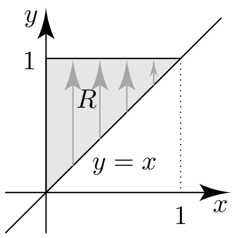
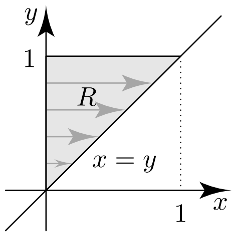
EMTH210 Tutorial 8: Double and Triple Integrals – Solutions

The homework questions this week are **2(a)** and **5**.

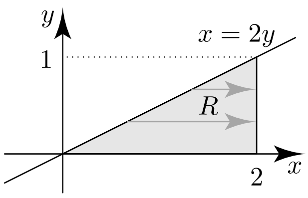
1. (a)



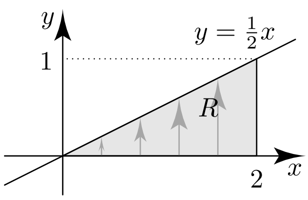
$$\begin{aligned}
\int_{x=0}^{1}\int_{y=x}^{1} x^2\sqrt{1+y^4}\,\mathrm{d}y\,\mathrm{d}x &= \int_{y=0}^{1}\int_{x=0}^{y} x^2\sqrt{1+y^4}\,\mathrm{d}x\,\mathrm{d}y\\
&= \int_{y=0}^{1}\left[\frac{1}{3}x^3\sqrt{1+y^4}\right]_{x=0}^{y}\mathrm{d}y\\
u = 1+y^4 \qquad &= \int_{y=0}^{1}\left(\frac{1}{3}y^3\sqrt{1+y^4}-0\right)\mathrm{d}y\\
\mathrm{d}u = 4y^3\,\mathrm{d}y \qquad &\\
&= \int_{y=0}^{1}\frac{1}{12}u^{\frac{1}{2}}\,\mathrm{d}u\\
&= \left[\frac{1}{12}\frac{2}{3}\left(1+y^4\right)^{\frac{3}{2}}\right]_{y=0}^{1}\\
&= \frac{1}{18}\left(2^{\frac{3}{2}}-1\right)
\end{aligned}$$



(b)



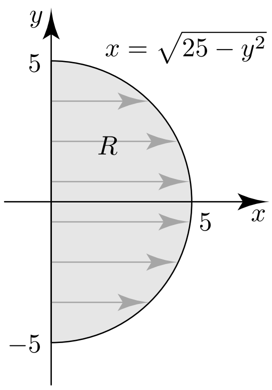
$$\begin{aligned}
\int_{y=0}^{1}\int_{x=2y}^{2} e^{-\frac{y}{x}}\,\mathrm{d}x\,\mathrm{d}y &= \int_{x=0}^{2}\int_{y=0}^{\frac{1}{2}x} e^{-\frac{y}{x}}\,\mathrm{d}y\,\mathrm{d}x\\
&= \int_{x=0}^{2}\left[-xe^{-\frac{y}{x}}\right]_{y=0}^{\frac{1}{2}x}\mathrm{d}x\\
&= \int_{x=0}^{2} -x\left(e^{-\frac{1}{2}}-1\right)\mathrm{d}x\\
&= \left(1-e^{-\frac{1}{2}}\right)\left[\frac{1}{2}x^2\right]_{x=0}^{2}\\
&= 2-2e^{-\frac{1}{2}}
\end{aligned}$$

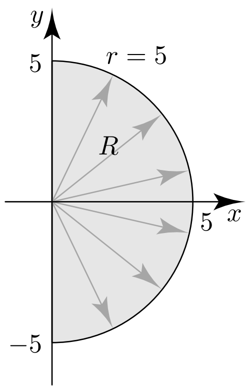


For these polar coordinates questions, keep in mind the relations between coordinate systems,

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad \text{and } r^2 = x^2+y^2.$$

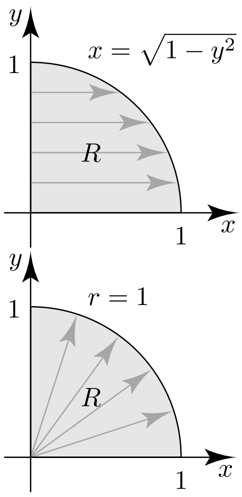
2. (a)





$$\begin{aligned}
&\int_{y=-5}^{5}\int_{x=0}^{\sqrt{25-y^2}} \sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y\\
&= \iint_R \sqrt{x^2+y^2}\,\mathrm{d}A\\
&= \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{r=0}^{5} rr\,\mathrm{d}r\,\mathrm{d}\theta\\
&= \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\frac{1}{3}r^3\right]_{r=0}^{5}\mathrm{d}\theta\\
&= \frac{125}{3}\int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{d}\theta\\
&= \frac{125\pi}{3}
\end{aligned}$$

(b)

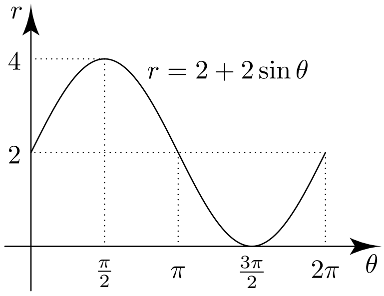


$$\int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} e^{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y = \iint_R e^{x^2+y^2}\,\mathrm{d}A$$

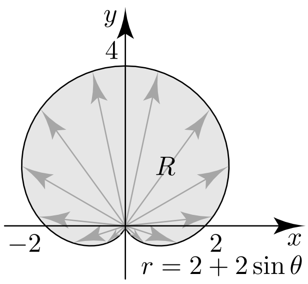
$$u = r^2 \qquad \mathrm{d}u = 2r\,\mathrm{d}r$$

$$\begin{aligned}
&= \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{1} re^{r^2}\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \frac{1}{2}\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{1} e^u\,\mathrm{d}u\,\mathrm{d}\theta \\
&= \frac{1}{2}\int_{\theta=0}^{\frac{\pi}{2}}\left[e^{r^2}\right]_{r=0}^{1}\mathrm{d}\theta \\
&= \frac{1}{2}(e-1)\int_{\theta=0}^{\frac{\pi}{2}}\mathrm{d}\theta \\
&= \frac{\pi}{4}(e-1)
\end{aligned}$$

3. (a) To find out what shape this region is (if you don't recognise it as a cardioid), start with a plot of r versus θ in the *rectangular* coordinate system θ–r:



So the border starts out at 2 when $\theta = 0$, and goes out to 4, back to 2, and in to 0 at $\theta = \frac{\pi}{2}$, π and $\frac{3\pi}{2}$, respectively. This doesn't tell us exactly how it goes between those points, but we can see that they are the extremes, and that the curve is smooth. The only place anything strange can be happening is near $\theta = \frac{3\pi}{2}$, since r going to zero means the r–θ coordinates get condensed.



Now we can see that, somewhat surprisingly, the curve has a sharp point!

$$\begin{aligned}
A &= \int_{\theta=0}^{2\pi}\int_{r=0}^{2+2\sin\theta} r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_{\theta=0}^{2\pi}\left[\frac{1}{2}r^2\right]_{r=0}^{2+2\sin\theta}\mathrm{d}\theta \\
&= \int_{\theta=0}^{2\pi} 2\left(1+\sin\theta\right)^2\,\mathrm{d}\theta \\
&= \int_{\theta=0}^{2\pi}\left(2+4\sin\theta+2\sin^2\theta\right)\mathrm{d}\theta \\
&= \int_{\theta=0}^{2\pi}\left(2+4\sin\theta+1-\cos 2\theta\right)\mathrm{d}\theta \\
&= \left[3\theta - 4\cos\theta - \frac{1}{2}\sin 2\theta\right]_{\theta=0}^{2\pi} \\
&= (6\pi - 4 - 0) - (0 - 4 - 0) = 6\pi
\end{aligned}$$

$$\begin{aligned}
\cos 2\theta &= \cos^2\theta - \sin^2\theta \\
&= 1 - 2\sin^2\theta
\end{aligned}$$

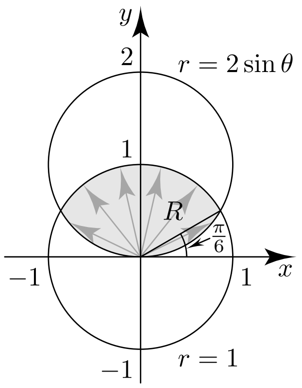
$$2\sin^2\theta = 1 - \cos 2\theta$$

(b) Here we have an intersection between two regions. $0 \le r \le 1$ is easy, but what shape is $0 \le r \le 2\sin\theta$? Using the same technique as before we find that $r \ge 0$ on $0 \le \theta \le \pi$, $r = 0$ at the end points of that interval, and r reaches a maximum of 2 at $\theta = \frac{\pi}{2}$. In fact, it's a circle of radius 1 centred at $(0, 1)$ (this is something you should know). To see why, we can substitute the x–y forms of r and $\sin\theta$ into the boundary equation:

$$\begin{aligned} r &= 2\sin\theta \\ \sqrt{x^2+y^2} &= \frac{2y}{\sqrt{x^2+y^2}} \\ x^2+y^2-2y &= 0 \\ x^2+(y-1)^2 &= 1. \end{aligned}$$

These two circles intersect when

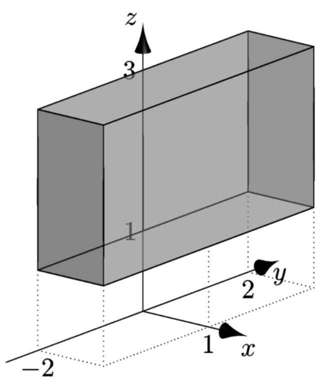
$$\begin{aligned} x^2+(y-1)^2 &= x^2+y^2 \qquad (\text{since } r=1 \text{ is } x^2+y^2=1) \\ y^2-2y+1 &= y^2 \\ y &= \frac{1}{2} \implies \theta = \frac{\pi}{6},\ \pi-\frac{\pi}{6}. \end{aligned}$$



We will need to split the integral up at $\theta = \frac{\pi}{6}$, since the r limit changes form there. We can also just do one half and double it.

$$\begin{aligned} A &= 2\int_{\theta=0}^{\frac{\pi}{6}}\int_{r=0}^{2\sin\theta} r\,\mathrm{d}r\,\mathrm{d}\theta + 2\int_{\theta=\frac{\pi}{6}}^{\frac{\pi}{2}}\int_{r=0}^{1} r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= 2\int_{\theta=0}^{\frac{\pi}{6}}\left[\frac{1}{2}r^2\right]_{r=0}^{2\sin\theta}\mathrm{d}\theta + 2\int_{\theta=\frac{\pi}{6}}^{\frac{\pi}{2}}\left[\frac{1}{2}r^2\right]_{r=0}^{1}\mathrm{d}\theta \\ &= 2\int_{\theta=0}^{\frac{\pi}{6}} 2\sin^2\theta\,\mathrm{d}\theta + 2\int_{\theta=\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{1}{2}\,\mathrm{d}\theta \\ &= 2\int_{\theta=0}^{\frac{\pi}{6}}(1-\cos 2\theta)\,\mathrm{d}\theta + [\theta]_{\theta=\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= 2\left[\theta-\frac{1}{2}\sin 2\theta\right]_{\theta=0}^{\frac{\pi}{6}} + [\theta]_{\theta=\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= 2\left(\left(\frac{\pi}{6}-\frac{1}{2}\sin\frac{\pi}{3}\right)-(0)\right)+\left(\frac{\pi}{2}-\frac{\pi}{6}\right) \\ &= \frac{\pi}{3}-\frac{\sqrt{3}}{2}+\frac{\pi}{3} \\ &= \frac{2\pi}{3}-\frac{\sqrt{3}}{2} \end{aligned}$$

4. (a)



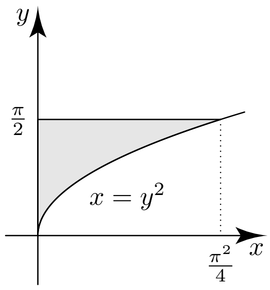
Note: The 3D diagrams like this are rotatable in Adobe Reader 9+.

$$\begin{aligned} &\int_{z=1}^{3}\int_{y=-2}^{2}\int_{x=0}^{1}(x+y+z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \int_{z=1}^{3}\int_{y=-2}^{2}\left[\frac{1}{2}x^2+yx+zx\right]_{x=0}^{1}\mathrm{d}y\,\mathrm{d}z \\ &= \int_{z=1}^{3}\int_{y=-2}^{2}\left(\frac{1}{2}+y+z\right)\mathrm{d}y\,\mathrm{d}z \\ &= \int_{z=1}^{3}\left[\frac{1}{2}y+\frac{1}{2}y^2+zy\right]_{y=-2}^{2}\mathrm{d}z \\ &= \int_{z=1}^{3}(1+2+2z+1-2+2z)\,\mathrm{d}z \\ &= \int_{z=1}^{3}(2+4z)\,\mathrm{d}z \\ &= \left[2z+2z^2\right]_{z=1}^{3} \\ &= 6+18-2-2 = 20 \end{aligned}$$

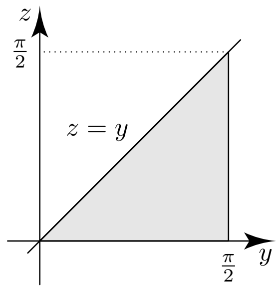
(b)

$$\int_{y=0}^{\frac{\pi}{2}}\int_{x=0}^{y^2}\int_{z=0}^{y}\cos\frac{x}{y}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$$

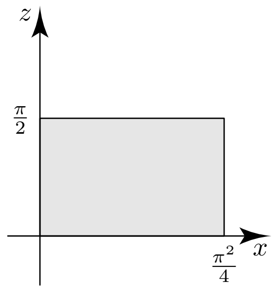
Here's how to draw these 3D regions. Pick some planes where one of the variables equals a constant (0 is often good), and draw the parts of the region that are on these planes (slices through the 3D region).



$z = 0$

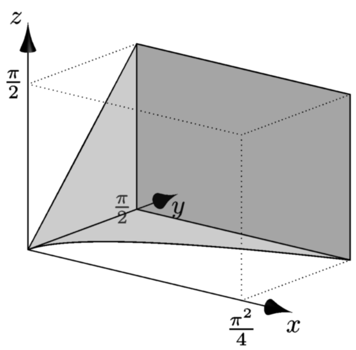


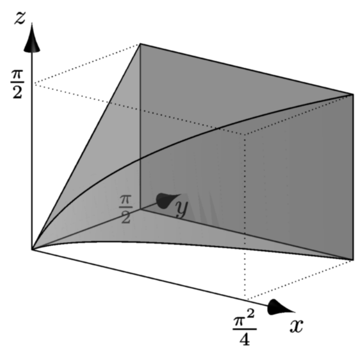
$x = 0$



$y = \frac{\pi}{2}$
($y = 0$ is just a point)

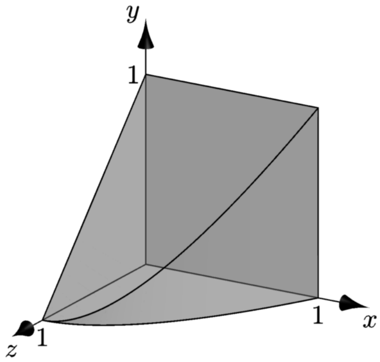
Then draw (and label!) some 3D axes from a view that seems suitable, and draw the planar plots you just did on the appropriate planes. Finally, draw the other edges. Here the region is a wedge (ignoring the x limits) with a parabola (the x limits) chopping out a big chunk.





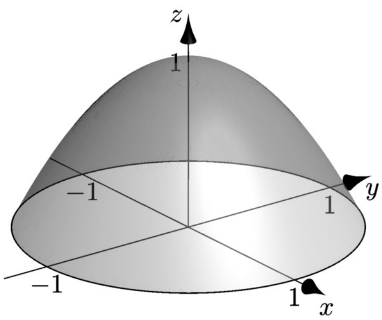
$$
\begin{aligned}
\int_{y=0}^{\frac{\pi}{2}} \int_{x=0}^{y^2} \int_{z=0}^{y} \cos \frac{x}{y} \, \mathrm{d}z \, \mathrm{d}x \, \mathrm{d}y &= \int_{y=0}^{\frac{\pi}{2}} \int_{x=0}^{y^2} \left[z \cos \frac{x}{y} \right]_{z=0}^{y} \mathrm{d}x \, \mathrm{d}y \\
&= \int_{y=0}^{\frac{\pi}{2}} \int_{x=0}^{y^2} y \cos \frac{x}{y} \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_{y=0}^{\frac{\pi}{2}} \left[y^2 \sin \frac{x}{y} \right]_{x=0}^{y^2} \mathrm{d}y \\
&= \int_{y=0}^{\frac{\pi}{2}} y^2 \sin y \, \mathrm{d}y \qquad \begin{matrix} u = y^2 & \mathrm{d}v = \sin y \, \mathrm{d}y \\ \mathrm{d}u = 2y \, \mathrm{d}y & v = -\cos y \end{matrix} \\
&= \left[-y^2 \cos y \right]_{y=0}^{\frac{\pi}{2}} - \int_{y=0}^{\frac{\pi}{2}} -2y \cos y \, \mathrm{d}y \qquad \begin{matrix} u = y & \mathrm{d}v = \cos y \, \mathrm{d}y \\ \mathrm{d}u = \mathrm{d}y & v = \sin y \end{matrix} \\
&= \left[-y^2 \cos y \right]_{y=0}^{\frac{\pi}{2}} + 2 \left(\left[y \sin y \right]_{y=0}^{\frac{\pi}{2}} - \int_{y=0}^{\frac{\pi}{2}} \sin y \, \mathrm{d}y \right) \\
&= \left[-y^2 \cos y \right]_{y=0}^{\frac{\pi}{2}} + 2 \left[y \sin y \right]_{y=0}^{\frac{\pi}{2}} - 2 \left[-\cos y \right]_{y=0}^{\frac{\pi}{2}} \\
&= (0 - 0) + 2 \left(\frac{\pi}{2} - 0 \right) - 2(0 - -1) \\
&= \pi - 2
\end{aligned}
$$

5.



$$
\begin{aligned}
\iiint_R z\,\mathrm{d}V &= \int_{x=0}^{1}\int_{z=0}^{1-x^2}\int_{y=0}^{1-z} z\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x \\
&= \int_{x=0}^{1}\int_{z=0}^{1-x^2} [zy]_{y=0}^{1-z}\,\mathrm{d}z\,\mathrm{d}x \\
&= \int_{x=0}^{1}\int_{z=0}^{1-x^2} \left(z-z^2\right)\,\mathrm{d}z\,\mathrm{d}x \\
&= \int_{x=0}^{1} \left[\frac{1}{2}z^2-\frac{1}{3}z^3\right]_{z=0}^{1-x^2}\,\mathrm{d}x \\
&= \int_{x=0}^{1} \left(\frac{1}{2}(1-x^2)^2-\frac{1}{3}(1-x^2)^3\right)\,\mathrm{d}x \\
&= \int_{x=0}^{1} \Bigg(\frac{1}{2}\left(1-2x^2+x^4\right)- \\
&\qquad \frac{1}{3}\left(1-3x^2+3x^4-x^6\right)\Bigg)\,\mathrm{d}x \\
&= \int_{x=0}^{1} \left(\frac{1}{6}-\frac{1}{2}x^4+\frac{1}{3}x^6\right)\,\mathrm{d}x \\
&= \left[\frac{1}{6}x-\frac{1}{10}x^5+\frac{1}{21}x^7\right]_{x=0}^{1} \\
&= \frac{1}{6}-\frac{1}{10}+\frac{1}{21}=\frac{4}{35}
\end{aligned}
$$

6.

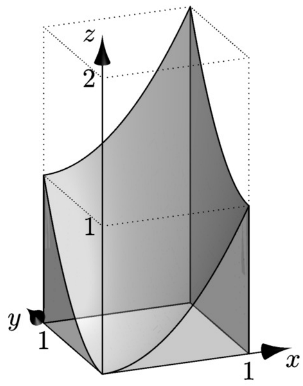


$$
\begin{aligned}
\iiint_R z\,\mathrm{d}V &= \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=0}^{1-r^2} zr\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \left[\frac{1}{2}z^2r\right]_{z=0}^{1-r^2}\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \frac{1}{2}\int_{\theta=0}^{2\pi}\int_{r=0}^{1} \left(r-2r^3+r^5\right)\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \frac{1}{2}\int_{\theta=0}^{2\pi} \left[\frac{1}{2}r^2-\frac{1}{2}r^4+\frac{1}{6}r^6\right]_{r=0}^{1}\,\mathrm{d}\theta \\
&= \frac{1}{12}\int_{\theta=0}^{2\pi}\mathrm{d}\theta \\
&= \frac{\pi}{6}
\end{aligned}
$$

7. This is just a sphere – you know what that looks like!

$$
\begin{aligned}
I &= \frac{3}{4\pi} \iiint_S (x^2 + y^2)\, dV \\
&= \frac{3}{4\pi} \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (r^2 \sin^2\phi \,\cos^2\theta \,+\, r^2 \sin^2\phi \,\sin^2\theta)\,.\, r^2 \sin\phi \;\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \frac{3}{4\pi} \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r^2 \sin^2\phi\,(\cos^2\theta \,+\, \sin^2\theta)\, r^2 \sin\phi \;\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \frac{3}{4\pi} \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{r=0}^{1} r^4 \sin^3\phi \;\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}\theta \quad \text{(note the change of order)} \\
&= \frac{3}{4\pi} \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \left[\frac{1}{5} r^5\right]_{r=0}^{1} \sin^3\phi \;\mathrm{d}\phi\,\mathrm{d}\theta \\
&= \frac{3}{20\pi} \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \sin^3\phi \;\mathrm{d}\phi\,\mathrm{d}\theta \\
&= \frac{3}{20\pi} \int_{\theta=0}^{2\pi} \left[-\cos\phi + \frac{1}{3}\cos^3\phi\right]_{\phi=0}^{\pi} \mathrm{d}\theta \quad \text{(by the given integral)} \\
&= \frac{3}{20\pi} \int_{\theta=0}^{2\pi} [(1 - \frac{1}{3}) \,-\, (-1 + \frac{1}{3})]\,\mathrm{d}\theta \\
&= \frac{3}{20\pi} \int_{\theta=0}^{2\pi} \frac{4}{3}\,\mathrm{d}\theta \\
&= \frac{1}{5\pi}\, [\theta]_{\theta=0}^{2\pi} \\
&= \frac{2}{5}
\end{aligned}
$$

8.



The average temperature is

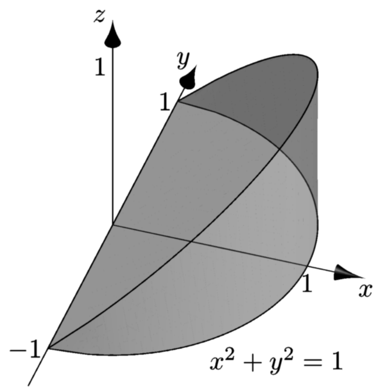
$$\frac{1}{V}\iiint_R T\,\mathrm{d}V.$$

This is analogous to the discrete case, where we are finding the average temperature at a set of n points. There, we add up all the temperatures and divide by the number of measurements. Here, we "add up" an infinite number of temperatures with an integral, then divide by the "number" of measurements, the volume.

$$\begin{aligned}
V &= \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{x^2+y^2} \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{x=0}^{1}\int_{y=0}^{1} \left(x^2+y^2\right)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{x=0}^{1} \left[x^2y+\frac{1}{3}y^3\right]_{y=0}^{1}\mathrm{d}x \\
&= \int_{x=0}^{1} \left(x^2+\frac{1}{3}\right)\mathrm{d}x \\
&= \left[\frac{1}{3}x^3+\frac{1}{3}x\right]_{x=0}^{1} \\
&= \frac{2}{3}
\end{aligned}$$

$$\begin{aligned}
T_{\text{avg}} &= \frac{1}{2/3}\int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{x^2+y^2}(25-3z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\
&= \frac{3}{2}\int_{x=0}^{1}\int_{y=0}^{1}\left[25z-\frac{3}{2}z^2\right]_{z=0}^{x^2+y^2}\mathrm{d}y\,\mathrm{d}x \\
&= \frac{3}{2}\int_{x=0}^{1}\int_{y=0}^{1}\left(25x^2+25y^2-\frac{3}{2}x^4-3x^2y^2-\frac{3}{2}y^4\right)\mathrm{d}y\,\mathrm{d}x \\
&= \frac{3}{2}\int_{x=0}^{1}\left[25x^2y+\frac{25}{3}y^3-\frac{3}{2}x^4y-x^2y^3-\frac{3}{10}y^5\right]_{y=0}^{1}\mathrm{d}x \\
&= \frac{3}{2}\int_{x=0}^{1}\left(24x^2-\frac{3}{2}x^4-\frac{241}{30}\right)\mathrm{d}x \\
&= \frac{3}{2}\left[8x^3-\frac{3}{10}x^5-\frac{241}{30}x\right]_{x=0}^{1} \\
&= \frac{3}{2}\,\frac{240-9+241}{30} = \frac{118}{5}
\end{aligned}$$

9.

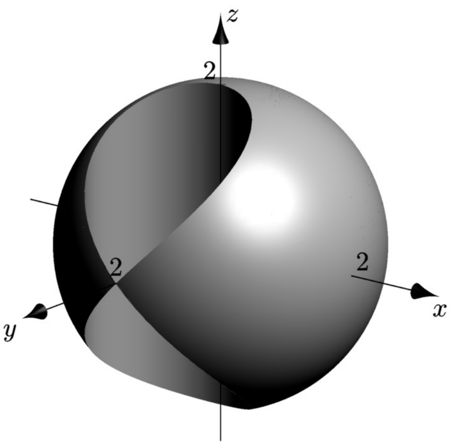


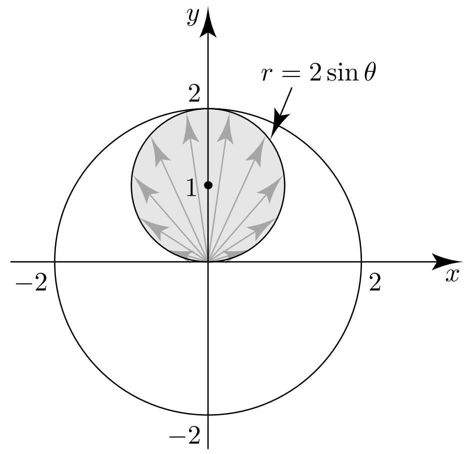
Let $\rho =$ density (constant)

$$
\begin{aligned}
\text{mass } m &= \rho V \\
&= \rho \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=0}^{1} \int_{z=0}^{r\cos\theta} r \,\mathrm{d}z \,\mathrm{d}r \,\mathrm{d}\theta \\
&= \rho \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=0}^{1} r^2 \cos\theta \,\mathrm{d}r \,\mathrm{d}\theta \\
&= \rho \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{1}{3} r^3 \cos\theta\right]_{r=0}^{1} \mathrm{d}\theta \\
&= \frac{1}{3}\rho \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \,\mathrm{d}\theta \\
&= \frac{1}{3}\rho \left[\sin\theta\right]_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{2}{3}\rho
\end{aligned}
$$

$$
\begin{aligned}
z_{\text{COM}} &= \frac{1}{m} \iiint_R \rho z \,\mathrm{d}V \\
&= \frac{3}{2} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=0}^{1} \int_{z=0}^{r\cos\theta} zr \,\mathrm{d}z \,\mathrm{d}r \,\mathrm{d}\theta \\
&= \frac{3}{2} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=0}^{1} \frac{1}{2} r^3 \cos^2\theta \,\mathrm{d}r \,\mathrm{d}\theta \\
&= \frac{3}{2} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{1}{8} r^4 \cos^2\theta\right]_{r=0}^{1} \mathrm{d}\theta \\
&= \frac{3}{16} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta \,\mathrm{d}\theta \\
&= \frac{3}{16} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2\theta) \,\mathrm{d}\theta \\
&= \frac{3}{32} \left[\theta + \frac{1}{2}\sin 2\theta\right]_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{3\pi}{32}
\end{aligned}
$$

10. The sphere's radius is twice that of the cylinder, but the actual sizes are unknown. Since we only need to find the proportion removed, to keep things simple we can assume that the cylinder's radius is 1.





There is quite a bit of symmetry in the removed section, so we can just do the volume of the bit where $x > 0$ and $z > 0$ and multiply the result by 4, giving us θ limits of 0 and $\frac{\pi}{2}$. From question 6(b) at the end of Tutorial 8, we know that the equation for the circular cross-section of the cylinder is $r = 2\sin\theta$, giving us r limits of 0 and $2\sin\theta$. The sphere is defined by $x^2 + y^2 + z^2 = 2^2$, so the upper hemisphere is $z = \sqrt{4 - x^2 - y^2} = \sqrt{4 - r^2}$, giving us z limits of 0 and $\sqrt{4 - r^2}$.

volume removed

$$
\begin{aligned}
&= 4\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{2\sin\theta}\int_{z=0}^{\sqrt{4-r^2}} r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta \\
&= 4\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{2\sin\theta} r\sqrt{4-r^2}\,\mathrm{d}r\,\mathrm{d}\theta && u = 4 - r^2 \\
& && \mathrm{d}u = -2r\,\mathrm{d}r \\
&= -2\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{2\sin\theta} u^{\frac{1}{2}}\,\mathrm{d}u\,\mathrm{d}\theta \\
&= -2\int_{\theta=0}^{\frac{\pi}{2}}\left[\frac{2}{3}\left(4-r^2\right)^{\frac{3}{2}}\right]_{r=0}^{2\sin\theta}\mathrm{d}\theta \\
&= -\frac{4}{3}\int_{\theta=0}^{\frac{\pi}{2}}\left(\left(4-(2\sin\theta)^2\right)^{\frac{3}{2}} - \left(4-(0)^2\right)^{\frac{3}{2}}\right)\mathrm{d}\theta && \left(4-(2\sin\theta)^2\right)^{\frac{3}{2}} = \left(4\cos^2\theta\right)^{\frac{3}{2}} \\
& && = 8\cos^3\theta \\
&= -\frac{4}{3}\int_{\theta=0}^{\frac{\pi}{2}}\left(8\cos^3\theta - 8\right)\mathrm{d}\theta \\
&= \frac{32}{3}\int_{\theta=0}^{\frac{\pi}{2}}\left(1-\cos^3\theta\right)\mathrm{d}\theta && \int\cos^3\theta\,\mathrm{d}\theta = \tfrac{1}{3}\cos^2\theta\sin\theta + \tfrac{2}{3}\int\cos\theta\,\mathrm{d}\theta \\
& && = \tfrac{1}{3}\cos^2\theta\sin\theta + \tfrac{2}{3}\sin\theta \\
&= \frac{32}{3}\left[\theta - \cancelto{0}{\frac{1}{3}\cos^2\theta\sin\theta} - \frac{2}{3}\sin\theta\right]_{\theta=0}^{\frac{\pi}{2}} \\
&= \frac{32}{3}\left(\left(\frac{\pi}{2} - \frac{2}{3}\right) - 0\right) \\
&= \frac{16\pi}{3} - \frac{64}{9}
\end{aligned}
$$

So the proportion removed is

$$\frac{\frac{16\pi}{3} - \frac{64}{9}}{\frac{4}{3}\pi(2)^3} = \frac{\frac{16\pi}{3} - \frac{64}{9}}{\frac{32}{3}\pi} = \frac{1}{2} - \frac{2}{3\pi}.$$