2021 PAGMO P6

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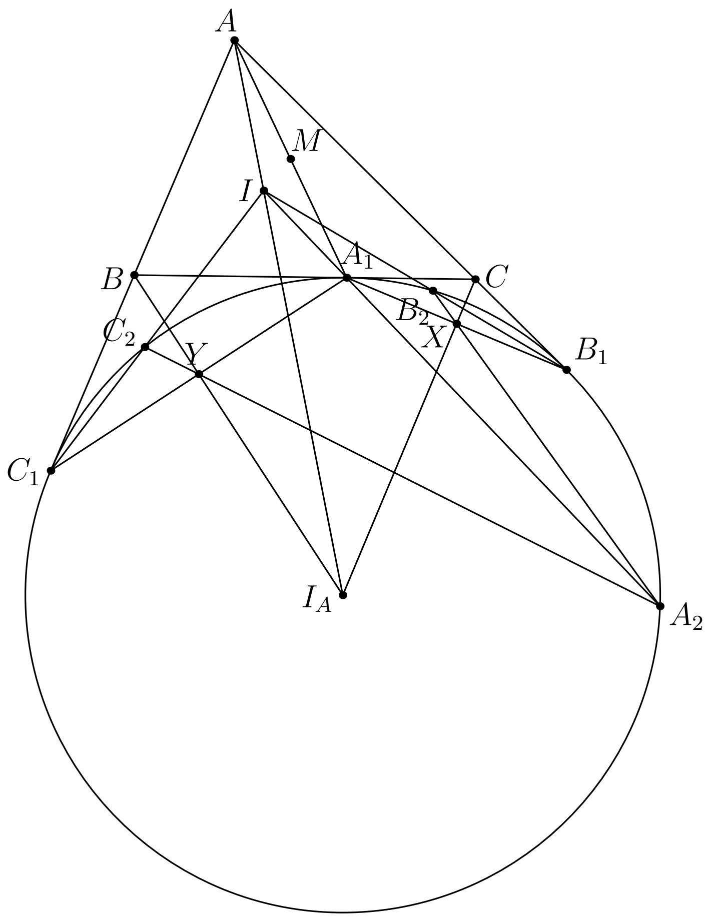
26 August 2025

Problem (PAGMO 2021/6). $\triangle ABC$ has incenter I and A-excircle Γ. Let A_1, B_1, C_1 be the points of tangency of Γ with BC, AC, and AB, respectively. Suppose IA_1, IB_1, and IC_1 intersect Γ for the second time at points A_2, B_2, C_2, respectively. M is the midpoint of AA_1. If $X = A_1B_1 \cap A_2B_2$ and $Y = A_1C_1 \cap A_2C_2$, prove that $MX = MY$.



¶ Main Idea Notice the homothety of ratio 2 centered at A_1. The rest is proving the very motivated claim of the midpoint conditions, using pole-polar chase and projective geometry(brokard's theorem)

¶ Solution(Projective Geometry) It suffices to prove that X and Y are the midpoints, as noted above. To do so, we pole-polar chase. A_1B_1 is the polar of C, hence C lies on the polar of X. I lies on the polar of X, by Brokard's theorem on $A_1A_2B_1B_2$. This implies that IC is in fact the polar of X, which means $IC \perp I_AX$, and we're done.(symmetric argument for Y)