CS 70 Discrete Mathematics and Probability Theory
Spring 2015 Vazirani HW 9

**Reminder:** Deadline for HW9 self-grade is Thursday, April 2 at noon.

**Reminder:** Deadline for HW8 self-grade is Thursday, March 19 at noon.

*Reading:* Note 12, Note 13, and Note 14.

# Due Monday March 23

1. **Counting practice**

   Leave your answers as (tidy) expressions involving factorials, binomial coefficients, etc., rather than evaluating them as decimal numbers (though you are welcome to perform this last step as well for your own interest if you like, provided it is clearly separated from your main answer). Explain clearly how you arrived at your answers.

   (a) How many 75-bit strings are there that contain more ones than zeros?

   (b) How many bridge hands contain exactly 4 hearts?

   (c) How many anagrams of OKLAHOMA are there? (An anagram of OKLAHOMA is any re-ordering of the letters of OKLAHOMA; i.e., any string made up of the eight letters O, K, L, A, H, O, M and A, in any order. The anagram does not have to be an English word.)

   (d) How many different anagrams of MISSISSIPPI are there?

   (e) How many rolls of 6 dice with at most 5 distinct values?

   (f) How many increasing sequences of $k$ numbers from $\{1,\ldots,n\}$? ($2,3,5,5,7$ is not an increasing sequence, but $2,3,5,7$ is.)

   (g) How many distinct degree $\leq d$ polynomials modulo $p$, where $d \leq p-1$?

   (h) How many distinct degree $\leq d$ polynomials with exactly $d$ roots modulo $p$ where $d \leq p-1$? (Brief explanation ok.)

   (i) How many values of $x \in \{0,\ldots,11\}$ satisfy $4x = 6 \pmod{12}$?

   (j) How many values of $x \in \{0,\ldots,35\}$ satisfy $8x = 4 \pmod{36}$?

   **Answer:**

   (a) $\boxed{2^{74}}$. Since 75 is odd, there are either more zeros than ones or more ones than zeros. By symmetry, the number of strings in each case is equal. Thus there are $2^{75}/2$ strings with more ones than zeros.

   (b) $\boxed{\binom{13}{4}\binom{39}{9}}$ We first choose the 4 hearts, then the rest of the hand, which should not contain any more hearts.

(c) $\boxed{\frac{8!}{2!2!}}$ O and A are both repeated twice.

(d) $\boxed{\frac{11!}{4!4!2!}}$ Same reasoning as above.

(e) $\boxed{6^6 - 6!}$ This is the same as all possible rolls of 6 dice minus the rolls with 6 distinct values.

(f) $\boxed{\binom{n}{k}}$ We choose any $k$ distinct numbers and arrange them in increasing order.

(g) $\boxed{p^{d+1}}$ A polynomial of degree $\leq d$ is uniquely described by $d+1$ coefficients (for $x^0, x^1, \ldots, x^d$). For each coefficient there are $p$ choices. Thus the total number is $p^{d+1}$.

(h) $\boxed{(p-1)\binom{p}{d}}$ We can pick the roots in $\binom{p}{d}$ ways. For any choice of $d$ roots, there are exactly $p-1$ polynomials of degree $\leq d$ with those roots, because given the value of the polynomial on any additional point, the polynomial is determined uniquely. Note that the value of the polynomial on the additional point cannot be zero (since that would result in the polynomial being identical to zero, which has more than $d$ roots). Therefore the value of the polynomial on the additional point can be any of $1, 2, \ldots, p-1$.

(i) $\boxed{0}$ If there was such an $x$, then 12 would divide $4x - 6 = 2(2x-3)$. But note that $2x-3$ is an odd number, therefore $2(2x-3)$ is not divisible by 4 and therefore it's not divisible by 12 either.

(j) $\boxed{4}$ We can use the Chinese remainder theorem. Note that $36 = 4 \times 9$. So we just need to find out the number of values mod 4 that satisfy the equation and the number of values mod 9 that satisfy the equation and multiply the two together (because values mod 36 correspond to pairs of values mod 4 and mod 9). When looking at the equation mod 4, it becomes $0 = 0 \pmod 4$ which always holds. So all values mod 4 are answers (there are 4 of them). When looking at the equation mod 9, note that 8 has a multiplicative inverse (itself). So there the equation becomes $8x = 4 \pmod 9$ which is equivalent to $x = 5 \pmod 9$. So there is one answer mod 9. Therefore the total number of answers mod 36 is $4 \times 1 = 4$.

2. **Combinatorial proofs**

Give a combinatorial proof that $\sum_{k=1}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$.

(*Hint:* count in two ways the number of ways to select a committee and a leader of the committee.)

**Answer:**

**Left-hand side:** We select a committee of size $\geq 1$ from a group of $n$ people and then a leader of the committee. For each committee size $1 \leq k \leq n$, we select $k$ people from the group ($\binom{n}{k}$ ways) and then select the committee's leader ($k$ ways). Together, there are $\sum_{k=1}^{n} k\binom{n}{k}$ ways.

**Right-hand side:** Each committee has a leader, so we first select the leader ($n$ possible ways), and then select the rest of the committee. This is the same as selecting a subset of $n-1$ elements, or selecting an $(n-1)$-bit string where the value of each digit indicates whether the corresponding person is included in the committee. In total there are $n \cdot 2^{n-1}$ ways.

3. **Sample Space and Events**

Consider the sample space $\Omega$ of all outcomes from flipping a coin 3 times.

(a) List all the outcomes in $\Omega$. How many are there?

(b) Let $A$ be the event that the first flip is a heads. List all the outcomes in $A$. How many are there?

(c) Let $B$ be the event that the third flip is a heads. List all the outcomes in $B$. How many are there?

(d) Let $C$ be the event that the first and third flip are heads. List all outcomes in $C$. How many are there?

(e) Let $D$ be the event that the first or the third flip is heads. List all outcomes in $D$. How many are there?

(f) Are the events $A$ and $B$ disjoint? Express $C$ in terms of $A$ and $B$. Express $D$ in terms of $A$ and $B$.

(g) Suppose now the coin is flipped $n \geq 3$ times instead of 4 flips. Compute $|\Omega|, |A|, |B|, |C|, |D|$.

(h) Your gambling buddy found a website online where he could buy trick coins that are heads or tails on both sides. He puts three coins into a bag: one coin that is heads on both sides, one coin that is tails on both sides, and one that is heads on one side and tails on the other side. You shake the bag, draw out a coin at random, put it on the table without looking at it, then look at the side that is showing. Suppose you notice that the side that is showing is heads. What is the probability that the other side is heads? Show your work. (Hint: the answer is NOT 1/2).

**Answer:**

(a) Each flip results in either heads (H) or tails (T). So in total the total number of outcomes is 8, which we represent by length 3 strings of H's and T's. We have

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

(b) These are the strings that start with $H$. We have $A = \{HHH, HHT, HTH, HTT\}$. There are 4 such outcomes.

(c) These are the strings that end with an $H$. We have $B = \{HHH, HTH, THH, TTH\}$. There are 4 such outcomes.

(d) These are the strings that start and end with an $H$. We have $C = \{HHH, HTH\}$. There are 2 such outcomes.

(e) We have $D = \{HHH, HHT, HTH, HTT, THH, TTH\}$. There are 6 such outcomes.

(f) No, $A$ and $B$ are not disjoint. For example $HHH$ belongs to both of them.

The event $C$ is the intersection of $A$ and $B$, because in $C$ we require exactly both $A$ (the first coin being heads) and $B$ (the third coin being heads) to happen. So $C = A \cap B$.

The event $D$ is the union of $A$ and $B$, because in $D$ we require at least one of $A$ (the first coin being heads) or $B$ (the second coin being heads) to happen. So $D = A \cup B$.

(g) First, obviously $|\Omega| = 2^n$.

Note that for each outcome in the three-coin case, there are now $2^{n-3}$ outcomes, each corresponding to a possible configuration of the 4th flip and beyond. Since $A$, $B$, $C$, and $D$ do not care about the outcomes of the 4th flip and beyond, this means that the size of each set is simply multiplied by $2^{n-3}$. Therefore we have $|A| = 4 \times 2^{n-3} = 2^{n-1}$, $|B| = 4 \times 2^{n-3} = 2^{n-1}$, $|C| = 2 \times 2^{n-3} = 2^{n-2}$, and $|D| = 6 \times 2^{n-3} = 3 \times 2^{n-2}$.

(h) There are 6 possible outcomes which are all equally likely. We have 3 choices for the coin that we draw (which we represent by $HH$, $HT$ and $TT$). Then for each coin we have two choices, we either see the first side or the second side (which we represent by 1 and 2). So the outcomes are $\Omega = \{(HH,1), (HH,2), (HT,1), (HT,2), (TT,1), (TT,2)\}$. Now given that we saw a heads, we can get rid of 3 of the outcomes, and the possible remaining outcomes are

$\{(HH,1),(HH,2),(HT,1)\}$ which are all equally likely. In this space, the event that the coin has two heads is $\{(HH,1),(HH,2)\}$ which consists of two equally likely outcomes. So the probability is $2/3$.

4. **Parking Lots**

   Your start-up company has twelve employees and twelve parking spaces arranged in a row. You may assume that each day all orderings of the twelve cars are equally likely.

   (a) What is the probability that you park next to the CEO on any given day?

   (b) What is the probability that there are exactly three cars between yours and the CEO's?

   (c) Suppose that, on some given day, you park in a space that is not at one of the ends of the row. As you leave your office, you know that exactly five of your colleagues have left work before you. Assuming that you remember nothing about where these colleagues had parked, what is the probability that you will find both spaces on either side of your car unoccupied?

   **Answer:**

   (a) There are 12! possible ways in which the cars get parked (all possible permutations). To count the number of permutations in which your car and the CEO's get parked next to each other, do this thought experiment: assume that your car wasn't even there and there were 11 spaces. Now there are 11! ways for the cars to get parked; after this you see where the CEO has parked and you create a space (this is a thought experiment, remember), either to the left or to the right of the CEO's car, and park your car there. You have two choices, which means that the total number of such arrangements is $11! \times 2$. Therefore the probability is $\frac{11! \times 2}{12!} = \frac{2}{12} = \frac{1}{6}$.

   (b) Again we have to count the number of arrangements and divide by 12!. Either your car is parked to the left of the CEO's or to its right. In the first case your car can be in any of these spots (assuming they're numbered from left to right) $1,2,\ldots,8$, and the CEO's car will always be in the spot whose number is higher by 4. This gives your car and the CEO's car 8 ways to be parked with your car being on the left. By symmetry, there are also 8 ways in which your car is to the right of the CEO's car with three cars in between. So in total there are 16 ways for your car and the CEO's car to be parked with 3 spaces in between. After the spaces for these two cars are determined, the remaining ones can park in any of the 10! arrangements. So the total number of arrangements is $16 \times 10!$. The probability is therefore $\frac{16 \times 10!}{12!} = \frac{16}{12 \cdot 11} = \frac{4}{33}$.

   (c) We know that 5 spots from the 11 that are not occupied by your car have been freed. All choices of these 5 spots are equally likely to happen. So we can count the arrangements of the 5 free spots to compute the probability. In total there are $\binom{11}{5}$ possible ways to choose the free spots. To count to number of ways that free both spaces next to your car, we pick those two spaces and then choose 3 more spots from the remaining 9. So there are $\binom{9}{3}$ such arrangements. Therefore the desired probability is $\binom{9}{3}/\binom{11}{5}$ which is equal to

   $$\frac{\frac{9!}{3!6!}}{\frac{11!}{5!6!}} = \frac{5 \times 4}{11 \times 10} = \frac{2}{11}.$$

5. **Rolling dice (conditional probability)**

   We roll two fair 6-sided dice. Each one of the 36 possible outcomes is assumed to be equally likely.

   (a) Find the probability that doubles were rolled.

(b) Given that the roll resulted in a sum of 4 or less, find the conditional probability that doubles were rolled.

(c) Find the probability that at least one die is a 6.

(d) Given that the two dice land on different numbers, find the conditional probability that at least one die is a 6.

**Answer:**

(a) There are 6 outcomes in which doubles are rolled (uniquely identified by which number showed up on the two dice). So the probability is $6/36 = \frac{1}{6}$.

(b) If we represent outcomes as $(a,b)$, where $a$ is the number on the first die and $b$ is the number on the second die, then the event that the sum is 4 or less is $\{(1,1),(1,2),(1,3),(2,1),(2,2),(3,1)\}$. This event contains 6 equally likely possibilities. Out of these six, two are doubles (namely $(1,1)$ and $(2,2)$). Therefore the conditional probability that we have doubles given that the sum is 4 or less is $\frac{2}{36}/\frac{6}{36} = \frac{1}{3}$.

(c) Since the number of rolls that do not contain a six is $5^2 = 25$, the number of rolls that contain at least one six is $36-25 = 11$. The desired probability is therefore $\frac{11}{36}$.

(d) Let $A$ be the event that at least one die is a 6, and let $B$ be the event that the two dice land on different numbers. Then we are after $\Pr[A \mid B]$, which is equal to the probability of $A \cap B$ divided by the probability of $B$. Note that the probability of $B$ is $\frac{5}{6}$, because we know there is a $\frac{1}{6}$ chance for having doubles and $1 - \frac{1}{6} = \frac{5}{6}$. To find $\Pr[A \cap B]$, we count the number of ways that two dice land on different numbers and one of them is a six. We do this by choosing which die is a six (2 ways) and then choosing the number on the other die (5 ways): $2 \cdot 5 = 10$.
Hence,
$$\Pr[A \mid B] = \frac{10}{36} \Big/ \frac{5}{6} = \frac{1}{3}$$

6. **Happy families**

Consider a collection of families, each of which has exactly two children. Each of the four possible combinations of boys and girls, $bg, gb, bb, gg$ (where the two children are listed in order by age), occurs with the same frequency. A family is chosen uniformly at random, and we are told that it contains at least one boy. What is the (conditional) probability that the other child is a boy? Justify your answer with a precise calculation. Can you give an intuitive explanation for your answer?

**Answer:** The outcomes $bg, gb, bb, gg$ are all equally likely. The information that the family has at least one boy eliminates one of these outcomes, namely $gg$. So now we have three outcomes, all of which are still equally likely $bb, gb, bg$. Exactly one of these three has two boys $bb$. So the conditional probability of having two boys given that there is at least one is $\frac{1}{3}$.

We can also formally write it this way: assume that $A$ is the event of having at least one boy, and $B$ is the event of having two boys. Then $B \cap A = B$, and so $\Pr[B \cap A] = \Pr[B] = \frac{1}{4}$. We also have $\Pr[A] = \frac{3}{4}$. Therefore
$$\Pr[B \mid A] = \frac{\Pr[A \cap B]}{\Pr[A]} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.$$

Note that if we were told which child (the younger or the older) was a boy, this conditional probability would be different (it would be $\frac{1}{2}$). The reason it is not $\frac{1}{2}$ in our case is that we are not told which child is a boy and therefore given less information. The predicate "there is at least a boy" conveys less information than "the younger child is a boy", and so it eliminates less possibilities.

7. **(Extra Credit) Monty Hall revisited**

In this variant of the Monty Hall problem, after the contestant has chosen a door, Monty asks another contestant to open one of the other two doors. That contestant, who also has no idea where the prize is, opens one of those two remaining doors at random, and (as it happens) you both see that there is no prize there. Monty now asks you if you wish to switch or stick with your original choice. What is your best strategy? Why?

**Answer:** Let us compute the conditional probability that the prize is behind the door you have chosen, given that the other contestant saw no prize. Let $A$ be the event that the prize is behind the door you originally chose, and $B$ be the event that the door the other contestant chose had no prize in it. We want to compute $\Pr[A \mid B] = \Pr[A \cap B] / \Pr[B]$.

First, we note that $\Pr[A \cap B] = \Pr[A] \cdot \Pr[B \mid A] = \frac{1}{3} \cdot 1 = \frac{1}{3}$, because if the prize was behind the door your originally chose, the other contestant's door will always have no prize behind it.

Now, to find $\Pr[B]$, observe that $\Pr[B] = \Pr[A \cap B] + \Pr[\bar{A} \cap B] = \frac{1}{3} + \Pr[\bar{A}] \cdot \Pr[B \mid \bar{A}] = \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$.

Hence, we have $\Pr[A \mid B] = \Pr[A \cap B] / \Pr[B] = \left(\frac{1}{3}\right) / \left(\frac{2}{3}\right) = \frac{1}{2}$.

In other words, given that the other contestant's door was empty, there is $\frac{1}{2}$ chance that the original door you chose has the prize. The prize has to be behind one of the two remaining doors, so the probability that the other door has the prize is also $\frac{1}{2}$. So it does not make a difference whether you switch or not.