1. **Sundry**

   Before you start your homework, write down your team. Who else did you work with on this homework? List names and email addresses. (In case of hw party, you can also just describe the group.) How did you work on this homework? Working in groups of 3-5 will earn credit for your "Sundry" grade.

2. (5 points) Prove that $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$ for all integers $n > 0$.

   Solution: We will do an induction on n:

   - Proposition: $P(n) =$ "$1^3 + \cdots + n^3 = (1 + \cdots + n)^2$".
   - Base case: $P(1)$ is the proposition $1^3 = (1)^2$, which is true, since $1 = 1$.
   - Inductive step: prove $P(n) \implies P(n+1)$ for all $n > 0$.

   (a) The inductive hypothesis is $1^3 + \cdots + n^3 = (1 + \cdots + n)^2$, where $n > 0$.

   (b) To prove: $1^3 + \cdots + n^3 + (n+1)^3 = (1 + \cdots + n + (n+1))^2$.

   (c) Taking the expression on the right hand side, we get

$$
\begin{aligned}
&((1+\cdots+n)+(n+1))^2 \\
&= (1+\cdots+n)^2 + 2(1+\cdots+n)(n+1) + (n+1)^2 \\
&\qquad\qquad \text{(since } (x+y)^2 = x^2+2xy+y^2 \text{ for all } x,y) \\
&= (1^3+\cdots+n^3) + 2(1+\cdots+n)(n+1) + (n+1)^2 \\
&\qquad\qquad \text{(using the inductive hypothesis)} \\
&= (1^3+\cdots+n^3) + 2\frac{n(n+1)}{2}(n+1) + (n+1)^2 \\
&\qquad\qquad \text{(since } 1+\cdots+n = n(n+1)/2 \text{ for all } n) \\
&= 1^3+\cdots+n^3+(n+1)^3 \qquad \text{(simplifying the right-hand side)}
\end{aligned}
$$

3. (10 points) **A Tricky Game**

   (a) (10 points) CS 70 course staff invite you to play a game: Suppose there are n^2 coins in a $n \times n$ grid ($n > 0$), each with their heads side up. In each move, you can pick one of the n rows or columns and flip over all of the coins in that row or column. However, you are not allowed to re-arrange them in any other way. You have an unlimited number of moves. If you happen to reach a configuration where there is exactly one coin with its tails side up, you will get an extra credit. Are you able to win this game? Does that apply to all n? Prove your answer.

Solution: When $n = 1$, the answer is trivial. Let's then analyze the base case when $n = 2$. We will prove the following lemma.

Lemma: The 2×2 puzzle is unwinnable.

Proof: Let P be the property that the number of coins in a configuration with heads side up on the 2×2 grid is even. Note that P is true initially, and moreover it is not disturbed by flipping of any row or column. Hence is a P is an invariant of the configuration. By induction on the number of moves, P holds after any number of moves, and so we can only reach configurations where P is true.

(In other words, we let $Q(k)$ denote the proposition that P holds after any sequence of k moves. The base case $Q(0)$ holds trivially. Also, $Q(k) \implies Q(k+1)$ holds for all $k \geq 0$, since every sequence of $k+1$ moves can be decomposed into a sequence of k moves followed by one more move, and if P holds before this last move, it holds after the last move, too, since P is not disturbed by flipping of any single row or column. Therefore by induction $Q(k)$ holds for all $k \geq 0$.)

But now the configuration with exactly one coins tails side up is incompatible with P and consequently is unreachable by any finite set of moves. Therefore the 2×2 puzzle is unwinnable.

Now, we will use the lemma to prove the following theorem.

Theorem: The $n \times n$ puzzle is winnable if and only if $n = 1$.

Proof: We show that the puzzle is unwinnable when $n \geq 3$. Suppose not, i.e., there is some winning sequence of moves that leaves just a single coin with "tails" side up at some location L. Consider any 2×2 sub-grid containing location L. Then we have found a sequence of moves which takes this 2×2 sub-grid from the initial all-heads position to a position containing 3 heads and 1 tail. But this is impossible, by the previous result. Hence our assumption that there exists a winning sequence of moves must have been impossible, which proves the theorem.

(b) (Optional) Now, suppose we change the rules: If the number of "tails" is between 1 and $n-1$, you win. Are you able to win this game? Does that apply to all n? Prove your answer.

Solution: Similarly, when $n = 1$, the answer is trivial. We will prove the following theorem:

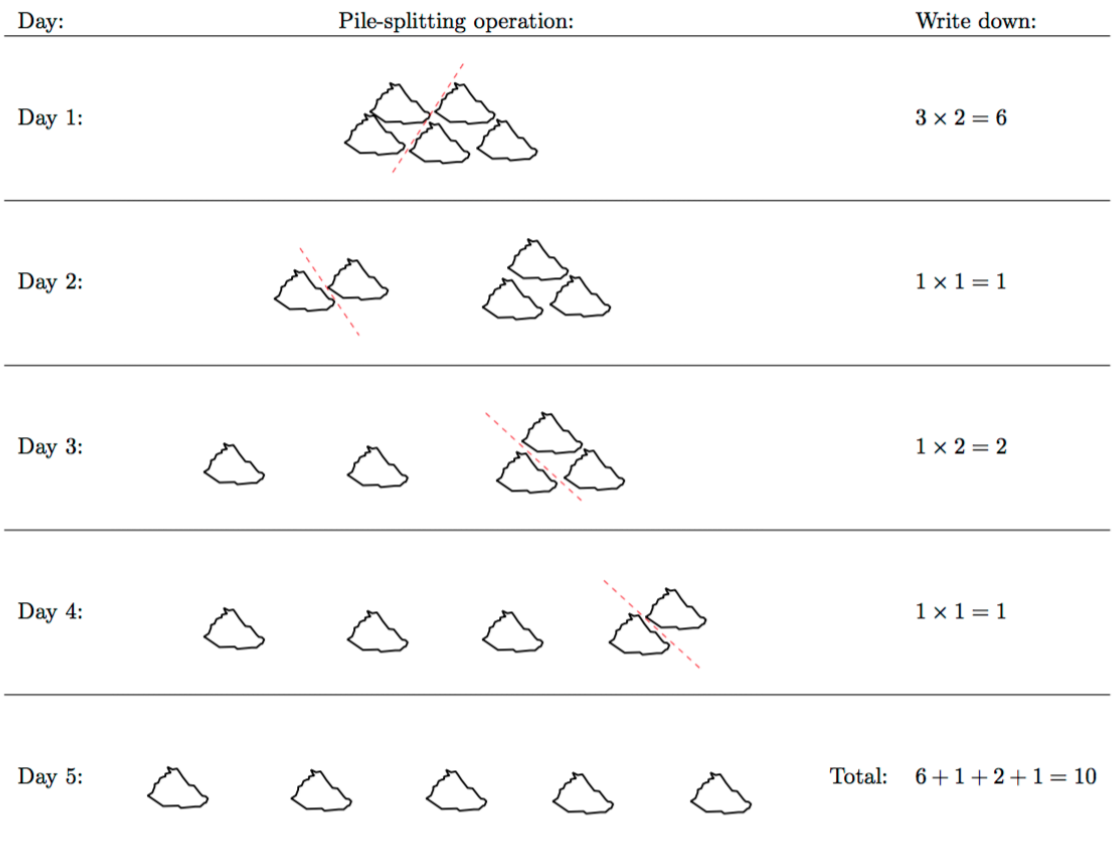
Theorem: The game remains unwinnable for all $n \geq 2$.

Proof: Consider any pair of rows of coins. Let P be the property that the two rows are identically configured, and Q be the property that these two rows are exact inverses of each other (i.e., each head in the first row corresponds to a tail in the second row, and vice versa). Then (for every pair of rows) $P \vee Q$ is an invariant of the game, since it holds initially and is not disturbed by any move.

Now there are only two cases: either there is some winning sequence of moves, or there is not. In the former case, in the winning final configuration some rows must have t tails, for some $1 \leq t \leq n-1$. This implies that every other row has either t or $n-t$ tails in it (according to whether it is P or Q that is true for this pair of rows), and since $t \geq 1$ and $n-t \geq 1$, this means that every row must have at least one coin with "tails" side up, for a total of at least n tails in the winning configuration. This is an absurdity. But this means the first case is impossible, hence the second case must always hold, and there can be no winning the puzzle when $n \geq 2$.

4. (10 pts) **Making big rocks into little rocks**
One day, you find a shady-looking flyer on Sproul that advertises an undergraduate research position in the Stanford geology department. After you make your way down to the Farm, Dr. Hoover, the professor in charge, hands you the following assignment to keep you busy for the first year, complaining that none of the Stanford undergrads have been able to help him:

You are given a single large pile of pebbles to start with. Every day, you have pick one pile of pebbles and split it into two smaller piles any way you wish. Then, you have to count the number of pebbles in each of the two new piles you just created (k and m), and write down the product $k \times m$ in your lab notebook. Of course, over time, you end up with more and more piles which get smaller and smaller. You get to stop when each pebble is in its own pile. Once you stop, add up all the numbers you wrote down and report them to Dr. Hoover. The following figure shows an example of the process.



Convince Dr. Hoover (and his funding agency) that this [1] is a pointless exercise: no matter how you split the piles, your answer will come out to $n(n-1)/2$, where n is the number of pebbles in the original pile (which you can just count on the first day).

Solution: Proof by strong induction on n. Let $P(n) =$ "following the given protocol, the final sum for n pebbles is $n(n-1)/2$."

Base case: $P(1)$ is clearly true: the final sum is zero since no steps are required, and $1 \times (1-1)/2 = 0$.

Inductive step: We need to prove $P(1) \wedge \ldots \wedge P(n) \implies P(n+1)$ for all $n \geq 1$. Suppose we already know that that for every j with $1 \leq j \leq n$, if we start with a mound of j pebbles then we will write down a sequence of numbers summing to $j(j-1)/2$.

Consider what happens with a pile of $n+1$ pebbles. At the first move, we split it into two non-empty mounds containing k and $m = n+1-k$ pebbles for some $0 < k < n+1$ (k cannot be 0 or $n+1$ since both mounds are non-empty). The sequence of numbers we write down afterwards may be divided into three categories: (1) the first number we write down, i.e., $k \times (n+1-k)$; (2) numbers arising from moves applied to the mound of k pebbles (or one of its sub-mounds); (3) numbers arising from moves applied to the mound of $n+1-k$ pebbles (or one of its sub-mounds). By the induction hypothesis, the second sequence of numbers sums to $k(k-1)/2$, and the third sums to $(n+1-k)(n-k)/2$. Hence, the total sum when starting with $n+1$ pebbles will be

$$
\begin{aligned}
k(n+1-k) + \frac{k(k-1)}{2} + \frac{(n+1-k)(n-k)}{2} &= \\
&= \frac{k(k-1)}{2} + \frac{k(n+1-k)}{2} + \frac{k(n+1-k)}{2} + \frac{(n+1-k)(n-k)}{2} \\
&= \frac{k(k-1) + k(n+1-k)}{2} + \frac{(n+1-k)(n-k) + k(n+1-k)}{2} \\
&= \frac{k(k-1+n+1-k)}{2} + \frac{(n+1-k)(n-k+k)}{2} \\
&= \frac{kn}{2} + \frac{(n+1-k)n}{2} = \frac{(k+n+1-k)n}{2} = \frac{(n+1)n}{2}.
\end{aligned}
$$

This proves the proposition $P(n+1)$. Thus, by strong induction, $P(n)$ holds for all $n \geq 1$.

5. (10 points) Prove that for every positive integer k, the following is true:

> For every real number $r > 0$, there are only finitely many solutions in positive integers to $\frac{1}{n_1} + \cdots + \frac{1}{n_k} = r$.

In other words, there exists some number m (that depends on k and r) such that there are at most m ways of choosing a positive integer n_1, and a (possibly different) positive integer n_2, etc., that satisfy the equation.

Solution: We will first transfer the problem to mathematical notation:

[1] The research assignment, not this homework problem!

Claim: $\forall k \in \mathbf{Z}\ \forall r \in \mathbf{R}\ \big((k > 0 \land r > 0) \Rightarrow (\text{There are finitely many solutions to } \frac{1}{n_1} + \cdots + \frac{1}{n_k} = r, n_i \in \mathbf{Z}, n_i > 0)\big)$

Proof: We will prove this by induction on k. For our base case, $k = 1$. In the base case, iff r can be written as $\frac{1}{n_1}$ when n_1 is a positive integer, then there is exactly one solution, $n_1 = \frac{1}{r}$. If r cannot be written in that form, then there are exactly zero solutions. In all cases, there is a finite number of solutions. For the inductive hypothesis, assume that there are finitely many solutions for some $k \geq 1$ for all r. Each real number r_1 either can or cannot be written as the sum of $k+1$ integers' inverses. If r_1 cannot be written in that form, then there are exactly zero solutions. If r_1 can be written in that form, then the integers' inverses can be ordered. Since r_1 is the sum of $k+1$ integers' inverses, the largest $\frac{1}{n_i}$ must be at least $\frac{r_1}{k+1}$. This means that the smallest n_i must be at most $\frac{k+1}{r_1}$, which means that the smallest n_i has finitely many possible values. For each of the possible smallest n_i values, there is a real number $r_1 - \frac{1}{n_i}$ that can be written as the sum of k integers' inverses in finitely many ways (using the induction hypothesis). This means that there are only finitely many possible solutions for $k+1$ (combining all solutions (finitely many) for each possible smallest n_i values (finitely many)). By the principle of induction, there are finitely many solutions for all k for all r.

6. (15 points) **Be a Judge**

   For each of the following statements about the traditional stable marriage algorithm with men proposing, indicate whether the statement is True or False and justify your answer with a short 2-3 line explanation:

   (a) (3 points) In a stable marriage algorithm execution which takes n days, there is a woman who did not receive a proposal on the $(n-1)$th day.

   Solution: True: The algorithm will terminate once there is exactly one man proposing to each woman. This means that there will be at least one woman who is not proposed to every day before the algorithm terminates. This includes day $(n-1)$.

   (b) (3 points) There is a set of preferences for n men and n women, such that in a stable marriage algorithm execution every man ends up with his least preferred woman.

   Solution: False: If this were to occur it would mean that at the end of the algorithm, every man would have proposed to every woman on his list and has been rejected $n-1$ times. This would also require every woman to reject $n-1$ suitors. We know this is impossible though if we consider what we learned in parts (a) and (b). There must be at least one woman who is not proposed to until the very last day.

   (c) (3 points) In a stable marriage instance, if man M and woman W each put each other at the top of their respective preference lists, then M must be paired with W in every stable pairing.

   Solution: True: We give a simple proof by contradiction. Assume that M and W can put each other at the top of their respective preference lists, but M and W are not paired with each other in some stable pairing. Then we have a stable pairing which includes the pairings (M, W'), (M', W), for some man M' and woman W'. However, M prefers W over his partner in this pairing, since W is at the top of his preference list. Similarly

W prefers M over her partner. Thus (M,W) form a rogue couple, so the pairing is not stable. We have arrived at a contradiction.

Therefore if man M and woman W put each other at the top of their respective preference lists, then M must be paired with W in a stable pairing.

(d) (3 points) In a stable marriage instance with at least two men and two women, if man M and woman W each put each other at the bottom of their respective preference lists, then M cannot be paired with W in any stable pairing.

Solution: False: The key here is to realize that this is possible if man M and woman W are at the bottom of everybody else's preference list as well. Consider the following example with the men m and M and the women w and W. Suppose that their preference lists are as follows:

$$\begin{aligned} m &: w, W \\ M &: w, W \\ w &: m, M \\ W &: m, M \end{aligned}$$

It is clear that M and W are at the bottom of each other's preference lists; however, it is also true that (m,w) and (M,W) is a stable pairing (indeed, it is the only stable pairing). So, we have a contradiction to the statement: here is a stable marriage instance with at least two men and two women, and man M and woman W put each other at the bottom of their respective preference lists, but yet M and W are paired together in a stable pairing.

(e) (3 points) For every $n > 1$, there is a stable marriage instance with n men and n women which has an unstable pairing in which every unmatched man-woman pair is a rogue couple.

Solution: True: Suppose $n > 1$ and we have men $M_1,\ldots,M_n$ and women $W_1,\ldots,W_n$. Further, assume that for $1 \le i \le n$, preference lists are as follows for every man M_i and woman W_i:

$$\begin{array}{ll} & \textit{highest} \Longrightarrow \textit{lowest} \\ M_i: & W_i \;\; W_{i+1} \;\; W_{i+2} \;\; \ldots \;\; W_{i-1} \\ & \textit{highest} \Longrightarrow \textit{lowest} \\ W_i: & M_i \;\; M_{i-1} \;\; M_{i-2} \;\; \ldots \;\; M_{i+1} \end{array}$$

Note that the indices are taken modulo n, so if i refers to $n+1$ in the preference lists above, it is really referring to 1. The idea in this construction is that there is a fixed ordering of men into a cycle, and a fixed ordering of women into another cycle. Every man's preference list complies to the ordering of women into the cycle, with the only difference between different men's preferences being where in the ordering the preference list begins. The analogous situation holds for women's preference lists.

Now consider the unstable pairing in which each man M_i, $1 \le i \le n$ is paired as (M_i, W_{i-1}). (M_1 is paired to W_n.) We claim every unmatched man-woman pair is a rogue couple.

In this pairing, every man M_i is paired with woman W_{i-1} at the bottom of his preference list, and every woman W_i is paired with man M_{i+1} at the bottom of her preference list.

Thus every man prefers any woman he has not been matched to over his partner, and likewise for women. So any unmatched pair (M,W) is a rogue couple.

7. (18 points) **Stable Marriage**

Consider a set of four men and four women with the following preferences:

men	preferences	women	preferences
A	1>2>3>4	1	D>A>B>C
B	1>3>2>4	2	A>B>C>D
C	1>3>2>4	3	A>B>C>D
D	3>1>2>4	4	A>B>D>C

(a) (3 points) Run on this instance the stable matching algorithm presented in class. Show each stage of the algorithm, and give the resulting matching, expressed as $\{(M,W),\dots\}$.

Solution: The situations on the successive days are:
Day 1: Proposals: $\{(A,1),(B,1),(C,1),(D,3)\}$, B and C are rejected.
Day 2: Proposals: $\{(A,1),(B,3),(C,3),(D,3)\}$, C and D are rejected.
Day 3: Proposals: $\{(A,1),(B,3),(C,2),(D,1)\}$, A is rejected.
Day 4: Proposals: $\{(A,2),(B,3),(C,2),(D,1)\}$, C is rejected.
Day 5: Proposals: $\{(A,2),(B,3),(C,4),(D,1)\}$, no one is rejected.
Final matching: $(A,2),(B,3),(C,4),(D,1)$.

(b) (5 points) We know that there can be no more than n^2 stages of the algorithm, because at least one woman is deleted from at least one list at each stage. Can you construct an instance with n men and n women so that $c \cdot n^2$ stages are required for some respectably large constant c? (We are looking for a *general pattern* here, one that results in $c \cdot n^2$ stages for any n.)

Solution: Consider the case where the preference lists have the following structure:

men	preferences	women	preferences
1	$1>2>\cdots>n-1>n$	1	$2>3>\cdots>n>1$
2	$2>3>\cdots>1>n$	2	$3>4>\cdots>1>2$
3	$3>4>\cdots>2>n$	3	$4>5>\cdots>2>3$
	$\cdots$		$\cdots$
$n-1$	$n-1>1>\cdots>n-2>n$	$n-1$	$n>1>\cdots>n-2>n-1$
n	$1>2>\cdots>n-1>n$	n	$1>2>\cdots>n-1>n$

In this case, man 1 and n go to woman 1 on the first day (while any other man i goes to woman i), and woman 1 rejects man 1. He then goes to woman 2 the next day, who rejects man 2, and so on. It can be shown that there is exactly one man rejected every day, and on the i-th day, the $((i-1) \mod n)+1$ man is rejected by woman $((i-1)$

mod $(n-1)) + 1$. This continues until the man 1 proposes to woman n. The number of days required for this to terminate is $n^2 - 2n + 2 \geq n^2/2$.

(c) (10 points) Suppose we relax the rules for the men, so that each unpaired man proposes to the next woman on his list at a time of his choice (some men might procrastinate for several days, while others might propose and get rejected several times in a single day). Can the order of the proposals change the resulting pairing? Give an example of such a change or prove that the pairing that results is the same.

Solution: Assume, that when a proposal is made and an answer is received, we write down it on a list L and enumerate them. Now the proof is similar to the proof in class that the algorithms finds male optimal pairing.

We should prove that the pairing P' that results is the same as pairing P found by the regular algorithm. Assume the opposite, so either there is a woman that rejects a man who was her pair in P or there is a woman who do not receive a proposal from a man who was her pair in P. By Well Ordering Principle, there is the first entry in the list L for one of that happens, let W denote the woman for whom it happens and let M denote her pair in P.

First, if W rejected M then it is because she get proposal from other man M' who is higher in her preference list. But M' proposed her only because he was rejected by other woman W' who is higher in his preference list and did not rejects him in P. In other words, even before W rejected M, W' rejected her previous pair M'. Contradiction.

Second, if W did not get a proposal from M then it is because M was not rejected by some other woman W' who his higher in his preference list and who rejects him in P. Contradiction.

Therefore, the resulting pairing P' is the same as P.

8. (18 points) **Combining Stable Marriages**

In this problem we examine a simple way to *combine* two different solutions to a stable marriage problem. Let R, R' be two distinct stable matchings. Define the new matching $R \wedge R'$ as follows:

> For every man m, m's date in $R \wedge R'$ is whichever is better (according to m's preference list) of his dates in R and R'.

Also, we will say that a man/woman *prefers* a matching R to a matching R' is he/she prefers his/her date in R to his/her date in R'. We will use the following example:

men	preferences	women	preferences
A	1>2>3>4	1	D>C>B>A
B	2>1>4>3	2	C>D>A>B
C	3>4>1>2	3	B>A>D>C
D	4>3>2>1	4	A>B>D>C

(a) (3 points) $R = \{(A,4),(B,3),(C,1),(D,2)\}$ and $R' = \{(A,3),(B,4),(C,2),(D,1)\}$ are stable matchings for the example given above. Calculate $R \wedge R'$ and show that it is also stable.

Solution: $R \wedge R' = \{(A,3),(B,4),(C,1),(D,2)\}$.

(b) (5 points) Prove that, for any matchings R, R', no man prefers R or R' to $R \wedge R'$.

Solution: Let m be a man, and let his dates in R and R' be w and w' respectively, and without loss of generality, let $w > w'$ in m's list. Then his date in $R \wedge R'$ is w, whom he prefers over w'. However, for m to prefer R or R' over $R \wedge R'$, he must prefer w or w' over w, which is not possible (since $w > w'$ in his list).

(c) (5 points) Prove that, for any stable matchings R, R' where m and w are dates in R but not in R', one of the following holds:

- m prefers R to R' and w prefers R' to R; or
- m prefers R' to R and w prefers R to R'.

[Hint: Let M and W denote the sets of mens and women respectively that prefer R to R', and M' and W' the sets of men and women that prefer R' to R. Note that $|M| + |M'| = |W| + |W'|$, where $|S|$ denotes the size of set S. (Why is this?) Show that $|M| \leq |W'|$ and that $|M'| \leq |W|$. Deduce that $|M'| = |W|$ and $|M| = |W'|$. The claim should now follow quite easily.]

(You may assume this result in subsequent parts even if you don't prove it here)

Solution: Let M and W denote the sets of men and women respectively that prefer R to R', and M' and W' the sets of men and women that prefer R' to R. Note that $|M| + |M'| = |W| + |W'|$, since the left-hand side is the number of men who have different partners in the two matchings, and the right-hand side is the number of women who have different partners.

Now, in R there cannot be a pair (m,w) such that $m \in M$ and $w \in W$, since this will be a rogue couple in R'. Hence the partner in R of every man in M must lie in W', and hence $|M| \leq |W'|$. A similar argument shows that every man in M' must have a partner in R' who lies in W, and hence $|M'| \leq |W|$.

Since $|M| + |M'| = |W| + |W'|$, both these inequalities must actually be tight, and hence we have $|M'| = |W|$ and $|M| = |W'|$. The result is now immediate: if the man m does not date the woman w in one but not both matchings, then:

- Either $m \in M$ and $w \in W'$, i.e., m prefers R to R' and w prefers R' to R.
- Or $m \in M'$ and $w \in W$, i.e., m prefers R' to R and w prefers R to R'.

(d) (5 points) Prove an interesting result: for any stable matchings R, R', (i) $R \wedge R'$ is a matching [Hint: use the results from (c)], and (ii) it is also stable.

Solution: (i) If $R \wedge R'$ is not a matching, then it is because two men get the same woman, or two women get the same man. Without loss of generality, assume it is the former case, with $(m,w) \in R$ and $(m',w) \in R'$ causing the problem. Hence m prefers R to R', and m' prefers R' to R. Using the results of the previous part would imply that w would prefer R' over R, and R over R' respectively, which is a contradiction.

(ii) Now suppose $R \wedge R'$ has a rogue couple (m, w). Then m strictly prefers w to his partners in both R and R'. Further, w prefers m to her partner in $R \wedge R'$. But w is matched to the better of her partners in R and R'. Let w's partners in R and R' be m_1 and m_2. If she is finally matched to m_1, then (m, w) is a rogue couple in R; on the other hand, if she is matched to m_2, then (m, w) is a rogue couple in R'. Since these are the only two choices for w's partner, we have a contradiction in either case.