Truncation errors in finite difference formulas

The backwards finite difference is given by:

$$[D_t^- u]^n = \frac{u(t_n) - u(t_{n-1})}{\Delta t} \approx u'(t_n).$$

The error between the true and finite difference is:

$$R^n = [D_t^- u]^n - u'(t_n).$$

A Taylor series approximation a t_{n-1} yields:

$$
\begin{aligned}
u(t_{n-1}) &= u(t - \Delta t) \\
&= \sum_{i=0}^{\infty} \frac{1}{i!} \frac{d^i u}{dt^i}(t_n)(-\Delta t)^i \\
&= u(t_n) - u'(t_n)\Delta t + \frac{1}{2} u''(t_n)\Delta t^2 + \mathcal{O}(\Delta t^3).
\end{aligned}
$$

Insert result into error formula:

$$
\begin{aligned}
[D_t^- u]^n - u'(t_n) &= \frac{u(t_n) - u(t_{n-1})}{\Delta t} - u'(t_n) \\
&= \frac{u(t_n) - \left\{ u(t_n) - u'(t_n)\Delta t + \frac{1}{2} u''(t_n)\Delta t^2 + \mathcal{O}(\Delta t^3) \right\}}{\Delta t} - u'(t_n) \\
&= -\frac{1}{2} u''(t_n)\Delta t + \mathcal{O}(\Delta t^2).
\end{aligned}
$$

Thus, the truncation error is

$$R_n = -\frac{1}{2} u''(t_n)\Delta t + \mathcal{O}(\Delta t^2),$$

and when Δt is small, $-\frac{1}{2} u''(t_n)\Delta t$ dominates. Thus, the truncation error is first order in Δt.

The central finite difference is given by:

$$[D_t u]^n = \frac{u(t_{n+1/2}) - u(t_{n-1/2})}{\Delta t} \approx u'(t_n).$$

The error between the true and finite difference is:

$$R^n = [D_t u]^n - u'(t_n).$$

A Taylor series approximation at $t_{n\pm 1/2}$ yields:

$$u(t_{n-1/2}) = u(t-\Delta t)$$

$$= \sum_{i=0}^{\infty} \frac{1}{i!}\frac{d^i u}{dt^i}(t_n)\left(-\frac{\Delta t}{2}\right)^i$$

$$= u(t_n) - u'(t_n)\frac{\Delta t}{2} + \frac{1}{2}u''(t_n)\left(\frac{\Delta t}{2}\right)^2 - \frac{1}{6}u'''(t_n)\left(\frac{\Delta t}{2}\right)^3$$

$$+ \frac{1}{24}u''''(t_n)\left(\frac{\Delta t}{2}\right)^4 - \frac{1}{120}u'''''(t_n)\left(\frac{\Delta t}{2}\right)^5 + \mathcal{O}(\Delta t^6)$$

$$u(t_{n+1/2}) = u(t-\Delta t)$$

$$= \sum_{i=0}^{\infty} \frac{1}{i!}\frac{d^i u}{dt^i}(t_n)\left(\frac{\Delta t}{2}\right)^i$$

$$= u(t_n) + u'(t_n)\left(\frac{\Delta t}{2}\right) + \frac{1}{2}u''(t_n)\left(\frac{\Delta t}{2}\right)^2 + \frac{1}{6}u'''(t_n)\left(\frac{\Delta t}{2}\right)^3$$

$$+ \frac{1}{24}u''''(t_n)\left(\frac{\Delta t}{2}\right)^4 + \frac{1}{120}u'''''(t_n)\left(\frac{\Delta t}{2}\right)^5 + \mathcal{O}(\Delta t^6)$$

Now,

$$u(t_{n+1/2}) - u(t_{n-1/2}) = u'(t_n)\Delta t + \frac{1}{24}u'''(t_n)\Delta t^3 + \frac{1}{960}u'''''(t_n)\Delta t^5 + \mathcal{O}(\Delta t^7).$$

Write the truncation error as:

$$R^n = [D_t u]^n - u'(t_n).$$

$$= \frac{u(t_{n+1/2}) - u(t_{n-1/2})}{\Delta t} - u'(t_n)$$

$$= \frac{1}{24}u'''(t_n)\Delta t^2 + \mathcal{O}(\Delta t^4)$$

using only the even powers of Δt. Since $R^n \sim \Delta t^2$, the centered difference is of second order in Δt.