EMTH210 Tutorial 4:
Divergence, Curl, and Lagrange Multipliers – Solutions

Preparation problems (homework)

1.
$$\mathbf{u} = \begin{pmatrix} x-y \\ y-z \\ z-x \end{pmatrix} \qquad \nabla \times \mathbf{u} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} x-y \\ y-z \\ z-x \end{pmatrix}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-y & y-z & z-x \end{vmatrix} = \begin{pmatrix} 0+1 \\ 0+1 \\ 0+1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The direction and magnitude of curl($\mathbf{u}$) are the axis and size of the circulation, respectively. Since curl($\mathbf{u}$) is constant (doesn't depend on position), $\mathbf{u}$ has constant circulation.

$$\nabla \cdot \mathbf{u} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} x-y \\ y-z \\ z-x \end{pmatrix} = 1+1+1 = 3$$

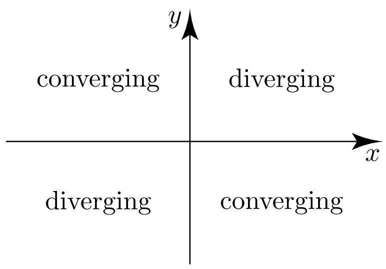
$$\mathbf{v} = \begin{pmatrix} xz \\ yz \\ xy \end{pmatrix} \qquad \nabla \times \mathbf{v} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} xz \\ yz \\ xy \end{pmatrix} = \begin{pmatrix} x-y \\ x-y \\ 0-0 \end{pmatrix} = \begin{pmatrix} x-y \\ x-y \\ 0 \end{pmatrix}$$

$$\nabla \cdot \mathbf{v} = z + z + 0 = 2z$$

$$\mathbf{w} = \begin{pmatrix} y \\ -x \\ xyz \end{pmatrix} \qquad \nabla \times \mathbf{w} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} y \\ -x \\ xyz \end{pmatrix} = \begin{pmatrix} xz-0 \\ 0-yz \\ -1-1 \end{pmatrix} = \begin{pmatrix} xz \\ -yz \\ -2 \end{pmatrix}$$

$$\nabla \cdot \mathbf{w} = 0 + 0 + xy = xy$$

$\mathbf{w}$ is diverging when $\operatorname{div} \mathbf{w} = xy > 0$, so $x > 0$ and $y > 0$, or $x < 0$ and $y < 0$. $\mathbf{w}$ is converging when $\operatorname{div} \mathbf{w} = xy < 0$, so $x < 0$ and $y > 0$, or $x > 0$ and $y < 0$. Both of these are true for all z.



2. Each element of $\nabla \times \mathbf{v}$ is of the form $\frac{\partial}{\partial x_i} v_j - \frac{\partial}{\partial x_j} v_i$. Since v_i has units m/s ($\forall i$), and x_i has units m ($\forall i$), the units of the elements of $\nabla \times \mathbf{v}$ have units

$$\left[\frac{1}{\mathrm{m}} \frac{\mathrm{m}}{\mathrm{s}} \right] = \left[\frac{1}{\mathrm{s}} \right].$$

3.
$$\nabla \cdot \mathbf{v} = \nabla \cdot \nabla F = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} \end{pmatrix} = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}$$

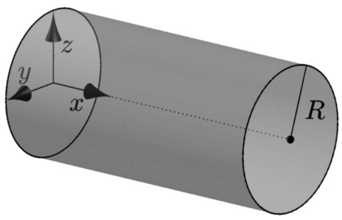
If $\mathbf{v} = (f, g)$, then $\operatorname{rot}(\mathbf{v}) = g_x - f_y$.

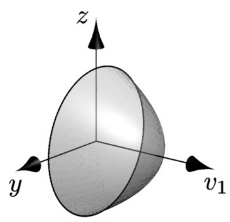
$$\operatorname{rot}(\mathbf{v}) = (F_y)_x - (F_x)_y = F_{yx} - F_{xy} = 0$$

4. Since $\mathbf{F} = (P, Q, R)$,

$$\begin{aligned}\nabla \cdot (\nabla \times \mathbf{F}) &= (R_y - Q_z)_x + (P_z - R_x)_y + (Q_x - P_y)_z \\ &= R_{yx} - R_{xy} - Q_{zx} + Q_{xz} + P_{zy} - P_{yz} \\ &= 0\end{aligned}$$

5.





Note: The 3D diagrams like this are rotatable in Adobe Reader 9+.

$$\nabla \times \mathbf{v} = k \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} R^2 - y^2 - z^2 \\ 0 \\ 0 \end{pmatrix} = k \begin{pmatrix} 0 - 0 \\ -2z - 0 \\ 0 - -2y \end{pmatrix} = k \begin{pmatrix} 0 \\ -2z \\ 2y \end{pmatrix}$$

So $\mathbf{v}$ is irrotational when $y = 0$ and $z = 0$, i.e. on the x-axis. This is the exact centre of the pipe, so the flow on all sides is the same, and so there is not tendency to rotate. Everywhere else, the flow is faster on side towards the centre than the other side, an imbalance which creates a tendency to rotate

Problems for the tutorial

6. (a)

$$\mathbf{f} = -k\nabla c = -k \begin{pmatrix} -2x \\ -2y \end{pmatrix}$$

(b)

$$\nabla \cdot \mathbf{f} = 4k$$

The concentration is not changing over time, so $\frac{\partial c}{\partial t} = 0$. Taking $\rho = c$ in the conservation of mass equation, the rate of toxin production is given by $s = \nabla \cdot \mathbf{f} = 4k$.

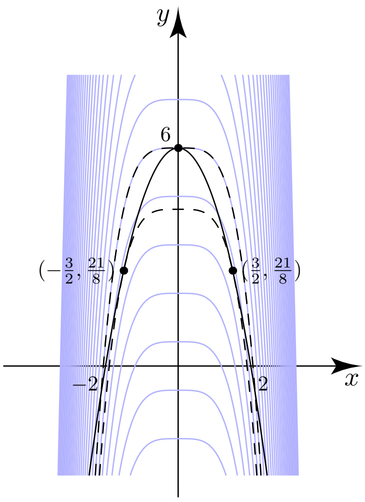
(c)

$$\text{density} \propto \text{production} = 4k$$

The rate of production is constant (independent of position), so the distribution of bacteria is constant (uniform).

7. If it were true, then preparation problem 4 shows that $\nabla \cdot (\nabla \times \mathbf{F}) = 0$. In preparation problem 1, we showed that $\nabla \cdot (x - y, y - z, z - x) = 3 \neq 0$. Hence $(x - y, y - z, z - x)$ is not the curl of some vector field $\mathbf{F}$.

8.



You don't really need to draw a picture for this question, but here's one to show what's going on.

$$f = x^4 + 3y \qquad c = 3x^2 + 2y - 12$$
$$\nabla f = (4x^3, 3) \qquad \nabla c = (6x, 2)$$

$$4x^3 = 6\lambda x \tag{1}$$
$$3 = 2\lambda \tag{2}$$
$$3x^2 + 2y - 12 = 0 \tag{3}$$

From (2), $\lambda = \frac{3}{2}$. From (1), either $x = 0$, in which case $y = 6$, or we can divide by x and get

$$\begin{aligned} 2x^2 &= 3\lambda \\ &= \frac{9}{2}, \\ x &= \pm\frac{3}{2}. \end{aligned}$$

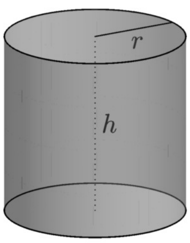
Putting this into (3), we get

$$\begin{aligned} 3\left(\frac{3}{2}\right)^2 + 2y - 12 = 0 \\ y &= \frac{1}{2}\left(12 - \frac{27}{4}\right) \\ &= 6 - \frac{27}{8} \\ &= \frac{21}{8}. \end{aligned}$$

So our critical points are

$$(0,6), \left(-\frac{3}{2}, \frac{21}{8}\right), \left(\frac{3}{2}, \frac{21}{8}\right).$$

9.



The constraint is:

$$c = \pi r^2 h - V$$
$$\nabla c = (2\pi r h, \pi r^2).$$

Within this constraint, we want to minimise area, $A(r,h)$:

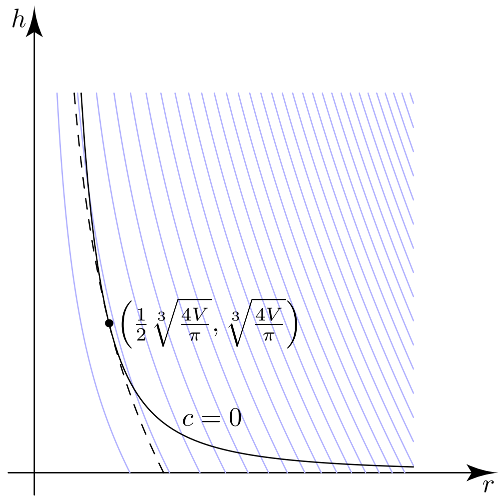
$$A = 2\pi r^2 + 2\pi r h$$
$$\nabla A = (4\pi r + 2\pi h, 2\pi r)$$

Note: If you did it with r and h in the other order, that's fine; the biggest difference will be that (1) and (2) will be swapped.

$$4\pi r + 2\pi h = 2\lambda \pi r h$$
$$2r + h = \lambda r h \tag{1}$$
$$2\pi r = \lambda \pi r^2$$
$$2r = \lambda r^2 \tag{2}$$
$$\pi r^2 h - V = 0 \tag{3}$$

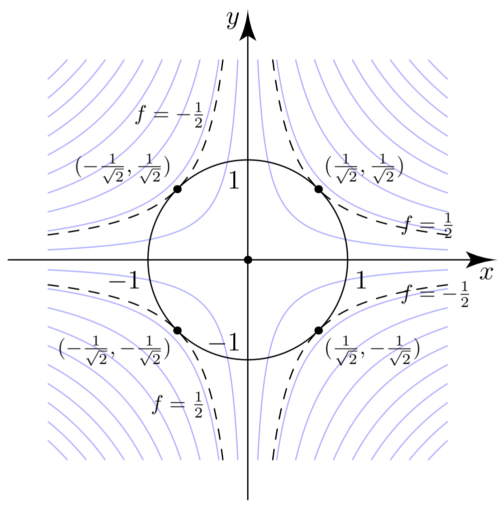
From (2), either $r = 0$, which contradicts (3) (and so can be safely ignored), or $\lambda \neq 0$ (since that would mean $r = 0$). So we can divide by λ:

$$\begin{aligned}
r &= \frac{2}{\lambda} && && (4)\\
2\frac{2}{\lambda} + h &= 2h && \text{(substituting (4) into (1))}\\
\frac{4}{\lambda} &= h\\
\lambda &= \frac{4}{h} && \text{(okay since } h \text{ can't be 0)} && (5)\\
r &= \frac{2}{4/h} && \text{(substituting (5) into (4))}\\
&= \frac{h}{2} && && (6)\\
\pi\left(\frac{h}{2}\right)^2 h - V &= 0 && \text{(substituting (6) into (3))}\\
h &= \sqrt[3]{\frac{4V}{\pi}}\\
r &= \frac{1}{2}\sqrt[3]{\frac{4V}{\pi}} && \text{(from (6))}
\end{aligned}$$



Again, drawing a picture is not essential for this one.

10.



$$f = xy \qquad c = x^2 + y^2 - 1$$

$$\nabla f = \begin{pmatrix} y \\ x \end{pmatrix} \qquad \nabla c = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

Internal stationary points (where $\nabla f = \mathbf{0}$): $(0, 0)$ is a candidate. We substitute it into the constraint inequality to see if it is inside the region:

$$0^2 + 0^2 = 0 \leq 1,$$

so the constraint is satisfied and $(0, 0)$ is confirmed as an internal stationary point.

Using a Lagrange multiplier to find critical points on the boundary:

$$\nabla f = \lambda \nabla c$$
$$\begin{pmatrix} y \\ x \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$
$$y = 2\lambda x \tag{1}$$
$$x = 2\lambda y \tag{2}$$
$$x^2 + y^2 = 1 \qquad \text{(= not $\leq$ because we are looking for points \textit{on} the boundary)} \tag{3}$$
$$x = 4\lambda^2 x \qquad \text{(substituting (1) into (2))} \tag{4}$$
$$x^2 + 4\lambda^2 x^2 = 1 \qquad \text{(substituting (1) into (3))}$$
$$(1 + 4\lambda^2)x^2 = 1 \tag{5}$$

From (5) $x \neq 0$, so it's safe to divide by it in (4). This gives us

$$1 = 4\lambda^2,$$
$$\lambda = \pm\frac{1}{2}.$$

Substituting this into (5) tells us that

$$x = \pm\frac{1}{\sqrt{2}},$$

and putting $\lambda = \pm\frac{1}{2}$ into (1) gives

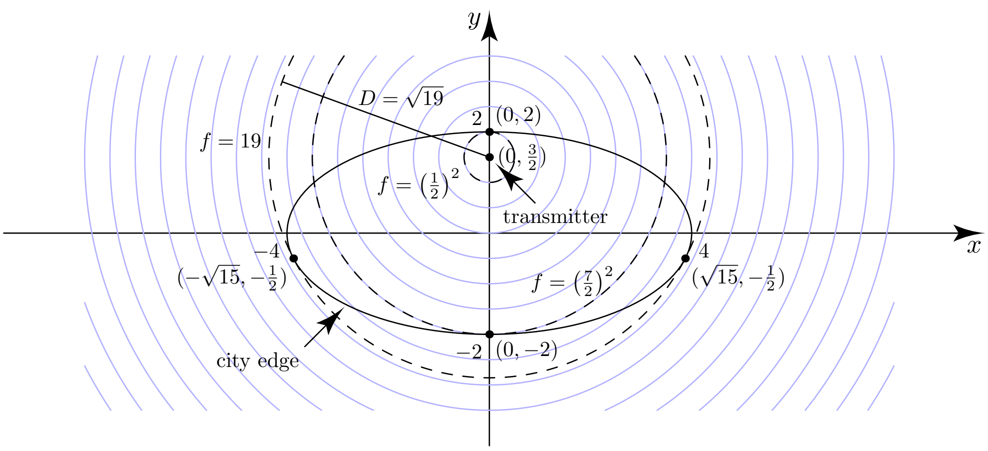
$$y = \pm x = \pm\frac{1}{\sqrt{2}}.$$

In summary, we have one internal stationary point, $(0, 0)$, and four critical points on the boundary:

$$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

Warning: As you can probably see, $\pm$ can be critically important in this type of problem. Make sure you remember! Another thing that often catches people out here is dividing by a variable without checking if it could be zero, or without splitting the zero situation off as a separate case.

11.



This is a complicated looking diagram, but much of it is stuff we will figure out as we go through. We have an elliptical city and a transmitter located at $(0, \frac{3}{2})$. So that the transmitter can be made efficient, we want to know how far you can get from it but still be in the city. Clearly that will be on the boundary somewhere, and it will be a maximum of the distance function, or, equivalently and easier, the distance-squared function

$$f = x^2 + \left(y - \frac{3}{2}\right)^2.$$

So, things on the diagram we currently know are the transmitter location, the city boundary, and the faint circular contours, which are level curves of the distance-squared function f.

Since we know what we're looking for is on the boundary, we can ignore internal stationary points and start straight away with the Lagrange multiplier method.

$$f = x^2 + \left(y - \frac{3}{2}\right)^2 \qquad c = x^2 + 4y^2 - 16$$

$$\nabla f = \begin{pmatrix} 2x \\ 2\left(y - \frac{3}{2}\right) \end{pmatrix} \qquad \nabla c = \begin{pmatrix} 2x \\ 8y \end{pmatrix}$$

$$\nabla f = \lambda \nabla c$$

$$2x = 2\lambda x \tag{1}$$

$$2\left(y - \frac{3}{2}\right) = 8\lambda y \tag{2}$$

$$x^2 + 4y^2 = 16 \tag{3}$$

From (1), either $\lambda = 1$ or $x = 0$. When $x = 0$, (3) gives us $y = \pm 2$. When $\lambda = 1$ we have

$$\begin{aligned}
(2-8)y &= 3, \qquad \text{(from (2))} \\
y &= \frac{3}{-6} \\
&= -\frac{1}{2}, \\
x^2 + 4\left(-\frac{1}{2}\right)^2 &= 16, \qquad \text{(from (3))} \\
x &= \pm\sqrt{15}.
\end{aligned}$$

So we have four critical points on the boundary:

$$(0,2),\ (0,-2),\ \left(-\sqrt{15}, -\frac{1}{2}\right),\ \left(\sqrt{15}, -\frac{1}{2}\right).$$

Plotting these on the diagram, we can see that the two with $x = 0$ are local minima (as we move along the boundary), and that the other two are the maxima we are after (both are the same distance from the transmitter). The level curves corresponding to these points are the dashed circles on the diagram. The maximum distance from the transmitter we can be that is still in the city is

$$\begin{aligned}
D &= \sqrt{\sqrt{15}^2 + \left(-\frac{1}{2} - \frac{3}{2}\right)^2} \\
&= \sqrt{15 + 4} \\
&= \sqrt{19}.
\end{aligned}$$

This is the minimum range for the transmitter.