EMTH210 Tutorial 2: Partial derivatives, chain rule, directional derivatives

For the week starting Monday 24 February.

Preparation problems (homework)

1. Find the first partial derivatives of the following functions:
   (a) $f(x, y) = x^4y^2 - \sin(xy) + 6x^5 - 4y$
   (b) $f(x, y, z) = \exp(x + z) - \cos(xy^2z^3)$
   (c) $f(x, y, z, t) = x^2 \ln(t^2) - xz \tan(y)$

2. Verify that the function $z = \ln\left(x^2 + y^2\right)$ satisfies Laplace's equation, where Laplace's equation is
$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

3. A string lies along the x axis, and is tied at its ends at $x = \pm\pi$. The string is vibrating, and its displacement f satisfies the wave equation:
$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}.$$
Show that $f(x, t) = \sin(ct)\sin(x)$ satisfies the wave equation. Also show that the displacement is zero at the endpoints $x = \pm\pi$ for all time t.

4. The diffusion of heat along a one dimensional insulated rod lying on the x axis is governed by the heat equation
$$\frac{\partial^2 \theta}{\partial x^2} = k\frac{\partial \theta}{\partial t},$$
where $\theta(x, t)$ is the temperature of the rod and k is a positive constant. A student is looking for a solution of the form $\theta = \exp(at)\sin(x)$. Find the value of a for which this solution satisfies the heat equation.

5. Compute the gradients of the following functions at the given points:
   (a) $f(x, y) = y - \ln\left(2x^2y\right)$ at $(2, 1)$.
   (b) $F(x, y, z) = xy\cos(yz)$ at $(2, 1, \pi)$.

6. An electrically charged particle is held fixed at the origin. Let $V = 1/r$ be the potential energy of a second particle with unit electric charge located at (x, y, z), where $r = \sqrt{x^2 + y^2 + z^2}$. The force on this second particle is given by $-\nabla V$. Calculate the force on the particle when it is at $(1, 2, -2)$.

P.T.O.

7. For each function and given point, sketch the level curve passing through the given point, and sketch the gradient vector at that point. Comment on the last function.

   (a) $f(x, y) = y - x^2$ at $(2, 5)$

   (b) $f(x, y) = \frac{x^2}{4} + \frac{y^2}{9}$ at $(-2, -3)$

   (c) $f(x, y) = x^2 - y^2$ at $(0, 0)$.

8. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by these equations.

   (a) $x^3 + y^3 + z^3 + 6xyz \;=\; 1$ (b) $yz + x \ln y \;=\; z^2$

Problems for the tutorial

9. Find the directional derivative $D_{\mathbf{u}}f(x, y)$ for $f = x^2 + y^2$ when $\mathbf{u}$ is the unit vector making an angle of $30°$ with the x–axis.

10. The volume V of a box of height h, width w, and length ℓ is $V = hw\ell$. If V is held constant, and w and ℓ are independent, calculate $\partial h/\partial w$ when $h = 4$, $w = 3$, and $\ell = 6$.

11. An ant is crawling across a warm metal strip. The ant is crawling along the x-axis, and its location at time t is given by $x_{\text{ant}} = t/3$. The temperature T on the strip is given by $T = 20e^{-t}\sin(x) + 10$.

    (a) Calculate
    $$\left(\frac{\partial T}{\partial x}\right)_t \quad \text{and} \quad \left(\frac{\partial T}{\partial t}\right)_x$$
    and hence calculate dT/dt for the ant using the chain rule for several variables.

    (b) Explain the difference between $\left(\frac{\partial T}{\partial t}\right)_x$ and the expression for the ant's $\frac{\mathrm{d}T}{\mathrm{d}t}$ you found in part (a).

12. Given
    $$zt = x^2 - y\sin(y) \quad \text{and} \quad xyt = x^2 + y^2$$
    calculate $\left(\frac{\partial z}{\partial t}\right)_x$ and $\left(\frac{\partial t}{\partial z}\right)_y$ in terms of x, y, z, and t using implicit partial differentiation. For each partial derivative state which variables are independent and which are dependent.

13. Given
    $$F = y^3 - xy \quad \text{and} \quad G = xye^y$$
    calculate $\left(\frac{\partial F}{\partial x}\right)_G$ in terms of F, G, x, and y. Which variables are independent?