1. **Strange Dilution**

You have a jar of red marbles and blue marbles. At each time step, you draw a marble, and you note the color of the marble. Then, you dilute the proportion of the opposite-colored marbles by a factor of $\gamma$, where $0 < \gamma < 1$. (For example: if you pick a red marble, then the proportion of blue marbles is reduced by a factor of $\gamma$.) If $p$ is the fraction of marbles that started off as red, what is the expected proportion of red marbles at time $n$?

**Solution:**

Let $X_n$ be the fraction of marbles that are red at time $n$. With probability $X_n$, you choose a red marble, so the fraction of blue marbles is now $\gamma(1 - X_n)$. Therefore, the new fraction of red marbles is $1 - \gamma(1 - X_n) = 1 - \gamma + \gamma X_n$. With probability $1 - X_n$, you choose a blue marble, so the fraction of red marbles is now $\gamma X_n$. We have

$$E[X_{n+1} \mid X_n] = X_n(1 - \gamma + \gamma X_n) + (1 - X_n)\gamma X_n = X_n - \gamma X_n + \gamma X_n = X_n$$

By the law of iterated expectation, $E[X_{n+1}] = E[E[X_{n+1} \mid X_n]] = E[X_n]$, so $E[X_n] = p$.

2. **A meeting of three**

James, Jon, and Mike each arrive at the movie theater at times uniformly distributed in the interval $(3:00, 3:20)$. Their arrival times are independent. Each person waits 5 minutes after their arrival before heading into the theater. What is the probability they all see each other before going into the theater?

**Solution:** Denote Jon, James, and Mike's arrival times as $X, Y$, and $Z$ respectively. Note that they are i.i.d $Uniform([0,1])$. (Here 1 represents 20 minutes, so 5 minutes is 1/4).
Consider the region where $X > Y$. For $X, Y$, and $Z$ to all be within the same 1/4 length interval, $Y$ can be between $X - 1/4$ and $X$. However since $X, Y$ cannot take negative values, for $X \in [0, 1/4]$, $Y$ can range from 0 to $X$. Now, for $Z$ to be within 1/4 of the other two, it can go from $X - 1/4$ to $Y + 1/4$ since $X > Y$. However, again, since $X, Y, Z$ must take values in $[0,1]$, for $X \in [0, 1/4]$, $Z$ goes from 0 to $Y + 1/4$, and for $Y \in [3/4, 1]$, $Z$ goes from $X - 1/4$ to 1. The case where $X < Y$ is symmetric so we multiply by 2. Using the joint pdf $f_{X,Y,Z} = 1$, $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$, we compute the following integrals:

$$\begin{aligned}
Pr(max(X,Y,Z) - min(X,Y,Z) < 1/4) &= 2 * Pr(X > Y \cap max(X,Z) - Y < 1/4) \\
&= 2 * \left( \int_0^{1/4} \int_0^x \int_0^{y+1/4} dzdydx + \int_{1/4}^{3/4} \int_{x-1/4}^{x} \int_{x-1/4}^{y+1/4} dzdydx + \right. \\
&\left. \int_{3/4}^{1} \int_{x-1/4}^{3/4} \int_{x-1/4}^{y+1/4} dzdydx + \int_{3/4}^{1} \int_{3/4}^{x} \int_{x-1/4}^{1} dzdydx \right) \\
&= 2 * (1/96 + 3/64 + 1/96 + 1/96) \\
&= 5/32
\end{aligned}$$

3. **Exponential Median**

   What is the expected value of the median of three i.i.d exponential variables with parameter $\lambda$?

   **Solution:** The minimum of exponential variables is another exponential variable with the sum of the originals' parameters as its parameter. Denote $M$ as the median of the exponential variables, $X_1, X_2, X_3$. Suppose $Min(X_1, X_2, X_3) = m$. Then, viewing the exponential variables as lightbulbs running simultaneously, the median is the next lightbulb to go out, which is the minimum of the next two exponential variables, which is an exponential with parameter $2\lambda$. Use the memoryless property. Therefore $E[M|Min(X_1, X_2, X_3)] = Min(X_1, X_2, X_3) + \frac{1}{2\lambda}$. Therefore $E[M] = \frac{1}{3\lambda} + \frac{1}{2\lambda} = \frac{5}{6\lambda}$.
   You can also try finding the pdf of the median. $Pr(M \in [t, t+dt]) = \int Pr(Min(X_1, X_2, X_3) \in [s, s+ds]) * Pr(Min(\{X_1, X_2, X_3\} - \{Min(X_1, X_2, X_3)\} \in [0, t-s+dt]) = \int_0^t (3e^{-3s}) * 2e^{-2(t-s)} ds dt = e^{-2t} dt \int_0^t 6e^{-s} ds = 6(e^{-2t} - e^{-3t}) dt$. So the pdf is $6(e^{-2t} - e^{-3t})$ which should give the same expectation.

4. **Random walk**

   Alice starts at vertex 0 and wishes to get to vertex $n$. When she is at vertex 0 she has a probability of 1 of transitioning to vertex 1. For any other vertex $i$, there is a probability of $1/2$ of transitioning to $i+1$ and a probability of $1/2$ of transitioning to $i-1$. What is the expected number of steps Alice takes to reach vertex $n$?

   **Solution:** Formulate hitting time equations; the hard part is solving them. $R_i$ represents the expected number of steps to get to vertex $n$ starting from vertex $i$. In particular, $R_n = 0$ and we are interested in calculating $R_0$. We have equations:
   $R_0 = 1 + R_1$
   $R_1 = 1 + 1/2 R_0 + 1/2 R_2$
   $\ldots$
   $R_i = 1 + 1/2 R_{i-1} + 1/2 R_{i+1}$
   $\ldots$
   $R_{n-1} = 1 + 1/2 R_{n-2} + 1/2 R_n$
   Plugging in $R_0 = 1 + R_1$ to the second equation: $R_1 = 1 + 1/2 + 1/2 R_1 + 1/2 R_2$ which implies $R_1 = 3 + R_2$. In fact, if $R_i = k + R_{i+1}$, then $R_{i+1} = 1 + 1/2 R_i + 1/2 R_{i+2} = 1 + k/2 + 1/2 R_{i+1} + 1/2 R_{i+2}$ which after moving $1/2 R_{i+1}$ to the left and multiplying by two implies $R_{i+1} = k + 2 + R_{i+2}$.
   Therefore, $R_0 = 1 + R_1 = 1 + 3 + R_2 = 1 + 3 + 5 + R_3 = \ldots = 1 + 3 + \ldots + 2n - 1 + R_n$ and since $R_n = 0$, we have $R_0 = 2n * \frac{n}{2} = n^2$.

5. **Exponential LLSE**

   Let $X \sim U[0, a]$ and let $Y = e^X$. Compute $L[Y \mid X]$. What does $L[Y \mid X]$ approach as $a \to 0$?

   **Solution:**

Compute all of the necessary terms.

$$\begin{aligned}
E[X] &= \frac{a}{2} \\
\text{var}(X) &= \frac{a^2}{12} \\
E[Y] &= \int_0^a e^x \cdot \frac{1}{a}\,\mathrm{d}x = \frac{1}{a}(e^a - 1) \\
E[XY] &= \int_0^a xe^x \cdot \frac{1}{a}\,\mathrm{d}x = e^a - \frac{1}{a}(e^a - 1) \\
\text{cov}(X,Y) &= e^a - (e^a - 1)\left(\frac{1}{2} + \frac{1}{a}\right) = \frac{2ae^a - (a+2)(e^a - 1)}{2a}
\end{aligned}$$

Therefore, one has

$$L[Y \mid X] = \frac{1}{a}(e^a - 1) + 6\frac{2ae^a - (a+2)(e^a - 1)}{a^3}\left(X - \frac{a}{2}\right)$$

As $a \to 0$, the LLSE should approach the tangent line to the curve $e^x$ at the point $x = 0$. Hence, as $a \to 0$, $L[Y \mid X] \to 1 + X$. (This can be verified by explicitly computing the limits, but this is tedious.

6. **First Exponential to Die**

Let $X$ and $Y$ be Expo$(\lambda_1)$ and Expo$(\lambda_2)$ respectively. What is $P(\min(X,Y) = X)$, the probability that the first of the two to die is $X$?

**Solution:**

Consider $P(\min(X,Y) = X \in (x, x+\delta))$. In order for $X \in (x, x+\delta)$, the probability is $f_X(x) \cdot \delta = \lambda_1 e^{-\lambda_1 x} \cdot \delta$. In order for $Y > X$ (for $X$ to be the minimum), we must have $P(Y > x) = e^{-\lambda_2 x}$. Therefore,

$$P(\min(X,Y) = X) = \int_0^\infty \lambda_1 e^{-\lambda_1 x} \cdot e^{-\lambda_2 x}\,\mathrm{d}x = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$