CS 70 Fall 2015	Discrete Mathematics and Probability Theory Rao HW 3

Due Wednesday Sept 16 at 10PM

1. **(Be a Judge)** For each of the following statements about the traditional stable marriage algorithm with men proposing, indicate whether the statement is True or False and justify your answer with a short 2-3 line explanation:

   (a) (3 points) In a stable marriage algorithm execution which takes n days, there is a woman who did not receive a proposal on the $(n-1)$th day.

   Answer: True: The algorithm will terminate once there is exactly one man proposing to each woman. This means that there will be at least one woman who is not proposed to every day before the algorithm terminates. This includes day $(n-1)$.

   (b) (3 points) In a stable marriage algorithm execution, if a woman receives a proposal on day k, she receives a proposal on every subsequent day until termination.

   Answer: True: This is true by the improvement lemma, since once a woman has a suitor on a string, she always has some suitor on a string and therefore must receive a proposal each day.

   (c) (3 points) There is a set of preferences for n men and n women, such that in a stable marriage algorithm execution every man ends up with his least preferred woman.

   Answer: False: If this were to occur it would mean that at the end of the algorithm, every man would have proposed to every woman on his list and has been rejected $n-1$ times. This would also require every woman to reject $n-1$ suitors. We know this is impossible though if we consider what we learned in parts (a) and (b). There must be at least one woman who is not proposed to until the very last day.

   (d) (3 points) If man M does not propose to woman W in the propose-and-reject algorithm, then there can be a stable pairing in which M is matched with W.

   Answer: True: The propose-and-reject algorithm produces a male-optimal matching, and M only proposes to women he likes at least as much as his final partner. If there exists a stable pairing that is not male-optimal, then M would be paired with someone he does not propose to in the algorithm. This is of course possible in general.

   (e) (3 points) In a stable marriage instance, there can be two women with the same optimal man.

   Answer: False: If we run the propose and reject algorithm with women proposing, then the resulting pairings will have every woman paired with her optimal man. A single man can not be paired with two women, so any two women must have different optimal men

2. **(Proposals)** Please first read the stable marriage algorithm carefully in Lecture Note 4. Then answer the following questions.

   (a) (3 points) Run the traditional propose and reject algorithm on the following example.

   Men's preference list:

1	A	B	C	D
2	B	C	A	D
3	C	A	B	D
4	A	B	C	D

Women's preference list:

A	2	3	4	1
B	3	4	1	2
C	4	1	2	3
D	1	2	3	4

Answer: Here is the overall structure of the proposals: On the first day, each man proposes to the first woman on his list. In the next four days, the men propose to the women from their second column, because each man in turn gets rejected by his first choice. In the next four days the men propose to the women from the third column. On the last (10-th) day, man 1 proposes to D, the last woman on his list, and the algorithm halts.

The table below shows the execution of the algorithm. Every entry shows which woman holds that man on a string. A bold letter indicates a proposal. A $-$ sign indicates the man is not on a string.

	Day 1		Day 2		Day 3		Day 4		Day 5		Day 6		Day 7		Day 8		Day 9		Day 10	
	M	A	M	A	M	A	M	A	M	A	M	A	M	A	M	A	M	A	M	A
1	**A**	-	**B**	B	B	B	B	B	B	-	**C**	C	C	C	C	C	C	-	**D**	D
2	**B**	B	B	-	**C**	C	C	C	C	C	C	-	**A**	A	A	A	A	A	A	A
3	**C**	C	C	C	C	-	**A**	A	A	A	A	A	A	-	**B**	B	B	B	B	B
4	**A**	A	A	A	A	A	A	-	**B**	B	B	B	B	B	B	-	**C**	C	C	C

(b) (5 points) In class we showed that the propose and reject algorithm must terminate after at most n^2 proposals. Prove a sharper bound showing that the algorithm must terminate after at most $n(n-1)+1$ proposals. Conclude that the above example is a worst case instance for $n = 4$. How many days does the algorithm take on this instance?

Answer: Let us prove that there is at most one man who proposes to the last woman in his list. On the day when a man, M, proposes to the last woman in his list W, we claim that every other woman must have some man on the string. This is because M proposed to each of these women, and by the Improvement Lemma, once a woman has been proposed to she always has a man on the string. Since there are $n-1$ other women, they must be paired with all $n-1$ remaining men. Thus there is only one proposal on this day, and since it is accepted, the algorithm halts.

Therefore, at most one man proposes to his last choice, and thus there are at most $n^2-(n-1) = n(n-1)+1$ proposals.

In the example above there was $13 = 4(4-1)+1$ proposals in 10 days.

3. **(Quantitative Stable Marriage Algorithm)** Once you have practiced the basic algorithm, let's quantify stable marriage problem a little bit. Here we define the following notation: on day j, let $P_j(M)$ be the rank of the woman that man M proposes to (where the first woman on his list has rank 1 and the last has rank n). Also, let $R_j(W)$ be the total number of men that woman W has rejected up through day $j-1$ (i.e. not including the proposals on day j). Please answer the following questions using the notation above.

(a) (5 points) Prove or disprove the following claim: $\sum_M P_j(M) - \sum_W R_j(W)$ is independent of j. If it is true, please also give the value of $\sum_M P_j(M) - \sum_W R_j(W)$. The notation, $\sum_M$ and $\sum_W$, simply means that we are summing over all men and all women.

Answer: On day $j = 1$, each man proposes to the first woman on his list so $\sum_M P_1(M) = n$, and no woman rejected any man through day 0, and therefore $\sum_M P_1(M) - \sum_W R_1(W) = n$. In general, each time a woman rejects a man on day $j-1$, it increases $\sum_W R_j(W)$ by exactly 1. It also increases $\sum_M P_j(M)$ by exactly 1, since the rejected man proposes to the next woman on his list on day j. Therefore $\sum_M P_j(M) - \sum_W R_j(W)$ stays constant and is independent of j. □

More formally, we can prove this by induction on j, with $j = 1$ as base case.
Induction Hypothesis: Assume $\sum_M P_j(M) - \sum_W R_j(W) = n$.
Induction Step: The quantity $\sum_W R_{j+1}(W) - \sum_W R_j(W)$ is exactly the number of men rejected by women on day j. But each of the rejected men propose to the next woman on their list on day $j+1$, and so this quantity is also equal to $\sum_M P_{j+1}(M) - \sum_M P_j(M)$. Equating the two, we get

$$\sum_W R_{j+1}(W) - \sum_W R_j(W) = \sum_M P_{j+1}(M) - \sum_M P_j(M).$$

Therefore,

$$\sum_M P_{j+1}(M) - \sum_W R_{j+1}(W) = \sum_M P_j(M) - \sum_W R_j(W)$$

and the right hand side is equal to n by the induction hypothesis. □

(b) (5 points) Prove or disprove the following claim: one of the **men or women** must be matched to someone who is ranked in the top half of their preference list. You may assume that n is even.

Answer: Assume that no man is matched with a woman in the top half of his preference list. Each of them must have been rejected at least $\frac{n}{2}$ times, for a total of at least $\frac{n^2}{2}$ rejections. This implies that at least one woman must have rejected at least $\frac{n}{2}$ men (because if not, then the total number of rejections must be less than $\frac{n}{2} \times n$, contradiction). But now, by the improvement lemma, this woman must be matched with a man she likes more than the $\frac{n}{2}$ men she rejected, meaning that the man she is matched with is in the top half of her preference list. □

Alternative proof:

Assume towards contradiction that every man and every woman is matched to someone who is ranked in the bottom half of their preference list.

Observe that a man M is matched to someone in the top half of his preference list if and only if $P_m(M) \leq \frac{n}{2}$, where m is the last day of the algorithm. Therefore, if M is matched to someone in the bottom half of his preference list, then $P_m(M) > \frac{n}{2}$, i.e., $P_m(M) \geq \frac{n}{2} + 1$. Summing over the men gives us $\sum_M P_m(M) \geq \frac{n^2}{2} + n$. By part (a), it follows that $\sum_W R_m(W) = \sum_M P_m(M) - n \geq \frac{n^2}{2}$. Observe also that if $R_m(W) \geq \frac{n}{2}$, then by the improvement lemma, W must be matched to someone in the top half of her preference list. Therefore, from our assumption that W is matched to someone in the bottom half of her preference list, we get $R_m(W) < \frac{n}{2}$. Summing over the women gives us $\sum_W R_m(W) < \frac{n^2}{2}$. But this contradicts our earlier result above! □

4. **(Better Off Alone)** In the stable marriage problem, suppose that some men and women have standards and would not just settle for anyone. In other words, in addition to the preference orderings they have, they prefer being alone to being with some of the lower-ranked individuals (in their own preference list). A pairing could ultimately have to be partial, i.e., some individuals would remain single.

The notion of stability here should be adjusted a little bit. A pairing is stable if

- there is no paired individual who prefers being single over being with his/her current partner,
- there is no paired man and paired woman that would both prefer to be with each other over their current partners, and
- there is no single man and single woman that would both prefer to be with each other over being single.
- there is no paired man and single woman (or single man and paired woman) that would both prefer to be with each other over the current choice (the current partner or being alone).

(a) (10 points) Prove that a stable pairing still exists in the case where we allow single individuals. You can approach this by introducing imaginary mates that people "marry" if they are single. How should you adjust the preference lists of people, including those of the newly introduced imaginary ones for this to work?

Answer: Following the hint, we introduce an imaginary mate (let's call it a robot) for each person. Note that we introduce one robot for each individual person, i.e. there are as many robots as there are people. For simplicity let us say each robot is owned by the person we introduce it for.

Each robot is in love with its owner, i.e. it puts its owner at the top of its preference list. The rest of its preference list can be arbitrary. The owner of a robot puts it in his/her preference list exactly after the last person he/she is willing to marry. i.e. owners like their robots more than people they are not willing to marry, but less than people they like to marry. The ordering of people who someone does not like to marry as well as robots he/she does not own is irrelevant as long as they all come after their robot.

To illustrate, consider this simple example: there are three men $1,2,3$ and three women A,B,C. The preference lists for men is given below:

Man	Preference List
1	$A > B$
2	$B > A > C$
3	C

and the following depicts the preference lists for women:

Woman	Preference List
A	1
B	$3 > 2 > 1$
C	$2 > 3 > 1$

In this example, 1 is willing to marry A and B and he likes A better than B, but he'd rather be single than to be with C. On the other side B has a low standard and does not like being single at all. She likes 3 first, then 2, then 1 and if there is no option left she is willing to be forced into singleness. On the other hand, A has pretty high standards. She either marries 1 or remains single.

According to our explanation we should introduce a robot for each person. Let's name the robot owned by person X as R_X. So we introduce male robots R_A, R_B, R_C and female robots R_1, R_2, R_3. Now we should modify the existing preference lists and also introduce the preference lists for robots.

According to our method, 1's preference list should begin with his original preference list, i.e. $A > B$. Then comes the robot owned by 1, i.e. R_1. The rest of the ordering, which should include C and R_2, R_3 does not matter, and can be arbitrary.

For B, the preference list should begin with $3 > 2 > 1$ and continue with R_B, but the ordering between the remaining robots (R_A and R_C) does not matter.

What about robots' preference lists? They should begin with their owners and the rest does not matter. So for example R_A's list should begin with A, but the rest of the humans/robots (B, C, R_1, R_2, and R_3) can come in any arbitrary order.

So the following is a list of preference lists that adhere to our method. There are arbitrary choices which are shown in bold (everything in bold can be reordered within the bold elements).

Man	Preference List
1	$A > B > R_1 > \mathbf{3} > \mathbf{R_3} > \mathbf{R_2}$
2	$B > A > C > R_2 > \mathbf{R_1} > \mathbf{R_3}$
3	$C > R_3 > \mathbf{R_1} > \mathbf{R_3} > \mathbf{A} > \mathbf{B}$
R_A	$A > \mathbf{B} > \mathbf{C} > \mathbf{R_1} > \mathbf{R_2} > \mathbf{R_3}$
R_B	$B > \mathbf{R_1} > \mathbf{R_2} > \mathbf{R_3} > \mathbf{A} > \mathbf{C}$
R_C	$C > \mathbf{A} > \mathbf{R_2} > \mathbf{B} > \mathbf{R_1} > \mathbf{R_3}$

and the following depicts the preference lists for women and female robots:

Woman	Preference List
A	$1 > R_A > \mathbf{3} > \mathbf{R_B} > \mathbf{2} > \mathbf{R_C}$
B	$3 > 2 > 1 > R_B > \mathbf{R_C} > \mathbf{R_A}$
C	$2 > 3 > 1 > R_C > \mathbf{R_A} > \mathbf{R_B}$
R_1	$1 > \mathbf{R_B} > \mathbf{2} > \mathbf{R_C} > \mathbf{3} > \mathbf{R_A}$
R_2	$2 > \mathbf{R_A} > \mathbf{R_C} > \mathbf{1} > \mathbf{3} > \mathbf{R_B}$
R_3	$3 > \mathbf{2} > \mathbf{1} > \mathbf{R_A} > \mathbf{R_C} > \mathbf{R_B}$

Now let us prove that a stable pairing between robots and owners actually corresponds to a stable pairing (with singleness as an option). This will finish the proof, since we know that in the robots and owners case, the propose and reject algorithm will give us a stable matching.

It is obvious that to extract a pairing without robots, we should simply remove all pairs in which there is at least one robot (two robots can marry each other, yes). Then each human who is not matched is declared to be single. It remains to check that this is a stable matching (in the new, modified sense). Before we do that, notice that a person will never be matched with another person's robot, because if that were so he/she and his/her robot would form a rogue couple (the robot's love is there, and the owner actually likes his/her robot more than other robots).

i. No one who is paired would rather break out of his/her pairing and be single. This is because if that were so, that person along with its robot would have formed a rogue couple in the original pairing. Remember, the robot loves its owner more than anything, so if the owner likes it more than his/her mate too, they would be a rogue couple.

ii. There is no rogue couple. If a rogue couple m and w existed, they would also be a rogue couple in the pairing which includes robots. If neither m nor w is single, this is fairly obvious. If one or both of them are single, they prefer the other person over being single, which in the robots scenario means they prefer being with each other over being with their robot(s) which is their actual match.

This shows that each stable pairing in the robots and humans setup gives us a stable pairing in the humans-only setup. It is noteworthy that the reverse direction also works. If there is a stable pairing in the humans-only setup, one can extend it to a pairing for robots and humans setup by first creating pairs of owners who are single and their robots, and then finding an arbitrary stable matching between the unmatched robots (i.e. we exclude everything other than the unmatched robots and find a stable pairing between them). To show why this works, we have to refute the possibility of a rogue pair. There are three cases:

i. A human-human rogue pair. This would also be rogue pair in the humans-only setup. The humans prefer each other over their current matches. If their matches are robots, that translates to them preferring each other over being single in the humans-only setup.
ii. A human-robot rogue pair. If the human is matched to his/her robot, our pair won't be a rogue pair since a human likes his/her robot more than any other robot. On the other hand if the human is matched to another human, he/she prefers being with that human over being single which places that human higher than any robot. Again this refutes the human-robot pair being rogue.
iii. A robot-robot rogue pair. If both robots are matched to other robots, then by our construction, this won't be a rogue couple (we explicitly selected a stable matching between left-alone robots). On the other hand, if either robot is matched to a human, that human is its owner, and obviously a robot loves its owner more than anything, including other robots. So again this cannot be a rogue pair.

This completes the proof.

(b) (10 points) As you saw in the lecture, we may have different stable pairings. But interestingly, if a person remains single in one stable pairing, s/he must remain single in any other stable pairing as well (there really is no hope for some people!). Prove this fact by contradiction.

Answer: We will perform proof by contradiction. Assume that there exists some man m_1 who is paired with a woman w_1 in stable pairing S and unpaired in stable pairing T. Since S is a stable pairing and m_1 is unpaired, w_1 must be paired in T with a man m_2 whom she prefers over m_1. (If w_1 were unpaired or paired with a man she does not prefer over m_1, then (m_1, w_1) would be a rouge couple, which is a contradiction.)

Since m_2 is paired with w_1 in T, he must be paired in S with some woman w_2 whom m_2 prefers over w_1. This process continues (w_2 must be paired with some m_3 in T, m_3 must be paired with some w_3 in S, etc.) until all persons are paired. Since this requires m_1 to be paired in T, where he is known to be unpaired, we have reached a contradiction. Therefore, our assumption must be false, and there cannot exist some man who is paired in a stable pairing S and unpaired in a stable pairing T. A similar argument can be used for women.

Since no man or woman can be paired in one stable pairing and unpaired in another, every man or woman must be either paired in all stable pairings or unpaired in all stable pairings.

Here is another possible proof:

We know that some male-optimal stable pairing exists. Call this pairing M. We first establish two lemmas.

Lemma 1. If a man is single in male-optimal pairing M, then he is single in all other stabling pairings.

Proof. Assume there exists a man that is single in M but not single in some other stable pairing M'. Then M would not be a male-optimal pairing, so this is a contradiction.

Lemma 2. If a woman is paired in male-optimal pairing M, she is paired in all other stable pairings.

Proof. Assume there exists a woman that is paired in M but single in some other stable pairing M'. Then M would not be female-pessimal, so this is a contradiction.

Let there be k single men in M. Let M' be some other stable pairing. Then by Lemma 1, we know single men in M' will be greater than or equal to k. We also know that there are $n-k$ paired men and women in M. Then by Lemma 2, we know that the number of paired women in M' will be greater than or equal to $n-k$.

Now, we want to prove that if a man is paired in M, then he is paired in every other stable pairing. We prove this by contradiction. Assume that there exists a man m that is paired in M but is single in some other stable pairing M'. Then there must be strictly greater than k single men in M', and thus strictly greater than k single women in M'. Since there are strictly greater than k single women in M', there must be strictly less than $n-k$ paired women in M'. But this contradicts that the number of paired women in M' will be greater than or equal to $n-k$.

We also have to prove that if a woman is single in M, then she must be single every other stable pairing. We again prove this by contradiction. Assume that there exists a woman w that is single in M and paired in some other stable pairing M'. Then there are strictly greater than $n-k$ paired women in M', which means there are strictly greater than $n-k$ paired men in M'. This means there must be strictly less than k single men in M'. But this contradicts that the number of single men in M' will be greater than or equal to k.

Since we have proved both 1) If a man is single in M then he is single in every other stable pairing and 2) If a man is paired in M then he is paired in every other stable pairing (note that the contrapositive of this is if a man is single in any other stable pairing, then this man is single in M), we know that a man is single in M if and only if he is single in every other stable pairing. Similarly, since we have proved both 1) If a woman is single in M then she is single in every other stable pairing and 2) If a woman is paired in M then she is paired in every other stable pairing, we know that a woman is single in M if and only if she is single in every stable pairing. Thus we have proved that if a person is single in one stable pairing, s/he is single in every stable pairing.

5. **(Networks and Tours)** After struggling with stable marriage problems for a while, let's play around with another application of proof technique – Graph. Please prove or disprove the following claims.

   (a) (8 points) Suppose we have n websites such that for every pair of websites A and B, either A has a link to B or B has a link to A. Prove or disprove that there exists a website that is reachable from every other website by clicking at most 2 links. (*Hint: Induction*)

   Answer: We prove this by induction on the number of websites n.

   Base case For $n = 2$, there's always a link from one website to the other.

   Induction Hypothesis When there are k websites, there exists a website w that is reachable from every other website by clicking at most 2 links.

   Induction Step Let A be the set of websites with a link to w, and B be the set of websites two links away from w. The induction hypothesis states that the set of k websites $W = \{w\} \cup A \cup B$. Now suppose we add another website v. Between this website and every website in W, there must be a link from one to the other. If there is at least one link from v to $\{w\} \cup A$, w would still be reachable from v with at most 2 clicks. Otherwise, if all links from $\{w\} \cup A$ point to v, v will be reachable from every website in B with at most 2 clicks, because every website in B can click one link to go to a website in A, then click on one more link to go to v. In either case there exists a website in the new set of $k+1$ websites that is reachable from every other website by clicking at most 2 links.

(b) (8 points) We have shown in the lecture (or you have read Lecture Note 5) that a connected undirected graph has an Eulerian tour if and only if every vertex has even degree.

Prove or disprove that if a connected graph G on n vertices has exactly $2d$ vertices of odd degree, then there are d walks that *together* cover all the edges of G (i.e., each edge of G occurs in exactly one of the d walks; and each of the walks should not contain any particular edge more than once).

Answer: We split the $2d$ odd-degree vertices into d pairs, and join each pair with an edge, adding d more edges in total. Notice that now all vertices in this graph are of even degree. Now by Euler's theorem the resulting graph has an Eulerian tour. Removing the d added edges breaks the tour into d walks covering all the edges in the original graph, with each edge belonging to exactly one walk.

(c) (8 points) An alternative type of tour to Euler Tour in graph is a Rudrata Tour: a tour that visits every vertex exactly once. Prove or disprove that the hypercube contains a Rudrata cycle, for hypercubes of dimension $n \geq 2$.

Answer: We are trying to prove it by strong induction.

Stronger Inductive Claim: There exists a tour in an n-dimensional hypercube that uses the edge: $(0^n, 0^{n-1}1)$.

Base Case: $n = 2$, the hypercube is just a four cycle, which is a cycle that contains the edge $(00, 01)$ as required.

Inductive Hypothesis: We take the statement for dimension n.

Inductive Step: The recursive definition of an $n+1$ dimensional hypercube is to take two n dimensional hypercubes, relabel each vertex x in one "subcube" as $0x$ and relabel each vertex in the other "subcube" as $1x$ and add edges $(0x, 1x)$ for each $x \in \{0,1\}^n$.

Use the inductive hypothesis to form seperate tours of each subcube, and then replace the edges $(0^{n+1}, 0^n1)$ and $(10^n, 10^{n-2}1)$ with the edges between the subcubes; $(0^{n+1}, 10^n)$ and $(0^n1, 10^{n-2}1)$.

The degree of all nodes are two so we do have a set of cycles that touch all the nodes. And the tour is connected as one can reach every node from all zeros in the first cube using the inductive tour, and in the second cube using the edge to the second cube and the rest of the inductive tour. $\square$

6. **(Build-up Error?)**

(8 points) What is wrong with the following "proof"?

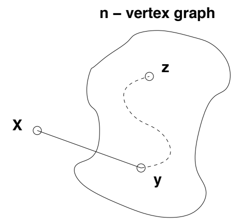
False Claim: If every vertex in an undirected graph has degree at least 1, then the graph is connected.

Proof: We use induction on the number of vertices $n \geq 1$.

Base case: There is only one graph with a single vertex and it has degree 0. Therefore, the base case is vacuously true, since the if-part is false.

Inductive hypothesis: Assume the claim is true for some $n \geq 1$.

Inductive step: We prove the claim is also true for $n+1$. Consider an undirected graph on n vertices in which every vertex has degree at least 1. By the inductive hypothesis, this graph is connected. Now add one more vertex x to obtain a graph on $(n+1)$ vertices, as shown below.



All that remains is to check that there is a path from x to every other vertex z. Since x has degree at least 1, there is an edge from x to some other vertex; call it y. Thus, we can obtain a path from x to z by adjoining the edge $\{x,y\}$ to the path from y to z. This proves the claim for $n+1$. $\square$

Answer: The mistake is in the argument that *"every $(n+1)$-vertex graph with minimum degree 1 can be obtained from an n-vertex graph with minimum degree 1 by adding 1 more vertex."* Instead of starting by considering an arbitrary $(n+1)$-vertex graph, this proof only considers an $(n+1)$-vertex graph that you can make by starting with an n-vertex graph with minimum degree 1. As a counterexample, consider a graph on four vertices $V = \{1,2,3,4\}$ with two edges $E = \{\{1,2\},\{3,4\}\}$. Every vertex in this graph has degree 1, but there is no way to build this 4-vertex graph from a 3-vertex graph with minimum degree 1.

More generally, this is an example of *build-up error* in proof by induction. Usually this arises from a faulty assumption that every size $n+1$ graph with some property can be "built up" from a size n graph with the same property. (This assumption is correct for some properties, but incorrect for others, such as the one in the argument above.)

One way to avoid an accidental build-up error is to use a *"shrink down, grow back"* process in the inductive step: start with a size $n+1$ graph, remove a vertex (or edge), apply the inductive hypothesis $P(n)$ to the smaller graph, and then add back the vertex (or edge) and argue that $P(n+1)$ holds.

Let's see what would have happened if we'd tried to prove the claim above by this method. In the inductive step, we must show that $P(n)$ implies $P(n+1)$ for all $n \geq 1$. Consider an $(n+1)$- vertex graph G in which every vertex has degree at least 1. Remove an arbitrary vertex v, leaving an n-vertex graph G' in which every vertex has degree... uh-oh! The reduced graph G' might contain a vertex of degree 0, making the inductive hypothesis $P(n)$ inapplicable! We are stuck — and properly so, since the claim is false!