MACM 316 – Computing Assignment 6

- Read the *Guidelines for Assignments* first.
- Write a one-page PDF report summarizing your finding. This report must also include all your figures.
- Submit the one-page PDF report using Crowdmark and add your Matlab code to it (as extra pages). You must use the link sent by Crowdmark to your SFU email address. Follow the instructions on the Crowdmark to upload your file. If the report is hand-written then use the CamScanner app with your cellphone to scan the report and every page of the code. You will lose marks for poor quality pictures.
- You must acknowledge any collaborations/assistance from fellow students, discussion forums, TAs, instructors, etc.

Deblurring and least squares

In many applications, one deals with blurry images or signals. *Deblurring* is the process or restoring a sharper image or signal from its blurred version.

A simple mathematical model for blurring is the following. The image or signal is represented as a function $f : [0, 1] \to \mathbb{R}$, and the blurred version of $f(x)$ is the function $g(x)$, defined by

$$g(x) = \int_0^1 k(x - y) f(y)\,\mathrm{d}y, \qquad 0 \le x \le 1. \tag{1}$$

The function $k(x)$ is known as a *blurring kernel*. In this assignment, we will take $k(x)$ to be a *Gaussian blur*:

$$k(x) = \frac{1}{\gamma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\gamma^2}\right).$$

Here $\gamma > 0$ is a parameter that determines the amount of blurring. The figure on the next page illustrates this type of blurring when applied to the function

$$f(x) = \begin{cases} 0.75 & 0.1 \le x \le 0.2 \\ 16(x - 0.75)^2 - 1 & 0.5 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

The deblurring problem is to recover $f(x)$ from its blurred version $g(x)$. We shall attempt to solve this by first converting it to a linear algebra problem. Consider a grid of n equally-spaced points $\{i/n\}_{i=1}^{n}$ in $[0, 1]$. You may take $n = 1024$ for the purposes of this assignment. Let $\boldsymbol{g} \in \mathbb{R}^n$ be the vector of values of g on the grid, and $\boldsymbol{f}$ be the unknown vector of values of f. Let $\boldsymbol{A} = (a_{ij})$ be the matrix with entries

$$a_{ij} = \frac{1}{n} k(i/n - j/n) = \frac{1}{n} \frac{1}{\gamma\sqrt{2\pi}} \exp\left(-\frac{(i - j)^2}{2n^2\gamma^2}\right), \quad i, j = 1, \dots, n.$$

Then, replacing the integral in (1) by a sum (see Part 6 of the course), we arrive at the $n \times n$ linear system

$$\boldsymbol{A}\boldsymbol{f} = \boldsymbol{g}. \tag{2}$$

Hence, deblurring involves recovering $\boldsymbol{f}$ by solving this linear system.

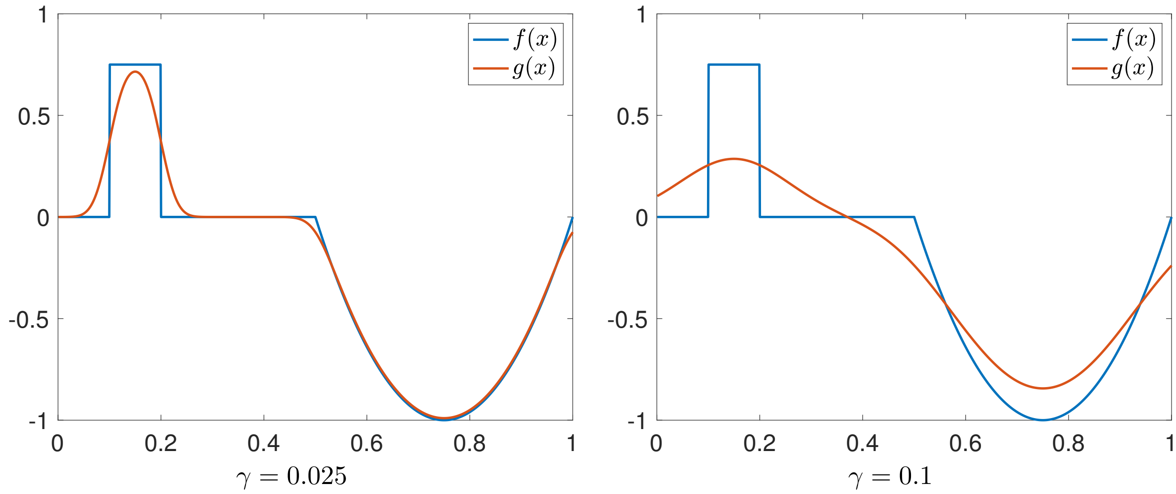

Figure 1: The function $f(x)$ and its blurred version $g(x)$.

(a) Set $\gamma = 0.1$. Write a code that (i) creates the matrix $\boldsymbol{A}$, (ii) creates the true vector $\boldsymbol{f}$, (iii) generates the right-hand side $\boldsymbol{g} = \boldsymbol{A}\boldsymbol{f}$, and (iv) computes the numerical solution $\tilde{\boldsymbol{f}}$ of the linear system using backslash. Plot the recovered function $\tilde{\boldsymbol{f}}$.

(b) You will hopefully have seen that $\tilde{\boldsymbol{f}}$ is a truly terrible reconstruction of $\boldsymbol{f}$! Compute the condition number of $\boldsymbol{A}$ to see why. Deblurring is a famously ill-conditioned problem. Next, explore how the condition number of $\boldsymbol{A}$ changes with γ. Report several values values of the condition number for several different values of γ. When is the condition number minimized? Briefly explain why this is the case.

(c) Fortunately, it is possible to deblur much more effectively than simply by solving (2). One way to do this is Tikhonov regularization. In Tikhonov regularization we replace (2) by the problem:

$$\text{find } \boldsymbol{f} \in \mathbb{R}^n \text{ that minimizes } \|\boldsymbol{A}\boldsymbol{f} - \boldsymbol{g}\|^2 + \alpha^2 \|\boldsymbol{f}\|^2.$$

Here $\alpha > 0$ is a parameter. This turns out to be equivalent to the least-squares problem

$$\text{find } \boldsymbol{f} \in \mathbb{R}^n \text{ that minimizes } \left\| \begin{bmatrix} \boldsymbol{A} \\ \alpha \boldsymbol{I} \end{bmatrix} \boldsymbol{f} - \begin{bmatrix} \boldsymbol{g} \\ \boldsymbol{0} \end{bmatrix} \right\|^2, \tag{3}$$

where $\boldsymbol{I} \in \mathbb{R}^{n \times n}$ is the identity matrix and $\boldsymbol{0} \in \mathbb{R}^n$ is the zero vector. You may wish to think about why this is the case.

Using $\gamma = 0.1$, write a code that computes the solution of (3) (recall from Part 5 of the lecture notes that backslash can be used to solve algebraic least-squares problems). Using the values $\alpha = 0.1$, $\alpha = 0.7$ and $\alpha = 1.5$, plot the true $\boldsymbol{f}$ and the reconstructions obtained from (3) in a single figure. Discuss how changing α affects the reconstruction, and briefly explain why.

(d) Finally, manually adjust α until you obtain what you think is the best deblurring. Include a figure showing the true $\boldsymbol{f}$ and the deblurred version and report the value of α you chose, along with a brief justification.