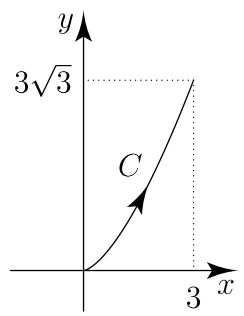
EMTH210 Tutorial 3:
Line Integrals and Differentials – Solutions

Preparation problems (homework)

1. Suppose S is a surface defined *explicitly* by the equation $z = g(x, y)$, and (x_0, y_0, z_0) is a point on S at which the tangent plane exists. Since we have an explicit formula for the surface (z in terms of x and y), an easy way to get equation of the tangent plane to S at (x_0, y_0, z_0) is from 2D Taylor series:
$$z = z_0 + g_x(x_0, y_0)(x - x_0) + g_y(x_0, y_0)(y - y_0).$$
Here, the partial derivatives of g are
$$\frac{\partial g}{\partial x} = 12x^2y^2 \quad \text{and} \quad \frac{\partial g}{\partial y} = 8x^3y + 2$$
so at the point $(1, -2)$
$$\left.\frac{\partial g}{\partial x}\right|_{(1,-2)} = 48 \quad \text{and} \quad \left.\frac{\partial g}{\partial y}\right|_{(1,-2)} = -14.$$
So the equation of the tangent plane to z at $(1, -2, 12)$ is
$$z = 12 + 48(x - 1) - 14(y + 2) \quad \text{that is,} \quad z = -64 + 48x - 14y.$$
A completely different way to get the same result is to write the surface *implicitly*, by moving the z to the right and defining a new function of 3 variables: Let $f(x, y, z) = 4x^3y^2 + 2y - z$. Then the surface is all the points satisfying $f(x, y, z) = 0$, and its normal vector is ∇f (see Question 2).

2. This surface is defined *implicitly*, so we can't use the same Taylor series approximation as before. The surface is a level surface of f. As such, it has normal vector ∇f.
$$\nabla f(x, y, z) = \begin{pmatrix} 2x \\ 4y \\ 6z \end{pmatrix} \qquad \text{and at our point this is} \qquad \nabla f(2, 2, 1) = \begin{pmatrix} 4 \\ 8 \\ 6 \end{pmatrix}.$$
Therefore the point normal form of the tangent plane is
$$\begin{aligned} (\mathbf{r} - \mathbf{r}_0) \cdot \nabla f &= 0 \\ \mathbf{r} \cdot \nabla f &= \mathbf{r}_0 \cdot \nabla f \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 8 \\ 6 \end{pmatrix} &= \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 8 \\ 6 \end{pmatrix} \\ 4x + 8y + 6z &= 8 + 16 + 6 = 30 \\ 2x + 4y + 3z &= 15. \end{aligned}$$

3. Arclength is given by
$$s = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\, \mathrm{d}t.$$
Let $x = t$, then $a = 0$, $b = 3$, and $\frac{\mathrm{d}x}{\mathrm{d}t} = 1$. Also
$$\begin{aligned} y &= t\sqrt{t} = t^{\frac{3}{2}} \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{3}{2}t^{\frac{1}{2}}. \end{aligned}$$



A very steep hill!

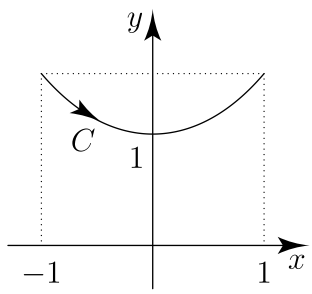
So the length of the fence is

$$
\begin{aligned}
s &= \int_0^3 \sqrt{1+\frac{9}{4}t}\,\mathrm{d}t && \text{let } u = 1+\frac{9}{4}t \\
&= \int_{t=0}^3 \frac{4}{9}u^{\frac{1}{2}}\,\mathrm{d}u && \mathrm{d}u = \frac{9}{4}\,\mathrm{d}t \\
&= \frac{4}{9}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_{t=0}^3 \\
&= \frac{4}{9}\frac{2}{3}\left(\left(1+\frac{9}{4}(3)\right)^{\frac{3}{2}} - \left(1+\frac{9}{4}(0)\right)^{\frac{3}{2}}\right) \\
&= \frac{8}{27}\left(\left(1+\frac{27}{4}\right)^{\frac{3}{2}} - 1\right) \\
&= \frac{8}{27}\left(\left(\frac{31}{4}\right)^{\frac{3}{2}} - 1\right) \\
&= \frac{1}{27}\left(31^{\frac{3}{2}} - 8\right)
\end{aligned}
$$

4. Let $x = t$, then the integral's limits are ± 1, and $\frac{\mathrm{d}x}{\mathrm{d}t} = 1$.

$$
\begin{aligned}
y &= \cosh t \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= \sinh t \\
s &= \int_{-1}^1 \sqrt{1+\sinh^2 t}\,\mathrm{d}t \\
&= \int_{-1}^1 \cosh t\,\mathrm{d}t \\
&= [\sinh t]_{-1}^1 \\
&= \sinh 1 - \sinh(-1) \\
&= 2\sinh 1
\end{aligned}
$$

$$
\begin{aligned}
&\cosh^2 t - \sinh^2 t = 1 \\
\implies \quad &1+\sinh^2 t = \cosh^2 t
\end{aligned}
$$

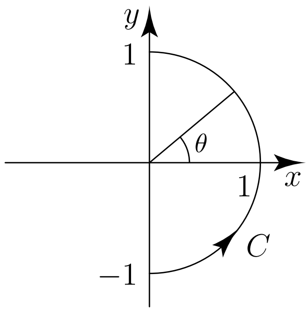


5.

$$
\begin{aligned}
W &= \int_C F\cdot\,\mathrm{d}\mathbf{r} \\
&= \int_{t=0}^1 \mathrm{d}x - y\,\mathrm{d}y + xyz\,\mathrm{d}z \\
&= \int_0^1 (1-2t^3-t^4)\,\mathrm{d}t \\
&= \left[t - \frac{1}{2}t^4 - \frac{1}{5}t^5\right]_0^1 \\
&= 1 - \frac{1}{2} - \frac{1}{5} \\
&= \frac{3}{10}
\end{aligned}
$$

$$
\begin{aligned}
x &= t & y &= -t^2 & z &= t \\
\mathrm{d}x &= \mathrm{d}t & \mathrm{d}y &= -2t\,\mathrm{d}t & \mathrm{d}z &= \mathrm{d}t
\end{aligned}
$$

6. Circles are almost always best parametrised with angle, θ, which is usually measured from the x axis anticlockwise.



$$x = \cos\theta \qquad y = \sin\theta \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\mathrm{d}x = -\sin\theta\,\mathrm{d}\theta \qquad \mathrm{d}y = \cos\theta\,\mathrm{d}\theta$$

$$\begin{aligned}\int_C x\,\mathrm{d}y - y\,\mathrm{d}x &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\cos^2\theta + \sin^2\theta\right)\,\mathrm{d}\theta \\ &= \pi\end{aligned}$$

7. (a)

$$\begin{aligned} f &= x^3 + y^2 \\ \mathrm{d}f &= 3x^2\,\mathrm{d}x + 2y\,\mathrm{d}y \end{aligned}$$

(b)

$$\begin{aligned} f &= (x-y)\cos(x+y) \\ \mathrm{d}f &= ((1)\cos(x+y) - (x-y)\sin(x+y))\,\mathrm{d}x + ((-1)\cos(x+y) - (x-y)\sin(x+y))\,\mathrm{d}y \end{aligned}$$

(c)

$$\begin{aligned} f &= e^{x/y}\cos z^2 y \\ \mathrm{d}f &= \frac{1}{y}e^{x/y}\cos z^2 y\,\mathrm{d}x + \left(-\frac{x}{y^2}e^{x/y}\cos z^2 y - z^2 e^{x/y}\sin z^2 y\right)\,\mathrm{d}y - 2zye^{x/y}\sin z^2 y\,\mathrm{d}z \end{aligned}$$

(d)

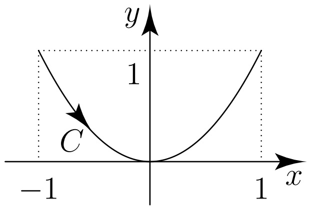
$$\begin{aligned} f &= \tan^{-1}\frac{y}{x} + \ln z \\ \mathrm{d}f &= \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)\,\mathrm{d}x + \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)\,\mathrm{d}y + \frac{1}{z}\,\mathrm{d}z \end{aligned}$$

8. (a) $J_{\mathbf{F}} = \begin{pmatrix} 2e^x & 6y \\ \frac{1}{yx}y & \frac{1}{yx}x \end{pmatrix} = \begin{pmatrix} 2e^x & 6y \\ \frac{1}{x} & \frac{1}{y} \end{pmatrix}$

(b) $J_{\mathbf{F}} = \begin{pmatrix} \cos(xy)y + z^2 & \cos(xy)x & 2xz \\ -1 & \cos(z) & -y\sin(z) \end{pmatrix}$

Problems for the tutorial

9. (a)

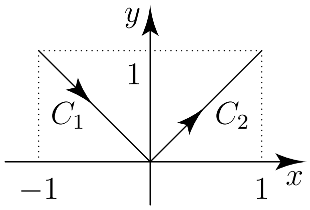


Let $x = t \qquad y = t^2 \qquad -1 \le t \le 1$

$\mathrm{d}x = \mathrm{d}t \qquad \mathrm{d}y = 2t\,\mathrm{d}t.$

$$\begin{aligned}\int_C y\,\mathrm{d}x + x\,\mathrm{d}y &= \int_{-1}^{1} (t^2 + 2t^2)\,\mathrm{d}t \\ &= \left[t^3\right]_{-1}^{1} \\ &= 2\end{aligned}$$

(b)



We split C into two straight lines, C_1 and C_2, so we can differentiate to get $\mathrm{d}y$.

For C_1, $y = -x$, so

let $x = t \qquad y = -t \qquad -1 \le t \le 0$

$\mathrm{d}x = \mathrm{d}t \qquad \mathrm{d}y = -\mathrm{d}t.$

For C_2, $y = x$, so

let $x = t \qquad y = t \qquad 0 \le t \le 1$

$\mathrm{d}x = \mathrm{d}t \qquad \mathrm{d}y = \mathrm{d}t.$

$$\begin{aligned}\int_C y\,\mathrm{d}x + x\,\mathrm{d}y &= \int_{C_1} y\,\mathrm{d}x + x\,\mathrm{d}y + \int_{C_2} y\,\mathrm{d}x + x\,\mathrm{d}y \\ &= \int_{-1}^{0} (-t - t)\,\mathrm{d}x + \int_{0}^{1} (t + t)\,\mathrm{d}x \\ &= \left[-t^2\right]_{-1}^{0} + \left[t^2\right]_{0}^{1} \\ &= 2\end{aligned}$$

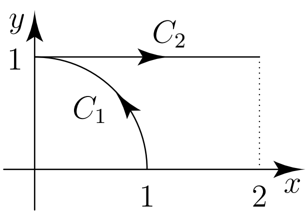
In general for $\int_C y\,\mathrm{d}x + x\,\mathrm{d}y$,

$$P = y \qquad\qquad Q = x$$

$$\frac{\partial P}{\partial y} = 1 \qquad\qquad \frac{\partial Q}{\partial x} = 1 = \frac{\partial P}{\partial y}.$$

We expect the answers to be the same because $y\,\mathrm{d}x + x\,\mathrm{d}y$ is an exact differential $\left(\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}\right)$, so line integrals are path-independent.

10.



C_1:

$x = \cos\theta \qquad y = \sin\theta \qquad 0 < \theta < \frac{\pi}{2}$

$\mathrm{d}x = -\sin\theta\,\mathrm{d}\theta \qquad \mathrm{d}y = \cos\theta\,\mathrm{d}\theta$

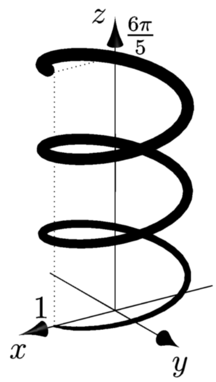
C_2:

$x = t \qquad y = 1 \qquad 0 < t < 2$

$\mathrm{d}x = \mathrm{d}t \qquad \mathrm{d}y = 0\,\mathrm{d}t$

$$\begin{aligned}\int_C xy\,\mathrm{d}x + y^2\,\mathrm{d}y &= \int_{C_1} xy\,\mathrm{d}x + y^2\,\mathrm{d}y + \int_{C_2} xy\,\mathrm{d}x + y^2\,\mathrm{d}y \\ &= \int_0^{\frac{\pi}{2}} (-\cos\theta\sin\theta\sin\theta + \sin^2\theta\cos\theta)\,\mathrm{d}\theta + \int_0^2 \big(t(1) + 0\big)\,\mathrm{d}t \\ &= 0 + \left[\frac{t^2}{2}\right]_0^2 \\ &= 2\end{aligned}$$

11.



Note: The 3D diagrams like this are rotatable in Adobe Reader 9+.

We want the total mass, but what we have is a formula for the mass per unit length, which is changing.

If the mass per unit length was constant, we could just multiply it by the total arclength to get the mass.

Since the mass per unit length is changing, we have to add up lots of infinitesimal snippets of arclength multiplied by the mass per unit length at that spot. I.e., we need to integrate mass per unit length w.r.t. arclength.

$$x = \cos\theta \qquad y = \sin\theta \qquad z = \frac{1}{5}\theta$$
$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -\sin\theta \qquad \frac{\mathrm{d}y}{\mathrm{d}\theta} = \cos\theta \qquad \frac{\mathrm{d}z}{\mathrm{d}\theta} = \frac{1}{5}$$

$$\begin{aligned}
\text{mass} &= \int_0^{6\pi} \left(1 + \frac{1}{10}\theta\right) \sqrt{(-\sin\theta)^2 + (\cos\theta)^2 + \frac{1}{25}}\,\mathrm{d}\theta \\
&= \int_0^{6\pi} \left(1 + \frac{1}{10}\theta\right) \sqrt{\frac{26}{25}}\,\mathrm{d}\theta \\
&= \sqrt{\frac{26}{25}} \left[\theta + \frac{1}{20}\theta^2\right]_0^{6\pi} \\
&= \sqrt{\frac{26}{25}} \left(6\pi + \frac{36}{20}\pi^2\right)
\end{aligned}$$

12.

$$\begin{aligned}
F = -\nabla V &= -\nabla\left(-\frac{GMm}{r}\right) \\
&= -\nabla\left(-GMm(x^2+y^2+z^2)^{-\frac{1}{2}}\right) \\
&= -\frac{1}{2}GMm(x^2+y^2+z^2)^{-\frac{3}{2}}(2x, 2y, 2z) \\
&= -\frac{GMm}{r^3}(x, y, z)
\end{aligned}$$

Let $U = -V$. Then

$$\begin{aligned}
\mathrm{d}U &= \nabla(-V)\cdot \mathrm{d}\mathbf{r} \\
&= (-\nabla V)\cdot \mathrm{d}\mathbf{r} \\
&= F \cdot \mathrm{d}\mathbf{r}.
\end{aligned}$$

I.e. there is a function $U(x, y, z)$ such that $F \cdot \mathrm{d}\mathbf{r} = \mathrm{d}U$, so $F \cdot \mathrm{d}\mathbf{r}$ is exact. Therefore line integrals are path independent. At infinity (in any direction), $\frac{1}{r} = 0$, so the work done is

$$\begin{aligned}
W &= U|_{\mathbf{r}} - U|_{\infty} \\
&= \frac{GMm}{r} - 0 \\
&= \frac{GMm}{r}
\end{aligned}$$

13. $\mathrm{d}f$ is exact if there exists a differentiable function F such that $\mathrm{d}f = \nabla F \cdot \mathrm{d}\mathbf{r}$. In more detail, say we are in 3D and have the differential

$$\mathrm{d}f = P(x,y,z)\,\mathrm{d}x + Q(x,y,z)\,\mathrm{d}y + R(x,y,z)\,\mathrm{d}z.$$

Then $\mathrm{d}f$ is exact if there exists a differentiable function F such that

$$\mathrm{d}f = \nabla F \cdot \mathrm{d}\mathbf{r} = \frac{\partial F}{\partial x}\,\mathrm{d}x + \frac{\partial F}{\partial y}\,\mathrm{d}y + \frac{\partial F}{\partial z}\,\mathrm{d}z,$$

and inexact if no such F exists. We can test for exactness by requiring simultaneously that

$$\begin{aligned} P_y &= Q_x && \text{(since both would equal } F_{xy}\text{)} \\ P_z &= R_x && \text{(since both would equal } F_{xz}\text{)} \\ Q_z &= R_y && \text{(since both would equal } F_{yz}\text{)} \end{aligned}$$

14. (a)

$$\begin{aligned} \mathrm{d}f &= \ln y\,\mathrm{d}x + \frac{x}{y}\,\mathrm{d}y \\ \frac{\partial}{\partial y}\ln y &= \frac{1}{y} \\ \frac{\partial}{\partial x}\frac{x}{y} &= \frac{1}{y} \implies \text{exact} \end{aligned}$$

$$\begin{aligned} \text{Since } \frac{\partial f}{\partial x} &= \ln y, \\ f &= \int \ln y\,\mathrm{d}x \\ &= x\ln y + G(y), \\ f_y &= \frac{x}{y} + G'(y) = \frac{x}{y}, \quad \text{(from d}f\text{)} \\ \implies G'(y) &= 0, \\ G(y) &= c, \\ \therefore f &= x\ln y + c. \end{aligned}$$

(b)

$$\begin{aligned} \mathrm{d}f &= (y+z)\,\mathrm{d}x + (x+z)\,\mathrm{d}y - 2z\,\mathrm{d}z \\ \frac{\partial}{\partial y}(y+z) &= 1 = \frac{\partial}{\partial x}(x+z) \\ \frac{\partial}{\partial y}(-2z) &= 0 \neq 1 = \frac{\partial}{\partial z}(x+z) \implies \text{not exact} \end{aligned}$$

(c)

$$\begin{aligned} \mathrm{d}f &= (y+x)\,\mathrm{d}x + (y-x)\,\mathrm{d}y \\ \frac{\partial}{\partial y}(y+x) &= 1 \neq -1 = \frac{\partial}{\partial x}(y-x) \implies \text{not exact} \end{aligned}$$

(d)

$$\begin{aligned} \mathrm{d}f &= (y+z)\,\mathrm{d}x + (x+z)\,\mathrm{d}y + (x+y)\,\mathrm{d}z \\ \frac{\partial}{\partial y}(y+z) &= 1 = \frac{\partial}{\partial x}(x+z) \\ \frac{\partial}{\partial z}(y+z) &= 1 = \frac{\partial}{\partial x}(x+y) \\ \frac{\partial}{\partial z}(x+z) &= 1 = \frac{\partial}{\partial y}(x+y) \implies \text{exact} \end{aligned}$$

$$\begin{aligned} f &= \int (y+z)\,\mathrm{d}x \\ &= (y+z)x + G(y,z) \\ f_y &= x + G_y \implies G_y = z \\ f_z &= x + G_z \implies G_z = y \\ &\implies G(y,z) = yz + c \\ \therefore f &= (y+z)x + yz + c \end{aligned}$$

15.

$$I = \int_{(0,0)}^{(2,-1+\frac{\pi}{6})} (\sin(x+2y) + x\cos(x+2y))\,\mathrm{d}x + 2x\cos(x+2y)\,\mathrm{d}y$$

$$\begin{aligned} P &= \sin(x+2y) + x\cos(x+2y) & Q &= 2x\cos(x+2y) \\ P_y &= 2\cos(x+2y) - 2x\sin(x+2y) & Q_x &= 2\cos(x+2y) - 2x\sin(x+2y) \\ & & &= P_y \implies \text{exact} \end{aligned}$$

(a) Since we now know $Q = f_y$,

$$\begin{aligned} f &= \int Q\,\mathrm{d}y \\ &= \int 2x\cos(x+2y)\,\mathrm{d}y \\ &= x\sin(x+2y) + G(x) \\ f_x &= \sin(x+2y) + x\cos(x+2y) + G'(x) \implies G'(x) = 0 \implies G(x) = c, \\ \therefore f &= x\sin(x+2y) + c. \\ I &= f(2, -1+\tfrac{\pi}{6}) - f(0,0) \\ &= 2\sin\tfrac{\pi}{3} \\ &= \sqrt{3} \end{aligned}$$

(b) The simplest choices of path are a straight line between the points, or a path made of one horizontal piece and one vertical piece. Let's use a straight line.

Here's how to parametrise a straight line between any two points (x_0, y_0) and (x_1, y_1):

$$(x, y) = (1-t)(x_0, y_0) + t\,(x_1, y_1) \qquad 0 \le t \le 1.$$

This linearly blends between the two points: When $t = 0$, the weighting given to (x_0, y_0) is $1 - t = 1$, and the weighting given to (x_1, y_1) is $t = 0$, so we're at (x_0, y_0). When $t = 1$, the weighting given to (x_0, y_0) is $1 - t = 0$, and the weighting given to (x_1, y_1) is $t = 1$, so we're at (x_1, y_1). When $0 < t < 1$, we're in between, because we're combining both end points.

In our situation, we have

$$(x, y) = (1-t)(0,0) + t\left(2, -1 + \frac{\pi}{6}\right) \qquad 0 \le t \le 1,$$

so $x = 2t$ and $y = \left(-1 + \frac{\pi}{6}\right)t$. Then $\mathrm{d}x = 2\mathrm{d}t$ and $\mathrm{d}y = \left(-1 + \frac{\pi}{6}\right)\mathrm{d}t$.

$$I = \int_0^1 \left(\sin(x+2y) + x\cos(x+2y)\right)2\,\mathrm{d}t + 2x\cos(x+2y)\left(-1 + \tfrac{\pi}{6}\right)\,\mathrm{d}t$$

Since $x + 2y = 2t + 2\left(-1 + \frac{\pi}{6}\right)t = 2t - 2t + \frac{\pi}{3}t = \frac{\pi}{3}t$,

$$\begin{aligned} I &= \int_0^1 \left(2\sin\tfrac{\pi}{3}t + 4t\cos\tfrac{\pi}{3}t + 4t\cos\tfrac{\pi}{3}t\left(-1 + \tfrac{\pi}{6}\right)\right)\,\mathrm{d}t \\ &= 2\int_0^1 \left(\sin\tfrac{\pi}{3}t + \tfrac{\pi}{3}t\cos\tfrac{\pi}{3}t\right)\,\mathrm{d}t \end{aligned}$$

We need to use integration by parts for the second term:
$$\begin{aligned} u &= t & \mathrm{d}v &= \tfrac{\pi}{3}\cos\tfrac{\pi}{3}t \\ \mathrm{d}u &= \mathrm{d}t & v &= \sin\tfrac{\pi}{3}t \end{aligned}$$

$$\begin{aligned} I &= 2\left(\left[-\tfrac{3}{\pi}\cos\tfrac{\pi}{3}t\right]_0^1 + \left[t\sin\tfrac{\pi}{3}t\right]_0^1 - \int_0^1 \sin\tfrac{\pi}{3}t\,\mathrm{d}t\right) \\ &= 2\left(\left[-\tfrac{3}{\pi}\cos\tfrac{\pi}{3}t\right]_0^1 + \left[t\sin\tfrac{\pi}{3}t\right]_0^1 - \left[-\tfrac{3}{\pi}\cos\tfrac{\pi}{3}t\right]_0^1\right) \\ &= 2\left[t\sin\tfrac{\pi}{3}t\right]_0^1 \\ &= 2\sin\tfrac{\pi}{3} \\ &= \sqrt{3} \end{aligned}$$