CS 70	Discrete Mathematics and Probability Theory	
Fall 2015	Rao	HW 1

Due Wednesday Oct 21 at 10PM

1. **Rationality**

   (a) Show that if A and B are countably infinite sets, then $A \cup B$ is also countably infinite.

   Answer: If we have two countable sets A and B, we can count their union by counting A with even numbers and B with odd numbers.

   Since A and B are countably infinite, there exist bijections $f_A : \mathbb{N} \to A$ and $f_B : \mathbb{N} \to B$. We construct $f : \mathbb{N} \to A \cup B$ as follows:

   $$f(n) = \begin{cases} f_A(\frac{n}{2}) & \text{if } n \text{ is even;} \\ f_B(\frac{n-1}{2}) & \text{if } n \text{ is odd.} \end{cases}$$

   Observe that f hits every element of $A \cup B$, because f_A and f_B hit every element of A and B respectively. Therefore $A \cup B$ is countably infinite.

   (b) A real number which is not a rational number is said to be *irrational*. Using (a), prove or disprove that the set of all irrational numbers is countable.

   Answer: Assume for the sake of contradiction that the set of irrational numbers $\mathbb{I}$ is countable. Then, since the set of rational numbers, $\mathbb{Q}$ is also countable, by part (a), we have that $\mathbb{I} \cup \mathbb{Q}$ should also be countable. However, note that the union of $\mathbb{I}$ and $\mathbb{Q}$ is precisely the set of all real numbers, which we know to be uncountable by Cantor's diagonalization. We have come to a contradiction, and therefore our initial assumption that $\mathbb{I}$ is countable must have been false.

2. **Countability**

   We say that a set S is **countable** if there is a bijection from $\mathbb{N}$, or a subset of $\mathbb{N}$, to S. Equivalently, S has the same cardinality as $\mathbb{N}$ or a subset of $\mathbb{N}$.

   For the following, you may find these facts useful (you should be able to prove them from the definition above, **but you don't need to**).

   If there exists a surjection from $\mathbb{N}$ to S, then S is countable. If there exists an injection from S to $\mathbb{N}$, then S is countable.

   (a) Given two countable sets A and B, prove that the Cartesian Product $A \times B = \{(a,b) : a \in A, b \in B\}$ is countable.

   (Hint: This should be reminiscent of the argument to show why rational numbers are countable.)

   Answer:

   First, let's show that $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$ (all pairs of natural numbers) have the same cardinality.

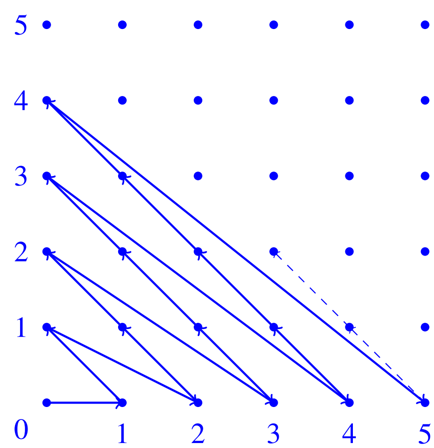
   In the lecture notes, an injection was constructed from $\mathbb{Z} \times \mathbb{Z}$ (all pairs of integers) to $\mathbb{N}$ using a spiral map. We notice that $\mathbb{N} \times \mathbb{N}$ is a subset of $\mathbb{Z} \times \mathbb{Z}$ (all pairs of natural numbers are also pairs of integers).

Thus, the same spiral map can be considered, acting just on $\mathbb{N} \times \mathbb{N}$. As in, we consider the same map, but now, the inputs are just taken from $\mathbb{N} \times \mathbb{N}$. The outputs, as before, are still natural numbers. Since the entire spiral map was injective to begin with, even this map must be injective. That is, two distinct pairs of natural numbers must be mapped to distinct natural numbers. Thus, we have constructed an injection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$.

We can construct an injection from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$ very easily. Consider the map which takes every natural number n to $(0,n)$. Since each distinct natural number is mapped to a distinct pair in $\mathbb{N} \times \mathbb{N}$, this is injective.

Thus, from Cantor-Bernstein's theorem, we see that $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$ have the same cardinality.

Another way to show this is to explicitly define a bijection from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$. Consider the diagram below:



This diagram shows us a way to explicitly construct a bijection by walking along the counter diagonals of $\mathbb{N} \times \mathbb{N}$ written as a grid. This, again, shows that we have a bijection from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$. For an explicit formula of this, look at the Cantor pairing function:
http://en.wikipedia.org/wiki/Cantor_pairing_function

Let Ψ be some bijection from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$ (for example, the Cantor pairing function).

Now, since A and B are countable, there must be surjections from $\mathbb{N}$ to A and from $\mathbb{N}$ to B. This is because every bijection is a surjection. If there is a bijection from a subset of $\mathbb{N}$ to A (or B), it can be made into a surjection from $\mathbb{N}$ to A by just making all the elements outside this subset map to the same element in A.

Let these surjections be Φ_A and Φ_B. With these, we can construct a surjection Φ from $\mathbb{N} \times \mathbb{N}$ to $A \times B$. This will just be the individuals surjections Φ_A and Φ_B mapping the two coordinates of an element in $\mathbb{N} \times \mathbb{N}$ independently.

I.e, $\Phi(i,j) = (\Phi_A(i), \Phi_B(j)) \in A \times B$. This is certainly a surjection as every element in $A \times B$ will be mapped to by some natural number under Φ, as Φ_A and Φ_B are surjections.

Thus, finally, we can create the surjective map $f : \mathbb{N} \to A \times B$ such that $f(n) = \Phi(\Psi(n))$. This takes it to the intermediate space of $\mathbb{N} \times \mathbb{N}$ and then maps it to $A \times B$. Since both Ψ and Φ are surjective, we know that the composite map f is also surjective.

Now that we have found a surjection from $\mathbb{N}$ to $A \times B$, this set is countable.

Note: You need not be this detailed in your answer. The important points to note are that we can find a surjection from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$ and then one from $\mathbb{N} \times \mathbb{N}$ to $A \times B$. You can briefly, but precisely construct these.

This allows us to find a composite map which is a surjection from $\mathbb{N}$ to $A \times B$ and thus, this set is countable.

(b) Prove that the countable union of countable sets is countable; i.e, prove the following is countable:

$$A_1 \cup A_2 \cup A_3 \dots$$

where each A_i is countable and there are countably many of them.

If there are countably infinite (also known as denumerable) sets A_i, could the above union still be finite? If yes, explain briefly what kind of sets A_i for which this holds. If not, prove that union is always infinite.

(Hint: How does this relate to Cartesian Products?)

Answer:

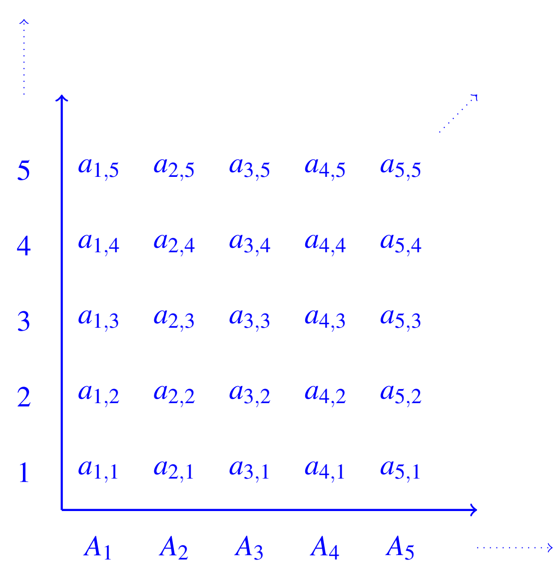
The trick here is to re-label the elements in the sets and show that the set of all possible labels is countable. Our labelling should be injective in that, if two elements are distinct, then they must have different labels. Then, we can find a surjective mapping from labels to elements in the union. So, if we show that there is a surjection from $\mathbb{N}$ to the set of labels, we can show that there is a surjection from $\mathbb{N}$ to the elements in the union.

Let's adopt a labelling scheme with two natural-number indices as follows: $a_{i,j}$ is j^{th} element from A_i. The label indices look like a Cartesian product of natural numbers.

We have a small problem with our second index, j. If a countable set is not ordered, what is its j^{th} element? This problem is not present with the first index because our sets are already labelled by natural number indices. So we already know how to find the i^{th} set, from just the way they are defined.

To deal with unordered sets A_i, we can just induce an ordering from the natural numbers. Since each A_i is countable, there must be a bijection from some subset of $\mathbb{N}$ (possibly $\mathbb{N}$ itself) to A_i. We can now number each element in A_i by the natural number which maps to it under this bijection. Thus, we have induced an ordering over each set A_i and referring to the j^{th} element now makes sense.

Our labelling scheme just looks like a grid (like $\mathbb{N} \times \mathbb{N}$) where column i contains the ordered elements of set A_i. Each label is just a coordinate on the grid. Look at the illustration below.



Now that we have our completed labelling scheme, let's see if it works. We see that we can represent every single element in every single set A_i uniquely. If any two elements are different, they would be on different coordinates and thus their labels will differ. Since every element is mapped to by some label (each element must appear somewhere on this grid), the map from labels to elements is a surjection. Can two labels map to the same element? Yes, that's possible. This happens if two or more sets share an element. This element would appear multiple times on the grid. But this is okay! We still have a valid surjection from labels to elements.

Now, we can find a surjection from $\mathbb{N}$ to the elements in the union, by using the labels as an intermediate space. the set of all labels is just a set of pairs of natural numbers (i, j). This is a subset of $\mathbb{N} \times \mathbb{N}$. Why is it a subset and not the entire thing? Some sets A_i could be finite. So elements will not exist at some coordinates (and thus, these coordinates will not be labels).

We can certainly find a surjection from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$ (as we did in part (a)). And since our set of labels is a subset of $\mathbb{N} \times \mathbb{N}$, we can easily construct another surjection from this, mapping elements from some subset of $\mathbb{N}$ to the set of labels. (Just take all the set of all elements in $\mathbb{N}$ which map to valid labels; the same surjective as above, with only these elements as input, will be a surjective map onto just the set of labels).

Finally, we have a surjection from $\mathbb{N}$ (or a subset of $\mathbb{N}$) to the set of labels and another surjection from the set of labels to the set of all elements in the union. Thus, the composite map of the two above is a surjection from $\mathbb{N}$ to the elements in the union. And so, the countable union of countable sets is countable.

Can the union be finite?

Yes, it can! It is finite only if there is some natural number N for which the following holds:

$$A_m \subset A_1 \cup A_2 \cup A_3 \dots A_N, \qquad \forall m > N$$

And, of course $A_1 \cup A_2 \cup A_3 \dots A_N$ itself must be finite. This is basically saying that only finite number of unique elements exist in all the sets together.

An example of this would be if each $A_i = \{1, 2, 3, 4, 5\}$. Then the union would be the same set.

3. **Hello World**

(a) You want to determine whether a program P on input x outputs "Hello World" before executing any other statement. Is there a computer program that can perform this task? Justify your answer.

Answer: There is a computer program that can perform this task. Simply run the first operation of program P, and check if that operation outputs "Hello World".

(b) You want to determine whether a program P on input x outputs "Hello World". Is there a computer program that can perform this task? Justify your answer.

Answer: There is no computer program that can perform this task, because we can reduce the halting problem to this problem.

To do this, suppose for sake of contradiction that we could determine whether any program outputs "Hello World". Then we could solve the halting problem as follows: for any program Q, modify it to obtain another program Q' such that:

i. Q' emulates Q except that it suppresses all printing instructions.

ii. If Q halts, Q' prints "Hello World" before halting.

Then determining whether Q' prints "Hello World" is equivalent to determining whether an arbitrary program Q halts, which is exactly the halting problem. Since the halting problem is uncomputable, this is a contradiction. Thus, we conclude that there is no computer program that can determine whether an arbitrary program eventually outputs "Hello World".

4. **Self-Grading Shuffle**

A TA likes the students to perform self-grading on their homework in section. 20 students walk in and sit down with their homework in front of them. At the beginning of class, the TA walk around and randomly shuffle the homework. For Parts (b)–(d), assume there is no tie in age.

(a) What is the total number of possible (student and homework) configurations the TA could have after shuffling?

(b) How many configurations are there where the oldest student gets his/her own homework back?

(c) How many configurations are there where at least 1 of the 2 oldest students get their own homework back?

(d) How many configurations are there where at least 1 of the 3 oldest students get their own homework back?

Answer:

(a) We permute the homeworks and assign them to students, so there are $(20)\cdot(20-1)\cdot\ldots\cdot(1) = 20!$ configurations.

(b) We fix the first homework and permute the remaining 19 homeworks, so there are $(1)\cdot(19)\cdot(19-1)\cdot(19-2)\cdot\ldots\cdot(1) = (19)!$ configurations.

(c) $19! + 19! - 18!$. We fix only the oldest student's homework and permute the remaining 19 homeworks. We fix only the the second oldest student's homework and permute the remaining 19 homeworks. We have now double counted the scenarios where the oldest and the second oldest get their own homework back, so we subtract out fixing the oldest and the second oldest students' homeworks and permuting the remaining 18 homeworks. (Inclusion-Exclusion Principle iterated once.)

(d) $3(19!) - 3(18!) + (17!)$. There are $3(19!)$ ways to fix the oldest, second oldest, or third oldest students' homeworks. Then we have double counted the $3(18!)$ scenarios where two of these students' homeworks are given back to them, so we subtract that amount. Finally, we've now subtracted the scenario where all three of these students get their own homework back an extra time, so we have to add the 17! ways (fix first three, permute the remaining 17 homeworks) to do that back in. (Inclusion-Exclusion Principle iterated twice.)

5. **You do not need to hand in this problem, but make sure you know how to do it. (This question is taken from Midterm 2 review.)**

A function $f\colon \mathbb{N} \to \mathbb{N}$ is said to be computable if there exists a program that takes x as input and produces $f(x)$ as output.

(a) Prove or disprove: every increasing function $f\colon \mathbb{N} \to \mathbb{N}$ (i.e., if $x \geq y$, then $f(x) \geq f(y)$) is computable.

Answer: No, there exists an increasing function that is uncomputable.

We prove this by an explicit reduction of the halting problem to this problem. The trick is simple; we know that the set of all programs is countable (Homework 7 Problem 5(f)) and therefore we can list all programs and index them by natural numbers. Thus we simply define an increasing function f such that $f(x)$ is equal to $f(x-1)$ if and only if program #x halts. If this function were computable, we can determine whether program #i halts by computing $f(i)$ and $f(i-1)$ and comparing them. This is a contradiction to the uncomputability of the halting problem, and therefore f must be uncomputable.

(b) Prove or disprove: every decreasing function $f\colon \mathbb{N} \to \mathbb{N}$ (i.e., if $x \geq y$, then $f(x) \leq f(y)$) is computable.

Answer: Yes. Let $m = f(0)$. Since f is decreasing, for any natural number $n > 0$, $f(n+1) \leq f(n)$. The key idea is that $f(n+1)$ is strictly less than $f(n)$ at most at m places, because $f(x)$ can never be negative, and $f(0) = m$. After those decreases, f will be constant. More formally, there exists some k such that for all $x \geq k$, $f(x) = f(k)$.

Because of this, we can hardcode function f's behavior in a finite space. Let program P contain an array storing all values of f up to $f(k)$. To compute $f(x)$, if $x < k$, lookup the value of $f(x)$; otherwise, lookup $f(k)$.

(More efficiently, we do not need to store all values up to k. We only need to store the locations where f decreases, and the values of $f(x)$ at those locations.)