CS 70 Discrete Mathematics and Probability Theory
Spring 2015 Vazirani HW 12

Reminder: Deadline for HW11 self-grade is Thursday, April 16 at noon.

Reading: Note 16, 17, 18

Due Monday April 20

1. **Elevator**

   Eight people get into an empty elevator on the ground floor, G. There are 10 floors labeled 1 through 10. Each person gets off at a randomly selected floor. Each person's destination is independent of everyone else's destination.

   (a) What is the probability that the elevator stops at floor 3?

   (b) Let X be the number of floors the elevator stops on. What is $E[X]$? (Hint: Use linearity of expectation. Create indicator random variables X_i which indicate whether the elevator stops on the i-th floor.)

   Answer:

   (a) The elevator skips floor 3 if and only if every person picks a floor other than 3, which happens with probability $(9/10)^8$. Therefore, The elevator stops at floor 3 with probability $1-(9/10)^8$.

   (b) Let $X_i = 1$ if the elevator stops on floor i, and 0 otherwise. Then, $X = X_1 + X_2 + \cdots + X_{10}$. By (a), we know $E[X_i] = 1-(9/10)^8$ for all i. By linearity of expectation,

   $$E[X] = \sum_{i=1}^{10} E[X_i] = 10 \cdot (1-(9/10)^8).$$

2. **Soulmates**

   We call a couple a soulmate couple if each of them is the first on the other person's preference list. If there are n men and n women, and all of their preference lists are random (independently and uniformly generated over all permutations), what is the expected number of soulmate couples?

   Answer: Let X be the number of soulmate couples. If X_{ij} is the indicator for the event that the i-th man and the j-th woman are a soulmate couple, we have $X = \sum_{1\le i,j\le n} X_{ij}$. Since the probability that the i-th man is the first on the j-th woman's preference list and the probability that the j-th woman is the first on the i-th man's preference list are both $\frac{1}{n}$, the probability of the event X_{ij} is $\frac{1}{n}\cdot\frac{1}{n}$. By linearity of expectation,

   $$E[X] = \sum_{1\le i,j\le n} E[X_{ij}] = \sum_{1\le i,j\le n} \frac{1}{n^2} = 1.$$

3. **Variance**

(a) X and Y are $0/1$-valued random variables, i.e., indicator variables. They satisfy $\Pr[X=0] = \Pr[X=1] = \Pr[Y=0] = \Pr[Y=1] = 0.5$ and $\mathrm{E}[XY] = 0.4$.
What is the smallest that $\mathrm{Var}[X+Y]$ could be? What is the largest that $\mathrm{Var}[X+Y]$ could be?

(b) Prove or give a counterexample: if X, Y are independent random variables, then $\mathrm{Var}[2X+Y] = \mathrm{Var}[X] + \mathrm{Var}[X+Y]$ is guaranteed to hold.

Answer:

(a) Note that

$$\begin{aligned}\mathrm{Var}[X+Y] &= \mathrm{E}[(X+Y)^2] - (\mathrm{E}[X+Y])^2 \\ &= \mathrm{E}[X^2 + 2XY + Y^2] - (\mathrm{E}[X+Y])^2 \\ &= \mathrm{E}[X^2] + 2\mathrm{E}[XY] + \mathrm{E}[Y^2] - (\mathrm{E}[X] + \mathrm{E}[Y])^2.\end{aligned}$$

From the given information we know:

$$\mathrm{E}[X^2] = \mathrm{E}[X] = \Pr[X=1] = 0.5, \qquad \mathrm{E}[Y^2] = \mathrm{E}[Y] = \Pr[Y=1] = 0.5$$

Hence,

$$\mathrm{Var}[X+Y] = \mathrm{E}[X^2] + 2\mathrm{E}[XY] + \mathrm{E}[Y^2] - (\mathrm{E}[X] + \mathrm{E}[Y])^2 = 0.5 + 2 \cdot 0.4 + 0.5 - (0.5+0.5)^2 = 0.8.$$

The smallest value $\mathrm{Var}[X+Y]$ could be and the largest value $\mathrm{Var}[X+Y]$ could be are both 0.8.

(b) Suppose X is any random variable with nonzero variance and $Y = 0$. Then,

$$\begin{aligned}&\mathrm{Var}[2X+Y] = \mathrm{Var}[2X] = \mathrm{E}[(2X)^2] - (\mathrm{E}[2X])^2 = 4(\mathrm{E}[X^2] - \mathrm{E}[X]^2) = 4\mathrm{Var}[X] \\ &\mathrm{Var}[X] + \mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[X] = 2\mathrm{Var}[X].\end{aligned}$$

Therefore the claim is false.

4. **Markov and Chebyshev**

A random variable X has expectation $\mathrm{E}[X] = 2$ and variance $\mathrm{Var}[X] = 9$, and the value of X never exceeds 10. Which three of the following statements must be true about X? Justify your answer.

- $\Pr[X=2] > 0$.
- $\Pr[X \geq 2] = \Pr[X \leq 2]$.
- $\mathrm{E}[X^2] = 13$.
- $\Pr[X \geq 6] \leq 1/3$.
- $\Pr[X \geq 6] \leq 9/16$.
- $\Pr[X \leq 1] \leq 8/9$.

Answer:

- $\mathrm{E}[X^2] = 13$.
Since $9 = \mathrm{Var}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2 = \mathrm{E}[X^2] - 2^2$, we have $\mathrm{E}[X^2] = 9 + 4 = 13$.

- $\Pr[X \geq 6] \leq 9/16$.

  Chebyshev's inequality says $\Pr[|X - \mathrm{E}[X]| \geq a] \leq \frac{\mathrm{Var}[X]}{a^2}$. If we set $a = 4$, we have

  $$\Pr[|X-2| \geq 4] \leq \frac{9}{16}.$$

  Now we simply observe that $\Pr[X \geq 6] \leq \Pr[|X-2| \geq 4]$, because the event $X \geq 6$ is a subset of the event $|X-2| \geq 4$.

- $\Pr[X \leq 1] \leq 8/9$.

  Let $Y = 10 - X$. Since X is never exceeds 10, Y is a nonnegative random variable. By Markov's inequality,

  $$\Pr[10 - X \geq a] = \Pr[Y \geq a] \leq \frac{\mathrm{E}[Y]}{a} = \frac{\mathrm{E}[10-X]}{a} = \frac{8}{a}.$$

  Setting $a = 9$, we get $\Pr[X \leq 1] = \Pr[10 - X \geq 9] \leq \frac{8}{9}$.

5. **Random bit strings**

   Let S be a random bit string of length n. For example $S = 101111001001$ is a bit string of length 12, and we say that it has a run of four 1's starting in position 3.

   (a) For a given position j in S, what is the probability that it is a starting point of a run of at least m 1's?

   (b) What is the expected number of positions j at which runs of at least m 1's start?

   (c) Use Markov's inequality to show that the probability that there exists a run of at least $5\lceil \log n \rceil$ 1's is at most $\frac{1}{n^4}$.

   (d) We now consider runs of alternating 1-0 that start with 1 (e.g., 101010101). What is the expected number of places j at which alternating runs (beginning with 1) of at least m bits start?

   Answer:

   (a) In order for position j to be the starting point of a run of at least m 1's, positions j through $j+m-1$ must all be 1's. The probability of this is $(\frac{1}{2})^m = \frac{1}{2^m}$.

   (b) Let X denote the number of starting positions for runs of at least m 1's. Let X_j be the indicator random variable for the event that position j is the starting point of a run of at least m 1's. We see from part (a) that $\mathrm{E}[X_j] = \frac{1}{2^m}$. Then $X = \sum_{j=1}^{n-m+1} X_j$, and

   $$\mathrm{E}[X] = \sum_{j=1}^{n-m+1} \mathrm{E}[X_j] = \frac{n-m+1}{2^m}.$$

   (c) We now use $m = 5\lceil \log n \rceil$, and we want to find the probability $\Pr[X \geq 1]$. By Markov's inequality, we have that

   $$\Pr[X \geq 1] \leq \frac{\mathrm{E}[X]}{1} = \frac{n-m+1}{2^m} = \frac{n - 5\lceil \log n \rceil + 1}{2^{5\lceil \log n \rceil}} \leq \frac{n}{2^{5\log n}} = \frac{n}{2^{\log n^5}} = \frac{n}{n^5} = \frac{1}{n^4}.$$

   (d) This is the same computation as in parts (a) and (b). The probability that position j is the starting point of an alternating run of at least m bits is $(\frac{1}{2})^m = \frac{1}{2^m}$, and again, the expected number of such places is $\frac{n-m+1}{2^m}$.

6. **Display**

Each pixel in a 10×10 display is turned on or off with equal probability. The display shows a horizontal line if all the pixels in a given row are turned on. Let X denote the number of horizontal lines that the display shows.

(a) What is $\mathrm{E}[X]$?

(b) What is $\mathrm{Var}[X]$?

(c) Show that $\Pr[X \geq 1] \leq 1/100$.

Answer:

(a) Let X_i be the indicator for the event that the i-th row shows a horizontal line. Clearly, $\Pr[X_i] = \frac{1}{2^{10}}$. By linearity of expectation,

$$\mathrm{E}[X] = \sum_{i=1}^{10} \mathrm{E}[X_i] = \sum_{i=1}^{10} \frac{1}{2^{10}} = \frac{5}{512}.$$

(b) Since $\mathrm{Var}[X_i] = \mathrm{E}[X_i^2] - \mathrm{E}[X_i]^2 = \frac{1}{2^{10}} - \frac{1}{2^{20}} = \frac{1}{2^{10}}(1 - \frac{1}{2^{10}})$ and the X_i's are independent,

$$\mathrm{Var}[X] = \sum_{i=1}^{10} \mathrm{Var}[X_i] = 10 \cdot \frac{1}{2^{10}}\left(1 - \frac{1}{2^{10}}\right) = \frac{5115}{524288}.$$

(c) First, note that $X \geq 1$ if and only if $|X - 5/512| \geq 507/512$. Then, using Chebyshev's inequality,

$$\Pr[|X - 5/512| \geq 507/512] \leq \frac{\frac{5115}{524288}}{\left(\frac{507}{512}\right)^2} = \frac{1705}{171366} \leq \frac{1}{100}.$$

Alternatively, since X is a non-negative random variable, we can also apply Markov's inequality:

$$\Pr[X \geq 1] \leq \frac{\mathrm{E}[X]}{1} = \frac{5}{512} < \frac{1}{100}.$$

(Note that here Markov's inequality gives an easier and better result than Chebyshev. But in general, Chebyshev's inequality usually gives a tighter bound than Markov. Moreover, Chebyshev's inequality applies to any random variable, while Markov only applies to non-negative random variables.)

7. **Which door?**

Each customer arriving at a bank chooses to enter either via the front door (with probability $3/4$) or via the back door (with probability $1/4$), independently of the other customers. On a given day, 1200 customers arrive at the bank. Let random variables F and B denote the numbers of customers that arrive via the front and back doors, respectively, on that day.

(a) Write down the expectation $\mathrm{E}[F]$ and the variance $\mathrm{Var}[F]$.

(b) Now consider the random variable $D = F - B$, which measures the difference between the numbers of customers who choose the front and back doors. What are $\mathrm{E}[D]$ and $\mathrm{Var}[D]$?
(Hint: Note that $F + B = 1200$.)

(c) For security purposes, the bank would like to know a typical value for the difference D. Using Chebyshev's inequality, calculate a value α so that the probability that D deviates by more than $\pm\alpha$ from its expectation is at most $\frac{1}{9}$.

Answer:

(a) Let F_j be the indicator random variable for the event that customer j enters through the front door. We know that $\mathrm{E}[F_j] = \frac{3}{4}$, and $\mathrm{Var}[F_j] = E[F_j^2] - E[F_j]^2 = E[F_j] - E[F_j]^2 = \frac{3}{4} - (\frac{3}{4})^2 = \frac{3}{16}$. We have $F = \sum_{j=1}^{1200} F_j$, so $\mathrm{E}[F] = \sum_{j=1}^{1200} \mathrm{E}[F_j] = 1200 \times \frac{3}{4} = 900$. Since the F_j are independent, we also have $\mathrm{Var}[F] = \sum_{j=1}^{1200} \mathrm{Var}[F_j] = 1200 \times \frac{3}{16} = 225$.

(b) Since $F + B = 1200$, we have $D = F - B = 2F - 1200$. Hence,

$$\mathrm{E}[D] = 2\mathrm{E}[F] - 1200 = 600.$$

$$\mathrm{Var}[D] = \mathrm{E}[(D - \mathrm{E}[D])^2] = \mathrm{E}[(2F - 1200 - \mathrm{E}[2F - 1200])^2] = \mathrm{E}[4(F - \mathrm{E}[F])^2] = 4\mathrm{Var}[F] = 900.$$

(c) We want to compute α so that $\Pr[|D - E[D]| \geq \alpha] \leq \frac{1}{9}$. By Chebyshev's inequality, we have

$$\Pr[|D - E[D]| \geq \alpha] \leq \frac{\mathrm{Var}[D]}{\alpha^2} = \frac{900}{\alpha^2}.$$

Therefore, we solve $\frac{900}{\alpha^2} = \frac{1}{9}$ to get that $\alpha = 90$.

8. **Extra Credit: Non-transitive Dice**

Consider three six-sided dice A, B, and C, whose sides are labeled with any numbers of your choice. Say that A beats B if $\Pr[A \text{ shows a number greater than } B] > 1/2$.

Choose numbers to label each of the sides of A, B and C such that A beats B, B beats C, and C beats A. Try to create your labels to maximize the probabilities in each case.

Notice that armed with a set of three non-transitive dice, you could challenge someone to a game of dice, where to ensure fairness, you would give your opponent first choice of die to play with!

Answer: There are many ways to do this. For one example:

$$A : 2, 2, 5, 5, 5, 5$$

$$B : 4, 4, 4, 4, 4, 4$$

$$C : 3, 3, 3, 3, 6, 6$$

Then,

$$\Pr[A \text{ shows a number greater than } B] = \frac{2}{3}$$

$$\Pr[B \text{ shows a number greater than } C] = \frac{2}{3}$$

$$\Pr[C \text{ shows a number greater than } A] = \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{3} = \frac{5}{9}$$