EMTH210 Tutorial 6: Fourier Series

For the week starting Monday 20 April.

The homework questions this week are **2** and **5**.

1. Find the Fourier series of the following functions. In each case sketch the function over two periods. (Note that notation like $f(x+2\pi) = f(x)$ means the function is periodic with period 2π: "if you go 2π units further along you get the same thing".)

   (a) $f(x) = \begin{cases} 0 & -\pi \le x < 0 \\ 1 & 0 \le x < \pi, \end{cases} \qquad f(x+2\pi) = f(x).$

   (b) $f(x) = \begin{cases} 0 & -\pi < x \le 0 \\ x & 0 < x \le \pi, \end{cases} \qquad f(x+2\pi) = f(x).$

2. **(Homework)** Identify which of the following are Fourier series. Where possible, state the fundamental frequency of each Fourier series.

   (a) $f(x) = \sin(3x) + \cos(5x)$

   (b) $f(x) = \sin(x) + \cos(\pi x)$

   (c) $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} \sin(nx)$, where the summation ranges over even values of n only

   (d) $f(x) = 4$

   (e) $f(x) = 4 + x + \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} \cos(nx)$

3. Find the Fourier series of the following functions. In each case sketch the function over two periods, and state whether the function is even, odd, or neither.

   (a) $f(x) = \begin{cases} 1 & -1 < x < 0 \\ x & 0 \leqslant x < 1, \end{cases} \qquad f(x+2) = f(x).$

   (b) $f(x) = x^2$ on $-\pi \leqslant x < \pi$, with $f(x+2\pi) = f(x)$.

   (c) $f(x) = x$ on $0 \leqslant x < \pi$, with $f(x+\pi) = f(x)$.

4. Find the Fourier series of

   $$f(x) = \sin^4(x)$$

   using the double angle formula $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 1 - 2\sin^2(\theta)$ twice.

   What is the fundamental frequency (that is, 1/shortest period) of f?

P.T.O.

5. **(Homework)** Find the general solution to

$$m\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + ky = f(t)$$

when $m = 1$, $k = 10$ and

$$f(t) = \begin{cases} 5 & 0 < t < \pi \\ -5 & \pi \leqslant t < 2\pi, \end{cases} \qquad f(x + 2\pi) = f(x).$$

6. Repeat the previous question with the new forcing term

$$f(t) = t^2 \quad \text{on} \quad -\pi \leqslant t < \pi, \quad \text{with } f(x + 2\pi) = f(x).$$

[HINT: your answer to question 3(b) of the preparation problems may be useful.]

7. The 2π-periodic function $f(x)$ is defined by $f(x) = x$ on $-\pi < x < \pi$ and by $f(x) = f(x + 2\pi)$ elsewhere. The Fourier series for $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{2}{n}\right)(-1)^{n+1}\sin(nx).$$

Using this Fourier series, find the Fourier series for the following functions, and sketch each function over two periods:

$$g(x) = \pi + f(x + \pi) \qquad \text{and} \qquad h(x) = f(\pi x)/2.$$

8. Sketch the following function

$$f(x) = \begin{cases} \cos(\pi x) & -1 < x < 1 \\ 0 & 1 < x < 3. \end{cases}$$

Does $f(x)$ have an odd or an even extension to a larger domain? Find the Fourier series for $f(x)$ with period 4.