Advanced Programming Practice
Cross Product and Convex Hull

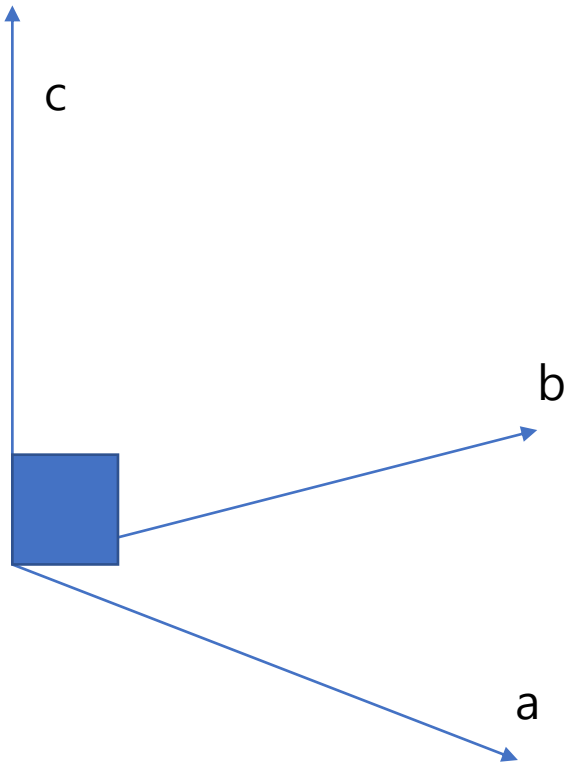
2022 Fall

Sogang University

Cross product

$$\mathbf{c} = \mathbf{a} \times \mathbf{b}$$

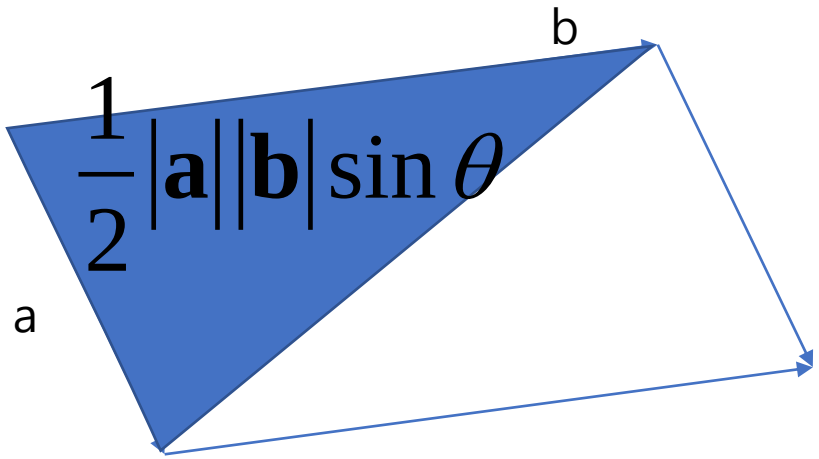
$$|\mathbf{c}| = |\mathbf{a}||\mathbf{b}| \sin\theta$$



Cross product

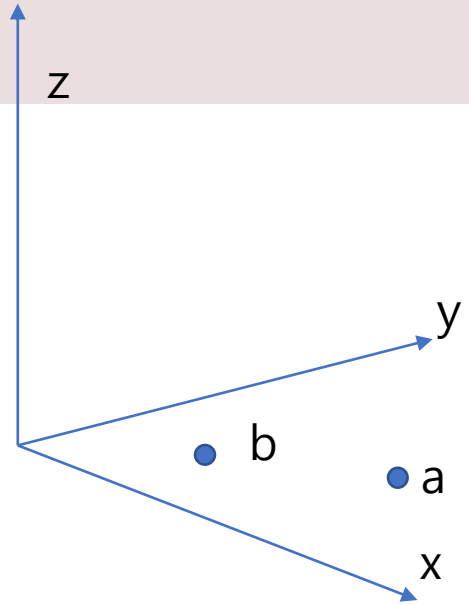
- An area of an triangle of two vectors a and b

$$\frac{1}{2}|\mathbf{a}||\mathbf{b}|\sin\theta = \frac{1}{2}\mathbf{a}\times\mathbf{b}$$



Cross product

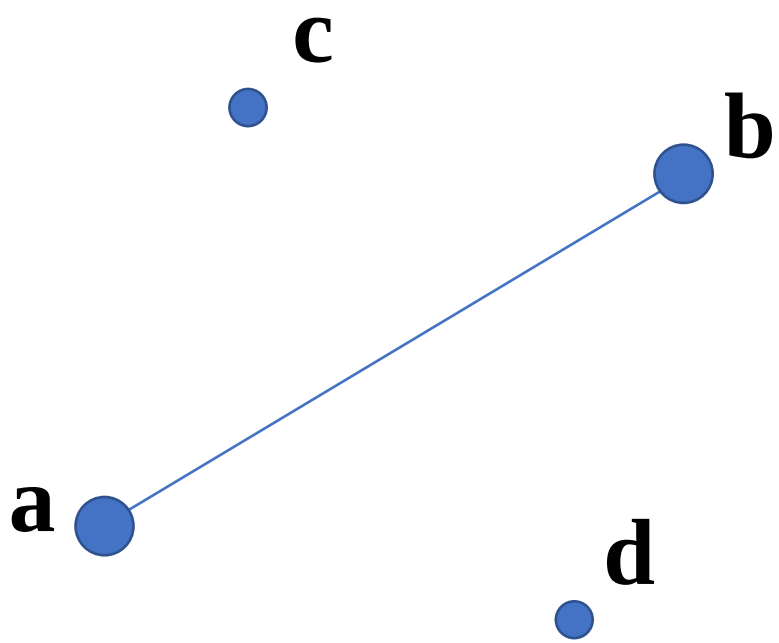
- Let vector **a** and **b** be on the **xy** plane.



$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{vmatrix} = \left(a_1 b_2 - a_2 b_1\right)\vec{k}$$

Clockwise or Counter-clockwise

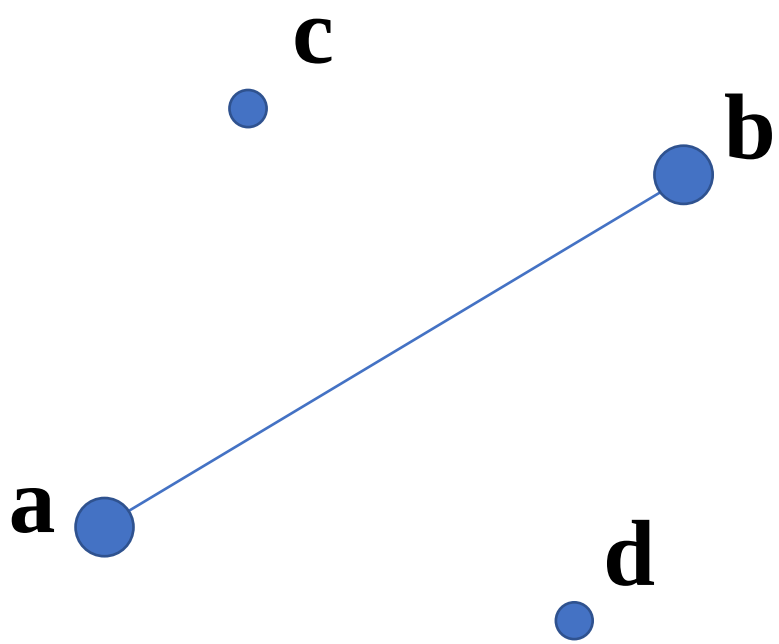
- Check whether points are on the clockwise or the counter-clockwise of line a-b with cross product.



Clockwise or Counter-clockwise

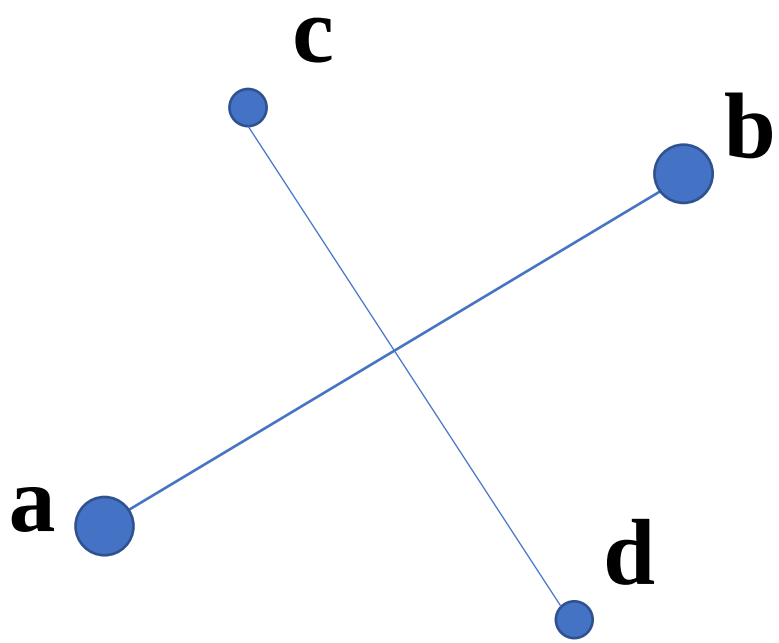
- Since the cross product (b-a)x(c-a) directs to the forward direction of the display, point C is on the counter-clockwise.

- Since the cross product of (b-a)x(d-a) points the backward direction of the display, point d is on the clockwise.



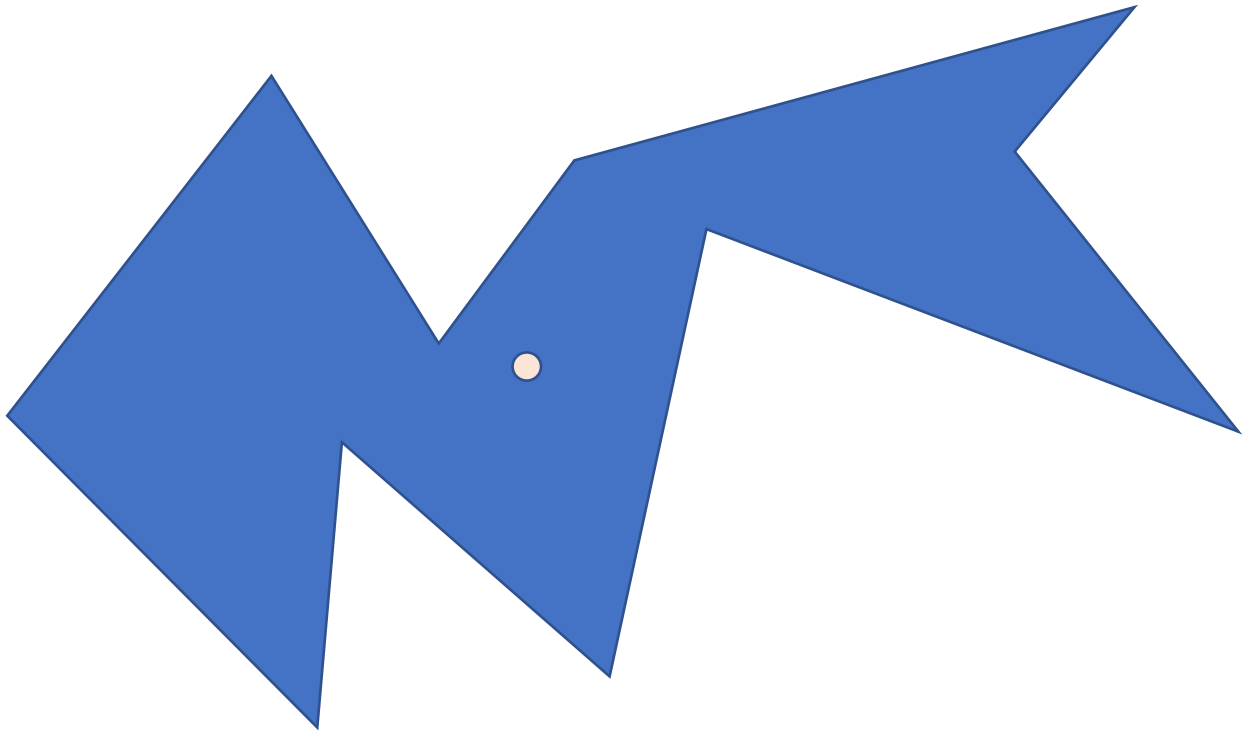
Intersection

- Two lines are crossed when end points of each line are on different sides of the other line.



Inner points

- How could we check whether a 2D point is in the polygon or not.

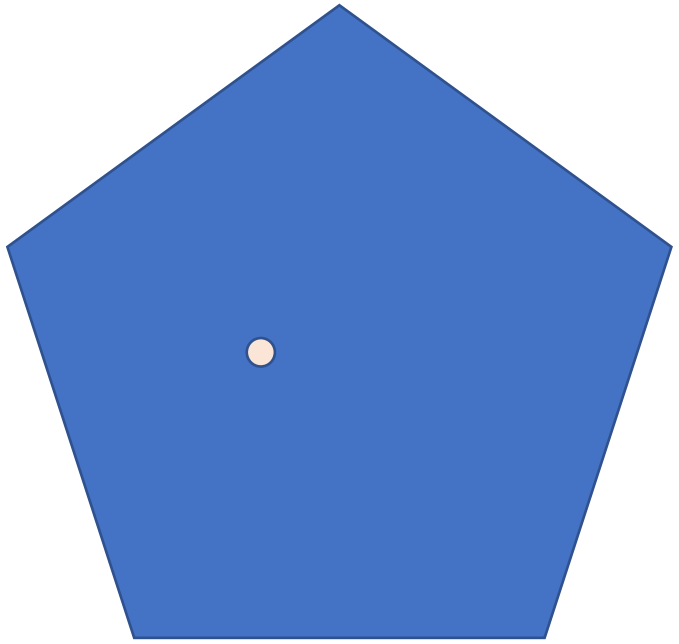


Approach 1. only on the one side

- Approach 1: A point is inside of the polygon when the point is on one side (clockwise or counter-clockwise) of every polygon edges.

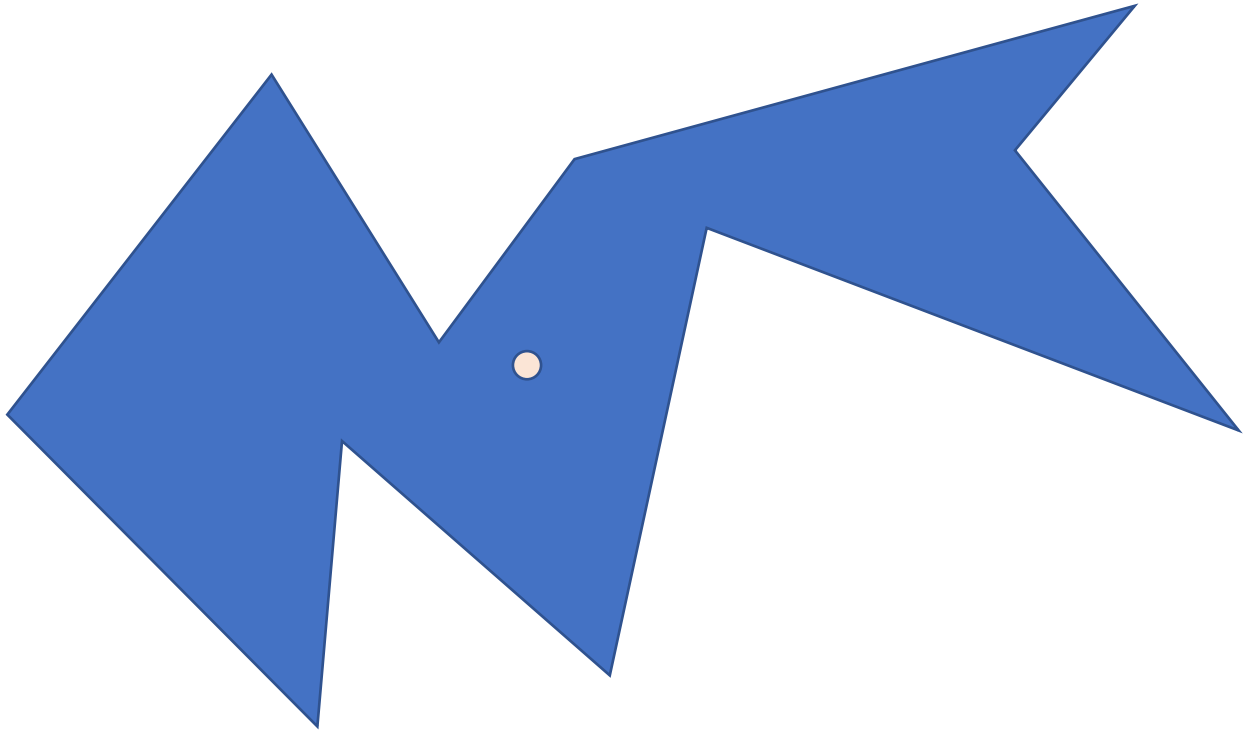
Solving the problem

- Satisfied when the polygon is convex.



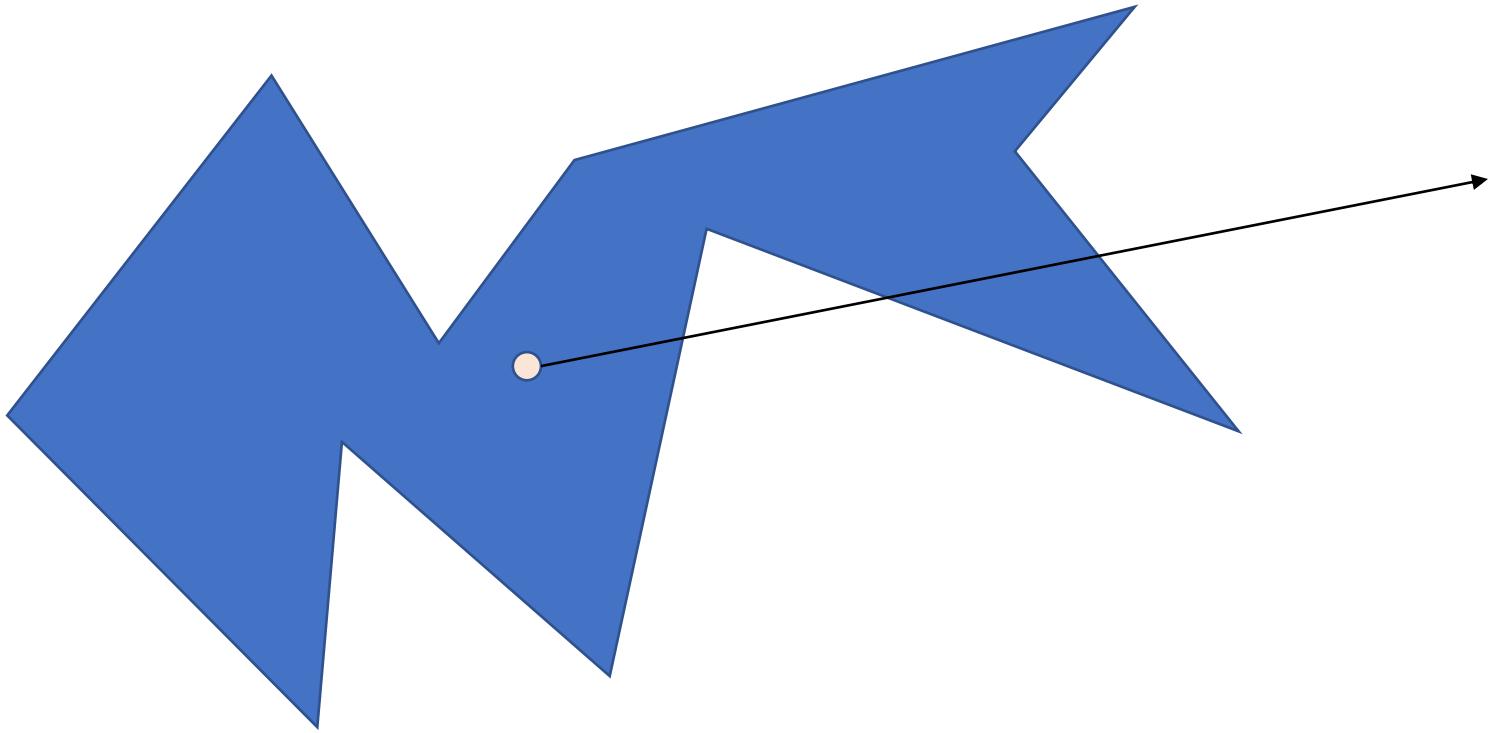
Solving the problem

- Not satisfied for non-convex polygons.

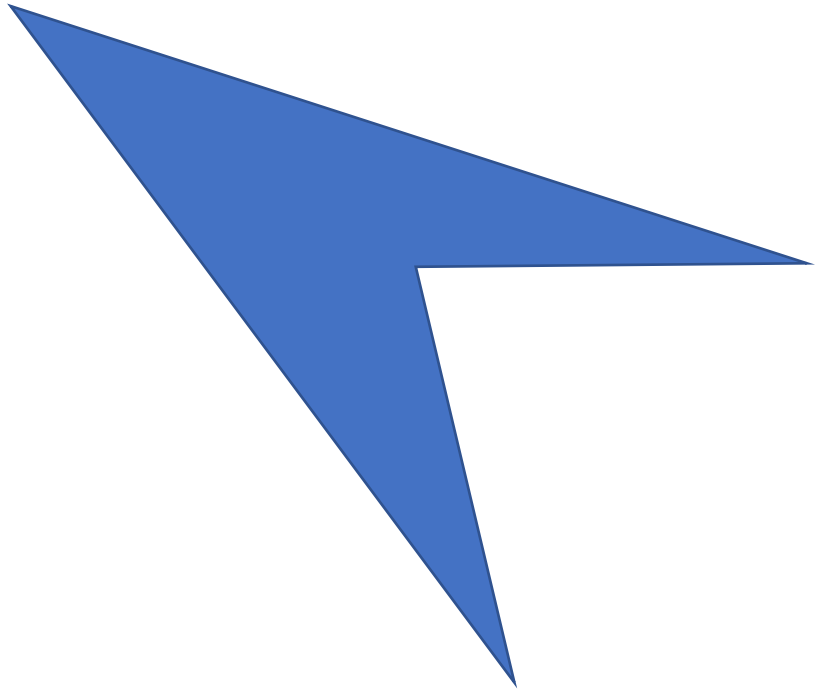


Approach 2. draw a infinite line

- The infinite line starting from every inner point has the odd number of intersection points.

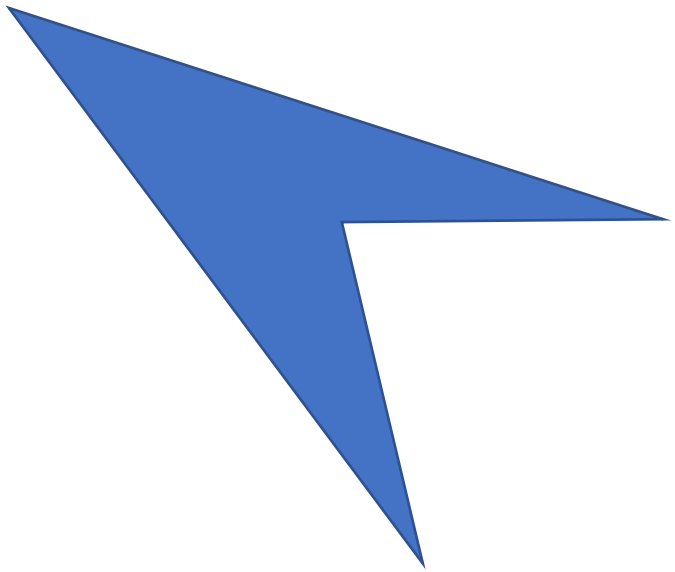


Computing the area of a polygon



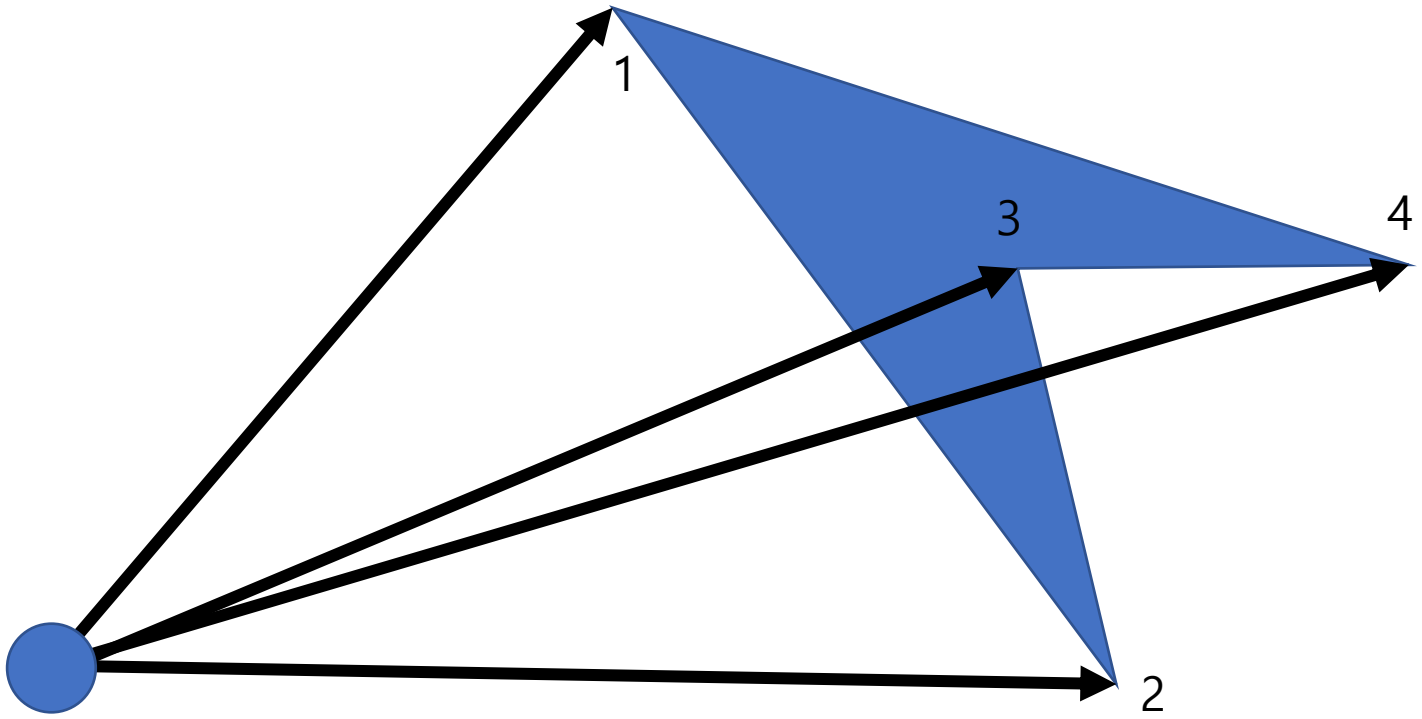
Computing the area of a polygon (cont.)

- Sort vertices in counter-clock wise.
- Find triangles where (i+1) th vertex is on clockwise of the edge of the i-th node and i+1-th node and accumulate areas of triangles.
- Delete triangles and repeat this process again until no vertex is left.



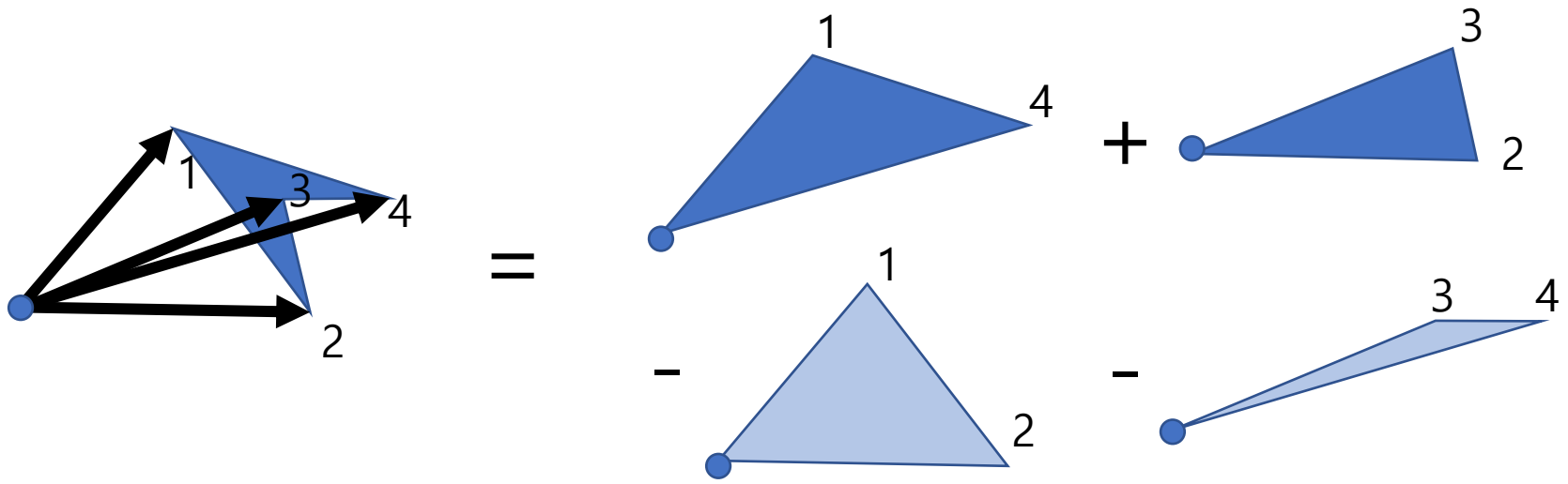
Computing the area of a polygon (cont.)

- Add up all cross products of edges and divide it by 2.



Computing the area of a polygon (cont.)

- Depending the direction of an edge, the area of the triangle is added or subtracted.
- Surprisingly, only the polygon area is left.



Computing Sum of Cross Products

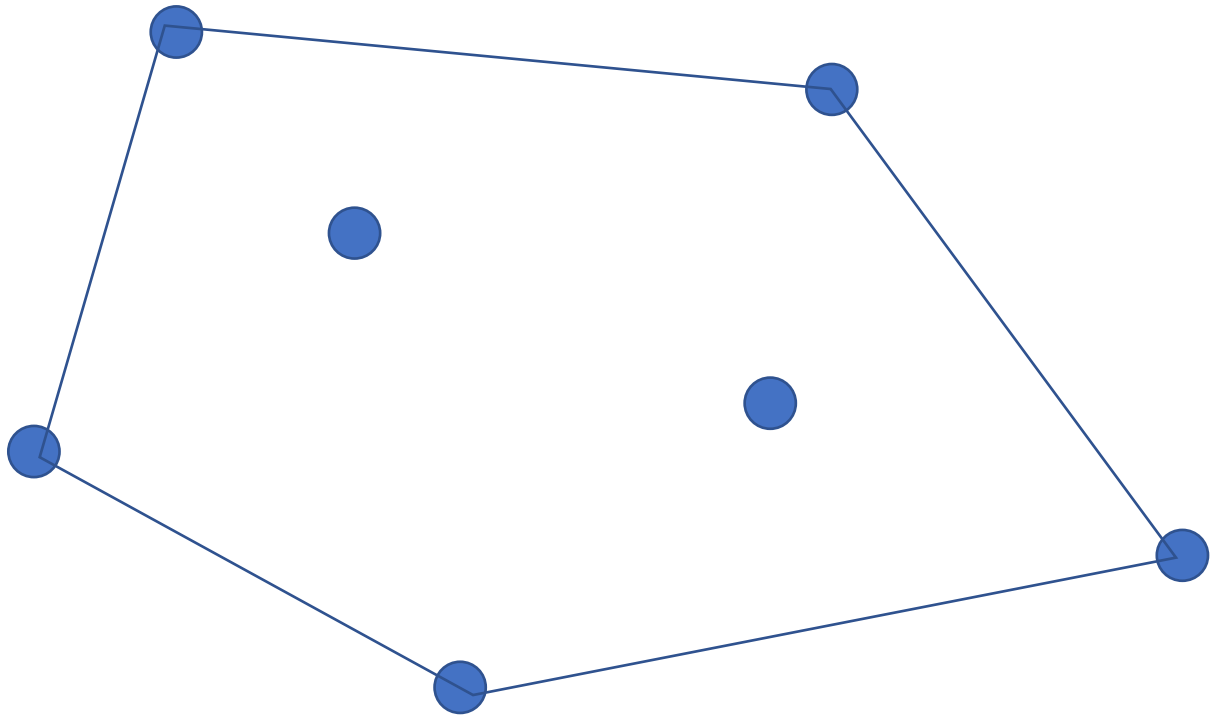
- It is analogous to sum of cross products of edges.

$$\mathbf{p}_1 \times \mathbf{p}_2 + \mathbf{p}_2 \times \mathbf{p}_3 + \mathbf{p}_3 \times \mathbf{p}_4 + \mathbf{p}_4 \times \mathbf{p}_1$$

$$= \begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix} = \begin{matrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{matrix} - \begin{matrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{matrix}$$

Computing a convex hull

- 2D points are given.



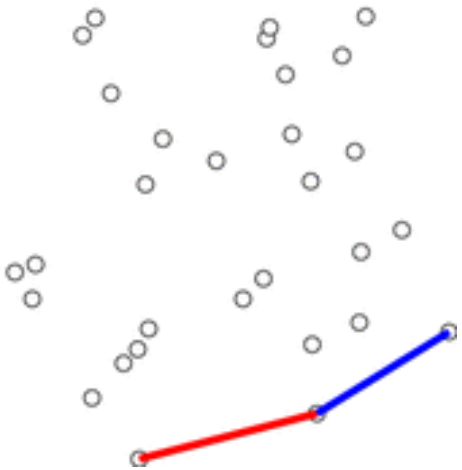
Graham Scan

- Algorithm to find a convex hull.

```
let points be the list of points
let stack = empty_stack()

find the lowest y-coordinate and leftmost point, called P0
sort points by polar angle with P0

for point in points:
    # pop the last point from the stack if we turn clockwise to reach this point
    while count stack > 1 and ccw(next_to_top(stack), top(stack), point) <= 0:
        pop stack
    push point to stack
end
```



https://en.wikipedia.org/wiki/Graham_scan

Assignment

Computing the area

For given N 2D points that composes a polygon, compute the area of the polygon.

Input
5 // N points
0 0 // (x1, y1)
2 0 // (x2, y2)
2 2
1 1
0 2

Output
3

Checking whether inside or outside of the polygon

For given N 2D points that composes a polygon and M 2D points, please check whether each point is in the polygon or not.

Points on lines is considered as inside.

Input	**Output**
5 // N points	Inside
0 0 // (x1, y1)	Inside
2 0 // (x2, y2)	Outside
2 2	
1 1	
0 2	
2	
0 0	
0.5 0.5	
-1 -1	

Computing the area of the convex hull

For given N 2D points, compute a convex hull and its area.

Input
5 // N points
0 0 // (x1, y1)
2 0 // (x2, y2)
2 2
1 1
0 2

Output
4
(0, 0), (2, 0), (2, 2), and (0, 2) are points of convex hull.