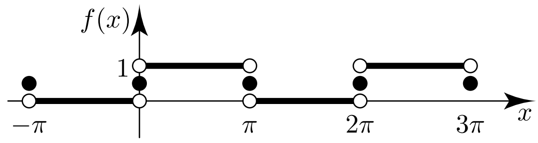
EMTH210 Tutorial 6: Fourier Series – Solutions

Preparation problems (homework)

1. (a)

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 1 & 0 \le x < \pi \end{cases}$$



The graph of the Fourier series of $f(x)$.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_{0}^{\pi} \mathrm{d}x \\ &= 1 \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_{0}^{\pi} \cos nx\,\mathrm{d}x \\ &= \frac{1}{\pi}\left[\frac{1}{n}\sin nx\right]_0^{\pi} \\ &= \frac{1}{\pi}\left(\frac{1}{n}\sin n\pi - 0\right) \\ &= 0 \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_{0}^{\pi} \sin nx\,\mathrm{d}x \\ &= \frac{1}{\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi} \\ &= \frac{1}{\pi}\left(-\frac{1}{n}\cos n\pi + \frac{1}{n}(1)\right) \\ &= \frac{1}{n\pi}\left(1 - (-1)^n\right) \\ &= \begin{cases} \frac{2}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \end{aligned}$$

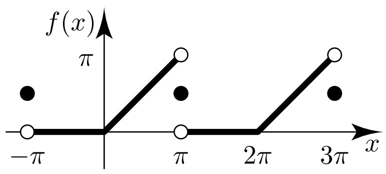
$$f(x) = \frac{1}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{2}{n\pi}\sin nx$$

$$\left(= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(2n-1)\pi}\sin\left((2n-1)x\right)\right)$$

(Note that $2n-1$ is always odd.)

(b)

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 \le x < \pi \end{cases}$$



The graph of the Fourier series of $f(x)$.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_{0}^{\pi} x\,\mathrm{d}x \\ &= \frac{\pi}{2} \end{aligned}$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{0}^{\pi} x\cos nx \,\mathrm{d}x \\
&\qquad \begin{aligned} u &= x & \mathrm{d}v &= \cos nx\,\mathrm{d}x \\ \mathrm{d}u &= \mathrm{d}x & v &= \tfrac{1}{n}\sin nx \end{aligned} \\
&= \frac{1}{\pi}\left(\left[\frac{x}{n}\sin nx\right]_0^{\pi} - \int_0^{\pi}\frac{1}{n}\sin nx\,\mathrm{d}x\right) \\
&= \frac{1}{\pi}\left(\left[\frac{x}{n}\sin nx\right]_0^{\pi} + \left[\frac{1}{n^2}\cos nx\right]_0^{\pi}\right) \\
&= \frac{1}{\pi}\left(0 + \frac{1}{n^2}(\cos n\pi - 1)\right) \\
&= \frac{1}{n^2\pi}((-1)^n - 1)
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{0}^{\pi} x\sin nx \,\mathrm{d}x \\
&\qquad \begin{aligned} u &= x & \mathrm{d}v &= \sin nx\,\mathrm{d}x \\ \mathrm{d}u &= \mathrm{d}x & v &= -\tfrac{1}{n}\cos nx \end{aligned} \\
&= \frac{1}{\pi}\left(\left[-\frac{x}{n}\cos nx\right]_0^{\pi} + \int_0^{\pi}\frac{1}{n}\cos nx\,\mathrm{d}x\right) \\
&= \frac{1}{\pi}\left(\left[-\frac{x}{n}\cos nx\right]_0^{\pi} + \left[\frac{1}{n^2}\sin nx\right]_0^{\pi}\right) \\
&= \frac{1}{\pi}\left(\left(-\frac{\pi}{n}\cos n\pi - 0\right) + (0)\right) \\
&= \frac{1}{n}(-1)^{n+1}
\end{aligned}
$$

$$
f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1}{n^2\pi}((-1)^n - 1)\cos nx + \frac{(-1)^{n+1}}{n}\sin nx\right)
$$

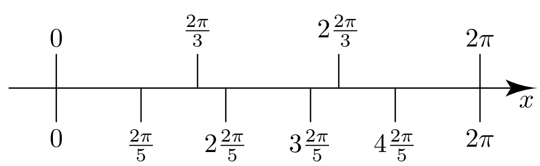
You've probably gathered that there are a few different ways of writing these terms, especially the alternating ones. Just pick the clearest for each situation.

2. (a) Yes. The periods of the two terms are $k\frac{2\pi}{3}$ and $\ell\frac{2\pi}{5}$ for $k, \ell \in \mathbb{Z}^+$.

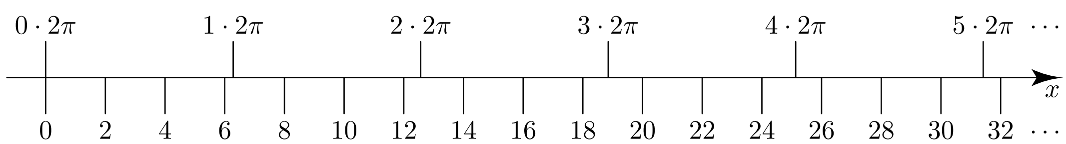
$$
\begin{aligned}
k\frac{2\pi}{3} &= \ell\frac{2\pi}{5} \\
5k &= 3\ell
\end{aligned}
$$

The least common multiple of 3 and 5 is 15, so $k = 3$, $\ell = 5$, and so the shortest common period is 2π, and the fundamental frequency is $\frac{1}{2\pi}$.

Alternatively, we can see it graphically. $\sin(3x)$ repeats every $\frac{2\pi}{3}$ units, and $\cos(5x)$ repeats every $\frac{2\pi}{5}$ units. They line up at 0, but where do they next line up? That tells us the shortest common period.



(b) No. The periods of $\sin x$ and $\cos(\pi x)$ are, respectively, $2k\pi$ and 2ℓ for $k, \ell \in \mathbb{Z}^+$. Since π is irrational, there are no common periods, so f is not periodic and is therefore not a Fourier series. The periods of the two terms never line up again:



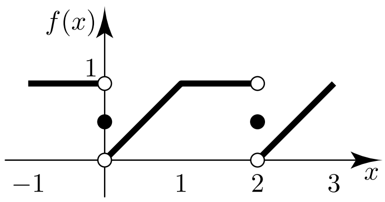
(c) Yes. The periods of each term $\sin(nx)$ are of the form $k\frac{2\pi}{2n} = k\frac{\pi}{n}$ for $k \in \mathbb{Z}^+$ (write out a few terms to see it), so the shortest period common to all n is π, and the fundamental frequency is $\frac{1}{\pi}$.

(d) Yes! All the coefficients are 0 except for a_0. Indeterminate fundamental frequency, because there is no *shortest* period.

(e) No. The x term is not periodic so neither is f, which cannot be a Fourier series.

3. (a)

$$f(x) = \begin{cases} 1 & -1 < x < 0 \\ x & 0 \le x < 1 \end{cases}$$



The graph of the Fourier series of $f(x)$. It has period 2 and is neither even nor odd.

$$\begin{aligned} a_0 &= \frac{2}{2}\int_{-1}^{1} f(x)\,\mathrm{d}x \\ &= \int_{-1}^{0} \mathrm{d}x + \int_{0}^{1} x\,\mathrm{d}x \\ &= 1 + \frac{1}{2} \\ &= \frac{3}{2} \end{aligned}$$

$$u = x \qquad \mathrm{d}v = \cos n\pi x\,\mathrm{d}x$$
$$\mathrm{d}u = \mathrm{d}x \qquad v = \tfrac{1}{n\pi}\sin n\pi x$$

$$\begin{aligned} a_n &= \frac{2}{2}\int_{-1}^{1} f(x)\cos n\pi x\,\mathrm{d}x \\ &= \int_{-1}^{0}\cos n\pi x\,\mathrm{d}x + \int_{0}^{1} x\cos n\pi x\,\mathrm{d}x \\ &= \left[\frac{1}{n\pi}\sin n\pi x\right]_{-1}^{0} + \left[\frac{x}{n\pi}\sin n\pi x\right]_{0}^{1} - \int_{0}^{1}\frac{1}{n\pi}\sin n\pi x\,\mathrm{d}x \\ &= \cancelto{0}{\left[\frac{1}{n\pi}\sin n\pi x\right]_{-1}^{0}} + \cancelto{0}{\left[\frac{x}{n\pi}\sin n\pi x\right]_{0}^{1}} - \left[-\frac{1}{n^2\pi^2}\cos n\pi x\right]_{0}^{1} \\ &= \frac{1}{n^2\pi^2}(\cos n\pi - 1) \\ &= \frac{1}{n^2\pi^2}((-1)^n - 1) \end{aligned}$$

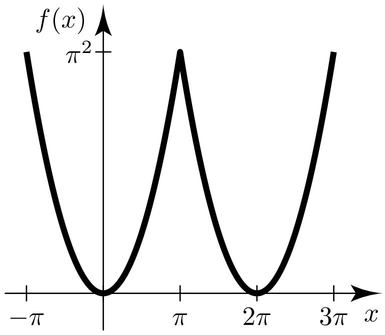
$$u = x \qquad \mathrm{d}v = \sin n\pi x\,\mathrm{d}x$$
$$\mathrm{d}u = \mathrm{d}x \qquad v = \tfrac{-1}{n\pi}\cos n\pi x$$

$$\begin{aligned} b_n &= \frac{2}{2}\int_{-1}^{1} f(x)\sin n\pi x\,\mathrm{d}x \\ &= \int_{-1}^{0}\sin n\pi x\,\mathrm{d}x + \int_{0}^{1} x\sin n\pi x\,\mathrm{d}x \\ &= \left[-\frac{1}{n\pi}\cos n\pi x\right]_{-1}^{0} + \left[-\frac{x}{n\pi}\cos n\pi x\right]_{0}^{1} + \int_{0}^{1}\frac{1}{n\pi}\cos n\pi x\,\mathrm{d}x \\ &= \left[-\frac{1}{n\pi}\cos n\pi x\right]_{-1}^{0} + \left[-\frac{x}{n\pi}\cos n\pi x\right]_{0}^{1} + \cancelto{0}{\left[\frac{1}{n^2\pi^2}\sin n\pi x\right]_{0}^{1}} \\ &= -\frac{1}{n\pi}(1 - \cos(-n\pi)) - \frac{1}{n\pi}(\cos n\pi - 0) \\ &= -\frac{1}{n\pi}\left(1 - (-1)^n + (-1)^n\right) \\ &= -\frac{1}{n\pi} \end{aligned}$$

$$\therefore f(x) = \frac{3}{4} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{n^2\pi^2}\cos n\pi x - \frac{1}{n\pi}\sin n\pi x\right)$$

(b)

$$
\begin{aligned}
f(x) &= x^2, \text{ on } -\pi < x < \pi \\
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,\mathrm{d}x \\
&= \frac{1}{\pi}\left[\frac{1}{3}x^3\right]_{-\pi}^{\pi} \\
&= \frac{\pi^2}{3} + \frac{\pi^2}{3} \\
&= \frac{2\pi^2}{3}
\end{aligned}
$$



The graph of the Fourier series of $f(x)$. It has period 2π and is even.

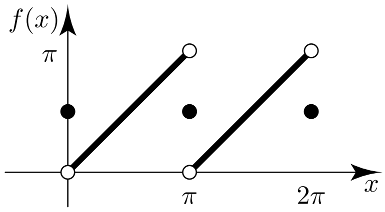
$$
\begin{aligned}
u &= x^2 & \mathrm{d}v &= \cos nx\,\mathrm{d}x \\
\mathrm{d}u &= 2x\,\mathrm{d}x & v &= \tfrac{1}{n}\sin nx \\
u &= x & \mathrm{d}v &= \sin nx\,\mathrm{d}x \\
\mathrm{d}u &= \mathrm{d}x & v &= -\tfrac{1}{n}\cos nx
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\left(\cancelto{0}{\left[\frac{x^2}{n}\sin nx\right]_{-\pi}^{\pi}} - \int_{-\pi}^{\pi} 2x\frac{1}{n}\sin nx\,\mathrm{d}x\right) \\
&= -\frac{2}{n\pi}\left(\left[-\frac{x}{n}\cos nx\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} -\frac{1}{n}\cos nx\,\mathrm{d}x\right) \\
&= -\frac{2}{n\pi}\left(\left[-\frac{x}{n}\cos nx\right]_{-\pi}^{\pi} - \cancelto{0}{\left[-\frac{1}{n^2}\sin nx\right]_{-\pi}^{\pi}}\right) \\
&= -\frac{2}{n\pi}\left(-\frac{\pi}{n}\cos n\pi - \frac{\pi}{n}\cos(-n\pi)\right) \\
&= -\frac{2}{n^2}\left(-(-1)^n - (-1)^n\right) \\
&= \frac{4(-1)^n}{n^2} \\
b_n &= 0, \text{since } f(x) \text{ is even}
\end{aligned}
$$

$$
\therefore f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx
$$

(c)

$$
\begin{aligned}
f(x) &= x, \text{ on } 0 \le x < \pi \\
a_0 &= \frac{1}{\pi/2}\int_0^{\pi} f(x)\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi} x\,\mathrm{d}x \\
&= \frac{2}{\pi}\left[\frac{1}{2}x^2\right]_0^{\pi} \\
&= \pi
\end{aligned}
$$



The graph of the Fourier series of $f(x)$. It has period π and is neither even nor odd.

$$
\begin{aligned}
u &= x & \mathrm{d}v &= \cos 2nx\,\mathrm{d}x \\
\mathrm{d}u &= \mathrm{d}x & v &= \tfrac{1}{2n}\sin 2nx
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} x\cos 2nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\left(\cancelto{0}{\left[\frac{x}{2n}\sin 2nx\right]_0^{\pi}} - \int_0^{\pi}\frac{1}{2n}\sin 2nx\,\mathrm{d}x\right) \\
&= -\frac{2}{\pi}\left[-\frac{1}{4n^2}\cos 2nx\right]_0^{\pi} \\
&= -\frac{2}{\pi}\left(-\frac{1}{4n^2} + \frac{1}{4n^2}\right) \\
&= 0
\end{aligned}
$$

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} x\sin 2nx\,\mathrm{d}x & u = x \quad \mathrm{d}v = \sin 2nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\left(\left[-\frac{x}{2n}\cos 2nx\right]_0^{\pi} - \int_0^{\pi} -\frac{1}{2n}\cos 2nx\,\mathrm{d}x\right) & \mathrm{d}u = \mathrm{d}x \quad v = \tfrac{-1}{2n}\cos 2nx \\
&= \frac{2}{\pi}\left(\left[-\frac{x}{2n}\cos 2nx\right]_0^{\pi} - \cancelto{0}{\left[-\frac{1}{4n^2}\sin 2nx\right]_0^{\pi}}\right) \\
&= \frac{2}{\pi}\left(-\frac{\pi}{2n}\cos 2n\pi - 0\right) \\
&= -\frac{1}{n}
\end{aligned}$$

$$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} -\frac{1}{n}\sin 2nx$$

4. This question is not really hard, but there's a trick. A periodic function $f(x)$ has one, and only one, Fourier series, and it can be written in the standard form. This means that if we can rewrite $f(x)$ into the Fourier series form, *no matter how we do it*, we have found its Fourier series. So here, we will use the double angle formula $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2(\theta)$ to find something that equals $f(x)$ but is in the Fourier series form.

$$\begin{aligned}
\cos 2x &= \cos^2 x - \sin^2 x \\
&= 1 - 2\sin^2 x \\
\sin^2 x &= \frac{1}{2}(1 - \cos 2x) \\
\sin^4 x &= \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x) \qquad (1)
\end{aligned}$$

$$\begin{aligned}
\cos 2x &= \cos^2 x - \sin^2 x \\
\cos 4x &= \cos^2 2x - \sin^2 2x \\
&= \cos^2 2x - (1 - \cos^2 2x) \\
&= 2\cos^2 2x - 1 \\
\cos^2 2x &= \frac{1}{2}(1 + \cos 4x) \qquad (2)
\end{aligned}$$

Substituting (2) into (1),

$$\begin{aligned}
\sin^4 x &= \frac{1}{4}\left(1 - 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right) \\
&= \frac{1}{4} + \frac{1}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x \\
&= \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x.
\end{aligned}$$

This is the Fourier series of $f(x)$. (Yes, it's okay that it's finite; the coefficients can be any real number, and any number of them can be 0.) The periods of $\cos 2x$ are π, 2π, etc., and the periods of $\cos 4x$ are $\frac{\pi}{2}$, π, etc., so the shortest common period is π, and the fundamental frequency is $\frac{1}{\pi}$.

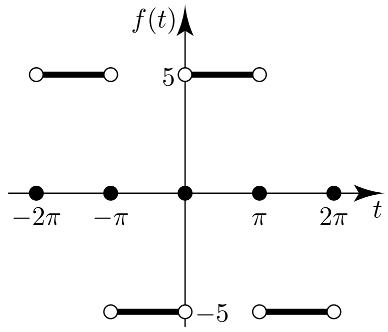
Problems for the tutorial

5. Now come to the main reason we are doing Fourier series: using the method of undetermined coefficients (that we were doing last tutorial) to solve differential equations with arbitrary periodic right hand sides. First, we rewrite the right hand side, $f(t)$, as its Fourier series.

$$f(t) = \begin{cases} 5 & 0 < t < \pi \\ -5 & \pi \le t < 2\pi \end{cases}$$

$$a_0 = a_n = 0, \text{ since } f(x) \text{ is odd.}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\left(\int_0^\pi 5\sin nt\,\mathrm{d}t + \int_\pi^{2\pi} -5\sin nt\,\mathrm{d}t\right) \\ &= \frac{1}{\pi}\left(\left[-\frac{5}{n}\cos nt\right]_0^\pi + \left[\frac{5}{n}\cos nt\right]_\pi^{2\pi}\right) \\ &= \frac{5}{n}\frac{1}{\pi}\big(-(-1)^n + 1 + 1 - (-1)^n\big) \\ &= \frac{10}{n\pi}(1-(-1)^n) \\ \therefore f(t) &= \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nt \end{aligned}$$



The graph of the Fourier series of $f(t)$. It has period 2π and is odd.

Next, we apply the method of undetermined coefficients using this Fourier series as the RHS. The α_n and β_n are the unknown coefficients, like A, B, etc. before, only there are an infinite number of them now. By convention we put α_n in front of cos and β_n in front of sin, like a_n and b_n.

$$\text{DE is} \quad y'' + 10y = f(t)$$

$$\text{aux. eq. is} \quad l^2 + 10 = 0 \implies l = \pm\sqrt{10}i \qquad (m \text{ was used in the DE})$$

$$y_c = c_1\cos\sqrt{10}t + c_2\sin\sqrt{10}t$$

Rule 1: $y_p = \sum_{n=1}^{\infty}\beta_n\sin nt$

Rule 2: no change

Rule 3: $y_p = \sum_{n=1}^{\infty}(\alpha_n\cos nt + \beta_n\sin nt)$

Rule 4: no change

$$y_p' = \sum_{n=1}^{\infty}(-\alpha_n n\sin nt + \beta_n n\cos nt), \quad y_p'' = \sum_{n=1}^{\infty}\big(-\alpha_n n^2\cos nt - \beta_n n^2\sin nt\big)$$

$$\begin{aligned} y_p'' + 10y_p &= \sum_{n=1}^{\infty}\big(\alpha_n(10-n^2)\cos nt + \beta_n(10-n^2)\sin nt\big) \\ &= \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nt \\ \alpha_n(10-n^2) &= 0 \implies \alpha_n = 0,\ \forall n \\ \beta_n(10-n^2) &= \frac{10(1-(-1)^n)}{n\pi} \\ \beta_n &= \frac{10(1-(-1)^n)}{n\pi(10-n^2)} \\ y_p &= \sum_{n=1}^{\infty}\frac{10(1-(-1)^n)}{n\pi(10-n^2)}\sin nt \end{aligned}$$

$$\therefore y = c_1\cos\sqrt{10}t + c_2\sin\sqrt{10}t + \sum_{n=1}^{\infty}\frac{10(1-(-1)^n)}{n\pi(10-n^2)}\sin nt$$

6.

$$
\begin{aligned}
f(t) &= t^2, \text{ on } -\pi \le t < \pi \\
&= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nt, \text{ from } 3\text{(b)} \\
y_c &= c_1 \sin \sqrt{10}t + c_2 \cos \sqrt{10}t, \text{ from } 5
\end{aligned}
$$

Rule 1: $y_p = A + \sum_{n=1}^{\infty} \alpha_n \cos nt$

Rule 2: no change

Rule 3: $y_p = A + \sum_{n=1}^{\infty} (\alpha_n \cos nt + \beta_n \sin nt)$

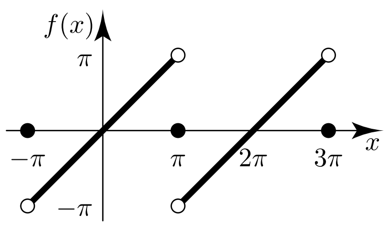
Rule 4: no change

$$
y_p' = \sum_{n=1}^{\infty} (-\alpha_n n \sin nt + \beta_n n \cos nt), \quad y_p'' = \sum_{n=1}^{\infty} \left(-\alpha_n n^2 \cos nt - \beta_n n^2 \sin nt\right)
$$

$$
\begin{aligned}
y_p'' + 10y_p &= 10A + \sum_{n=1}^{\infty} \left(\alpha_n(10-n^2)\cos nt + \beta_n(10-n^2)\sin nt\right) \\
&= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nt \\
10A &= \frac{\pi^2}{3} \implies A = \frac{\pi^2}{30} \\
\alpha_n(10-n^2) &= \frac{4(-1)^n}{n^2} \\
\alpha_n &= \frac{4(-1)^n}{n^2(10-n^2)} \\
\beta_n(10-n^2) &= 0 \implies \beta_n = 0, \ \forall n \\
y_p &= \frac{\pi^2}{30} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2(10-n^2)} \cos nt \\
\therefore y &= c_1 \cos \sqrt{10}t + c_2 \sin \sqrt{10}t + \frac{\pi^2}{30} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2(10-n^2)} \cos nt
\end{aligned}
$$

7.

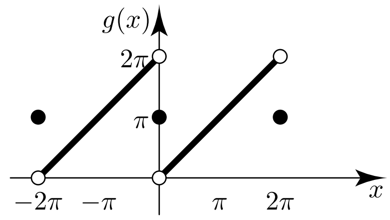
$$
\begin{aligned}
f(x) &= x, \text{ on } -\pi < x < \pi \\
&= \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx
\end{aligned}
$$



The graph of the Fourier series of $f(x)$.

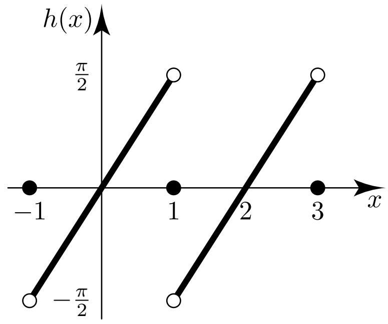
$$
\begin{aligned}
g(x) &= \pi + f(x+\pi) \\
&= \pi + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin\left(n(x+\pi)\right) \\
&= \pi + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} (-1)^n \sin nx \\
&= \pi - \sum_{n=1}^{\infty} \frac{2}{n} \sin nx
\end{aligned}
$$

$$
\begin{aligned}
\sin(A+B) &= \sin A \cos B + \cos A \sin B \\
\sin(nx+n\pi) &= \sin nx \cos n\pi + \cos nx \cancelto{0}{\sin n\pi} \\
&= (-1)^n \sin nx
\end{aligned}
$$



The graph of the Fourier series of $g(x)$. The whole graph of $f(x)$ is shifted up π and left π.

$$
\begin{aligned}
h(x) &= \frac{f(\pi x)}{2} \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin n\pi x \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\pi x
\end{aligned}
$$

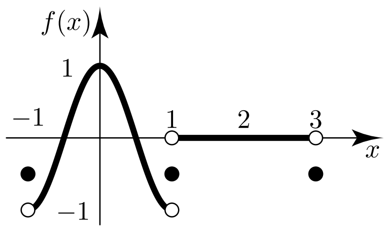


The graph of the Fourier series of $h(x)$. The whole graph of $f(x)$ is scaled by $\frac{1}{\pi}$ in the x direction and $\frac{1}{2}$ in the y direction.

8.

$$
f(x) = \begin{cases} \cos \pi x & -1 < x < 1 \\ 0 & 1 < x < 3 \end{cases}
$$

$$
\begin{aligned}
a_0 &= \frac{1}{2} \int_{-1}^{1} \cos \pi x \,\mathrm{d}x + \int_{1}^{3} 0 \,\mathrm{d}x \\
&= \frac{1}{2} \cancelto{0}{\left[\frac{1}{\pi} \sin \pi x\right]_{-1}^{1}} \\
&= 0
\end{aligned}
$$



The graph of the Fourier series of $f(x)$ with period 4. It is even.

$$
\begin{aligned}
a_n &= \frac{1}{2} \int_{-1}^{1} \cos \pi x \cos \frac{n\pi x}{2} \,\mathrm{d}x + \int_{1}^{3} 0 \cos \frac{n\pi x}{2} \,\mathrm{d}x \\
&= \frac{1}{4} \int_{-1}^{1} \left(\cos\left(\pi x + \frac{n\pi x}{2}\right) + \cos\left(\pi x - \frac{n\pi x}{2}\right) \right) \mathrm{d}x \\
&= \frac{1}{4} \int_{-1}^{1} \left(\cos\left(\left(1+\frac{n}{2}\right)\pi x\right) + \cos\left(\left(1-\frac{n}{2}\right)\pi x\right) \right) \mathrm{d}x
\end{aligned}
$$

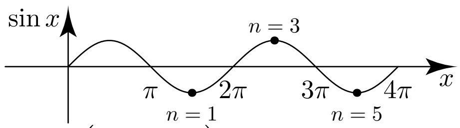
Since we have a $\cos\left(\left(1-\frac{n}{2}\right)\pi\right)x$ term, we have to treat the $n=2$ case separately.

$$\begin{aligned}
a_2 &= \frac{1}{4}\int_{-1}^{1}(\cos 2\pi x + 1)\,\mathrm{d}x \\
&= \frac{1}{4}\left[\frac{1}{2\pi}\sin 2\pi x + x\right]_{-1}^{1} \\
&= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\text{For } n \neq 2,\ a_n &= \frac{1}{4}\left[\frac{1}{\left(1+\frac{n}{2}\right)\pi}\sin\left(\left(1+\frac{n}{2}\right)\pi x\right) + \frac{1}{\left(1-\frac{n}{2}\right)\pi}\sin\left(\left(1-\frac{n}{2}\right)\pi x\right)\right]_{-1}^{1} \\
&= \frac{1}{4}\left(\frac{\sin\left(\left(1+\frac{n}{2}\right)\pi\right)}{\left(1+\frac{n}{2}\right)\pi} + \frac{\sin\left(\left(1-\frac{n}{2}\right)\pi\right)}{\left(1-\frac{n}{2}\right)\pi} - \right. \\
&\qquad \left. \frac{\sin\left(-\left(1+\frac{n}{2}\right)\pi\right)}{\left(1+\frac{n}{2}\right)\pi} - \frac{\sin\left(-\left(1-\frac{n}{2}\right)\pi\right)}{\left(1-\frac{n}{2}\right)\pi}\right) \\
&= \frac{2}{4}\left(\frac{\sin\left(\left(1+\frac{n}{2}\right)\pi\right)}{\left(1+\frac{n}{2}\right)\pi} + \frac{\sin\left(\left(1-\frac{n}{2}\right)\pi\right)}{\left(1-\frac{n}{2}\right)\pi}\right).
\end{aligned}$$

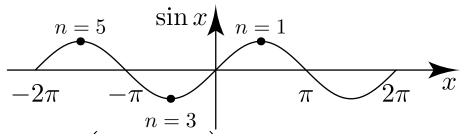
When n is even, the sin terms will have whole numbers times π, and so the whole thing will be 0. When n is odd, the sin terms will contain $\frac{\pi}{2}$ or $\frac{3\pi}{2}$, modulo 2π, and so they will be ± 1, depending on n.

For $\sin\left(\left(1+\frac{n}{2}\right)\pi\right)$:



So $\sin\left(\left(1+\frac{n}{2}\right)\pi\right) = (-1)^{\frac{n+1}{2}}$.

For $\sin\left(\left(1-\frac{n}{2}\right)\pi\right)$:



So $\sin\left(\left(1-\frac{n}{2}\right)\pi\right) = (-1)^{\frac{n-1}{2}} = -(-1)^{\frac{n+1}{2}}$ (to match the other one).

Putting this all together, for odd n,

$$\begin{aligned}
a_n &= \frac{2}{4}\left(\frac{\sin\left(\left(1+\frac{n}{2}\right)\pi\right)}{\left(1+\frac{n}{2}\right)\pi} + \frac{\sin\left(\left(1-\frac{n}{2}\right)\pi\right)}{\left(1-\frac{n}{2}\right)\pi}\right) \\
&= \frac{1}{2}\left(\frac{(-1)^{\frac{n+1}{2}}}{\left(1+\frac{n}{2}\right)\pi} - \frac{(-1)^{\frac{n+1}{2}}}{\left(1-\frac{n}{2}\right)\pi}\right) \\
&= \frac{1}{2\pi}\left(\frac{(-1)^{\frac{n+1}{2}}\left(1-\frac{n}{2}\right) - (-1)^{\frac{n+1}{2}}\left(1+\frac{n}{2}\right)}{1-\frac{n^2}{4}}\right) \\
&= \frac{1}{2\pi}\left(\frac{-n(-1)^{\frac{n+1}{2}}}{1-\frac{n^2}{4}}\right) = \frac{n(-1)^{\frac{n+1}{2}}}{2\pi\left(\frac{n^2}{4}-1\right)},
\end{aligned}$$

and so for any n,

$$\begin{aligned}
a_n &= \begin{cases} \frac{1}{2} & \text{for } n = 2 \\ \frac{n(-1)^{\frac{n+1}{2}}}{2\pi\left(\frac{n^2}{4}-1\right)} & \text{for odd } n \\ 0 & \text{otherwise,} \end{cases} \\
b_n &= 0 \qquad \text{(since the Fourier series for } f(x) \text{ with period 4 is even).} \\
\therefore f(x) &= \frac{1}{2}\cos\pi x + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{n(-1)^{\frac{n+1}{2}}}{2\pi\left(\frac{n^2}{4}-1\right)}\cos\frac{n\pi x}{2}.
\end{aligned}$$