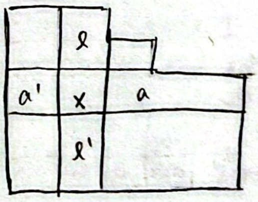
# Haiman: $t,q$ - Catalan Numbers and the Hilbert Scheme

recieved April 1995
revised submission August 1995
accepted March 1998

Original Catalan Numbers: $C_n = \frac{1}{n+1}\binom{2n}{n}$

To define Haiman's $t,q$ Catalan numbers, we need a few tools.




- We will use the French convention for Young diagrams.
- The leg of $x$ is the set of boxes above $x$.
- The coleg of $x$ is the set of boxes below $x$.
- The arm and coarm are defined analogously for boxes to the right and left of $x$ respectively.

- We write $l'(x)$ for the number of boxes in the coleg of $x$. We define $l(x)$, $a(x)$, and $a'(x)$ similarly.
- Note that $l'(x)$ and $a'(x)$ are the row and column coordinates of $x$ indexed from $(0,0)$.
- Define $n(\mu) = \sum_{x\in\mu} l(x)$.

Haiman's $t,q$ Catalan Numbers: $C_n^{(m)}(t,q)$

$$= \sum_{\mu \vdash n} \frac{t^{m\cdot n(\mu)} q^{m\, n(\mu')} (1-t)(1-q) \left(\prod_{x\in\mu\setminus(0,0)} \left(1-t^{l'(x)} q^{a'(x)}\right)\right)\left(\sum_{x\in\mu} t^{l'(x)} q^{a'(x)}\right)}{\prod_{x\in\mu} \left(1-t^{1+l(x)} q^{-a(x)}\right)\left(1-t^{-l(x)} q^{1+a(x)}\right)}$$

This paper works out the necessary geometry to explain how this formula came about and proves that

- for all integers $m \geq 0$ and $n \geq 1$, $C_n^{(m)}(t,q)$ is a polynomial in $t$ and $q$, and
- for $m$ sufficiently large ($n$ fixed), this polynomial has non-negative coefficients.

Hint: This involves diagonal harmonics, Hilbert polynomials, and Hilbert schemes.

I'll talk about the later today.

# A Combinatorialist's View of the Hilbert Scheme of Points in the Plane

For this talk, all fields will be characteristic 0. We could be more general if we wished, though.

Def'n The punctual Hilbert scheme of the plane, $H^n = \mathrm{Hilb}^n(\mathbb{A}^2)$, is the set of all ideals $I \subseteq \mathbb{k}[x,y]$ such that $\dim_{\mathbb{k}}(\mathbb{k}[x,y]/I) = n$.

Notes: Our field is $\mathbb{k}$ and $\mathbb{k}[x,y]$ is the polynomial ring in the variables $x$ and $y$.

Let's describe the scheme structure of $H^n$ via explicit coordinates on open affine subsets indexed by partitions $\mu$ of $n$.

Given $\mu \vdash n$, let

$$B_\mu = \{ x^h y^k \mid (h,k) \in \mu \}.$$

Ex. $B_{(4,4,2,2)}$

$$\begin{matrix} x^3 & x^3y & & \\ x^2 & x^2y & & \\ x & xy & xy^2 & xy^3 \\ 1 & y & y^2 & y^3 \end{matrix}$$

Ex. $B_{(2,1)}$

$$\begin{matrix} x & \\ 1 & y \end{matrix}$$

Now define

$$U_\mu = \{ I \in H^n \mid B_\mu \text{ spans } \mathbb{k}[x,y]/I \}.$$

Ex. $\langle x^2, xy, y^2 \rangle \in U_{(2,1)}$

$\langle x^2 - y, xy, y^2 \rangle \in U_{(2,1)}$

$\langle x^2 - y, xy, y^2 - x^2 \rangle \in U_{(2,1)}$

Since $\dim_{\mathbb{k}}(\mathbb{k}[x,y]/I) = n$ by definition of $H^n$, this make $B_\mu$ a basis modulo $I$ of $\mathbb{k}[x,y]/I$. Since $B_\mu$ is a basis, for each monomial $x^r y^s$ and ideal $I \in U_\mu$, there is a unique expansion

$$x^r y^s \equiv \sum_{(h,k)\in\mu} c^{rs}_{hk}(I)\, x^h y^k \pmod{I}, \qquad (*)$$

whose coefficients depend on $I$ and thus define a collection of functions

$$c^{rs}_{hk} : U_\mu \to \mathbb{k}.$$

**Prop (2.1)** The sets $U_\mu$ are open affine subvarieties which cover $H^n$. The affine coordinate ring $\mathcal{O}_{U_\mu}$ is generated by the functions $c_{hk}^{rs}$ for $(h,k) \in \mu$ and all $(r,s)$.

Pf. The sets $U_\mu$ cover $H^n$ because:

Fact from Gordan, often attributed to Gröbner basis theory

for every ideal $I$ in a polynomial ring there is a basis $\mathcal{B}$ modulo $I$, consisting of monomials, such that every divisor of a monomial in $\mathcal{B}$ is also in $\mathcal{B}$.

For $I$ in $H^n$, it is clear that such a basis must be $B_\mu$ for some $\mu \vdash n$.

We now black box some facts. There is an injective map

$$H^n \longrightarrow G^n(\Bbbk M_N)$$

Grassmann variety of $n$-dim quotients of the linear span of $M_N$, the set of all monomials of degree at most $N$

which defines the structure of $H^n$ as a scheme.

The sets $U_\mu$ are the preimages under this embedding of standard affines on $G^n(\Bbbk M_N)$ and the standard coordinates on these affines reduce to the functions $c_{hk}^{rs}$. The image of $U_\mu$ is closed in the corresponding standard affine on $G^n(\Bbbk M_N)$, so the functions $c_{hk}^{rs}$ generate $\mathcal{O}_{U_\mu}$. □

For each $I \in H^n$, the scheme $S = \mathrm{Spec}(\Bbbk[x,y]/I)$ has a finite number of points. If we assign each point $p \in S$ a multiplicity $m_p$ equal to the length of the local ring $\mathcal{O}_{p,S} = (\Bbbk[x,y]/I)_p$ then these multiplicities sum to $n$. In this way, we associate with $I$ an $n$-element multiset $\pi(I) \in \mathbb{A}^2$.

Ex. Let $I = (x^3, y) \subseteq \Bbbk[x,y]$.

On what points does this ideal vanish? Only at $(0,0)$. "The subscheme $S$ is concentrated at $(0,0)$."

Note that $\Bbbk[x,y]/I$ has basis $\{1, x, x^2\}$ so it is dimension 3 as a $\Bbbk$ vector space. The length at point $(0,0)$ is three, so $\pi(I)$ is $\{(0,0), (0,0), (0,0)\}$.

Ex. Let $I = (x(x-1), y) \subseteq \Bbbk[x,y]$.

Points vanishing on $I$: $(0,0)$ and $(1,0)$

Basis for $\Bbbk[x,y]/I$: $\{1, x\}$.

Multiset $\pi(I)$: $\{(0,0), (1,0)\}$.

[The n-element multisets contained in $\mathbb{A}^2$ form an affine variety $\mathrm{Sym}^n(\mathbb{A}^2)$.

So we have a map $\pi: H^n \to \mathrm{Sym}^n(\mathbb{A}^2)$ which is called the Chow morphism.

Prop (2.2) The Chow morphism $\pi: H^n \to \mathrm{Sym}^n(\mathbb{A}^2)$ is a projective morphism.

(In other words, it factors through a projective space.

Pf. Omitted.

The two dimensional torus group

$$T^2 = \{(t,q) \mid t,q \in \mathbb{k}^*\}$$

acts algebraically on $\mathbb{A}^2$ by $(t,q)\cdot(\xi,\zeta) = (t\xi, q\zeta)$, or equivalently on $\mathbb{k}[x,y]$ by $(t,q)\cdot x = tx$, $(t,q)\cdot y = qy$. There is an induced action on $H^n$ which is given by $(t,q)\cdot c_{hk}^{rs} = t^{r-h} q^{s-k} c_{hk}^{rs}$. Why? Because $(*)$ must remain invariant.
?

Note $(t,q)\cdot I = \{p(t^{-1}x, q^{-1}y) \mid p(x,y) \in I\}$.

An ideal $I \in H^n$ is a $T^2$ fixed point iff $I$ is spanned by monomials. Why? IDK

Such an ideal must be of the form

$$I_\mu = (x^h y^k \mid (h,k) \notin \mu)$$

for some partition $\mu$ of $n$. Note that the subscheme of $\mathbb{A}^2$ defined by such an ideal $I_\mu$ is concentrated at the origin, the sole $T^2$ fixed point of $\mathbb{A}^2$.

Lemma (2.3) Every ideal $I \in H^n$ has a torus fixed point in the closure of its orbit.

We have $\lim_{a\to 0} \lim_{b\to 0} (a,b)\cdot I = I_\mu$ for some partition $\mu$ of $n$.

Pf. Omitted.

<u>Prop (2.4)</u> The punctual Hilbert scheme $H^n$ of $\mathbb{A}^2$ is smooth and irreducible of dimension $2n$.

same dimension everywhere

Remark: This is special to the two dimensional setting. In three dimensions, we can't say much besides the parity.

<u>Pf.</u> It suffices to verify smoothness locally near each $T^2$ fixed point $I_\mu$. Why? IDK 2.3

Under $\pi: H^n \to \mathrm{Sym}^n(\mathbb{A}^2)$ we can see that the image $\pi(U_\mu)$ is dense in $\mathrm{Sym}^n(\mathbb{A}^2)$. So $U_\mu$ has at least dimension $2n$.

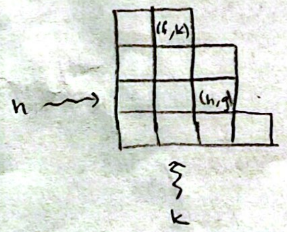
The maximal ideal of $I_\mu$ in $\mathcal{O}_{U_\mu}$ is given by

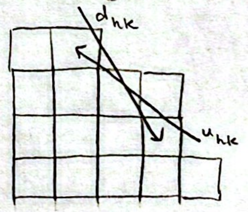
$$m = (c_{hk}^{rs} \mid (h,k) \in \mu, (r,s) \notin \mu).$$

Why? For $(r,s) \in \mu$ we have $c_{hk}^{rs} = 0$ when $(h,k) \neq (r,s)$ and $c_{rs}^{rs} = 1$, so we omit these $c_{hk}^{rs}$ from the ideal.

Plan: Find $2n$ of the coordinate functions $c_{hk}^{rs}$ which span the cotangent space $m/m^2$. This will show that $\dim m/m^2 = 2n$, so $H^n$ is smooth at $I_\mu$. *

We single out two special coordinate functions at each square $(h,k) \in \mu$. Let $(f,k)$ be the top square in column $k$ and let $(h,g)$ be the last square in row $h$.

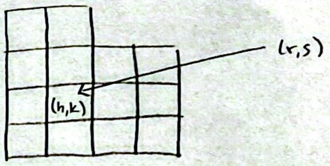



Now, let $u_{hk} = c_{f,k}^{h,g+1}$ and $d_{hk} = c_{h,g}^{f+1,k}$. These will be our $2n$ spanning parameters for $m/m^2$.

* It is convenient to depict each $c_{hk}^{rs}$ by an arrow from $(r,s)$ to $(h,k)$.

<u>Ex.</u>



Multiply (*) through by $x$ and then expand both sides by (*) again.

$$\sum_{(h,k)\in \mathcal{U}} c^{r+1,s}_{hk}\, x^h y^k \equiv \overset{\text{START}}{x^{r+1} y^s} \equiv \sum_{(f,g)\in \mathcal{U}} c^{rs}_{fg}\, x^{f+1} y^g = \sum_{(f,g)\in \mathcal{U}} c^{rs}_{fg} \left( \sum_{(h,k)\in \mathcal{U}} c^{f+1,g}_{hk}\, x^h y^k \right)$$

This yields $\quad c^{r+1,s}_{hk} = \sum_{(f,g)\in \mathcal{U}} c^{rs}_{fg}\, c^{f+1,g}_{hk}$.

Modulo $m^2$, the terms on the RHS reduce to zero besides one: $c^{rs}_{h-1,k}$.
(This is zero if $h=0$.)

So in $m/m^2$ we have

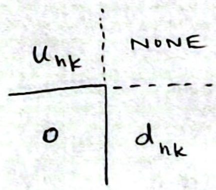
$$c^{r+1,s}_{hk} = c^{rs}_{h-1,k} \quad (\text{or } 0 \text{ if } h=0). \qquad (1)$$

We can do something analogous by multiplying (*) through by $y$ and we get

$$c^{r,s+1}_{hk} = c^{rs}_{h,k-1} \quad (\text{or } 0 \text{ if } k=0). \qquad (2)$$

Equations (1) and (2) say that we can slide arrows in our picture up, down, left, and right without changing their values modulo $m^2$ provided we keep the arrow head inside $\mathcal{U}$ and the tail outside. More generally, as long as we keep the the tail in the first quadrant outside of $\mathcal{U}$, we may even move the head across the $x$- or $y$- axis. When this is possible, the value of the arrow is zero.

What arrows survive? Only the $u_{hk}$ and $d_{hk}$.




Since we have our $2n$ spanning elements, we can conclude that $H^n$ is smooth as claimed.

The lemma above shows that $H^n$ is connected. [Smooth and connected → irreducible.
? □