EMTH210 Tutorial 5: Differential Equations

For the week starting Monday 16 March.

Preparation problems (homework)

1. Find all solutions to each of the following initial value problems.
   (a) $y'' + 6y' + 5y = 0$ with $y(0) = 0$ and $y'(0) = 3$.

   (b) $y'' + 16y = 0$ with $y(0) = 2$ and $y'(0) = -2$.

   (c) $2y'' - 2y' + y = 0$ with $y(0) = -1$ and $y'(0) = 0$.

2. For what positive values of k does the boundary value problem
$$y'' + k^2 y = 0 \qquad \text{with} \qquad y(0) = y(\pi) = 0$$
have non-trivial solutions (i.e. solutions that are not zero everywhere)?

3. Find the form of the particular integral for each of the following differential equations. Do not solve the DE and do not include any unnecessary terms.

   (a) $y'' + y = e^x \sin(x)$

   (b) $y'' - y = x^2(e^x + e^{2x})$

4. Find the general solutions to the following differential equations using the method of undetermined coefficients.

   (a) $y'' + 4y' + 3y = 8$

   (b) $y'' + y = 45xe^{2x}$

   (c) $y'' - 4y' = 8x + 6$

Problems for the tutorial

5. Find all solutions to each of the following boundary value problems, or state why no solution exists.
   (a) $y'' - 2y' + y = 0$ with $y(0) = 1$ and $y(1) = 1$.
   (b) $y'' + \pi^2 y = 0$ with $y(0) = 1$ and $y(1) = 0$.
   (c) $y'' - 2y' + 2y = 0$ with $y(0) = 0$ and $y(\pi) = 0$.

6. Find the general solutions to the following differential equations using the method of undetermined coefficients.

   (a) $4y'' - 4y' + y = 12 + 8e^{x/2}$

   (b) $y''' + 4y' = 8 + 3\sin(x)$

   (c) $y'' + 4y = 3\sin(2x)$

P.T.O.

7. Find the form of the particular integral for each of the following differential equations. Do not solve the DE and do not include any unnecessary terms.

   (a) $y' - 2y = x^2 e^{2x} \sin(x)$ (b) $y''' - y = e^x$

8. The motion of a object with mass m at the end of a spring affected by an external force $f(t)$ may be described by the equation

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \rho\frac{\mathrm{d}x}{\mathrm{d}t} + kx = f(t),$$

   where $k > 0$ is the spring constant, $\rho \geq 0$ is the damping constant, and t is time.

   Suppose that the damping constant $\rho = 0$ and an external force $f(t) = f_0 \cos \omega t$ is applied, where $\omega = \sqrt{\frac{k}{m}}$. Use the method of undetermined coefficients to show that the motion of the mass is given by

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{f_0}{2m\omega} t \sin \omega t.$$

9. Suppose an electric circuit contains an electromotive force $E(t)$ (supplied by a battery or generator), a resistor with resistance R (in ohms), an inductor with inductance L (in henries), and a capacitor with capacitance C (in farads), in series. Then the charge $Q(t)$ (in coulombs) on the capacitor at time t may be described by the differential equation

$$L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + R\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{1}{C}Q = E(t).$$

   Find the charge $Q(t)$ in the circuit if

$$R = 40\Omega, \quad L = 1\mathrm{H}, \quad C = 16 \times 10^{-4}\mathrm{F}, \quad E(t) = 100 \cos 10t,$$

   and the charge at the times $t = 0$ and $t = \frac{\pi}{10}$ are both 0.