Chapter 3.

Growth of Functions

Growth of functions

- A way to describe behavior of functions *in the limit*
  -- ***asymptotic*** efficiency.
- *Growth* of functions.
- Focus on what's important by abstracting away low-order terms and constant factors.
- How to indicate running times of algorithms?
- A way to compare "sizes" of functions:

$O \approx \leq$ $\qquad$ $\Omega \approx \geq$

$\Theta \approx =$

$o \approx <$ $\qquad$ $\omega \approx >$

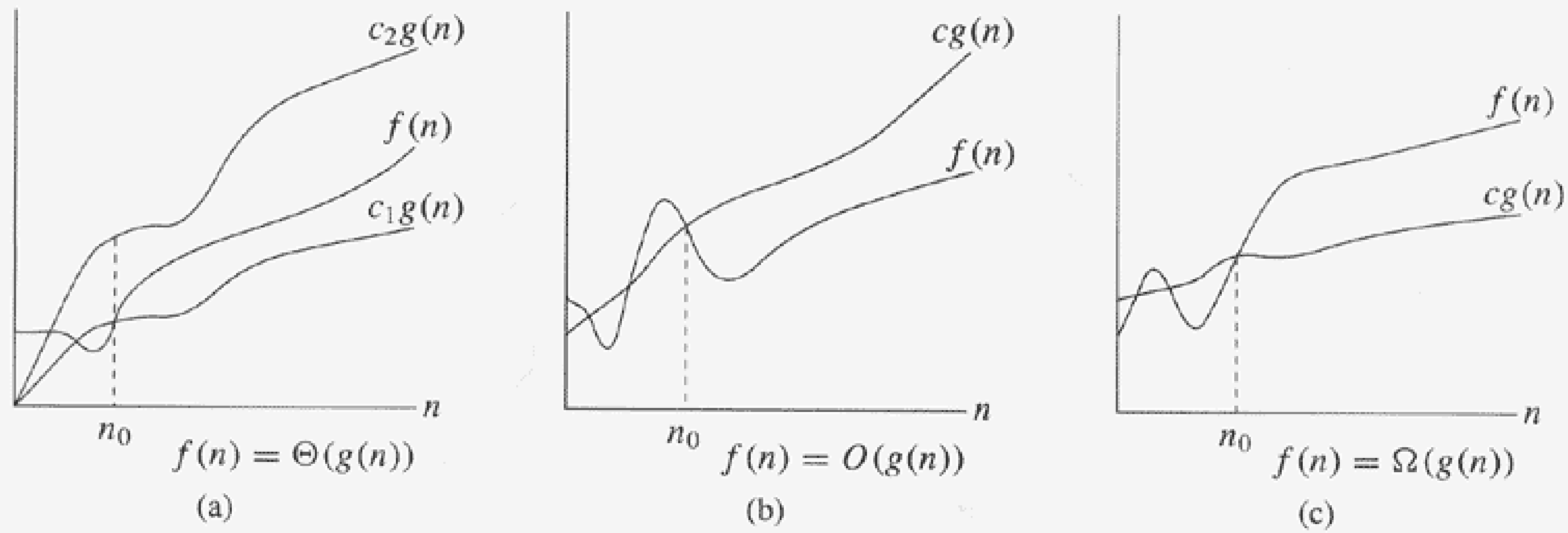

Figure 3.1 Graphic examples of the Θ, O, and Ω notations. In each part, the value of n_0 shown is the minimum possible value; any greater value would also work. **(a)** Θ-notation bounds a function to within constant factors. We write $f(n) = \Theta(g(n))$ if there exist positive constants n_0, c_1, and c_2 such that to the right of n_0, the value of $f(n)$ always lies between $c_1 g(n)$ and $c_2 g(n)$ inclusive. **(b)** O-notation gives an upper bound for a function to within a constant factor. We write $f(n) = O(g(n))$ if there are positive constants n_0 and c such that to the right of n_0, the value of $f(n)$ always lies on or below $cg(n)$. **(c)** Ω-notation gives a lower bound for a function to within a constant factor. We write $f(n) = \Omega(g(n))$ if there are positive constants n_0 and c such that to the right of n_0, the value of $f(n)$ always lies on or above $cg(n)$.

Asymptotic Notation

O-notation

$O(g(n)) = \{f(n)$: there exist positive constants c and n_0 such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0\}$.

$g(n)$ is an ***asymptotic upper bound*** for $f(n)$.

Example:

$2n^2 = O(n^3)$, with $c=1$ and $n_0=2$.

also, $2n^2 = O(n^2)$, with $c=2$ and $n_0=0$.

Examples of functions in $O(n^2)$:

n^2, $n^2 + n$, $n^2 + 1000n$, $1000n^2 + 1000n$

Also,

n, $n/1000$, $n^{1.9999}$, $n^2/\lg \lg \lg n$

.. continued

Ω-notation

$\Omega(g(n)) = \{f(n)$: there exist positive constants c and n_0 such that $0 \le c\,g(n) \le f(n)$ for all $n \ge n_0\}$.

$g(n)$ is an ***asymptotic lower bound*** for $f(n)$.

Example:

$\sqrt{n} = \Omega(\lg n)$, with $c=1$ and $n_0=16$.

Examples of functions in $\Omega(n^2)$:

n^2, $n^2 + n$, $n^2 - n$, $1000n^2 + 1000n$, $1000n^2 - 1000n$,

Also,

n^3, $n^{2.0000}$, $n^2 \lg\lg\lg n$, 2^{2^n}

.. continued

Θ-notation

$\Theta(g(n)) = \{f(n)$: there exist positive constants c_1, c_2 and n_0 such that $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_0\}$.

$g(n)$ is an ***asymptotic tight bound*** for $f(n)$.

Example:

$n^2/2 - 2n = \Theta(n^2)$, with $c_1=1/4$, $c_2=1/2$ and $n_0=8$.

Also, $2n^2 = \Theta(n^2)$, with $c_1=1$, $c_2=3$ (or $c_1=c_2=2$) and $n_0=0$.

Theorem:

$f(n) = \Theta(g(n))$ iff $f = O(g(n))$ and $f = \Omega(g(n))$.

.. continued

o-notation

$o(g(n)) = \{f(n)$: for all constants $c > 0$, there exist a constant $n_0 > 0$ such that $0 \leq f(n) < cg(n)$ for all $n \geq n_0\}$.

$g(n)$ is an ***asymptotic strict upper bound*** for $f(n)$.

Another view: $\lim_{n\to\infty} \frac{f(x)}{g(x)} = 0$

Example:

$n^{1.9999} = o(n^2)$, $n^2 / \lg n = o(n^2)$,

$n^2 \neq o(n^2)$ (just like $2 \nless 2$), $n^2/1000 \neq o(n^2)$

ω-notation

$\omega(g(n)) = \{f(n)$: for all constants $c > 0$, there exist a constant $n_0 > 0$ such that $0 \leq cg(n) < f(n)$ for all $n \geq n_0\}$.

$g(n)$ is an ***asymptotic strict lower bound*** for $f(n)$.

Another view: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty.$

Example:

$n^{2.0001} = \omega(n^2)$, $n^2 \lg n = \omega(n^2)$, $n^2 \neq \omega(n^2)$

Comparisons of Functions

Related Properties:

Transitivity:

$f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n)) \Rightarrow \boldsymbol{f(n) = \Theta(h(n))}$.

Same for O, Ω, o, and ω.

Reflexivity:

$\boldsymbol{f(n) = \Theta(f(n))}$.

Same for O and Ω.

Symmetry:

$\boldsymbol{f(n) = \Theta(g(n))}$ if and only if $\boldsymbol{g(n) = \Theta(f(n))}$.

Transpose symmetry:

$\boldsymbol{f(n) = O(g(n))}$ if and only if $\boldsymbol{g(n) = \Omega(f(n))}$.

$\boldsymbol{f(n) = \omega(g(n))}$ if and only if $\boldsymbol{g(n) = \omega(f(n))}$.

Comparisons:

$f(n)$ is ***asymptotically smaller*** than $g(n)$ if $f(n) = o(g(n))$.

$f(n)$ is ***asymptotically larger*** than $g(n)$ if $f(n) = \omega(g(n))$.

Standard notations and common functions

Monotonicity:

$f(n)$ is ***monotonically increasing*** if $m \leq n \Rightarrow f(m) \leq f(n)$.

$f(n)$ is ***monotonically decreasing*** if $m \leq n \Rightarrow f(m) \geq f(n)$.

$f(n)$ is ***strictly increasing*** if $m < n \Rightarrow f(m) < f(n)$.

$f(n)$ is ***strictly decreasing*** if $m < n \Rightarrow f(m) > f(n)$.

Floor and Ceilings: $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x+1$

Modular arithmetic: $a \bmod n = a - \lfloor a/n \rfloor n$

Polynomials: $p(n) = \sum_{i=0}^{d} a_i n^i$

Exponentials:

Logarithms:

Factorials: -- refer to the textbook (p52-55).