PDE NOTES

Notes on Inequalities and Embeddings

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Preface

These are notes from the seminar with the authors that are heavily based on the book Lecture Notes on Functional Analysis With Applications to Linear Partial Differential Equations by Alberto Bressan, together with other sources that are mostly listed in the Bibliography. Usually, we conform the contents of the book as well as the numberings.

Abbreviation

Since these notes is not for formal research, we almost always employ the following abbreviations:
◇ iff : if and only if
◇ s.t. : s.t.
◇ TFAE : The following are equivalent
◇ v.s. : vector space
◇ n.v.s. : normed vector space
◇ f.n.v.s. : finite dimensional normed vector space

Notation

Let a set X. We employ the following notations:
◇ $\mathbb{R}^n$: n-dimensional real Euclidean space, $\mathbb{R} = \mathbb{R}^1$
◇ $\mathbb{C}^n$: n-dimensional complex space, $\mathbb{C} = \mathbb{C}^1$
◇ $\mathbb{K}$: either $\mathbb{R}$ or $\mathbb{C}$.
◇ ∂X : boundary of X.
◇ $\mathcal{C}(X) = \{f : X \to \mathbb{R} : f$ is continuous (possibly unbounded) on $X\}$.
◇ $d(x, y)$: Euclidean distance.

Chapter 1

Introduction

Chapter 2

Definitions and notations

In this note, we always denote $x = (x_1, x_2, \dots, x_n)$ to be a point in $\mathbb{R}^n$.

2.1 Definitions

Definition 2.1.1. Let $p > 1$ and define $q \in \mathbb{R}$ by

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then p and q are called *conjugate exponents*.

Remark 2.1.2. A simple calculation shows the following:

(i) $pq = p + q$,

(ii) $1 = \dfrac{p+q}{pq}$,

(iii) $(p-1)(q-1) = 1$,

(iv) $q = \dfrac{p}{p-1}$.

Chapter 3

Basic properties

3.1 Averages of a function

Definition 3.1.1. Let $f \in L^1(U)$ with an open set $U \subset \mathbb{R}^n$.

(a) An average of f over set E is

$$\fint_E f dx := \frac{1}{\text{meas}(E)} \int_E f(x)dx.$$

(b) An average of f over the ball $B_r(x_0)$ is

$$\fint_{B_r(x_0)} f dx := \frac{1}{\alpha(n)r^n} \int_{B_r(x_0)} f(x)dx.$$

(c) An average of f over the sphere $\partial B_r(x_0)$ is

$$\fint_{\partial B_r(x_0)} f dS := \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B_r(x_0)} f(x)dS.$$

3.1.1 Change of variable

Theorem 3.1.2. *Let $f : U \to \mathbb{R}$ with an open set $U \subset \mathbb{R}^n$.*

(a) $\displaystyle\int_{B_r(x_0)} f(ax+b)dx = \frac{1}{r^n} \int_{B_{ar}(x_0+b)} f(x)dx.$

(b) $\displaystyle\fint_{B_r(x_0)} f(ax+b)dx = \fint_{B_{ar}(x_0+b)} f(x)dx.$

Proof. TODO

To show (b), let $\tilde{x} := ax + b$. Then $d\tilde{x} = r^n dx$. Also, since $|\tilde{x} - b| = |ax| < |a|\,r$, we get $\tilde{x} \in B_{|a|r}(x_0 + b)$. Hence,

$$\begin{aligned}
\fint_{B_r(x_0)} f(ax+b)dx &= \frac{1}{\alpha(n)r^n} \int_{B_r(x_0)} f(x)dx \\
&= \fint_{B_{ar}(x_0+b)} f(\tilde{x})d\tilde{x} \\
&= \fint_{B_{ar}(x_0+b)} f(\tilde{x})d\tilde{x}.
\end{aligned}$$

□

3.1.2 Polar coordinates

Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous and summable. Then

(a) $\displaystyle\int_{\mathbb{R}^n} f dx = \int_0^\infty \left(\int_{\partial B_r(x_0)} f(x) dS \right) dr \quad \forall x_0 \in \mathbb{R}^n.$

(b) $\displaystyle\frac{d}{dr} \left(\int_{B_r(x_0)} f(x) dx \right) = \int_{\partial B_r(x_0)} f(x) dS$ for each $r > 0$.

Chapter 4

Inequalities

4.1 Power inequalities

Theorem 4.1.1. *The following statements holds.*

(a) $1+x \leq e^x \quad \forall x \in \mathbb{R}$.

(b) (Cauchy's inequality)
$$xy \leq \frac{x^2}{2} + \frac{y^2}{2} \quad \forall x, y \in \mathbb{R}$$

(c) $e^{(x+y)/2} < \dfrac{e^y - e^x}{y-x} \quad \forall x, y \in \mathbb{R}$ with $x \neq y$.

Theorem 4.1.2. *The following statements holds.*

(a) $\left(\dfrac{1}{e}\right)^{\frac{1}{e}} \leq x^x \quad \forall x > 0$.

(b) $x \leq x^{x^x} \quad \forall x > 0$.

(c) $1 < x^y + y^x \quad \forall x, y > 0$

(d) $x^y + y^x \leq x^x + y^y \quad \forall x, y > 0$

(e) $x^{ey} + y^{ex} \leq x^{ex} + y^{ey} \quad \forall x, y > 0$

(f) $\dfrac{1 - \frac{1}{x^y}}{y} \leq \ln(x) \leq \dfrac{x^y - 1}{y} \quad \forall x, y > 0$. The upper and lower bounds converge to $\ln(x)$ as $y \to 0$.

(g) $2 < (x+y)^z + (x+z)^y + (y+z)^x \quad \forall x, y, z > 0$.

(h) $x^{2y} + y^{2z} + z^{2x} \leq x^{2x} + y^{2y} + z^{2z} \quad \forall x, y, z > 0$.

(i) $(xyz)^{(x+y+z)/3} \leq x^x y^y z^z \quad \forall x, y, z > 0$

Theorem 4.1.3. *The following statements holds.*

(a) (Cauchy's inequality with ε)
$$xy \leq \epsilon x^2 + \frac{y^2}{4\varepsilon} \quad \forall x, y > 0\ ,\ \forall \varepsilon > 0.$$

Theorem 4.1.4. *The following statements holds.*

(a) $(x+y)^p < x^p + y^p \quad \forall x, y > 0$, $\forall p \in (0,1)$.

(b) $(x+y)^p \leq 2^{p-1}(x^p + y^p) \quad \forall x, y \geq 0$, $\forall p \in [1, \infty)$.

Chapter 5

Embeddings

5.1 Sobolev Embedding

In this section, we deal with embeddings of diverse Sobolev spaces into others. Given a Sobolev space, it automatically belongs to certain other space, depending on the relationship between the integrability p and the dimension n.[1] There are three cases:

$$\begin{aligned} &p \in [1, n), \\ &p = n, \\ &p \in (n, \infty]. \end{aligned}$$

In particular, the second case $p = n$ is called the *borderline case*. What are we trying to obtain from the Sobolev embedding theory? Broadly speaking, given a Sobolev space $W^{k,p}$, imbeddings of $W^{k,p}$ target two types of Banach spaces: either another Sobolev space $W^{j,q}$ or Hölder spaces $C^{j,\alpha}$ for some constants $j \leq k$, $q \geq p$, and $0 \leq \alpha \leq 1$.

5.1.1 The case $1 \leq p < n$

Definition 5.1.1. If $1 \leq p < n$, the *Sobolev conjugate* p^* of p is defined by

$$p^* := \frac{np}{n-p}.$$

Remark 5.1.2. A simple calculation shows the following:

(i) $p^* > p$,

(ii) $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}$,

(iii) $p^* \to \infty$ as $p \to n$.

Theorem 5.1.3. (Gagliardo-Nirenberg-Sobolev inequality) *If $1 \leq p < n$, then there exists a constant C, depending only on p and n, such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)},$$

for all $u \in C_c^1(\mathbb{R}^n)$.

Theorem 5.1.4. (Estimates for $W^{1,p}, 1 \leq p < n$) *Let U be a bounded, open subset of $\mathbb{R}^n$, and suppose ∂U is C^1. Assume $1 \leq p < n$ and $u \in W^{1,p}(U)$. Then $u \in L^{p^*}(U)$, with the estimate*

$$\|u\|_{L^{p^*}(U)} \leq C\|u\|_{W^{1,p}(U)},$$

the constant C depending only on $p, n,$ and U.

[1] The regularity of domains also affects the inclusion.

5.1.2 The case $p = n$

5.1.3 The case $n < p \leq \infty$

For convenience as in the notation for the Sobolev conjugate, we write

$$\gamma := 1 - \frac{n}{p},$$

whenever $n < p \leq \infty$.

Theorem 5.1.5. (Morrey's inequality) *If $n < p \leq \infty$, then there exists a constant C, depending only on p and n, such that*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

for all $u \in C^1(\mathbb{R}^n)$.

Theorem 5.1.6. (Estimates for $W^{1,p}$, $n < p \leq \infty$) *Let U be a bounded, open subset of $\mathbb{R}^n$, and suppose ∂U is C^1. Assume $n < p \leq \infty$ and $u \in W^{1,p}(U)$. Then u has a version $u^* \in C^{0,\gamma}(\overline{U})$ with with the estimate*

$$\|u^*\|_{C^{0,\gamma}(\bar{U})} \leq C\|u\|_{W^{1,p}(U)},$$

the constant C depending only on p, n and U.

5.1.4 The case $p = \infty$

Theorem 5.1.7. (Characterization of $W^{1,\infty}$) *Let U be open and bounded, with ∂U of class C^1. Then $u : U \to \mathbb{R}$ is Lipschitz continuous if and only if $u \in W^{1,\infty}(U)$*