2013 January TST P2

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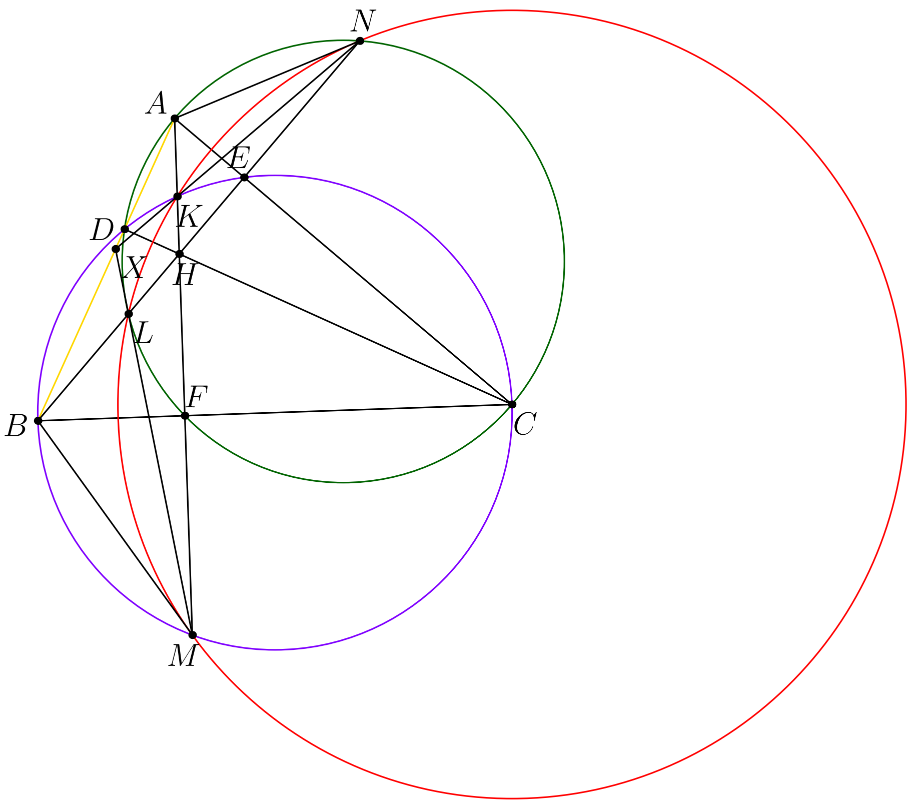
Problem (January TST 2013/2). Let ABC be an acute triangle. Circle ω_1, with diameter $\overline{AC}$, intersects side $\overline{BC}$ at F (other than C). Circle ω_2, with diameter $\overline{BC}$, intersects side $\overline{AC}$ at E (other than C). Ray AF intersects ω_2 at K and M with $AK < AM$. Ray BE intersects ω_1 at L and N with $BL < BN$. Prove that lines AB, ML, NK are concurrent.



¶ Main Idea The main idea of the problem was to notice a nice cyclic quadrilateral with juggling power of a point and the auspicious diameter conditions. After that, we see that we can in fact rephrase the concurrency condition in terms of Pascal's theorem applied to the found circle, which turns out to be cleanly done. Although my initial thought process wasn't bad, I don't think it was the best way to think about the problem. Instead, I should've focused more on utilizing **Brokard's Theorem**, because it is more motivated here. It also helps a lot in naturally yielding the tangency results. Main solutions are Harmonic(Brokard's theorem), Inversion, and Pascal's theorem.

¶ Solution(Pascal's Theorem) Let H be the orthocenter of ABC, and D the foot from C to AB.

Claim — $NKLM$ is cyclic

Proof.

$$\begin{aligned} LH \cdot HN &= AH \cdot HF \\ &= DH \cdot HC \\ &= KH \cdot HM \end{aligned}$$

□

Claim — C is the center of $(NKLM)$

Proof. Trivial upon noting that C obviously lies on the perpendicular bisectors of LN and KM, by symmetry. □

This also implies that AN and BM are tangent to $(NKLM)$.

Claim — AB, ML, NK concur

Proof. By Pascal's thoerem on $MMKLNN$, A, $ML \cap NK$, and B are collinear; this directly implies the result. □

¶ Solution 2(Harmonic & Brokard's theorem) By Brokard's thoerem, it suffices to prove that AB is the polar of H wrt $(NKML)$. This follows from appying Brokard's theorem twice in $(BDEC)$ and $(ADFC)$.