CS 70 Discrete Mathematics and Probability Theory
Fall 2012 Vazirani **HW 8**

1 How to steal the GSI's car

One of the CS70 GSIs lives in a building with an electronic garage door that is opened by a remote control.

One day, you overhear his room-mate talking about how, after losing his remote control and having the landlord demand $60 to replace it, he discovered that the remote controls can be bought online for $12, and all you need to do after buying one is to open it up and set ten tiny on-off switches to match the "code" of the door. These settings, of course, can be obtained by looking at the switches in another remote for the same door.

After a particularly nasty homework, you decide to take revenge by getting one of these remotes and breaking into the garage. Of course, unlike the room-mate, you dont have access to another remote from which to copy the code. It takes 5 seconds to flip a single switch in either direction (theyre tiny, and CS students aren't known for fine motor control), and 1 second to test whether the current switch settings work by just pressing the button. What is the shortest amount of time that you need to definitely open the garage door?

Explain precisely how you would go about achieving this minimum time. Keep in mind that, per Murphy's law, even if you try all but one of the combinations, it is possible that the right combination is the one you haven't tried.

–Solution–

In the worst case, you must try every combination. You can minimize the time it takes to try every combination by following a Hamiltonian path on a Hypercube, visiting every vertex exactly once. Encode each switch in binary, 1 for up, and 0 for down, leading to a Hypercube of size n, for n switches.

By the Theorem proved in the lecture notes, every Hypercube $n \geq 2$ has a Hamiltonian path.

Assuming there are $n \geq 2$ switches, there are 2^n combinations of switch positions. Following a Hamiltonian path, $2^n - 1$ changes (flipping a single switch in each change) must be made to try every combination. The minimum worst-case time is $2^n + 5(2^n - 1) = 6(2^n) - 5$ seconds.

2 An intro to shortest paths

One of the most practical graph-theoretic concepts is the "shortest path" between two vertices u and v. The definition is natural — a path P between u and v is the shortest path if there are no paths that start at u and end at v and have fewer edges than P. Notice that there might be multiple shortest paths.

Suppose the graph G has a Hamiltonian path P starting at vertex u and ending at vertex v. Describe precisely the structure of G if P is the shortest path from u to v. This means you must both prove that graphs matching your description always have this property, and graphs not matching it never do.

–Solution–

Re-label u, v as v_1 and v_n respectively.
The graph is a chain, where the vertexes $v_1, v_2, \ldots, v_n$ are connected by edges $(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n)$. That is, $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{(v_i, v_{i+1}) \mid i \in [1, n-1]\}$, and $G = (V, E)$.

Theorem 0.1: *The shortest path between vertexes v_1 and v_n in a graph G is a Hamiltonian path if and only if G is a simple chain starting from v_1 and ending at v_n, i.e. $V = \{v_1, v_2, \ldots, v_n\}$, $E = \{(v_i, v_{i+1}) \mid i \in [1, n-1]\}$, and $G = (V, E)$.*

Proof:

- ($\Leftarrow$)
  Given $G = (V, E)$ satisfying properties of theorem. The only path between v_1 and v_n is a path reaching $v_1, v_2, \ldots, v_n$ in sequence. This is both a Hamiltonian path and the shortest path between v_1 and v_n.

- ($\Rightarrow$)
  Given a graph $\hat{G} = (\hat{V}, \hat{E})$ where the shortest path between v_1 and v_n is a Hamiltonian path. This implies that the graph is connected, so $\hat{V} = V$.
  Without loss of generality, the sequences of vertexes along this path are labeled $v_1, v_2, \ldots, v_n$. This implies that the edge set $\hat{E} \supset E$. What remains is to show that there are no extra edges in the graph, i.e. $\hat{E} \subset E$. Suppose $(v_i, v_j) \in \hat{E}$ and $(v_i, v_j) \notin E$. Without loss of generality, we can assume $j > i + 1$, i.e. the two vertexes connected by the edge are not reached consecutively along the Hamiltonian path. Contradiction: introduction of this edge creates a shorter path between v_1 and v_n than the Hamiltonian path, as it skips every vertex between v_i and v_j. Therefore, $\hat{E} \subset E$, and therefore, $\hat{G} = (V, E) = G$.

$\square$

[1]Solution Note: Sets A and B are equal (i.e. $A = B$) if and only if $A \subset B$ and $A \supset B$.

3 This problem counts more than the rest

This problem lets you practice counting things. You should leave your answers as (tidy) expressions involving factorials, binomial coefficients, etc., rather than evaluating them as decimal numbers (though you are welcome to perform this last step as well for your own interest if you like, provided it is clearly separated from your main answer). Also, you should explain clearly how you arrived at your answers; solutions provided without appropriate explanation will receive no credit.

1. How many 13-bit strings are there that contain exactly 5 ones?

–Solution–

Choose 5 bits from 13, so $\binom{13}{5}$.

2. How many 55-bit strings are there that contain more ones than zeros?

–Solution–

Add the appropriate binomial coefficients: $\sum_{i=0}^{27} \binom{55}{i}$.

Alternatively, there are 2^{55} possible 55-bit strings, and half of those will have more ones than zeros, so 2^{54} possible 55-bit strings with more ones than zeros.

Alternatively,

$$\begin{aligned} 2^{55} &= \sum_{i=0}^{27} \binom{55}{k} + \sum_{i=28}^{55} \binom{55}{k} \\ &= \sum_{i=0}^{27} \binom{55}{k} + \sum_{i=28}^{55} \binom{55}{55-k} \\ &= \sum_{i=0}^{27} \binom{55}{k} + \sum_{i=0}^{27} \binom{55}{k} \\ &= 2\sum_{i=0}^{27} \binom{55}{k} \end{aligned}$$

This implies that $\sum_{i=0}^{27} \binom{55}{k} = 2^{55}/2 = 2^{54}$.

3. How many different 13-card bridge hands are there? (A bridge hand is obtained by selecting 13 cards from a standard 52-card deck. The order of the cards within a bridge hand is considered irrelevant.)

–Solution–

Each card in a 52-card deck is unique. Thus, we are choosing 13 from 52, or $\binom{52}{13}$.

4. How many different 13-card bridge hands contain no aces? [Recall that there are four aces in a standard 52-card deck]

–Solution–

We will choose our 13 cards from a deck of $(52 - 4) = 48$, the deck of cards minus the aces. Therefore, $\binom{48}{13}$.

5. How many different 13-card bridge hands contain all four aces?

–Solution–

We fix four cards in the hand, and choose $13 - 4 = 9$ cards from a deck of $52 - 4 = 48$. Therefore, there are $\binom{48}{9}$ possible hands containing all four aces.

6. How many different 13-card bridge hands contain exactly 5 spades? [Recall that there are 13 spades in a standard 52-card deck.

–Solution–

We first choose a $13 - 5 = 8$ card hand of non-spades, i.e. from a deck of $52 - 13 = 39$ cards. That sub-hand has $\binom{39}{8}$ possibilities. The sub-hand of only spades has $\binom{13}{5}$ possibilities. In total, there are $\binom{39}{8}\binom{13}{5}$ possibilities.

7. How many ways are there to order a standard 52-card deck?

–Solution–

First, pick one card. There are 52 choices. Pick a second card; there are 51 choices. A third; 50 choices. For the first two cards, there are $52 \cdot 51$ possibilities. We can continue this argument recursively to conclude that there are 52! ways to order a deck.

8. Two identical decks of 52 distinct cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?

–Solution–

Mark each card in one deck with a marker. (i.e. consider a deck with 104 distinct cards.) Then, there are 104! orderings. However, in this case, each card can be swapped with its markered dual. There are 2^{52} previously distinct orderings that all collapse to one when the markings are erased. Therefore, there are $\frac{104!}{2^{52}}$ orderings.

9. How many different anagrams of "KENTUCKY" are there? (An anagram of "KENTUCKY" is any re-ordering of the letters of "KENTUCKY"; i.e., any string made up of the eight letters K, E, N, T, U, C, K, and Y, in any order. The anagram does not have to be an English word.)

–Solution–

There are two K's, so if we consider the distinct number of possibilities, 8!, we have over-counted by a factor of 2 (i.e. swapping the K's). There are $\frac{8!}{2}$ possibilities.

10. How many different anagrams of "ALASKA" are there?

–Solution–

A is repeated 3 times, Alaska has 6 letters. In any given ordering, the A's can be re-arranged $3! = 6$ different ways. Thus, there are $\frac{6!}{6} = 5!$ total anagrams.

11. How many different anagrams of "MINNESOTA" are there?

–Solution–

With two N's, there are $\frac{9!}{2}$ anagrams.

12. How many different anagrams of "MISSISSIPPI" are there?

–Solution–

In any given ordering, the repeated letters may be swapped. 4 I's, 4 S's, 2 P's. Total orderings considering every letter unique: 11!. Overcounted by a factor of 4!4!2!. Total of $\frac{11!}{4!4!2!}$ anagrams.
Alternatively, we can pick the location of the I's first: $\binom{11}{4}$. Second, fix the location of the S's: $\binom{11-4}{4}$. Finally, fix the location of the P's: $\binom{11-4-4}{2}$. Then, we have 1 location left for the M, so there are a total of $\binom{11}{4}\binom{7}{4}\binom{3}{2}$ anagrams, which you can confirm is exactly the same as the first answer.

13. Suppose you are given 8 distinguishable balls (numbered 1 through 8) and 24 bins. How many different ways are there to distribute these 8 balls among these 24 bins?

–Solution–

Each ball can independently be placed in 24 distinct bins. For each ball placement you have 24 options. So, there are 24^8 possibilities.

14. Suppose you are given 8 indistinguishable balls and 24 bins. How many distinguishable ways are there to distribute these 8 balls among these 24 bins?

–Solution–

Representing as a binary string of length $n + k - 1$, or $24 + 8 - 1 = 31$. Must choose 8 to have 0's, or $\binom{31}{8}$ possibilities.

15. Suppose you are given 8 indistinguishable balls and 5 bins. How many distinguishable ways are there to distribute these 8 balls among these 5 bins such that no bin is empty?

–Solution–

We require 5 balls to be in 5 separate bins. That leaves 3 to distribute among 5 bins. There are $\binom{5+3-1}{3} = \binom{7}{3}$ ways to distribute the balls.

16. There are 30 students currently enrolled in a class. How many different ways are there to pair up the 30 students (that is, to split the class into groups of 2 students)?

–Solution–

Shuffle the students like a deck of cards. There are 30! ways to do this.
Then, pair students 1 and 2, 3 and 4, 5 and 6, etc. Within each pair, order doesn't matter, so we have over-counted this arrangement by a factor of 2^{15}.
Further, the order of the pairs in the "deck" doesn't matter, and there are 15! ways to shuffle the pairs of students around.
Thus, there are a total of

$$\frac{30!}{15!2^{15}}$$

ways to pair the students.

4 Fermat's necklace

In the following, let p be a prime number and let k be a positive integer.

1. We have an endless supply of beads. The beads come in k different colors. All beads of the same color are indistinguishable. We have a piece of string. We want to make a pretty decoration by threading p many beads onto the string (from left to right). We can choose any sequence of colors, subject only to one rule: the p beads must not all have the same color.

   How many different ways are there construct such a sequence of beads?

–Solution–

This is sampling with replacement, with a constraint. Consider each bead to be a k-nary digit of length p. There are k^p possible colorings of the bead sequence, but we restrict the case where every k-nary digit (or bead) has the same value (or color). There are k of these cases. Thus, there are $k^p - k$ total possibilities.

2. Now we tie the two ends of the string together, forming a circular necklace. This lets us freely rotate the beads around the necklace. We'll consider two necklaces equivalent if the sequence of colors on one can be obtained by rotating the beads on the other. (For instance, if we have

$k = 3$ colors — red (R), green (G), and blue (B) — then the length p = 5 necklaces RGGBG, GGBGR, GBGRG, BGRGG, and GRGGB are all equivalent, because these are cyclic shifts of each other.)

Count how many non-equivalent necklaces there are, if the p beads must not all have the same color. [Hint: What can you conclude if rotating all the beads on a necklace to another position produces an identical looking necklace?]

–Solution–

Keep in mind that p is prime. To see what happens when p is not prime, consider an alternating sequence of beads, i.e. $R-G-R-G$. Rotating yields one other combination, $G-R-G-R$. Rather than there being 4 distinct bead sequences that end up being considered equivalent, we only have 2 in this case. When p is prime, however, there are exactly p distinct sequences that are considered equivalent.
If a sequence is of length q, where q is not prime, and m divides q, we could have subsequences of length m, repeating $m|q$ times. Any rotation by m elements yields precisely the same bead sequence, rather than one that is distinct.
The solution to the previous part counts each equivalent sequence in this scenario p times (since p is prime). We have over-counted, then, by a factor of p, and the result is $\frac{k^p-k}{p}$.

3. How can you use the above reasoning to prove Fermat's little theorem? (Recall that the theorem says that if $a \neq 0$, $a^{p-1} \equiv 1 \bmod p$.)

–Solution–

From the above, we know that there are $k^p - k$ ways to arrange the necklace. From the second part, we know this to be a multiple of p, since we have over-counted cyclic arrangements by a factor of p. Therefore, $k^p - k \equiv 0 \mod p$. Since $k < p$ and p prime, k^{-1} exists, so $k^{p-1} \equiv 1 \mod p$.

5 Extra credit: candy

You have been hired as an actuary by a candy store, and have just been informed that an eccentric millionaire has distributed ten golden tickets to his newly re-opened chocolate factory among the 100 candy bars in your store (so there are 90 bars with no tickets, and 10 bars with one ticket each). Given that the store has twenty customers, each of whom will purchase five candy bars uniformly at random, calculate the probabilities for any particular customer of:

1. not receiving any tickets to the chocolate factory

–Solution–

Reserve 5 of the 100 tickets for the unlucky customer.
Distribute the 10 tickets among the remaining 95 bars - there are $\binom{95}{10}$ ways to do this.
The total number of distributions includes the subset of bars given to the unlucky customer, so in total there are $\binom{100}{10}$ ways to distribute the tickets.
Therefore, the probability of not receiving any tickets is:

$$P(\text{loser}) = \frac{\binom{95}{10}}{\binom{100}{10}}$$

2. receiving exactly one ticket to the chocolate factory

–Solution–

Consider the subset of 5 bars given to the customer. One ticket is distributed among their bars; there are 5 ways to distribute the ticket.
The remaining 9 bars are distributed among the rest of the bars. There are $\binom{95}{9}$ ways to do this. Thus, there are $5\binom{95}{9}$ ways to distribute tickets such that a customer receives exactly one ticket.
Therefore, the probability is:

$$P(1 \text{ ticket}) = \frac{5\binom{95}{9}}{\binom{100}{10}} = \frac{50 \cdot 90 \cdot 89 \cdot 88 \cdot 87}{100 \cdot 99 \cdot 98 \cdot 97 \cdot 96}$$

3. receiving at least one ticket to the chocolate factory

–Solution–

Count every possibility except those where you receive no tickets, or: $1 - P(\text{loser})$.

$$P(\text{winner}) = \frac{\binom{100}{10} - \binom{95}{10}}{\binom{100}{10}}$$