1 de Méré and the Earl of Yarborough

(a) Historically, the theory of modern probability began as a result of a famous correspondence between mathematicians Blaise Pascal and Pierre de Fermat. The correspondence came about as a result of problems posed to Pascal by the Chevalier de Méré, a noted gambler of the time. One of these problems concerned how to divide the stake in a game that is interrupted. Assume that the winner of the game was to be the one who first won 4 deals out of 7. One gambler has so far won 1 deal and the other 2 deals. They agree to divide the stake according to the probability each had of winning the game. Assuming that each player had an equal probability of winning a given deal, what is each player's probability of winning the game?

–Solution–

Suppose Alice has won 1 deal, and Bob has won 2 deals.
For Alice to win, she must win 3 out of 5. If Bob wins 2 before Alice has won 3, she loses.
Encode a win for Alice as 1 and a loss as 0. The following scenarios are wins for Alice:

- 111, probability 0.5^3
- 0111, probability 0.5^4
- 1011, probability 0.5^4
- 1101, probability 0.5^4

Any other scenario requires Bob to win 2 before Alice wins 3.
In total, $P(\text{Alice wins}) = 0.5^3 + 3(0.5)^4 = \frac{5}{16}$.
$P(\text{Bob wins}) = 1 - P(\text{Alice wins}) = \frac{11}{16}$.

(b) It is said that the Earl of Yarborough used to bet 1000 to 1 against being dealt a hand of 13 cards containing no card higher than 9 (the order is 2 through 10 followed by Jack, Queen, King, Ace). Did he have a good bet?

–Solution–

In a deck of 52 cards, there are 8 ranks 9 or lower, and of each rank, one card for each of the four suits. Thus, there are 32 cards of rank 9 or lower.
Therefore, there are $\binom{32}{13}$ ways to be dealt the low-ranking hand, while there are $\binom{52}{13}$ total

ways of being dealt a 13-card hand.

$$\begin{aligned}
\frac{\binom{32}{13}}{\binom{52}{13}} &= \frac{\frac{32!}{13!19!}}{\frac{52!}{13!39!}} \\
&= \frac{39!32!}{52!19!} \\
&= \frac{5394}{9860459} \\
&\approx \frac{5.5}{10000} \approx \frac{1}{1800}
\end{aligned}$$

Since the Earl was the person offering the thousand pounds to anyone who could draw such a hand, the bet was good for him and bad for the person accepting it. It didn't pay well enough to counter-act the incredible odds against the hand.

2 Calculating Probabilities

badly

1. Compute the probability of a full house assuming every five card poker hand is equally likely. A full house consists of three cards of one denomination and two of another (i.e. three of a kind and a pair). You may leave your answer in terms of factorials unless you are curious about the numerical value.

–Solution–

We need to select a rank for the pair and a rank for the triple, and then the suits of the cards in each set.

$$\begin{aligned}
\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} &= 13 \times 4 \times 12 \times \frac{4!}{2!2!} \\
&= 13 \times 4 \times 12 \times 6 \qquad = 3744
\end{aligned}$$

The probability is

$$\begin{aligned}
\frac{13 \times 12 \times 4 \times 6}{\frac{52!}{47!5!}} &= \frac{13 \times 12 \times 4 \times 6 \times 5!}{52 \times 51 \times 50 \times 49 \times 48} \\
&= \frac{13 \times 12 \times 4 \times 6 \times 5 \times 4 \times 3 \times 2}{4 \cdot 13 \times 3 \cdot 17 \times 2 \cdot 5 \cdot 5 \times 49 \times 4 \cdot 12} \\
&= \frac{6}{17 \times 5 \times 49} \\
&= \frac{6}{4165} \qquad \approx .00144
\end{aligned}$$

2. Although Robin Hood gets a bullseye with probability 0.9, he finds himself facing stiff competition in the tournament. To win he must get at least four bullseyes with his next five arrows. However, if he gets five bullseyes, he risks exposing his identity to the sheriff. Assume that if he wishes to, he can miss the bullseye with probability 1. What is the probability that Robin wins the tournament without risking exposing his identity?

–Solution–

Robin Hood's optimal strategy is to aim at the bullseye the first four times. Then, on the fifth time, if he has four hits, he will intentionally miss, and if he has fewer, he will aim this time as well. This strategy can never result in Robin Hood being exposed, and it is more likely to succeed than any strategy that intentionally misses before the fifth shot.

Let 1 represent a hit and 0 represent a miss, e.g., 11111 is a sequence of five hits that exposes Robin Hood. There are five possible scenarios compatible with the above strategy in which Robin Hood wins: 1111 (followed by an intentional miss), and then the four scenarios where Robin Hood accidentally misses one of his first four shots, e.g., 10111. The first scenario has probability 0.9^4, and each of the scenarios of the second type has probability $0.9^4 \cdot 0.1$. Thus, $P(\text{win}) = .09^4 + 4 \cdot (0.9^4 \cdot 0.1) = .91854$. Robin Hood wins 92% of the time.

3 Monty Hall Variants

(a) Consider the following variant of the Monty Hall problem. The contestant picks a door, but instead of revealing this door to Monty, he/she writes it down on a piece of paper. Monty then opens one of the two goat doors (chosen uniformly at random). If the door opened by Monty is the one chosen by the contestant, then the contestant knows he was wrong and picks one of the remaining two doors at random. Otherwise (i.e., when the door opened by Monty is not his chosen door) the contestant is given the option of switching to the other unopened door. (Note that the switching option comes into play only in this second case.)

Should the contestant switch? First answer this question using probabilistic reasoning.

Then write down (in tree form) the entire sample space for this version of the game, and compute the probability that the contestant wins under both the switching and sticking stategies.

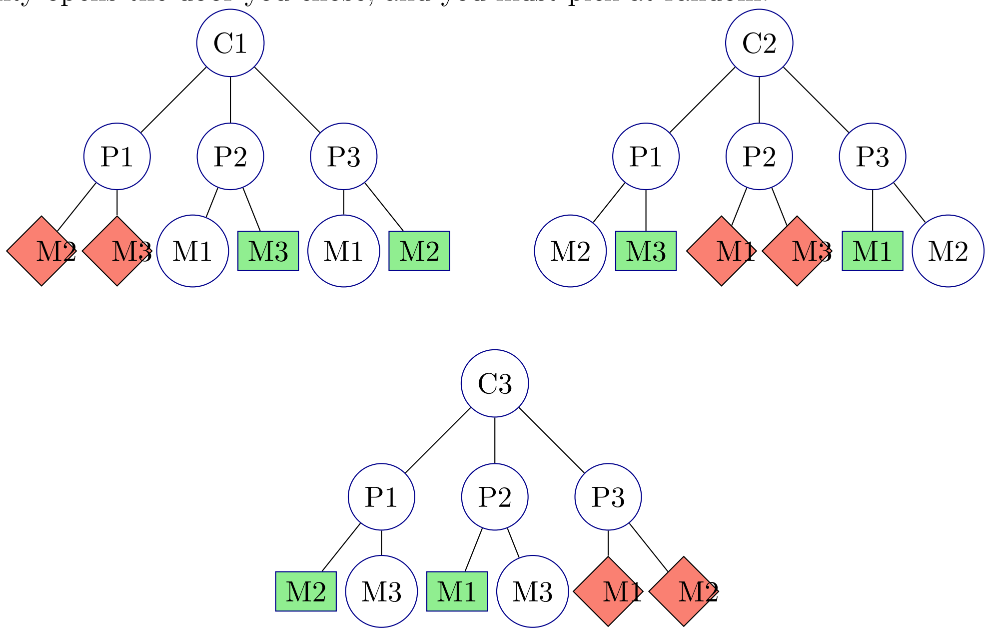
–Solution–

The contestant has 3 choices, the prize has 3 choices, and Monty has 2 choices. Thus, there are a total of 18 possible games.
$P(\text{Pick correct door}) = 1/3$.
$P(\text{Pick wrong door}) = 2/3$.
$P(\text{Monty picks other door \& you chose wrong}) = \frac{1}{2}\frac{2}{3} = \frac{1}{3}$.

When you pick correctly, Monty will open another door, and it is against your interest to switch. This happens with probability 1/3.
When you pick incorrectly and Monty opens another door, it is in your best interest to switch. This happens with probability 1/3.

The instance when switching is not an option occurs with probability $1/3$, so conditioned on the switching strategy coming into play, each outcome has probability $1/2$.
Thus, the benefit of switching disappears, and the best you can do is to choose randomly.

What follows is a tree illustrating the possible game outcomes. The top layer indicates the door opened by the contestant (C). The second layer indicates the door the prize is behind (P). The third layer indicates the door opened by Monty (note the constraint - Monty cannot reveal the prize). Red diamonds indicate outcomes where switching causes you to lose. Green rectangles indicate where switching causes you to win. Circles are the “null” case, when Monty opens the door you chose, and you must pick at random.



(b) After many years, the standard version of the game with three doors becomes a little boring, so Monty decides to increase the number of doors to four (with one prize and three goats). After the contestant picks a door, Monty opens two of the remaining doors to reveal goats. What is the probability now that the contestant wins under the switching strategy?

–Solution–

At your initial choice, you have probability $\frac{1}{4}$ of guessing correctly. Monty then opens two doors, and you are left with one alternative. With probability $\frac{3}{4}$, your initial guess is wrong, so therefore, with probability $\frac{3}{4}$ the other door has the prize. Here, switching results in a win with probability $\frac{3}{4}$.

4 Office Space

Your start-up company has twelve employees and twelve parking spaces arranged in a row. You may assume that each day all orderings of the twelve cars are equally likely.

(a) What is the probability that you park next to the CEO on any given day?

–Solution–

You park in a given space with probability $1/12$, as we assume an even distribution among parking spaces.
Then, 11 employees park in 11 remaining spaces, and there is a $1/11$ probability a given space adjacent to you will have the CEO.
There is a $1/6$ probability you park on the end; in that case, there is a $1/11$ chance the CEO parks next to you.
There is a $5/6$ probability you park somewhere in the middle; in that case, there is a $2/11$ chance the CEO parks next to you.
This is an application of the Law of Total Probability :
In total, there is $\frac{1}{6}\frac{1}{11}+\frac{5}{6}\frac{2}{11}=\frac{1}{66}+\frac{5}{33}=\frac{1}{6}$ probability of parking next to the CEO.

(b) What is the probability that there are exactly two cars between yours and the CEO's?

–Solution–

Number the spaces 0 through 11. If we park in spaces $0,1,2,9,10$, or 11, there is only one space for the CEO to park in so there are two cars between ours and the CEO's. Any other space, there are two ways the CEO can park with two cars in between.
Given you park at the "end" spaces: probability $1/11$ the CEO parks with two cars in between.
Given you park in the "middle" spaces: probability $2/11$ the CEO parks with two cars in between.
Note that parking at the middle or the end has equal probability; there are 6 spaces in each category.
Thus, the total probability is $\frac{1}{22}+\frac{1}{11}=\frac{3}{22}$.

(c) Suppose that, on some given day, you park in a space that is not at one of the ends of the row. As you leave your office, you know that exactly four of your colleagues have left work before you. Assuming that you remember nothing about where these colleagues had parked, what is the probability that you will find both spaces on either side of your car unoccupied?

–Solution–

The sample space is the choices, from the 11 other people, of the 4 of them that left the office. Every such choice is equally probable, and there are $\binom{11}{4}$ of them.

How many of these choices satisfy the desired condition, that the two spaces on either side of you being empty? The group of 4 must contain the 2 people who parked on either side of you, then after that, the 2 remaining people can be chosen arbitrarily from the 9 remaining people in the office. Thus there are $\binom{9}{2}$ such choices. The probability of finding both spaces unoccupied is therefore

$$\frac{\binom{9}{2}}{\binom{11}{4}}=\frac{6}{55}\approx 0.109$$

5 Lost Marbles

Box A contains 3 black and 1 white marbles, and box B contains 2 black and 2 white marbles. A box is selected at random, and a marble is drawn at random from the selected box.

(a) What is the probability that the marble is black?

–Solution–

Law of total probability:
$P(\text{Black}) = P(\text{Black}|A)P(A) + P(\text{Black}|B)P(B) = \frac{3}{4} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{5}{8}$.

(b) Given that the marble is white, what is the probability that it came from the box A?

–Solution–

Bayes' Rule for inverse probability:
$P(A|\text{White}) = \frac{P(\text{White}|A)P(A)}{1-P(\text{Black})} = \frac{1/8}{3/8} = \frac{1}{3}$.

6 Money bags

I have a bag containing either a \$1 or \$10 bill (with equal probability assigned to both possibilities). I then add a \$1 bill to the bag, so it now contains two bills. The bag is shaken, and you draw out a bill at random, and it turns out to be a \$1 bill. If a second student draws the remaining bill from the bag, what is the chance he gets a \$10 bill? Show your work.

–Solution–

Let Bag A be the bag with 2 \$1 bills and let Bag B represent the bag with one \$1 bill and one \$10 bill.

$$P(A|\$1) = \frac{P(\$1|A)P(A)}{P(\$1|A)P(A) + P(\$1|B)P(B)} = \frac{\frac{1}{2}}{1 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{4}} = \frac{2}{3}$$

So, $P(B|\$1) = \frac{1}{3}$. The second draw, for bag B has probability 1 of being a \$10 bill, and for bag A has probability 1 of being a \$1 bill.
Thus, there is a probability $\frac{1}{3}$ of drawing a \$10 bill.

7 How to beat the heat

It's a hot summer day in the Central Valley. Two children Alice and Bob are engaged in a water balloon fight. Alice hits her target with 75% probability while Bob is 50% accurate. They take turns throwing water balloons, with Alice going first, until one of them is hit by a balloon. What is the chance that Bob is the winner (scores the first hit)?

–Solution–

Let A be the event that Alice wins the game. Consider the possible ways Alice can win: she can win immediately, by making her first shot, with probability $\frac{3}{4}$. Then, she can miss her first shot (with probability $\frac{1}{4}$), then Bob can miss his first shot (probability $\frac{1}{2}$), at which point the game is back in its original state — i.e., Alice has the same probability of winning that she did to start with.

It follows that $P(A)$ must be governed by the following recurrence:

$$P(A) = \frac{3}{4} + \frac{1}{4} \cdot \frac{1}{2} \cdot P(A)$$

which yields $\frac{7}{8}P(A) = \frac{3}{4}$ and thus $P(A) = \frac{6}{7}$. The probability that Bob wins is then $1 - \frac{6}{7} = \frac{1}{7}$.

Alternately, the probability can be calculated as the sum of a geometric series:
$P(\text{Bob Wins time } 1) = 0.25 \cdot 0.5$. (Alice misses, Bob hits).
$P(\text{Bob Wins time } 2) = (0.25 \cdot 0.5)^2$. (Alice misses, Bob misses, Alice misses, Bob hits).
$P(\text{Bob Wins time } n) = (\frac{1}{8})^n$.
$P(\text{Bob Wins}) = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{8}\right)^n$. This is a geometric sequence which converges to $\frac{1/8}{7/8} = \frac{1}{7}$.

8 Extra Credit: Three person duel

A three person duel is much harder to analyze. We consider a simple case. Three children Alice, Bob, and Carlos are engaged in a "duel to the death" with water balloons. They stand at the corners of an equilateral triangle, and take turns (in cyclic order) throwing water balloons, until one of them is hit and must "play dead." The remaining two then continue with a two person duel. Assume that in each turn the child throwing the balloon can choose which of the other two children to target.

All three know that Alice is 75% accurate, Bob is 50% accurate, and Carlos always hits his target. Of course, if for some reason any of them deliberately decides to miss they can do so with certainty. Suppose that Bob has drawn the first shot, and Carlos second. What is Alice's best strategy, and what is the chance that she comes out the eventual winner? What about Carlos?

–Solution–

Somewhat surprisingly, Alice has the best chance of winning out of all 3 players. Her chance is $\frac{3}{7} = \frac{12}{28}$, Carlos's is $\frac{1}{4} = \frac{7}{28}$, and Bob's is $\frac{9}{28}$.

First, consider the optimum strategy for each player. Bob's optimum strategy is to attack Carlos first, then attack Alice; if he aims at Alice and hits her while Carlos is still alive, then Carlos will immediately finish him off. By the same argument, Alice's optimum strategy is to attack Carlos first, then Bob. As for Carlos, his optimum strategy is to attack Alice first, then Bob — either way, any surviving player will attack him, and Alice has a better chance of succeeding.

What is the probability that Carlos wins (call this event C)? Given the strategies above, there is only one way for Carlos to win: Bob misses Carlos, Carlos hits Alice, Bob misses Carlos, Carlos hits Bob. The probability of this is $\frac{1}{2} \cdot 1 \cdot \frac{1}{2} \cdot 1 = \frac{1}{4}$.

Now, what is the probability that Alice wins? For Alice to win (call this event A), Bob must hit Carlos in the first round (otherwise Carlos will hit her next). Call this event X; we know that $P(X) = \frac{1}{2}$. After that, Alice can hit Bob, with probability $\frac{3}{4}$, in which case she has won. Alternately, she can miss him with probability $\frac{3}{4}$, then Bob can miss her with probability $\frac{1}{2}$, and then the game has looped back to the moment just after X took place. Thus:

$$P(A|X) = \frac{3}{4} + \frac{1}{4} \cdot \frac{1}{2} \cdot P(A|X)$$

solving this recurrence, as before, we get $P(A|X) = \frac{6}{7}$, so $P(A) = P(A|X)P(X) = \frac{6}{7} \cdot \frac{1}{2} = \frac{3}{7}$.

As before, we can also analyze Alice's probability of winning as the sum of an infinite series:

$$\frac{1}{2} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{3}{4} \ldots = \frac{1}{2} \cdot \frac{3}{4} \cdot \sum_{i=0}^{\infty} (\frac{1}{8})^i = \frac{3}{8} \cdot \frac{8}{7} = \frac{3}{7}$$

At this point, Bob's probability of winning (call this event B) must be $1 - P(A) - P(C) = \frac{9}{28}$. But to check our work, we can compute it anyway. There are two distinct ways in which Bob can win. One way is for him to hit Carlos (event X) and then for Alice to lose her duel with him; but $P(B|X) = 1 - P(A|X) = 1 - \frac{6}{7} = \frac{1}{7}$. The other way is for him to miss Carlos (event $\neg X$), and then for Carlos to hit Alice, and him to hit Carlos (which happens with probability $\frac{1}{2}$). Thus

$$P(B) = P(B|X)P(X) + P(B|\neg X)P(\neg X) = \frac{1}{7} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{9}{28}$$