EMTH210 Tutorial 3: Line Integrals and Differentials

For the week starting Monday 02 March.

Preparation problems (homework)

1. Find the equation of the tangent plane to the surface
$$z = g(x, y) = 4x^3y^2 + 2y$$
at the point $(x_0, y_0, z_0) = (1, -2, 12)$.

2. Find the equation of the tangent plane to the surface
$$f(x, y, z) = x^2 + 2y^2 + 3z^2 = 15$$
at the point $(x_0, y_0, z_0) = (2,\ 2,\ 1)$.

3. A farmer's fence runs Eastwards up a hill. Using x to denote horizontal distance, the fence starts at $x = 0$ and ends at $x = 3$. The height of the fence above the end at $x = 0$ is given by the function $y = x\sqrt{x}$.

   Calculate the length of the farmer's fence.

4. Find the length of the curve given by
$$y = \cosh(x)$$
from $x = -1$ to $x = 1$. (There are notes on the hyperbolic functions in the Resources and Revision material section of Learn.)

5. Calculate the work done by the force
$$F = \mathbf{i} - y\mathbf{j} + xyz\mathbf{k}$$
in moving a particle from $(0, 0, 0)$ to $(1, -1, 1)$ along the curve $x = t$, $y = -t^2$, $z = t$ for $0 \leq t \leq 1$.

6. Evaluate
$$\int (x\,\mathrm{d}y - y\,\mathrm{d}x)$$
along the part of the unit circle from $(x, y) = (0, -1)$ to $(0, 1)$ which lies in the half plane satisfying $x \geq 0$.

7. For each of the following functions f, find the differential $\mathrm{d}f$

   (a) $f = x^3 + y^2$

   (b) $f = (x - y)\cos(x + y)$

   (c) $f = e^{x/y}\cos(z^2y)$

   (d) $f = \tan^{-1}(y/x) + \ln(z)$

8. Find the Jacobian matrix for each of the following functions.

   (a) $\mathbf{F}(x, y) = \begin{pmatrix} 3y^2 + 2e^x \\ \ln(yx) \end{pmatrix}$

   (b) $\mathbf{F}(x, y, z) = \begin{pmatrix} \sin(xy) + xz^2 \\ y\cos(z) - x \end{pmatrix}$

Problems for the tutorial

9. Evaluate
$$\int (y\,\mathrm{d}x + x\,\mathrm{d}y)$$
along each of the following curves from $(-1, 1)$ to $(1, 1)$:

   (a) $y = x^2$ (b) $y = |x|$

   Briefly state why you would expect the answer to be the same for both integrals.

10. Let C be the curve consisting of the quarter circle from $(1, 0)$ to $(0, 1)$ followed by the line segment from $(0, 1)$ to $(2, 1)$ Calculate
$$\int xy\,\mathrm{d}x + y^2\,\mathrm{d}y$$
along C.

11. A helical wire is given parametrically by $x = \cos(\theta)$, $y = \sin(\theta)$ and $z = 0.2\theta$. The mass per unit length is given by $1 + 0.1\theta$. Calculate the mass of 3 turns of the helix from $\theta = 0$ to 6π.

12. A planet of mass M has its centre fixed at the origin. The gravitational potential energy of a mass m at a point $\mathbf{r} = (x, y, z)$ is $V = -GMm/r$, where $r = \sqrt{x^2 + y^2 + z^2}$ and where G is a positive constant. The force on m is given by $F = -\nabla V$.

    Calculate F and show that $F \bullet d\mathbf{r}$ is exact.

    Hence calculate the work done by F on a particle which travels from infinity to $\mathbf{r}$.

13. What is meant when a differential is said to be exact, or not exact (inexact)?

14. For each of the following differentials $\mathrm{d}f$, find f or show that $\mathrm{d}f$ is not exact.

    (a) $\mathrm{d}f = \ln(y)\,\mathrm{d}x + (x/y)\,\mathrm{d}y$
    (b) $\mathrm{d}f = (y + z)\,\mathrm{d}x + (x + z)\,\mathrm{d}y - 2z\,\mathrm{d}z$
    (c) $\mathrm{d}f = (y + x)\,\mathrm{d}x + (y - x)\,\mathrm{d}y$
    (d) $\mathrm{d}f = (y + z)\,\mathrm{d}x + (x + z)\,\mathrm{d}y + (x + y)\,\mathrm{d}z$

15. First, show that the following integral is independent of the path.
$$\int_{(0,0)}^{(2,-1+\frac{\pi}{6})} \big(\sin(x + 2y) + x\cos(x + 2y)\big)\,\mathrm{d}x + 2x\cos(x + 2y)\,\mathrm{d}y.$$

    Next, evaluate it in two ways by:

    (a) finding a function $f(x, y)$ such that $\mathrm{d}f$ is the integrand; and by
    (b) integrating along any convenient path between the endpoints.