CS 70	Discrete Mathematics and Probability Theory
Fall 2015	Rao HW 1

Due Wednesday Sept 2 at 10PM

1. (12 points) Determine whether the following equivalences hold, by writing out truth tables. Clearly state whether or not each pair is equivalent.

   (a) (3 points) $P \wedge (Q \vee P) \equiv P \wedge Q$

   Answer: **Answer**: Not equivalent.

P	Q	$P \wedge (Q \vee P)$	$P \wedge Q$
T	T	T	T
T	F	T	F
F	T	F	F
F	F	F	F

   (b) (3 points) $(P \Rightarrow Q) \Rightarrow R \equiv P \Rightarrow (Q \Rightarrow R)$

   Answer: **Answer**: Not equivalent.

P	Q	R	$(P \Rightarrow Q) \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	F	T

   (c) (3 points) $(P \Rightarrow Q) \Rightarrow (P \Rightarrow R) \equiv P \Rightarrow (Q \Rightarrow R)$

   Answer:

   Answer: Equivalent.

P	Q	R	$(P \Rightarrow Q) \Rightarrow (P \Rightarrow R)$	$P \Rightarrow (Q \Rightarrow R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

   (d) (3 points) $(P \wedge \neg Q) \Leftrightarrow (\neg P \vee Q) \equiv (Q \wedge \neg P) \Leftrightarrow (\neg Q \vee P)$

   Answer: **Answer**: Equivalent.

P	Q	$(P \wedge \neg Q) \Leftrightarrow (\neg P \vee Q)$	$(Q \wedge \neg P) \Leftrightarrow (\neg Q \vee P)$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	F	F

2. (9 points) Decide whether each of the following propositions is true, when the domain for x and y is the real numbers $\mathbb{R}$. Prove your answers.

   (a) (3 points) $\forall x \exists y\ (xy \geq x^2)$

   Answer: **Claim**: $\forall x \exists y\ (xy \geq x^2)$
   Answer: True.
   Proof: Let $y = x$. It is trivially true that $\forall x\ (x^2 \geq x^2)$. ■

   (b) (3 points) $\exists y \forall x\ (xy \geq x^2)$

   Answer: **Claim**: $\exists y \forall x\ (xy \geq x^2)$
   Answer: False.
   Proof: The proposition cannot be true for some $y < 0$, since $x^2 \geq 0$ and $xy < 0$ for $x > 0$ and $y < 0$. The proposition similarly cannot be true for some $y > 0$, since $x^2 \geq 0$ and $xy < 0$ for $x < 0$ and $y < 0$. The proposition is obviously not true for $y = 0$, since $x^2 > 0$ for $x \neq 0$. Since the proposition cannot be true for any real number y, the proposition is false. ■

   (c) (3 points) $\neg \forall x \exists y\ (xy > 0 \Rightarrow y > 0)$

   Answer: **Answer**: False.
   Proof: This is easiest to approach by looking at the proposition before negation, then applying negation. The proposition before negation is $\forall x \exists y\ (xy > 0 \Rightarrow y > 0)$. The implication in this proposition is vacuously true for $y = 0$. Because of this, the proposition before negation is true, so the negation of that proposition is false. ■

3. (9 points) Determine whether the following equivalences hold, and give brief justifications for your answers. Clearly state whether or not each pair is equivalent.

   (a) (3 points) $\neg \forall x\ \exists y\ \big(P(x) \Rightarrow \neg Q(x,y)\big) \ \equiv\ \exists x\ \forall y\ \big(P(x) \wedge Q(x,y)\big)$

   Answer: **Claim**: $\neg \forall x\ \exists y\ \big(P(x) \Rightarrow \neg Q(x,y)\big) \ \equiv\ \exists x\ \forall y\ \big(P(x) \wedge Q(x,y)\big)$
   Answer: The equivalence holds.
   Justification: Truth tables show that $P(x) \Rightarrow \neg Q(x,y) \ \equiv\ \neg P(x) \vee \neg Q(x,y)$. Using De Morgan's Law to distribute the negation on the left side yields $\exists x\ \forall y\ \big(\neg\neg P(x) \wedge \neg\neg Q(x,y)\big)$, which is equivalent to the right side.

   (b) (3 points) $\forall x\ \exists y\ \big(P(x) \Rightarrow Q(x,y)\big) \ \equiv\ \forall x\ \big(P(x) \Rightarrow (\exists y\ Q(x,y))\big)$

   Answer: **Claim**: $\forall x\ \exists y\ \big(P(x) \Rightarrow Q(x,y)\big) \ \equiv\ \forall x\ \big(P(x) \Rightarrow (\exists y\ Q(x,y))\big)$
   Answer: The equivalence holds.
   Justification: We can rewrite the claim as $\forall x\ \exists y\ \big(\neg P(x) \vee Q(x,y)\big) \ \equiv\ \forall x\ \big(\neg P(x) \vee (\exists y\ Q(x,y))\big)$. Clearly, the two sides are the same if $\neg P(x)$ is true. If $\neg P(x)$ is false, then the two sides are still the same, because $\forall x\ \exists y\ \big(\text{False} \vee Q(x,y)\big) \ \equiv\ \forall x\ \big(\text{False} \vee (\exists y\ Q(x,y))\big)$.

   (c) (3 points) $\forall x\ \exists y\ \big(Q(x,y) \Rightarrow P(x)\big) \ \equiv\ \forall x\ \big((\exists y\ Q(x,y)) \Rightarrow P(x)\big)$

   Answer: **Claim**: $\forall x\ \exists y\ \big(Q(x,y) \Rightarrow P(x)\big) \ \equiv\ \forall x\ \big((\exists y\ Q(x,y)) \Rightarrow P(x)\big)$
   Answer: The equivalence does not hold.

Justification: We can rewrite the claim as $\forall x\ \exists y\ \big(\neg Q(x,y) \vee P(x)\big) \equiv \forall x\ \big((\neg(\exists y\ Q(x,y))) \vee P(x)\big)$. By De Morgan's Law, distributing the negation on the right side of the equivalence changes the $\exists y$ to $\forall y$, and the two sides are clearly not the same. Another approach to the problem is to consider by linguistic example. Let x and y span the universe of all people, and let $Q(x,y)$ mean "Person x is Person y's offspring", and let $P(x)$ mean "Person x likes tofu". The left side claims that, for all Persons x, there exists some Person y such that either Person x is not Person y's offspring or that Person x likes tofu. The right side claims that, for all Persons x, if there exists a parent of Person x, then Person x likes tofu. Obviously, these are not the same.

4. (7 points) Here is an extract from Lewis Carroll's treatise *Symbolic Logic* of 1896:

    (I) No one, who is going to a party, ever fails to brush his or her hair.

    (II) No one looks fascinating, if he or she is untidy.

    (III) Opium-eaters have no self-command.

    (IV) Everyone who has brushed his or her hair looks fascinating.

    (V) No one wears kid gloves, unless he or she is going to a party.

    (VI) A person is always untidy if he or she has no self-command.

    (a) (3 points) Write each of the above six sentences as a quantified proposition over the universe of all people. You should use the following symbols for the various elementary propositions: $P(x)$ for "x goes to a party", $B(x)$ for "x has brushed his or her hair", $F(x)$ for "x looks fascinating", $U(x)$ for "x is untidy", $O(x)$ for "x is an opium-eater", $N(x)$ for "x has no self-command", and $K(x)$ for "x wears kid gloves".

    Answer:

    (I) "No one, who is going to a party, ever fails to brush his or her hair"
    Answer: $\forall x\ \big(P(x) \Rightarrow B(x)\big)$

    (II) "No one looks fascinating, if he or she is untidy."
    Answer: $\forall x\ \big(U(x) \Rightarrow \neg F(x)\big)$

    (III) "Opium-eaters have no self-command."
    Answer: $\forall x\ \big(O(x) \Rightarrow N(x)\big)$

    (IV) "Everyone who has brushed his or her hair looks fascinating"
    Answer: $\forall x\ \big(B(x) \Rightarrow F(x)\big)$

    (V) "No one wears kid gloves, unless he or she is going to a party"
    Answer: $\forall x\ \big(K(x) \Rightarrow P(x)\big)$

    (VI) "A person is always untidy if he or she has no self-command."
    Answer: $\forall x\ \big(N(x) \Rightarrow U(x)\big)$

    (b) (2 points) Now rewrite each proposition equivalently using the contrapositive.

    Answer:

    (I) $\forall x\ \big(\neg B(x) \Rightarrow \neg P(x)\big)$

    (II) $\forall x\ \big(F(x) \Rightarrow \neg U(x)\big)$

    (III) $\forall x\ \big(\neg N(x) \Rightarrow \neg O(x)\big)$

    (IV) $\forall x\ \big(\neg F(x) \Rightarrow \neg B(x)\big)$

    (V) $\forall x\ \big(\neg P(x) \Rightarrow \neg K(x)\big)$

(VI) $\forall x\ (\neg U(x) \Rightarrow \neg N(x))$

(c) (2 points) You now have twelve propositions in total. What can you conclude from them about a person who wears kid gloves? Explain clearly the implications you used to arrive at your conclusion.

Answer: **Answer**: A person who wears kid gloves is not an opium-eater.
Derivation: $K(x) \Rightarrow P(x) \Rightarrow B(x) \Rightarrow F(x) \Rightarrow \neg U(x) \Rightarrow \neg N(x) \Rightarrow \neg O(x)$

5. (20 points) Karnaugh Maps

Below is the truth table where F is encoded as 0 and T is encoded as 1 for the boolean function

$$Y = (\neg A \wedge \neg B \wedge C) \vee (\neg A \wedge B \wedge \neg C) \vee (A \wedge \neg B \wedge C) \vee (A \wedge B \wedge C).$$

A	B	C	Y
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

In this question, we will explore a different way of representing a truth table, the *Karnaugh map*. A Karnaugh map is just a grid-like representation of a truth table, but as we will see, the mode of presentation can give more insight. The values inside the squares are copied from the output column of the truth table, so there is one square in the map for every row in the truth table.

Around the edge of the Karnaugh map are the values of the input variables, where again F is encoded as 0 and T is encoded is 1. Note that the sequence of numbers across the top of the map is not in binary sequence, which would be 00, 01, 10, 11. It is instead 00, 01, 11, 10, which is called *Gray code* sequence. Gray code sequence only changes one binary bit as we go from one number to the next in the sequence. That means that adjacent cells will only vary by one bit, or Boolean variable. In other words, *cells sharing common Boolean variable values are adjacent*.

For example, here is the Karnaugh map for Y:

		BC			
		00	01	11	10
A	0	0	1	0	1
	1	0	1	1	0

The Karnaugh map provides a simple and straight-forward method of minimizing boolean expressions by visual inspection. The technique is to examine the Karnaugh map for any groups of adjacent ones that occur, which can be combined to simplify the expression. Note that "adjacent" here means in the modular sense, so adjacency wraps around the top/bottom and left/right of the Karnaugh map; for example, the top-most cell of a column is adjacent to the bottom-most cell of the column.

For example, the ones in the second column in the Karnaugh map above can be combined because $(\neg A \wedge \neg B \wedge C) \vee (A \wedge \neg B \wedge C)$ simplifies to $(\neg B \wedge C)$. Applying this technique to the Karnaugh map (illustrated below), we obtain the following simplified expression for Y:

$$Y = (\neg B \wedge C) \vee (A \wedge C) \vee (\neg A \wedge B \wedge \neg C).$$

A \ BC	00	01	11	10
0	0	1	0	1
1	0	1	1	0

Answer:

(a) Write the truth table for the boolean function

$$Z = (\neg A \wedge \neg B \wedge \neg C \wedge \neg D) \vee (\neg A \wedge \neg B \wedge C \wedge \neg D) \vee (A \wedge \neg B \wedge \neg C \wedge \neg D) \vee (A \wedge \neg B \wedge C \wedge \neg D).$$

A	B	C	D	Z
0	0	0	0	1
0	0	0	1	0
0	0	1	0	1
0	0	1	1	0
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	0
1	0	0	0	1
1	0	0	1	0
1	0	1	0	1
1	0	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

(b) Using your truth table from Part 1, fill in the Karnaugh map for Z below.

AB \ CD	00	01	11	10
00				
01				
11				
10				

	CD			
AB	00	01	11	10
00	1	0	0	1
01	0	0	0	0
11	0	0	0	0
10	1	0	0	1

(c) Using your Karnaugh map from Part 2, write down a simplified expression for Z.

The four corners can be combined to get

$$Z = \neg B \wedge \neg D.$$

The entire map can be wrapped onto a torus (a donut shape - the way that video games [like Pac-Man or Asteroids] sometimes wrap around so if you move off the right side, you come out the left side, and if you move past the top, you come out the bottom). The ones form a square with only B and D remaining unchanged at 0 and 0 whereas A and C takes on the values $(00, 01, 10, 11)$ which constitutes all possible combinations AC can take.

	CD			
AB	00	01	11	10
00	1	0	0	1
01	0	0	0	0
11	0	0	0	0
10	1	0	0	1

(d) Show that this simplification could also be found algebraically by factoring the expression for Z in (1).

$$Z = (\neg A \wedge \neg B \wedge \neg C \wedge \neg D) \vee (\neg A \wedge \neg B \wedge C \wedge \neg D) \vee (A \wedge \neg B \wedge \neg C \wedge \neg D) \vee (A \wedge \neg B \wedge C \wedge \neg D)$$

By using the distributive law $(A \wedge B) \vee (A \wedge C) = A \wedge (B \vee C)$, we get the following.

$$Z = (\neg B \wedge \neg D) \wedge ((\neg A \wedge \neg C) \vee (\neg A \wedge C) \vee (A \wedge \neg C) \vee (A \wedge C))$$

As $((\neg A \wedge \neg C) \vee (\neg A \wedge C) \vee (A \wedge \neg C) \vee (A \wedge C))$ is a tautology (fancy word meaning always true no matter what), we get the following simplification.

$$Z = (\neg B \wedge \neg D) \wedge (1)$$

$$Z = (\neg B \wedge \neg D)$$

6. (5 points) Proof by?

   (a) (3 points) Prove that if $x, y \in \mathbb{Z}$, if 10 does not divide xy, then 10 does not divide x and 10 does not divide y. In notation: $(\forall x, y \in \mathbb{Z})\ \ 10 \nmid xy \implies (10 \nmid x \ \wedge \ 10 \nmid y)$. What proof technique did you use?

   Answer: We will use proof by contraposition. For any arbitrary given x and y, the statement $10 \nmid xy \implies (10 \nmid x \wedge 10 \nmid y)$ is equivalent using contraposition to $\neg(10 \nmid x \wedge 10 \nmid y) \implies \neg(10 \nmid xy)$. Moving the negations inside, this becomes equivalent to $(10 \mid x \vee 10 \mid y) \implies 10 \mid xy$.

   Now for this part, we give a proof by cases. Assuming that $10 \mid x \vee 10 \mid y$, one of the two cases must be true.

   i. $10 \mid x$: in this case $x = 10k$ for some $k \in \mathbb{Z}$. Therefore $xy = 10ky$ which is a multiple of 10. So $10 \mid xy$.
   ii. $10 \mid y$: in this case $y = 10k$ for some $k \in \mathbb{Z}$. Therefore $xy = 10kx$ which is a multiple of 10. So $10 \mid xy$.

   Therefore assuming $10 \mid x \vee 10 \mid y$ we proved $10 \mid xy$.

   We used proof by cases and proof by contraposition.

   (b) (1 point) Prove or disprove the contrapositive.

   Answer: We proved the statement. The contrapositive of a statement has logically equivalent to the statement. So we are done.

   (c) (1 point) Prove or disprove the converse.

   Answer: Its not true! The converse is that if 10 does not divide x and does not divide y than 10 does not divide xy. We can choose $x = 2$ and $y = 5$ and see a counterexample to the statement.

7. (18 points) Prove or disprove each of the following statements. For each proof, state which of the proof types (as discussed in Note 2) you used.

   (a) (3 points) For all natural numbers n, if n is odd then $n^2 + 2n$ is odd. **Answer:** **Claim**: For all natural numbers n, if n is odd then $n^2 + 2n$ is odd.
   Answer: True.
   Proof: We will use a direct proof. Assume n is odd. By the definition of odd numbers, $n = 2k + 1$ for some natural number k. Substituting into the expression $n^2 + 2n$, we get $(2k+1)^2 + 2 \times (2k+1)$. Simplifying the expression yields $4k^2 + 8k + 3$. This can be rewritten as $2 \times (2k^2 + 4k + 1) + 1$. Since $2k^2 + 4k + 1$ is a natural number, by the definition of odd numbers, $n^2 + 2n$ is odd. ■

   (b) (3 points) For all natural numbers n, $n^2 + 7n + 1$ is odd.

   Answer: **Claim**: For all natural numbers n, $n^2 + 7n + 1$ is odd.
   Answer: True.
   Proof: We will use a proof by cases. Let n be an even number. By the definition of even numbers, $n = 2k$ for some natural number k. Substituting into the expression $n^2 + 7n + 1$, we get $(2k)^2 + 7 \times (2k) + 1$. Simplifying the expression yields $4k^2 + 14k + 1$. This can be rewritten as $2 \times (2k^2 + 7k) + 1$, which is an odd number. Therefore, if n is even, then $n^2 + 7n + 1$ is odd. Now let n be an odd number. By the definition of odd numbers, $n = 2k + 1$ for some natural number k. Substituting into the expression $n^2 + 7n + 1$, we get $(2k+1)^2 + 7 \times (2k+1) + 1$. Simplifying the expression yields $4k^2 + 18k + 9$. This can be rewritten as $2 \times (2k^2 + 9k + 4) + 1$, which is an odd number. Therefore, if n is odd, then $n^2 + 7n + 1$ is odd. Since $n^2 + 7n + 1$ is odd when n is even or when n is odd, $n^2 + 7n + 1$ is odd for all natural numbers n. ■

(c) (3 points) For all real numbers a,b, if $a+b \leq 10$ then $a \leq 7$ or $b \leq 3$.

Answer: **Claim**: For all real numbers a, b, if $a+b \leq 10$ then $a \leq 7$ or $b \leq 3$.
Answer: True.
Proof: We will use a proof by contraposition. Suppose that $a > 7$ and $b > 3$ (note that this is equivalent to $\neg(a \leq 7 \vee b \leq 3)$). Since $a > 7$ and $b > 3$, $a+b > 10$ (note that $a+b > 10$ is equivalent to $\neg(a+b \leq 10)$). Thus, if $a+b \leq 10$, then $a \leq 7$ or $b \leq 3$ (or both, as "or" is not "exclusive or" in this case). $\blacksquare$

(d) (3 points) For all real numbers r, if r is irrational then $r+1$ is irrational.

Answer: **Claim**: For all real numbers r, if r is irrational then $r+1$ is irrational.
Answer: True.
Proof: We will use a proof by contraposition. Assume that $r+1$ is rational. Since $r+1$ is rational, it can be written in the form a/b where a and b are integers. Then r can be written as $(a-b)/b$. By the definition of rational numbers, r is a rational number, since both $a-b$ and b are integers. By contraposition, if r is irrational, then $r+1$ is irrational. $\blacksquare$

(e) (3 points) For all natural numbers n, $10n^2 > n!$.

Answer: **Claim**: For all natural numbers n, $10n^2 > n!$.
Answer: False.
Proof: We will use proof by counterexample. Let $n = 6$. $10 \times 6^2 = 360$. $6! = 720$. Since $10n^2 < n!$, the claim is false. $\blacksquare$

(f) (3 points) For all natural numbers a where a^2 is even, then a is even. **Answer:** **Claim: For all natural numbers if a^2 is even, then a is even.**
Answer: True.
Proof: This will be proof by contrapositive. The contrapositive is "If a is odd, then a^2 is odd." Let a be odd. By the definition of odd, $a = 2k+1$. Then $a^2 = (2k)^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which implies a odd. $\blacksquare$

8. (0 points) Just for fun!

There are two doors, one leads to heaven and one leads to hell. There are two guards, one for each door. One always lies and the other always tells the truth. You are allowed to ask one of the guards a single question and must determine which door is which. What question would you ask?

Answer: We can pick one of the guards, point to a door, and ask "if I asked the other guard whether this door leads to heaven would he say yes?" The door leads to heaven if and only if the answer is no.

To prove this, we use proof by cases.

(a) The guard we asked the question from is the liar. In this case, the other guard would have told us the truth, so the other guard would have said yes to "whether this door leads to heaven" if and only if that door led to heaven. But since we are asking the liar, the answer will get flipped. So the door leads to heaven if and only if the answer is no.

(b) The guard we asked the question from is the truth-teller. In this case, the other guard would have answered "whether this door leads to heaven" with no if and only if it led to heaven. Since we are asking the truth-teller, we get this directly from him, so the final answer is no if and only if the door leads to heaven.