# EMTH210 Tutorial 7: Eigenproblems, and Basic Double Integrals – Solutions

The homework questions this week are **7** and **11(d)**.

1. We look to see which vectors are (non-zero) solutions to $A\mathbf{x} = \lambda\mathbf{x}$.

   $\mathbf{u}$ is an eigenvector of $A$, since $A\mathbf{u} = (4, 4, 4) \implies \lambda = 4$.

   $\mathbf{v}$ is not an eigenvector of $A$, since $A\mathbf{v} = (3, 2, 3)$.

2. Let's go down the last column, since that has the most zeros and the smallest numbers (the last row would also be good). The signs are $-$, $+$, $-$, $+$.

$$\begin{aligned}
\begin{vmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 3 & 2 \\ 1 & -5 & -3 & 1 \\ 2 & 7 & 0 & 0 \end{vmatrix} &= -0 + 2\begin{vmatrix} 1 & -2 & 1 \\ 1 & -5 & -3 \\ 2 & 7 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 2 & 7 & 0 \end{vmatrix} + 0 \\
&= 2\left(\begin{vmatrix} 1 & -5 \\ 2 & 7 \end{vmatrix} + 3\begin{vmatrix} 1 & -2 \\ 2 & 7 \end{vmatrix}\right) - \left(\begin{vmatrix} 0 & 1 \\ 2 & 7 \end{vmatrix} - 3\begin{vmatrix} 1 & -2 \\ 2 & 7 \end{vmatrix}\right) \\
&= 2\big((7+10) + 3(7+4)\big) - \big((0-2) - 3(7+4)\big) \\
&= 2(17+33) - (-2-33) \\
&= 135
\end{aligned}$$

3.

$$\begin{aligned}
C = \begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix} \implies |\, C - \lambda I \,| &= \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} \\
&= (-5-\lambda)(-2-\lambda) - 4 \\
&= \lambda^2 + 7\lambda + 6 \\
&= (\lambda+1)(\lambda+6) \implies \text{Let } \lambda_1 = -1, \lambda_2 = -6.
\end{aligned}$$

   Note that this labelling of $\lambda_1$ and $\lambda_2$ is arbitrary, and letting $\lambda_1 = -6$ and $\lambda_2 = -1$ is okay (it just means the labelling of your eigenvectors will be different to match).

   For $\lambda_1 = -1$, solve $\big(C - (-1)I\big)\mathbf{x}_1 = \mathbf{0}$ for $\mathbf{x}_1 \neq \mathbf{0}$.

$$\left(\begin{array}{cc|c} -4 & 2 & 0 \\ 2 & -1 & 0 \end{array}\right) \quad R_2 \leftarrow R_1 + 2R_2$$

$$\left(\begin{array}{cc|c} -4 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \implies x_2 \text{ free, let } x_2 = t \neq 0\text{, then } x_1 = \tfrac{1}{2}t \implies \mathbf{x}_1 = t\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix},\ t \neq 0.$$

   Make sure you underline your vector names! $\mathbf{x}_1$ must look different from $x_1$.

   For $\lambda_2 = -6$, solve $\big(C - (-6)I\big)\mathbf{x}_2 = \mathbf{0}$ for $\mathbf{x}_2 \neq \mathbf{0}$.

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right) \quad R_2 \leftarrow R_2 - 2R_1$$

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \implies x_2 \text{ free, let } x_2 = t \neq 0\text{. Then } x_1 = -2t \implies \mathbf{x}_2 = t\begin{pmatrix} -2 \\ 1 \end{pmatrix},\ t \neq 0.$$

   In practice, we then choose a value for $t$ to get a particular vector. We choose a $t$ value that is "nice", i.e. avoids fractions etc. Here, $t = 1$ would be simple.

$$D = \begin{pmatrix} -1 & 1 & 3 \\ 1 & 2 & 0 \\ 3 & 0 & 2 \end{pmatrix} \implies \left| D - \lambda I \right| = \begin{vmatrix} -1-\lambda & 1 & 3 \\ 1 & 2-\lambda & 0 \\ 3 & 0 & 2-\lambda \end{vmatrix}$$

$$= 3\begin{vmatrix} 1 & 2-\lambda \\ 3 & 0 \end{vmatrix} + (2-\lambda)\begin{vmatrix} -1-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix}$$

Watch out for common factors like $(2-\lambda)$ here.

$$= -9(2-\lambda) + (2-\lambda)\big((-1-\lambda)(2-\lambda) - 1\big)$$
$$= (2-\lambda)(\lambda^2 - \lambda - 12)$$
$$= (2-\lambda)(\lambda+3)(\lambda-4) \implies \text{Let } \lambda_1 = 2, \lambda_2 = -3, \lambda_3 = 4.$$

For $\lambda_1 = 2$, solve $(D - 2I)\mathbf{x}_1 = \mathbf{0}$ for $\mathbf{x}_1 \neq \mathbf{0}$.

$$\left(\begin{array}{ccc|c} -3 & 1 & 3 & 0 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \\ R_2 \leftarrow R_1 + 3R_2 \\ R_3 \leftarrow R_1 + R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} -3 & 1 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right) \begin{array}{l} \\ \\ R_3 \leftarrow R_3 - R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} -3 & 1 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \implies x_3 \text{ free, let } x_3 = t, \text{ then } x_2 = -3t,\ x_1 = -\tfrac{1}{3}(-3t + 3t) = 0.$$

$$\implies \mathbf{x}_1 = t\begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix},\ t \neq 0.$$

For $\lambda_2 = -3$, solve $\big(D - (-3)I\big)\mathbf{x}_2 = \mathbf{0}$ for $\mathbf{x}_2 \neq \mathbf{0}$.

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & 5 & 0 & 0 \\ 3 & 0 & 5 & 0 \end{array}\right) \begin{array}{l} \\ R_2 \leftarrow R_1 - 2R_2 \\ R_3 \leftarrow 3R_2 - R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 0 & -9 & 3 & 0 \\ 0 & 15 & -5 & 0 \end{array}\right) R_3 = -\tfrac{5}{3}R_2 \implies x_3 \text{ free, let } x_3 = 3t \neq 0, \text{ then } x_2 = t,\ x_1 = \tfrac{1}{2}(-t - 9t) = -5t.$$

$$\implies \mathbf{x}_2 = t\begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix},\ t \neq 0.$$

Why let $x_3 = 3t$? Looking ahead to avoid fractions.

For $\lambda_3 = 4$, solve $(D - 4I)\mathbf{x}_3 = \mathbf{0}$ for $\mathbf{x}_3 \neq \mathbf{0}$.

$$\left(\begin{array}{ccc|c} -5 & 1 & 3 & 0 \\ 1 & -2 & 0 & 0 \\ 3 & 0 & -2 & 0 \end{array}\right) \begin{array}{l} \\ R_2 \leftarrow R_1 + 5R_2 \\ R_3 \leftarrow R_3 - 3R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} -5 & 1 & 3 & 0 \\ 0 & -9 & 3 & 0 \\ 0 & 6 & -2 & 0 \end{array}\right) R_3 = -\tfrac{2}{3}R_2 \implies x_3 \text{ free, let } x_3 = 3t, \text{ then } x_2 = t,\ x_1 = \tfrac{1}{5}(t + 9t) = 2$$

$$\implies \mathbf{x}_3 = t\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix},\ t \neq 0.$$

$$E = \begin{pmatrix} -1 & 2 & 1 \\ 2 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix} \implies \left| \, E - \lambda I \, \right| = \begin{vmatrix} -1-\lambda & 2 & 1 \\ 2 & 3-\lambda & 0 \\ 1 & 0 & 3-\lambda \end{vmatrix}$$

$$\begin{aligned} &= \begin{vmatrix} 2 & 3-\lambda \\ 1 & 0 \end{vmatrix} + (3-\lambda) \begin{vmatrix} -1-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} \\ &= -(3-\lambda) + (3-\lambda)\big((-1-\lambda)(3-\lambda) - 4\big) \\ &= (3-\lambda)(\lambda^2 - 2\lambda - 8) \\ &= (3-\lambda)(\lambda+2)(\lambda-4) \implies Let \lambda_1 = -2, \lambda_2 = 3, \lambda_3 = 4 \end{aligned}$$

For $\lambda_1 = -2$, solve $\big(E - (-2)I\big)\mathbf{x}_1 = \mathbf{0}$ for $\mathbf{x}_1 \neq \mathbf{0}$.

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 5 & 0 & 0 \\ 1 & 0 & 5 & 0 \end{array}\right) \begin{array}{l} \\ R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_2 - 2R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 5 & -10 & 0 \end{array}\right) \; R_3 = 5R_2 \implies x_3 \text{ free, let } x_3 = t \neq 0 \text{, then } x_2 = 2t,\; x_1 = -2(2t) - t = -5t$$

$$\implies \mathbf{x}_1 = t \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix}, \; t \neq 0.$$

For $\lambda_2 = 3$, solve $(E - 3I)\mathbf{x}_2 = \mathbf{0}$ for $\mathbf{x}_2 \neq \mathbf{0}$.

$$\left(\begin{array}{ccc|c} -4 & 2 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \\ R_2 \leftarrow 2R_2 + R_1 \\ R_3 \leftarrow 4R_3 + R_1 \end{array}$$

$$\left(\begin{array}{ccc|c} -4 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right) \; R_3 = R_2 \implies x_3 \text{ free, let } x_3 = 2t \text{, then } x_2 = -t,\; x_1 = -\tfrac{1}{4}\big(-2(-t) - 2t\big) = 0$$

$$\implies \mathbf{x}_2 = t \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}, \; t \neq 0.$$

For $\lambda_3 = 4$, solve $(E - 4I)\mathbf{x}_3 = \mathbf{0}$ for $\mathbf{x}_3 \neq \mathbf{0}$.

$$\left(\begin{array}{ccc|c} -5 & 2 & 1 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right) \begin{array}{l} \\ R_2 \leftarrow R_2 - 2R_3 \\ R_3 \leftarrow R_1 + 5R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} -5 & 2 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -4 & 0 \end{array}\right) \; R_3 = -2R_2 \implies x_3 \text{ free, let } x_3 = t \text{, then } x_2 = 2t,\; x_1 = -\tfrac{1}{5}\big(-2(2t) - t\big) = t$$

$$\implies \mathbf{x}_3 = t \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \; t \neq 0.$$

4. A matrix $A$ is orthogonal if its transpose is its inverse, i.e. $A^T = A^{-1}$, so $A^T A = A^{-1}A = I$.

$$B^T B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \implies \text{yes}$$

$$C^T C = \frac{1}{9} \begin{pmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{4}{9} \\ & & \\ & & \end{pmatrix} \implies \text{no}$$

$$E^T E = \frac{1}{9} \begin{pmatrix} -2 & -1 & -2 \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 & 2 \\ -1 & 2 & 2 \\ -2 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{8}{9} & \\ & & \\ & & \end{pmatrix} \implies \text{no}$$

$$F^T F = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \implies \text{yes}$$

$$G^T G = \frac{1}{18^2} \begin{pmatrix} -2 & 7 \\ 7 & 2 \end{pmatrix} \begin{pmatrix} -2 & 7 \\ 7 & 2 \end{pmatrix} = \frac{1}{18^2} \begin{pmatrix} 53 & \\ & \end{pmatrix} \implies \text{no}$$

When we're doing this $A^T A = I$ check, we can stop as soon as we get an element that tells us the answer won't be $I$.

5. We don't need to go through all the work of deriving the eigenvectors again, we just need to check that $F\mathbf{x}_i = \lambda_i \mathbf{x}_i$, for some scalar $\lambda_i$, the eigenvalue.

$$F\mathbf{x}_1 = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 6 & 0 \\ 3 & 0 & -8 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix} = 6\mathbf{x}_1 \implies \lambda_1 = 6$$

$$F\mathbf{x}_2 = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 6 & 0 \\ 3 & 0 & -8 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \mathbf{x}_2 \implies \lambda_2 = 1$$

$$F\mathbf{x}_3 = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 6 & 0 \\ 3 & 0 & -8 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} -9 \\ 0 \\ 27 \end{pmatrix} = -9\mathbf{x}_3 \implies \lambda_3 = -9$$

To check that the eigenvectors are all orthogonal, we check that each possible dot product between them is 0:

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = 0 \quad \checkmark \qquad \mathbf{x}_1 \cdot \mathbf{x}_3 = 0 \quad \checkmark \qquad \mathbf{x}_2 \cdot \mathbf{x}_3 = 0 \quad \checkmark$$

6. The general solution will be in the form

$$\mathbf{y} = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2.$$

We just need to find the eigenvectors $\mathbf{x}_i$ and eigenvalues $\lambda_i$ of the coefficient matrix.

$$A = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}, \ \left| \, A - \lambda I \, \right| = \begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = (-3-\lambda)^2 - 1 = 0, \text{ so } \lambda_1 = -2, \ \lambda_2 = -4.$$

For $\lambda_1 = -2$, solve $(A + 2I)\mathbf{x}_1 = \mathbf{0}$.

$$\left(\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right) \ R_2 = -R_1 \implies x_2 \text{ is free, choose } x_2 = 1, \text{ then } x_1 = 1 \text{ and } \mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = -4$, solve $(A + 4I)\mathbf{x}_2 = \mathbf{0}$.

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right) \ R_2 = R_1 \implies x_2 \text{ is free, choose } x_2 = 1, \text{ then } x_1 = -1 \text{ and } \mathbf{x}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Therefore the general solution is

$$\mathbf{y} = c_1 e^{-2t}\begin{pmatrix}1\\1\end{pmatrix} + c_2 e^{-4t}\begin{pmatrix}-1\\1\end{pmatrix}.$$

The general solution will be in the form

$$\mathbf{y} = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 + c_3 e^{\lambda_3 t}\mathbf{x}_3.$$

$$B = \begin{pmatrix}3&0&0\\5&4&0\\3&6&1\end{pmatrix}, \left| B-\lambda I \right| = \begin{vmatrix}3-\lambda&0&0\\5&4-\lambda&0\\3&6&1-\lambda\end{vmatrix} = (3-\lambda)(4-\lambda)(1-\lambda) = 0$$

So $\lambda_1 = 3$, $\lambda_2 = 4$, $\lambda_3 = 1$.

For $\lambda_1 = 3$, $x_1$ is free, choose $x_1 = 2$, then $x_2 = -10$, $x_3 = -\frac{1}{2}\big(-3(2) - 6(-10)\big) = -27$, and $\mathbf{x}_1 = (2, -10, -27)$.

For $\lambda_2 = 4$, $x_1 = 0$, $x_2$ is free, choose $x_2 = 1$, then $x_3 = -\frac{1}{3}\big(-3(0) - 6(1)\big) = 2$, and $\mathbf{x}_2 = (0, 1, 2)$.

For $\lambda_3 = 1$, $x_1 = 0$, $x_2 = 0$, $x_3$ is free, choose $x_3 = 1$, then, and $\mathbf{x}_3 = (0, 0, 1)$.

Therefore the general solution is

$$\mathbf{y} = c_1 e^{3t}\begin{pmatrix}2\\-10\\-27\end{pmatrix} + c_2 e^{4t}\begin{pmatrix}0\\1\\2\end{pmatrix} + c_3 e^{t}\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

7. First we need to write this system of DEs in matrix form.

$$\text{Let } \mathbf{y} = \begin{pmatrix}y\\m\\\theta\end{pmatrix}, \text{then } \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \begin{pmatrix}\frac{\mathrm{d}y}{\mathrm{d}t}\\\frac{\mathrm{d}m}{\mathrm{d}t}\\\frac{\mathrm{d}\theta}{\mathrm{d}t}\end{pmatrix} = \begin{pmatrix}-y+m\\y-m\\m-\theta\end{pmatrix} = \underbrace{\begin{pmatrix}-1&1&0\\1&-1&0\\0&1&-1\end{pmatrix}}_{A}\begin{pmatrix}y\\m\\\theta\end{pmatrix} = A\mathbf{y}.$$

Our general solution will have the form

$$\mathbf{y} = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 + c_3 e^{\lambda_3 t}\mathbf{x}_3.$$

$$\begin{aligned}0 = \left| A-\lambda I \right| &= \begin{vmatrix}-1-\lambda&1&0\\1&-1-\lambda&0\\0&1&-1-\lambda\end{vmatrix}\\ &= -(\lambda+1)((\lambda+1)^2-1)\\ &= -(\lambda+1)(\lambda^2+2\lambda+1-1)\\ &= -(\lambda+1)(\lambda)(\lambda+2) \implies \lambda = -1, 0, -2\end{aligned}$$

For $\lambda_1 = -1$, solve $(A+I)\mathbf{x}_1 = \mathbf{0}$.

$$\left(\begin{array}{ccc|c}0&1&0&0\\1&0&0&0\\0&1&0&0\end{array}\right) R_3 = R_1 \implies x_1 = x_2 = 0,\ x_3 \text{ free, choose } x_3 = 1, \text{ then } \mathbf{x}_1 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

For $\lambda_2 = 0$, solve $A\mathbf{x}_2 = \mathbf{0}$.

$$\left(\begin{array}{ccc|c}-1&1&0&0\\1&-1&0&0\\0&1&-1&0\end{array}\right) R_2 = -R_1 \implies x_2 \text{ free, choose } x_2 = 1, \text{ then } x_1 = x_3 = 1, \text{ and } \mathbf{x}_2 = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

For $\lambda_3 = -2$, solve $(A+2I)\mathbf{x}_3 = \mathbf{0}$.

$$\left(\begin{array}{ccc|c}1&1&0&0\\1&1&0&0\\0&1&1&0\end{array}\right) R_2 = R_1 \implies x_2 \text{ free, choose } x_2 = 1, \text{ then } x_1 = x_3 = -1, \text{ and } \mathbf{x}_3 = \begin{pmatrix}-1\\1\\-1\end{pmatrix}$$

The general solution is then

$$\mathbf{y} = c_1 e^{-t}\begin{pmatrix}0\\0\\1\end{pmatrix} + c_2\begin{pmatrix}1\\1\\1\end{pmatrix} + c_3 e^{-2t}\begin{pmatrix}-1\\1\\-1\end{pmatrix}.$$

Substituting in the initial condition gives

$$\begin{pmatrix}4\\2\\1\end{pmatrix} = \mathbf{y}(0) = c_1\begin{pmatrix}0\\0\\1\end{pmatrix} + c_2\begin{pmatrix}1\\1\\1\end{pmatrix} + c_3\begin{pmatrix}-1\\1\\-1\end{pmatrix} = \begin{pmatrix}0 & 1 & -1\\0 & 1 & 1\\1 & 1 & -1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}$$

$$\left(\begin{array}{ccc|c}0 & 1 & -1 & 4\\0 & 1 & 1 & 2\\1 & 1 & -1 & 1\end{array}\right) \quad \begin{array}{l}R_1 \leftarrow R_3\\ \\ R_3 \leftarrow R_1 - R_2\end{array}$$

$$\left(\begin{array}{ccc|c}1 & 1 & -1 & 1\\0 & 1 & 1 & 2\\0 & 0 & -2 & 2\end{array}\right) \quad \Longrightarrow c_3 = -1,\ c_2 = 3,\ c_1 = -3$$

As $t \to \infty$, $e^{-t}$ and $e^{-2t}$ both go to 0, so only the middle term is left, and $\mathbf{y} \to (3,3,3)$. I.e. all three categories tend to 3000 individuals.

8. For these ellipse problems, we have a complicated equation for the ellipse in $x$–$y$ coordinates, and we want to find some coordinates in which the equation is simple and easily drawable. We rewrite the ellipse equation $ax^2 + 2bxy + cy^2 = d$ in matrix form

$$ax^2 + 2bxy + cy^2 = \begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}a & b\\b & c\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x & y\end{pmatrix} A \begin{pmatrix}x\\y\end{pmatrix} = d,$$

then use eigenvalues and eigenvectors to rewrite $A$ as $PDP^T$, where $P$ and $D$ are orthogonal and diagonal, respectively,

$$\begin{pmatrix}x & y\end{pmatrix} PDP^T \begin{pmatrix}x\\y\end{pmatrix} = d,$$

and finally define new coordinates $u$ and $v$ as $\begin{pmatrix}u\\v\end{pmatrix} = P^T\begin{pmatrix}x\\y\end{pmatrix}$, giving

$$\begin{pmatrix}u & v\end{pmatrix} D \begin{pmatrix}u\\v\end{pmatrix} = d,$$

which is very simple when expanded out since $D$ is diagonal:

$$\begin{pmatrix}u & v\end{pmatrix}\begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix} = \lambda_1 u^2 + \lambda_2 v^2 = d.$$

These new coordinate axes $u$ and $v$ go in the directions of the eigenvectors. Because the labelling of which eigenvector is which is arbitrary, you may end up swapping $u$ and $v$. The ellipse will be the same no matter what.

(a)

$$3x^2 - 2xy + 3y^2 = 20$$

$$\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}3 & -1\\-1 & 3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = 20 \Longrightarrow A = \begin{pmatrix}3 & -1\\-1 & 3\end{pmatrix}$$

(The coefficients of $x^2$ and $y^2$ will always be on the diagonal, and half the coefficient of $xy$ goes into each off-diagonal element.)

Then we find the eigenvalues and eigenvectors of $A$ to get $P$ and $D$.

$$\begin{aligned}
\mid A - \lambda I \mid &= \begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} \\
&= (3-\lambda)^2 - 1 \\
&= \lambda^2 - 6\lambda + 8 \\
&= (\lambda - 2)(\lambda - 4) \implies \lambda_1 = 2,\ \lambda_2 = 4
\end{aligned}$$

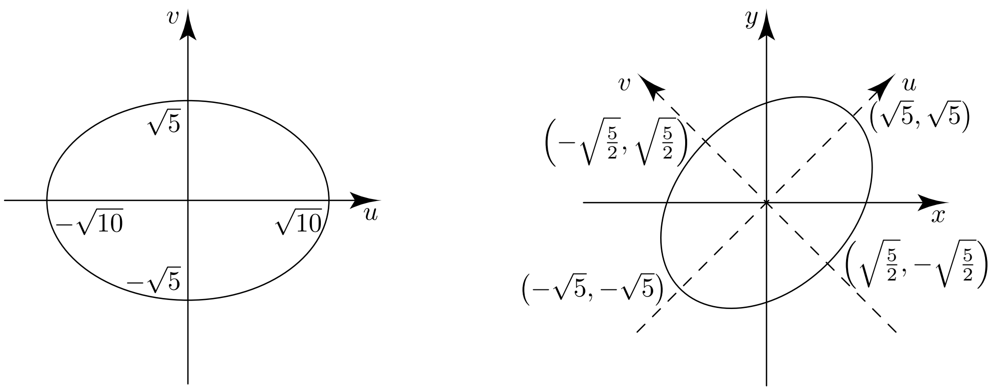
For $\lambda_1 = 2$, $\quad \left(\begin{array}{rr|r} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right) \implies x_2$ free, choose $x_2 = 1$, then $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\hat{\mathbf{x}}_1 = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda_2 = 4$, $\quad \left(\begin{array}{rr|r} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right) \implies x_2$ free, choose $x_2 = 1$, then $\mathbf{x}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\hat{\mathbf{x}}_2 = \dfrac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

$$\begin{aligned}
\therefore P &= \begin{pmatrix} \hat{\mathbf{x}}_1 & \hat{\mathbf{x}}_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \\
D &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}.
\end{aligned}$$

$$\begin{aligned}
20 = 3x^2 - 2xy + 3y^2 &= \begin{pmatrix} x & y \end{pmatrix} A \begin{pmatrix} x \\ y \end{pmatrix} \\
&= \begin{pmatrix} x & y \end{pmatrix} PDP^T \begin{pmatrix} x \\ y \end{pmatrix} \\
&= \begin{pmatrix} u & v \end{pmatrix} D \begin{pmatrix} u \\ v \end{pmatrix} \\
20 &= 2u^2 + 4v^2 \\
1 &= \frac{u^2}{10} + \frac{v^2}{5}
\end{aligned}$$

This is now easy to draw in $(u, v)$-coordinates, and also in $(x, y)$-coordinates, since we know the $u$ axis is in the direction of $\mathbf{x}_1$ and the $v$ axis is in the direction of $\mathbf{x}_2$.




To see why the axes go in the directions of the eigenvectors, note that $P^T = P^{-1}$ (since $P$ is orthogonal). Then,

$$\begin{aligned}
P^T \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} u \\ v \end{pmatrix} \\
\begin{pmatrix} x \\ y \end{pmatrix} &= P \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{x}}_1 & \hat{\mathbf{x}}_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.
\end{aligned}$$

Now substitute $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to get the $u$-axis direction and $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to get the $v$-axis direction.

If desired, the $(x, y)$-coordinates of the vertices $\left(\sqrt{5}, \sqrt{5}\right)$ and $\left(-\sqrt{5}, -\sqrt{5}\right)$, and co-vertices $\left(-\sqrt{\frac{5}{2}}, \sqrt{\frac{5}{2}}\right)$ and $\left(\sqrt{\frac{5}{2}}, -\sqrt{\frac{5}{2}}\right)$, can be obtained by substituting in the corresponding $(u, v)$-coordinates.

(b)

$$A = \begin{pmatrix} 10 & 3 \\ 3 & 2 \end{pmatrix}$$

$$\begin{aligned} \mid A - \lambda I \mid &= \begin{vmatrix} 10-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix} \\ &= (10-\lambda)(2-\lambda) - 9 \\ &= \lambda^2 - 12\lambda + 11 \\ &= (\lambda - 1)(\lambda - 11) \implies \lambda_1 = 1,\, \lambda_2 = 11 \end{aligned}$$

For $\lambda_1 = 1$,

$$\left(\begin{array}{cc|c} 9 & 3 & 0 \\ 3 & 1 & 0 \end{array}\right) \; R_1 = 3R_2 \implies x_2 \text{ free, choose } x_2 = 3, \text{ then } \mathbf{x}_1 = \begin{pmatrix} -1 \\ 3 \end{pmatrix},\; \hat{\mathbf{x}}_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} -1 \\ 3 \end{pmatrix}.$$

For $\lambda_2 = 11$,

$$\left(\begin{array}{cc|c} -1 & 3 & 0 \\ 3 & -9 & 0 \end{array}\right) \; R_2 = -3R_1 \implies x_2 \text{ free, choose } x_2 = 1, \text{ then } \mathbf{x}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix},\; \hat{\mathbf{x}}_2 = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

(To choose values for these free variables we're looking ahead to the next step and choosing something that avoids fractions. If you choose a value and it leads to something messy, you can go back and choose a new one, as long as you change everything over to the new value. The unit vectors ($\hat{\mathbf{x}}_i$) will end up the same anyway, except maybe for signs.)

$$\therefore P = \begin{pmatrix} \hat{\mathbf{x}}_1 & \hat{\mathbf{x}}_2 \end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix} -1 & 3 \\ 3 & 1 \end{pmatrix},$$

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 11 \end{pmatrix}.$$

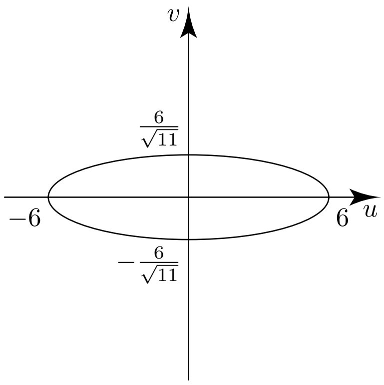
$$36 = 10x^2 + 6xy + 2y^2 = u^2 + 11v^2$$

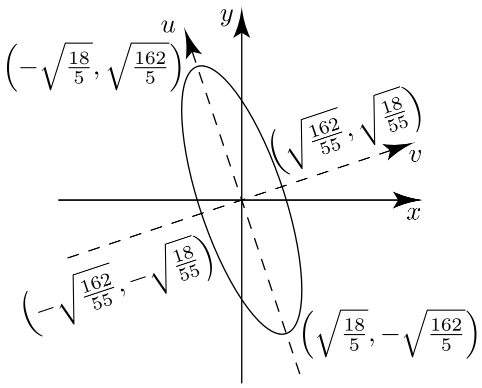
$$1 = \frac{u^2}{36} + \frac{v^2}{36/11}$$

We can now sketch the ellipse in the $(u, v)$-plane.

The direction of the $u$-axis is $\mathbf{x}_1 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$, and the direction of the $v$-axis is $\mathbf{x}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

If desired, the $(x, y)$-coordinates of the vertices can be obtained by substituting the corresponding $(u, v)$-coordinates into $\begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} u \\ v \end{pmatrix}$: $\left(-\sqrt{\frac{18}{5}}, \sqrt{\frac{162}{5}}\right)$ and $\left(\sqrt{\frac{18}{5}}, -\sqrt{\frac{162}{5}}\right)$, and co-vertices $\left(\sqrt{\frac{162}{55}}, \sqrt{\frac{18}{55}}\right)$ and $\left(-\sqrt{\frac{162}{55}}, -\sqrt{\frac{18}{55}}\right)$. They are pretty horrible! The $u$–$v$ coordinate system is much nicer.



9.
$$\begin{aligned}
|\, A-\lambda I \,| &= \begin{vmatrix} 5-\lambda & \sqrt{10} \\ \sqrt{10} & 8-\lambda \end{vmatrix} \\
&= (5-\lambda)(8-\lambda)-10 \\
&= \lambda^2 - 13\lambda + 30 \\
&= (\lambda-3)(\lambda-10) \implies \lambda_1 = 3,\ \lambda_2 = 10
\end{aligned}$$

For $\lambda_1 = 3$,

$$\left(\begin{array}{cc|c} 2 & \sqrt{10} & 0 \\ \sqrt{10} & 5 & 0 \end{array}\right) \quad R_2 \leftarrow \sqrt{10}R_2$$
$$\left(\begin{array}{cc|c} 2 & \sqrt{10} & 0 \\ 10 & 5\sqrt{10} & 0 \end{array}\right) \quad R_2 = 5R_1 \implies x_2 \text{ free, let } x_2 = t \neq 0, \text{ then } \mathbf{x}_1 = t\begin{pmatrix} -\frac{\sqrt{10}}{2} \\ 1 \end{pmatrix}.$$

If we let $t = 2$, we get $\mathbf{x}_1 = \begin{pmatrix} -\sqrt{10} \\ 2 \end{pmatrix}$ and then $\hat{\mathbf{x}}_1 = \frac{1}{\sqrt{14}}\begin{pmatrix} -\sqrt{10} \\ 2 \end{pmatrix}$. Note that there are other equal but very different-looking versions of this, e.g. letting $t = \sqrt{10}$ leads to $\hat{\mathbf{x}}_1 = \frac{1}{\sqrt{35}}\begin{pmatrix} -5 \\ \sqrt{10} \end{pmatrix}$, which is actually the same!

For $\lambda_2 = 10$,

$$\left(\begin{array}{cc|c} -5 & \sqrt{10} & 0 \\ \sqrt{10} & -2 & 0 \end{array}\right) \quad R_2 \leftarrow \sqrt{10}R_2$$
$$\left(\begin{array}{cc|c} -5 & \sqrt{10} & 0 \\ 10 & -2\sqrt{10} & 0 \end{array}\right) \quad R_2 = -2R_1 \implies x_2 \text{ free, let } x_2 = t \neq 0, \text{ then } \mathbf{x}_2 = t\begin{pmatrix} \frac{\sqrt{10}}{5} \\ 1 \end{pmatrix}.$$

If we let $t = \sqrt{10}$, we get $\mathbf{x}_2 = \begin{pmatrix} 2 \\ \sqrt{10} \end{pmatrix}$ and then $\hat{\mathbf{x}}_2 = \frac{1}{\sqrt{14}}\begin{pmatrix} 2 \\ \sqrt{10} \end{pmatrix}$. Letting $t = 5$ leads to $\hat{\mathbf{x}}_2 = \frac{1}{\sqrt{35}}\begin{pmatrix} \sqrt{10} \\ 5 \end{pmatrix}$, which is equal.

For the $P$ matrix, it is convenient if the multiplier in front of both $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ is the same, since we can then pull it out in front of the whole matrix. We'll use the ones with $\sqrt{14}$ here:

$$\begin{aligned}
P &= \begin{pmatrix} \hat{\mathbf{x}}_1 & \hat{\mathbf{x}}_2 \end{pmatrix} = \frac{1}{\sqrt{14}}\begin{pmatrix} -\sqrt{10} & 2 \\ 2 & \sqrt{10} \end{pmatrix}, \\
D &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 10 \end{pmatrix}. \\
\text{Check: } PDP^T &= \frac{1}{\sqrt{14}}\begin{pmatrix} -\sqrt{10} & 2 \\ 2 & \sqrt{10} \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 10 \end{pmatrix}\frac{1}{\sqrt{14}}\begin{pmatrix} -\sqrt{10} & 2 \\ 2 & \sqrt{10} \end{pmatrix} \\
&= \frac{1}{14}\begin{pmatrix} -\sqrt{10} & 2 \\ 2 & \sqrt{10} \end{pmatrix}\begin{pmatrix} -3\sqrt{10} & 6 \\ 20 & 10\sqrt{10} \end{pmatrix} \\
&= \frac{1}{14}\begin{pmatrix} 70 & 14\sqrt{10} \\ 14\sqrt{10} & 112 \end{pmatrix} \\
&= \begin{pmatrix} 5 & \sqrt{10} \\ \sqrt{10} & 8 \end{pmatrix} = A \quad \checkmark
\end{aligned}$$

10.
$$A = PDP^{-1} \implies A^2 = PDP^{-1}PDP^{-1} = PDDP^{-1} = PD^2P^{-1}$$

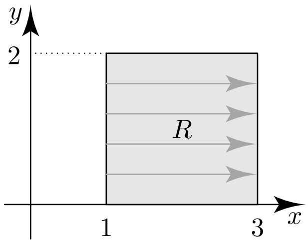
The same argument works for any positive integer power: $A^k = PD^kP^{-1}$ for any $k \in \mathbb{Z}^+$. That means powers of diagonalisable matrices can be calculated very efficiently.

11. For some of these questions, both orders of integration are equally easy. For others, one way is harder or impossible. Choose the order which seems easiest, but if you get stuck, go back and try the other way.

We are integrating functions over areas, so when the function is entirely positive, what we get at the end is the volume underneath the function surface—imagine the function sticking out of the page towards you. (When the integrand goes negative, we get the volume above the $x$–$y$ plane minus the volume below it.)

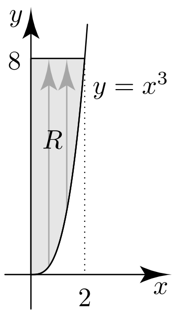
The arrows show the direction of the strips made by the inner integral, and the outer integral then adds up all the strips.

(a)



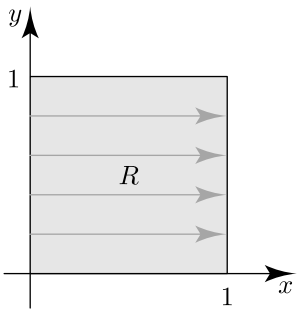

$$\begin{aligned}\iint_R 2xy\,\mathrm{d}A &= \int_{y=0}^{2}\int_{x=1}^{3} 2xy\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_{y=0}^{2} \left[x^2y\right]_{x=1}^{3}\,\mathrm{d}y\\ &= \int_{y=0}^{2} (9-1)y\,\mathrm{d}y\\ &= \left[4y^2\right]_{y=0}^{2} = 4(4) - 0 = 16\end{aligned}$$

(b)




$$\begin{aligned}\iint_R 2xy\,\mathrm{d}A &= \int_{x=0}^{2}\int_{y=x^3}^{8} 2xy\,\mathrm{d}y\,\mathrm{d}x\\ &= \int_{x=0}^{2} \left[xy^2\right]_{y=x^3}^{8}\,\mathrm{d}x\\ &= \int_{x=0}^{2} \left(64x - x^7\right)\,\mathrm{d}x\\ &= \left[32x^2 - \frac{1}{8}x^8\right]_{x=0}^{2}\\ &= \left(32(4) - \frac{256}{8}\right) - 0 = 96\end{aligned}$$

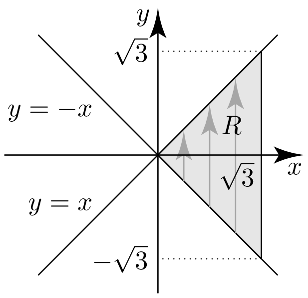
(c)




$$\begin{aligned}\iint_R \frac{y}{1+xy}\,\mathrm{d}A &= \int_{y=0}^{1}\int_{x=0}^{1} \frac{y}{1+xy}\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_{y=0}^{1}\int_{x=0}^{1} \frac{1}{u}\,\mathrm{d}u\,\mathrm{d}y\\ &= \int_{y=0}^{1} \left[\ln(1+xy)\right]_{x=0}^{1}\,\mathrm{d}y\\ &= \int_{y=0}^{1} \left(\ln(1+y) - \cancelto{0}{\ln 1}\right)\mathrm{d}y\\ &= \left[(1+y)\ln(1+y) - (1+y)\right]_{y=0}^{1}\\ &= \big((2)\ln(2) - (2)\big) - \big(\cancelto{0}{(1)\ln(1)} - (1)\big)\\ &= 2\ln 2 - 1\end{aligned}$$

$$u = 1 + xy$$
$$\mathrm{d}u = y\,\mathrm{d}x$$

(d)




$$\iint_R \sqrt{x^2+1}\,\mathrm{d}A = \int_{x=0}^{\sqrt{3}} \int_{y=-x}^{x} \sqrt{x^2+1}\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_{x=0}^{\sqrt{3}} \Big[y\sqrt{x^2+1}\Big]_{y=-x}^{x}\,\mathrm{d}x$$

$$u = x^2+1 \qquad \mathrm{d}u = 2x\,\mathrm{d}x$$

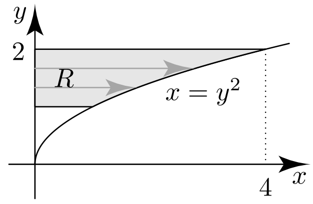
$$= \int_{x=0}^{\sqrt{3}} \left(x\sqrt{x^2+1} + x\sqrt{x^2+1}\right)\,\mathrm{d}x$$

$$= \int_{x=0}^{\sqrt{3}} u^{\frac{1}{2}}\,\mathrm{d}u$$

$$= \left[\frac{2}{3}\left(x^2+1\right)^{\frac{3}{2}}\right]_{x=0}^{\sqrt{3}}$$

$$= \frac{2}{3}(3+1)^{\frac{3}{2}} - \frac{2}{3} = \frac{2}{3}\sqrt{4^3} - \frac{2}{3} = \frac{14}{3}$$

(e)




$$\iint_R \sin\frac{\pi x}{y}\,\mathrm{d}A = \int_{y=1}^{2}\int_{x=0}^{y^2} \sin\frac{\pi x}{y}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_{y=1}^{2}\left[-\frac{y}{\pi}\cos\frac{\pi x}{y}\right]_{x=0}^{y^2}\mathrm{d}y$$

$$= \int_{y=1}^{2}\left(-\frac{y}{\pi}\cos\pi y + \frac{y}{\pi}\right)\mathrm{d}y$$

$$= \left[\frac{y^2}{2\pi}\right]_{y=1}^{2} - \int_{y=1}^{2}\left(\frac{y}{\pi}\cos\pi y\right)\mathrm{d}y$$

$$u = y \qquad \mathrm{d}u = \mathrm{d}y \qquad \mathrm{d}v = \tfrac{1}{\pi}\cos\pi y\,\mathrm{d}y \qquad v = \tfrac{1}{\pi^2}\sin\pi y$$

$$= \left[\frac{y^2}{2\pi}\right]_{y=1}^{2} - \left(\left[\frac{y}{\pi^2}\sin\pi y\right]_{y=1}^{2} - \int_{y=1}^{2}\frac{1}{\pi^2}\sin\pi y\,\mathrm{d}y\right)$$

$$= \left[\frac{y^2}{2\pi}\right]_{y=1}^{2} - \cancelto{0}{\left[\frac{y}{\pi^2}\sin\pi y\right]_{y=1}^{2}} + \left[-\frac{1}{\pi^3}\cos\pi y\right]_{y=1}^{2}$$

$$= \frac{4-1}{2\pi} + \frac{-1-1}{\pi^3}$$

$$= \frac{3}{2\pi} - \frac{2}{\pi^3}$$

12.

$$\int_{y=-x}^{1}\int_{x=y}^{y^2} x^2 y\,\mathrm{d}x\,\mathrm{d}y$$

The limits of the outside integral must be constant, so there are no $x$s or $y$s left at the end. The outer limits need to be in the form $\int_a^b$.

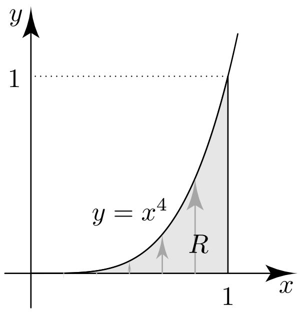
$$\int_{y=0}^{1}\int_{x=y}^{xy^2} \sin y\,\mathrm{d}x\,\mathrm{d}y$$

The limits of an integral cannot contain the variable of integration, so that it is not left at the end. The inner limits need to be in the form $\int_{f(y)}^{g(y)}$ ($y$ in this case).

$$\int_{y=1}^{2}\int_{y=x}^{x^2} (x+y)\,\mathrm{d}y\,\mathrm{d}y$$

The inner and outer integrals cannot both be with respect to the same variable for this to make sense as an integral over a region in the plane. $\mathrm{d}y^2$ is not a unit of area in the $x$–$y$ plane.

13. (a)

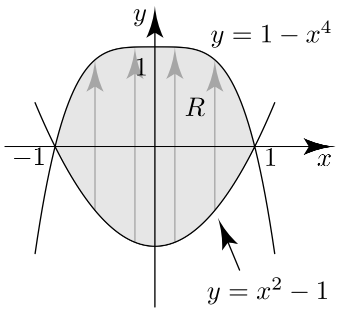



$0 \le x \le 1$

$0 \le y \le x^4$

$$\int_{x=0}^{1} \int_{y=0}^{x^4} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

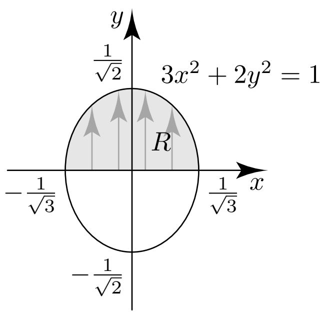
(b)




$x^2 - 1 \le y \le 1 - x^4$

$$\int_{x=-1}^{1} \int_{y=x^2-1}^{1-x^4} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

(c)




$3x^2 + 2y^2 \le 1$

$0 \le y$

$$\int_{x=\frac{-1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \int_{y=0}^{\sqrt{\frac{1}{2}(1-3x^2)}} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$