# MACM 316 – Computing Assignment 7

- Read the *Guidelines for Assignments* first.
- Write a one-page PDF report summarizing your finding. This report must also include all your figures.
- Submit the one-page PDF report using Crowdmark and add your Matlab code to it (as extra pages). You must use the link sent by Crowdmark to your SFU email address. Follow the instructions on the Crowdmark to upload your file. If the report is hand-written then use the CamScanner app with your cellphone to scan the report and every page of the code. You will lose marks for poor quality pictures.
- You must acknowledge any collaborations/assistance from fellow students, discussion forums, TAs, instructors, etc.

## Numerical integration of an unpleasant function

In this assignment you will be computing the integral

$$I = \int_0^1 \frac{\sin(x^{-1}\log(x))}{x}\,\mathrm{d}x. \tag{1}$$

(note: log is the natural logarithm, i.e. base e). It is known that $I \approx -0.46$. This is a challenging integral to compute, however, because the integrand $f(x) = x^{-1}\sin(x^{-1}\log(x))$ has infinitely-many oscillations in the interval $[0,1]$ and is also singular at $x = 0$. You may quickly want to check that any standard numerical quadrature (e.g. composite Trapezoidal, Simpson's, etc) gives you poor approximations to $I$.

To compute this integral, you will be using a subdivision scheme based on splitting $[0,1]$ into subintervals defined by the zeros of $f(x)$. Note that $f(x) = 0$ at the points

$$1 > a_1 > a_2 > a_3 > \ldots > 0,$$

where $a_i = \exp(-b_i)$ and $b_i$ is the unique solution to the equation

$$b\exp(b) - i\pi = 0, \quad 1 < b < \infty. \tag{2}$$

To compute $I$, you first need to find these roots. Recalling Part 3 of the course, this can be done in Matlab using the *fzero* command:

```
b = fzero(@(x) x*exp(x)-i*pi,0);
a(i) = exp(-b);
```

Given the values $a_1, a_2, \ldots, a_n$, we can now approximate $I$ as follows:

$$I \approx Q_n = \sum_{i=0}^{n} I_i, \qquad I_i = \int_{a_{i+1}}^{a_i} f(x)\,\mathrm{d}x, \qquad a_0 = 1.$$

Write a code that computes $Q_n$ by applying a standard numerical quadrature to each integral $I_i$. I recommend you use a built-in routine to evaluate each $I_i$, i.e. Matlab's `integral` command. List your results for $n = 50, 100, \ldots, 500$. You may find the `format long` command useful. Roughly, jow many digits of $I$ can you accurately compute?

You will hopefully have noticed that $Q_n$ is converging rather slowly to $I$. Fortunately, there's a way to get a faster converging approximation, known as Aitken's $\Delta^2$ Method (see Burden & Faires, Sec 2.5). Given the sequence $\{Q_n\}_{n=0}^{\infty}$ we define the new sequence $\{\hat{Q}_n\}_{n=0}^{\infty}$ by

$$\hat{Q}_n = Q_n - \frac{(Q_{n+1} - Q_n)^2}{Q_{n+2} - 2Q_{n+1} + Q_n}.$$

Compute this new sequence and use it to get as good an approximation to $I$ as you can (in reasonable computing time). Report your the values for $n = 50, 100, \ldots, 500$ as before. How many digits of $I$ can you accurately compute using this approach? Make sure to justify the number of digits you give.

Integrals such as (1) can arise in problems in quantum physics. More generally, one might consider an integral of the form

$$I(p) = \int_0^1 \frac{\sin(x^{-p} \log(x))}{x} \, \mathrm{d}x,$$

where $p \geq 1$ is a parameter. The goal is to understand how $I(p)$ varies with $p$. Modify your code from the previous part of the assignment to compute this integral for a range of values of $p$. Make sure to adapt the root-finding step (2) suitably to take into account this new integrand. Plot $I(p)$ versus $p$ using suitable axes and discuss their observed relationship.