REACHABILITY PROJECT LOG :)

1. Problem Description

Our goal is to learn *where the robot should stand* given a desired grasp.

1.1. **Problem inputs.** Formulating the problem, we are given as inputs:

- $x_e \in SE(3)$: desired end-effector pose in world coordinates
- $b \in SE(2)$: wheeled mobile base pose (x, y, γ)
- $q \in \mathbb{R}^n$: arm joint configuration.

1.2. **Problem outputs.** We want to find a range of feasible whole-body configurations that satisfy the following requirements:

(1) **IK feasibility:** forward kinematics$(b, q) = x_e$
(2) **Visibility/sensing constraints:** The end-effector should lie within the camera's field of view and range (and un-occluded by the robot itself), i.e. visible$(b, q, x_e) = 1$
(3) **Ball of feasible configurations:** We want a region in joint space around (b, q) that keeps the hand in approximately the right pose and satisfies the above constraints.

Our target is to model some distribution $p(b, q \mid x_e)$ subject to these constraints $\uparrow$, ideally with some notion of pose quality (visibility, manipulability, clearance, etc.).

2. Simplified problem

2.1. **Setup.** Simple robot is a round thing with a fixed "stick" attached to it: a disk on the floor with a rigid stick of fixed length L glued to it, and the stick rotates with the disk.

- Configuration: $Q = x, y, \theta$ $(\mathbb{R}^2 \times S^1)$
- Hand target: $H = h_x, h_y$ (point in $\mathbb{R}^2$)

In this setup, $H \in \mathbb{R}^2$ is the desired 2D point on the floor where we want the tip of the stick to be.

2.2. **Forward kinematics.**

$$\text{hand}(x, y, \theta) \coloneqq f(Q) = \begin{bmatrix} x \\ y \end{bmatrix} + L \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix}$$

So the IK constraint ("hand hits the target") is:

$$H = \begin{bmatrix} h_x \\ h_y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + L \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

Rearranging shows that as θ varies, the base center (x, y) traces out a circle of radius L around the target point H: the set of feasible base positions for the robot to reach H is a circle centered at H.

$$\text{Ground truth feasible set for a fixed } H \coloneqq \{Q : f(Q) = H\} = \{(h_x - L\cos\theta, h_y - L\sin\theta, \theta) : \theta \in [0, 2\pi)\}$$

2.3. **Approaches.** We try the following approaches to model $p(Q \mid H)$:

(1) *Nearest neighbors* (baseline):
  - Given a dataset $\mathcal{D} = \{(H_i, Q_i)\}_{i=1}^N$: for a query H',
    (a) Find indices of the k nearest H_i to H' (under $||H_i - H'||$)
    (b) Return their associated Q_i's (either all or sample one etc.)
  - kNN induces an empirical conditional: think about this more

$$\hat{p}_{\text{kNN}}(Q \mid H') = \sum_{j \in \mathcal{N}_k(H')} w_j(H')\delta(Q - Q_j) \qquad w_j \propto \exp(-||H_j - H'||^2/\sigma^2)$$

(2) *Conditional VAE:* Introduce latent $z \in \mathbb{R}^d$: $z \sim \mathcal{N}(0, I)$, $Q \sim p_\theta(Q \mid H, z)$

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- Train an encoder $q_\phi(z \mid Q, H)$ and decoder $p_\theta(Q \mid H, z)$ via ELBO loss:
$$\log p_\theta(Q \mid H) \geq \mathbb{E}_{z \sim q_\phi(z|Q,H)}[\log p_\theta(Q \mid H, z)] - \mathrm{KL}[q_\theta(z \mid Q, H) \,||\, \mathcal{N}(0, I)]$$
- Potential decoder choice: output $\mu_\theta(H, z)$ and diagonal $\Sigma_\theta(H, z)$; use Gaussian likelihood:
$$\log p_\theta(Q \mid H, z) = \log \mathcal{N}(Q; \mu_\theta(H, z), \Sigma_\theta(H, z))$$

Read into theory behind this way more

(3) *Invertible NN* (FrEIA): Read into theory behind this way more
- Invertibility requires equal dimensions $\rightarrow$ in the toy problem, $\dim(Q) = 3 \neq \dim(H) = 2$, so add a latent $z \in \mathbb{R}^1$ s.t. $(H, z) \in \mathbb{R}^3$
- Learn an invertible map: $F_\theta : Q \leftrightarrow (\hat{H}, z)$ where z is encouraged to be $\mathcal{N}(0, 1)$ and $\hat{H}$ matches H
- $\uparrow$ Training objectives:
  - Forward consistency: $\hat{H} \approx H \rightarrow \mathcal{L}_H = ||\hat{H} - H||^2$
  - Latent regularization: make $z \sim \mathcal{N}(0, 1)$
- At test time, the inverse we want is:
$$Q = F_\theta^{-1}(H, z), \quad z \sim \mathcal{N}(0, 1)$$

2.4. **Evaluation.** Assuming that the ground-truth $p^*(Q \mid H)$ is a uniform distribution over the feasible circle set, i.e.: $\theta \sim \mathrm{Unif}[0, 2\pi) \rightarrow x = h_x - L\cos\theta$, $y = h_y - L\sin\theta$:
- Accuracy: $e_{\mathrm{hand}}(Q, H) \coloneqq ||f(Q) - H||$
- Diversity/coverage: convert each sample to its implied angle on the circle:
$$\theta^{(s)}_{\mathrm{implied}} \coloneqq \mathrm{atan2}\left(h_y - y^{(s)}, h_x - x^{(s)}\right)$$
Check how well the empirical distribution over θ matches the target (here we assume $\theta \sim \mathrm{Unif}[0, 2\pi)$): for histogram $\hat{p}(\theta)$, we can compute:
  - KL divergence to uniform: $D_{\mathrm{KL}}(\hat{p}(\theta) \,||\, \mathrm{Unif})$
  - Max angle gap: sort angles $\theta^{(s)}$ around the circle, compute max gap $\Delta_{\max} = \max_i(\theta_{i+1} - \theta_i)$ Large $\Delta_{\max}$ = model is missing big arcs (collapse)
- "Uses latent" test (for cVAE/cINN): fix H, sample many z's, then measure output variance $\mathrm{Var}(Q \mid H)$. If the model ignores z, this variance will be small.
- Generalization in H: make a test set where H is (1) in-distribution, (2) near boundary, (3) out-of-distribution (slightly outside training workspace). Compare success and coverage!
- Inference speed per sample

2.5. **Extending to more general cases.** Ideally, would be nice if this toy problem gave us an idea of the advantages/disadvantages of different approaches in regards to the following challenges:
- *Mode collapse:* Test: fixed H, sample 1k solultions, compute $\Delta_{\max}$, KL-to-uniform, etc.
- *Generalization:* Test: distribution of $e_{\mathrm{hand}} = ||f(Q) - H||$
- *Bias from data collection* (forward vs. inverse sampling):

2.6. **Potential next-step extensions.**
- Disconnected feasible sets: modify toy with a "visibility" constraint such as
$$\mathrm{visible}(Q, H) = 1 \iff \theta \in [\alpha_1, \beta_1] \cup [\alpha_2, \beta_2]$$
Now the true $p^*(Q \mid H)$ is two separated modes. **Test:** cluster sampled angles into arcs and compute mode recall: fraction of samples that land in each arc.
- Scalability with dimension: add DoF to the toy (e.g., 2-link arm, variable stick length). **Test:** track how coverage/error degrades with dimension.

3. Extra notes

3.1. **Mode collapse.** A generative model has mode collapse if, for a fixed condition H, the samples $Q \sim \hat{p}(Q \mid H)$ cover only a small subset of the true support of $p^*(Q \mid H)$.

- In this toy problem, the support is the full circle parameterized by θ. Mode collapse looks like the model always outputting $\theta \approx 0$ (i.e. stands in one favorite location) or outputting only a few discrete angles, ignoring the rest the circle, etc.

3.2. **cVAE theory.**

3.2.1. *ELBO loss.*

3.3. **FrEIA INN theory.** They claim that their invertible neural net is essentially normalized flow – here is the math behind this!

3.4. **Normalized flow.**

References