2019 RMM SL G1

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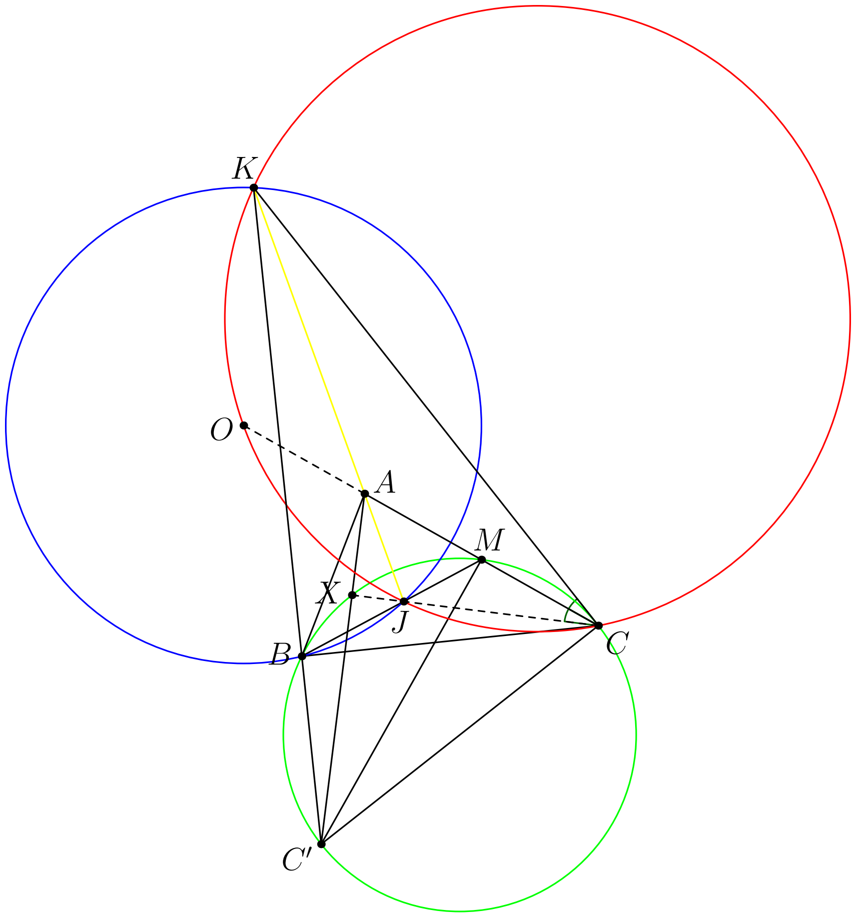
Problem (RMM Shortlist 2019 G1). Let BM be a median in an acute-angled triangle ABC. A point K is chosen on the line through C tangent to the circumcircle of $\triangle BMC$ so that $\angle KBC = 90°$. The segments AK and BM meet at J. Prove that the circumcenter of $\triangle BJK$ lies on the line AC.



Let C' be the antipode of C wrt to (ABC) and O the circumcenter of $\triangle KBJ$. Note that $\triangle AC'C$ is isosceles, as $\angle C'MC = 90$ and M is the midpoint of AC.

Claim — $AC' \cap JC \in (ABC)$

Proof. Let X be the second intersection of AC' and (ABC). By Pascal's theorem on $CCMBC'X$, we have that $K = CC \cap BC'$, $A = CM \cap C'X$, and $MB \cap CX$ is collinear.

Note that this implies $J = MB \cap CX$, as J lies both on MB and KA. This proves that $AC' \cap JC \in (ABC)$, as desired. $\square$

Claim — AC bisects $\angle KCJ$

Proof. $\angle KCM = \angle CC'M = \angle AC'M = \angle XC'M = \angle XCM$ $\square$

Claim — $KOJC$ is cyclic, and the arc midpoint of KJ is O

Proof. $\angle KOJ = 2\angle KBJ = 180 - 2\angle MBC = 180 - 2\angle MC'C = 180 - \angle AC'C = 180 - \angle XCK = 180 - \angle JCK$ And since O lies on both the circumcircle of $\triangle KJC$ and perpendicular bisector KJ, it is the arc midpoint of KJ not containing C. $\square$

by the second and third claim, A, C, O are collinear, as O must lie on the angle bisector of $\angle JCK$.