Linear Algebra Review

$$\begin{cases} 3x_1 + 2x_2 + x_3 = 6 \\ -x_1 + 4x_2 + 5x_3 = 8 \\ 2x_1 - 8x_2 + 10x_3 = 4 \end{cases} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 6 \\ -1 & 4 & 5 & 8 \\ 2 & -8 & 10 & 4 \end{array}\right] (1/20)(2R_2 + R_3) \rightarrow R_3$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 6 \\ -1 & 4 & 5 & 8 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} R_1 - R_3 \rightarrow R_1 \\ -R_2 + 5R_3 \rightarrow R_2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 0 & 5 \\ 1 & -4 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right] 1/7(2R_1 + R_2) \rightarrow R_2$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 0 & 5 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} R_1 \leftrightarrow R_2 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 3 & 2 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right] (1/2)(R_2 - 3R_1) \rightarrow R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\therefore \begin{cases} x_1 = 1 \\ x_2 = 1 \\ x_3 = 1 \end{cases}$$

Eigenvalues

$$A = \begin{pmatrix} 0 & -4 & 6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{pmatrix}$$

$$A\underline{v} = \lambda \underline{v}$$

$$A\underline{v} - \lambda I \underline{v}$$

$$(A - \lambda \underline{I})\, \underline{v} = 0$$

$$\det |A - \lambda I| = 0$$

$$\begin{vmatrix} -\lambda & -4 & -6 \\ -1 & -\lambda & -3 \\ 1 & 2 & 5-\lambda \end{vmatrix}$$

$$= (-\lambda)(-\lambda)(5-\lambda) + (-4)(-3)(1) + (-6)(-1)(2)$$
$$-(-6)(-\lambda)(1) - (-\lambda)(-3)(2) - (-4)(-1)(5-\lambda)$$

$$= \lambda^2(5-\lambda) + 12 + 12 - 6\lambda - 6\lambda - 20 + 4\lambda$$

$$= -\lambda^3 + 5\lambda^2 - 8\lambda + 4$$

roots of characteristic equation are the eigenvalues

$$\lambda_1 = 1$$
$$\left.\begin{matrix} \lambda_2 = 2 \\ \lambda_3 = 2 \end{matrix}\right\} \text{repeated roots}$$

Convexity

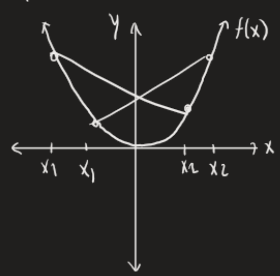
A function is convex if it obeys Jensen's inequality:

$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2) \ \forall \alpha \in [0,1] \wedge \{x_1, x_2\} \in \mathcal{X}$

or

$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2) \ \forall \alpha \in (0,1) \wedge \{x_1, x_2 \mid x_1 \neq x_2\} \in \mathcal{X}$

1) $x^2 + ax + b$



generalized form: $f(\underline{x}) = \frac{1}{2}\underline{x}^T A \underline{x} + b^T \underline{x} + c$

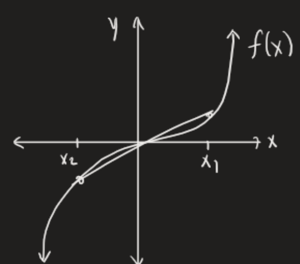
$f(\underline{x})$ is convex when $\underline{x}^T A \underline{x} \geq 0$

In scalar form, $f(x) = ax^2 + x + c$ where $x^2 \geq 0 \ \forall \ x \in \mathbb{R}$.

$\therefore f(x)$ is convex if $a \geq 0$ (see right)

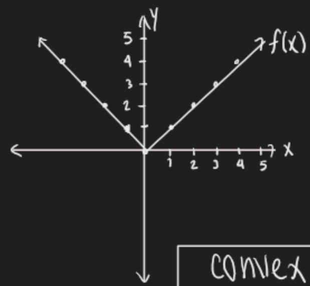
2) $x^3, x \in \mathbb{R}$

$\boxed{\text{not convex}}$



(also not concave)

3) $|x|, x \in [-5, 5]$

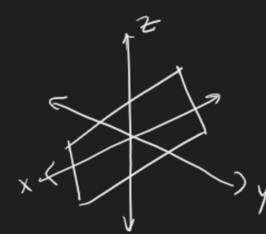


$\boxed{\text{convex}}$

$$
\begin{aligned}
f(\alpha x_1 + (1-\alpha) x_2) &= |\alpha x_1 + (1-\alpha) x_2| \\
&\leq |\alpha x_1| + |(1-\alpha) x_2| \\
&= |\alpha| |x_1| + |1-\alpha| |x_2| \\
&= \alpha |x_1| + (1-\alpha) |x_2| \\
&= \alpha f(x_1) + (1-\alpha) f(x_2)
\end{aligned}
$$

4) $x+y, x \in \mathbb{R}, y \in \mathbb{R}$

$$
\begin{aligned}
f(\alpha \underline{z}_1 + (1-\alpha) \underline{z}_2) &= f\left(\alpha \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + (1-\alpha) \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right) \\
&= f\left(\begin{bmatrix} \alpha x_1 + (1-\alpha) x_2 \\ \alpha y_1 + (1-\alpha) y_2 \end{bmatrix} \right) \\
&= \alpha (x_1 + y_1) + (1-\alpha)(x_2 + y_2) \\
&= \alpha f(\underline{z}) + (1-\alpha) f(\underline{z}_2)
\end{aligned}
$$



$\boxed{\text{convex and concave}}$

5) xy

$$f(\alpha \underline{z} + (1-\alpha)\underline{z}) = f\left(\alpha \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + (1-\alpha)\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)$$

$$= f\left(\begin{bmatrix} \alpha x_1 + (1-\alpha)x_2 \\ \alpha y_1 + (1-\alpha)y_2 \end{bmatrix}\right)$$

$$= (\alpha x_1 + (1-\alpha)x_2)(\alpha y_1 + (1-\alpha)y_2)$$

$$= \alpha^2 x_1 y_1 + \alpha(1-\alpha)x_1 y_2 + \alpha(1-\alpha)x_2 y_1 + (1-\alpha)^2 x_2 y_2$$

$$\not\leq \alpha f(\underline{z}_1) + (1-\alpha) f(\underline{z}_2)$$

$\boxed{\text{not convex}}$ also not concave

6) $\log(x)$, $x \in (0,1]$

$$f(\alpha x_1 + (1-\alpha)x_2) = \log(\alpha x_1 + (1-\alpha)x_2)$$

$$\alpha f(x_1) + (1-\alpha) f(x_2) = \alpha \log(x_1) + (1-\alpha)\log(x_2)$$

$$f(\alpha x_1 + (1-\alpha)x_2) \neq \alpha f(x_1) + (1-\alpha) f(x_2)$$

but $f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2)$

$\boxed{\text{not convex}}$

Convexity

Prove the following properties

Consider twice differentiable function $f(x): \mathbb{R}^n \mapsto \mathbb{R}$

$f(x)$ is convex on $x \in \mathbb{R}^n$ iff

$f(x+p) \geq f(x) + \nabla f(x)^T p \quad \forall (x, p) \in \mathbb{R}^n$

PSD ⇒ convexity

Taylor expansion at $\underline{x}$:

$$f(\underline{y}) = f(\underline{x}) + \nabla f(\underline{x})^T (\underline{y} - \underline{x}) + \frac{1}{2}\left[(\underline{y}-\underline{x})^T H_f (\underline{y}-\underline{x}) \right].$$

If H_f is PSD, then

$(\underline{y}-\underline{x})^T H_f (\underline{y}-\underline{x}) \geq 0 \quad \forall (\underline{y} - \underline{x})$,

Which implies

$f(\underline{y}) \geq f(\underline{x}) + \nabla f(\underline{x})^T (\underline{y}-\underline{x})$.

$\therefore$ ~Q.E.D.

Convexity ⇒ PSD

For a small $\lambda > 0$,

$$f(\underline{x} + \lambda \underline{d}) = f(\underline{x}) + \lambda \nabla f(\underline{x})^T \underline{d} + \frac{\lambda^2}{2} \underline{d}^T H_f(\underline{x}) + \mathcal{O}(\|\lambda \underline{d}\|^2)$$

If f is convex, then

$f(\underline{x} + \lambda \underline{d}) \geq f(\underline{x}) + \lambda \nabla f(\underline{x})^T \underline{d}$

$\therefore$ for any $\underline{d} \in \mathbb{R}^n$,

$$\frac{\lambda^2}{2} \underline{d}^T H_f(\underline{x}) + \mathcal{O}(\|\lambda \underline{d}\|^2) \geq 0.$$

Divide by λ^2:

$$\frac{1}{2} \underline{d}^T H_f(\underline{x}) + \frac{1}{\lambda^2} \mathcal{O}(\|\lambda d\|^2) \geq 0,$$

and evaluate the limit:

$$\lim_{\lambda \to 0^+} \frac{1}{2} \underline{d}^T H_f(\underline{x}) + \frac{1}{\lambda^2} O(\|\lambda d\|^2) \geq 0.$$

$$\therefore \underline{d}^T H_f(\underline{x}) \geq 0.$$

PD ⇛ Strictly convex

Positive definite if

$$(\underline{y}-\underline{x})^T H_f (\underline{y}-\underline{x}) \geq 0 \quad \forall (\underline{y}-\underline{x}),$$

and strictly convex if

$$f(\underline{y}) > f(\underline{x}) + \nabla f(\underline{x})^T (\underline{y}-\underline{x}).$$

Taylor series expansion:

$$f(\underline{y}) = f(\underline{x}) + \nabla f(\underline{x})^T (\underline{y}-\underline{x}) + \frac{1}{2}\left[(\underline{y}-\underline{x})^T H_f (\underline{y}-\underline{x}) \right].$$

$$H_f > 0 \Rrightarrow$$

$$f(y) > f(\underline{x}) + \nabla f(\underline{x})^T (\underline{y}-\underline{x}).$$

$\therefore$ ~Q.E.D.