EMTH210 Tutorial 7:
Eigenproblems, and Basic Double Integrals

For the week starting Monday 27 April.

The homework questions this week are **7** and **11(d)**.

1. Which of the vectors u and v are eigenvectors of the matrix A? State the corresponding eigenvalue.
$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \quad u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

2. Find the determinant of the following matrix by cofactor expansion.
$$B = \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 3 & 2 \\ 1 & -5 & -3 & 1 \\ 2 & 7 & 0 & 0 \end{pmatrix}.$$

3. Find the eigenvalues and eigenvectors of the following matrices.
$$C = \begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 1 & 3 \\ 1 & 2 & 0 \\ 3 & 0 & 2 \end{pmatrix}, \quad E = \begin{pmatrix} -1 & 2 & 1 \\ 2 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix}.$$

4. Determine which of the following matrices are orthogonal
$$B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad C = \frac{1}{3}\begin{pmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ 1 & 2 & 2 \end{pmatrix}, \quad E = \frac{1}{3}\begin{pmatrix} -2 & 1 & 2 \\ -1 & 2 & 2 \\ -2 & 2 & 1 \end{pmatrix},$$
$$F = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad G = \frac{1}{18}\begin{pmatrix} -2 & 7 \\ 7 & 2 \end{pmatrix}.$$

5. Show that the vectors
$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$$
are eigenvectors of the matrix F, where
$$F = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 6 & 0 \\ 3 & 0 & -8 \end{pmatrix}.$$
Find the eigenvalue corresponding to each eigenvector, and show that the eigenvectors are orthogonal to one another.

6. Find the general solutions of the following systems of equations.
$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 6 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

7. **(Homework)** A population is divided into three categories (young, mature, and old) by age. The numbers (in thousands) in each category are given by $y(t)$, $m(t)$, and $\theta(t)$ respectively, where t is time. Research has shown that the numbers in each age group are governed by the following equations

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -y + m, \qquad \frac{\mathrm{d}m}{\mathrm{d}t} = y - m, \qquad \text{and} \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = m - \theta.$$

Start by writing this system in matrix form ($\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = A\mathbf{y}$, where $\mathbf{y} = (y, m, \theta)$). Find the general solution to this system of equations, then find the particular solution for the case when the initial numbers in each age group are $y(0) = 4$, $m(0) = 2$, and $\theta(0) = 1$ at time 0. What happens to the population as $t \to \infty$?

8. Use eigenvalues and eigenvectors to identify the principal axes of the following ellipses. Graph each ellipse.

   (a) $3x^2 - 2xy + 3y^2 = 20$

   (b) $10x^2 + 6xy + 2y^2 = 36$

9. For the following **symmetric** matrix A, find a diagonal matrix D and an orthogonal matrix P such that $A = PDP^T$.

$$A = \begin{pmatrix} 5 & \sqrt{10} \\ \sqrt{10} & 8 \end{pmatrix}.$$

10. Let A be diagonalisable, with $A = PDP^{-1}$ where D is a diagonal matrix and P is an invertible matrix. Show that $A^2 = PD^2P^{-1}$.

11. Evaluate each of the following integrals, choosing the most convenient order of integration. In each case, first sketch the region of integration, the one satisfying the inequalities listed with the integral.

   (a)
$$\iint_{\mathcal{R}} 2xy\,\mathrm{d}A \qquad 1 \le x \le 3, \quad 0 \le y \le 2.$$

   (b)
$$\iint_{\mathcal{R}} 2xy\,\mathrm{d}A \qquad x \ge 0, \quad y \ge x^3, \quad y \le 8.$$

   (c)
$$\iint_{\mathcal{R}} \frac{y}{1+xy}\,\mathrm{d}A \qquad 0 \le x \le 1, \quad 0 \le y \le 1.$$

   (d) **(Homework)**
$$\iint_{\mathcal{R}} \sqrt{x^2+1}\,\mathrm{d}A \qquad x \le \sqrt{3}, \quad y \le x, \quad y \ge -x.$$

   (e)
$$\iint_{\mathcal{R}} \sin\left(\frac{\pi x}{y}\right)\,\mathrm{d}A \qquad 0 \le x \le y^2 \quad 1 \le y \le 2.$$

P.T.O.

12. State why each of the following is NOT an iterated integral.

$$\int_{y=-x}^{1} \left(\int_{x=y}^{y^2} x^2 y \, \mathrm{d}x \right) \mathrm{d}y, \qquad \int_{y=0}^{1} \left(\int_{x=y}^{xy^2} \sin(y) \, \mathrm{d}x \right) \mathrm{d}y,$$

$$\int_{y=1}^{2} \left(\int_{y=x}^{x^2} x + y \, \mathrm{d}y \right) \mathrm{d}y \ .$$

13. For each of the following regions R, write the integral $\iint_R f(x, y) \, \mathrm{d}A$ as an iterated integral, where $\mathrm{d}A = \mathrm{d}x \, \mathrm{d}y$.

(a) $0 \leq x \leq 1$ and $0 \leq y \leq x^4$

(b) $x^2 - 1 \leq y \leq 1 - x^4$

(c) The intersection of the ellipse $3x^2 + 2y^2 \leq 1$ and the half plane $y \geq 0$.