2017 USA January TST P2

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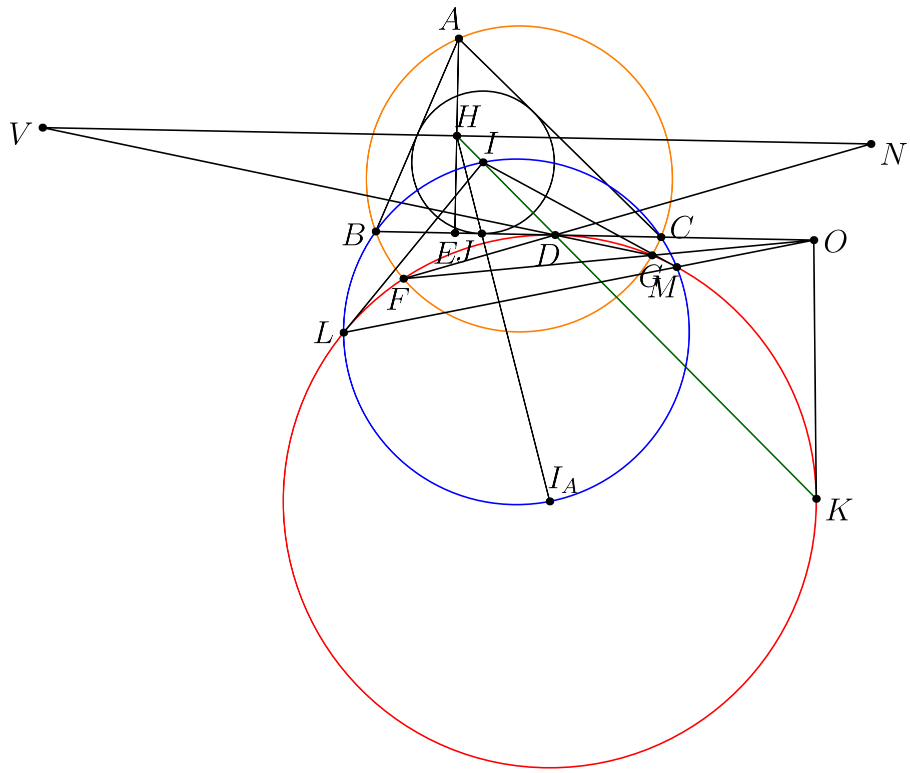
Problem (2017 USA Januart TST P2). Let ABC be a triangle with altitude AE. The A-excircle touches BC at D, and intersects the circumcircle at two points F and G. Prove that one can select points V and N on lines DG and DF such that quadrilateral $EVAN$ is a rhombus.



¶ Main Idea Use midpoint of altitude lemma, and the rest is just standard projective(harmonic) geometry - pole-polar, tangents, harmonics, perspectivity, adding random tangencies, proving concurrency, and (important) using $(BICI_A)$!

¶ Solution Note that BC, FG, and LM concur at a point, say O by radical center theorem. Let K be the tangent from O. It follows that H, I, D, K are collinear as IM and IL are in fact tangents.(We can easily pole-polar chase or just symmedians). This also obviously implies that $DFKG$ is harmonic, and perspectivity at D finishes as

$$-1 = (G, F, K, D) \stackrel{D}{=} (V, N, H, \infty)$$