University of Canterbury

EMTH210

ENGINEERING MATHEMATICS

Topic 6: Laplace transforms

6.1 LAPLACE TRANSFORM

The most important use of the *Laplace Transform* is in solving differential equations; we will focus on solving second order inhomogeneous ordinary differential equations (ODEs). The Laplace Transform, or LT, turns the problem of solving an ODE into one of solving an algebraic equation, then looking things up in a table.

Here is the definition and notation for the Laplace Transform of a function $f(t)$ when $t \geqslant 0$:

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \mathcal{L}\{f(t)\} \ .$$

Example 6.1.1

Find the Laplace Transform of $f(t) = 1$.

Solution.

♦

NB: If $s \leqslant 0$ the integral is infinite and so this $F(s)$ is only defined for $s > 0$.

The basic idea behind our use of LTs to solve ODEs is that we transform the ODE (which has an unknown function of t and its derivatives) into an algebraic equation (in terms of an unknown function of s), rearrange it a bit, and then use the *inverse* Laplace transform to turn the algebraic equation (in s) into an algebraic equation (in t) — the solution to the ODE. Moreover, the method automatically inserts initial conditions.

A function $f(t)$ has a Laplace transform provided:

1. $f(t)$ is piecewise continuous on $t \geqslant 0$;
2. $f(t)$ is exponentially bounded for $t \geqslant 0$. Formally, this means that there exists M, c such that

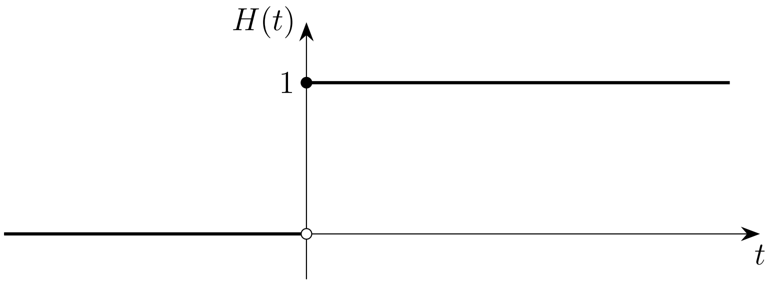
$$|f(t)| < Me^{ct}, \qquad \forall t \geqslant 0 \ .$$

Under these conditions the Laplace Transform exists for $s > c$. An example of a function that fails the second condition is e^{t^2}.

6.2 HEAVISIDE STEP FUNCTION

We will see that it is convenient to define a function that is 0 or "off" up to a certain time and then 1 or "on" after that time. The function is called the *Heaviside step function*[1]:

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geqslant 0 \end{cases}$$



An immediate use of the Heaviside step function is to deal with functions $f(t)$ which are defined for $t < 0$ as well as for $t \geqslant 0$, because the definition of the LT only considers $f(t)$ in $t \geqslant 0$, and ignores anything that happens in $t < 0$. So for a function which has other undesired behaviour in $t < 0$, we use the Heaviside step function to "switch it off" for negative t. It is always safest to consider the LT of $f(t)H(t)$ rather than just $f(t)$. For example

$$\mathcal{L}\{1 \cdot H(t)\} = \int_0^\infty 1 \cdot H(t)\, e^{-st} dt = \frac{1}{s} \ ,$$

as before.

[1] Sometimes called the *unit* step function, $u(t)$.

Similarly,

$$\begin{aligned}
\mathcal{L}\{e^{at}H(t)\} &= \int_0^\infty e^{at}\, e^{-st} H(t)dt \\
&= \underline{\hspace{6cm}} \\
&= \underline{\hspace{6cm}} \quad \text{provided } s > a, \\
&= \underline{\hspace{6cm}} \\
&= \frac{1}{s-a}\ .
\end{aligned}$$

The fact that s must be greater than a in this example is crucial, and is related to the second of the two conditions on page 3.

One of the most useful results is the LT of $\frac{df}{dt}$, which we now derive.

Example 6.2.1

Show that

$$\mathcal{L}\left\{\frac{df}{dt}H(t)\right\} = sF(s) - f(0)\ ,$$

where $\mathcal{L}\{f(t)H(t)\} = F(s)$.

Solution.

♦

We will see how important this is soon, but for now just notice that the LT of the first derivative of a function $f(t)$ is s times the LT of $f(t)$ — a hint already that derivatives are being converted to products, ie, that differential equations may become algebraic equations. Note also that when we transform the derivative of a function we also have to supply an initial value — $f(0)$ — of that function. The final piece of the puzzle which will transform ODEs into algebraic equations is to prove the *linearity* of Laplace transforms.

6.3 LINEARITY

If a, b are constants and $f(t), g(t)$ are functions, then

$$\begin{aligned}\mathcal{L}\{af(t)+bg(t)\} &= \int_0^\infty \Big(af(t)+bg(t)\Big)e^{-st}dt \\ &= a\int_0^\infty f(t)e^{-st}dt + b\int_0^\infty g(t)e^{-st}dt \\ &= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}\ .\end{aligned}$$

The above is the proof of the *linearity* of the Laplace transform:

$$\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}\ .$$

In the next section, this enables us to say that if we want the LT of an equation, then we go through it transforming each term in turn. Together, linearity and the transform of derivatives of functions enable ODEs to be solved quite easily, as we now see.

6.4 SOLVING AN ODE USING LTs

1. LT the ODE using linearity, inserting the initial conditions.
2. Rearrange the resulting algebraic equation, solving for the transformed function (usually $Y(s)$).
3. Take the inverse LT of the resulting algebraic equation, by looking things up in a table and using linearity.

Example 6.4.1

Solve the following first order ODE using Laplace transforms[2]:

$$\frac{dy}{dt} + y = 1 \ , \qquad y(0) = 4 \ .$$

Solution.

[2]We could also solve it using the integrating factor method, of course, and it would be good if you could check that you get the same answer.

♦

Note that the Heaviside step function $H(t)$ was inserted here for the reasons outlined above, namely that we want our solution to be defined only for $t \geqslant 0$. Always include it in your solutions if it does not appear there automatically as it does when the second shift theorem is used (we learn this later).

6.5 MORE LAPLACE TRANSFORMS

LAPLACE TRANSFORMS OF POLYNOMIALS

Example 6.5.1

If $n > 0$ is an integer, show that

$$\mathcal{L}\{t^n H(t)\} = \frac{n!}{s^{n+1}} \ .$$

Solution.

♦

This result, combined with linearity, enables us to find the LT of any polynomial.

LAPLACE TRANSFORMS OF TRIG FUNCTIONS

Example 6.5.2

Show that

$$\mathcal{L}\{\sin(\omega t)H(t)\} = \frac{\omega}{s^2+\omega^2}\ ,$$

and hence show that

$$\mathcal{L}\{\cos(\omega t)H(t)\} = \frac{s}{s^2+\omega^2}\ .$$

Solution.

$\blacklozenge$

LAPLACE TRANSFORMS OF HIGHER DERIVATIVES

We find $\mathcal{L}\{\frac{d^2f}{dt^2}H(t)\}$ and the LT of higher derivatives by an iterative process.

$$\mathcal{L}\left\{\frac{d^2f}{dt^2}H(t)\right\} = \mathcal{L}\left\{\frac{d}{dt}\left(\frac{df}{dt}\right)H(t)\right\} = s\mathcal{L}\left\{\frac{df}{dt}H(t)\right\} - \left.\frac{df}{dt}\right|_{t=0}$$

$$= s\big(s\mathcal{L}\left\{f(t)H(t)\right\} - f(0)\big) - f'(0)$$

$$\mathcal{L}\left\{\frac{d^2f}{dt^2}H(t)\right\} = s^2F(s) - sf(0) - f'(0)\ .$$

In general:

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)\ .$$

We can now tackle some harder problems, such as the following second order ODE.

Example 6.5.3

Solve the initial value problem

$$y'' - y' - 6y = 2\ , \quad y(0) = 1\ , \quad y'(0) = 0\ .$$

***Solution*.**

♦

Recall once more that we have already inserted the initial conditions: this solution satisfies the ODE *and* the initial conditions.

LAPLACE TRANSFORM OF AN INTEGRAL

From time to time, we may wish to take the LT of an integral. We can derive this in a roundabout but clever way, as follows. Suppose that

$$f(t) = \int_{\tau=0}^{t} g(\tau)d\tau$$

is the integral whose LT we wish to find. The fundamental theorem of calculus[3] then tells us that

$$\underline{\hspace{6cm}} \ ,$$

while the definition of $f(t)$ shows us that $f(0) = 0$. We can put all of this together with our knowledge that

$$\mathcal{L}\left\{\frac{df}{dt}H(t)\right\} = s\mathcal{L}\{f(t)\} - f(0)$$

to get

$$\mathcal{L}\{g(t)H(t)\} = s\mathcal{L}\left\{\int_{\tau=0}^{t} g(\tau)d\tau\right\} - 0 \ ,$$

which yields

$$\mathcal{L}\left\{\int_{\tau=0}^{t} g(\tau)d\tau\right\} = \frac{1}{s}G(s) \ ,$$

where $G(s) = \mathcal{L}\{g(t)\}$.

[3] Do you need to look this up?

6.6 FIRST SHIFT THEOREM

This fact will enable us to tackle a wider range of problems.

Example 6.6.1

Prove the *First Shift Theorem*, namely that $\boxed{\mathcal{L}\{e^{at}f(t)H(t)\} = F(s-a).}$

Solution.

♦

This result is called the first *shift* theorem because it says that the LT of $e^{at}f(t)$ is the LT of $f(t)$ *shifted* by a to the right. For example, earlier we saw that

$$\mathcal{L}\{\sin(\omega t)H(t)\} = \frac{\omega}{s^2+\omega^2}\ .$$

The first shift theorem tells us that

$$\mathcal{L}\{e^{at}\sin(\omega t)H(t)\} = \underline{\hspace{6cm}}\ . \tag{1}$$

We also saw that

$$\mathcal{L}\{\cos(\omega t)H(t)\} = \frac{s}{s^2+\omega^2}$$

and so in a similar manner the first shift theorem tells us that

$$\mathcal{L}\{e^{at}\cos(\omega t)H(t)\} = \underline{\hspace{6cm}}\ . \tag{2}$$

The first shift theorem enables us to find the inverse LT of a wider variety of fractions which may result from running through the four steps of solving an ODE by LTs. Consider the following example.

Example 6.6.2

Find $y(t)$ when

$$Y(s) = \frac{s}{s^2 - 10s + 29} \; .$$

Solution.

♦

When using the first shift theorem related to sine and cosine functions — results (1) and (2) — one has to be extremely careful about the different role played by s and ω.

Example 6.6.3

Find the current in the electrical circuit containing, in series, a capacitor (0.025 farads), a resistor (4 ohms), and an inductor (1 henry), when the initial conditions at time $t = 0$ are that the current is 0 and that the charge on the capacitor is Q_0.

Solution.

♦

Let us continue to extend the range of functions for which we can find the LT.

Example 6.6.4

Show that

$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s) \ .$$

Solution.

♦

In general

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$$

— this is not in the table on the formula sheet, but is quite important here: we use it to build up the range of functions with which we are happy to work, as follows. We know that

$$f(t) = \sin(\omega t) \qquad \Longrightarrow \qquad F(s) = \frac{\omega}{s^2+\omega^2} \ .$$

So $t\sin(\omega t)$ transforms to

$$-\frac{d}{ds}\left(\frac{\omega}{s^2+\omega^2}\right) = \rule{4cm}{0.4pt} = \rule{4cm}{0.4pt} \ .$$

Similarly

$$\mathcal{L}\{t\cos(\omega t)\} = \rule{4cm}{0.4pt} = \rule{4cm}{0.4pt} \ .$$

Also

$$\mathcal{L}\{te^{at}\} = \underline{\hspace{4cm}} = \underline{\hspace{4cm}} \; .$$

More generally

$$\mathcal{L}\{t^n e^{at}\} = \underline{\hspace{4cm}} = \underline{\hspace{4cm}} \; ,$$

which can also be proved using the first shift theorem and the fact that

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \; .$$

These results together with the first shift theorem can be used to take the inverse Laplace Transform of any proper rational function[4]. A partial fractions expansion will yield terms like:

$\dfrac{1}{s^{n+1}}$	with inverse transform	$\dfrac{t^n}{n!} H(t)$
$\dfrac{1}{(s-a)^{n+1}}$	"	$\dfrac{t^n}{n!}\, e^{at} H(t)$
$\dfrac{(s-a)}{(s-a)^2+w^2}$	"	$e^{at}\, \cos(wt) H(t)$
$\dfrac{w}{(s-a)^2+w^2}$	"	$e^{at}\, \sin(wt) H(t)$
$\dfrac{2(s-a)w}{((s-a)^2+w^2)^2}$	"	$e^{at}\, t\, \sin(wt) H(t)$

Here is an example with a slightly trickier RHS, but really it just comes down to (a) being able to do partial fractions, and (b) thinking ahead to what you want in terms of entries in the table of Laplace transforms on the formula sheet.

[4]Proper means that the numerator is a polynomial of lower degree than the denominator.

Example 6.6.5

Solve

$$y'' + 4y' + 5y = 40\sin(3t) \ , \quad y(0) = y'(0) = 0 \ .$$

Solution.

Example 6.6.6

Find $y(t)$ when

$$Y = \frac{s^2+3s+1}{(s+1)(s+2)^2} \ .$$

Solution.

♦

6.7 SECOND SHIFT (DELAY) THEOREM

This is the final LT theorem we will look at in this course. It is very important, as it lets us work with piecewise functions, which happen a lot in the real world where we have *sudden changes* (e.g. a circuit being switched on or off, a camera shutter opening and closing, a collision suddenly occurring). The theorem states that if

$$\mathcal{L}\{f(t)H(t)\} = F(s)$$

then

$$\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}F(s)$$

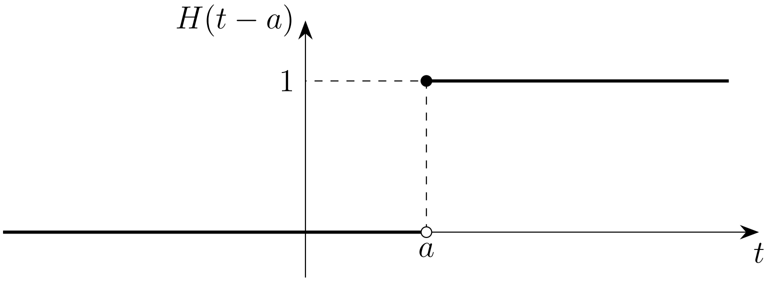
provided $a > 0$. (This is invalid for $a < 0$.)

Before we prove this formula, let's consider its meaning. First, sketch any function $f(t)$:

Now draw on the same axes $f(t-a)$ for $a > 0$. We must also have

$$H(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geqslant a \end{cases}$$

which looks like



Therefore $f(t-a)H(t-a)$ is your function shifted to the right by a then switched off for all $t < a$ — in other words, it is your function delayed by a (it starts at a rather than 0).

It won't hurt to see where the second shift formula comes from.

$$\mathcal{L}\{f(t-a)H(t-a)\} = \int_{t=0}^{\infty} H(t-a)f(t-a)e^{-st}dt$$

by definition. So, using the fact that $a > 0$:

$$\mathcal{L}\{f(t-a)H(t-a)\} = \int_{t=a}^{\infty} f(t-a)e^{-st}dt\ .$$

Now do a change of variables $u = t - a$, so that

$$\begin{aligned}
\mathcal{L}\{f(t-a)H(t-a)\} &= \int_{u=0}^{\infty} f(u)e^{-s(u+a)}du \\
&= \int_{u=0}^{\infty} f(u)e^{-su}e^{-sa}du \\
&= e^{-sa}\int_{u=0}^{\infty} f(u)e^{-su}du
\end{aligned}$$

$$\implies \mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}F(s)$$

as stated.

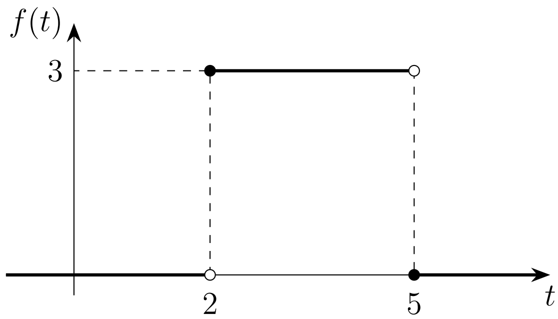
One of the major uses of the second shift theorem is to construct strange start-stop functions. In practical engineering-type applications, these functions are perhaps more common than the usual "nice" functions we deal with in mathematics. For example, current in a circuit might be discharged from a capacitor for a fixed length of time, or a building might be subject to forced oscillations for a period of time. Most sudden changes are best represented with piecewise functions.

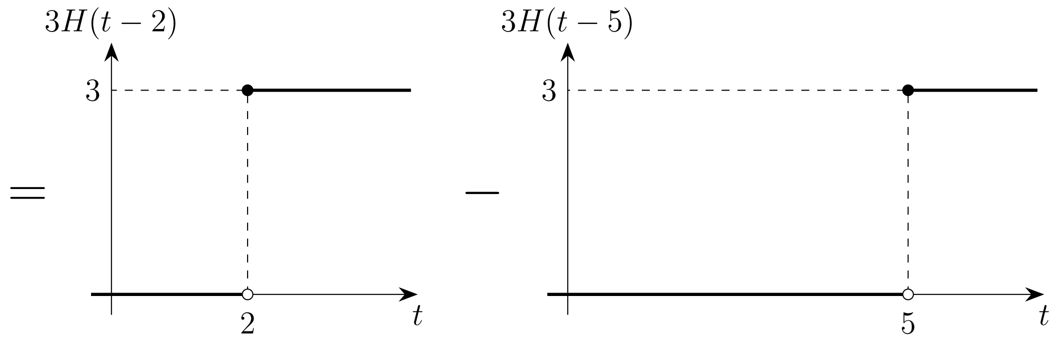
6.8 PIECEWISE FUNCTIONS

The following is an example of a *piecewise function*[5]

$$f(t) = \begin{cases} 3 & \text{if } 2 \leqslant t < 5 \\ 0 & \text{otherwise} \end{cases}$$

We can sketch the function itself, then draw as the difference between two graphs.





meaning that symbolically we have

$$f(t) = 3H(t-2) - 3H(t-5)$$

and in words: "$f(t)$ is 3 delayed by 2, minus 3 delayed by 5". We've written $f(t)$ without any "if"s at all! Hence, using the second shift theorem and the fact that $\mathcal{L}\{3\} = 3\mathcal{L}\{1\} = 3/s$, we see that

$$F(s) = \underline{\hspace{5cm}} \; .$$

[5]So called because it is made of multiple pieces, and within each piece it is often quite nice.

Example 6.8.1

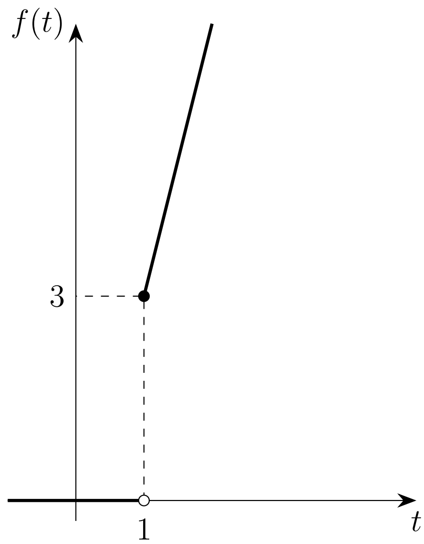
Find the Laplace Transform of

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 \leqslant t < 2 \\ 2 & \text{if } t \geqslant 2 \end{cases}$$

Solution.

♦

Example 6.8.2



Find the LT of

$$f(t) = \begin{cases} 0 & t < 1 \\ 2t + 1 & t \geqslant 1 \end{cases}$$

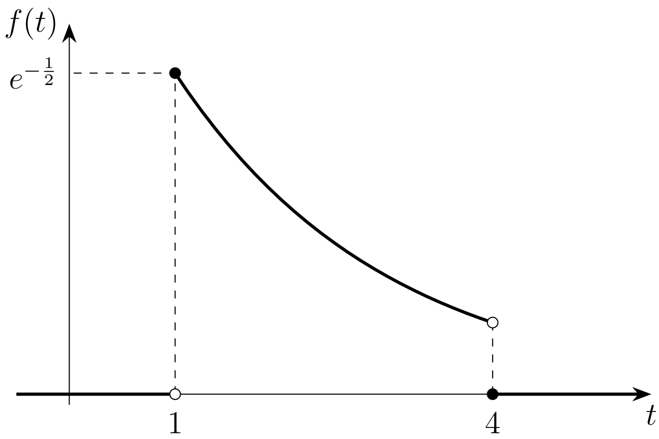
Solution.

♦

Example 6.8.3

Find the LT of

$$f(t) = \begin{cases} e^{-t/2} & 1 \leqslant t < 4 \\ 0 & \text{otherwise} \end{cases}$$



***Solution*.**

♦

Let's tie all this together in a weird example, in which we will need to take inverse LTs using the second shift theorem. Hold on to your hats!

Example 6.8.4

Solve

$$y' + y = f(t)$$

when $y(0) = 0$ and $f(t) = H(t-3) + e^{-(t-7)}H(t-7)$.

Solution.