EMTH210 Tutorial 9: Laplace Transforms – Solutions

The homework questions this week are **6(a)** and **9(a)**.

1. (a)

$$
\begin{aligned}
f(t) &= \begin{cases} -1 & 0 \le t < 1 \\ 1 & t \ge 1 \end{cases} \\
F(s) &= \int_0^\infty e^{-st} f(t)\,\mathrm{d}t \\
&= -\int_0^1 e^{-st}\,\mathrm{d}t + \int_1^\infty e^{-st}\,\mathrm{d}t \\
&= -\left[-\frac{1}{s}e^{-st}\right]_{t=0}^{1} + \left[-\frac{1}{s}e^{-st}\right]_{t=1}^{\infty} \\
&= -\left(-\frac{1}{s}e^{-s} + \frac{1}{s}\right) + \left(0 + \frac{1}{s}e^{-s}\right) \\
&= \frac{2}{s}e^{-s} - \frac{1}{s}
\end{aligned}
$$

(b)

$$
\begin{aligned}
f(t) &= \begin{cases} t & 0 \le t < 1 \\ 1 & t \ge 1 \end{cases} \\
F(s) &= \int_0^\infty e^{-st} f(t)\,\mathrm{d}t \\
&= \int_0^1 te^{-st}\,\mathrm{d}t + \int_1^\infty e^{-st}\,\mathrm{d}t \\
&= \left[-\frac{t}{s}e^{-st}\right]_{t=0}^{1} - \int_0^1 -\frac{1}{s}e^{-st}\,\mathrm{d}t + \left[-\frac{1}{s}e^{-st}\right]_{t=1}^{\infty} \\
&= \left[-\frac{t}{s}e^{-st}\right]_{t=0}^{1} - \left[\frac{1}{s^2}e^{-st}\right]_{t=0}^{1} + \left[-\frac{1}{s}e^{-st}\right]_{t=1}^{\infty} \\
&= \left(-\frac{1}{s}e^{-s} - 0\right) - \left(\frac{1}{s^2}e^{-s} - \frac{1}{s^2}\right) + \left(0 - -\frac{1}{s}e^{-s}\right) \\
&= \frac{1}{s^2} - \frac{1}{s^2}e^{-s}
\end{aligned}
$$

$$
\begin{aligned}
u &= t & \mathrm{d}v &= e^{-st}\,\mathrm{d}t \\
\mathrm{d}u &= \mathrm{d}t & v &= -\tfrac{1}{s}e^{-st}
\end{aligned}
$$

From now on, we don't do Laplace transforms from the integral definition like this, we just use the table and the transform rules, which make them much less work. The basic strategy is to rearrange expressions to look like things in the table, then apply the rules.

2. (a)

$$
\begin{aligned}
f(t) &= e^{t+8} \\
&= e^8 e^t \\
F(s) &= e^8 \mathcal{L}\{e^t\} \\
&= e^8 \frac{1}{s-1} \quad \text{(using linearity and the } e^{at} \text{ rule)}
\end{aligned}
$$

(b)

$$
\begin{aligned}
f(t) &= 1 + 4t - 2e^t \\
F(s) &= \frac{1}{s} + \frac{4}{s^2} - \frac{2}{s-1}
\end{aligned}
$$

(c)

$$
\begin{aligned}
f(t) &= 12t^5 \\
F(s) &= 12\frac{5!}{s^6} \\
&= \frac{1440}{s^6}
\end{aligned}
$$

3. (a)

$$
\begin{aligned}
F(s) &= \frac{1}{s^3} \\
f(t) &= \frac{1}{2}t^2
\end{aligned}
$$

(b)

$$
\begin{aligned}
F(s) &= \frac{(s+1)^2}{s^3} \\
&= \frac{s^2}{s^3} + \frac{2s}{s^3} + \frac{1}{s^3} \\
&= \frac{1}{s} + \frac{2}{s^2} + \frac{1}{s^3} \\
f(t) &= 1 + 2t + \frac{1}{2}t^2
\end{aligned}
$$

(c)

$$
\begin{aligned}
F(s) &= \frac{1}{2s+1} \\
&= \frac{1}{2}\frac{1}{s+\frac{1}{2}} \\
f(t) &= \frac{1}{2}e^{-\frac{1}{2}t}
\end{aligned}
$$

(d) $F(s) = \dfrac{7}{s^2+36}$

$= \dfrac{7}{6}\dfrac{6}{s^2+36}$

$f(t) = \dfrac{7}{6}\sin 6t$

(e) $F(s) = \dfrac{7s+3}{s^2+9}$

$= 7\dfrac{s}{s^2+9} + \dfrac{3}{s^2+9}$

$f(t) = 7\cos 3t + \sin 3t$

(f) $F(s) = \dfrac{1}{s^2-16}$

$= \dfrac{1}{4}\dfrac{4}{s^2-16}$

$f(t) = \dfrac{1}{4}\sinh 4t$

4. (a) $f(t) = t^2 e^{3t}$

$\mathcal{L}\{t^2\} = \dfrac{2}{s^3}$

$F(s) = \dfrac{2}{(s-3)^3}$

(using the first shift theorem and the t^n rule)

(b) $f(t) = e^{3t}\sin 2t$

$\mathcal{L}\{\sin 2t\} = \dfrac{2}{s^2+2^2}$

$F(s) = \dfrac{2}{(s-3)^2+4}$

(using the first shift theorem and the $\sin kt$ rule)

(c) $f(t) = e^{3t}\cos 2t$

$\mathcal{L}\{\cos 2t\} = \dfrac{s}{s^2+2^2}$

$F(s) = \dfrac{s-3}{(s-3)^2+4}$

5. (a)

$$\begin{aligned} F(s) &= \frac{1}{(s-3)^2} \\ &= G(s-3) \qquad \text{where } G(s) = \frac{1}{s^2} \implies g(t) = t. \\ \therefore f(t) &= e^{3t}t \qquad \text{by the first shift theorem.} \end{aligned}$$

(b)

$$\begin{aligned} F(s) &= \frac{1}{(s-2)^2-4} \\ &= G(s-2) \qquad \text{where } G(s) = \frac{1}{s^2-2^2} = \frac{1}{2}\frac{2}{s^2-2^2} \implies g(t) = \frac{1}{2}\sinh 2t. \\ \therefore f(t) &= e^{2t}\frac{1}{2}\sinh 2t \qquad \text{by the first shift theorem.} \end{aligned}$$

(c)

$$\begin{aligned} F(s) &= \frac{s}{(s-2)^2-4} \\ &= \frac{s-2+2}{(s-2)^2-4} \qquad \text{(make every } s \text{ have a } -2 \text{ for 1st shift theorem)} \\ &= \frac{s-2}{(s-2)^2-4} + \frac{2}{(s-2)^2-4} \\ &= G(s-2) \qquad \text{where } G(s) = \frac{s}{s^2-2^2} + \frac{2}{s^2-2^2} \implies g(t) = \cosh 2t + \sinh 2t. \\ \therefore f(t) &= e^{2t}(\cosh 2t + \sinh 2t) \qquad \text{by the first shift theorem.} \end{aligned}$$

(d)

$$\begin{aligned} F(s) &= \frac{2s+5}{s^2+6s+10} \\ &= \frac{2s+5}{(s+3)^2+1} \qquad \text{(completing the square)} \\ &= \frac{2(s+3)}{(s+3)^2+1} + \frac{-6+5}{(s+3)^2+1} \qquad \text{(make every } s \text{ have a } +3 \text{ for 1st shift theorem)} \\ &= G\big(s-(-3)\big) \qquad \text{where } G(s) = \frac{2s}{s^2+1} - \frac{1}{s^2+1} \implies g(t) = 2\cos t - \sin t. \\ \therefore f(t) &= e^{-3t}(2\cos t - \sin t) \qquad \text{by the first shift theorem.} \end{aligned}$$

(e)

$$\begin{aligned}
F(s) &= \frac{4}{s^2(s+1)} \\
&= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} \\
&= -\frac{4}{s} + \frac{4}{s^2} + \frac{4}{s+1} \\
f(t) &= -4 + 4t + 4e^{-t}
\end{aligned}$$

$$\begin{aligned}
& 4 = (As+B)(s+1) + Cs^2 \\
\text{constant terms:}\quad & 4 = B \\
s \text{ terms:}\quad & 0 = A + B \implies A = -4 \\
s^2 \text{ terms:}\quad & 0 = A + C \implies C = 4
\end{aligned}$$

(f)

$$\begin{aligned}
F(s) &= \frac{s^2+1}{s(s+1)(s-1)} \\
&= \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s-1} \\
&= -\frac{1}{s} + \frac{1}{s+1} + \frac{1}{s-1} \\
f(t) &= -1 + e^{-t} + e^{t}
\end{aligned}$$

$$\begin{aligned}
s^2 + 1 &= A(s+1)(s-1) + Bs(s-1) + Cs(s+1) \\
&= (A+B+C)s^2 + (-B+C)s + (-A)
\end{aligned}$$

$$\begin{aligned}
\text{constant terms:}\quad & 1 = -A \implies A = -1 \\
s \text{ terms:}\quad & 0 = -B + C \implies B = C \\
s^2 \text{ terms:}\quad & 1 = A + B + C \\
& 2 = 2C \implies C = 1,\ B = 1
\end{aligned}$$

6. To solve differential equations, we take the Laplace transform of the entire DE, taking $y(t)$ to $Y(s)$, rearrange for Y, and take the inverse Laplace transform. The rearrangement usually involves partial fractions decomposition.

(a)

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} + y &= 1, \qquad y(0) = 0 \\
sY(s) - y(0) + Y(s) &= \frac{1}{s} \\
(s+1)Y(s) - 0 &= \frac{1}{s} \\
Y(s) &= \frac{1}{s(s+1)} \\
&= \frac{A}{s} + \frac{B}{s+1} \\
&= \frac{1}{s} - \frac{1}{s+1} \\
y(t) &= 1 - e^{-t}
\end{aligned}$$

$$\begin{aligned}
& 1 = A(s+1) + Bs \\
\text{constant terms:}\quad & 1 = A \\
s \text{ terms:}\quad & 0 = A + B \implies B = -1
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} + 3y &= e^{-3t}, \qquad y(0) = 6 \\
sY(s) - y(0) + 3Y(s) &= \frac{1}{s+3} \\
(s+3)Y(s) - 6 &= \frac{1}{s+3} \\
Y(s) &= \frac{1}{(s+3)^2} + \frac{6}{s+3} \\
y(t) &= te^{-3t} + 6e^{-3t} \qquad \text{(using the first shift theorem)}
\end{aligned}$$

(c)

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 4y = 0, \qquad y(0) = 1,\, y'(0) = 0$$

$$\begin{aligned}
s^2Y(s) - sy(0) - y'(0) + 5sY(s) - 5y(0) + 4Y(s) &= 0 \\
s^2Y(s) - s - 0 + 5sY(s) - 5 + 4Y(s) &= 0 \\
\left(s^2 + 5s + 4\right) Y(s) - s - 5 &= 0 \\
Y(s) &= \frac{s+5}{(s+1)(s+4)} \\
&= \frac{A}{s+1} + \frac{B}{s+4} \\
s + 5 &= A(s+4) + B(s+1) \\
\text{constant terms:} \quad 5 &= 4A + B \implies B = 5 - 4A \\
s \text{ terms:} \quad 1 &= A + B \\
-4 &= -3A \implies A = \frac{4}{3},\, B = 5 - \frac{16}{3} = -\frac{1}{3} \\
Y(s) &= \frac{4}{3}\frac{1}{s+1} - \frac{1}{3}\frac{1}{s+4} \\
y(t) &= \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t}
\end{aligned}$$

7. (a)

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = t, \qquad y(0) = 0,\, y'(0) = 1$$

$$\begin{aligned}
s^2Y(s) - 1 - 6sY(s) + 9Y(s) &= \frac{1}{s^2} \\
\left(s^2 - 6s + 9\right) Y(s) - 1 &= \frac{1}{s^2} \\
Y(s) &= \frac{1}{s^2(s-3)^2} + \frac{1}{(s-3)^2}
\end{aligned}$$

The second term here is okay as is, but we need to use partial fraction decomposition for the first one.

$$\begin{aligned}
\frac{1}{s^2(s-3)^2} &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-3} + \frac{D}{(s-3)^2} \\
1 &= As(s-3)^2 + B(s-3)^2 + Cs^2(s-3) + Ds^2
\end{aligned}$$

This may be a bit big for the method used above, but since it must be true for all s, and it's already quite factorised, we can take a shortcut by substituting in specific values of s that simplify the equation.

$$\begin{aligned}
&\text{When } s = 0, \quad 1 = B(-3)^2 \implies B = \frac{1}{9}. \\
&\text{When } s = 3, \quad 1 = D(3)^2 \implies D = \frac{1}{9}.
\end{aligned}$$

Now,

$$\begin{aligned}
1 - \frac{1}{9}\left(s^2 - 6s + 9\right) - \frac{1}{9}s^2 &= A\left(s^3 - 6s^2 + 9s\right) + C\left(s^3 - 3s^2\right) \\
s \text{ terms:} \quad \frac{6}{9} &= 9A \implies A = \frac{2}{27} \\
s^3 \text{ terms:} \quad 0 &= \frac{2}{27} + C \implies C = -\frac{2}{27} \\
Y(s) &= \frac{2}{27}\frac{1}{s} + \frac{1}{9}\frac{1}{s^2} - \frac{2}{27}\frac{1}{s-3} + \frac{1}{9}\frac{1}{(s-3)^2} + \underbrace{\frac{1}{(s-3)^2}}_{\text{from } Y(s) \text{ above}} \\
y(t) &= \frac{2}{27} + \frac{1}{9}t - \frac{2}{27}e^{3t} + \frac{1}{9}te^{3t} + te^{3t} \quad \text{(using 1st shift thm.)} \\
y(t) &= \frac{2}{27} + \frac{1}{9}t - \frac{2}{27}e^{3t} + \frac{10}{9}te^{3t}.
\end{aligned}$$

(b)

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} + 5y = 1 + t, \qquad y(0) = 0,\, y'(0) = 4$$

$$\begin{aligned}
s^2Y(s) - 4 - 2sY(s) + 5Y(s) &= \frac{1}{s} + \frac{1}{s^2} \\
\left(s^2 - 2s + 5\right)Y(s) - 4 &= \frac{1}{s} + \frac{1}{s^2} \\
Y(s) &= \frac{1}{s\left(s^2 - 2s + 5\right)} + \frac{1}{s^2\left(s^2 - 2s + 5\right)} + \frac{4}{s^2 - 2s + 5}
\end{aligned}$$

Partial fractions decomposition for the first term:

$$\begin{aligned}
\frac{1}{s\left(s^2 - 2s + 5\right)} &= \frac{A}{s} + \frac{Bs + C}{s^2 - 2s + 5} \\
1 &= A\left(s^2 - 2s + 5\right) + (Bs + C)s \\
\text{constant terms:} \quad 1 &= 5A \implies A = \frac{1}{5} \\
s \text{ terms:} \quad 0 &= -2A + C \implies C = \frac{2}{5} \\
s^2 \text{ terms:} \quad 0 &= A + B \implies B = -\frac{1}{5} \\
\therefore \frac{1}{s\left(s^2 - 2s + 5\right)} &= \frac{1}{5}\frac{1}{s} + \frac{-\frac{1}{5}s + \frac{2}{5}}{s^2 - 2s + 5}.
\end{aligned}$$

And for the second term:

$$\begin{aligned}
\frac{1}{s^2\left(s^2 - 2s + 5\right)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 - 2s + 5} \\
1 &= As\left(s^2 - 2s + 5\right) + B\left(s^2 - 2s + 5\right) + (Cs + D)s^2 \\
&= (A + C)s^3 + (-2A + B + D)s^2 + (5A - 2B)s + 5B \\
\text{constant terms:} \quad 1 &= 5B \implies B = \frac{1}{5} \\
s \text{ terms:} \quad 0 &= 5A - 2B \implies A = \frac{1}{5}(2B) = \frac{2}{25} \\
s^2 \text{ terms:} \quad 0 &= -2A + B + D \implies D = 2A - B = -\frac{1}{25} \\
s^3 \text{ terms:} \quad 0 &= A + C \implies C = -A = -\frac{2}{25} \\
\therefore \frac{1}{s^2\left(s^2 - 2s + 5\right)} &= \frac{2}{25}\frac{1}{s} + \frac{1}{5}\frac{1}{s^2} - \frac{\frac{2}{25}s + \frac{1}{25}}{s^2 - 2s + 5}.
\end{aligned}$$

Putting this all together, we get

$$\begin{aligned}
Y(s) &= \frac{1}{5}\frac{1}{s} + \frac{-\frac{1}{5}s + \frac{2}{5}}{s^2 - 2s + 5} + \frac{2}{25}\frac{1}{s} + \frac{1}{5}\frac{1}{s^2} - \frac{\frac{2}{25}s + \frac{1}{25}}{s^2 - 2s + 5} + \frac{4}{s^2 - 2s + 5} \\
&= \frac{7}{25}\frac{1}{s} + \frac{1}{5}\frac{1}{s^2} + \frac{-\frac{7}{25}s + \frac{109}{25}}{s^2 - 2s + 5} \\
&= \frac{7}{25}\frac{1}{s} + \frac{1}{5}\frac{1}{s^2} + \frac{-\frac{7}{25}(s - 1) + \frac{102}{25}}{(s - 1)^2 + 4} \\
&= \frac{7}{25}\frac{1}{s} + \frac{1}{5}\frac{1}{s^2} - \frac{7}{25}\frac{(s - 1)}{(s - 1)^2 + 2^2} + \frac{51}{25}\frac{2}{(s - 1)^2 + 2^2}, \\
y(t) &= \frac{7}{25} + \frac{1}{5}t - \frac{7}{25}e^t\cos 2t + \frac{51}{25}e^t\sin 2t,
\end{aligned}$$

using the first shift theorem.

8. Here $\mathrm{H}(t)$ is the Heaviside step function (and yes it really is named after a person who just happened to have an appropriate name for a function with one heavy side!). To draw a function that involves it, say $g(t-a)\,\mathrm{H}(t-a)$, think about what $g(t)$ looks like, shift it right a units, and then the graph looks like that except "switched on" at a by $\mathrm{H}(t-a)$, which is zero for $t < a$ and one for $t \geq a$. If there are multiple things added, subtracted, or multiplied, it can help to draw little plots of each of them individually.

   The second shift theorem says that if

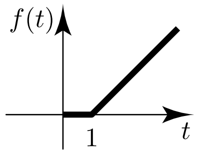
   $$\mathcal{L}\{g(t)\,\mathrm{H}(t)\} = G(s),$$

   then

   $$\mathcal{L}\{g(t-a)\,\mathrm{H}(t-a)\} = e^{-as}G(s).$$

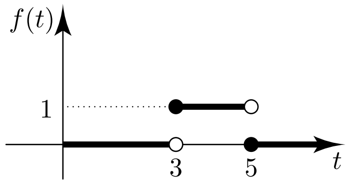
   To use it, we often need to rearrange g to be in a form where all mentions of t are in the form $(t-a)$, where a is the shift in $\mathrm{H}(t-a)$.

   (a)



$$\begin{aligned} f(t) &= (t-1)\,\mathrm{H}(t-1) \\ g(t-1) &= (t-1) \\ \implies g(t) &= t \\ \implies G(s) &= \frac{1}{s^2} \\ \therefore F(s) &= e^{-s}\frac{1}{s^2} \end{aligned}$$

   (b)

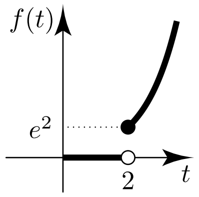


$$f(t) = \mathrm{H}(t-3) - \mathrm{H}(t-5)$$

$$\begin{aligned} g_1(t-3) &= 1 \\ \implies g_1(t) &= 1 \\ \implies G_1(s) &= \frac{1}{s} \end{aligned} \qquad \begin{aligned} g_2(t-5) &= 1 \\ \implies g_2(t) &= 1 \\ \implies G_2(s) &= \frac{1}{s} \end{aligned}$$

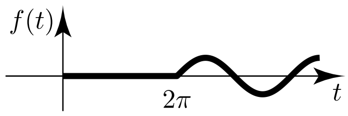
$$\therefore F(s) = e^{-3s}\frac{1}{s} - e^{-5s}\frac{1}{s}$$

   (c)



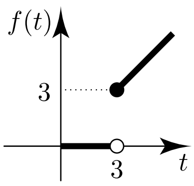
$$\begin{aligned} f(t) &= e^t\,\mathrm{H}(t-2) \\ &= e^{(t-2)+2}\,\mathrm{H}(t-2) \qquad \text{(now all } t\text{s have a " } - 2\text{"; a useful trick)} \\ &= e^2e^{t-2}\,\mathrm{H}(t-2) \\ g(t-2) &= e^2e^{t-2} \\ \implies g(t) &= e^2e^t \\ \implies G(s) &= e^2\frac{1}{s-1} \\ \therefore F(s) &= e^{-2s}e^2\frac{1}{s-1} \end{aligned}$$

   (d)



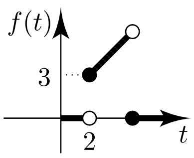
$$\begin{aligned} f(t) &= \sin t\,\mathrm{H}(t-2\pi) \\ &= \sin(t-2\pi)\,\mathrm{H}(t-2\pi) \qquad (\text{as } \sin t = \sin(t-2\pi)) \\ g(t-2\pi) &= \sin(t-2\pi) \\ \implies g(t) &= \sin(t) \\ \implies G(s) &= \frac{1}{s^2+1} \\ \therefore F(s) &= e^{-2\pi s}\frac{1}{s^2+1} \end{aligned}$$

(e)



$$\begin{aligned} f(t) &= t\,\mathrm{H}(t-3) \\ &= \big((t-3)+3\big)\,\mathrm{H}(t-3) \\ g(t-3) &= (t-3)+3 \\ \implies g(t) &= t+3 \\ \implies G(s) &= \frac{1}{s^2}+\frac{3}{s} \\ \therefore F(s) &= e^{-3s}\left(\frac{1}{s^2}+\frac{3}{s}\right) \end{aligned}$$

(f)

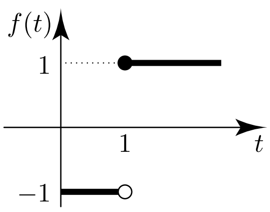


$$\begin{aligned} f(t) &= (1+t)\big(\mathrm{H}(t-2)-\mathrm{H}(t-5)\big) \\ &= \big((t-2)+3\big)\,\mathrm{H}(t-2)-\big((t-5)+6\big)\,\mathrm{H}(t-5) \end{aligned}$$

$$\begin{aligned} g_1(t-2) &= (t-2)+3 & g_2(t-5) &= (t-5)+6 \\ \implies g_1(t) &= t+3 & \implies g_2(t) &= t+6 \\ \implies G_1(s) &= \frac{1}{s^2}+\frac{3}{s} & \implies G_2(s) &= \frac{1}{s^2}+\frac{6}{s} \end{aligned}$$

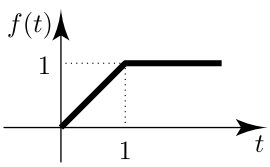
$$\therefore F(s) = e^{-2s}\left(\frac{1}{s^2}+\frac{3}{s}\right)-e^{-5s}\left(\frac{1}{s^2}+\frac{6}{s}\right)$$

(g)



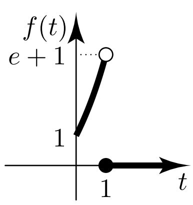
$$\begin{aligned} f(t) &= \begin{cases} -1 & 0 \le t < 1 \\ 1 & t \ge 1 \end{cases} \\ &= -1\big(\mathrm{H}(t-0)-\mathrm{H}(t-1)\big)+1\,\mathrm{H}(t-1) \\ &= -1\,\mathrm{H}(t)+2\,\mathrm{H}(t-1) \\ F(s) &= -\frac{1}{s}+e^{-s}\frac{2}{s} \end{aligned}$$

(h)



$$\begin{aligned} f(t) &= \begin{cases} t & 0 \le t < 1 \\ 1 & t \ge 1 \end{cases} \\ &= t\big(\mathrm{H}(t-0)-\mathrm{H}(t-1)\big)+1\,\mathrm{H}(t-1) \\ &= t\,\mathrm{H}(t)-(t-1)\,\mathrm{H}(t-1) \\ g(t-1) &= t-1 \\ \implies g(t) &= t \\ \implies G(s) &= \frac{1}{s^2} \\ \therefore F(s) &= \frac{1}{s^2}-e^{-s}\frac{1}{s^2} \end{aligned}$$

(i)



$$\begin{aligned} f(t) &= \begin{cases} e^t+t & 0 \le t < 1 \\ 0 & t \ge 1 \end{cases} \\ &= \big(e^t+t\big)\big(\mathrm{H}(t-0)-\mathrm{H}(t-1)\big) \\ &= \big(e^t+t\big)\,\mathrm{H}(t)-\Big(e^{(t-1)+1}+(t-1)+1\Big)\,\mathrm{H}(t-1) \\ g(t-1) &= e^{(t-1)}e^1+(t-1)+1 \\ \implies g(t) &= ee^t+t+1 \\ \implies G(s) &= e\frac{1}{s-1}+\frac{1}{s^2}+\frac{1}{s} \\ \therefore F(s) &= \frac{1}{s-1}+\frac{1}{s^2}-e^{-s}\left(\frac{e}{s-1}+\frac{1}{s^2}+\frac{1}{s}\right) \end{aligned}$$

9. (a)

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} + y &= 4t\,\mathrm{H}(t-2), \qquad y(0) = 0 \\
&= 4\big((t-2)+2\big)\,\mathrm{H}(t-2) \\
sY(s) - y(0) + Y(s) &= 4e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right) \\
(s+1)Y(s) &= 4e^{-2s}\left(\frac{1+2s}{s^2}\right) \\
Y(s) &= 4e^{-2s}\left(\frac{1+2s}{s^2(s+1)}\right)
\end{aligned}$$

Partial fractions decomposition gives

$$\begin{aligned}
&= 4e^{-2s}\left(\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}\right) \\
1 + 2s &= As(s+1) + B(s+1) + Cs^2 \\
\text{constant terms:}\quad & 1 = B \\
s \text{ terms:}\quad & 2 = A + B \implies A = 2 - B = 1 \\
s^2 \text{ terms:}\quad & 0 = A + C \implies C = -A = -1. \\
Y(s) &= 4e^{-2s}\left(\frac{1}{s} + \frac{1}{s^2} - \frac{1}{s+1}\right) \\
y(t) &= 4\left(1 + (t-2) - e^{-(t-2)}\right)\mathrm{H}(t-2) \\
&= 4\left(-1 + t - e^{-t+2}\right)\mathrm{H}(t-2)
\end{aligned}$$

by the second shift theorem.

(b)

$$\begin{aligned}
\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 3\frac{\mathrm{d}y}{\mathrm{d}t} + 2y &= \mathrm{H}(t-2) - \mathrm{H}(t-3), \quad y(0) = 0,\ y'(0) = 0 \\
\left(s^2 + 3s + 2\right)Y(s) &= \left(e^{-2s} - e^{-3s}\right)\frac{1}{s} \\
Y(s) &= \left(e^{-2s} - e^{-3s}\right)\frac{1}{s(s+1)(s+2)}
\end{aligned}$$

Using partial fractions decomposition:

$$\begin{aligned}
&= \left(e^{-2s} - e^{-3s}\right)\left(\frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}\right) \\
1 &= A(s+1)(s+2) + Bs(s+2) + Cs(s+1) \\
\text{constant terms:}\quad 1 &= 2A \implies A = \frac{1}{2} \\
s^2 \text{ terms:}\quad 0 &= A + B + C \implies B = -A - C = -\frac{1}{2} - C \\
s \text{ terms:}\quad 0 &= 3A + 2B + C \\
&= \frac{3}{2} - 1 - 2C + C \implies C = \frac{3}{2} - 1 = \frac{1}{2} \\
&\qquad\qquad\qquad \implies B = -\frac{1}{2} - \frac{1}{2} = -1. \\
Y(s) &= \left(e^{-2s} - e^{-3s}\right)\left(\frac{1}{2}\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\frac{1}{s+2}\right) \\
&= e^{-2s}\left(\frac{1}{2}\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\frac{1}{s+2}\right) - e^{-3s}\left(\frac{1}{2}\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\frac{1}{s+2}\right) \\
y(t) &= \left(\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)}\right)\mathrm{H}(t-2) - \\
&\quad \left(\frac{1}{2} - e^{-(t-3)} + \frac{1}{2}e^{-2(t-3)}\right)\mathrm{H}(t-3)
\end{aligned}$$

by the second shift theorem.