EECS 70 Discrete Mathematics and Probability Theory
Fall 2015 Jean Walrand

Homework 14

You do not need to hand in this homework.

1. Continuous random variable

Let X be a continuous random variable whose pdf is cx^3 (for some constant c) in the range $0 \le x \le 1$, and is 0 outside this range.

a Find c.

Solution: Since our total probability must be equal to 1, $\int_0^1 cx^3 dx = 1 = [\frac{1}{4}cx^4]_{x=0}^1 = \frac{c}{4}$, so $c = 4$.

b Find $\Pr\left[\frac{1}{3} \le X \le \frac{2}{3} \;\middle|\; X \le \frac{1}{2}\right]$.

Solution:

$$\begin{aligned}
\Pr\left[\frac{1}{3} \le X \le \frac{2}{3} \;\middle|\; X \le \frac{1}{2}\right] &= \Pr\left[\frac{1}{3} \le X \le \frac{2}{3} \cap X \le \frac{1}{2}\right] / \Pr\left[X \le \frac{1}{2}\right] \\
&= \Pr\left[\frac{1}{3} \le X \le \frac{1}{2}\right] / \Pr\left[X \le \frac{1}{2}\right] \\
&= \left(\int_{\frac{1}{3}}^{\frac{1}{2}} 4x^3 dx\right) / \left(\int_0^{\frac{1}{2}} 4x^3 dx\right) \\
&= \left([x^4]_{x=\frac{1}{3}}^{\frac{1}{2}}\right) / \left([x^4]_{x=0}^{\frac{1}{2}}\right) \\
&= ((\frac{1}{2})^4 - (\frac{1}{3})^4) / (\frac{1}{2})^4 \\
&= \frac{65}{81}
\end{aligned}$$

c. Find $E(X)$.

Solution:

$$\begin{aligned}
E(X) &= \int_0^1 x \cdot 4x^3 dx \\
&= \int_0^1 4x^4 dx \\
&= [\frac{4}{5}x^5]_{x=0}^1 \\
&= \frac{4}{5}
\end{aligned}$$

d Find $\mathrm{Var}(X)$.

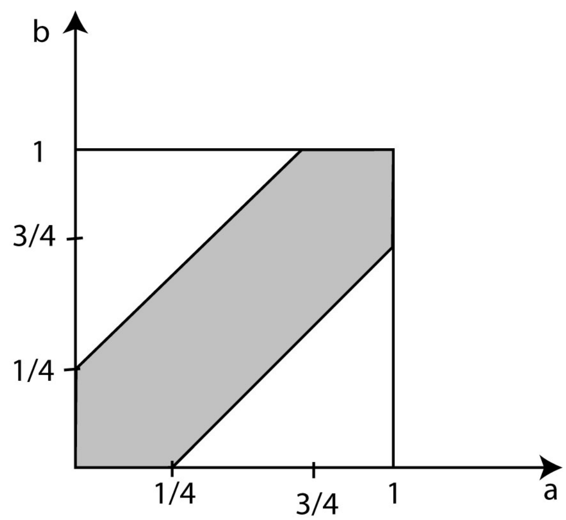
Solution:

$$\begin{aligned}\mathrm{Var}(X) &= \int_0^1 x^2 \cdot 4x^3 dx - \mathrm{E}(X)^2 \\ &= \int_0^1 4x^5 dx - (\frac{4}{5})^2 \\ &= [\frac{2}{3}x^6]_{x=0}^1 - \frac{16}{25} \\ &= \frac{2}{75}\end{aligned}$$

2. Joint continuous random variables

Alice and Bob agree to try to meet for lunch between 12pm and 1pm at their favorite sushi restaurant. Being extremely busy, they are unable to specify their arrival times exactly, and can say only that each of them will arrive (independently) at a time that is uniformly distributed within the hour. In order to avoid wasting precious time, if the other person is not there when they arrive they agree to wait exactly fifteen minutes before leaving. What is the probability that they will actually meet for lunch?

Solution: Let the r.v. A be the time that Alice arrives and the r.v. B be the time when Bob arrives. Consider the following picture, plotting the space of all outcomes (a,b):



The shaded region is the set of values (a,b) for which Alice and Bob will actually meet for lunch. Since all points in this square are equally likely, the probably they meet is the ratio of the shaded area to the area of the square. If the area of the square is 1, then the area of the shaded region is

$$1 - 2 \times \left(\frac{1}{2} \times \left(\frac{3}{4}\right)^2\right) = \frac{7}{16},$$

since the area of the white triangle on the upper-left is $\frac{1}{2} \times (\frac{3}{4})^2$, and the white triangle on the lower-right has the same area. Therefore, the probability that Alice and Bob actually meet is $\frac{7}{16}$.

3. **Geometric probability**

Given a circle. An inscribed equilateral triangle of the circle is an equilateral triangle with vertices on the circumference of the circle. Now, let us independently and uniformly randomly pick two points(A and B) on the circumference of the circle. Find the probability that the chord(line segment AB) is longer than the side of the inscribed equilateral triangle.

Solution: Without loss of generality, we can fix point A and let point B be uniformly randomly chosen. Also, let one of the vertices of the inscribed triangle coincides with point A and let the remaining two vertices be C and D. Therefore, for AB to be longer than AC(or equivalently AD), B has to lie in the arc of CD but not AC/AD. Since the length of the arc of AC, AD and CD are the same, this gives the probability required to be $\frac{1}{3}$.

4. **Conditional expectation**
Let $Y_0 = X_1^0, Y_1 = X_1^1 + X_2^1 + ... + X_{Y_0}^1, Y_2 = X_1^2 + X_2^2 + ... + X_{Y_1}^2$, where X_j^i are identically and independently distributed as Bernoulli distribution with parameter 0.5.

What is $E(Y_2)$?

Solution:

$$E(Y_0) = 0.5$$
$$E(Y_1) = E(E(Y_1 \mid Y_0)) = E(Binomial(Y_0, 0.5)) = E(0.5Y_0) = 0.5^2$$
$$E(Y_2) = E(E(Y_2 \mid Y_1)) = E(Binomial(Y_1, 0.5)) = E(0.5Y_1) = 0.5^3$$

5. **Central Limit Theorem**

Use the Central Limit Theorem to determine the number of people to poll to have an estimation error less than 3% w.p. 95%.

Solution:

Let $\bar{X} = \frac{1}{n}\sum_{1\le i\le n} X_i$, where X_i are iid Bernoulli random variable with parameter p.

We want to find n such that $P(|\bar{X} - p| < 0.03) = 0.95$, equivalently, $P(|\frac{\bar{X}-p}{\sqrt{p(1-p)}/\sqrt{n}}| < \frac{0.03}{\sqrt{p(1-p)}/\sqrt{n}}) = 0.95$

For standard normal distribution $P(N(0,1) \in [-1.96, 1.96]) = 0.95$.

By Central Limit Theorem, our problem translates to finding n such that for all $0 < p < 1$, $\frac{0.03}{\sqrt{p(1-p)}/\sqrt{n}} > 1.96$.

It means we want n such that for all $0 < p < 1$, $n > (\frac{1.96}{0.03})^2(p(1-p))$.

Since $max_{0\le p\le 1} p(1-p) = \frac{1}{4}$, we have $n > (\frac{1.96}{0.03})^2(\frac{1}{4})$