2022 INMO P1

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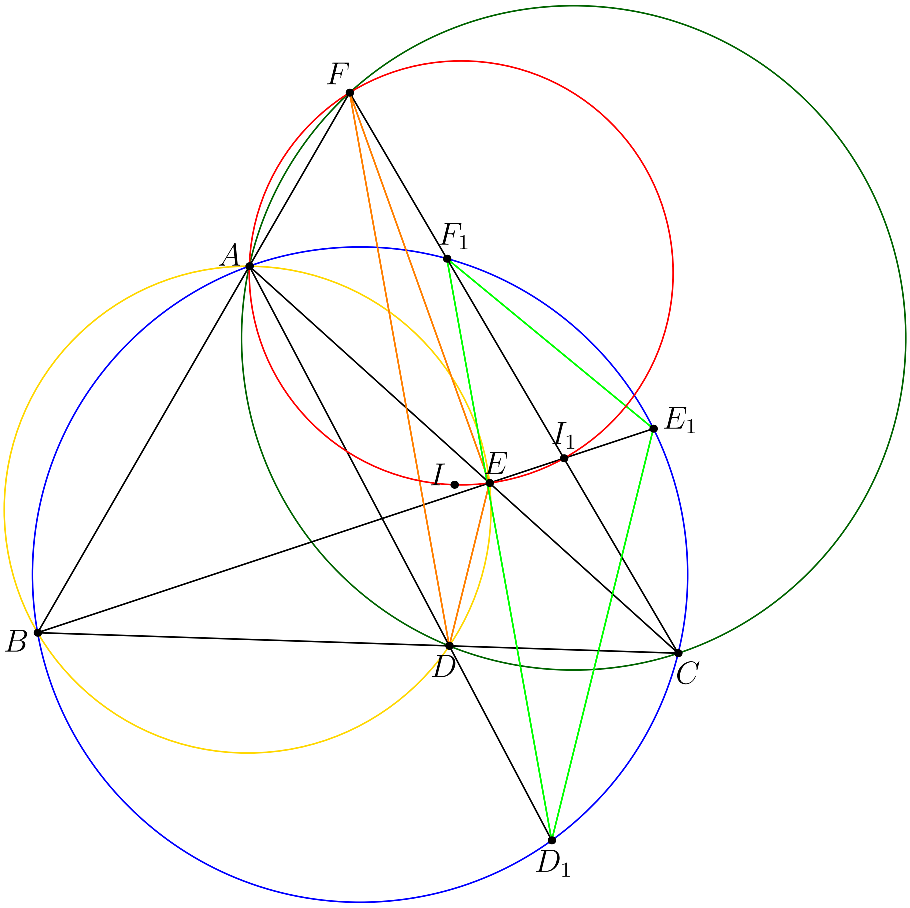
Problem (INMO 2022/1). Let D be an interior point on the side BC of an acute-angled triangle ABC. Let the circumcircle of triangle ADB intersect AC again at E and the circumcircle of triangle ADC intersect AB again at F. Let AD, BE, and CF intersect the circumcircle of triangle ABC again at D_1, E_1 and F_1, respectively. Let I and I_1 be the incenters of triangles DEF and $D_1E_1F_1$, respectively. Prove that E, F, I, I_1 are concyclic.



¶ **Main Idea** Find the incenters of the triangles and notice some favorable properties, by drawing a nice diagram. After, upon trying to prove the given concyclicity, show that a nice point lies on the circle too.

¶ Solution

Claim — $I_1 = BE_1 \cap CF_1$

Proof. $\angle BE_1D_1 = \angle BAD_1 = \angle BAD = 180 - \angle FAD = \angle DCF = \angle DCF_1 = \angle BCF_1 = \angle BE_1F_1$, and similarly $\angle D_1F_1C = \angle E_1F_1C$. □

Claim — $ID \perp BC$

Proof. Note that this naturally follows from $\angle FDB = \angle EDC$. This is true as

$$\angle FDB = \angle 180 - \angle FDC = 180 - \angle FAC = 180 - \angle BDE = \angle EDC$$

□

Claim — $FAIE$ is cyclic

Proof. $\angle FIE = 90 + \frac{1}{2}\angle FDE = \angle FDC = \angle FAC = \angle FAE$ □

Claim — $FAEI_1$ is cyclic

Proof.

$$\begin{aligned}\angle FI_1E &= \angle F_1I_1E \\ &= \frac{1}{2}\angle D_1F_1E_1 + \frac{1}{2}\angle F_1E_1D_1 \\ &= \angle EE_1D_1 + \angle D_1F_1C \\ &= \angle BAD_1 + \angle D_1AC \\ &= \angle BAC & = 180 - \angle FAE\end{aligned}$$

□