Discussion 2B Solution

1. (Contraposition) Prove that if $a+b<c+d$, then $a<c$ or $b<d$.

Assume $a \geq c$ and $b \geq d$ (note that this is the negation of $a<c \vee b<d$). Then, $a+b \geq c+b \geq c+d$, which is the negation of $a+b<c+d$.

2. (Problem formulation) Write the following statements using the notation covered in class. Use $\mathbb{N}$ to denote the set of natural numbers and $\mathbb{Z}$ to denote the set of integers. Also write $P(n)$ for the statement "n is odd".

a) For all natural numbers n, $2n$ is even.
$\forall\, n \in \mathbb{N}, \neg P(2n)$

b) For all natural numbers n, n is odd if n^2 is odd.
$\forall\, n \in \mathbb{N}, P(n^2) \implies P(n)$

c) There are no integer solutions to the equation $x^2 - y^2 = 10$.
$\neg(\exists\, x, y \in \mathbb{Z}, x^2 - y^2 = 10)$

3. (Induction) Prove that, for any positive integer n, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

- Base case: when $n=1$, $\sum_{i=1}^{1} i^2 = 1 = \frac{1(1+1)(2\cdot 1+1)}{6}$.
- Inductive hypothesis: assume for $n=k \geq 1$ that $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$.
- Inductive step:

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= \left(\sum_{i=1}^{k} i^2\right) + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{(by the inductive hypothesis)} \\
&= \frac{k(k+1)(2k+1)+6(k+1)^2}{6} \\
&= \frac{(k+1)(k(2k+1)+6(k+1))}{6} \\
&= \frac{(k+1)(2k^2+k+6k+6)}{6} \\
&= \frac{(k+1)(2k^2+7k+6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}.
\end{aligned}$$

By the principle of induction, the claim is proved.

4. (Problem solving) Prove that the length of the hypotenuse of a non-degenerate right triangle is strictly less than the sum of the two remaining sides.

1. Write down the definition of a right triangle and the claim to be proven in mathematical notation.
2. Prove the statement by contradiction.
3. Prove the statement directly.

Definition of a right triangle: $a^2+b^2=c^2$. Claim to be proven: $a+b>c$. We can prove this directly by adding $2ab$ (a positive number) to the LHS of the definition of a right triangle.

$$a^2+b^2=c^2 \implies a^2+2ab+b^2>c^2 \implies (a+b)^2>c^2 \implies a+b>c$$

We can prove the claim with contradiction by assuming it is not true. This is basically the reverse of the previous proof:

$$a+b\leq c \implies (a+b)^2\leq c^2 \implies a^2+2ab+b^2\leq c^2 \implies a^2+b^2<c^2$$

$$\implies \impliedby$$

5. (Problem solving) Let $H_j=\sum_{k=1}^{j}\frac{1}{k}$. Use mathematical induction to show that, for all integers $n\geq 0$, $H_{2^n}\geq 1+\frac{n}{2}$, thus showing that H_j must grow unboundedly as $j\to\infty$.

Base case: $H_{2^0}=H_1=1\geq 1+\frac{0}{2}$

Inductive Step: Assume that $H_{2^n}\geq 1+\frac{n}{2}$. Then:

$$\begin{aligned}
H_{2^{k+1}} &= 1+\frac{1}{2}+\ldots+\frac{1}{2^{k+1}} \\
&= (1+\frac{1}{2}+\ldots+\frac{1}{2^k})+(\frac{1}{2^k+1}+\ldots+\frac{1}{2^{k+1}}) \\
&= H_{2^k}+(\frac{1}{2^k+1}+\ldots+\frac{1}{2^{k+1}})
\end{aligned}$$

By noting that $(\frac{1}{2^k+1}+\ldots+\frac{1}{2^{k+1}})$ has 2^k terms, each of which is at least $\frac{1}{2^{k+1}}$

$$\geq H_{2^k}+2^k*\frac{1}{2^{k+1}}$$

By the inductive hypothesis:

$$\begin{aligned}
&\geq 1+\frac{k}{2}+2^k*\frac{1}{2^{k+1}} \\
&= 1+\frac{k}{2}+\frac{1}{2} \\
&= 1+\frac{k+1}{2}
\end{aligned}$$

Hence we have proved the statement by induction, and can conclude that H_{2^n} must go to infinity as $n\to\infty$, hence H_n must be diverging as $n\to\infty$.