CS 70 Discrete Mathematics and Probability Theory
Fall 2015 Jean Walrand Discussion 11A

Definition: A random variable X on a sample space Ω is a function that assigns to each sample point $\omega \in \Omega$ a real number $X(\omega)$

Definition: The distribution of a discrete random variable X is the collection of values $\{(a, Pr[X=a]) : a \in A\}$, where A is the set of all possible values taken by X.

1. Binary Fun

1. What is the sample space Ω generated by flipping two quarters ($H = 1, T = 0$)?
   For example, $(H,T) = (1,0)$.
   Solution: $\{(0,0),(1,0),(0,1),(1,1)\}$.
2. Define a random variable X to be the number of heads. What is the distribution of X?
   Solution: $(0,0.25),(1,0.5),(2,0.25)$.
3. Define a random variable Y to be 1 if $\omega = (1,0)$ and 0 otherwise. What is the distribution of Y?
   Solution: $(0,0.75),(1,0.25)$.

2. Locked out

You just rented a large house and the realtor gave you five keys, one for the front door and the other four for each of the four side and back doors of the house. Unfortunately, all keys look identical, so to open the front door, you are forced to try them at random.

Find the distribution and the expectation of the number of trials you will need to open the front door. (Assume that you can mark a key after you've tried opening the front door with it and it doesn't work.)

Solution: Let K be a random variable denoting the number of trials until we find the right key.

$$\begin{aligned}
Pr[K=1] &= \frac{1}{5} \\
Pr[K=2] &= \frac{4}{5} \times \frac{1}{4} = \frac{1}{5} \\
Pr[K=3] &= \frac{4}{5} \times \frac{3}{4} \times \frac{1}{3} = \frac{1}{5} \\
Pr[K=4] &= \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} = \frac{1}{5} \\
Pr[K=5] &= \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{1} \times = \frac{1}{5} \\
\mathbb{E}(K) &= \frac{1}{5} \sum_{i=1}^{5} i = 3
\end{aligned}$$

This result may seem surprising at first, but if we consider this experiment as follows: randomly line up keys, then try them in order, we see that this is equivalent to our earlier scheme. Furthermore, the right key is now equally likely to be in any of the five spots.

3. Parking

Reese Prosser never puts money in a 25-cent parking meter in Hanover. He assumes that there is a probability of 0.05 that he will be caught (each attempt is independent). Assume each offense that is caught costs him $10. Under his assumptions:

1. How does the expected cost of parking 10 times without paying the meter compare with the cost of paying the meter each time?

   Solution: If he pays each time, it costs $2.50. We can see that if he doesn't pay, this is a binomial distribution with $n = 10$ and $p = 0.05$, which means the expected number of times he'll get caught in 10 days is $np = 10 \times 0.05 = 0.5$. We multiply that by the cost to see how much he can expect to pay in 10 days if he never pays the meter, which is $0.5 \times \$10 = \$5$. Therefore, It looks like he's better off just paying each time.

2. If he parks at the meter 10 times without paying the meter, what is the probability he will end up paying more than he would have had he lawfully paid the meter 25-cents each time? Is this probability greater than or less than $1/2$?

   Solution: Reese will save $\$0.25 \times 10 = \$2.50$ by not paying all ten times, which means he only needs to get caught once to lose more money than he would have saved. To calculate the probability that he gets caught at least once, we take (hopefully, as you've already guessed) the complement of the probability that he never gets caught:

   $$1 - (1 - 0.05)^{10} \approx 0.4$$

   Note: You can see that the probability will be less than $\frac{1}{2}$ by noting that as you carry out the exponentiation, the result decreases by a little less than 0.05, meaning that 0.95^{10} will be greater than $\frac{1}{2}$. Thus, $1 - 0.95^{10}$ will be a little less than $\frac{1}{2}$.

4. Graph

Consider a random graph (undirected, no multi-edges, no self-loops) on n nodes, where each possible edge exists independently with probability p. Let X be the number of isolated nodes (nodes with degree 0). What is $\mathrm{E}(X)$? Why isn't X a binomial distribution?

Solution: Let's first pause and ask ourselves why X is not binomial. If we consider a trial as adding an edge, which happens with probability p, we will have $\frac{n(n-1)}{2}$ trials. If we were interested in the number of edges that the resulting graph has, then it would be binomial. But unfortunately, that is not the random variable we're looking for (Star Wars reference here).

Since we are interested in the number of isolated nodes, we must instead consider a trial creating an isolated node, which happens with probability $(1 - p)^{n-1}$. However, now our trials are not independent. For example, given that a node is not isolated, the conditional probability of all nodes connected to that node being isolated becomes 0.

So how can we solve this problem? Let's introduce some indicator variables $X_1, X_2, \ldots, X_n$, where $X_i = 1$ if node i is isolated.
Note that $\Pr[X_i = 1] = (1 - p)^{n-1}$, and thus $\mathbf{E}(X_i) = (1 - p)^{n-1}$.
Now, we can rewrite X as

$$X = X_1 + X_2 + \cdots + X_n$$

Using the Linearity of Expectation, we know

$$\begin{aligned}\mathbf{E}(X) &= \mathbf{E}(X_1 + X_2 + \cdots + X_n) \\ &= \mathbf{E}(X_1) + \mathbf{E}(X_2) + \cdots + \mathbf{E}(X_n) \\ &= (1-p)^{n-1} + (1-p)^{n-1} + \cdots + (1-p)^{n-1} \\ &= n(1-p)^{n-1}\end{aligned}$$

What happened here? We ended up with the same expectation as a binomial distribution, even though it wasn't binomial. In general, many different distributions can have the same expectation, but can vary greatly. This is one such example. Another is the two following distributions: one that is always $\frac{1}{2}$, and another is a fair coin toss. Both have the same expectation, but are very different.