EMTH210 Tutorial 8: Double and Triple Integrals

For the week starting Monday 04 May.

The homework questions this week are **2(a)** and **5**.

1. Evaluate the following integrals by reversing the order of integration. In each case sketch the region of integration first.

   (a) $\displaystyle\int_{x=0}^{1}\left(\int_{y=x}^{1} x^2\sqrt{1+y^4}\,\mathrm{d}y\right)\mathrm{d}x$

   (b) $\displaystyle\int_{y=0}^{1}\left(\int_{x=2y}^{2} e^{-y/x}\,\mathrm{d}x\right)\mathrm{d}y$

2. Evaluate each of the following iterated integrals by changing to polar coordinates.

   (a) **(Homework)** $\displaystyle\int_{y=-5}^{5}\left(\int_{x=0}^{\sqrt{25-y^2}} \sqrt{x^2+y^2}\,\mathrm{d}x\right)\mathrm{d}y$

   (b) $\displaystyle\int_{y=0}^{1}\left(\int_{x=0}^{\sqrt{1-y^2}} e^{x^2+y^2}\,\mathrm{d}x\right)\mathrm{d}y$

3. **Harder**

   Use a double integral in polar coordinates to find the area of each region satisfying the given inequalities. In each case sketch the region of integration.

   (a) $0 \le r \le 2 + 2\sin(\theta)$

   (b) $0 \le r \le 1$ and $0 \le r \le 2\sin(\theta)$

4. For each of the following iterated triple integrals, sketch the region of integration and evaluate the integral.

   (a) $\displaystyle\int_{z=1}^{3}\left(\int_{y=-2}^{2}\left(\int_{x=0}^{1}(x+y+z)\,\mathrm{d}x\right)\mathrm{d}y\right)\mathrm{d}z$

   (b) $\displaystyle\int_{y=0}^{\pi/2}\left(\int_{x=0}^{y^2}\left(\int_{z=0}^{y}\cos\left(\frac{x}{y}\right)\mathrm{d}z\right)\mathrm{d}x\right)\mathrm{d}y$

5. **(Homework)** Evaluate the following triple integral in Cartesian coordinates:

   $$\iiint_{\mathcal{R}} z\,\mathrm{d}V \qquad \text{over} \qquad 0 \le z \le 1-x^2, \quad 0 \le x \le 1, \quad \text{and} \quad 0 \le y \le 1-z.$$

6. Evaluate the following triple integral using cylindrical polar coordinates.

   $$\iiint_{\mathcal{R}} z\,\mathrm{d}V \qquad \text{over} \qquad 0 \le z \le 1-x^2-y^2.$$

7. A solid sphere S of radius 1 is centred on the origin. The moment of inertia of this sphere about the z-axis is given by the integral

$$I = \frac{3}{4\pi} \iiint_S \left(x^2 + y^2\right) \, \mathrm{d}V,$$

where dV is an element of volume. Find I by evaluating this integral using spherical polar coordinates. You may find the following indefinite integral useful:

$$\int \sin^3 \alpha \, \mathrm{d}\alpha = -\cos\alpha + \frac{1}{3}\cos^3 \alpha.$$

8. **Long**

   A solid is defined by the inequalities $0 \leq x \leq 1$, $0 \leq y \leq 1$, and $0 \leq z \leq x^2 + y^2$. The temperature of the solid is given by the function $T = 25 - 3z$.

   Find the average temperature of the solid.

9. **Longer**

   Find the mass and height of the centre of mass of the wedge defined by $0 \leq z \leq x$ and $x^2 + y^2 \leq 1$. Assume the density is constant throughout the wedge.

10. **HARDER, but FUN!**

    A circular hole of radius a is drilled through a sphere of radius $2a$ in such a way that the edge of the hole passes through the centre of the sphere. The sphere and cylindrical hole are given by the equations $x^2 + y^2 + z^2 = 4a^2$ and $x^2 + (y - a)^2 = a^2$.

    Use the polar coordinate system $\big(r\cos(\theta), r\sin(\theta), z\big)$ to find the volume of material removed from the sphere by the hole. Express your answer in terms of the volume of the sphere.