2008 IMO P6

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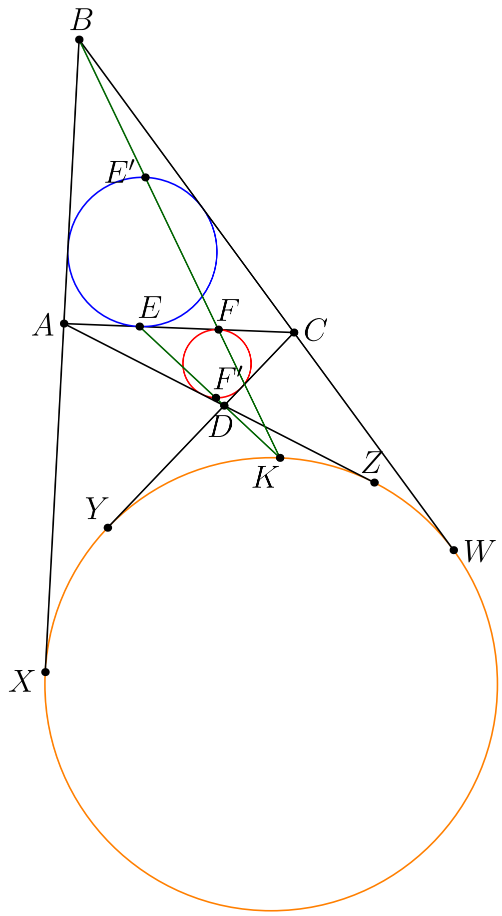
Problem (IMO 2008/6)**.** Let $ABCD$ be a convex quadrilateral with $BA \neq BC$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents to ω_1 and ω_2 intersect on ω.



¶ Main Idea First, note the ex-tangential condition and identify the symmetry of the tangency points. Second, Rephrase the problem by noting that the exsimilicenter of the two circles lie on the ex-tangential circle of $ABCD$.

¶ Solution Note that it suffices to prove that the two incircles are homothetic with respect to a point on the ex-tangential circle of $ABCD$.

Let K be the 'uppermost' point of the ex-tangential circle of $ABCD$.

Claim — $BA + AD = BC + CD$

Proof.

$$\begin{aligned} BA + AD &= (BX - AX) + (AZ - DZ) \\ &= BX - DZ \\ &= BW(-CW + CY) - YD \\ &= (BW - CW) + (CY - YD) \\ &= BC + CD \end{aligned}$$

□

Claim — $AE = FC$

Proof.

$$\begin{aligned} AE &= \frac{(AB - BC) + CA}{2} \\ &= \frac{(CD - AD) + CA}{2} \\ &= AF \end{aligned}$$

□

This implies that F is in fact the A-extouch point. Hence, we may deduce that B, E', F, K and E, F', D, K are collinear, by homothety centered at B and D, respectively. Moreover, since $E'E \parallel FF'$, E' and F, E and F' are corresponding points of the two incircles, and this implies that K is the desired exsimilicenter of the two incircles.