

PDE NOTES

Notes on Inequalities and Embeddings

Kiyuob Jung

(in progress)



Department of Mathematics,
Kyungpook National University

December 9, 2022

Contents

1	Definitions and notations	3
1.1	Definitions	3
2	Basic properties	4
2.1	Change of variable	4
2.2	Coordinates	4
2.2.1	Polar coordinates	4
3	Inequalities	5
3.1	Scalar	5
3.1.1	Power inequalities	5
3.1.2	product type	5
3.1.3	summation type	6
3.2	Function	6
4	Embeddings	7
4.1	Sobolev Embedding	7
4.1.1	The case $1 \leq p < n$	7
4.1.2	The case $p = n$	8
4.1.3	The case $n < p \leq \infty$	8
4.1.4	The case $p = \infty$	8
4.1.5	Summary	8

Preface

Todo.

Abbreviation

Since these notes is not for formal research, we almost always employ the following abbreviations:

- ◇ s.t. : s.t.
- ◇ TFAE : The following are equivalent

Notation

Let a set X . We employ the following notations:

- ◇ \mathbb{R}^n : n -dimensional real Euclidean space, $\mathbb{R} = \mathbb{R}^1$
- ◇ \mathbb{C}^n : n -dimensional complex space, $\mathbb{C} = \mathbb{C}^1$
- ◇ \mathbb{K} : either \mathbb{R} or \mathbb{C} .
- ◇ ∂X : boundary of X .
- ◇ \forall : for all.

Chapter 1

Definitions and notations

In this note, we always denote $x = (x_1, x_2, \dots, x_n)$ to be a point in \mathbb{R}^n .

1.1 Definitions

Here, we collect definitions for averages of a function.

Definition 1.1.1. Let $f \in L^1(U)$ with an open set $U \subset \mathbb{R}^n$.

(a) An average of f over set E is

$$\oint_E f dx := \frac{1}{\text{meas}(E)} \int_E f(x) dx.$$

(b) An average of f over the ball $B_r(x_0)$ is

$$\oint_{B_r(x_0)} f dx := \frac{1}{\alpha(n)r^n} \int_{B_r(x_0)} f(x) dx.$$

(c) An average of f over the sphere $\partial B_r(x_0)$ is

$$\oint_{\partial B_r(x_0)} f dS := \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B_r(x_0)} f(x) dS.$$

As for integrability, TODO.

Definition 1.1.2. Let $p > 1$ and define $q \in \mathbb{R}$ by

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then p and q are called *conjugate exponents*.

Remark 1.1.3. A simple calculation shows the following:

- (i) $pq = p + q$,
- (ii) $1 = \frac{p+q}{pq}$,
- (iii) $(p-1)(q-1) = 1$,
- (iv) $q = \frac{p}{p-1}$.

Chapter 2

Basic properties

2.1 Change of variable

Proposition 2.1.1. *Let $f : U \rightarrow \mathbb{R}$ with an open set $U \subset \mathbb{R}^n$.*

$$(a) \int_{B_r(x_0)} f(ax+b)dx = \frac{1}{r^n} \int_{B_{ar}(x_0+b)} f(x)dx.$$

$$(b) \int_{B_r(x_0)} f(ax+b)dx = \int_{B_{ar}(x_0+b)} f(x)dx.$$

Proof. TODO

To show (b), let $\tilde{x} := ax + b$. Then $d\tilde{x} = r^n dx$. Also, since $|\tilde{x} - b| = |ax| < |a|r$, we get $\tilde{x} \in B_{|a|r}(x_0 + b)$. Hence,

$$\begin{aligned} \int_{B_r(x_0)} f(ax+b)dx &= \frac{1}{\alpha(n)r^n} \int_{B_r(x_0)} f(x)dx \\ &= \int_{B_{ar}(x_0+b)} f(\tilde{x})d\tilde{x} \\ &= \int_{B_{ar}(x_0+b)} f(\tilde{x})d\tilde{x}. \end{aligned}$$

□

Remark 2.1.2. Convolution. TODO

2.2 Coordinates

2.2.1 Polar coordinates

Proposition 2.2.1. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuous and summable. Then*

$$(a) \int_{\mathbb{R}^n} f(x)dx = \int_0^\infty \left(\int_{\partial B_r(x_0)} f(x)d\mathcal{S} \right) dr \quad \forall x_0 \in \mathbb{R}^n.$$

$$(b) \frac{d}{dr} \left(\int_{B_r(x_0)} f(x)dx \right) = \int_{\partial B_r(x_0)} f(x)d\mathcal{S} \quad \forall r > 0.$$

Proposition 2.2.2. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuous and summable. Then*

$$(a) \int_{B_\varepsilon(0)} f(x)dx = \int_0^\varepsilon \left(\int_{\partial B_r(0)} f(x)d\mathcal{S}(x) \right) dr \quad \forall \varepsilon > 0.$$

Chapter 3

Inequalities

3.1 Scalar

3.1.1 Power inequalities

Theorem 3.1.1. *The following statements holds.*

- (a) $1 + x \leq e^x \quad \forall x \in \mathbb{R}.$
- (b) $e^{(x+y)/2} < \frac{e^y - e^x}{y - x} \quad \forall x, y \in \mathbb{R} \text{ with } x \neq y.$

Theorem 3.1.2. *The following statements holds.*

- (a) $\left(\frac{1}{e}\right)^{\frac{1}{e}} \leq x^x \quad \forall x > 0.$
- (b) $x \leq x^{x^x} \quad \forall x > 0.$
- (c) $1 < x^y + y^x \quad \forall x, y > 0.$
- (d) $x^y + y^x \leq x^x + y^y \quad \forall x, y > 0.$
- (e) $x^{ey} + y^{ex} \leq x^{ex} + y^{ey} \quad \forall x, y > 0.$
- (f) $\frac{1 - \frac{1}{x^y}}{y} \leq \ln(x) \leq \frac{x^y - 1}{y} \quad \forall x, y > 0.$ The upper and lower bounds converge to $\ln(x)$ as $y \rightarrow 0.$
- (g) $2 < (x + y)^z + (x + z)^y + (y + z)^x \quad \forall x, y, z > 0.$
- (h) $x^{2y} + y^{2z} + z^{2x} \leq x^{2x} + y^{2y} + z^{2z} \quad \forall x, y, z > 0.$
- (i) $(xyz)^{(x+y+z)/3} \leq x^x y^y z^z \quad \forall x, y, z > 0.$

3.1.2 product type

Theorem 3.1.3. *The following statements holds.*

- (a) (Cauchy's inequality)
$$xy \leq \frac{x^2}{2} + \frac{y^2}{2} \quad \forall x, y \in \mathbb{R}.$$
- (b) (Cauchy's inequality with ε)
$$xy \leq \varepsilon x^2 + \frac{y^2}{4\varepsilon} \quad \forall x, y > 0, \forall \varepsilon > 0.$$
- (c) (Young's inequality)
$$xy \leq \frac{x^p}{p} + \frac{y^q}{q} \quad \forall x, y > 0, \forall 1 < p < q < \infty, p \text{ and } q \text{ are conjugate exponents.}$$

3.1.3 summation type

Theorem 3.1.4. *The following statements holds.*

- (a) $(x + y)^p < x^p + y^p \quad \forall x, y > 0, \forall p \in (0, 1).$
- (b) $(x + y)^p \leq 2^{p-1} (x^p + y^p) \quad \forall x, y \geq 0, \forall p \in [1, \infty).$

3.2 Function

Theorem 3.2.1. *Let Ω be an open set in \mathbb{R}^n . The following statements holds.*

- (a) (Hölder's inequality)

If $u \in L^p(\Omega)$, $v \in L^q(\Omega)$ with $1 \leq p, q \leq \infty$, where p and q are conjugate exponents, then

$$\int_{\Omega} |uv| dx \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}.$$

- (b) (General version of Hölder's inequality)

If $u_i \in L^{p_i}(\Omega)$ for $i = 1, 2, \dots, m$ with $1 \leq p_1, p_2, \dots, p_m \leq \infty$ satisfying $\frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_m} = 1$, then

$$\int_{\Omega} |u_1 u_2 \cdots u_m| dx \leq \prod_{i=1}^m \|u_i\|_{L^{p_i}(\Omega)}.$$

- (c) (Minkowski's inequality)

If $u, v \in L^p(\Omega)$ with $1 \leq p \leq \infty$, then

$$\|u + v\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}.$$

- (d) (Interpolation inequality)

Let $1 \leq p \leq r \leq q \leq \infty$, with

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$$

for some $\theta \in [0, 1]$. If $u \in L^p(\Omega) \cap L^q(\Omega)$, then we also have $u \in L^r(\Omega)$ and

$$\|u\|_{L^r(\Omega)} \leq \|u\|_{L^p(\Omega)}^{\theta} \|u\|_{L^q(\Omega)}^{1-\theta}.$$

Chapter 4

Embeddings

4.1 Sobolev Embedding

In this section, we deal with embeddings of diverse Sobolev spaces into others. Given a Sobolev space, it automatically belongs to certain other space, depending on the relationship between the integrability p and the dimension n .¹ There are three cases:

$$\begin{aligned} p &\in [1, n), \\ p &= n, \\ p &\in (n, \infty]. \end{aligned}$$

In particular, the second case $p = n$ is called the *borderline case*. What are we trying to obtain from the Sobolev embedding theory? Broadly speaking, given a Sobolev space $W^{k,p}$, imbeddings of $W^{k,p}$ target two types of Banach spaces: either another Sobolev space $W^{j,q}$ or Hölder spaces $C^{j,\alpha}$ for some constants $j \leq k$, $q \geq p$, and $0 \leq \alpha \leq 1$.

4.1.1 The case $1 \leq p < n$

Definition 4.1.1. If $1 \leq p < n$, the *Sobolev conjugate* p^* of p is defined by

$$p^* := \frac{np}{n-p}.$$

Remark 4.1.2. A simple calculation shows the following:

- (i) $p^* > p$,
- (ii) $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$,
- (iii) $p^* \rightarrow \infty$ as $p \rightarrow n$.

Theorem 4.1.3. (Gagliardo-Nirenberg-Sobolev inequality) *If $1 \leq p < n$, then there exists a constant C , depending only on p and n , such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|Du\|_{L^p(\mathbb{R}^n)},$$

for all $u \in C_c^1(\mathbb{R}^n)$.

Theorem 4.1.4. (Estimates for $W^{1,p}$, $1 \leq p < n$) *Let U be a bounded, open subset of \mathbb{R}^n , and suppose ∂U is C^1 . Assume $1 \leq p < n$ and $u \in W^{1,p}(U)$. Then $u \in L^{p^*}(U)$, with the estimate*

$$\|u\|_{L^{p^*}(U)} \leq C \|u\|_{W^{1,p}(U)},$$

the constant C depending only on p, n , and U .

¹The regularity of domains also affects the inclusion.

4.1.2 The case $p = n$

4.1.3 The case $n < p \leq \infty$

For convenience as in the notation for the Sobolev conjugate, we write

$$\gamma := 1 - \frac{n}{p},$$

whenever $n < p \leq \infty$.

Theorem 4.1.5. (Morrey's inequality) *If $n < p \leq \infty$, then there exists a constant C , depending only on p and n , such that*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(\mathbb{R}^n)}$$

for all $u \in C^1(\mathbb{R}^n)$.

Theorem 4.1.6. (Estimates for $W^{1,p}$, $n < p \leq \infty$, [1]) *Let U be a bounded, open subset of \mathbb{R}^n , and suppose ∂U is C^1 . Assume $n < p \leq \infty$ and $u \in W^{1,p}(U)$. Then u has a version $u^* \in C^{0,\gamma}(\bar{U})$ with the estimate*

$$\|u^*\|_{C^{0,\gamma}(\bar{U})} \leq C \|u\|_{W^{1,p}(U)},$$

the constant C depending only on p, n and U .

4.1.4 The case $p = \infty$

Theorem 4.1.7. (Characterization of $W^{1,\infty}$, [1]) *Let U be open and bounded, with ∂U of class C^1 . Then $u : U \rightarrow \mathbb{R}$ is Lipschitz continuous if and only if $u \in W^{1,\infty}(U)$*

4.1.5 Summary

Theorem 4.1.8. (Sobolev, [2, Theorem 7.29]) *Let $u \in W_0^{1,p}(B_R(0))$, where $B_R(0) \subset \mathbb{R}^n$. Then there are universal constants c_1, c_2, c_3 and c_4 , depending on n , such that*

(a) if $1 < p < n$ then

$$\|u\|_{L^{p^*}} \leq c_1 \|Du\|_{L^p},$$

(b) if $p = n$ then

$$\int_{B_R(0)} \exp\left(c_2 \frac{|u|}{\|Du\|_{L^n}}\right)^{\frac{n}{n-1}} dx \leq c_3,$$

(c) if $p > n$ then

$$\|u\|_{L^\infty} \leq c_4 R^{1-\frac{n}{p}} \|Du\|_{L^p}.$$

Bibliography

- [1] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.
- [2] M. Giaquinta and L. Martinazzi. *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs*, volume 11 of *Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]*. Edizioni della Normale, Pisa, second edition, 2012.