# EMTH210 Tutorial 1: Revision – Solutions

1. (a)

$$y = (t-4)e^{3t}$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = e^{3t} + 3(t-4)e^{3t}$$

(using the product rule)

(b)

$$y = (t^2+t)\cos(5t)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = (2t+1)\cos(5t) - 5(t^2+t)\sin(5t)$$

(using the product and chain rules)

(c)

$$y = \sin(t^2)\ln(t^3)$$
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= 2t\cos(t^2)\ln(t^3) + 3t^2\sin(t^2)\frac{1}{t^3} \\ &= 2t\cos(t^2)\ln(t^3) + \frac{3\sin(t^2)}{t}\end{aligned}$$

(d)

$$y = \cos(3t)\sin(2t)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -3\sin(3t)\sin(2t) + 2\cos(3t)\cos(2t)$$

(e)

$$y = \big(\sin(\ln t)\big)^2$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2\sin(\ln t)\cos(\ln t)\frac{1}{t}$$

(using the chain rule twice)

2.

$$\begin{aligned}e^{x^2} &= 8e^{-x} \\ e^{x^2+x} &= 8 \\ x^2 + x - \ln 8 &= 0 \\ \therefore x &= \frac{-1 \pm \sqrt{1+4\ln 8}}{2}\end{aligned}$$

3. (a)

$$\begin{aligned}s^2 + 4s + 20 &= 0 \\ (s+2)^2 + 16 &= 0 \\ s &= -2 \pm 4i\end{aligned}$$

(b)

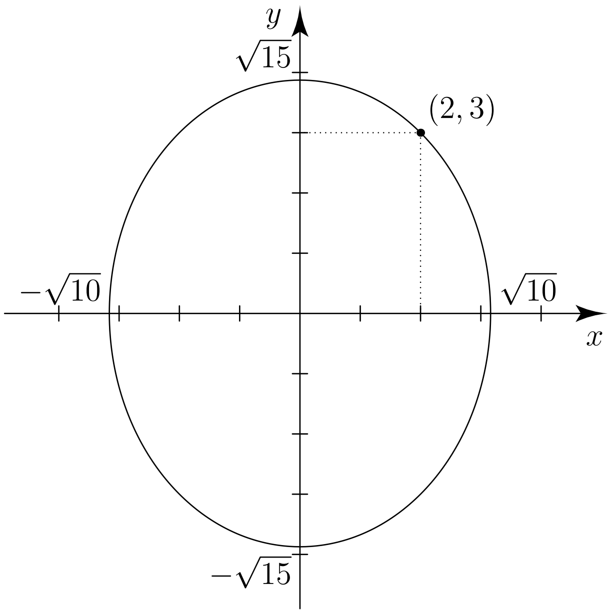
$$\begin{aligned}\frac{2s+1}{s-3} &= \frac{s+1}{s-1} \\ (2s+1)(s-1) &= (s+1)(s-3) \\ 2s^2 + s - 2s - 1 &= s^2 + s - 3s - 3 \\ s^2 + s + 2 &= 0 \\ \left(s+\frac{1}{2}\right)^2 + \frac{7}{4} &= 0 \\ s &= -\frac{1}{2} \pm \frac{\sqrt{7}}{2}i\end{aligned}$$

(c)

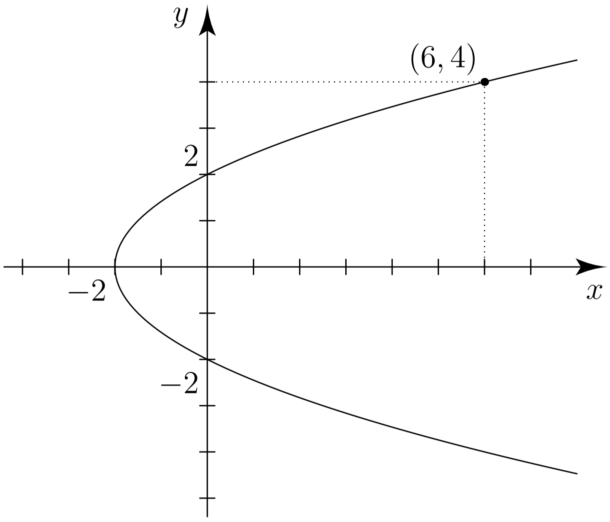
$$\begin{aligned}\ln(s+5) + \ln(s+1) &= \ln 5 \\ \ln\big((s+5)(s+1)\big) &= \ln 5 \\ (s+5)(s+1) &= 5 \\ s^2 + 6s + 5 &= 5 \\ s^2 + 6s &= 0 \\ (s+3)^2 - 9 &= 0 \\ s &= -3 \pm 3 \implies s = 0 \text{ or } s = -6.\end{aligned}$$

However $s = -6$ is not valid in the original equation, so the only solution is $s = 0$.

4. (a) At $(2,3)$, the function value is $f(2,3) = 3(2)^2 + 2(3)^2 = 12 + 18 = 30$, so the level curve is $3x^2 + 2y^2 = 30$. This is an ellipse centred on the origin. To find the $x$ intercepts, we set $y = 0$: $3x^2 + 2(0)^2 = 30 \implies x = \pm\sqrt{10}$. To find the $y$ intercepts, we set $x = 0$: $3(0)^2 + 2y^2 = 30 \implies y = \pm\sqrt{15}$. Therefore the level curve looks like this:




(b) At $(6,4)$, the function value is $f(6,4) = (4)^2 - 2(6) = 4$, so the level curve is $y^2 - 2x = 4$. This is a parabola (sideways, concave right). To find the $x$ intercepts, we set $y = 0$: $(0)^2 - 2x = 4 \implies x = -2$. To find the $y$ intercepts, we set $x = 0$: $y^2 - 2(0) = 4 \implies y = \pm 2$. Therefore the level curve looks like this:



5.

$$\begin{aligned}
s^2Y + 5sY + 4Y &= 2s + 5 \\
Y &= \frac{2s+5}{s^2+5s+4} \\
&= \frac{2s+5}{(s+1)(s+4)} \\
&= \frac{A}{s+1} + \frac{B}{s+4} \\
2s + 5 &= A(s+4) + B(s+1) \\
&= (A+B)s + (4A+B) \\
s \text{ terms: } 2 &= A + B \\
\text{constant terms: } 5 &= 4A + B \\
\implies A &= B = 1 \\
\therefore Y &= \frac{1}{s+1} + \frac{1}{s+4}
\end{aligned}$$

6. (a)

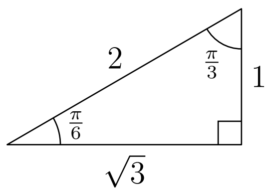
$$(1+2i)(1-2i) = 1 + 2i - 2i + 4 = 5$$

(b)

$$\begin{aligned}
\frac{i}{1-2i} &= \frac{i(1+2i)}{(1-2i)(1+2i)} \\
&= \frac{i-2}{5} \\
&= -\frac{2}{5} + \frac{1}{5}\,i
\end{aligned}$$

(c)

$$e^{\frac{\pi}{6}i} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{1}{2}\,i$$




($\frac{\pi}{6}$ is one of the angles you need to know the exact values of cos and sin for, as are $\frac{\pi}{3}$ and $\frac{\pi}{4}$, from the special triangles.)

7. (a)

$$\begin{aligned}
x^3 + 5x^2y + 2y^2 &= 0 \\
\implies \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(x^3 + 5x^2y + 2y^2\right) &= \frac{\mathrm{d}}{\mathrm{d}x}(0) \\
\implies \quad 3x^2 + 10xy + 5x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 4y\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\implies \quad \left(5x^2 + 4y\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= -\left(3x^2 + 10xy\right) \\
\implies \quad \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{3x^2 + 10xy}{5x^2 + 4y}
\end{aligned}$$

(b)

$$\begin{aligned} x^2y^3 - xy &= 6 \\ \implies \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2y^3 - xy\right) &= \frac{\mathrm{d}}{\mathrm{d}x}(6) \\ \implies \quad 2xy^3 + x^2\,3y^2\,\frac{\mathrm{d}y}{\mathrm{d}x} - y - x\,\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \implies \quad \left(3x^2y^2 - x\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= y - 2xy^3 \\ \implies \quad \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y - 2xy^3}{3x^2y^2 - x} \end{aligned}$$

(c)

$$\begin{aligned} xe^{xy} - y^2 &= 4x \\ \implies \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(xe^{xy} - y^2\right) &= \frac{\mathrm{d}}{\mathrm{d}x}(4x) \\ \implies \quad e^{xy} + xe^{xy}\left(y + x\,\frac{\mathrm{d}y}{\mathrm{d}x}\right) - 2y\,\frac{\mathrm{d}y}{\mathrm{d}x} &= 4 \\ \implies \quad e^{xy} + xye^{xy} + x^2e^{xy}\,\frac{\mathrm{d}y}{\mathrm{d}x} - 2y\,\frac{\mathrm{d}y}{\mathrm{d}x} &= 4 \\ \implies \quad \left(x^2e^{xy} - 2y\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 4 - e^{xy} - xye^{xy} \\ \implies \quad \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{4 - e^{xy} - xye^{xy}}{x^2e^{xy} - 2y} \end{aligned}$$

8. (a)

$$\left(\begin{array}{ccc|c} 2 & 1 & 2 & 2 \\ 4 & 2 & 0 & 8 \\ 1 & 1 & 0 & 3 \end{array}\right) \quad \begin{array}{l} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_1 - 2R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 2 & 2 \\ 0 & 0 & -4 & 4 \\ 0 & -1 & 2 & -4 \end{array}\right) \quad \begin{array}{r} \implies x_1 = \tfrac{1}{2}(2 - x_3 - x_2) = 1 \\ \implies x_3 = -1 \\ \implies x_2 = -(-4 - 2x_3) = 2 \end{array}$$

(b)

$$\left(\begin{array}{ccc|c} -1 & -1 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{r} \implies x_1 = 3t + 2t = 5t \\ \implies x_2 = -2t \\ \text{free variable, so let } x_3 = t, t \in \mathbb{R} \end{array}$$

9. (a)

$$\begin{aligned} \frac{1}{s^2(s+1)} &= \frac{As + B}{s^2} + \frac{C}{s+1} \\ 1 &= (As + B)(s + 1) + Cs^2 \\ &= (A + C)s^2 + (A + B)s + B \\ \text{constant terms: } 1 &= B \\ s \text{ terms: } 0 &= A + B \implies A = -1 \\ s^2 \text{ terms: } 0 &= A + C \implies C = 1 \\ \therefore \frac{1}{s^2(s+1)} &= \frac{-1}{s} + \frac{1}{s^2} + \frac{1}{s+1} \end{aligned}$$

(b)

$$\begin{aligned}
\frac{3s^2-1}{s(s^2-1)} &= \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s-1} \\
3s^2-1 &= A(s+1)(s-1) + Bs(s-1) + Cs(s+1) \\
\text{set s=-1: } 2B = 2 &\implies B = 1 \\
\text{set s=1: } 2C = 2 &\implies C = 1 \\
\text{set s=0: } -A = -1 &\implies A = 1 \\
\therefore \frac{3s^2-1}{s(s^2-1)} &= \frac{1}{s} + \frac{1}{s+1} + \frac{1}{s-1}
\end{aligned}$$

(c)

$$\begin{aligned}
\frac{s^2+6s}{(s+4)(s^2+4)} &= \frac{A}{s+4} + \frac{Bs+C}{s^2+4} \\
s^2+6s &= A(s^2+4) + (Bs+C)(s+4) \\
&= (A+B)s^2 + (4B+C)s + (4A+4C) \\
\text{constant terms: } 0 = 4A+4C &\implies A = -C \\
s \text{ terms: } 6 = 4B+C &\implies 4B - A = 6 \qquad (1) \\
s^2 \text{ terms: } 1 = A+B &\implies B + A = 1 \qquad (2) \\
(1) + (2) \text{ gives } 5B = 7 &\implies B = \tfrac{7}{5},\ A = 1 - \tfrac{7}{5} = -\tfrac{2}{5},\ C = \tfrac{2}{5} \\
\therefore \frac{s^2+6s}{(s+4)(s^2+4)} &= -\frac{2}{5}\left(\frac{1}{s+4}\right) + \frac{7}{5}\left(\frac{s}{s^2+4}\right) + \frac{2}{5}\left(\frac{1}{s^2+4}\right)
\end{aligned}$$

10. Using $\sin^2 x + \cos^2 x = 1$,

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 \qquad \therefore \quad \cos^2 x = \tfrac{1}{2} + \tfrac{1}{2}\cos 2x$$

$$\begin{aligned}
\int \cos^2 x \, \mathrm{d}x &= \int \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right) \mathrm{d}x \\
&= \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + C
\end{aligned}$$

11. (a) Here the integrand is a product, and integration by substitution won't work (neither factor is like the derivative of part of the other), so we need to use integration by parts.

$$\begin{aligned}
\int t\sin(2t)\,\mathrm{d}t &= uv - \int v\,\mathrm{d}u \\
&= t\left(-\tfrac{1}{2}\cos(2t)\right) - \int -\tfrac{1}{2}\cos(2t)\,\mathrm{d}t \\
&= -\tfrac{t}{2}\cos(2t) + \tfrac{1}{2}\left(\tfrac{1}{2}\sin(2t)\right) + c \\
&= -\tfrac{t}{2}\cos(2t) + \tfrac{1}{4}\sin(2t) + c
\end{aligned}$$

$$\begin{aligned}
u &= t & \mathrm{d}v &= \sin(2t)\,\mathrm{d}t \\
\mathrm{d}u &= \mathrm{d}t & v &= -\tfrac{1}{2}\cos(2t)
\end{aligned}$$

(b) This integral is not in the table of integrals for this course, so we need to be a bit clever. The integrand is not a product but we can think of it as one by writing $(\ln t)(1)$, so we can use integration by parts.

$$\begin{aligned}
\int \ln t\,\mathrm{d}t &= t\ln t - \int \tfrac{t}{t}\,\mathrm{d}t \\
&= t\ln t - t + c
\end{aligned}$$

$$\begin{aligned}
u &= \ln t & \mathrm{d}v &= \mathrm{d}t \\
\mathrm{d}u &= \tfrac{1}{t}\mathrm{d}t & v &= t
\end{aligned}$$

(c) Here the outermost expression, $4t$, is like the derivative of the innermost expression, $t^2 - 1$, so this is a good candidate for integration by substitution ($t^2 - 1$ is innermost because it is inside the reciprocal function $\frac{1}{x}$).

$$\begin{aligned}\int \frac{4t}{t^2-1}\,\mathrm{d}t &= \int \frac{2}{u}\,\mathrm{d}u & u &= t^2 - 1\\ &= 2\ln|u| + c & \mathrm{d}u &= 2t\mathrm{d}t\\ &= 2\ln\left|t^2-1\right| + c\end{aligned}$$

(d)

$$\begin{aligned}\int 6t\sqrt{4-t^2}\,\mathrm{d}t &= -3\int u^{\frac{1}{2}}\,\mathrm{d}u & u &= 4 - t^2\\ & & \mathrm{d}u &= -2t\mathrm{d}t\\ &= -3\left(\frac{2}{3}u^{\frac{3}{2}}\right) + c\\ &= -2\left(4-t^2\right)^{\frac{3}{2}} + c\end{aligned}$$

(e)

$$\begin{aligned}\int 12te^{3t}\,\mathrm{d}t &= 4te^{3t} - 4\int e^{3t}\,\mathrm{d}t & u &= 12t & \mathrm{d}v &= e^{3t}\,\mathrm{d}t\\ & & \mathrm{d}u &= 12\,\mathrm{d}t & v &= \tfrac{1}{3}e^{3t}\\ &= 4te^{3t} - \frac{4}{3}e^{3t} + c\end{aligned}$$

(f)

$$\begin{aligned}\int \frac{\cos t}{\sin t}\,\mathrm{d}t &= \int \frac{1}{u}\,\mathrm{d}u & u &= \sin t\\ & & \mathrm{d}u &= \cos t\,\mathrm{d}t\\ &= \ln|\sin t| + c\end{aligned}$$

(g) This one needs some rearrangement before we have something we can integrate.

$$\begin{aligned}\int \frac{(2+t)^2}{1+t^2}\,\mathrm{d}t &= \int \frac{4+4t+t^2}{1+t^2}\,\mathrm{d}t\\ &= \int \frac{4}{1+t^2}\,\mathrm{d}t + \int \frac{4t}{1+t^2}\,\mathrm{d}t + \int \frac{t^2}{1+t^2}\,\mathrm{d}t\end{aligned}$$

Next consider each of these integrals individually.

$$\begin{aligned}\int \frac{4}{1+t^2}\,\mathrm{d}t &= 4\tan^{-1} t + c_1\\ \int \frac{4t}{1+t^2}\,\mathrm{d}t &= 2\ln\left|1+t^2\right| + c_2 & u &= 1+t^2\\ & & \mathrm{d}u &= 2t\,\mathrm{d}t\end{aligned}$$

And now a clever trick you might not have seen before:

$$\begin{aligned}\int \frac{t^2}{1+t^2}\,\mathrm{d}t &= \int \frac{1+t^2-1}{1+t^2}\,\mathrm{d}t \qquad \text{(to get a } 1+t^2 \text{ on both top and bottom)}\\ &= \int \left(1 - \frac{1}{1+t^2}\right)\mathrm{d}t\\ &= t - \tan^{-1} t + c_3\end{aligned}$$

Therefore,

$$\begin{aligned}\int \frac{(2+t)^2}{1+t^2}\,\mathrm{d}t &= 4\tan^{-1} t + 2\ln\left|1+t^2\right| + t - \tan^{-1} t + c\\ &= 3\tan^{-1} t + 2\ln\left|1+t^2\right| + t + c\end{aligned}$$

12. (a)

$$\begin{aligned}
\sin x \sin 3x &= \tfrac{1}{2}\big(\cos(x-3x) - \cos(x+3x)\big) \\
&= \tfrac{1}{2}(\cos 2x - \cos 4x) \\
\int_{-\pi}^{\pi} \sin x \sin 3x \, \mathrm{d}x &= \tfrac{1}{2}\int_{-\pi}^{\pi} (\cos 2x - \cos 4x) \, \mathrm{d}x \\
&= \tfrac{1}{2}\left[\tfrac{1}{2}\sin 2x - \tfrac{1}{4}\sin 4x\right]_{-\pi}^{\pi} \\
&= 0 \qquad (\text{since } \sin n\pi = 0 \text{ for all } n \in \mathbb{Z})
\end{aligned}$$

(b)

$$\begin{aligned}
\cos x \sin 2x &= \tfrac{1}{2}\big(\sin(2x+x) + \sin(2x-x)\big) \\
&= \tfrac{1}{2}(\sin 3x + \sin x) \\
\int_{-\pi}^{\pi} \cos x \sin 2x \, \mathrm{d}x &= \tfrac{1}{2}\int_{-\pi}^{\pi} (\sin 3x + \sin x) \, \mathrm{d}x \\
&= \tfrac{1}{2}\left[-\tfrac{1}{3}\cos 3x - \cos x\right]_{-\pi}^{\pi} \\
&= \tfrac{1}{2}\left(-\tfrac{1}{3}\cos 3\pi - \cos\pi + \tfrac{1}{3}\cos(-3\pi) + \cos(-\pi)\right) \\
&= \tfrac{1}{2}\left(-\tfrac{1}{3}(-1) - (-1) + \tfrac{1}{3}(-1) + (-1)\right) \\
&= \tfrac{1}{2}\left(\tfrac{1}{3} - \tfrac{1}{3} + 1 - 1\right) \\
&= 0
\end{aligned}$$

13.

$$e^{ix} = \cos x + i \sin x \tag{1}$$

Substituting in $-x$ gives

$$\begin{aligned}
e^{-ix} &= \cos(-x) + i\sin(-x) \\
&= \cos x - i \sin x. \qquad (2)
\end{aligned}$$

Adding (1) and (2) together gives

$$e^{ix} + e^{-ix} = 2\cos x \qquad \therefore \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

By a similar method we can find a formula for $\sin x$ too. Try it!