# Notes on Inequalities and Embedding

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## 1 Introduction

In this note, we always denote $x = (x_1, x_2, \ldots, x_n)$ to be a point in $\mathbb{R}^n$.

## 2 Definition

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### 2.1 Definitions

**Definition 2.1.** Let $p > 1$ and define $q \in \mathbb{R}$ by

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then $p$ and $q$ are called *conjugate exponents*.

**Remark 2.2.** A simple calculation shows the following:

(i) $pq = p + q$,

(ii) $1 = \dfrac{p+q}{pq}$,

(iii) $(p-1)(q-1) = 1$,

(iv) $q = \dfrac{p}{p-1}$.

## 3 Inequalities

## 4 Sobolev Embedding

In this section, we deal with embeddings of diverse Sobolev spaces into others. Given a Sobolev space, it automatically belongs to certain other space, depending on the relationship between the integrability $p$ and the dimension $n$.[1] There are three cases:

$$\begin{aligned} &p \in [1, n), \\ &p = n, \\ &p \in (n, \infty]. \end{aligned}$$

In particular, the second case $p = n$ is called the *borderline case*. What are we trying to obtain from the Sobolev embedding theory? Broadly speaking, given a Sobolev space $W^{k,p}$, imbeddings of $W^{k,p}$ target two types of Banach spaces: either another Sobolev space $W^{j,q}$ or Hölder spaces $C^{j,\alpha}$ for some constants $j \leq k$, $q \geq p$, and $0 \leq \alpha \leq 1$.

[1] The regularity of domains also affects the inclusion.

## 4.1 The case $1 \leq p < n$

**Definition 4.1.** If $1 \leq p < n$, the *Sobolev conjugate* $p^*$ of $p$ is defined by

$$p^* := \frac{np}{n-p}.$$

**Remark 4.2.** A simple calculation shows the following:

(i) $p^* > p$,

(ii) $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}$,

(iii) $p^* \to \infty$ as $p \to n$.

**Theorem 4.3.** (Gagliardo-Nirenberg-Sobolev inequality) *If $1 \leq p < n$, then there exists a constant $C$, depending only on $p$ and $n$, such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)},$$

*for all $u \in C_c^1(\mathbb{R}^n)$.*

**Theorem 4.4.** (Estimates for $W^{1,p}, 1 \leq p < n$) *Let $U$ be a bounded, open subset of $\mathbb{R}^n$, and suppose $\partial U$ is $C^1$. Assume $1 \leq p < n$ and $u \in W^{1,p}(U)$. Then $u \in L^{p^*}(U)$, with the estimate*

$$\|u\|_{L^{p^*}(U)} \leq C\|u\|_{W^{1,p}(U)},$$

*the constant $C$ depending only on $p, n,$ and $U$.*

## 4.2 The case $p = n$

## 4.3 The case $n < p \leq \infty$

For convenience as in the notation for the Sobolev conjugate, we write

$$\gamma := 1 - \frac{n}{p},$$

whenever $n < p \leq \infty$.

**Theorem 4.5.** (Morrey's inequality) *If $n < p \leq \infty$, then there exists a constant $C$, depending only on $p$ and $n$, such that*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

*for all $u \in C^1(\mathbb{R}^n)$.*

**Theorem 4.6.** (Estimates for $W^{1,p}$, $n < p \leq \infty$) *Let $U$ be a bounded, open subset of $\mathbb{R}^n$, and suppose $\partial U$ is $C^1$. Assume $n < p \leq \infty$ and $u \in W^{1,p}(U)$. Then $u$ has a version $u^* \in C^{0,\gamma}(\bar{U})$ with with the estimate*

$$\|u^*\|_{C^{0,\gamma}(\bar{U})} \leq C\|u\|_{W^{1,p}(U)},$$

*the constant $C$ depending only on $p, n$ and $U$.*

## 4.4 The case $p = \infty$

**Theorem 4.7.** (Characterization of $W^{1,\infty}$) *Let $U$ be open and bounded, with $\partial U$ of class $C^1$. Then $u : U \to \mathbb{R}$ is Lipschitz continuous if and only if $u \in W^{1,\infty}(U)$*

### 4.5 The general case

$\Omega \subset \mathbb{R}^n$ to be open and

**Theorem 4.8** (Hypothesis)**.** *. The following are equivalent:*

(i) *text.*

(ii) *text.*

(iii) *text.*

**Lemma 4.9** (Hypothesis)**.** *. The following hold:*

(a) *text*

(b) *text*

(c) *text*

$$= \begin{cases} , & \text{if} \\ , & \text{if} \\ , & \text{if} \end{cases}$$

then

1. the direct scattery problem is to determine $u^s$ from $u^i$;
2. the inverse scattery problem is to determine the nature of inhomogeneity to reconstruct the differential equation and/or its domain from a knowledge of the asymptotic behavior $u^s$.