PDE Notes

Notes on Inequalities and Embeddings

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Preface

Todo.

Abbreviation

Since these notes is not for formal research, we almost always employ the following abbreviations:
◇ s.t. : s.t.
◇ TFAE : The following are equivalent

Notation

Let a set X. We employ the following notations:
◇ $\mathbb{R}^n$: n-dimensional real Euclidean space, $\mathbb{R} = \mathbb{R}^1$
◇ $\mathbb{C}^n$: n-dimensional complex space, $\mathbb{C} = \mathbb{C}^1$
◇ $\mathbb{K}$: either $\mathbb{R}$ or $\mathbb{C}$.
◇ ∂X : boundary of X.
◇ $\forall$: for all.

Chapter 1

Definitions and notations

In this note, we always denote $x = (x_1, x_2, \dots, x_n)$ to be a point in $\mathbb{R}^n$.

1.1 Definitions

1.1.1 Integration

Here, we collect definitions for averages of a function.

Definition 1.1.1. Let $f \in L^1(\Omega)$ with an open set $\Omega \subset \mathbb{R}^n$.

(a) An average of f over set E is

$$⨍_E f dx := \frac{1}{\text{meas}(E)} \int_E f(x) dx.$$

(b) An average of f over the ball $B_r(x_0)$ is

$$⨍_{B_r(x_0)} f dx := \frac{1}{\alpha(n) r^n} \int_{B_r(x_0)} f(x) dx.$$

(c) An average of f over the sphere $\partial B_r(x_0)$ is

$$⨍_{\partial B_r(x_0)} f dS := \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B_r(x_0)} f(x) dS.$$

When it comes to integrability, the following definition plays an important role.

Definition 1.1.2. Let $p > 1$ and define $p' \in \mathbb{R}$ by

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Then p and p' are called *conjugate exponents.*

Even if some authors use q instead of p', we stick to using p' in order to save letters and will use q to stand for other integrability.

Remark 1.1.3. A simple calculation shows the following:

(i) $pp' = p + p'$,

(ii) $1 = \dfrac{p + p'}{pp'}$,

(iii) $(p-1)(p'-1) = 1$,

(iv) $p' = \dfrac{p}{p-1}$.

1.1.2 Convex functions

Definition 1.1.4. A function $f : \mathbb{R}^n \to \mathbb{R}$ is *convex* if

$$f(\lambda x + (1-\lambda)y) \leq \tau f(x) + (1-\lambda)f(y)$$

for all $x, y \in \mathbb{R}^n$ and each $0 \leq \lambda \leq 1$.

Remark 1.1.5. If f is C^2, then f is convex if and only if $D^2 f \geq 0$.

Definition 1.1.6. A C^2 function $f : \mathbb{R}^n \to \mathbb{R}$ is *uniformly convex* if $D^2 g \geq \theta I$ for some constant $\theta > 0$, that is,

$$\sum_{i,j=1}^{n} f_{x_i x_j}(x)\xi_i \xi_j \geq \theta |\xi|^2$$

for all $x, \xi \in \mathbb{R}^n$.

Chapter 2

Basic properties

2.1 Change of variable

Proposition 2.1.1. *Let $f : \Omega \to \mathbb{R}$ with an open set $\Omega \subset \mathbb{R}^n$.*

(a) $\displaystyle\int_{B_r(x_0)} f(ax+b)dx = \frac{1}{r^n}\int_{B_{ar}(x_0+b)} f(x)dx.$

(b) $\displaystyle\fint_{B_r(x_0)} f(ax+b)dx = \fint_{B_{ar}(x_0+b)} f(x)dx.$

Proof. TODO

To show (b), let $\tilde{x} := ax+b$. Then $d\tilde{x} = r^n dx$. Also, since $|\tilde{x}-b| = |ax| < |a|\, r$, we get $\tilde{x} \in B_{|a|r}(x_0+b)$. Hence,

$$\begin{aligned}\fint_{B_r(x_0)} f(ax+b)dx &= \frac{1}{\alpha(n)r^n}\int_{B_r(x_0)} f(x)dx \\ &= \fint_{B_{ar}(x_0+b)} f(\tilde{x})d\tilde{x} \\ &= \fint_{B_{ar}(x_0+b)} f(\tilde{x})d\tilde{x}.\end{aligned}$$

□

Remark 2.1.2. Convolution. TODO

2.2 Coordinates

2.2.1 Polar coordinates

Proposition 2.2.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous and summable. Then*

(a) $\displaystyle\int_{\mathbb{R}^n} f(x)dx = \int_0^\infty \left(\int_{\partial B_r(x_0)} f(x)d\mathcal{S}\right) dr \quad \forall x_0 \in \mathbb{R}^n.$

(b) $\displaystyle\frac{d}{dr}\left(\int_{B_r(x_0)} f(x)dx\right) = \int_{\partial B_r(x_0)} f(x)d\mathcal{S} \quad \forall r > 0.$

Proposition 2.2.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous and summable. Then*

(a) $\displaystyle\int_{B_\varepsilon(0)} f(x)dx = \int_0^\varepsilon \left(\int_{\partial B_r(0)} f(x)d\mathcal{S}(x)\right) dr \quad \forall \varepsilon > 0.$

Chapter 3

Inequalities

3.1 Scalar

3.1.1 Power inequalities

Theorem 3.1.1. *The following statements hold.*

(a) $1 + x \le e^x \quad \forall x \in \mathbb{R}$.

(b) $e^{(x+y)/2} < \dfrac{e^y - e^x}{y - x} \quad \forall x, y \in \mathbb{R}$ with $x \neq y$.

Theorem 3.1.2. *The following statements hold.*

(a) $\left(\dfrac{1}{e}\right)^{\frac{1}{e}} \le x^x \quad \forall x > 0$.

(b) $x \le x^{x^x} \quad \forall x > 0$.

(c) $1 < x^y + y^x \quad \forall x, y > 0$.

(d) $x^y + y^x \le x^x + y^y \quad \forall x, y > 0$.

(e) $x^{ey} + y^{ex} \le x^{ex} + y^{ey} \quad \forall x, y > 0$.

(f) $\dfrac{1 - \frac{1}{x^y}}{y} \le \ln(x) \le \dfrac{x^y - 1}{y} \quad \forall x, y > 0$. The upper and lower bounds converge to $\ln(x)$ as $y \to 0$.

(g) $2 < (x + y)^z + (x + z)^y + (y + z)^x \quad \forall x, y, z > 0$.

(h) $x^{2y} + y^{2z} + z^{2x} \le x^{2x} + y^{2y} + z^{2z} \quad \forall x, y, z > 0$.

(i) $(xyz)^{(x+y+z)/3} \le x^x y^y z^z \quad \forall x, y, z > 0$.

3.1.2 product type

Theorem 3.1.3. *The following statements hold.*

(a) (Cauchy's inequality)

$$xy \le \frac{x^2}{2} + \frac{y^2}{2} \quad \forall x, y \in \mathbb{R}.$$

(b) (Cauchy's inequality with ε)

$$xy \le \varepsilon x^2 + \frac{y^2}{4\varepsilon} \quad \forall x, y > 0, \quad \forall \varepsilon > 0.$$

(c) (Young's inequality)

$$xy \leq \frac{x^p}{p} + \frac{y^{p'}}{p'} \quad \forall x, y > 0, \quad \forall 1 < p, p' < \infty,$$

where p and p' are conjugate exponents.

3.1.3 summation type

Theorem 3.1.4. *The following statements hold.*

(a) $(x+y)^p < x^p + y^p \quad \forall x, y > 0, \forall p \in (0,1)$.

(b) $(x+y)^p \leq 2^{p-1}(x^p + y^p) \quad \forall x, y \geq 0, \forall p \in [1, \infty)$.

3.2 Function

3.2.1 Convex functions

Theorem 3.2.1. (Jensen's inequality)
Assume $f : \mathbb{R}^m \to \mathbb{R}$ is convex and $\Omega \subset \mathbb{R}^n$ is open and bounded. Let $\mathbf{u} : \Omega \to \mathbb{R}^m$ be summable. Then

$$f\left(\fint_\Omega \mathbf{u} dx\right) \leq \fint_\Omega f(\mathbf{u}) dx.$$

3.2.2 Sobolev space

Theorem 3.2.2. *Let Ω be an open set in $\mathbb{R}^n$. The following statements hold.*

(a) (Hölder's inequality)
If $u \in L^p(\Omega)$, $v \in L^{p'}(\Omega)$ with $1 \leq p, p' \leq \infty$, where p and p' are conjugate exponents, then

$$\int_\Omega |uv| dx \leq \|u\|_{L^p(\Omega)} \|g\|_{L^{q'}(\Omega)}.$$

(b) (General version of Hölder's inequality)
If $u_i \in L^{p_i}(\Omega)$ for $i = 1, 2, \ldots, m$ with $1 \leq p_1, p_2, \ldots, p_m \leq \infty$ satisfying $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = 1$, then

$$\int_\Omega |u_1 u_2 \cdots u_m| dx \leq \prod_{i=1}^m \|u_i\|_{L^{p_i}(\Omega)}.$$

(c) (Minkowski's inequality)
If $u, v \in L^p(\Omega)$ with $1 \leq p \leq \infty$, then

$$\|u + v\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}.$$

(d) (Interpolation inequality)
Let $1 \leq p \leq r \leq q \leq \infty$, with

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$$

for some $\theta \in [0,1]$. If $u \in L^p(\Omega) \cap L^q(\Omega)$, then we also have $u \in L^r(\Omega)$ and

$$\|u\|_{L^r(\Omega)} \leq \|u\|_{L^p(\Omega)}^\theta \|u\|_{L^q(\Omega)}^{1-\theta}.$$

Chapter 4

Embeddings

4.1 Sobolev Embedding

In this section, we deal with embeddings of diverse Sobolev spaces into others. Given a Sobolev space, it automatically belongs to certain other space, depending on the relationship between the integrability p and the dimension n.[1] There are three cases:

$$\begin{aligned} &p \in [1, n), \\ &p = n, \\ &p \in (n, \infty]. \end{aligned}$$

In particular, the second case $p = n$ is called the *borderline case*. What are we trying to obtain from the Sobolev embedding theory? Broadly speaking, given a Sobolev space $W^{k,p}$, imbeddings of $W^{k,p}$ target two types of Banach spaces: either another Sobolev space $W^{j,q}$ or Hölder spaces $C^{j,\alpha}$ for some constants $j \leq k$, $q \geq p$, and $0 \leq \alpha \leq 1$.

4.1.1 The case $1 \leq p < n$

Definition 4.1.1. If $1 \leq p < n$, the *Sobolev conjugate* p^* of p is defined by

$$p^* := \frac{np}{n-p}.$$

Remark 4.1.2. A simple calculation shows the following:

(i) $p^* > p$,

(ii) $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}$,

(iii) $p^* \to \infty$ as $p \to n$.

Theorem 4.1.3. (Gagliardo-Nirenberg-Sobolev inequality, [1])
If $1 \leq p < n$, then there exists a constant C, depending only on p and n, such that

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)},$$

for all $u \in C_c^1(\mathbb{R}^n)$.

Theorem 4.1.4. (Estimates for $W^{1,p}, 1 \leq p < n$, [1])
Let U be a bounded, open subset of $\mathbb{R}^n$, and suppose ∂U is C^1. Assume $1 \leq p < n$ and $u \in W^{1,p}(U)$. Then $u \in L^{p^*}(U)$, with the estimate

$$\|u\|_{L^{p^*}(U)} \leq C\|u\|_{W^{1,p}(U)},$$

the constant C depending only on p, n, and U.

[1] The regularity of domains also affects the inclusion.

4.1.2 The case $p = n$

4.1.3 The case $n < p \leq \infty$

For convenience as in the notation for Hölder exponents, we write

$$\gamma := 1 - \frac{n}{p},$$

whenever $n < p \leq \infty$.

Theorem 4.1.5. (Morrey's inequality, [1])
If $n < p \leq \infty$, then there exists a constant C, depending only on p and n, such that

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

for all $u \in C^1(\mathbb{R}^n)$.

Theorem 4.1.6. (Estimates for $W^{1,p}$, $n < p \leq \infty$, [1])
Let U be a bounded, open subset of $\mathbb{R}^n$, and suppose ∂U is C^1. Assume $n < p \leq \infty$ and $u \in W^{1,p}(U)$. Then u has a version $u^* \in C^{0,\gamma}(\overline{U})$ with with the estimate

$$\|u^*\|_{C^{0,\gamma}(\bar{U})} \leq C\|u\|_{W^{1,p}(U)},$$

the constant C depending only on p, n and U.

4.1.4 The case $p = \infty$

Theorem 4.1.7. (Characterization of $W^{1,\infty}$, [1])
Let U be open and bounded, with ∂U of class C^1. Then $u : U \to \mathbb{R}$ is Lipschitz continuous if and only if $u \in W^{1,\infty}(U)$

4.1.5 Summary

Theorem 4.1.8. (Sobolev, [2, Theorem 7.29])
Let $u \in W_0^{1,p}(B_R(0))$, where $B_R(0) \subset \mathbb{R}^n$. Then there are universal constants c_1, c_2, c_3 and c_4, depending on n, such that

(a) if $1 < p < n$ then
$$\|u\|_{L^{p^*}} \leq c_1\|Du\|_{L^p},$$

(b) if $p = n$ then
$$\fint_{B_R(0)} \exp\left(c_2 \frac{|u|}{\|Du\|_{L^n}}\right)^{\frac{n}{n-1}} dx \leq c_3,$$

(c) if $p > n$ then
$$\|u\|_{L^\infty} \leq c_4 R^{1-\frac{n}{p}}\|Du\|_{L^p}.$$

Bibliography

[1] L. C. Evans. *Partial differential equations.* American Mathematical Society, Providence, RI, second edition, 2010.

[2] M. Giaquinta and L. Martinazzi. *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs.* Edizioni della Normale, Pisa, second edition, 2012.