Notes on Inequalities and Embeddings

Kiyuob Jung

(in progress)

Department of Mathematics,
Kyungpook National University

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Preface

Todo.

Abbreviation

Since these notes is not for formal research, we almost always employ the following abbreviations:
◇ s.t. : s.t.
◇ TFAE : The following are equivalent

Notation

Let a set X. We employ the following notations:
◇ $\mathbb{R}^n$: n-dimensional real Euclidean space, $\mathbb{R} = \mathbb{R}^1$
◇ $\mathbb{C}^n$: n-dimensional complex space, $\mathbb{C} = \mathbb{C}^1$
◇ $\mathbb{K}$: either $\mathbb{R}$ or $\mathbb{C}$.
◇ ∂X : boundary of X.
◇ $\forall$: for all.

Chapter 1

Definitions and notations

In this note, we always denote $x = (x_1, x_2, \ldots, x_n)$ to be a point in $\mathbb{R}^n$.

1.1 Definitions

Here, we collect definitions for averages of a function.

Definition 1.1.1. Let $f \in L^1(U)$ with an open set $U \subset \mathbb{R}^n$.

(a) An average of f over set E is

$$⨍_E f dx := \frac{1}{\mathrm{meas}(E)} \int_E f(x) dx.$$

(b) An average of f over the ball $B_r(x_0)$ is

$$⨍_{B_r(x_0)} f dx := \frac{1}{\alpha(n) r^n} \int_{B_r(x_0)} f(x) dx.$$

(c) An average of f over the sphere $\partial B_r(x_0)$ is

$$⨍_{\partial B_r(x_0)} f dS := \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B_r(x_0)} f(x) dS.$$

As for integrability, TODO.

Definition 1.1.2. Let $p > 1$ and define $q \in \mathbb{R}$ by

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then p and q are called *conjugate exponents.*

Remark 1.1.3. A simple calculation shows the following:

(i) $pq = p + q$,

(ii) $1 = \dfrac{p+q}{pq}$,

(iii) $(p-1)(q-1) = 1$,

(iv) $q = \dfrac{p}{p-1}$.

Chapter 2

Basic properties

2.0.1 Change of variable

Theorem 2.0.1. *Let $f : U \to \mathbb{R}$ with an open set $U \subset \mathbb{R}^n$.*

(a) $\displaystyle\int_{B_r(x_0)} f(ax+b)dx = \frac{1}{r^n}\int_{B_{ar}(x_0+b)} f(x)dx.$

(b) $\displaystyle⨍_{B_r(x_0)} f(ax+b)dx = ⨍_{B_{ar}(x_0+b)} f(x)dx.$

Proof. TODO

To show (b), let $\tilde{x} := ax+b$. Then $d\tilde{x} = r^n dx$. Also, since $|\tilde{x}-b| = |ax| < |a|\, r$, we get $\tilde{x} \in B_{|a|r}(x_0+b)$. Hence,

$$\begin{aligned} ⨍_{B_r(x_0)} f(ax+b)dx &= \frac{1}{\alpha(n)r^n}\int_{B_r(x_0)} f(x)dx \\ &= ⨍_{B_{ar}(x_0+b)} f(\tilde{x})d\tilde{x} \\ &= ⨍_{B_{ar}(x_0+b)} f(\tilde{x})d\tilde{x}. \end{aligned}$$

□

2.0.2 Polar coordinates

Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous and summable. Then

(a) $\displaystyle\int_{\mathbb{R}^n} f dx = \int_0^\infty \left(\int_{\partial B_r(x_0)} f(x)dS\right) dr \quad \forall x_0 \in \mathbb{R}^n.$

(b) $\displaystyle\frac{d}{dr}\left(\int_{B_r(x_0)} f(x)dx\right) = \int_{\partial B_r(x_0)} f(x)dS$ for each $r > 0$.

Chapter 3

Inequalities

3.1 Power inequalities

Theorem 3.1.1. *The following statements holds.*

(a) $1 + x \leq e^x \quad \forall x \in \mathbb{R}$.

(b) (Cauchy's inequality)

$$xy \leq \frac{x^2}{2} + \frac{y^2}{2} \quad \forall x, y \in \mathbb{R}$$

(c) $e^{(x+y)/2} < \dfrac{e^y - e^x}{y - x} \quad \forall x, y \in \mathbb{R}$ with $x \neq y$.

Theorem 3.1.2. *The following statements holds.*

(a) $\left(\dfrac{1}{e}\right)^{\frac{1}{e}} \leq x^x \quad \forall x > 0$.

(b) $x \leq x^{x^x} \quad \forall x > 0$.

(c) $1 < x^y + y^x \quad \forall x, y > 0$

(d) $x^y + y^x \leq x^x + y^y \quad \forall x, y > 0$

(e) $x^{ey} + y^{ex} \leq x^{ex} + y^{ey} \quad \forall x, y > 0$

(f) $\dfrac{1 - \frac{1}{x^y}}{y} \leq \ln(x) \leq \dfrac{x^y - 1}{y} \quad \forall x, y > 0$. The upper and lower bounds converge to $\ln(x)$ as $y \to 0$.

(g) $2 < (x+y)^z + (x+z)^y + (y+z)^x \quad \forall x, y, z > 0$.

(h) $x^{2y} + y^{2z} + z^{2x} \leq x^{2x} + y^{2y} + z^{2z} \quad \forall x, y, z > 0$.

(i) $(xyz)^{(x+y+z)/3} \leq x^x y^y z^z \quad \forall x, y, z > 0$

Theorem 3.1.3. *The following statements holds.*

(a) (Cauchy's inequality with ε)

$$xy \leq \epsilon x^2 + \frac{y^2}{4\varepsilon} \quad \forall x, y > 0\ ,\ \forall \varepsilon > 0.$$

Theorem 3.1.4. *The following statements holds.*

(a) $(x+y)^p < x^p + y^p \quad \forall x, y > 0, \forall p \in (0, 1)$.

(b) $(x+y)^p \leq 2^{p-1}(x^p + y^p) \quad \forall x, y \geq 0, \forall p \in [1, \infty)$.

Chapter 4

Embeddings

4.1 Sobolev Embedding

In this section, we deal with embeddings of diverse Sobolev spaces into others. Given a Sobolev space, it automatically belongs to certain other space, depending on the relationship between the integrability p and the dimension n.[1] There are three cases:

$$\begin{aligned} &p \in [1, n), \\ &p = n, \\ &p \in (n, \infty]. \end{aligned}$$

In particular, the second case $p = n$ is called the *borderline case*. What are we trying to obtain from the Sobolev embedding theory? Broadly speaking, given a Sobolev space $W^{k,p}$, imbeddings of $W^{k,p}$ target two types of Banach spaces: either another Sobolev space $W^{j,q}$ or Hölder spaces $C^{j,\alpha}$ for some constants $j \leq k$, $q \geq p$, and $0 \leq \alpha \leq 1$.

4.1.1 The case $1 \leq p < n$

Definition 4.1.1. If $1 \leq p < n$, the *Sobolev conjugate* p^* of p is defined by

$$p^* := \frac{np}{n-p}.$$

Remark 4.1.2. A simple calculation shows the following:

(i) $p^* > p$,

(ii) $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}$,

(iii) $p^* \to \infty$ as $p \to n$.

Theorem 4.1.3. (Gagliardo-Nirenberg-Sobolev inequality) *If $1 \leq p < n$, then there exists a constant C, depending only on p and n, such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|Du\|_{L^p(\mathbb{R}^n)},$$

for all $u \in C_c^1(\mathbb{R}^n)$.

Theorem 4.1.4. (Estimates for $W^{1,p}, 1 \leq p < n$) *Let U be a bounded, open subset of $\mathbb{R}^n$, and suppose ∂U is C^1. Assume $1 \leq p < n$ and $u \in W^{1,p}(U)$. Then $u \in L^{p^*}(U)$, with the estimate*

$$\|u\|_{L^{p^*}(U)} \leq C \|u\|_{W^{1,p}(U)},$$

the constant C depending only on $p, n,$ and U.

[1] The regularity of domains also affects the inclusion.

4.1.2 The case $p = n$

4.1.3 The case $n < p \leq \infty$

For convenience as in the notation for the Sobolev conjugate, we write

$$\gamma := 1 - \frac{n}{p},$$

whenever $n < p \leq \infty$.

Theorem 4.1.5. (Morrey's inequality) *If $n < p \leq \infty$, then there exists a constant C, depending only on p and n, such that*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

for all $u \in C^1(\mathbb{R}^n)$.

Theorem 4.1.6. (Estimates for $W^{1,p}$, $n < p \leq \infty$) *Let U be a bounded, open subset of $\mathbb{R}^n$, and suppose ∂U is C^1. Assume $n < p \leq \infty$ and $u \in W^{1,p}(U)$. Then u has a version $u^* \in C^{0,\gamma}(\overline{U})$ with with the estimate*

$$\|u^*\|_{C^{0,\gamma}(\bar{U})} \leq C\|u\|_{W^{1,p}(U)},$$

the constant C depending only on p, n and U.

4.1.4 The case $p = \infty$

Theorem 4.1.7. (Characterization of $W^{1,\infty}$) *Let U be open and bounded, with ∂U of class C^1. Then $u : U \to \mathbb{R}$ is Lipschitz continuous if and only if $u \in W^{1,\infty}(U)$*