm{dA}(\vec{a}_v, \vec{v}, \vec{d}), -\vec{u}, \vec{u})\\
\vec{a}_v &= \mathrm{rrM}\,A(\vec{a}_u, -\mathrm{rM}\,\vec{d}\,) = \mathrm{rrM}\,\mathrm{dA}(\vec{a}_u, -\mathrm{rM}\,\vec{d}, 0)\\
&= \mathrm{dA}(\mathrm{dA}(\vec{a}_u, \vec{u}, 0), -\vec{v}, \vec{u})\\
\vec{a}_v &= \mathrm{rrM}\,A(\mathrm{rA}(\vec{a}_r, \vec{u}), -\mathrm{rM}\,\vec{d}) = \mathrm{dA}(\vec{a}_r, -\vec{v}, \vec{u})
\end{aligned}$$

It was confirmed that the acceleration of Qb due to the electromagnetic field in the three inertial systems and the relativistically converted value of the acceleration in other inertial systems were consistent up to the limit of calculation precision in all possible paths.

3 Conclusion

The work I have undertaken may not appear as a conventional mathematical proof, but I am confident that it sufficiently instills unquestionable confidence in the relativistic consistency of the electromagnetic force. Furthermore, the theoretical underpinning of the Heaviside-Feynman formula has been further substantiated.

Bibliography

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