UNIVERSITY OF CANTERBURY

EMTH210

ENGINEERING MATHEMATICS

Topic 1: Partial differentiation

1.1 PARTIAL DERIVATIVES

Recall that for

$$f = f(x, y)\ ,$$

where x and y are independent variables,

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$\frac{\partial f}{\partial x} = f_x = \left(\frac{\partial f}{\partial x}\right)_y = \text{rate of change of } f \text{ with respect to } x \text{ with } y \text{ held constant.}$$

Example 1.1.1

Find the partial derivatives of

$$f = y^3\, e^{2x} + 5\, \sin(xy).$$

Solution.

♦

1.2 HIGHER PARTIAL DERIVATIVES

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \qquad \text{etc} \ldots$$

Example 1.2.1

Consider the Gas Law:

$$P(V,T) = \frac{nRT}{V}$$

where P = pressure, V = volume, T = temperature, and n and R are constants. Find all first and second partial derivatives. (Note that T and V are ***independent***).

Solution.

♦

Note:

$$\frac{\partial^2 P}{\partial T \partial V} = \frac{\partial^2 P}{\partial V \partial T}$$

This is called *equality of mixed partial derivatives*. This happens if and only if _______________

____

1.3 TAYLOR SERIES in 2D

$$f(x+\Delta x, y+\Delta y) = f(x, y+\Delta y) \ + \ \Delta x \frac{\partial f}{\partial x}(x, y+\Delta y) + \text{ higher order terms in } \Delta x$$

$$= f(x,y) + \Delta y \, \frac{\partial f}{\partial y}(x,y) + \cdots + \Delta x \left[\frac{\partial f}{\partial x}(x,y) + \dots \right]$$

$$\boxed{f(x+\Delta x, y+\Delta y) = f(x,y) + \Delta x \, \frac{\partial f}{\partial x}(x,y) + \Delta y \, \frac{\partial f}{\partial y}(x,y) + \text{higher order terms}}$$

Example 1.3.1

Find the 2D Taylor series up to the first order terms of the ideal gas law from example 1.2.1.

Solution.

♦

If ΔV and ΔT are small we therefore have the so-called *linear approximation*

$$P(V+\Delta V, T+\Delta T) \approx \frac{nRT}{V} - \frac{nRT}{V^2}\,\Delta V + \frac{nR}{V}\,\Delta T\ .$$

1.4 CHAIN RULE

Let $f(x,y) =$ temperature. Move along the path given by $x = x(t), y = y(t)$. What is $\frac{\partial f}{\partial t}$ along path?

The 2D Taylor series gives

$$\Delta f = f(x+\Delta x, y+\Delta y) - f(x,y)$$

$$= \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \text{higher order terms}$$

But

$$\Delta x \;=\; x(t+\Delta t) - x(t) \;=\; \frac{dx}{dt}\Delta t + \text{higher order terms}$$

and similarly for Δy, so

$$\Delta f \;=\; \left(\frac{\partial f}{\partial x}\frac{dx}{dt} \;+\; \frac{\partial f}{\partial y}\frac{dy}{dt}\right)\,\Delta t + \text{higher order terms}$$

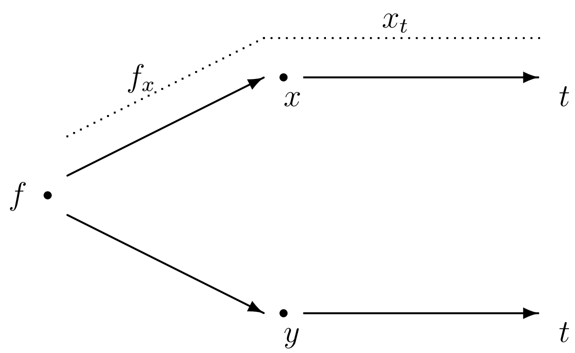
Therefore, since

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t}\ ,$$

we have the **2D chain rule**

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\ .}$$

The chain rule can be visualised by a *dependence tree*. The form of the tree depends on the form of the function. In the above case, it looks like



Example 1.4.1

Find $\frac{df}{dt}$ when

$$f = x^2 + y^2 \quad , \quad x = 3t \quad , \quad y = t^2\ .$$

Solution.

$\blacklozenge$

Example 1.4.2

A ship is sailing due East. Let t = time, x = distance East, s = ship speed, and s and t be independent. Furthermore, let the temperature measured by the ship at (x, t) be

$$f = f(x, t) = 20 + 10^{-3} x \sqrt{t} \ ,$$

and the ship's distance East be given by $x = st$. Find the time rate of change of temperature if the ship's speed is fixed.

Solution.

♦

Note that we can interpret the formula as follows.

$$\left(\frac{\partial f}{\partial t}\right)_s = \left(\frac{\partial f}{\partial x}\right)_t \left(\frac{\partial x}{\partial t}\right)_s + \left(\frac{\partial f}{\partial t}\right)_x$$

Total rate		rate of		speed		rate of change
of change	=	change with	×	going	+	of temp with
of temperature		distance		east		time at ship's
		east				current position

1.5 IMPLICIT PARTIAL DIFFERENTIATION

The van der Waals modification of the perfect gas law $PV = nRT$ is

$$\left(P + \frac{a}{v^2}\right)(v - b) = RT \ , \tag{1}$$

where v = volume per mole of gas ($= V/n$) and a and b are constants. We cannot find

$$\left(\frac{\partial v}{\partial T}\right)_P$$

explicitly (why not?). But we can do so, *implicitly*, by taking $\frac{\partial}{\partial T}$ of both sides of (1) (holding P fixed and assuming that P and v are ______________), to get

$$\frac{-2a}{v^3}(v - b)\frac{\partial v}{\partial T} + \left(P + \frac{a}{v^2}\right)\frac{\partial v}{\partial T} = R \ .$$

This expression is linear in $\frac{\partial v}{\partial T}$. Solve for $\frac{\partial v}{\partial T}$:

$$\left(\frac{\partial v}{\partial T}\right)_P = \frac{R}{P + \frac{a}{v^2} - \frac{2a}{v^3}(v - b)} \ ,$$

provided

$$P + \frac{a}{v^2} - \frac{2a}{v^3}(v - b) \neq 0.$$

Example 1.5.1

For the functions

$$F = r\, e^{r\theta} \tag{2}$$

$$H = r + \theta \sin(r) \tag{3}$$

find

$$\left(\frac{\partial F}{\partial H}\right)_\theta$$

and list the independent variables.

Solution.

♦

Example 1.5.2

The variables x, y, and z are linked via

$$F = F(x, y, z) = 0 \tag{4}$$

where F is a smooth function of x, y, and z. The function F is sufficiently complicated that $F(x, y, z) = 0$ can not be solved analytically for z in terms of x and y. Find

$$\left(\frac{\partial z}{\partial x}\right)_y$$

in terms of the partial derivatives of F. List the independent variables.

Solution.

♦

1.6 GRADIENT

In 2D the *gradient* ∇f of a function is

$$\nabla f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} = \begin{pmatrix} f_x \\ f_y \end{pmatrix} \ ,$$

while in 3D it is

$$\nabla f = f_x\boldsymbol{i} + f_y\boldsymbol{j} + f_z\boldsymbol{k} = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} \ ,$$

where $\boldsymbol{i} = (1,0,0)^T$ etc., as usual.

The symbol ∇ (commonly called 'del' or 'nabla') is

$$\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} \qquad (2D)$$
$$\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z} \qquad (3D)$$

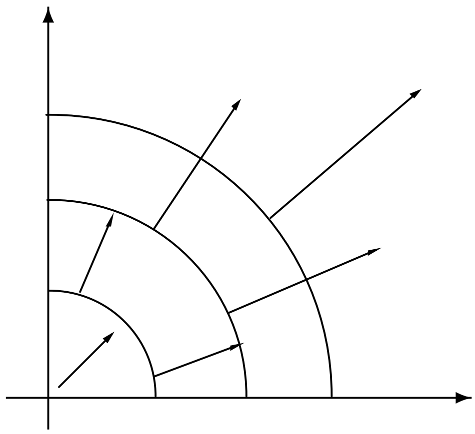
So for example, if

$$f = x^2 + y^2 \ ,$$

then

$$\nabla f = \begin{pmatrix} 2x \\ 2y \end{pmatrix} \qquad \text{and} \qquad \nabla f \,|_{(3,1)} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \ .$$

The latter is the gradient at the point $(3,1)$. We can represent the general cases in the upper right quadrant on the following diagram.



1.7 DIRECTIONAL DERIVATIVE

Consider $f(x,y)$ where

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t \begin{pmatrix} u \\ w \end{pmatrix}, \qquad \begin{pmatrix} u \\ w \end{pmatrix} = \boldsymbol{v} \quad \text{is a unit vector.}$$

Then

$$\frac{df}{dt} =$$

$$=$$

$$= \nabla f \cdot \boldsymbol{v} \ .$$

So

$$\boxed{\left(\frac{df}{dt}\right)\Bigg|_{(x_0,y_0)} = \nabla f \Bigg|_{(x_0,y_0)} \cdot \begin{pmatrix} u \\ w \end{pmatrix}}$$

This is called the *directional derivative of f in the direction $\boldsymbol{v}$*, and is often written as $D_{\boldsymbol{v}} f$ and $D_{\boldsymbol{v}} f\,\big|_{(x_0,y_0)}$. It represents the derivative of f along direction $(u,w)^T$ at the point $(x_0,y_0)^T$.

Example 1.7.1

Find the directional derivative of

$$f = x^2 - y^2$$

at $(1, 3)$ along

$$\boldsymbol{v} = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} .$$

Draw a sketch.

Solution.

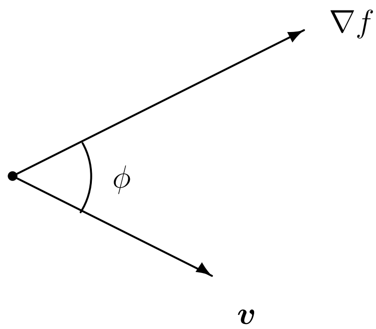
♦

1.8 STEEPEST ASCENT/DESCENT

Recalling the properties of the scalar product, we write

$$D_{\boldsymbol{v}} f \;=\; \nabla f \cdot \boldsymbol{v} \;=\; \|\nabla f\|\,\|\boldsymbol{v}\| \cos\phi$$

where ϕ is the angle between $\boldsymbol{v}$ and ∇f:



But $\|\boldsymbol{v}\| = 1$, (because ________________________), so

$$-\|\nabla f\| \leqslant \nabla f \cdot \boldsymbol{v} \leqslant \|\nabla f\| \; .$$

In particular, $\nabla f \cdot \boldsymbol{v} = \|\nabla f\|$ when ____

$$\boxed{\Longrightarrow\ \nabla f \text{ is the direction of steepest } \underline{\text{as}}\text{cent}}$$

Now $\phi =$ _____ gives $\cos\phi = -1$ and $\nabla f \cdot \boldsymbol{v} = -\|\nabla f\|$.

$$\boxed{\Longrightarrow\ -\nabla f \text{ is the direction of steepest } \underline{\text{de}}\text{scent.}}$$

Example 1.8.1

The temperature T of a plate is

$$T = 20 + 5\sin(x)\cos(x - y)\ .$$

In which direction should an ant at $(0, \pi)$ move in order to cool off as quickly as possible?

Solution.

♦

1.9 PERPENDICULAR TO LEVEL SURFACES

If $\nabla f \neq \mathbf{0}$ then

$$\nabla f \cdot \boldsymbol{v} = 0 \iff \nabla f \text{ and } \boldsymbol{v} \text{ are perpendicular.}$$

(where $\boldsymbol{v}$ is a unit vector).

That is, ____

In particular, ____

Hence $\boxed{\nabla f \text{ is perpendicular to the level curves/surfaces of } f.}$

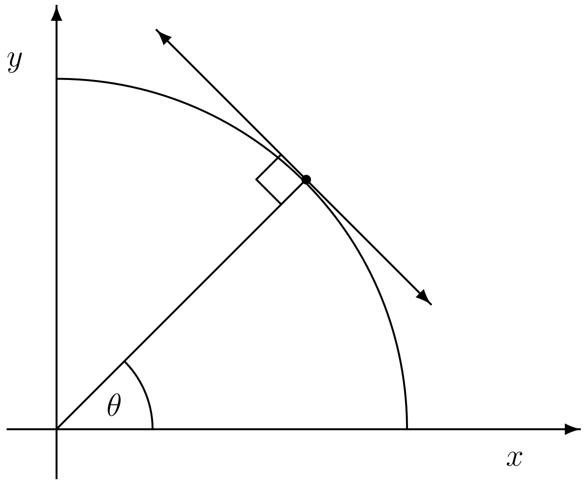
For example, for the function

$$f = x^2 + y^2 \ ,$$

the level curves are circles centered on the origin, and

$$\nabla f = \begin{pmatrix} 2x \\ 2y \end{pmatrix} \ .$$

We can draw a sketch:



It is a good exercise to use this sketch to show that at the point $(x, y) = (\cos\theta, \sin\theta)$, the tangents to the level curve and the gradient of f at that point are indeed perpendicular.

1.10 TANGENT PLANE

The equation of a plane through the point (x_0, y_0, z_0) with a normal vector $\boldsymbol{n}$ orthogonal to the plane is

$$\boldsymbol{n} \bullet \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} = 0.$$

Example 1.10.1

What is the tangent plane to the surface S given by

$$z = 4 - x^2 - y^2$$

at the point $(x_0, y_0, z_0) = (1, 1, 2)$?

Solution.

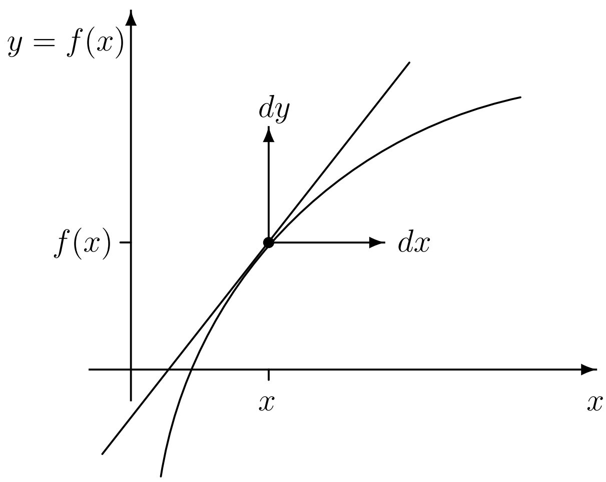
♦

1.11 DIFFERENTIALS

Take $y = f(x)$. Fix x and treat dx as an independent variable. Define dy via

$$dy = f'(x)dx. \tag{5}$$

We call dy the *differential* (in 1D). Expression (5) is the equation of the tangent line at $(x, f(x))$ with dx and dy as variables and the origin for dx, dy located at $(x, f(x))$:



Now the differential in 2D is

$$\boxed{dz = f_x dx + f_y dy\ ,}$$

which is the equation of the tangent plane at $(x, y, f(x, y))$. So for example when

$$z = f(x, y) = 1 - x^2 - y^2\ ,$$

the differential is

$$dz = -2xdx - 2ydy\ .$$

The 3D (and higher) version follows the pattern.

Example 1.11.1

Implicitly find the tangent plane of the sphere

$$f(x, y, z) = x^2 + y^2 + z^2 - 1 = 0.$$

Solution.

♦

1.12 TOTAL DIFFERENTIAL

If $f = f(x, y)$ then

$$\boxed{df \;=\; \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy}$$

is called the *total differential* of f. The 3D version follows similarly. It is easier to remember

$$\boxed{df = \nabla f \cdot \boldsymbol{dr}} \quad \text{where} \quad \boldsymbol{dr} = \begin{pmatrix} dx \\ dy \end{pmatrix},$$

in 2D, and similarly in 3D.

Example 1.12.1

Find the total differential of

$$f = x^2 + 4y^2 - 3xz\ .$$

Solution.

♦

1.13 THE JACOBIAN

Let $f = f(x, y, z)$ and $g = g(x, y, z)$. The chain rule gives

$$df = f_x\,dx + f_y\,dy + f_z\,dz$$

and

$$dg = g_x\,dx + g_y\,dy + g_z\,dz.$$

These two equations can be put in matrix form as

$$\begin{pmatrix} df \\ dg \end{pmatrix} = \begin{pmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} \nabla f^T \\ \nabla g^T \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}.$$

The matrix

$$J = \begin{pmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \end{pmatrix} = \begin{pmatrix} \nabla f^T \\ \nabla g^T \end{pmatrix}$$

is known as the Jacobian of the vector

$$\boldsymbol{F} = \begin{pmatrix} f(x,y,z) \\ g(x,y,z) \end{pmatrix}.$$

Using $\boldsymbol{F}$ the Jacobian can also be written as

$$J_{\boldsymbol{F}} = \begin{pmatrix} \frac{\partial}{\partial x}\boldsymbol{F} & \frac{\partial}{\partial y}\boldsymbol{F} & \frac{\partial}{\partial z}\boldsymbol{F} \end{pmatrix}$$

where the subscript on $J_{\boldsymbol{F}}$ indicates that it is the Jacobian of the vector $\boldsymbol{F}$.

Example 1.13.1

If

$$\boldsymbol{F} = \begin{pmatrix} f(x,y,z) \\ g(x,y,z) \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{r} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix},$$

express $d\boldsymbol{F}$ in terms of dt using Jacobians.

Solution.

♦

1.14 EXACT AND INEXACT DIFFERENTIALS

A differential

$$P(x, y)\, dx + Q(x, y)\, dy$$

is called *exact* if and only if there is a function $f(x, y)$ such that

$$df = f_x dx + f_y dy = P dx + Q dy \tag{6}$$

i.e. a differential is exact if and only if it is equal to the total differential of a function (called f here). Otherwise $P dx + Q dy$ is *inexact*. Combining (6) and equality of mixed partial derivatives gives a quick test for exactness:

$$\boxed{P dx + Q dy \text{ EXACT} \iff P_y = Q_x}$$

Test your understanding of this derivation by showing that in 3D with $P dx + Q dy + R dz$, exactness requires all of the following to hold: $P_y = Q_x$; $P_z = R_x$; and $Q_z = R_y$.

Example 1.14.1

Determine whether the following differentials are exact:

$$\text{(a)} \quad Pdx + Qdy = 2(x-y)dx - \left(2x + \frac{1}{y}\right) dy \ ;$$

$$\text{(b)} \quad Pdx + Qdy = x^2 dx + xdy \ .$$

Solution.

♦

Example 1.14.2

Find a function f whose total differential df is equal to the exact differential in example (1.14.1).

Solution.

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1.15 (ELEMENT OF) ARC LENGTH

On a curve $x = h(t)$ and $y = g(t)$, Pythagoras tells us that $ds^2 = dx^2 + dy^2$. Hence

$$ds^2 = \Big[(h')^2 + (g')^2\Big](dt)^2 \quad \text{or} \quad \boxed{ds = \pm\sqrt{(g')^2 + (h')^2}dt}\ \text{, or even}$$

$$\boxed{ds = \pm\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt}\ .$$

The sign is chosen to make the length ds positive. Sketch:

1.16 LINE INTEGRALS

Standard integral: area = sum of areas of rectangles (in limit). Sketch:

Area of rectangle at t is $f(t) \times dt$ = height $\times$ width. Sum these areas and take the limit as $dt \to 0$, to get

$$\text{area} = \int f(t)dt \ .$$

Now consider a curve C given by $x = x(t)$ and $y = y(t)$. This is called a parametric form of the curve, because the x and y coordinates are given in terms of a parameter, t. What is the area beneath C? Sketch:

$$\begin{aligned} \text{area} &= \text{(limit of) sum of areas of rectangles} \\ &= \int f\big(x(t), y(t)\big)\ ds \\ &= \text{(height)} \times \text{(width)} \end{aligned}$$

$$ds = \text{element of arc length}$$

$$\boxed{ds = \pm\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt\ .}$$

Note the $\pm$ here. The sign is chosen to make ds positive. So if dt is positive, we choose the $+$, while if dt is negative we choose the $-$. This is very important.

Example 1.16.1

The position of a particle at time t is $x = t^2$, $y = \frac{2}{3}t^3$. Find the distance the particle travels from $t = 0$ to $t = 1$.

Solution.

♦

Aside:

$$s = \int ds = \int \frac{ds}{dt} dt$$

$$\frac{ds}{dt} = \sqrt{\dot{x}^2 + \dot{y}^2} = \text{particle's speed}$$

In general, the basic solution strategy is as follows.

1. Write the curve in terms of a parameter (ie $x = x(t)$ and $y = y(t)$). (Called "parameterizing the curve" or "putting the curve in parametric form", etc.)
2. Write f, ds and limits in terms of the parameter t.
3. This yields an ordinary integral — do the integration to get the final answer.

Example 1.16.2

Find

$$I = \int_{C_0} xy \, ds \ ,$$

where C_0 is the arc of the unit circle from $\theta = 0$ to $\theta = \pi/2$.

Solution.

♦

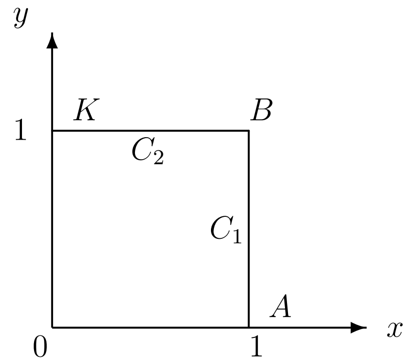
Suppose we are confronted with the integral along a non-smooth curve, as in the following example.

Example 1.16.3

Evaluate the integral I, where

$$\begin{aligned} I &= \int_A^K xy\, ds \\ &= \int_A^B xy\, ds + \int_B^K xy\, ds \ , \end{aligned}$$

where the curve of integration consists of the two straight line segments shown below:



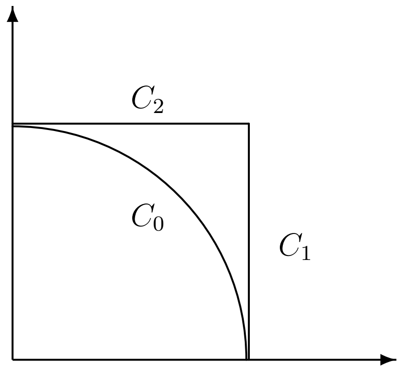
Splitting the integral up into a sum of integrals along"nice" (smooth) curves by making a finite number of breaks at "bad" (non-smooth) points is maybe a natural thing to do. But it is not obvious that the value of the integral is unchanged by doing this. The fact that the value is the same is a result of slightly deeper math than we have time for. But it is true!

For each of the smooth segments of the curve, we follow the basic solution strategy from the previous Lecture, then sum the results.

Solution.

♦

Important note:



$$\boxed{\int_{C_1+C_2} \neq \int_{C_0}} \qquad !!!$$

ie: line integrals are (usually) ***path*** dependent as well as __________________________.

Later on we will learn the circumstances under which they are path independent.

1.17 THINGS OTHER THAN ‘ds’

Can have line integrals with ‘dx’ or ‘dy’ rather than ‘ds’. Use exactly the same approach, (namely, parameterise and rewrite everything in terms of parameters), except:

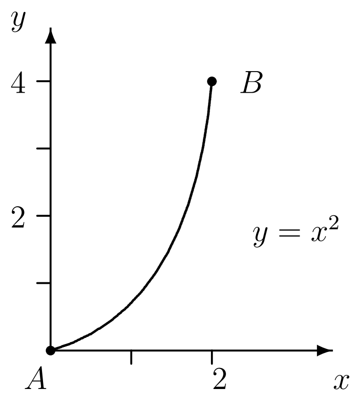
1. use $dx = \frac{dx}{dt}dt$ or $dy = \frac{dy}{dt}dt$;
2. dx or dy may be negative (unlike ds).

Example 1.17.1

Find

$$I = \int_A^B (x - y)dx$$

for A and B on the following curve:



Solution.

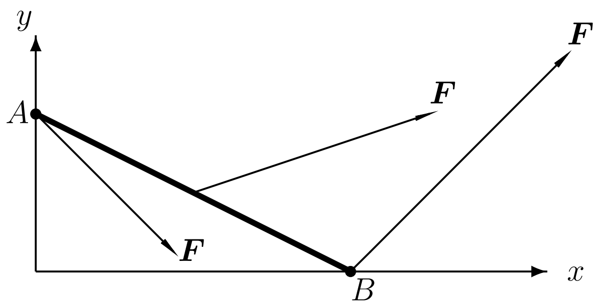
♦

Example 1.17.2

Suppose that a force $\boldsymbol{F}$ is applied along the curve C, which is the straight line segment from A to B, where

$$\boldsymbol{F} = \begin{pmatrix} x+y \\ x-y \end{pmatrix} ,$$

and $A = (0,1)$, $B = (2,0)$. The following diagram shows the curve C and the force vector $\boldsymbol{F}$ at $x = 0, 1,$ and 2.



Find

$$I = \int_C \boldsymbol{F} \cdot \boldsymbol{dr}$$

where

$$\boldsymbol{dr} = \begin{pmatrix} dx \\ dy \end{pmatrix} ,$$

and discuss the physical meaning of I.

Solution.

♦

Example 1.17.3

Imagine that we want to calculate the weight per unit length of a channel of water. "Per unit length" means that —assuming things are constant in the "length" direction — we can consider only a 2D problem, as follows. The sides of an elliptical channel are given by $x = 5\cos\theta$, $y = 3\sin\theta$ and $-\pi \leqslant \theta \leqslant 0$. Suppose that the channel is full of water of density ρ to height $y = 0$. Find the total downwards force of water on the ground.

Solution.

First a quick sketch:

♦

Note that the parameterisation of the ellipse was given to us here, but we won't always be so lucky. Be able to paramterise circles, ellipses, straight lines, polynomial curves (eg $y = ax^3 + bx$ — set $x = t$), and learn all the examples from class and tutorials.

1.18 LINE INTEGRALS OF EXACT DIFFERENTIALS

Suppose we want to find

$$I = \int_A^B 2xy\,dx + x^2\,dy$$

where $A = (1, 0)$ and $B = (2, 4)$.

Such a question immediately raises suspicions because we haven't been given the curve along which to integrate. Sallying bravely forth regardless, we spot something interesting: the integrand is an exact differential. Line integrals of exact differentials are easier to do: the methodology is given in the following solution, and briefly summarised at the end of it[1].

Consider a curve — any curve — $x = x(t)$ and $y = y(t)$, where

$$\Big(x(t_A), y(t_A)\Big) = (1, 0)$$

and

$$\Big(x(t_B), y(t_B)\Big) = (2, 4)$$

(The notation t_A just means "the value of t when we are at A".) The point here is that the curve is not specified, only two points on it. Then, following the standard procedure,

$$I = \int_{t_A}^{t_B} \left(2xy\frac{dx}{dt} + x^2\frac{dy}{dt}\right) dt$$

But now we exploit the fact that the integrand is an exact differential, by replacing it with something simpler. First, we can show that for $f = x^2y$ we have $df = 2xydx + x^2dy$. Next, we get

$$\frac{df}{dt} = 2xy\frac{dx}{dt} + x^2\frac{dy}{dt}$$

by the ______________________________. So

$$I = \int_{t_A}^{t_B} \frac{df}{dt}dt = f(t)\Big|_{t_A}^{t_B}$$

[1]If you do not spot that the integrand is exact and use the standard technique from lectures 5 and 6, you will still get the same answer! However, you may be *tested* on the following technique, so please learn it. Besides, something deep will happen with it as well.

by the ______________________________, where

$$f(t_B) = f(x(t_B), y(t_B)) = f(2,4) \ ,$$

and so

$$I = f(2,4) - f(1,0) = 16 \ .$$

The really important thing to note is that in calculating I we never used any information about the curve $x = x(t)$, $y = y(t)$! None was even available! This only happens for exact differentials. So *line integrals of exact differentials depend only on the endpoints, not the path — they are "path independent".* In summary, the technique for exact differentials is:

$$\boxed{P\ dx + Q\ dy \text{ exact} \quad \Longrightarrow \ I = \int_A^B P\ dx + Q\ dy = \int_{t_A}^{t_B} df = f(x_B, y_B) - f(x_A, y_A) \ .}$$

Example 1.18.1

Find

$$I = \int_{(0,0)}^{(2,3)} y dx + x dy$$

along the line $x = 2t$, $y = 3t$ for $0 \leqslant t \leqslant 1$ by:

1. using the techniques of previous lectures;

2. using the technique of this lecture.

Solution.

♦

We have already noted that $Pdx + Qdy$ exact $\implies \int_A^B Pdx + Qdy$ is path independent. Although we could never check in practice, it is possible to incontrovertibly prove mathematically that the converse holds, namely that if $\int_A^B Pdx + Qdy$ is path independent for all A and B then $Pdx + Qdy$ is exact.

Example 1.18.2

Use the notion of the integral of exact differentials to show that the work done by gravity on a mass m is mgh, where g is the acceleration due to gravity and h is the change in height of the mass.

Solution.

♦

Deep point: if $\exists\phi$ such that $\boldsymbol{F} = \nabla\phi$ for a force $\boldsymbol{F}$, we see that the work done by $\boldsymbol{F}$ is path independent. In this case, the work done along a closed path is zero. Such forces are called ____________________.

WARNING: some functions (including inverse trigonometric functions) are multivalued. When these functions occur, integrals of exact differentials can be path dependent, as in the following example.

Example 1.18.3

Discuss the value of

$$I = \int_C \frac{xdy - ydx}{x^2 + y^2} \ ,$$

where C is the unit circle in the xy-plane.

Solution.

♦

1.19 DIVERGENCE

We start with a definition:

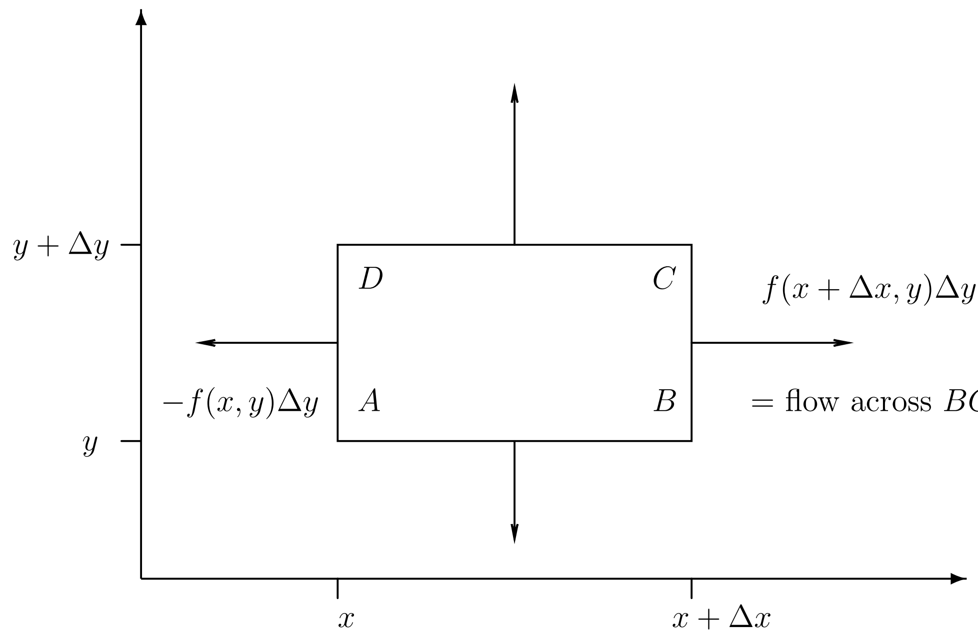
$$\nabla \cdot \boldsymbol{v} = \text{ divergence of vector field } \boldsymbol{v} \text{ where } \boldsymbol{v} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$$

$$\boxed{\nabla \cdot \boldsymbol{v} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} f \\ g \end{pmatrix} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}}$$

What does the divergence represent? To approach this question, first recall (Lecture 1):

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x,y)}{\Delta x}$$

Now let $\boldsymbol{v} = (f,g)^T$ be the velocity of a fluid in the xy-plane, and consider a small, fixed oblong $ABCD$ placed in the flow. The following diagram labels the oblong and the out-flow across the two sides perpendicular to the x axis.



So the net outflow across the two sides perpendicular to the x axis is

$$= \frac{\text{flow across } AD + \text{ flow across } BC}{\rule{6cm}{0.4pt}}$$

Since the oblong has area $\Delta x \Delta y$, the net outflow across edges AD and BC per unit area is

and so cancelling the Δy terms and taking the limit of smaller and smaller oblongs, we find that $\frac{\partial f}{\partial x}$ = net outflow across edges AD and BC per unit area. The $\frac{\partial g}{\partial y}$ term is the same for sides AB and CD. So $\boxed{\nabla \cdot \boldsymbol{v} = \text{net outflow of fluid (with velocity } \boldsymbol{v}\text{) per unit area}}$.

An incompressible fluid with no sources or sinks has $\nabla \cdot \boldsymbol{v} = 0$ where $\boldsymbol{v}$ = velocity. More generally, such a fluid has

$$\nabla \cdot \boldsymbol{v} > 0 \implies \text{net outflow (source) (near the point)}$$
$$\nabla \cdot \boldsymbol{v} < 0 \implies \text{net intflow (sink) (near the point)}$$

Example 1.19.1

The heat in an opaque 2D solid spreads by diffusion. If the temperature is T, then Fourier's law says that the heat flow is

$$\boldsymbol{q} = -k\nabla T \ ,$$

where k is a positive constant known as the thermal conductivity. Interpret Fourier's law in terms of the isotherms of the solid.

Solution.

♦

Example 1.19.2

Now suppose that the 2D solid has a temperature field $T = xy$. Discuss the temperature change at a point, and sketch the isotherms and heat flow lines.

Solution.

♦

Two things to note about this example.

1. Pointwise temperatures do not change even though heat is flowing.

2. The temperature field is time invariant; such a non-uniform time-invariant temperature field in a conducting medium must be maintained somehow, for example by heating and cooling at the boundaries in a specific way.

Example 1.19.3

Repeat the previous example, but now with a temperature field of $T = x^2 + y^2$ on the unit disc $x^2 + y^2 \leqslant 1$. There is no need to draw a sketch.

Solution.

♦

1.20 TWO BASIC RESULTS

Let

$$\boldsymbol{v} = \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} u(x,y) \\ w(x,y) \end{pmatrix}$$

be a 2D vector field, and $\phi = \phi(x, y)$ be a scalar field on the plane. The following hold[2]:

1.

$$\nabla \cdot (\phi \boldsymbol{v}) =$$

$$=$$

$$=$$

$$\boxed{\nabla \cdot (\phi \boldsymbol{v}) = \nabla \phi \cdot \boldsymbol{v} + \phi \nabla \cdot \boldsymbol{v}}$$

[2]They hold in higher dimensions too, with suitable changes in notation

2.

$$\nabla \cdot (\nabla \phi) =$$

$$=$$

$$=$$

This is called the ____________________ of ϕ and is often written $\nabla^2\phi$, or even $\triangle\phi$:

$$\boxed{\nabla \cdot (\nabla \phi) = \nabla^2 \phi = \triangle \phi = \phi_{xx} + \phi_{yy}}$$

1.21 CONSERVATION OF MASS

Let:

- ρ = density (mass per unit volume) of a chemical $= \rho(x, y, z, t)$;
- $\boldsymbol{f}$ = flow (mass velocity per unit volume) of the chemical $= \boldsymbol{f}(x, y, z, t)$;
- s = amount (mass per unit volume per unit time) of the chemical created per unit volume $= s(x, y, z, t)$.

Then:

- $\frac{\partial \rho}{\partial t}$ = time rate of change of density, ie, mass per unit volume per unit time;
- $\nabla \cdot \boldsymbol{f}$ = rate of outflow of chemical per unit volume, ie, mass per unit volume per unit time.

So conservation of mass is

$$\boxed{\frac{\partial \rho}{\partial t} = s - \nabla \cdot \boldsymbol{f}\ .}$$

Example 1.21.1

Derive the conservation of mass equation for a substance of density ρ being carried by a fluid flowing with velocity $\boldsymbol{v}$ with no sources or sinks.
Solution.

♦

Example 1.21.2

In the scenario of the previous example, suppose now that

$$\boldsymbol{v} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$$

and ρ is constant at time $t = 0$. Discuss the change in the density of the chemical at all locations at $t = 0$.
Solution.

♦

1.22 ROTATION (in 2D)

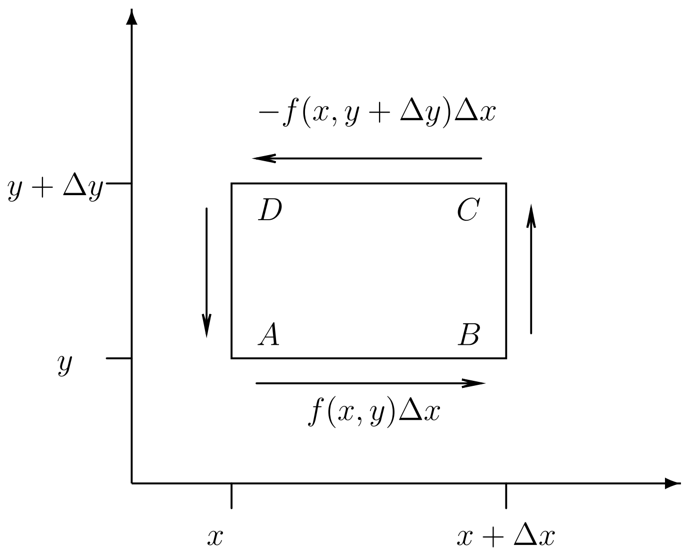
Starting with the 2D vector field

$$\boldsymbol{v} = \begin{pmatrix} f \\ g \end{pmatrix}$$

we define the *rotation of* $\boldsymbol{v}$ as

$$\boxed{\operatorname{rot}(\boldsymbol{v}) = \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}}\,.$$

The meaning of this formula is derived as follows. Let us suppose that $\boldsymbol{v}$ is a flow, and consider the flow along the edges of the small rectangle $ABCD$ placed in the flow:



So the total flow along edges AB and CD divided by the area of the rectangle is

Hence

$$\lim_{\Delta y \to 0} \frac{[f(x,y) - f(x, y+\Delta y)]\Delta x}{\Delta x \Delta y} = -\frac{\partial f}{\partial y} \quad \text{(by definition.)}$$

So we can say:

$$-\frac{\partial f}{\partial y} = \underline{\hspace{8cm}},$$

$$\frac{\partial g}{\partial x} = \underline{\hspace{8cm}},$$

$$\text{so} \quad \boxed{\text{rot}(v) = \text{net circulation of } \boldsymbol{v} \text{ per unit area.}}$$

Note that just as we have seen with the divergence, the "per unit area" is crucial.

1.23 CURL

The three-dimensional equivalent of rot is the curl which can be defined as

$$\boxed{\text{curl}(\boldsymbol{v}) = \nabla \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}},$$

where

$$\boldsymbol{v} = \begin{pmatrix} f(x,y,z) \\ g(x,y,z) \\ h(x,y,z) \end{pmatrix} \quad \text{and} \quad \boldsymbol{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{etc (the unit vectors).}$$

Expanding out the determinant gives

$$\boxed{\nabla \times \boldsymbol{v} = \begin{pmatrix} \hspace{4cm} \\ \\ \\ \end{pmatrix}}.$$

It is important to note that **the curl of a vector is a vector**.

An alternative notation is $\text{curl}(\boldsymbol{v}) = \nabla \wedge \boldsymbol{v}$.

Example 1.23.1

Find $\nabla \times \begin{pmatrix} z \\ x \\ y \end{pmatrix}$.

Solution.

♦

When the flow is 2D and independent of z then $\nabla \times \boldsymbol{v}$ becomes

$$\nabla \times \boldsymbol{v} = \begin{pmatrix} 0 - 0 \\ 0 - 0 \\ g_x - f_y \end{pmatrix} = (g_x - fy)\boldsymbol{k} = rot((f, g))\boldsymbol{k}$$

that is, a vector which points solely in the z-direction and whose magnitude is the rotation rot($\boldsymbol{v}$). This means that we can "forget" the definition of rot, and only remember that it is the 2D version of the curl. In fact, we get an extra piece of information for free: the curl of a 2D flow field is a vector perpendicular to the plane of the flow. In fact, we have the following important point in any number of dimensions: **the direction of $\nabla \times \boldsymbol{v}$ is the axis about which the circulation occurs and it's length is the size of the circulation (per unit area)**.

One not-so-obvious consequence of this is that if $\boldsymbol{v}$ is a fluid velocity and $\nabla \times \boldsymbol{v} = \mathbf{0}$, then each particle of fluid doesn't rotate *although the body of fluid can rotate as a whole.* Consider the following examples.

Example 1.23.2

Let

$$\boldsymbol{v}_1 = \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix} \quad , \quad \boldsymbol{v}_2 = \frac{1}{x^2+y^2} \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix} .$$

1. Show that $\nabla \times \boldsymbol{v}_1 \neq \mathbf{0}$ and sketch the vector field $\boldsymbol{v}_1$.

2. Show that $\nabla \times \boldsymbol{v}_2 = \mathbf{0}$ and sketch the vector field $\boldsymbol{v}_2$.

Solution.

♦

Example 1.23.3

There are many vector identities with curl; prove the following:

$$\nabla \times (\nabla f) = \mathbf{0} \ .$$

Solution.

♦

The converse of the above is also true: if $\nabla \times \boldsymbol{F} = \mathbf{0}$ then $\exists \phi$ such that $\boldsymbol{F} = \nabla \phi$. Consequence: if $\boldsymbol{F}$ is a force field for which $\text{curl}(\boldsymbol{F}) = \mathbf{0}$ then it is conservative and line integrals are path independent. It is also then true that $\nabla \times (\nabla F) = \mathbf{0}$.

We've now see three crucial definitions and interpretations involving the symbol ∇: grad (gradient), div (divergence), and curl. Grad acts on scalar fields, and gives you a ______________. Div acts on vector fields and gives you a ______________. Curl acts on vector fields and gives you a ______________. These three will appear very frequently over the next few years.

Example 1.23.4

A solid body rotates about the z axis with angular velocity ω radians per second. A particle of the solid body at the point (x, y, z) has velocity $\boldsymbol{v}$ given by

$$\boldsymbol{v} = \omega \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}.$$

Calculate the circulation of the velocity vector field using the curl. Perform a check by calculating the net circulation of $\boldsymbol{v}$ per unit area around a circle radius R centred on the origin and lying in the x, y plane.

Solution.

♦

1.24 LAGRANGE MULTIPLIERS (MAXIMA AND MINIMA)

In 2D a point (x_0, y_0) is called a *stationary point* of the function $f(x, y)$ if and only if $\nabla f|_{(x_0,y_0)} = \mathbf{0}$.

Why? First, if this is so then for any direction $\boldsymbol{u}$, the directional derivative $D_{\boldsymbol{u}} f = \nabla f \cdot \boldsymbol{u} = 0$ at (x_0, y_0):f is flat at (x_0, y_0). Conversely, if $\nabla f \neq \mathbf{0}$ then taking $\boldsymbol{u} = \nabla f / \|\nabla f\|$ gives $D_{\boldsymbol{u}} f > 0$ and $D_{-\boldsymbol{u}} f < 0$ — there is a direction in which f is decreasing and another in which it is increasing; f is not flat. Hence f is stationary ("flat") at (x_0, y_0) precisely when ("if and only if") the gradient of f at (x_0, y_0) is zero.

Some stationary points are *minimisers* (also called minima, singular: minimum). For example, the point $(x_0, y_0) = (0, 0)$ is a minimiser of the function $f = x^2 + y^2$. Stationary points also include maximisers and saddles. An example of a maximiser is $f = -(x^2 + y^2)$ at the point $(x_0, y_0) = (0, 0)$. Similarly $(x_0, y_0) = (0, 0)$ is a saddle point of $f = x^2 - y^2$.

1.25 WHEN A CONSTRAINT IS PRESENT

Often we seek maximizers (or minimizers) of functions when constraints are present. A simple example would be to find the highest point along a road over a hill. The road forms a constraint, and remaining on the road limits movement to two directions: either forwards or backwards along the road. If rather than looking for points at which the function is flat in all directions we instead look for points at which the function is flat in certain specified directions, then we are looking for so-called *critical points*[3].

Define a critical point on the constraint $c(x, y) = 0$ as a point at which $D_{\boldsymbol{u}} f = 0$ for

[3]There are several other uses of "critical point". For example, a point at which the gradient of a function is either zero (a stationary point) *or* undefined. However, we will in this course assume that all of our functions have well-behaved gradients, and so reserve the phrase "critical point" for constrained stationary points.

all directions $\boldsymbol{u}$ tangential to the constraint, the only directions in which we can move if we are constrained to move along c. So ∇f is a normal to the constraint (or is zero), by definition of the dot product. But assuming $\nabla c \neq \mathbf{0}$, ∇c is also a normal to the constraint, by the interpretation of grad which we have seen. So ∇f must be parallel to ∇c (or zero). In mathspeak: $\nabla f = \lambda \nabla c$ for some $\lambda \in \mathbb{R}$. Here, λ is known as a *Lagrange multiplier*.

Example 1.25.1

What are the critical points of $f = y^2 - x^2$ on the curve $c(x, y) = x^2 - y - 1 = 0$? Draw a sketch showing the contour lines of f, the curve c, and the critical points.

Solution.

Example 1.25.2

Find the critical points of $f = x^2 + 3y^2$ subject to the constraint $c = x + 2y + 1 \geqslant 0$

Solution.

This is a so-called *inequality constraint*, and must be tackled in two parts:

1. Find all stationary points ($\nabla f = 0$) which satisfy $c \geqslant 0$.

2. Find all the critical points of f subject to $c = 0$ (note the equals sign).

♦

Example 1.25.3

Find the critical points of $f = xy$ subject to the constraint $c = x^2 + y^2 - 2 = 0$

Solution.

♦

1.26 THE LAGRANGIAN

Suppose we want the critical points of the function $f(x, y)$ subject to the constraint $c(x, y) = 0$. If we look at the Lagrangian function

$$L(x, y, \lambda) = f(x, y) - \lambda c(x, y),$$

then any stationary point of L satisfies $\nabla f = \lambda \nabla c$ and $c = 0$.

Example 1.26.1

Use the Lagrangian function to find the critical points of $f = x + z^2$ subject to the constraint $c = x^2 + y^2 - 1 = 0$

Solution.

$\blacklozenge$

1.27 Two examples to try later

Example 1.27.1

Find the critical point of $f = xy$ on $x + y - 4 = 0$.

Example 1.27.2

$$\text{Maximise } \; f(x, y) = xy$$

$$\text{subject to } \; g(x, y) = 4x^2 + 8y^2 - 16 = 0 \; .$$