EMTH210 Tutorial 10:
Revision, Delta Method, and Random Vectors

For the week starting Monday 18 May.

The homework questions this week are **5(b)** and **7**.

1. (**Revision**) Let X be a continuous random variable with probability density function

$$f_X(x) = \begin{cases} kx^2 & \text{if } 0 < x < \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Determine the value for k (use this value for k in the following questions).
   (b) Find the distribution function, $F_X(x)$.
   (c) Find $\mathrm{P}\left(\frac{1}{4} \leq X \leq \frac{1}{2}\right)$.
   (d) Find x such that $\mathrm{P}(X \leq x) = \frac{4}{5}$.
   (e) Find the population mean, $\mathrm{E}(X)$.
   (f) Find the population variance, $\mathrm{Var}(X)$.
   (g) Find $\mathrm{E}(1/X - 1)$.
   (h) Find $\mathrm{Var}(1/X - 1)$.

2. (**Revision**) Let

$$f(x) = \begin{cases} \frac{1}{(1+x)^k} & \text{if } x \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

   where $k > 1$ is an integer. Find the value of k that makes $f(x)$ a PDF.

3. (**Revision**) Let X be a uniform random variable with probability density function

$$f_X(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [1, 3] \\ 0 & \text{otherwise.} \end{cases}$$

   Use $f_X(x)$ to show that $\mathrm{E}(1/X) \neq 1/\,\mathrm{E}(X)$ in this case.

4. Let $X \sim \mathrm{Uniform}(0, 0.5)$ (uniform random variable) with $Y = (X+2)^{-2}$. Use the delta method to find a two-term approximation to $\mathrm{E}(Y)$ and one-term approximation to $\mathrm{Var}(Y)$.

   Note: from EMTH119, for $X \sim \mathrm{Uniform}(a, b)$, $\mathrm{E}(X) = \frac{a+b}{2}$ and $\mathrm{Var}(X) = \frac{(a-b)^2}{12}$.

5. Let (X, Y) be a continuous bivariate random vector with joint probability density function

$$f_{X,Y}(x, y) = \begin{cases} 10e^{-5x-2y} & \text{if } x > 0,\ y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Verify that $f_{X,Y}(x, y)$ is a joint probability density function.
   (b) **(Homework)** Find $\mathrm{P}(X < 1/5, Y > 1/2)$.
   (c) Find $F_{X,Y}(x, y)$, the joint cumulative distribution function of (X, Y), for all x and y (hint: draw a picture of the support of $f_{X,Y}$ to figure out where $F_{X,Y}$ is non-zero).
   (d) Find $f_X(x)$, the marginal probability density of X.
   (e) Find $f_Y(y)$, the marginal probability density of Y.

6. Let (X, Y) be a continuous bivariate random vector with joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{4}x + \frac{1}{2}y & \text{if } 0 < x < 2,\ 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Verify that $f_{X,Y}(x,y)$ is a joint probability density function.

   (b) Find $\mathrm{P}(X > 1, Y < 3/4)$.

   (c) Find $F_{X,Y}(x,y)$, the joint cumulative distribution function of (X,Y), for $0 < x < 2$ and $0 < y < 1$ (Optional, fiddly: find $F_{X,Y}(x,y)$ for all x and y).

   (d) Find $f_X(x)$, the marginal probability density of X.

   (e) Find $f_Y(y)$, the marginal probability density of Y.

7. **(Homework)** Let X be an exponential random variable with probability density function

$$f_X(x) = \begin{cases} 2e^{-2x} & \text{if } x > 0 \\ 0 & \text{if } x \le 0. \end{cases}$$

   Let $Y = \ln(1 + 3X)$ and use the delta method to find a two-term approximation to $\mathrm{E}(Y)$ and a one-term approximation to $\mathrm{Var}(Y)$.

   Note: from EMTH119, for $X \sim \mathrm{Exp}(\lambda)$, $\mathrm{E}(X) = \frac{1}{\lambda}$ and $\mathrm{Var}(X) = \frac{1}{\lambda^2}$.

8. Let X be a continuous random variable with probability density function

$$f_X(x) = \begin{cases} \frac{3}{2} - \frac{3}{2}x^2 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Verify that $f_X(x)$ is a probability density function.

   (b) Let $Y = Xe^X$ and use the delta method to find a one-term approximation and a two-term approximation to $\mathrm{E}(Y)$, and a one-term approximation to $\mathrm{Var}(Y)$.

9. Let $X \sim \mathrm{Normal}(0, 1)$ (standard normal random variable) with $Y = (X - 2)^3$. Use the delta method to find a one-term approximation and a two-term approximation to $\mathrm{E}(Y)$, and a one-term approximation to $\mathrm{Var}(Y)$.

10. Let (X, Y, Z) be a continuous random vector with joint probability density function

$$f_{X,Y,Z}(x,y,z) = \begin{cases} k(x+y)e^{-z} & \text{if } 0 < x < 1,\ 0 < y < 2,\ z > 0 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find the value of k (use this value for k in the following questions).

   (b) Find the joint cumulative distribution function of (X, Y, Z) for $0 < x < 1, 0 < y < 2$ and $z > 0$.

   (c) Find $\mathrm{P}(X < 1, Y < 1, Z < 1)$.