EMTH210 Tutorial 2:
Partial derivatives, chain rule, directional derivatives – Solutions

Preparation problems (homework)

1. (a)
$$f(x,y) = x^4y^2 - \sin(xy) + 6x^5 - 4y$$
$$f_x = 4x^3y^2 - y\cos(xy) + 30x^4$$
$$f_y = 2x^4y - x\cos(xy) - 4$$

(b)
$$f(x,y,z) = \exp(x+z) - \cos(xy^2z^3)$$
$$f_x = \exp(x+z) + y^2z^3\sin(xy^2z^3)$$
$$f_y = 2xyz^3\sin(xy^2z^3)$$
$$f_z = \exp(x+z) + 3xy^2z^2\sin(xy^2z^3)$$

(c)
$$f(x,y,z,t) = x^2\ln(t^2) - xz\tan(y)$$
$$f_x = 2x\ln(t^2) - z\tan(y)$$
$$f_y = -xz\sec^2(y)$$
$$f_z = -x\tan(y)$$
$$f_t = \frac{2tx^2}{t^2} = \frac{2x^2}{t}$$

2.
$$z = \ln(x^2+y^2)$$
$$\frac{\partial z}{\partial x} = \frac{2x}{x^2+y^2} \qquad \frac{\partial z}{\partial y} = \frac{2y}{x^2+y^2}$$
$$\frac{\partial^2 z}{\partial x^2} = \frac{2}{x^2+y^2} - \frac{(2x)^2}{(x^2+y^2)^2} \qquad \frac{\partial^2 z}{\partial y^2} = \frac{2}{x^2+y^2} - \frac{(2y)^2}{(x^2+y^2)^2}$$
$$\therefore \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{4}{x^2+y^2} - \frac{4(x^2+y^2)}{(x^2+y^2)^2} = 0,$$
and Laplace's equation is satisfied.

3.
$$f(x,t) = \sin(ct)\sin(x)$$
$$\frac{\partial f}{\partial x} = \sin(ct)\cos(x) \qquad \frac{\partial f}{\partial t} = c\cos(ct)\sin(x)$$
$$\frac{\partial^2 f}{\partial x^2} = -\sin(ct)\sin(x) \qquad \frac{\partial^2 f}{\partial t^2} = -c^2\sin(ct)\sin(x)$$
$$\therefore \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = -\sin(ct)\sin(x) = \frac{\partial^2 f}{\partial x^2},$$
and the wave equation is satisfied. At the end point $\pm\pi$,
$$f(-\pi,t) = \sin(ct)\sin(-\pi) = 0$$
$$f(\pi,t) = \sin(ct)\sin(\pi) = 0.$$

This corresponds to e.g. the two fixed ends of a vibrating guitar string.

4.
$$\theta(x,t) = e^{at}\sin(x)$$

$$\theta_x = e^{at}\cos(x) \qquad \theta_t = ae^{at}\sin(x)$$
$$\theta_{xx} = -e^{at}\sin(x)$$

$$\therefore \theta_{xx} = k\theta_t \implies -e^{at}\sin(x) = kae^{at}\sin(x) \implies a = -1/k.$$

5. (a)
$$f(x,y) = y - \ln(2x^2y)$$
$$f_x = \frac{-4xy}{2x^2y} = \frac{-2}{x} \qquad f_y = 1 - \frac{2x^2}{2x^2y} = 1 - \frac{1}{y}$$
$$\nabla f = \left(\frac{-2}{x}, 1 - \frac{1}{y}\right)$$
$$\nabla f(2,1) = (-1,0) \quad (= -\mathbf{i})$$

(b)
$$F(x,y,z) = xy\cos(yz)$$

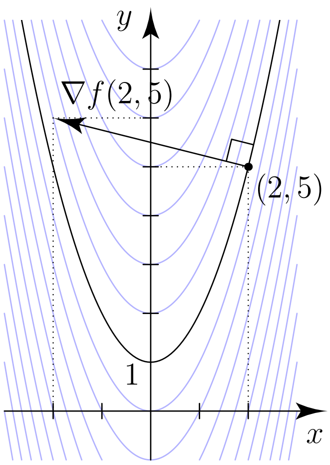
$$F_x = y\cos(yz) \qquad F_y = x\cos(yz) - xyz\sin(yz) \qquad F_z = -xy^2\sin(yz)$$
$$\nabla F = (y\cos(yz), x\cos(yz) - xyz\sin(yz), -xy^2\sin(yz))$$
$$\nabla F(2,1,\pi) = (\cos(\pi), 2\cos(\pi) - 2\pi\sin(\pi), -2\sin(\pi))$$
$$= (-1,-2,0) \quad (= -\mathbf{i} - 2\mathbf{j})$$

6.
$$V = \frac{1}{r} = (x^2+y^2+z^2)^{-\frac{1}{2}}$$
$$V_x = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}(2x) = -\frac{x}{r^3} \qquad V_y = -\frac{y}{r^3} \qquad V_z = -\frac{z}{r^3}$$
$$-\nabla V = \left(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\right) \qquad \text{or} \qquad -\nabla V = \frac{1}{(x^2+y^2+z^2)^{3/2}}\ (x,\ y,\ z)$$

At $(1,2,-2)$, $r=3$, and so

$$-\nabla V(1,2,-2) = \left(\frac{1}{3^3}, \frac{2}{3^3}, \frac{-2}{3^3}\right) = \frac{1}{27}(1,2,-2).$$

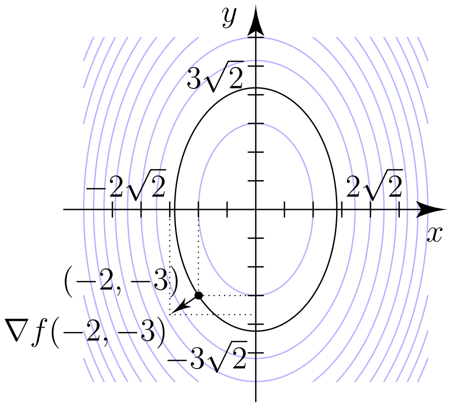
7. (a)



The level curves are $f(x,y) = c = y - x^2$, $c \in \mathbb{R}$ (parabolas). At $(2,5)$, $c = 5 - 2^2 = 1$, so $y = x^2 + 1$.

$$\nabla f = (-2x, 1)$$
$$\nabla f(2,5) = (-4,1)$$

(b)



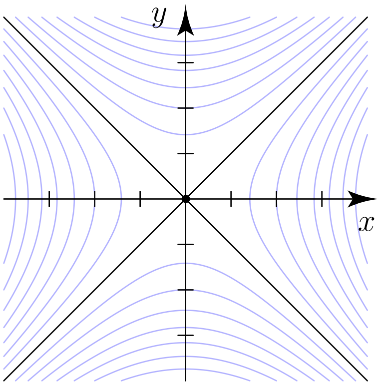
The level curves are

$$f(x, y) = c = \frac{x^2}{4} + \frac{y^2}{9}, \quad c \in \mathbb{R}$$

(ellipses: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the ellipse centred at the origin that goes out to $x = \pm a$ and $y = \pm b$). At $(-2, -3)$, $c = \frac{(-2)^2}{4} + \frac{(-3)^2}{9} = 2$, so $\frac{x^2}{(2\sqrt{2})^2} + \frac{y^2}{(3\sqrt{2})^2} = 1$.

$$\nabla f = \left(\frac{x}{2}, \frac{2y}{9}\right)$$
$$\nabla f(-2, -3) = \left(-1, -\frac{2}{3}\right)$$

(c)



The level curves are $f(x, y) = c = x^2 - y^2$, $c \in \mathbb{R}$ (hyperbolas). At $(0, 0)$, $c = 0$, so $y = \pm x$.

$$\nabla f = (2x, -2y)$$
$$\nabla f(0, 0) = (0, 0)$$

The gradient vanishes at the origin. (Note that these straight lines are the asymptotes of the family of hyperbolas.)

8. (a) To find $\frac{\partial z}{\partial x}$, we differentiate implicitly with respect to x, being careful to treat y as a constant:

$$3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy \frac{\partial z}{\partial x} \;=\; 0.$$

Solving this equation for $\frac{\partial z}{\partial x}$, we obtain

$$\frac{\partial z}{\partial x} \;=\; -\frac{x^2 + 2yz}{z^2 + 2xy}.$$

Similarly, implicit differentiation with respect to y gives

$$\frac{\partial z}{\partial y} \;=\; -\frac{y^2 + 2xz}{z^2 + 2xy}.$$

(b) To find $\frac{\partial z}{\partial x}$, we differentiate implicitly with respect to x, being careful to treat y as a constant:

$$y \frac{\partial z}{\partial x} + \ln y \;=\; 2z \frac{\partial z}{\partial x}.$$

Solving this equation for $\frac{\partial z}{\partial x}$, we obtain

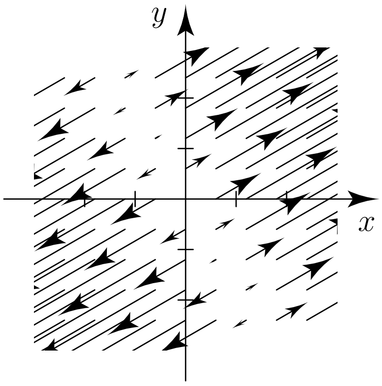
$$\frac{\partial z}{\partial x} \;=\; \frac{\ln y}{2z - y}.$$

Similarly, implicit differentiation with respect to y gives

$$\frac{\partial z}{\partial y} \;=\; \frac{z + (x/y)}{2z - y} \;=\; \frac{x + yz}{y(2z - y)}.$$

Problems for the tutorial

9.



$(D_{\mathbf{u}}f)\mathbf{u}$ for a few locations.

$$
\begin{aligned}
f &= x^2 + y^2 \\
f_x &= 2x \qquad f_y = 2y \qquad \nabla f = (2x, 2y) \\
\mathbf{u} &= (\cos 30^\circ, \sin 30^\circ) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \\
D_{\mathbf{u}}f &= \nabla f \cdot \mathbf{u} \\
&= (2x, 2y) \cdot \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \\
&= \sqrt{3}x + y
\end{aligned}
$$

10.

$$V = hw\ell$$

$$h = h(w, \ell) = \frac{V}{\ell\, w} = \frac{V}{\ell}\left(\frac{1}{w}\right)$$

$$\frac{\partial h}{\partial w} = -\frac{V}{\ell\, w^2} = -\frac{hw\ell}{\ell\, w^2} = -\frac{h}{w}$$

$$\left.\frac{\partial h}{\partial w}\right|_{(4,3,6)} = -\frac{4}{3}$$

11. (a) From the chain rule,

$$\frac{dT}{dt} = \left(\frac{\partial T}{\partial x}\right)_t \frac{dx}{dt} + \left(\frac{\partial T}{\partial t}\right)_x \frac{dt}{dt},$$

so we need to find $\left(\frac{\partial T}{\partial x}\right)_t$, $\left(\frac{\partial T}{\partial t}\right)_x$ and $\frac{dx}{dt}$ (as $\frac{dt}{dt}$ is just 1).

$$T = 20e^{-t}\sin x + 10 \qquad\qquad x = \frac{1}{3}t$$

$$\left(\frac{\partial T}{\partial x}\right)_t = 20e^{-t}\cos x \qquad \left(\frac{\partial T}{\partial t}\right)_x = -20e^{-t}\sin x \qquad \frac{dx}{dt} = \frac{1}{3}$$

$$\therefore \frac{dT}{dt} = \frac{20}{3}e^{-t}\cos x - 20e^{-t}\sin x$$

(b) With $\left(\frac{\partial T}{\partial t}\right)_x$, x is held constant, so $\left(\frac{\partial T}{\partial t}\right)_x$ is the rate of change of temperature with time at a particular fixed location. With $\frac{\mathrm{d}T}{\mathrm{d}t}$, x is the ant's position, which changes with time, so $\frac{\mathrm{d}T}{\mathrm{d}t}$ is the rate of change of temperature with time as experienced by the ant.

12. (Note that finding these two partial derivatives means two completely separate calculations.)

For $\left(\frac{\partial z}{\partial t}\right)_x$, z is being thought of as a function of t (and so is dependent), and x is held fixed so clearly depends on nothing (and so is independent). Because t is depended on, it is independent, and because y is only mentioned in the equations it can only be defined in terms of the other variables, and so y is dependent.

Since we want $\left(\frac{\partial z}{\partial t}\right)_x$, we start with the only equation in which z appears, and differentiate both sides with respect to t, holding x constant:

$$
\begin{aligned}
zt &= x^2 - y\sin y \\
\left(\frac{\partial z}{\partial t}\right)_x t + z &= -\left(\frac{\partial y}{\partial t}\right)_x \sin y - y\cos y\left(\frac{\partial y}{\partial t}\right)_x \\
\left(\frac{\partial z}{\partial t}\right)_x &= -\frac{1}{t}\left(z + (\sin y + y\cos y)\left(\frac{\partial y}{\partial t}\right)_x\right). \qquad (\text{for } t \neq 0) \qquad (1)
\end{aligned}
$$

We now have the derivative we want, but there's that other one still there on the right. Let's look at the second equation.

$$
\begin{aligned}
xyt &= x^2 + y^2 \\
x\left(\frac{\partial y}{\partial t}\right)_x t + xy &= 2y\left(\frac{\partial y}{\partial t}\right)_x \\
(xt - 2y)\left(\frac{\partial y}{\partial t}\right)_x &= -xy \\
\left(\frac{\partial y}{\partial t}\right)_x &= \frac{-xy}{xt - 2y}.
\end{aligned}
$$

Substituting this into (1), we obtain:

$$
\left(\frac{\partial z}{\partial t}\right)_x = -\frac{1}{t}\left(z - (\sin y + y\cos y)\frac{xy}{xt - 2y}\right).
$$

Finding $\left(\frac{\partial t}{\partial z}\right)_y$ is very similar. Here y and z are independent, and t and x are dependent.

$$
\begin{aligned}
zt &= x^2 - y\sin y \\
t + z\left(\frac{\partial t}{\partial z}\right)_y &= 2x\left(\frac{\partial x}{\partial z}\right)_y.
\end{aligned}
$$

$$
\begin{aligned}
xyt &= x^2 + y^2 \\
\left(\frac{\partial x}{\partial z}\right)_y yt + xy\left(\frac{\partial t}{\partial z}\right)_y &= 2x\left(\frac{\partial x}{\partial z}\right)_y \\
(yt - 2x)\left(\frac{\partial x}{\partial z}\right)_y &= -xy\left(\frac{\partial t}{\partial z}\right)_y \\
\left(\frac{\partial x}{\partial z}\right)_y &= \frac{-xy}{yt - 2x}\left(\frac{\partial t}{\partial z}\right)_y \qquad (2)
\end{aligned}
$$

From (2),

$$
\begin{aligned}
t + z\left(\frac{\partial t}{\partial z}\right)_y &= 2x\frac{-xy}{yt - 2x}\left(\frac{\partial t}{\partial z}\right)_y \\
\left(z + \frac{2x^2y}{yt - 2x}\right)\left(\frac{\partial t}{\partial z}\right)_y &= -t \\
\left(\frac{\partial t}{\partial z}\right)_y &= \frac{-t(yt - 2x)}{z(yt - 2x) + 2x^2y}.
\end{aligned}
$$

13. Note that x and G are the independent variables here.

$$F = y^3 - xy$$

$$\left(\frac{\partial F}{\partial x}\right)_G = 3y^2\left(\frac{\partial y}{\partial x}\right)_G - y - x\left(\frac{\partial y}{\partial x}\right)_G$$

$$= -y + (3y^2 - x)\left(\frac{\partial y}{\partial x}\right)_G$$

$$G = xye^y$$

$$0 = ye^y + x\left(\frac{\partial y}{\partial x}\right)_G e^y + xye^y\left(\frac{\partial y}{\partial x}\right)_G$$

$$0 = y + x\left(\frac{\partial y}{\partial x}\right)_G + xy\left(\frac{\partial y}{\partial x}\right)_G$$

$$-y = (x + xy)\left(\frac{\partial y}{\partial x}\right)_G$$

$$\left(\frac{\partial y}{\partial x}\right)_G = \frac{-y}{x + xy} \tag{1}$$

Substituting in (1),

$$\left(\frac{\partial F}{\partial x}\right)_G = -y + \frac{(3y^2 - x)(-y)}{x + xy}$$

$$= -y - \frac{y(3y^2 - x)}{x + xy}$$

$$= \frac{-y^2(x + 3y)}{x(1 + y)}.$$