Notes on Inequalities and Embeddings

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Preface

Todo.

Abbreviation

Since these notes is not for formal research, we almost always employ the following abbreviations:

⋄ s.t. : s.t.

⋄ TFAE : The following are equivalent

Notation

Let a set X. We employ the following notations:

⋄ $\mathbb{R}^n$: n-dimensional real Euclidean space, $\mathbb{R} = \mathbb{R}^1$

⋄ $\mathbb{C}^n$: n-dimensional complex space, $\mathbb{C} = \mathbb{C}^1$

⋄ $\mathbb{K}$: either $\mathbb{R}$ or $\mathbb{C}$.

⋄ ∂X : boundary of X.

⋄ $\forall$: for all.

Chapter 1

Definitions and notations

In this note, we always denote $x = (x_1, x_2, \ldots, x_n)$ to be a point in $\mathbb{R}^n$.

1.1 Definitions

Here, we collect definitions for averages of a function.

Definition 1.1.1. Let $f \in L^1(U)$ with an open set $U \subset \mathbb{R}^n$.

(a) An average of f over set E is

$$\fint_E f \, dx := \frac{1}{\text{meas}(E)} \int_E f(x) \, dx.$$

(b) An average of f over the ball $B_r(x_0)$ is

$$\fint_{B_r(x_0)} f \, dx := \frac{1}{\alpha(n)r^n} \int_{B_r(x_0)} f(x) \, dx.$$

(c) An average of f over the sphere $\partial B_r(x_0)$ is

$$\fint_{\partial B_r(x_0)} f \, dS := \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B_r(x_0)} f(x) \, dS.$$

As for integrability, TODO.

Definition 1.1.2. Let $p > 1$ and define $q \in \mathbb{R}$ by

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then p and q are called *conjugate exponents*.

Remark 1.1.3. A simple calculation shows the following:

(i) $pq = p + q$,

(ii) $1 = \dfrac{p + q}{pq}$,

(iii) $(p-1)(q-1) = 1$,

(iv) $q = \dfrac{p}{p-1}$.

Chapter 2

Basic properties

2.1 Change of variable

Proposition 2.1.1. *Let $f : U \to \mathbb{R}$ with an open set $U \subset \mathbb{R}^n$.*

(a) $\displaystyle\int_{B_r(x_0)} f(ax + b)dx = \frac{1}{r^n} \int_{B_{ar}(x_0+b)} f(x)dx.$

(b) $\displaystyle\fint_{B_r(x_0)} f(ax + b)dx = \fint_{B_{ar}(x_0+b)} f(x)dx.$

Proof. TODO

To show (b), let $\tilde{x} := ax + b$. Then $d\tilde{x} = r^n dx$. Also, since $|\tilde{x} - b| = |ax| < |a|\, r$, we get $\tilde{x} \in B_{|a|r}(x_0 + b)$. Hence,

$$\fint_{B_r(x_0)} f(ax + b)dx = \frac{1}{\alpha(n)r^n} \int_{B_r(x_0)} f(x)dx$$

$$= \fint_{B_{ar}(x_0+b)} f(\tilde{x})d\tilde{x}$$

$$= \fint_{B_{ar}(x_0+b)} f(\tilde{x})d\tilde{x}.$$

$\square$

Remark 2.1.2. Convolution. TODO

2.2 Coordinates

2.2.1 Polar coordinates

Proposition 2.2.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous and summable. Then*

(a) $\displaystyle\int_{\mathbb{R}^n} f(x)dx = \int_0^\infty \left(\int_{\partial B_r(x_0)} f(x)d\mathcal{S} \right) dr \quad \forall x_0 \in \mathbb{R}^n.$

(b) $\displaystyle\frac{d}{dr} \left(\int_{B_r(x_0)} f(x)dx \right) = \int_{\partial B_r(x_0)} f(x)d\mathcal{S} \quad \forall r > 0.$

Proposition 2.2.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous and summable. Then*

(a) $\displaystyle\int_{B_\varepsilon(0)} f(x)dx = \int_0^\varepsilon \left(\int_{\partial B_r(0)} f(x)d\mathcal{S}(x) \right) dr \quad \forall \varepsilon > 0.$

Chapter 3

Inequalities

3.1 Power inequalities

Theorem 3.1.1. *The following statements holds.*

(a) $1 + x \le e^x \quad \forall x \in \mathbb{R}.$

(b) (Cauchy's inequality)
$$xy \le \frac{x^2}{2} + \frac{y^2}{2} \quad \forall x, y \in \mathbb{R}.$$

(c) $e^{(x+y)/2} < \dfrac{e^y - e^x}{y - x} \quad \forall x, y \in \mathbb{R}$ with $x \ne y.$

Theorem 3.1.2. *The following statements holds.*

(a) $\left(\dfrac{1}{e}\right)^{\frac{1}{e}} \le x^x \quad \forall x > 0.$

(b) $x \le x^{x^x} \quad \forall x > 0.$

(c) $1 < x^y + y^x \quad \forall x, y > 0.$

(d) $x^y + y^x \le x^x + y^y \quad \forall x, y > 0.$

(e) $x^{ey} + y^{ex} \le x^{ex} + y^{ey} \quad \forall x, y > 0.$

(f) $\dfrac{1 - \frac{1}{x^y}}{y} \le \ln(x) \le \dfrac{x^y - 1}{y} \quad \forall x, y > 0.$ The upper and lower bounds converge to $\ln(x)$ as $y \to 0.$

(g) $2 < (x + y)^z + (x + z)^y + (y + z)^x \quad \forall x, y, z > 0.$

(h) $x^{2y} + y^{2z} + z^{2x} \le x^{2x} + y^{2y} + z^{2z} \quad \forall x, y, z > 0.$

(i) $(xyz)^{(x+y+z)/3} \le x^x y^y z^z \quad \forall x, y, z > 0.$

Theorem 3.1.3. *The following statements holds.*

(a) (Cauchy's inequality with ε)
$$xy \le \epsilon x^2 + \frac{y^2}{4\varepsilon} \quad \forall x, y > 0 \ , \ \forall \varepsilon > 0.$$

Theorem 3.1.4. *The following statements holds.*

(a) $(x + y)^p < x^p + y^p \quad \forall x, y > 0, \forall p \in (0, 1).$

(b) $(x + y)^p \le 2^{p-1} (x^p + y^p) \quad \forall x, y \ge 0, \forall p \in [1, \infty).$

Chapter 4

Embeddings

4.1 Sobolev Embedding

In this section, we deal with embeddings of diverse Sobolev spaces into others. Given a Sobolev space, it automatically belongs to certain other space, depending on the relationship between the integrability p and the dimension n.[1] There are three cases:

$$p \in [1, n),$$
$$p = n,$$
$$p \in (n, \infty].$$

In particular, the second case $p = n$ is called the *borderline case*. What are we trying to obtain from the Sobolev embedding theory? Broadly speaking, given a Sobolev space $W^{k,p}$, imbeddings of $W^{k,p}$ target two types of Banach spaces: either another Sobolev space $W^{j,q}$ or Hölder spaces $C^{j,\alpha}$ for some constants $j \leq k$, $q \geq p$, and $0 \leq \alpha \leq 1$.

4.1.1 The case $1 \leq p < n$

Definition 4.1.1. If $1 \leq p < n$, the *Sobolev conjugate* p^* of p is defined by

$$p^* := \frac{np}{n - p}.$$

Remark 4.1.2. A simple calculation shows the following:

(i) $p^* > p$,

(ii) $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}$,

(iii) $p^* \to \infty$ as $p \to n$.

Theorem 4.1.3. (Gagliardo-Nirenberg-Sobolev inequality) *If $1 \leq p < n$, then there exists a constant C, depending only on p and n, such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)},$$

for all $u \in C_c^1(\mathbb{R}^n)$.

Theorem 4.1.4. (Estimates for $W^{1,p}, 1 \leq p < n$) *Let U be a bounded, open subset of $\mathbb{R}^n$, and suppose ∂U is C^1. Assume $1 \leq p < n$ and $u \in W^{1,p}(U)$. Then $u \in L^{p^*}(U)$, with the estimate*

$$\|u\|_{L^{p^*}(U)} \leq C\|u\|_{W^{1,p}(U)},$$

the constant C depending only on $p, n,$ and U.

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[1] The regularity of domains also affects the inclusion.

4.1.2 The case $p = n$

4.1.3 The case $n < p \leq \infty$

For convenience as in the notation for the Sobolev conjugate, we write

$$\gamma := 1 - \frac{n}{p},$$

whenever $n < p \leq \infty$.

Theorem 4.1.5. (Morrey's inequality) *If $n < p \leq \infty$, then there exists a constant C, depending only on p and n, such that*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

for all $u \in C^1(\mathbb{R}^n)$.

Theorem 4.1.6. (Estimates for $W^{1,p}$, $n < p \leq \infty$, [1]) *Let U be a bounded, open subset of $\mathbb{R}^n$, and suppose ∂U is C^1. Assume $n < p \leq \infty$ and $u \in W^{1,p}(U)$. Then u has a version $u^* \in C^{0,\gamma}(\overline{U})$ with with the estimate*

$$\|u^*\|_{C^{0,\gamma}(\overline{U})} \leq C\|u\|_{W^{1,p}(U)},$$

the constant C depending only on p, n and U.

4.1.4 The case $p = \infty$

Theorem 4.1.7. (Characterization of $W^{1,\infty}$, [1]) *Let U be open and bounded, with ∂U of class C^1. Then $u : U \to \mathbb{R}$ is Lipschitz continuous if and only if $u \in W^{1,\infty}(U)$*

4.1.5 Summary

Theorem 4.1.8. (Sobolev, [2, Theorem 7.29]) *Let $u \in W_0^{1,p}(B_R(0))$, where $B_R(0) \subset \mathbb{R}^n$. Then there are universal constants c, c_1, c_2 and c_3, depending on n, such that*

(a) if $1 < p < n$ then
$$\|u\|_{L^{p^*}} \leq c\|Du\|_{L^p},$$

(b) if $p = n$ then
$$\fint_{B_R(0)} \exp\left(c_1 \frac{|u|}{\|Du\|_{L^n}}\right)^{\frac{n}{n-1}} dx \leq c_2,$$

(c) if $p > n$ then
$$\|u\|_{L^\infty} \leq c_3 R^{1-\frac{n}{p}} \|Du\|_{L^p}.$$

Bibliography

[1] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[2] M. Giaquinta and L. Martinazzi. *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs*, volume 11 of *Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]*. Edizioni della Normale, Pisa, second edition, 2012.