Notes on Inequalities and Embeddings

Kiyuob Jung

(in progress)

Department of Mathematics,
Kyungpook National University

December 15, 2022

Contents

Preface

Todo.

Abbreviation

Since these notes is not for formal research, we almost always employ the following abbreviations:
$\diamond$ s.t. : s.t.
$\diamond$ TFAE : The following are equivalent

Notation

Let a set X. We employ the following notations:
$\diamond$ $\mathbb{R}^n$: n-dimensional real Euclidean space, $\mathbb{R} = \mathbb{R}^1$
$\diamond$ $\mathbb{C}^n$: n-dimensional complex space, $\mathbb{C} = \mathbb{C}^1$
$\diamond$ $\mathbb{K}$: either $\mathbb{R}$ or $\mathbb{C}$.
$\diamond$ ∂X : boundary of X.
$\diamond$ $\forall$: for all.

Chapter 1

Definitions and notations

In this note, we always denote $x = (x_1, x_2, \dots, x_n)$ to be a point in $\mathbb{R}^n$.

1.1 Definitions

1.1.1 Integration

Here, we collect definitions for averages of a function.

Definition 1.1.1. Let $f \in L^1(\Omega)$ with an open set $\Omega \subset \mathbb{R}^n$.

(a) An average of f over set E is

$$⨍_E f dx := \frac{1}{\text{meas}(E)} \int_E f(x) dx.$$

(b) An average of f over the ball $B_r(x_0)$ is

$$⨍_{B_r(x_0)} f dx := \frac{1}{\alpha(n) r^n} \int_{B_r(x_0)} f(x) dx.$$

(c) An average of f over the sphere $\partial B_r(x_0)$ is

$$⨍_{\partial B_r(x_0)} f dS := \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B_r(x_0)} f(x) dS.$$

When it comes to integrability, the following definition plays an important role.

Definition 1.1.2. Let $p > 1$ and define $p' \in \mathbb{R}$ by

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Then p and p' are called *conjugate exponents*.

Even if some authors use q instead of p', we stick to using p' in order to save letters and will use q to stand for other integrability.

Remark 1.1.3. A simple calculation shows the following:

(i) $pp' = p + p'$,

(ii) $1 = \dfrac{p + p'}{pp'}$,

(iii) $(p-1)(p'-1) = 1$,

(iv) $p' = \dfrac{p}{p-1}$.

1.1.2 Convex functions

Definition 1.1.4. A function $f : \mathbb{R}^n \to \mathbb{R}$ is *convex* if

$$f(\lambda x + (1-\lambda)y) \leq \tau f(x) + (1-\lambda)f(y)$$

for all $x, y \in \mathbb{R}^n$ and each $0 \leq \lambda \leq 1$.

Remark 1.1.5. If f is C^2, then f is convex if and only if $D^2 f \geq 0$.

Definition 1.1.6. A C^2 function $f : \mathbb{R}^n \to \mathbb{R}$ is *uniformly convex* if $D^2 g \geq \theta I$ for some constant $\theta > 0$, that is,

$$\sum_{i,j=1}^{n} f_{x_i x_j}(x)\xi_i \xi_j \geq \theta |\xi|^2$$

for all $x, \xi \in \mathbb{R}^n$.

Chapter 2

Basic properties

2.1 Change of variable

Proposition 2.1.1. *Let $f : \Omega \to \mathbb{R}$ with an open set $\Omega \subset \mathbb{R}^n$.*

(a) $\displaystyle\int_{B_r(x_0)} f(ax+b)dx = \frac{1}{r^n}\int_{B_{ar}(x_0+b)} f(x)dx.$

(b) $\displaystyle\fint_{B_r(x_0)} f(ax+b)dx = \fint_{B_{ar}(x_0+b)} f(x)dx.$

Proof. TODO

To show (b), let $\tilde{x} := ax+b$. Then $d\tilde{x} = r^n dx$. Also, since $|\tilde{x}-b| = |ax| < |a|\, r$, we get $\tilde{x} \in B_{|a|r}(x_0+b)$. Hence,

$$\begin{aligned}\fint_{B_r(x_0)} f(ax+b)dx &= \frac{1}{\alpha(n)r^n}\int_{B_r(x_0)} f(x)dx \\ &= \fint_{B_{ar}(x_0+b)} f(\tilde{x})d\tilde{x} \\ &= \fint_{B_{ar}(x_0+b)} f(\tilde{x})d\tilde{x}.\end{aligned}$$

□

Remark 2.1.2. Convolution. TODO

2.2 Coordinates

2.2.1 Polar coordinates

Proposition 2.2.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous and summable. Then*

(a) $\displaystyle\int_{\mathbb{R}^n} f(x)dx = \int_0^\infty \left(\int_{\partial B_r(x_0)} f(x)d\mathcal{S}\right) dr \quad \forall x_0 \in \mathbb{R}^n.$

(b) $\displaystyle\frac{d}{dr}\left(\int_{B_r(x_0)} f(x)dx\right) = \int_{\partial B_r(x_0)} f(x)d\mathcal{S} \quad \forall r > 0.$

Proposition 2.2.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous and summable. Then*

(a) $\displaystyle\int_{B_\varepsilon(0)} f(x)dx = \int_0^\varepsilon \left(\int_{\partial B_r(0)} f(x)d\mathcal{S}(x)\right) dr \quad \forall \varepsilon > 0.$

Chapter 3

Inequalities

3.1 Scalar

3.1.1 Power inequalities

Theorem 3.1.1. *The following statements hold.*

(a) $1 + x \leq e^x \quad \forall x \in \mathbb{R}$.

(b) $e^{(x+y)/2} < \dfrac{e^y - e^x}{y - x} \quad \forall x, y \in \mathbb{R}$ with $x \neq y$.

Theorem 3.1.2. *The following statements hold.*

(a) $\left(\dfrac{1}{e}\right)^{\frac{1}{e}} \leq x^x \quad \forall x > 0$.

(b) $x \leq x^{x^x} \quad \forall x > 0$.

(c) $1 < x^y + y^x \quad \forall x, y > 0$.

(d) $x^y + y^x \leq x^x + y^y \quad \forall x, y > 0$.

(e) $x^{ey} + y^{ex} \leq x^{ex} + y^{ey} \quad \forall x, y > 0$.

(f) $\dfrac{1 - \frac{1}{x^y}}{y} \leq \ln(x) \leq \dfrac{x^y - 1}{y} \quad \forall x, y > 0$. The upper and lower bounds converge to $\ln(x)$ as $y \to 0$.

(g) $2 < (x+y)^z + (x+z)^y + (y+z)^x \quad \forall x, y, z > 0$.

(h) $x^{2y} + y^{2z} + z^{2x} \leq x^{2x} + y^{2y} + z^{2z} \quad \forall x, y, z > 0$.

(i) $(xyz)^{(x+y+z)/3} \leq x^x y^y z^z \quad \forall x, y, z > 0$.

3.1.2 product type

Theorem 3.1.3. *The following statements hold.*

(a) (Cauchy's inequality)

$$xy \leq \frac{x^2}{2} + \frac{y^2}{2} \quad \forall x, y \in \mathbb{R}.$$

(b) (Cauchy's inequality with ε)

$$xy \leq \varepsilon x^2 + \frac{y^2}{4\varepsilon} \quad \forall x, y > 0, \quad \forall \varepsilon > 0.$$

(c) (Young's inequality)

$$xy \leq \frac{x^p}{p} + \frac{y^{p'}}{p'} \quad \forall x, y > 0, \quad \forall 1 < p, p' < \infty,$$

where p and p' are conjugate exponents.

3.1.3 summation type

Theorem 3.1.4. *The following statements hold.*

(a) $(x+y)^p < x^p + y^p \quad \forall x, y > 0, \forall p \in (0,1)$.

(b) $(x+y)^p \leq 2^{p-1}(x^p + y^p) \quad \forall x, y \geq 0, \forall p \in [1, \infty)$.

3.2 Function

3.2.1 Convex functions

Theorem 3.2.1. (Jensen's inequality)
Assume $f : \mathbb{R}^m \to \mathbb{R}$ is convex and $\Omega \subset \mathbb{R}^n$ is open and bounded. Let $\mathbf{u} : \Omega \to \mathbb{R}^m$ be summable. Then

$$f\left(\fint_\Omega \mathbf{u} dx\right) \leq \fint_\Omega f(\mathbf{u}) dx.$$

3.2.2 Sobolev space

Theorem 3.2.2. *Let Ω be an open set in $\mathbb{R}^n$. The following statements hold.*

(a) (Hölder's inequality)
If $u \in L^p(\Omega)$, $v \in L^{p'}(\Omega)$ with $1 \leq p, p' \leq \infty$, where p and p' are conjugate exponents, then

$$\int_\Omega |uv| dx \leq \|u\|_{L^p(\Omega)} \|v\|_{L^{p'}(\Omega)}.$$

(b) (General version of Hölder's inequality)
If $u_i \in L^{p_i}(\Omega)$ for $i = 1, 2, \ldots, m$ with $1 \leq p_1, p_2, \ldots, p_m \leq \infty$ satisfying $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = 1$, then

$$\int_\Omega |u_1 u_2 \cdots u_m| dx \leq \prod_{i=1}^m \|u_i\|_{L^{p_i}(\Omega)}.$$

(c) (Minkowski's inequality)
If $u, v \in L^p(\Omega)$ with $1 \leq p \leq \infty$, then

$$\|u + v\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}.$$

(d) (Interpolation inequality)
Let $1 \leq p \leq r \leq q \leq \infty$, with

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$$

for some $\theta \in [0,1]$. If $u \in L^p(\Omega) \cap L^q(\Omega)$, then we also have $u \in L^r(\Omega)$ and

$$\|u\|_{L^r(\Omega)} \leq \|u\|^\theta_{L^p(\Omega)} \|u\|^{1-\theta}_{L^q(\Omega)}.$$

Chapter 4

Embeddings

4.1 Sobolev Embedding

In this section, we deal with embeddings of diverse Sobolev spaces into others. Given a Sobolev space, it automatically belongs to certain other space, depending on the relationship between the integrability p and the dimension n.[1] There are three cases:

$$\begin{aligned} &p \in [1, n), \\ &p = n, \\ &p \in (n, \infty]. \end{aligned}$$

In particular, the second case $p = n$ is called the *borderline case*. What are we trying to obtain from the Sobolev embedding theory? Broadly speaking, given a Sobolev space $W^{k,p}$, imbeddings of $W^{k,p}$ target two types of Banach spaces: either another Sobolev space $W^{j,q}$ or Hölder spaces $C^{j,\alpha}$ for some constants $j \leq k$, $q \geq p$, and $0 \leq \alpha \leq 1$.

4.1.1 The case $1 \leq p < n$

Definition 4.1.1. If $1 \leq p < n$, the *Sobolev conjugate* p^* of p is defined by

$$p^* := \frac{np}{n-p}.$$

Remark 4.1.2. A simple calculation shows the following:

(i) $p^* > p$,

(ii) $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}$,

(iii) $p^* \to \infty$ as $p \to n$.

Theorem 4.1.3. (Gagliardo-Nirenberg-Sobolev inequality, [1])
If $1 \leq p < n$, then there exists a constant C, depending only on p and n, such that

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)},$$

for all $u \in C_c^1(\mathbb{R}^n)$.

Theorem 4.1.4. (Estimates for $W^{1,p}, 1 \leq p < n$, [1])
Let U be a bounded, open subset of $\mathbb{R}^n$, and suppose ∂U is C^1. Assume $1 \leq p < n$ and $u \in W^{1,p}(U)$. Then $u \in L^{p^*}(U)$, with the estimate

$$\|u\|_{L^{p^*}(U)} \leq C\|u\|_{W^{1,p}(U)},$$

the constant C depending only on $p, n,$ and U.

[1] The regularity of domains also affects the inclusion.

4.1.2 The case $p = n$

4.1.3 The case $n < p \leq \infty$

For convenience as in the notation for Hölder exponents, we write

$$\gamma := 1 - \frac{n}{p},$$

whenever $n < p \leq \infty$.

Theorem 4.1.5. (Morrey's inequality, [1])
If $n < p \leq \infty$, then there exists a constant C, depending only on p and n, such that

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

for all $u \in C^1(\mathbb{R}^n)$.

Theorem 4.1.6. (Estimates for $W^{1,p}$, $n < p \leq \infty$, [1])
Let U be a bounded, open subset of $\mathbb{R}^n$, and suppose ∂U is C^1. Assume $n < p \leq \infty$ and $u \in W^{1,p}(U)$. Then u has a version $u^* \in C^{0,\gamma}(\overline{U})$ with with the estimate

$$\|u^*\|_{C^{0,\gamma}(\bar{U})} \leq C\|u\|_{W^{1,p}(U)},$$

the constant C depending only on p, n and U.

4.1.4 The case $p = \infty$

Theorem 4.1.7. (Characterization of $W^{1,\infty}$, [1])
Let U be open and bounded, with ∂U of class C^1. Then $u : U \to \mathbb{R}$ is Lipschitz continuous if and only if $u \in W^{1,\infty}(U)$

4.1.5 Summary

Theorem 4.1.8. (Sobolev, [2, Theorem 7.29])
Let $u \in W_0^{1,p}(B_R(0))$, where $B_R(0) \subset \mathbb{R}^n$. Then there are universal constants c_1, c_2, c_3 and c_4, depending on n, such that

(a) if $1 < p < n$ then

$$\|u\|_{L^{p^*}} \leq c_1\|Du\|_{L^p},$$

(b) if $p = n$ then

$$\fint_{B_R(0)} \exp\left(c_2 \frac{|u|}{\|Du\|_{L^n}}\right)^{\frac{n}{n-1}} dx \leq c_3,$$

(c) if $p > n$ then

$$\|u\|_{L^\infty} \leq c_4 R^{1-\frac{n}{p}}\|Du\|_{L^p}.$$

The Riesz potential of a function f with respect to a positive integer α is defined as

$$I_\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}, dy$$

where Γ is the gamma function. In your case, the value of α is 1, so the Riesz potential becomes

$$I_1 f(x) = \frac{1}{\Gamma(1)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-1}}, dy$$

Since $\Gamma(1) = 1$, the Riesz potential reduces to

$$I_1 f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-1}}, dy$$

This is the general formula for the Riesz potential with $\alpha = 1$. Note that the value of n (the dimension of the space) can be any positive integer. To compute the Riesz potential for a specific function f, you will need to specify the value of n and evaluate the integral for each value of x where the potential is desired.

Bibliography

[1] L. C. Evans. *Partial differential equations.* American Mathematical Society, Providence, RI, second edition, 2010.

[2] M. Giaquinta and L. Martinazzi. *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs.* Edizioni della Normale, Pisa, second edition, 2012.