University of Canterbury

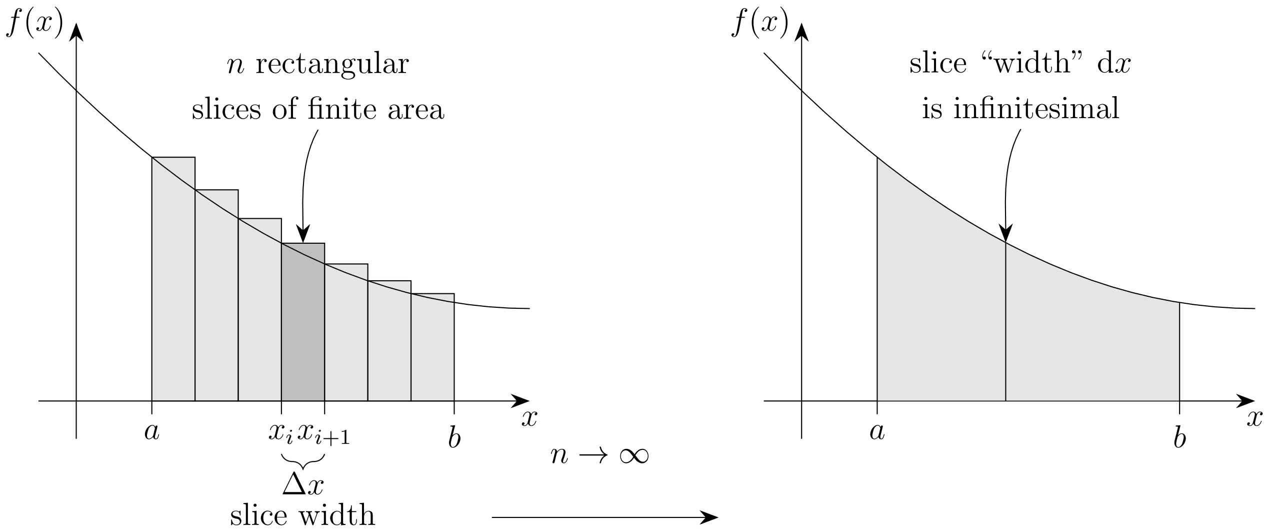
EMTH210

ENGINEERING MATHEMATICS

Topic 5: Double and triple integrals

5.1 DOUBLE AND TRIPLE INTEGRALS

The area under a curve $f(x)$ is found by dividing it into small strips whose width is Δx and whose height is $f(x)$. The area of each strip is height times width, ie $f(x)\Delta x$. Therefore an approximation to the area under the curve is the sum of all these areas, $\sum f(x)\Delta x$. Now we take the limit as $\Delta x \to 0$ and replace all things Greek ($\sum, \Delta$) with things Roman ($\int, d$), ending up with the area being equal to $\int f(x)dx$. (It is also common then to think of the small width as being dx rather than Δx, even though strictly speaking the usual understanding of dx is as part of a limiting process[1].)



$\Longrightarrow$ slice area = height × width $= f(x_i) \times \Delta x$

$\Longrightarrow$ total area under curve $\approx \sum_{i=1}^{n} f(x_i)\Delta x$

$\Delta x \to \mathrm{d}x$

$x_i \rightsquigarrow x$

"$\approx$" $\rightsquigarrow$ "$=$"

"$\sum$" $\rightsquigarrow$ "$\int$"

total area under curve $= \int_a^b f(x)\,\mathrm{d}x$ (height: $f(x)$; "width": $\mathrm{d}x$)

"add up" all the tiny slices of area

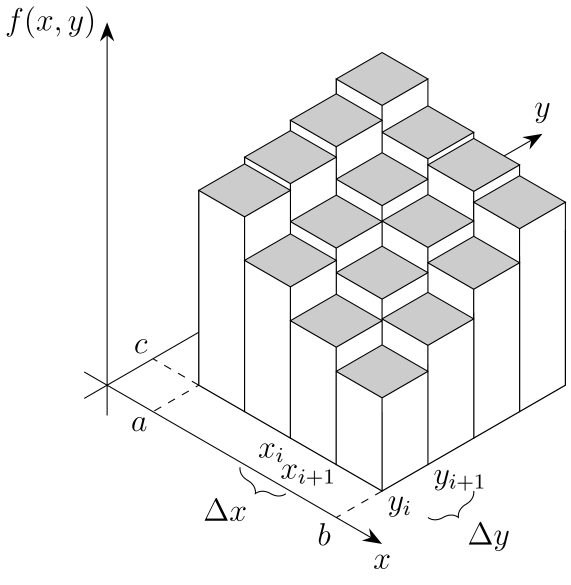
[1]A lot of the metaphysics of calculus was really only firmed up in modern times, mainly in the late 19th Century. In that formalisation, dx is not really a measurable width. However, in the 1960s there emerged another way of thinking about calculus, closer in some ways to how its early proponents thought of it, in which dx really is a small distance. This is called *non-standard analysis*.

Suppose now that we wish to find the volume of a solid shape sitting on the xy-plane, whose height is $f(x, y)$. We carry over our experience from the 1D case, dividing the shape into blocks whose base area is a little bit of the whole base area, ΔA, and whose height is $f(x, y)$. The volume of these blocks is height times base area, $f(x, y)\Delta A$, and the shape's total volume is well-approximated by the sum of these, $\sum f(x, y)\Delta A$. Now do the usual limiting process, namely $\Delta A \to 0$, and the Greek-to-Roman thing, and we end up with a so-called *double integral*:

$$\iint f(x, y)dA.$$

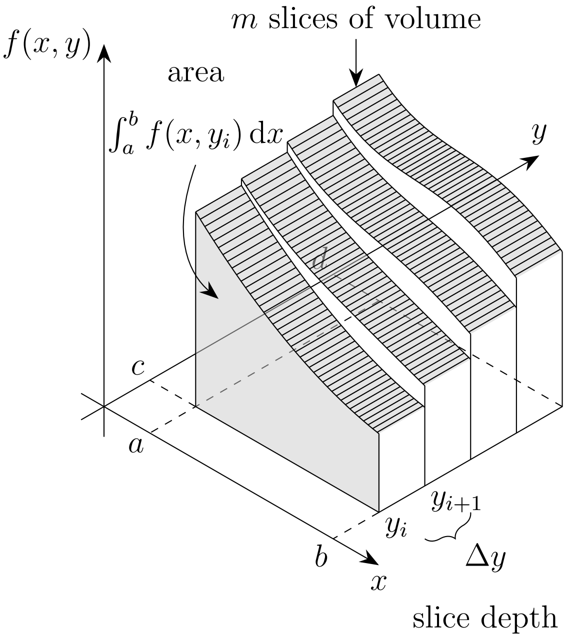
As before, it is common to now think of "a little bit of area" as being dA rather than ΔA.

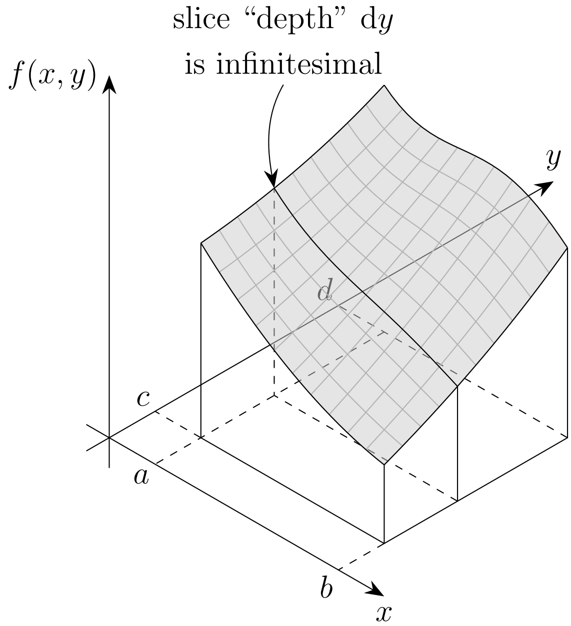
A simple double integral (constant limits = rectangular region) is like this:



Since the shape is sitting in the xy-plane, we can let that a little bit of area, dA, be a rectangle in the xy-plane with sides dx and dy, (from $\Delta A = \Delta x \Delta y$). So we can write $dA = dxdy$, the so-called *element of area* in the xy-plane. In different coordinate systems, dA takes different forms. Especially of note is the very different form it takes in polar coordinates, as we will see later.

There's another way to think about the same thing, building on our existing ability to do single integrals (using the same function on the same simple rectangular region):





$$\implies \text{slice volume} = \text{area} \times \text{depth}$$

$$= \int_a^b f(x,y_i)\,\mathrm{d}x \times \Delta y$$

$$\implies \begin{array}{l}\text{total volume}\\ \text{under surface}\end{array} \approx \sum_{i=1}^{m} \int_a^b f(x,y_i)\,\mathrm{d}x\Delta y$$

$$\xrightarrow{\;m\to\infty\;}$$

$\Delta y \to \mathrm{d}y$

$y_i \rightsquigarrow y$

"$\approx$" $\rightsquigarrow$ "$=$"

"$\sum$" $\rightsquigarrow$ "$\int$"

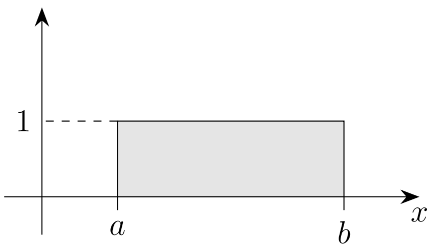
$$\begin{array}{l}\text{total volume}\\ \text{under surface}\end{array} = \underbrace{\int_c^d \int_a^b}_{\substack{\text{“add up” all the}\\ \text{tiny bits of volume}}} \overset{\text{height}}{f(x,y)}\, \overset{\text{“width”}}{\mathrm{d}x}\, \overset{\text{“depth”}}{\mathrm{d}y}$$

Of course, in practice, we're not limited to rectangular regions! This is controlled by the limits of integration, which take more care than in the 1D case; we'll consider these in more detail soon. More generally, before we get into specifics like that, we usually give the *region*[2] *of integration* a name, e.g. R, and write this at the foot of the innermost integral. Then our shape has volume

$$\iint_R f(x,y)\,dA.$$

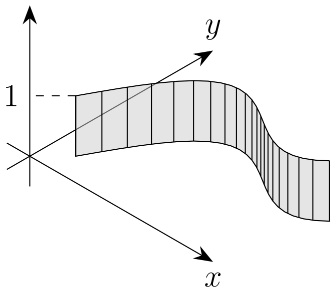
[2] Or *area*.

Consider the simple case when $f(x, y) = 1$. Integrating 1 with respect to something tells you how much of that thing your region has:

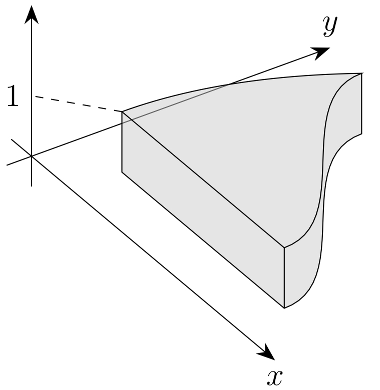


$$\int_a^b 1 \, \mathrm{d}x = \big[x\big]_a^b = b - a,$$

the length of the x interval.

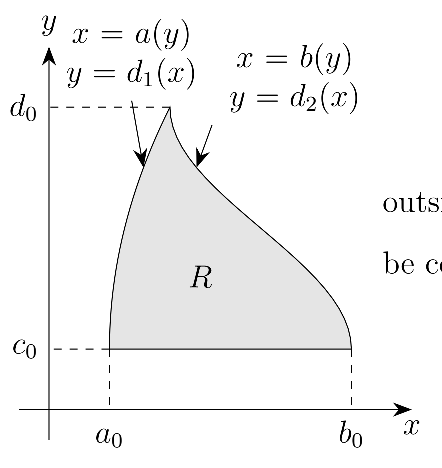


$\int_C 1 \, \mathrm{d}s$ is the arc length s of the curve C.



$\iint_R 1 \, \mathrm{d}A$ is the area A of the region R.

Usually sketching the function itself is not useful in finding its integral over a two dimensional region. What is useful is drawing the region of integration itself. More interesting regions are defined by more interesting limits, where the inner integral's limits can depend on the outer integral's variable. For example:



$$\int_{c_0}^{d_0} \int_{a(y)}^{b(y)} f(x, y) \, \mathrm{d}x \, \mathrm{d}y$$

outside limits must be constants

inside limits could depend on outer var. y

inside limits could depend on outer var. x

$$\int_{a_0}^{b_0} \int_{c(x)}^{d(x)} f(x, y) \, \mathrm{d}y \, \mathrm{d}x$$

In practice, doing a double integral involves doing two ordinary integrals. Likewise, triple integrals in practice are just three ordinary integrals in succession. The difficulty comes in (1) describing the region of integration, and (2) translating this into limits for the integrals. There are many ways of describing a region — in words, with a picture, as inequalities, etc. Sometimes the description is so explicit that difficulty (1) is no difficulty at all. Usually, we're not so lucky. This one-after-the-other way of doing double- and triple-integrals is called *iterated integration*, and we write the double- or triple-integral as an *iterated integral*.

The idea behind iterated integrals is as follows. Remember the picture: we have divided the solid into blocks whose height is the integrand $f(x, y)$ and whose base area is $dA = dxdy$. In the method of iterated integrals we first choose a direction, say parallel to the y-axis at a fixed x, and sum all of the block volumes in that direction. This gives the volume of a slice of the shape, where the slice has been made parallel to the y-axis. Now if we add together the volumes of all such slices we get the total volume. It's really a(n) (ac)counting exercise.

> We usually focus on the region of integration, R, when working through this process, because often thinking of the integrand as a height above a plane makes no sense. So a safer language is to think of a function $f(x, y)$ defined in a region R. To calculate the double integral of f over R we write the integral as an iterated integral, pick a direction for our strips to run (these give the inner integral) and then sum all the strips (the outer integral). We ask "where does each strip begin and end?" This give the limits of the inner integral. We ask "where is the first strip and where the last?" This gives the limits of the outer integral. So it is vital to **sketch the region of integration (NOT the integrand)** so as to more easily work out the integral limits.

In the box, "begin" *vs* "end" and "first" *vs* "last" are defined with reference to the direction of increasing x and y, ie the direction in which dx and dy are positive. This means that *begin/first* means "on the left or at the bottom" while *end/last* means "on the right or at the top". In polar coordinates, this will be different.

Example 5.1.1

Integrate the following function over the given region of integration.

$$f(x,y) = y \quad \text{over} \quad R : 1 \leqslant y \leqslant 2 \text{ and } y^2 \leqslant x \leqslant 6y.$$

Solution.

♦

Some important comments on this example are in order.

1. After doing the (inner) integral with respect to x, all the xs disappeared. This is a good check at each stage: having integrated with respect to a variable, that variable should no longer appear in the problem.

2. The first point is also an important thing to check in your limits: your limits for a given variable should not include that variable, or any variable already integrated. In the example, our first limits (for x) did not involve x, only y. The next limits (for y) only involved numbers (because they couldn't involve x, having already integrated this variable, and they couldn't involve y, the variable being integrated).

You should always end up with all xs and ys gone at the end.

Now let's sketch out how to do the same integral with vertical strips (obviously, we should get the same answer).

Solution.

♦

Example 5.1.2

Find

$$I = \iint_R \Big(4 - (x + y)\Big) dA$$

where R is bounded by $x, y \geqslant 1$, $x + y \leqslant 3$.

Solution.

$\blacklozenge$

In the example above, the way we described the upper boundary $y = 3 - x$ suggests writing the integral in the iterated form of y first (inside) then x:

$$I = \int_{x=1}^{2} \left(\int_{y=1}^{3-x} \dots \right.$$

But, looking at the region R, we could have described it just as well by $1 \leqslant x \leqslant 3-y$ and $1 \leqslant y \leqslant 2$. In that case, horizontal strips running parallel to x would have been a better choice. It all depends on the the shape of the region and the way in which you describe it (or have it described to you). This is why being able to draw the region of integration is so useful, because you can switch your choice of strip direction, as we will see in §5.2.

In addition to difficulties directly associated with the method, effort is sometimes required to extract an integral from the question, as in the following example.

Example 5.1.3

Find the volume of the pyramid $OBCD$ where

$$O = (0,0,0), \quad B = (1,0,0), \quad C = (0,1,0), \quad D = (0,0,1).$$

Solution.

5.2 REVERSING THE ORDER OF INTEGRATION

This means changing the order of integration; effectively, changing the choice of strip direction. In previous examples we saw that we can often choose either direction for our strips, although sometimes one choice is easier than the other as the region of integration can be covered by strips of one form.

The easiest situation in which to reverse the order of integration is when the region R is of the form

$$a \leqslant x \leqslant b \quad \& \quad c \leqslant y \leqslant d$$

where a, b, c and d are constants.

We have emphasised that it is the region of integration R which dictates the limits; this is always true. However, there are times when the form of the integrand suggests the order of integration, as in the following example.

Example 5.2.1

Find

$$I = \int_{y=0}^{1} \int_{x=y^2}^{1} \sqrt{x} \cos(x) dx dy \ .$$

Solution.

Example 5.2.2

Find the centroid of the shape defined by $y < 4 - x^2$ and $y > 0$.

Solution.

♦

As an exercise at home, show by reversing the order of integration that

$$\int_{x=0}^{1}\left(\int_{y=x}^{1}\frac{\sin(y)}{y}dy\right)dx = 1-\cos(1)\ .$$

5.3 DOUBLE INTEGRALS IN POLAR COORDINATES

For some reason, many of us seem to think of cartesian coordinates[3] as "natural", even though they were an astonishing innovation when they were introduced to the world in the middle of the 17th Century. We often, then, need or want to relate polar coordinates[4] to cartesian coordinates, through the familiar relations

$$x = r\cos\theta$$
$$y = r\sin\theta\ .$$

Again, because of the cumulative nature of mathematics, you only really need to remember how to define sine and cosine, then draw a quick sketch to derive these relations.

> If your double integral is in polar coordinates, then a strip direction still needs to be chosen, but now the two options are in relation to the $r\theta$-axes: either parallel to r, at fixed θ, or parallel to θ, at fixed r. On the $r\theta$-axes these directions really are horizontal and vertical. On the xy-axes, the strips become radial and annular strips, respectively. The strip direction will give the limits in terms of r and θ. The integrand must also be given in terms of r and θ via the formulae above. Further, and *absolutely crucially*, the element of area is $\boxed{dA = r dr d\theta}$. In summary, we put limits, integrand, and element of area in terms of r and θ.

[3]René Descartes called himself "Renatus Cartesius" in the *lingua franca* used for scholarly communication in Europe, Latin.

[4]Not invented by Mr or Mrs Polar.

Example 5.3.1

Find

$$I = \iint_R x \, dA \ ,$$

where R is bounded by $1 \leqslant r \leqslant 2$ and $0 \leqslant \theta \leqslant \pi/2$.

***Solution*.**

♦

Example 5.3.2

Find the area inside the cardioid $r = 1 + \cos\theta$, for $-\pi < \theta \leqslant \pi$.

Solution.

In the next example, we call the integral I^2 rather than I and the region of integration R^2 rather than R. While this might seem rather bizarre at the moment, you'll see soon that this enables us to do something very, very cool.

Example 5.3.3

Find

$$I^2 = \iint_{R^2} e^{-(x^2+y^2)} dxdy \ ,$$

where the region of integration R^2 is the whole xy-plane.

Solution.

♦

5.4 SEPARATING AN ITERATED INTEGRAL

Integrals of the form

$$J = \int_{x=a}^{b} \int_{y=c}^{d} f(x)g(y)\, dydx$$

where all the limits are constants and the integrand is the product of a function of x (only) and a function of y (only) can be rewritten as the product of two single integrals

$$J = \int_{x=a}^{b} f(x)\, dx \int_{y=c}^{d} g(y)\, dy \ .$$

Now for the coolness promised earlier.

Example 5.4.1

Switch to polar coordinates in the previous example, and see what you find.

$$I^2 = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} e^{-(x^2+y^2)}\, dydx \ .$$

Solution.

♦

This is very cool: we have found $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ in just a few steps. Other ways to find this result[5] are very hard!

5.5 TRIPLE INTEGRALS

Building cumulatively on our experience with double integrals, we now consider a function $f(x, y, z)$ defined in a volume of space, which we will still refer to as a region called R. To calculate the *triple integral* of f over R we write the integral as an iterated integral, pick a direction for our strips to run (these give the inner integral), then sum all the strips *in a given layer* (the middle integral), then sum up all the *layers* (the outer integral). We ask "where does each strip begin and end?" This give the limits of the inner integral. We ask "where is the first strip in the layer and where the last?" This gives the limits of the middle integral. We ask "where is the first layer and where the last?". This gives the limits of the outer integral. Once again it is vital to sketch the region of integration (NOT the integrand) so as to more easily work out the integral limits.

[5] Also cool: yet another mysterious link between the exponential function, infinity, and π. Note that this result also accounts for the normalising presence of $\sqrt{\pi}$ in the probability density function of the normal/Gaussian distribution.

For example, choosing the strips to run parallel to x means that we fix y and z and then look at a typical x-strip. Then we add all these up in, say, a layer at fixed z — an xy-layer. Then we add up all the layers (integrate with respect to z).

In Cartesian coordinates, the *element of volume* $dV = dxdydz$. We will see what it is in other coordinate systems soon.

Example 5.5.1

Find

$$I = \iiint_R z \, dV \ ,$$

where R is the region given by $x, y, z \geqslant 0$ and $x + y + z \leqslant 1$.

Solution.

♦

$\boxed{\text{To find the volume of a region, integrate 1 over the region.}}$ This is just like when we found the area of a 2D region by integrating 1 over the region. So for a 3D region, you can find its volume either by doing a double integral of the height function over the 2D footprint region **or** doing a triple integral of 1 over the entire 3D region.

Example 5.5.2

Find the volume of the region satisfying

$$0 \leqslant x \leqslant 1 \ , \quad x^2 \leqslant y \leqslant 1 \ , \quad z \geqslant 0 \ , \quad x + y + z \leqslant 2 \ .$$

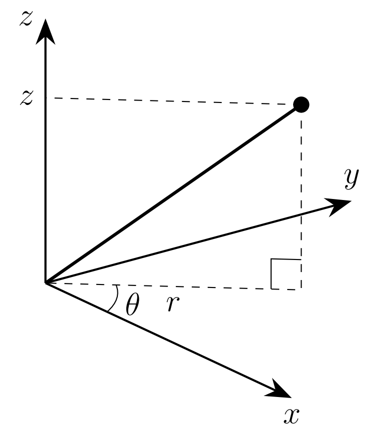
Solution.

5.6 CYLINDRICAL COORDINATES

Use the following to write the integrand, limits, and element of area in terms of cylindrical polar coordinates (r, θ, z):

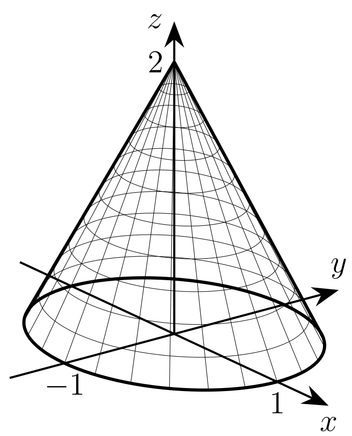
$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z$$

$$dV = r\,dr\,d\theta\,dz$$



Example 5.6.1

Find the centre of mass of the cone of constant density ρ, radius 1 and height 2, oriented as follows:



Solution.

5.7 SPHERICAL POLAR COORDINATES

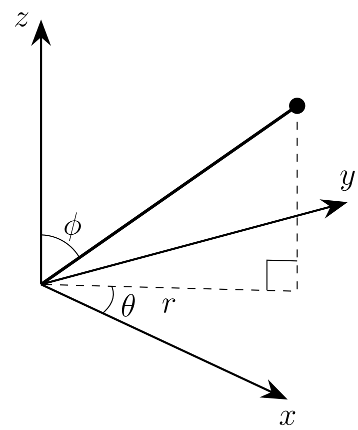
Use the following to write the integrand, limits, and element of area in terms of spherical polar coordinates (r, θ, ϕ), where r is the radial length, θ is the azimuthal angle in the xy-plane, and ϕ is the inclination or zenith angle (the angle the radius line makes with the positive z-axis. Sketch the diagram to the right, and think about what surfaces of constant r, θ, and ϕ look like.

$$x = r\sin\phi\cos\theta, \quad y = r\sin\phi\sin\theta, \quad z = r\cos\phi$$

where

$$0 \leqslant r \leqslant \infty, \quad 0 \leqslant \theta \leqslant 2\pi, \quad 0 \leqslant \phi \leqslant \pi$$

$$dV = r^2 \sin\phi \, dr \, d\theta \, d\phi.$$



Example 5.7.1

Find the volume and the centroid of the sector of a sphere of constant density 1, radius 1, and semi-angle β.

Solution.

♦