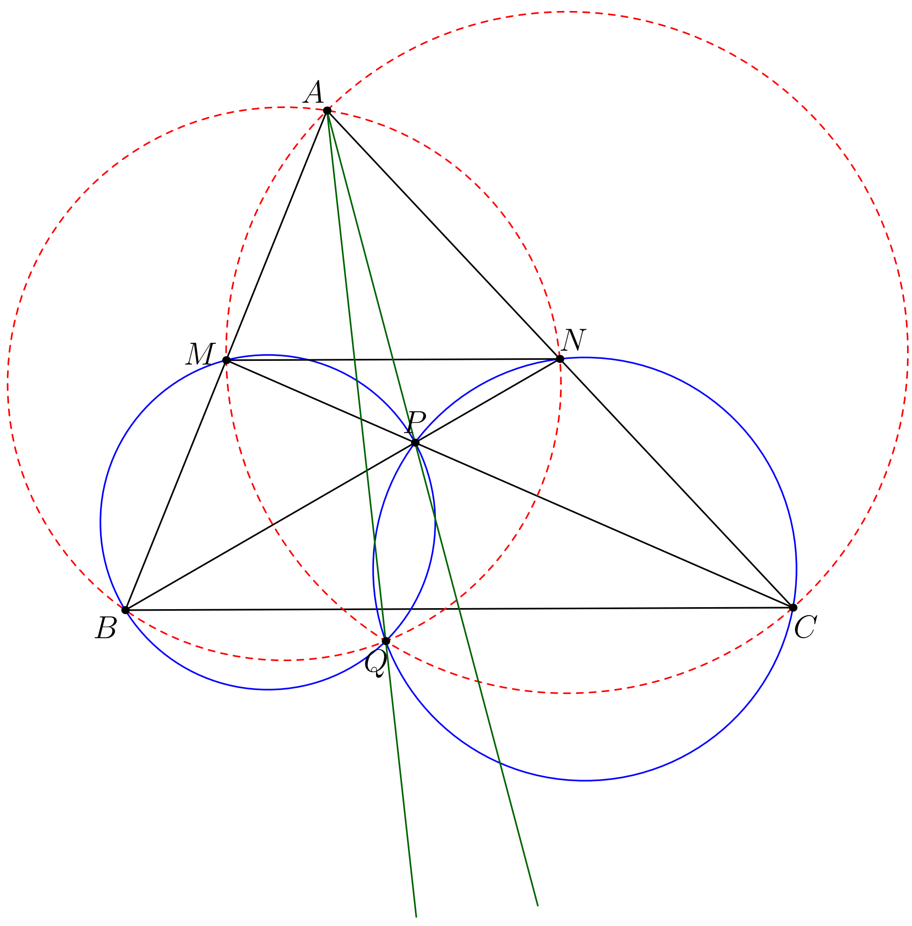
2009 Balkan MO P2

Kim Yong Joon

8 September 2025

Problem (Balkan 2009/2)**.** Let MN be a line parallel to BC of a triangle ABC, with M on side AB and N on side AC. The lines BN and CM meet at a point P. The circumcircles of $\triangle BMP$ and $\triangle CNP$ meet at two distinct points P and Q. Prove that $\angle BAQ = \angle CAP$.



¶ Main Idea The main idea of the problem is to notice that it suffices to prove that AQ is a symmedian. This follows from the following three ways. We're motivated to length-chase and try to use symmedian properties, $\sqrt{bc}$ inversion which always work well with symmedians, or harmonics which doesn't quite work, and lastly Miquel point and complete quadrilaterals after noticing two more cyclic quadrilaterals.

¶ Solution 1: Ratio + Symmedian Properties Notice the spiral similarity $Q : BM \leftrightarrow NC$. With this, we have

$$\frac{d(Q, AB)}{d(Q, AC)} = \frac{BM}{NC} = \frac{AB}{AC}$$

This is enough to imply that Q lies on the symmedian, as desired.

Note that

- The locus of a point X on the symmedian is $\frac{d(X,AB)}{d(X,AC)} = \frac{AB}{AC}$

¶ Solution 2: $\sqrt{bc}$ inversion

Claim — $ABQN$, $AMQC$ are cyclic

Proof. $\angle QCA = \angle QCN = \angle QPB = \angle QMB = 180 - \angle AMQ$ and similarly for $ABQN$. □

Perform an inversion centered at A with radius $\sqrt{AB \cdot AN}$, followed by a reflection wrt the angle bisector of $\angle BAC$.

By the parallel condition, we have $\frac{AM}{AB} = \frac{AN}{AC} \implies AB \cdot AN = AC \cdot AM$, hence $B \leftrightarrow N$ and $C \leftrightarrow M$. This also means $BN \leftrightarrow (AMC)$ and $CM \leftrightarrow (ABN)$, so $P = AM \cap BN \leftrightarrow (AMC) \cap (ABN) = Q$, hence P and Q are isogonal, as desired.

¶ Solution 3: Miquel Point & Complete Quadrilaterals It is known that this is possible with angle chasing.