EMTH210 Tutorial 5: Differential Equations – Solutions

Preparation problems (homework)

1. (a) The auxiliary equation is

$$m^2 + 6m + 5 = 0,$$
$$(m+1)(m+5) = 0 \implies m = -1, -5.$$

The general solution is

$$y = c_1 e^{-x} + c_2 e^{-5x}, \qquad y(0) = 0 \implies c_1 = -c_2$$
$$y' = -c_1 e^{-x} - 5c_2 e^{-5x} \qquad y'(0) = 3 \implies -c_1 + 5c_1 = 3,$$
$$\implies c_1 = \frac{3}{4},\ c_2 = -\frac{3}{4}.$$

So the solution is

$$y = \frac{3}{4}e^{-x} - \frac{3}{4}e^{-5x}.$$

(b) The auxiliary equation is

$$m^2 + 16 = 0 \implies m = \pm 4i.$$

The general solution is

$$y = c_1 \cos 4x + c_2 \sin 4x \qquad y(0) = 2 \implies c_1 = 2$$
$$y' = -4c_1 \sin 4x + 4c_2 \cos 4x \qquad y'(0) = -2 \implies c_2 = -\frac{1}{2}.$$

So the solution is

$$y = 2\cos 4x - \frac{1}{2}\sin 4x.$$

(c) The auxiliary equation is

$$2m^2 - 2m + 1 = 0,$$
$$m^2 - m + \frac{1}{2} = 0$$
$$(m - \frac{1}{2})^2 + \frac{1}{4} = 0 \implies m = \frac{1}{2} \pm \frac{1}{2}i.$$

The general solution is

$$y = c_1 e^{\frac{x}{2}} \cos\frac{x}{2} + c_2 e^{\frac{x}{2}} \sin\frac{x}{2},$$
$$y(0) = -1 \implies c_1 = -1$$
$$y' = -\frac{1}{2}e^{\frac{x}{2}}\cos\frac{x}{2} + \frac{1}{2}e^{\frac{x}{2}}\sin\frac{x}{2} + \frac{1}{2}c_2 e^{\frac{x}{2}}\sin\frac{x}{2} + \frac{1}{2}c_2 e^{\frac{x}{2}}\cos\frac{x}{2}$$
$$y'(0) = 0 = -\frac{1}{2} + 0 + 0 + \frac{1}{2}c_2 \implies c_2 = 1.$$

So the solution is

$$y = -e^{\frac{x}{2}}\cos\frac{x}{2} + e^{\frac{x}{2}}\sin\frac{x}{2}.$$

2. The auxiliary equation is

$$m^2 + k^2 = 0 \implies m = \pm ki.$$

The general solution is

$$\begin{aligned} y &= c_1 \cos kx + c_2 \sin kx, \\ 0 = y(0) &= c_1 \\ 0 = y(\pi) &= c_2 \sin k\pi. \end{aligned}$$

For nontrivial solutions $c_2 \neq 0$, so $k \in \mathbb{Z}$, and, since we only want positive values for k, k is a positive integer: $k \in \mathbb{Z}^+$.

3. (a) We follow the four rules (in your lecture notes!) to get the particular solution.

$$\begin{aligned} y'' + y &= e^x \sin x \\ \text{aux. eq. is} \quad m^2 + 1 &= 0 \\ m &= \pm i \\ y_c &= c_1 \cos x + c_2 \sin x \end{aligned}$$

Rule 1: $y_p = Ae^x \sin x$
Rule 2: no change
Rule 3: $y_p = Ae^x \cos x + Be^x \sin x$
Rule 4: no change

(b)

$$\begin{aligned} y'' - y &= x^2(e^x + e^{2x}) \\ \text{aux. eq. is} \quad m^2 - 1 &= 0 \\ m &= \pm 1 \\ y_c &= c_1 e^x + c_2 e^{-x} \end{aligned}$$

Rule 1: $y_p = Ax^2e^x + Bx^2e^{2x}$
Rule 2:
$y_p = (Ax^2+Bx+C)e^x + (Dx^2+Ex+F)e^{2x}$
Rule 3: no change
Rule 4:
$y_p = (Ax^2+Bx+C)xe^x + (Dx^2+Ex+F)e^{2x}$

Warning: It's okay to write $Ax^2e^x + Bx^2e^{2x}$ as $Ax^2(e^x + B_1e^{2x})$, because then $B_1 = B/A$, and we still have all the same information (for $A \neq 0$). It's *not* okay to write $(Ax^2 + Bx + C)e^x + (Dx^2 + Ex + F)e^{2x}$ as $(Ax^2 + Bx + C)(e^x + D_1e^{2x})$, since that would make $D_1 = D/A$, $D_1 = E/B$ and $D_1 = F/C$, and we don't know enough to say that those three ratios are equal (for this question, for example, they're not). Some information has been lost.

4. (a)

$$y'' + 4y' + 3y = 8$$

The auxiliary equation is

$$\begin{aligned} m^2 + 4m + 3 &= 0, \\ (m+1)(m+3) &= 0 \implies m = -1, -3. \end{aligned}$$

So the complementary solution is

$$y_c = c_1e^{-x} + c_2e^{-3x}.$$

Next we follow the four rules to get the particular solution.

Rule 1:	$y_p = A$
Rules 2–4:	no change
	$y_p' = y_p'' = 0$

Substituting these into the DE gives

$$\begin{aligned} 3A = 8 &\implies A = \frac{8}{3}, \\ y_p &= \frac{8}{3}, \\ \therefore y = y_c + y_p &= c_1e^{-x} + c_2e^{-3x} + \frac{8}{3}. \end{aligned}$$

This is a general solution. If we had initial or boundary conditions we could find out what c_1 and c_2 are for some specific situation.

(b)

$$y'' + y = 45xe^{2x}$$

$$\text{aux. eq. is} \quad m^2 + 1 = 0 \implies m = \pm i$$

$$y_c = c_1 \cos x + c_2 \sin x$$

$$\begin{aligned}
&\text{Rule 1:} && y_p = Axe^{2x} \\
&\text{Rule 2:} && y_p = (Ax+B)e^{2x} \\
&\text{Rules 3–4:} && \text{no change} \\
& && y_p' = Ae^{2x} + 2(Ax+B)e^{2x} \\
& && y_p'' = 2Ae^{2x} + 2Ae^{2x} + 4(Ax+B)e^{2x} \\
& && \quad = 4Axe^{2x} + 4(A+B)e^{2x}
\end{aligned}$$

$$\begin{aligned}
45xe^{2x} = y_p'' + y_p &= 4Axe^{2x} + 4(A+B)e^{2x} + (Ax+B)e^{2x} \\
&= 5Axe^{2x} + (4A+5B)e^{2x}
\end{aligned}$$

$$\begin{aligned}
xe^{2x} \text{ terms:} &\quad 45 = 5A \implies A = 9 \\
e^{2x} \text{ terms:} &\quad 0 = 4A + 5B \implies B = -\frac{4A}{5} = -\frac{36}{5} \\
\therefore y = y_c + y_p &= c_1 \cos x + c_2 \sin x + \left(9x - \frac{36}{5}\right) e^{2x}
\end{aligned}$$

(c)

$$y'' - 4y' = 8x + 6$$

$$\text{aux. eq. is} \quad m^2 - 4m = 0 \implies m = 0, 4$$

$$y_c = c_1 + c_2 e^{4x}$$

$$\begin{aligned}
&\text{Rule 1:} && y_p = Ax + B \\
&\text{Rules 2–3:} && \text{no change} \\
&\text{Rule 4:} && y_p = Ax^2 + Bx \\
& && y_p' = 2Ax + B \\
& && y_p'' = 2A
\end{aligned}$$

$$\begin{aligned}
8x + 6 = y_p'' - 4y_p' &= 2A - 4(2Ax + B) \\
&= -8Ax + (2A - 4B) \\
x \text{ terms:} &\quad 8 = -8A \implies A = -1 \\
\text{constant terms:} &\quad 6 = 2A - 4B \implies B = \frac{6 - 2A}{-4} = \frac{8}{-4} = -2 \\
\therefore y = y_c + y_p &= c_1 + c_2 e^{4x} - x^2 - 2x
\end{aligned}$$

Problems for the tutorial

5. (a) The auxiliary equation is

$$\begin{aligned} m^2 - 2m + 1 &= 0, \\ (m-1)^2 &= 0 \implies m = 1. \end{aligned}$$

The general solution is

$$\begin{aligned} y &= c_1 e^x + c_2 x e^x, \\ 1 = y(0) &= c_1 \\ 1 = y(1) &= e + c_2 e \implies c_2 = \frac{1-e}{e} = e^{-1} - 1. \end{aligned}$$

So the solution is

$$y = e^x + (e^{-1} - 1) x e^x.$$

(b) The auxiliary equation is

$$m^2 + \pi^2 = 0 \implies m = \pm \pi i.$$

The general solution is

$$\begin{aligned} y &= c_1 \cos \pi x + c_2 \sin \pi x, \\ 1 = y(0) &= c_1 \\ 0 = y(1) &= \cos \pi + c_2 \sin \pi = -1 + 0 \implies \text{contradiction, no solutions exist.} \end{aligned}$$

(c) The auxiliary equation is

$$\begin{aligned} m^2 - 2m + 2 &= 0, \\ (m-1)^2 + 1 &= 0 \implies m = 1 \pm i. \end{aligned}$$

The general solution is

$$\begin{aligned} y &= e^x (c_1 \cos x + c_2 \sin x), \\ 0 = y(0) &= c_1 \\ 0 = y(\pi) &= e^\pi (0) \implies \text{no new information, so no restriction on } c_2. \end{aligned}$$

So the solution is $y = c_2 e^x \sin x$, for any $c_2 \in \mathbb{R}$.

6. (a)

$$\begin{aligned} 4y'' - 4y' + y &= 12 + 8e^{\frac{x}{2}} \\ \text{aux. eq. is} \quad 4m^2 - 4m + 1 &= 0 \\ m^2 - m + \frac{1}{4} &= 0 \\ \left(m - \frac{1}{2}\right)^2 &= 0 \implies m = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} & y_c = c_1 e^{\frac{x}{2}} + c_2 x e^{\frac{x}{2}} \\ \text{Rule 1:} \quad & y_p = A + B e^{\frac{x}{2}} \\ \text{Rules 2–3:} \quad & \text{no change} \\ \text{Rule 4:} \quad & y_p = A + B x^2 e^{\frac{x}{2}} \\ & y_p' = 2Bx e^{\frac{x}{2}} + \frac{1}{2} B x^2 e^{\frac{x}{2}} \\ & y_p'' = 2B e^{\frac{x}{2}} + Bx e^{\frac{x}{2}} + Bx e^{\frac{x}{2}} + \frac{1}{4} B x^2 e^{\frac{x}{2}} \\ & \quad = 2B e^{\frac{x}{2}} + 2Bx e^{\frac{x}{2}} + \frac{1}{4} B x^2 e^{\frac{x}{2}} \end{aligned}$$

$$
\begin{aligned}
12 + 8e^{\frac{x}{2}} = 4y_p'' - 4y_p' + y &= 8Be^{\frac{x}{2}} + 8Bxe^{\frac{x}{2}} + Bx^2e^{\frac{x}{2}} - 8Bxe^{\frac{x}{2}} - 2Bx^2e^{\frac{x}{2}} + A + Bx^2e^{\frac{x}{2}} \\
&= 8Be^{\frac{x}{2}} + A
\end{aligned}
$$

$$
\begin{aligned}
\text{constant terms:}\quad & 12 = A \\
e^{\frac{x}{2}} \text{ terms:}\quad & 8 = 8B \implies B = 1 \\
\therefore y = y_c + y_p &= c_1e^{\frac{x}{2}} + c_2xe^{\frac{x}{2}} + 12 + x^2e^{\frac{x}{2}} \\
&= \left(c_1 + c_2x + x^2\right)e^{\frac{x}{2}} + 12
\end{aligned}
$$

(b)

$$
\begin{aligned}
y''' + 4y' &= 8 + 3\sin x \\
\text{aux. eq. is}\quad m^3 + 4m &= 0 \\
m(m^2+4) &= 0 \implies m = 0, \pm 2i \\
y_c &= c_1 + c_2\cos 2x + c_3\sin 2x \\
\text{Rule 1:}\quad y_p &= A + B\sin x \\
\text{Rule 2:}\quad & \text{no change} \\
\text{Rule 3:}\quad y_p &= A + B\cos x + C\sin x \\
\text{Rule 4:}\quad y_p &= Ax + B\cos x + C\sin x \\
y_p' &= A - B\sin x + C\cos x \\
y_p'' &= -B\cos x - C\sin x \\
y_p''' &= B\sin x - C\cos x
\end{aligned}
$$

$$
\begin{aligned}
8 + 3\sin x = y_p''' + 4y_p' &= B\sin x - C\cos x + 4A - 4B\sin x + 4C\cos x \\
&= 4A - 3B\sin x + 3C\cos x \\
\text{constant terms:}\quad & 8 = 4A \implies A = 2 \\
\sin x \text{ terms:}\quad & 3 = -3B \implies B = -1 \\
\cos x \text{ terms:}\quad & 0 = 3C \implies C = 0 \\
\therefore y = y_c + y_p &= c_1 + c_2\cos 2x + c_3\sin 2x + 2x - \cos x
\end{aligned}
$$

(c)

$$
\begin{aligned}
y'' + 4y &= 3\sin 2x \\
\text{aux. eq. is}\quad m^2 + 4 &= 0 \implies m = \pm 2i
\end{aligned}
$$

$$
\begin{aligned}
y_c &= c_1\cos 2x + c_2\sin 2x \\
\text{Rule 1:}\quad y_p &= A\sin 2x \\
\text{Rule 2:}\quad & \text{no change} \\
\text{Rule 3:}\quad y_p &= A\cos 2x + B\sin 2x \\
\text{Rule 4:}\quad y_p &= Ax\cos 2x + Bx\sin 2x \\
y_p' &= A\cos 2x - 2Ax\sin 2x + B\sin 2x + 2Bx\cos 2x \\
y_p'' &= -2A\sin 2x - 2A\sin 2x - 4Ax\cos 2x + 2B\cos 2x + 2B\cos 2x - 4Bx\sin 2x \\
&= -4A\sin 2x - 4Ax\cos 2x + 4B\cos 2x - 4Bx\sin 2x
\end{aligned}
$$

$$
\begin{aligned}
3\sin 2x &= y_p'' + 4y_p \\
&= -4A\sin 2x - 4Ax\cos 2x + 4B\cos 2x - 4Bx\sin 2x \\
&\quad + 4Ax\cos 2x + 4Bx\sin 2x \\
&= -4A\sin 2x + 4B\cos 2x \\
\sin 2x \text{ terms:}\quad & 3 = -4A \implies A = -\frac{3}{4} \\
\cos 2x \text{ terms:}\quad & 0 = 4B \implies B = 0 \\
\therefore y = y_c + y_p &= c_1\cos 2x + c_2\sin 2x - \frac{3}{4}x\cos 2x
\end{aligned}
$$

7. (a)

$$y' - 2y = x^2 e^{2x} \sin x$$

aux. eq. is $m - 2 = 0$

$$m = 2$$

$$y_c = c_1 e^{2x}$$

Rule 1: $y_p = Ax^2 e^{2x} \sin x$
Rule 2: $y_p = (Ax^2 + Bx + C)e^{2x} \sin x$
Rule 3: $y_p = (Ax^2 + Bx + C)e^{2x} \cos x + (Dx^2 + Ex + F)e^{2x} \sin x$
Rule 4: no change

(b)

$$y''' - y = e^x$$

aux. eq. is $m^3 - 1 = 0$

$$m_1 = 1, m_2 = e^{\frac{2\pi}{3}i}, m_3 = e^{-\frac{2\pi}{3}i}$$

$$m_1 = 1, m_2 = -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i, m_3 = -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i$$

$$y_c = c_1 e^x + c_2 e^{-\frac{x}{2}} \sin \frac{\sqrt{3}}{2}x + c_3 e^{-\frac{x}{2}} \cos \frac{\sqrt{3}}{2}x$$

Rule 1: $y_p = Ae^x$
Rule 2: no change
Rule 3: no change
Rule 4: $y_p = Axe^x$

If you haven't met complex roots before, here's what's going on. $m = re^{\theta i}$ for some r and θ, and $m^3 = 1 = 1e^{0i} = 1e^{(0+k2\pi)i}$ for any $k \in \mathbb{Z}$ (since going around any whole number of revolutions ($k2\pi$) means you stay in the same place). Then,

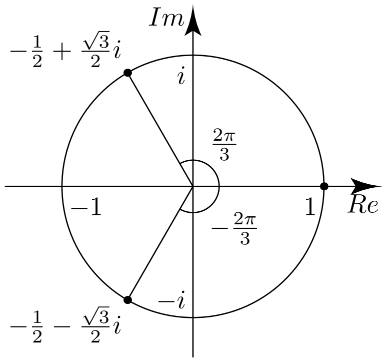
$$m^3 = \left(re^{\theta i}\right)^3 = 1 = 1e^{(0+k2\pi)i}$$

$$m = re^{\theta i} = \left(1e^{(0+k2\pi)i}\right)^{\frac{1}{3}}$$

$$= 1e^{\left(0+k\frac{2\pi}{3}\right)i}$$

$$\implies r = 1, \qquad \theta = k\frac{2\pi}{3}.$$

The simplest choices for k are $k = 0, \pm 1$, so $\theta = 0, \pm\frac{2\pi}{3}$. (Other choices of k will just give you the same 3 roots.)



8. The differential equation is

$$mx'' + kx = f_0 \cos \omega t.$$

We can't use m in the auxiliary equation since it's already used in the DE, so let's use r. We have

$$mr^2 + k = 0,$$

$$\implies r = \pm\sqrt{\frac{k}{m}}i.$$

So the complementary solution is

$$x_c = c_1 \cos \omega t + c_2 \sin \omega t.$$

Next we follow the four rules to get the particular solution.

Rules 1–2: $x_p = B \cos \omega t$.
Rule 3: $x_p = A \sin \omega t + B \cos \omega t$.
Rule 4: $x_p = t(A \sin \omega t + B \cos \omega t)$.

$$x_p' = \omega t(A \cos \omega t - B \sin \omega t) + A \sin \omega t + B \cos \omega t = (A\omega t + B)\cos \omega t + (A - B\omega t)\sin \omega t.$$

$$x_p'' = A\omega \cos \omega t - \omega(A\omega t + B)\sin \omega t + \omega(A - B\omega t)\cos \omega t - B\omega \sin \omega t$$

$$= (2A\omega - B\omega^2 t)\cos \omega t - (A\omega^2 t + 2B\omega)\sin \omega t.$$

Substituting these into the DE gives

$$
\begin{aligned}
& m[(2A\omega - B\omega^2 t)\cos\omega t - (A\omega^2 t + 2B\omega)\sin\omega t] + kt(A\sin\omega t + B\cos\omega t) = f_0\cos\omega t, \\
& \implies 2Am\omega = f_0, \quad 2Bm\omega = 0 \\
& \text{(you may also check that } \omega^2 = \tfrac{k}{m} \text{ implies that the other coefficients are zero)} \\
& \implies A = \frac{f_0}{2m\omega}, \quad B = 0. \\
& \therefore x = x_c + x_p = c_1\cos\omega t + c_2\sin\omega t + \frac{f_0}{2m\omega} t\sin\omega t.
\end{aligned}
$$

This is a general solution. If we had initial or boundary conditions we could find out what c_1 and c_2 are for some specific situation.

9. With the given values of L, R, C, and $E(t)$, the differential equation becomes

$$Q'' + 40Q' + 625Q = 100\cos 10t$$

$$\text{aux. eq. is} \quad m^2 + 40m + 625 = 0 \implies m = -20 \pm 15i$$

$$
\begin{aligned}
\therefore \qquad & Q_c = e^{-20t}(c_1\cos 15t + c_2\sin 15t) \\
\text{Rules 1–2: } & Q_p = A\cos 10t \\
\text{Rules 3–4: } & Q_p = A\cos 10t + B\sin 10t \\
& Q_p' = -10A\sin 10t + 10B\cos 10t \\
& Q_p'' = -100A\cos 10t - 100B\sin 10t
\end{aligned}
$$

$$
\begin{aligned}
100\cos 10t =& Q'' + 40Q' + 625Q \\
=& -100A\cos 10t - 100B\sin 10t + 40(-10A\sin 10t + 10B\cos 10t) \\
& + 625(A\cos 10t + B\sin 10t) \\
=& (525A + 400B)\cos 10t + (525B - 400A)\sin 10t \\
\text{Comparing } \cos 10t \text{ terms: } & \quad 525A + 400B = 100 \\
\sin 10t \text{ terms: } & \quad 525B - 400A = 0 \\
\therefore A = \frac{84}{697}, B = \frac{64}{697} & \implies Q_p = \frac{1}{697}(84\cos 10t + 64\sin 10t),
\end{aligned}
$$

and the general solution is

$$Q = Q_c + Q_p = e^{-20t}(c_1\cos 15t + c_2\sin 15t) + \frac{1}{697}(84\cos 10t + 64\sin 10t).$$

The boundary conditions

$$
\begin{aligned}
Q(0) = 0 &\implies Q(0) = c_1 + \frac{84}{697} = 0 \implies c_1 = -\frac{84}{697}, \\
Q\left(\frac{\pi}{10}\right) = 0 &\implies Q\left(\frac{\pi}{10}\right) = -e^{-2\pi}c_2 - \frac{84}{697} = 0 \implies c_2 = -\frac{84e^{2\pi}}{697}.
\end{aligned}
$$

Hence the formula for the charge is

$$Q(t) = \frac{1}{697}(84\cos 10t + 64\sin 10t) - \frac{84e^{-20t}}{697}(\cos 15t + e^{2\pi}\sin 15t).$$