University of Canterbury

EMTH210

ENGINEERING MATHEMATICS

Topic 2: Differential equations

2.1 FIRST ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

Equations of the following form are called *first order linear ODEs*:

$$\frac{dy}{dt} + p(t)y = q(t)$$

First order because ____.

Linear because ____.

Ordinary as opposed to ____.

Note that there is no function of t in front of the dy/dt term. If you come across a 1st order linear ODE with a term in front of the dy/dt term, then divide through by that term to obtain the *general* or *standard form* given above.

To solve ODEs of this type we multiply by a so-called *integrating factor* $\mu(t)$:

$$\mu\frac{dy}{dt} + \mu p(t)y = \mu q(t)\ . \tag{1}$$

This seems to have made things worse! But now choose μ so that

$$\frac{d\mu}{dt} = \mu p\ . \tag{2}$$

Doing so makes the LHS of (1) an exact derivative:

$$\frac{d}{dt}(\mu y) =$$

So (1) becomes:

and we can integrate both sides[1].

At this stage, we can find the solution to (1) by doing the integration then dividing through by μ. But how is μ found? We defined μ in (2), and this is separable:

We only need one μ that works, so we can use $c \equiv 0$ without loss of generality, giving:

Again, we only need one working μ, so we define the integrating factor to be

$$\mu = e^{\int p dt} .$$

The steps for solving an ODE by the integrating factor method are as follows.

1. Ensure the ODE is 1st order, linear, and in standard form.
2. Find the integrating factor using the above expression.
3. Multiply the ODE in standard form by the integrating factor.
4. Rewrite the LHS as a derivative.
5. Integrate both sides.
6. Divide by μ to get the solution y.
7. If you have any extra information (initial conditions, say) then use this to find the constant of integration.

[1] Assuming that μq is integrable. Everything you see in this course will be, but in general you may have to settle for a numerical approximation.

Example 2.1.1

Solve the following ODE, and check your solution by substitution into the ODE.

$$t\frac{dy}{dt} - 3y = t^2 \quad , \quad t > 0 \; .$$

Solution.

♦

Example 2.1.2

A chain is coiled near the edge of a smooth table with its end hanging vertically over the edge a distance $x(t)$. As it falls and uncoils, the velocity of the hanging part of the chain is $v(t)$ at time t. The constant 1D density (ie mass per unit length) of the hanging chain is ρ. Initially $x(0) = 0.1$, $v(0) = 0$, and throughout the acceleration due to gravity is $g = 9.81m^{-2}$. Find v as a function of x.

Solution.

♦

2.2 SECOND ORDER HOMOGENEOUS LINEAR CONSTANT COEFFICIENT ODEs

These are things of the form:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \ ,$$

where a, b and c are constants with $a \neq 0$.

Second order because ____.

Homogeneous because ____.

Linear because ____.

Constant coefficient because ____.

We solve such equations by analogy with first order ODEs. We saw that first order ODEs like $y' + 2y = 0$ have exponential solutions, ($y = ke^{-2t}$, k arbitrary, in this case). So let's try $y = e^{mt}$ in the 2nd order ODE, (where m is a constant).

Example 2.2.1

Solve the following DE:

$$y'' + 3y' + 2y = 0 \ .$$

Solution.

♦

The linearity of the equation let us take a linear combination of the two component solutions to get the general solution. So this method will not work for non-linear equations!

We should check that our general solution does indeed solve the DE. I'll set this out in a way which makes the linear dependence clear.

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \frac{d^2}{dt^2}(c_1e^{-t} + c_2e^{-2t}) + 3\frac{d}{dt}(c_1e^{-t} + c_2e^{-2t}) + 2(c_1e^{-t} + c_2e^{-2t})$$

$$= c_1\left[\frac{d^2}{dt^2}e^{-t} + 3\frac{d}{dt}e^{-t} + 2e^{-t}\right] + c_2\left[\frac{d^2}{dt^2}e^{-2t} + 3\frac{d}{dt}e^{-2t} + 2e^{-2t}\right]$$

$$= c_1 \cdot 0 + c_2 \cdot 0 = 0 \ .$$

The steps for solving a second order homogeneous linear constant coefficient ODE are:

1. Ensure the ODE is 2nd order, homogeneous, linear, constant coefficients.
2. Create the *ansatz* $y = e^{mt}$ (or $y = e^{mx}$, etc, depending on the variable names).
3. Substitute the ansatz into the ODE.
4. Obtain and solve the *auxiliary equation*, which will be a second order **algebraic** equation like $am^2 + bm + c = 0$, to get two values m_1, m_2.
5. Form the *component solutions* $y_1 = e^{m_1 t}$ and $y_2 = e^{m_2 t}$.
6. Write down the *general solution* of the ODE, which is a linear combination of the component solutions:

$$y = c_1 y_1 + c_2 y_2 \ .$$

7. If you have any extra information (initial or boundary conditions, say) then use these to find the constants c_1, c_2.

Note that the solution of a first order ODE has one arbitrary constant, which can be set if one initial or boundary condition is known, while our second order ODE solution has two unknowns which can be set with two conditions.

Example 2.2.2

Find c_1 and c_2 for the previous example if the initial conditions are

$$y(0) = 1 \quad , \quad y'(0) = 3 \ .$$

Solution.

♦

The power of this method of solution lies in reducing the problem of solving an ODE to the problem of solving an algebraic equation, namely the auxiliary equation $am^2+bm+c=0$. Recall that when the discriminant b^2-4ac of this quadratic equation is positive, zero, or negative, we get two distinct real roots, one repeated real root, or two complex conjugate roots, respectively. In the example above we saw what happens when the discriminant is positive and we get two real roots. In the next example we see what happens when the discriminant is negative and we get two complex roots. In the next lecture, we will discover what happens when the discriminant is zero and the real root is repeated.

When you are familiar with all three cases, you will be able to look at a second order homogeneous linear constant coefficient ODE and quickly say something about how the solutions will behave before you solve the ODE. This can be tremendously useful.

Example 2.2.3

Solve

$$y'' + 6y' + 13y = 0 \ .$$

Solution.

♦

Two comments.

1. The general solution is real. Here, that hitherto mysterious link between complex numbers and trigonometric functions becomes essential to a very real application. Of course a physical situation modelled by this ODE would have to have real, not complex, solutions. So even though the method involves complex numbers, the solution is real.

2. From now on *do not* go through the rigmarole of writing $A = (a + ib)/2$ etc — just take the shortcut and **go directly** from $m = -3 \pm 2i$ to

$$y = e^{-3t} \left[a \cos(2t) + b \sin(2t)\right] \; .$$

At this point, we can glance at the coefficients of an ODE, imagine they were the coefficients of the auxiliary equation, and if the discriminant thereof is positive we can predict that the solution will involve exponential growth, decay, or a combination of the two, while if the discriminant is negative we can predict the existence of cyclic solutions (combinations of sines and cosines). Next, we learn what happens when the discriminant is zero.

2.3 REPEATED ROOTS

This is what we call the situation when the discriminant of the auxiliary equation is zero, as in this example:

$$y'' + 2y' + y = 0.$$

As usual, we put $y = e^{mt}$, and obtain the auxiliary equation

$$m^2 + 2m + 1 = 0$$
$$(m + 1)^2 = 0$$

so $m = -1, -1$. This gives only ONE linearly independent component solution $y_1 = e^{-t}$. But we expect a second component solution as the ODE is second order. The general rule for repeated roots is that $\boxed{\text{if } y_1 = e^{m_1 t} \text{ is the first component solution then the second component solution is simply } y_2 = te^{m_1 t}}$, that is, t times the first solution. For the current example, we can see that this works:

$$y_2(t) = te^{-t},$$

$$\implies y_2'(t) = \underline{\hspace{6cm}}$$

$$\implies y_2''(t) = \underline{\hspace{12cm}},$$

and so

$$y_2'' + 2y_2' + y_2' = \underline{\hspace{12cm}}.$$

So as y_2 satisfies the original DE we know that this is the second solution. Thus the general solution for this example is

$$y = c_1 e^{-t} + c_2 te^{-t},$$

where c_1 and c_2 are arbitrary constants.

> In general, (i.e. for higher roots), if a root $m = \lambda$ occurs k times then a DE has k linearly independent component solutions,
>
> $$e^{\lambda t},\ te^{\lambda t},\ t^2 e^{\lambda t},\ \ldots,\ t^{k-1} e^{\lambda t},$$
>
> plus solutions from any other roots.

We have now seen the three types of solution possible for second order homogeneous linear constant coefficient ODEs, and how they occur depending on the roots of the auxiliary equation. We now move on to finding values of the constants c_1, c_2 depending on extra information with which we may be supplied.

2.4 BOUNDARY VALUE PROBLEMS

These differ from initial value problems in that the extra information involves *more than one* value of t.

Example 2.4.1

Solve the boundary value problem:

$$y'' + \omega^2 y = 0$$

$$y(0) = 0 \quad \text{and} \quad y(L) = 0 \quad (L \neq 0)$$

Solution.

Another way to look at those two cases is to think about what must happen in order for $A\sin(\omega L)$ to be zero, namely either $A = 0$ or $\sin(\omega L) = 0$.

There is another possibility that can occur with different boundary conditions, as we see in the next example.

Example 2.4.2

As before,

$$y'' + \omega^2 y = 0 \ ,$$

but now

$$y(0) = 0 \quad \text{and} \quad y(L) = 1, \quad L \neq 0$$

Solve the new boundary value problem.

Solution.

♦

So in both examples, we needed to think about the zeroes of the function $\sin(x)$. In the first example, this was to identify the two possible situations: the trivial solution, or an infinite family of (non-trivial) solutions. In the second example, the two possible situations were: a unique solution, or no solutions. In both cases, the number and type of solution depends on the value of ωL. You should think about what this means.

In contrast, and you should check this, an initial value problem $y''+\omega y = 0$ with $y(t_0) = y_0$ and $y'(t_0) = y'_0$ *always* has a unique solution.

2.5 APPLICATION: BUCKLING OF A THIN BEAM

Sketch of the situation:

This is not a time-dependent problem, but rather $y = y(x)$ where y is the downwards displacement of the beam at horizontal position x. The beam is pinned freely at the ends, so that $y(0) = y(L) = 0$ but $y'(0)$ and $y'(L)$ are free. The beam is under compression by a force P.

The governing equation and boundary conditions of the beam are

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = 0 \quad , \quad y(0) = y(L) = 0 \ ,$$

where E is Young's modulus and I is the second moment of the cross-sectional area of the beam[2].

This has general solution

[2] The derivation is as follows. If $M(x)$ is the moment on the left-hand part of the beam exerted by the right-hand part of the beam at a point C, then taking moments about C gives

$$Py(x) + M(x) = 0 \ .$$

For small displacements

$$M(x) = EI\frac{d^2y}{dx^2} \ .$$

Substituting the latter into the former gives the governing equation.

Then the boundary conditions give:

so that either $A = 0$, in which case we have the trivial solution $y = 0$ (beam stays straight), or $kL = n\pi$ for some integer n, in which case we have the solution

$$y = A \sin\left(\frac{n\pi x}{L}\right) ,$$

where A is arbitrary. In particular, A can be arbitrarily large — the beam buckles.

We may now wish to ask under which conditions the beam will buckle. Well,

Hence the beam buckles for these values of P, which are known as *critical loads*. Note that these are steady-state results, not dynamic, so some care is required when interpreting these results.The smallest force P to cause buckling is

which is the so-called "Euler load".

 [3]

Good old Euler; just look at that hat.

[3]Portrait by Jakob Emanuel Handmann (1718-1781). File downloaded from commons.wikimedia.org, licensed under the Creative Commons Attribution ShareAlike 3.0. Details available from url and address on the front page.

2.6 INHOMOGENEOUS LINEAR ODEs

These are DEs of the form

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t) \;, \tag{3}$$

where you will note that we are no longer requiring that the coefficients are constants, and likewise the right-hand side (RHS) can now be non-zero.

The inhomogeneous linear ODE (3) has a general solution $y(t) = y_c(t) + y_p(t)$ where $y_p(t)$ is *any* solution to (3) and $y_c(t)$ is the general solution to

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = 0 \;.$$

This general solution is valid on any interval on which $a_2(t) \neq 0$. This last condition is very important! Anywhere that $a_2 = 0$ the DE becomes first order, or even algebraic, and this changes the solution. Terminology:

- $y_p(t)$ is called the *particular integral* (PI);
- $y_c(t)$ is called the *complementary function* (CF).

The general strategy for solving an inhomogeneous linear ODE is

1. Set the RHS equal to zero, and solve the resulting homogeneous ODE (by the methods we have learned) to find the complementary function $y_c(t)$.
2. Find any solution to the full inhomogeneous equation (by the methods to be outlined below) — this is the particular integral $y_p(t)$.
3. The general solution is $y(t) = y_c(t) + y_p(t)$.

Before discussing the general method for finding y_p in §2.7, let's see the ideas in action.

Example 2.6.1

Find the general solution of

$$y'' + 3y' + 2y = 4t + 8 \ .$$

Solution.

♦

This method of "guessing" the form of the RHS but with unknown constant is known as the *method of undetermined coefficients*, because the method involves coefficients (A and B in the previous example) which are undetermined[4]

2.7 BASIC STRATEGY FOR "GUESSING" FORM OF y_p — THE *METHOD OF UNDETERMINED COEFFICIENTS*

Follow the subsequent four rules carefully and *in order* and you won't go wrong. When I say "in order" I really mean it: do not apply rule 3, then rule 4, then go back to rule 2. Do 1, then 2, then 3, then 4, and never look back.

[4] And because "big fat guess" doesn't sound impressive enough.

Rule 1: Take $f(t)$ and multiply each term by an unknown constant.

Rule 2: Make sure polynomials do not miss out powers below the highest power. For example, if

$$f(t) = 3t^4 + 8t^2 \ ,$$

then y_p should be

$$y_p = At^4 + Bt^3 + Ct^2 + Dt + E \ .$$

Rule 3: Wherever there is a sine/cosine term, insert a matching cosine/-sine term. For example, if

$$f(t) = \sin(3t) + 2\cos(5t) \ ,$$

then y_p should be

$$y_p = A\sin(3t) + B\cos(3t) + C\sin(5t) + D\cos(5t) \ .$$

Rule 4: For terms in y_p which also appear in y_c, multiply these "overlapping" terms (only) by t until they are no longer part of y_c. Multiply by t to remove repeated terms in y_p as well. We will see an example of this in the first example of the next lecture.

One application of homogeneous ODEs is to model (some) systems which are being *forced* externally. For example, while a freely-swinging pendulum is modelled by the solutions of a second order homogeneous ODE, a pendulum which is being moved by the application of a force is modelled by the solutions of a second order inhomogeneous ODE. Tall buildings (inverted pendulums) must be designed to withstand oscillations induced by strong winds (the forcing). These oscillations are modelled by an inhomogeneous ODE. When the wind stops, the damped decay of the oscillations is modelled by a homogeneous ODE.

When your lecturer goes through the following examples in class s/he will condense the working for finding the complementary function, in order to focus on the new thing we are learning, the method of undetermined coefficients for finding the particular integral. Under test conditions, please show all working for finding y_c as well as for finding y_p. Also, your lecturer will explicitly label the relevant steps "Rule 1" and so on, in line with the boxed methodology in the previous lecture. I encourage you to always do the same.

Example 2.7.1

Find the general solution of

$$y'' - 2y' + y = t^2 e^t + 5 \ .$$

Solution.

Example 2.7.2

Find the general solution of

$$y'' + y = e^t \sin(t) \ .$$

Solution.

♦

Example 2.7.3

Find the general solution of

$$y'' - y = te^t \ .$$

Solution.

♦

Example 2.7.4

Find the general solution of

$$y'' + y = \sin(t) + 3te^{-t} \ .$$

Solution.

♦

If in doubt about applying the method of undetermined coefficients, it is safer to include extra terms. If these terms are not needed the method will automatically set their unknown coefficients to zero except for terms which overlap with y_c; the coefficients of these terms will be arbitrary. On the other hand if necessary terms are omitted then the method will almost always fail. Consider the final example below.

Example 2.7.5

Find the general solution of

$$\frac{d^3y}{dt^3} + \frac{dy}{dt} = t^2 + e^{2t} \ .$$

Solution.

♦

Note: the method of undetermined coefficients only really works for constant coefficient DEs when the RHS of $f(t)$ consists of terms with are polynomial, exponential, sine, cosine, or products thereof.