EMTH210 Tutorial 4:
Divergence, Curl, and Lagrange Multipliers

For the week starting Monday 09 March.

Preparation problems (homework)

1. Find the curl and divergence of the three vector fields:

$$\mathbf{u} = \begin{pmatrix} x-y \\ y-z \\ z-x \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} xz \\ yz \\ xy \end{pmatrix} \qquad \text{and} \qquad \mathbf{w} = \begin{pmatrix} y \\ -x \\ xyz \end{pmatrix}$$

   Show that $\mathbf{u}$ has a constant circulation, and state the axis about which this circulation occurs. Where is $\mathbf{w}$ converging, and where is it diverging ?

2. Let $\mathbf{v}$ be a velocity vector with units metres/second. What are the units of $\nabla \times \mathbf{v}$?

3. Let $\mathbf{v} = \nabla F$, where F is a smooth function of x and y. Show that

$$\nabla \cdot \mathbf{v} = F_{xx} + F_{yy}$$

   and also show that $\mathrm{rot}(\mathbf{v}) = 0$.

   (The equality of mixed partial derivatives of F could be useful for the second part).

4. If $\mathbf{F}(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$ and P, Q, R are smooth functions of x, y, z, show that the divergence of the curl of $\mathbf{F}$ is zero (i.e. show that $\nabla \cdot (\nabla \times \mathbf{F}) = 0$).

5. A circular pipe of inside radius R lies with its axis along the x axis. Water is flowing steadily through the pipe. The flow is laminar, and so the velocity vector $\mathbf{v}$ is given by the Poiseuille formula

$$\mathbf{v} = k \begin{pmatrix} R^2 - y^2 - z^2 \\ 0 \\ 0 \end{pmatrix} ,$$

   where k is a positive constant.

   At what points is the flow irrotational (i.e. at what points is $\nabla \times \mathbf{v} = \mathbf{0}$)?

P.T.O.

Problems for the tutorial

6. The mass concentration $c(x, y)$ of toxin being produced by bacteria in a disc of radius 2 is given by $c = 4 - x^2 - y^2$. This concentration is not changing with time. The toxin diffuses through the disc to the edge where it is immediately removed. The flow $\mathbf{f}$ of toxin at any point in the disc is given by $\mathbf{f} = -k\nabla c$, where k is a positive constant.

   (a) Find $\mathbf{f}$.

   (b) Find $\nabla \cdot \mathbf{f}$ and explain why it gives the rate at which toxin is produced at each point in the disc.
   (Hint: Consider the conservation of mass equation $\dfrac{\partial \rho}{\partial t} = s - \nabla \cdot \mathbf{f}$.)

   (c) If the rate of toxin production at each point is proportional to the density of bacteria at that point, what is the distribution of bacteria in the disc?

7. Is there a vector field $\mathbf{F}$ such that $\nabla \times \mathbf{F} = (x - y, y - z, z - x)$? Explain.
   (Hint: See preparation problems 1 and 4.)

8. Find the critical points of $f = x^4 + 3y$ subject to $c = 3x^2 + 2y - 12 = 0$.

9. A cylindrical can has radius r and height h. The amount of steel in the can is proportional to the surface area $A = 2\pi r(h + r)$ of the can. For a fixed volume V of the can, find the critical points of the area A. Using your answer find the dimensions of the cylindrical can which use the least steel (for a fixed volume).

10. Find the critical and stationary points of $f = xy$ subject to the inequality constraint $x^2 + y^2 \leq 1$.

11. The edge of a city is given by the equation

$$x^2 + 4y^2 = 16.$$

A radio station is located inside the city at $x = 0$, $y = 3/2$. The output power of the station is a function of the maximum distance D at which the station can be heard. Using

$$f = x^2 + (y - 3/2)^2\,,$$

find the critical points of f subject to the constraint

$$c = x^2 + 4y^2 - 16 = 0$$

and hence find the minimum possible value of D if the station can be heard anywhere in the city.