University of Canterbury

EMTH210

ENGINEERING MATHEMATICS

Topic 3: Fourier series

3.1 FOURIER SERIES

A function $f(x)$ can have a Fourier Series (F.S.) associated with it:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos(nx) + b_n \sin(nx) \Big) \ ,$$

where the coefficients $a_0, \{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ depend in some way on $f(x)$ (for a preview, check out the box on page 5). The important point is that for a "nice" function, its associated Fourier series is actually equal to the function itself, so we can write:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos(nx) + b_n \sin(nx) \Big) \ .$$

When we encounter an inhomogeneous ODE with a RHS which we can't tackle by the method of undetermined coefficients, we will use the FS to rewrite the RHS in terms of sines and cosines, which *can* be tackled with the method of undetermined coefficients. There are many other uses of FS, which in turn is just one part of the wider subject of Fourier analysis. Joseph Fourier[1] (1768–1830) invented FS in order to solve the heat equation in a metal plate.

It might be that only a finite number of the coefficients a_n and b_n are non-zero, in which case we have a finite Fourier series. On the other hand, it may be that an infinite number of the coefficients are non-zero.

Note that because $\sin(nx)$ and $\cos(nx)$ are 2π-periodic, (or rather 2π is the shortest period common to all), the FS is also 2π periodic.

An example of a FS is that if $f(x) = x$ on $-\pi < x < \pi$ then

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin(nx) \ ,$$

[1] Interesting chap. An important part of the French Revolution, later chummy with Napoleon Bonaparte. Discovered the greenhouse effect. Was keen on heat flow (Fourier's law) and vibrations (Fourier analysis) in general: ironically, tripped on the blanket he was wearing to keep warm and died after falling down the stairs.

ie $a_0 =$ ________, $a_n =$ ________ for all n, and $b_n =$ ________ for all n.

For the idea of FS to be useful, we need to know how to get a_n and b_n. We do so by use of the *orthogonality property* of the set $\mathcal{S}$ of functions

$$\mathcal{S} = \{1, \sin(x), \cos(x), \sin(2x), \cos(2x), \dots\} \ ,$$

which is that

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for any two DIFFERENT functions $\phi_1(x)$ and $\phi_2(x)$ in $\mathcal{S}$.

Example 3.1.1

Show that the orthogonality property holds for the pair of functions $\phi_1(x) = \sin(nx)$, $\phi_2(x) = \sin(kx)$ where k and n are natural numbers, and $k \neq n$.

Solution.

♦

Now, when $\phi_1(x) = \phi_2(x)$ we get either

$$\int_{-\pi}^{\pi} \sin^2(kx)dx = \pi \quad \text{or}$$

$$\int_{-\pi}^{\pi} \cos^2(kx)dx = \pi \quad \text{or}$$

$$\int_{-\pi}^{\pi} 1.1dx = 2\pi \ .$$

We use orthogonality and these last three properties to get the FS coefficients, as follows.

In summary:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos(nx) + b_n \sin(nx) \Big) \ ,$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx \ .$$

Note that some books omit the 1/2 from the $a_0/2$ term, in which case the factor in front of this last integral becomes $1/2\pi$. Either way, you get the same answer, of course. But it is best to stick with one definition, and we will use the one in the box.

The above concerns functions on the symmetric interval $(-\pi, \pi)$. Later, we will generalise to any interval, at which point our definition of the FS and its coefficients will change.

Example 3.1.2

Find the Fourier series of

$$f(x) = x \text{ on } -\pi < x < \pi \ .$$

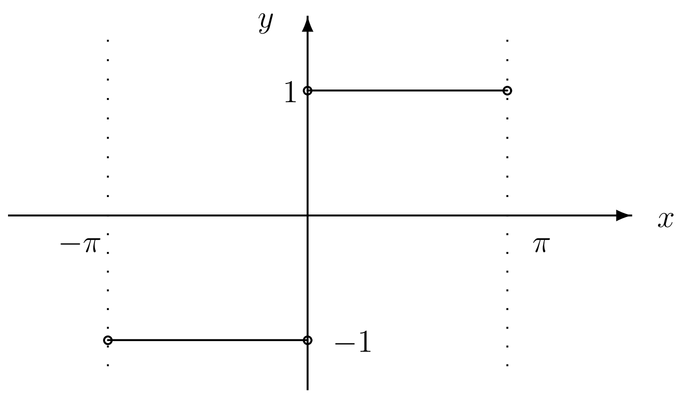
Solution.

♦

Example 3.1.3

Find the Fourier series of the following function:

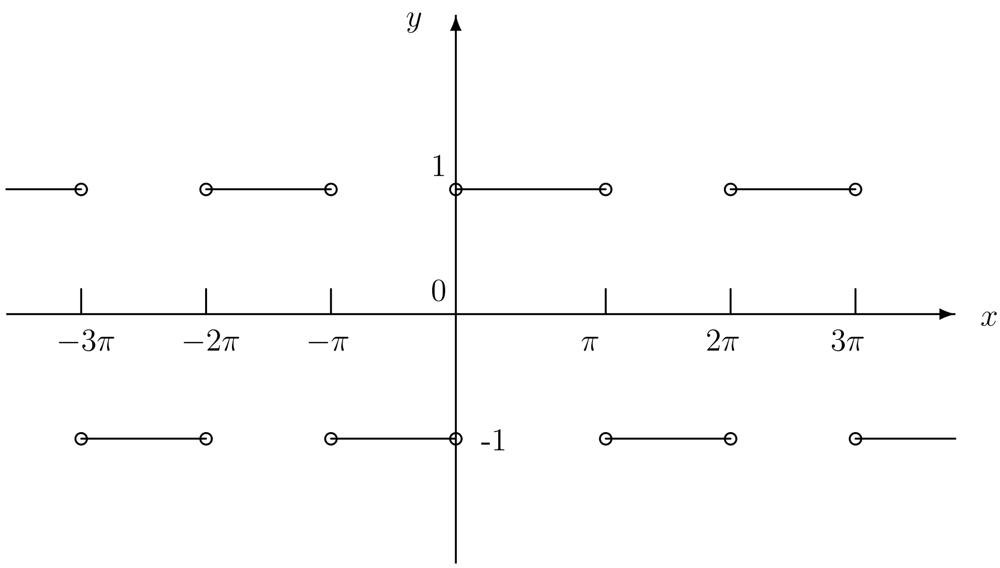
$$f(x) = \begin{cases} 1 & \text{for } 0 < x < \pi \ ; \\ -1 & \text{for } -\pi < x < 0 \ . \end{cases}$$



Solution.

♦

Were we also asked to sketch the FS, we would draw something like this:



3.2 FUNDAMENTAL HARMONICS

Consider the following finite FS:

$$f(x) = 4 + \sin(x) + 12\sin(2x) + 35\sin(5x) \ .$$

The '4' is the constant term which is the $\boxed{\text{average value of } f(x) \text{ over one period}}$. The shortest period common to all terms is 2π. The $\boxed{\textit{fundamental frequency} \text{ is } 1/(\text{shortest period})}$. Here,

$$\text{fundamental frequency } = \frac{1}{2\pi} \ .$$

- The $\sin(x)$ term is the *fundamental frequency term*[2], because its frequency (the inverse of its period) is equal to the fundamental frequency.
- The $12\sin(2x)$ term is the *2nd harmonic*[3], because its frequency is twice the fundamental frequency (so not just because there is a '2' in there).
- The $35\sin(5x)$ term is the *5th harmonic*[4], because its frequency is five times the fundamental frequency.

In the above example, the FS is missing some harmonics, namely the _______, _______, and everything above the _______. Likewise, a FS may also miss a fundamental term, although it will always have a fundamental frequency, as in the following example.

Example 3.2.1

Find the fundamental frequency and identify the harmonic terms of the FS

$$f(x) = 4 + \sin(2x) + \cos(3x) \ .$$

[2]Musicians among you may know this as the *fundamental tone.*

[3]Musicians: *1st overtone.*

[4]Musicians: *4th overtone.*

Solution.

♦

3.3 GIBBS'S PHENOMENON

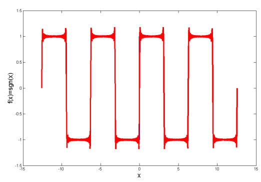
We have already seen that the function

$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases}$$

has FS

$$f(x) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} \sin(nx) \ .$$

If we include only the first N terms as an approximation to $f(x)$ we get erroneous oscillations close to corners of the square wave. The output from a quick MATLAB® m-file which you should all be able to write looks like



If the file is played as sound (or if the phenomenon is a naturally aural one) then these oscillations make a ringing sound. This phenomenon is named not after its discoverer (typical), but rather its populariser, JW Gibbs. The maximum height of the oscillations near the jump discontinuities do not go to zero as $N \to \infty$. This causes all manner of problems in signal processing, for example in MRI scans.

3.4 NOT A FOURIER SERIES

Example 3.4.1

Why is the following not a FS?

$$f(x) = \cos(x) + \frac{1}{2}\cos(\sqrt{2}x) + \frac{1}{4}\cos(2x) + \frac{1}{8}\cos(\sqrt{8}x) + \ldots$$

Solution.

♦

3.5 ODD AND EVEN FUNCTIONS

Here we formalise the shortcuts for odd and even functions which we saw in Lecture 17.

ODD FUNCTIONS

For an odd function,

$$f(-x) = \underline{\hspace{3cm}}$$

so

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \underline{\hspace{3cm}} .$$

Now $f(x)\cos(nx)$ is __________, so

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx) = \underline{\hspace{3cm}} ,$$

for all n, but $f(x)\sin(nx)$ is __________, so

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)dx ,$$

for all n.

EVEN FUNCTIONS

For an even function

$$f(-x) = \_\_\_\_\_\_\_\_\_\_ \ ,$$

so

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^{\pi} f(x)dx \\ a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx \ , \end{aligned}$$

for all n, as $f(x)\cos(nx)$ is __________, but $f(x)\sin(nx)$ is __________ so

$$b_n = \_\_\_\_\_\_\_\_\_\_$$

for all n.

Example 3.5.1

Find the Fourier series of the function

$$f(x) = \pi - |x| \qquad -\pi < x < \pi \ .$$

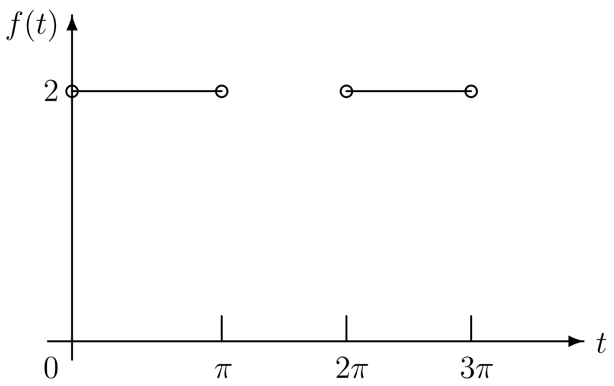
Solution.

3.6 F.S. IN UNDETERMINED COEFFICIENTS

The main use of FS in this course will be to rewrite unusual right hand sides of ODEs in terms of sines and cosines, which we know how to deal with in the method of undetermined coefficients. Here is an illustrative example.

Example 3.6.1

A mass $m = 1$kg is suspended on a spring which hangs vertically. The spring obeys Hooke's law with a spring constant k whose square root is not a natural number. Let t be time and y be the displacement of the mass from the equilibrium position, where downwards is positive, and is the direction in which gravity acts. The mass-spring system is subject to forcing from the force $f(t)$. Find the position of the mass at all positive times when the forcing takes the form



and continues in this way for all positive times.

Solution.

♦

The following is very important, because we generalise what we have learned about FS of 2π-periodic functions to enable us to find the FS of a p-periodic function[5].

So let us take a 2π-periodic function which we have already studied, namely

$$f(x) = f(x + 2\pi) \quad \text{and} \quad f(x) = x \quad \text{on} \quad -\pi < x < \pi \ .$$

Although initially we didn't say that the function was periodic, its FS *is* periodic, so both $f(x)$ defined only on $(-\pi, \pi)$ and $f(x)$ defined as above have the same FS, namely

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin(nx) \ .$$

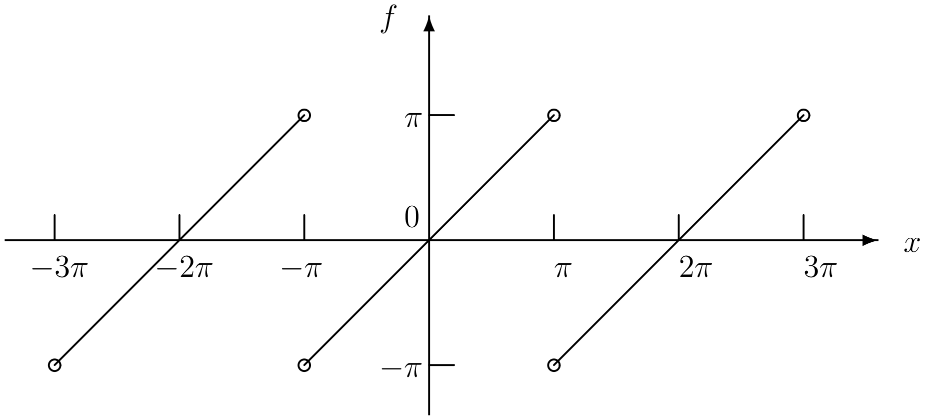
Now suppose that in both the original definition and the FS, we put $x = 2\pi t/p$, where p is some non-zero number, and t is a variable:

$$f(x) = f\left(\frac{2\pi t}{p}\right) = g(t) \quad \text{say.}$$

Now our new function $g(t)$ has period p. The Fourier series for $g(t)$ is simply

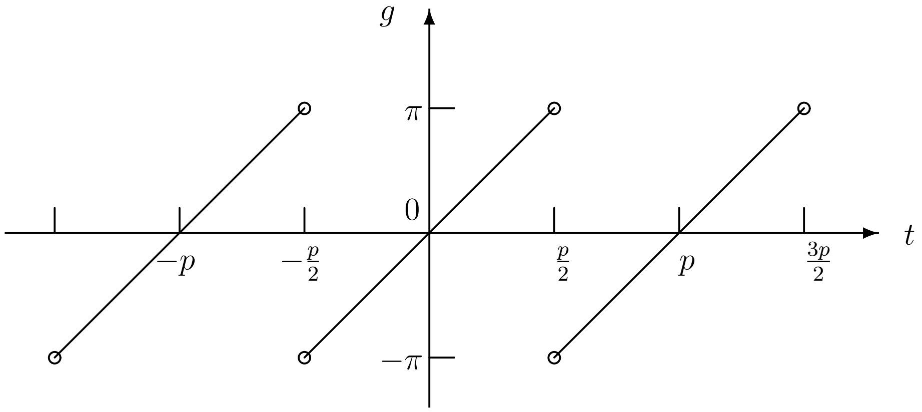
$$g(t) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin\left(\frac{2n\pi t}{p}\right) ,$$

after making the substitution. Graphically, we have gone from this



[5]What we don't get into here, but which is nevertheless important, is that almost any function can be thought of as p-periodic for some p, if you truncate it, focus on only a certain range of interest, or chop it into suitable pieces.

to this



We can generalise the definition of Fourier series to work with any periodic function $g(t)$ of period p:

$$g(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi t}{p}\right) + b_n \sin\left(\frac{2n\pi t}{p}\right)$$

where

$$a_0 = \frac{2}{p}\int_{\text{one period}} g(t)dt$$

$$a_n = \frac{2}{p}\int_{\text{one period}} g(t)\cos\left(\frac{2n\pi t}{p}\right)dt$$

$$b_n = \frac{2}{p}\int_{\text{one period}} g(t)\sin\left(\frac{2n\pi t}{p}\right)$$

Note that we do not have to integrate from $-p/2$ to $p/2$. It is sufficient to integrate over any interval $[A, A+p]$, with A arbitrary.

The above can by proved by orthogonality on the set

$$\left\{1, \sin\left(\frac{2n\pi t}{p}\right), \cos\left(\frac{2n\pi t}{p}\right), \dots\right\}.$$

If we set $p = 2\pi$ then we retrieve exactly the definition we have been working with. You are much more likely to be tested on the general form of the FS than the FS of a 2π-periodic function. Also, it is this general form which appears on the formula sheet.

Example 3.6.2

Find the FS of the function

$$f(x) = \begin{cases} 0 & -1 \leqslant x < 0 \\ \sin(\pi x) & 0 \leqslant x < 1 \end{cases}.$$

Solution.