CS 70 Discrete Mathematics and Probability Theory
Summer 2015 Chung-Wei Lin HW 4

Due ~~Monday July 20~~ Tuesday July 21 at Noon

Please write your answer of each question on a separated page.

1. **Self-Grading Shuffle** (12 points, 3 points for each part)

   A TA likes the students to perform self-grading on their homework in section. 20 students walk in and sit down with their homework in front of them. At the beginning of class, the TA walk around and randomly shuffle the homework. For Parts (b)–(d), assume there is no tie in age.

   (a) What is the total number of possible (student and homework) configurations the TA could have after shuffling?

   (b) How many configurations are there where the oldest student gets his/her own homework back?

   (c) How many configurations are there where at least 1 of the 2 oldest students get their own homework back?

   (d) How many configurations are there where at least 1 of the 3 oldest students get their own homework back?

   Answer:

   (a) We permute the homeworks and assign them to students, so there are $(20)\cdot(20-1)\cdot\ldots\cdot(1) = 20!$ configurations.

   (b) We fix the first homework and permute the remaining 19 homeworks, so there are $(1)\cdot(19)\cdot(19-1)\cdot(19-2)\cdot\ldots\cdot(1) = (19)!$ configurations.

   (c) $19!+19!-18!$. We fix only the oldest student's homework and permute the remaining 19 homeworks. We fix only the the second oldest student's homework and permute the remaining 19 homeworks. We have now double counted the scenarios where the oldest and the second oldest get their own homework back, so we subtract out fixing the oldest and the second oldest students' homeworks and permuting the remaining 18 homeworks. (Inclusion-Exclusion Principle iterated once.)

   (d) $3(19!)-3(18!)+(17!)$. There are $3(19!)$ ways to fix the oldest, second oldest, or third oldest students' homeworks. Then we have double counted the $3(18!)$ scenarios where two of these students' homeworks are given back to them, so we subtract that amount. Finally, we've now subtracted the scenario where all three of these students get their own homework back an extra time, so we have to add the 17! ways (fix first three, permute the remaining 17 homeworks) to do that back in. (Inclusion-Exclusion Principle iterated twice.)

2. **Combinatorial Proof** (30 points, 10 points for each part)

(a) n males and n females apply for the CS major at UC Berkeley. The CS department only has n seats available. How many ways can it admit students? Use the above story for a combinatorial argument to prove the following identity:

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2.$$

(b) Following Part (a), the department decides to provide two fellowships — one for a male student and the other one for a female student. Use the above story for a combinatorial argument to prove the following identity:

$$\sum_{k=1}^{n-1} k \cdot (n-k) \cdot \binom{n}{k}^2 = n^2 \cdot \binom{2n-2}{n-2}.$$

(c) Now, come up with your own story for a combinatorial argument to prove the following identity.

$$n2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + \cdots + (n-1)\binom{n}{n-1} + n\binom{n}{n}$$

Answer:

(a) One way (LHS) of counting is simply $\binom{2n}{n}$, since we must pick n students from $2n$. The other way (RHS) is to first pick i males, then $n-i$ females. Equivalently, choose i males to admit, and i females to NOT admit. For a fixed i, this yields $\binom{n}{i}\binom{n}{n-i} = \binom{n}{i}^2$ choices. Thus, over all choices of i:

$$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2.$$

(b) Out of the n males and n females who applied, count the number of ways that admitting students and giving fellowships.

RHS: First pick one male recipient and one female recipient from the n male and n female applicants (n^2). Then pick the other $n-2$ students from the pool of $2n-2$ remaining applicants.

LHS: Pick k males and $n-k$ females that are admitted: $\binom{n}{k}\binom{n}{n-k} = \binom{n}{k}^2$. Then pick a recipient among the k males and the other recipient among the $n-k$ females. Lastly, take summation over all choices of k (note that k cannot be 0 or n because at least one male and one female must be picked).

Since LHS and RHS count the same thing, they are equal.

(c) We want to count the number of ways that we can select a team with a single leader out of n people. We don't allow an empty team because we insist on having a single leader.

LHS: From n people, pick one team leader (n choices) and then some (possibly empty) subset of other people into the team.

RHS: First decide the number of people k in the team, pick k people into the team ($\binom{n}{k}$ choices), and then pick the leader among the k people (k choices). For example, when k is 2, we first choose our team of 2 people in $\binom{n}{2}$ ways, then pick a leader in 2 ways because there are only 2 members to choose from. Lastly, take summation over all choices of k (note that k cannot be 0 because we don't allow an empty team).

Since LHS and RHS count the same thing, they are equal.

3. **Monty Hall Problem** (24 points, 6/6/3/3/3/3 points for each part)

(a) Explain the Monty Hall problem in your own words. Pretend you are teaching yourself as you were at the beginning of this term.

(b) Define the outcome of the game (up to the point where the contestant makes the final decision) by a triple of the form (i, j, k), where $i, j, k \in \{1,2,3\}$. The values i, j, k respectively specify the location of the prize, the initial door chosen by the contestant, and the door opened by Carol. What is the sample space of this problem? Draw it as a tree structure.

(c) What is the probability of each sample point?

(d) What is the event of interest?

(e) If the contestant is using the "sticking strategy," which sample points are in this event? What is the probability of this event?

(f) If the contestant is using the "switching strategy," which sample points are in this event? What is the probability of this event?

Answer:

(a) The Monty Hall problem is a probability puzzle based on the show Let's Make a Deal (the show's host was Monty Hall). In this puzzle, a contestant is present with three doors and is asked to choose one. Behind two of the doors are goats and behind the remaining door is a car. Once the contestant chooses a door, Monty Hall's assistant Carol opens one of the doors *not* chosen by the contestant, revealing (always) a goat. Finally, the contestant is given the choice of keeping his original door vs. picking the remaining closed door. In most formulations of the problem, we assume that the contestant is rational and prefers the car to either of the goats (after all, he could just sell the car and buy more goats). The goal is therefore to find a strategy for the contestant which will maximize his odds of winning the car.

Suggestion: Give your solution to one of your friends (or a relative) who is not in the class. Let them read it and then ask them to play the game a few times (you'll play the roles of both Monty Hall and Carol).

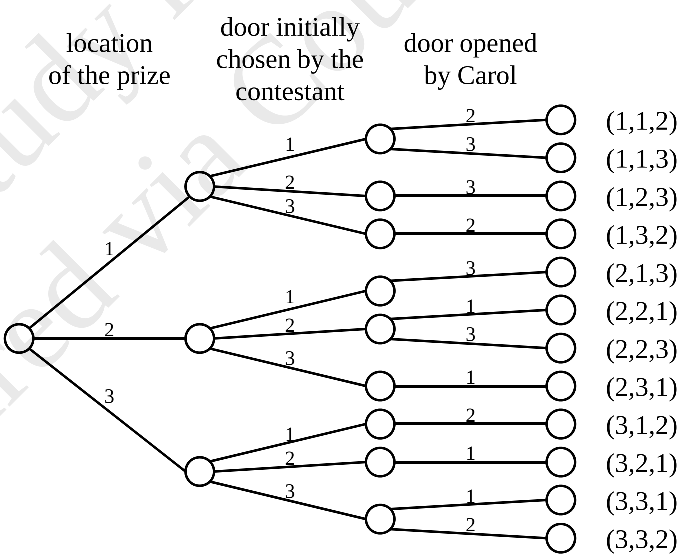

Figure 1: The tree structure of all sample points.

(b) As suggested, we will describe each possible instance of the game with a triplet of numbers: (prize location, door initially chosen, Carol's door). If we number the doors 1-3, we see that there are 12 sample points in the sample space: {(1,1,2), (1,1,3), (1,2,3), (1,3,2), (2,1,3), (2,2,1), (2,2,3), (2,3,1), (3,1,2), (3,2,1), (3,3,1), (3,3,2)}. The tree of sample points is shown in Figure 1.

(c) We assume that each time a node branches (*e.g.*, $(1,X,X)$ branches to the "child" events $(1,1,X)$, $(1,2,X)$, and $(1,3,X)$) that the children have equivalent probabilities. The probabilities are $\frac{1}{9}$ for nodes {(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)} and $\frac{1}{18}$ for nodes {(1,1,2), (1,1,3), (2,2,1), (2,2,3), (3,3,1), (3,3,2)}.

(d) The event is "the contestant wins the car."

(e) A sample point (i,j,k) is in this event if $i = j$, so {(1,1,2), (1,1,3), (2,2,1), (2,2,3), (3,3,1), (3,3,2)} are in this event, and the total probability is $\frac{1}{3}$.

(f) A sample point (i,j,k) is in this event if $i \neq j$, so {(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)} and in this event, and the total probability is $\frac{2}{3}$.

4. **Profound Experimentation** (18 points, 3 points for each part)

You perform an exhilarating and groundbreaking experiment upon the outcome of flipping an unbiased coin 5 times (order matters).

(a) How many total outcomes are there?

(b) What is the probability that there is exactly 1 Head?

(c) What is the probability that there is at least 1 Head?

(d) What is the probability that there are more Heads than Tails?

(e) What is the probability that at least one of the first two flips is a Tail?

(f) What is the probability that there are exactly 4 Heads in a row?

Answer:

(a) $2^5 = 32$. Each of the 5 flips is with 2 options.

(b) $\frac{5}{2^5} = \frac{5}{32}$. The 5 outcomes are $HTTTT$, $THTTT$, $TTHTT$, $TTTHT$, and $TTTTH$.

(c) $1 - \frac{1}{2^5} = \frac{31}{32}$. All outcomes but $TTTTT$.

(d) $\frac{\binom{5}{3} + \binom{5}{4} + \binom{5}{5}}{2^5} = \frac{1}{2}$. It can also be observed that there is a one-to-one correspondence between the set of outcomes where there are more heads than tails and the set where there are more tails than heads. This combined with the fact that 5 is odd (there can't be an equal number of Heads and Tails), means that these two events cover the entire sample space and are equal.

(e) $1 - \frac{1}{2^2} = \frac{3}{4}$. All outcomes but HH.

(f) $\frac{2}{2^5} = \frac{2}{32}$. Having 4 Heads in a row is the same as having 1 Tail at either the first or last spot but not both. The 2 outcomes are $THHHH$ and $HHHHT$.

5. **Wedding Chaos** (9 points, 3 points for each part)

You have been invited to your cousin's wedding celebration. She has ordered special suits and dresses, custom fitted, for you and her favorite friends from college. All together, there are 8 men and 8 women. Each man has a custom suit and each woman has a custom dress. Unfortunately, when you arrive you realize the tailor has forgotten to label which belongs to whom.

(a) The event starts soon, and you are all running out of time. The men all decide to randomly grab a suit and put it on. What is the probability that all the men are wearing the right suit?

(b) The women have a better plan. By comparing the dresses, they first separate them into two piles; one for the smaller four dresses, the other for the larger four dresses. Then, the smaller four women choose a dress from the small pile and the larger from the large pile. What is the chance that the women will all end up wearing the right dress?

(c) Everyone is finally dressed, but now a dance is starting. Your cousin has told you that she expects four couples to dance, but hasn't said who. Four of the men and four of the women randomly agree to dance. It turns out she really only wanted people who don't wear glasses dancing (she isn't your kindest cousin), because she thinks glasses would look bad in her wedding photographs. If one of the women wears glasses, and two of the men wear glasses, how likely is it that you didn't ruin her wedding photos?

Answer:

(a) This is really a permutation of suits. Order matters because it determines who gets which suit. There is only 1 correct assignment of suits and there are $8!$ ways to assign suits, so the probability is $\frac{1}{8!}$.

(b) The smaller four dresses are now permuted independently of the larger four dresses. There are $4!$ ways to permute four dresses. There is a $\frac{1}{4!}$ chance that each half is assigned correctly, and they are independent, so we multiply the probabilities together: $\frac{1}{4!4!}$.

(c) There are $\binom{8}{4}$ ways to choose either the women or men dancing, so there are $\binom{8}{4}\binom{8}{4}$ total ways for four couples to dance, since order doesn't matter on the dance floor. There are $1 \cdot \binom{7}{3}$ ways to include the woman with glasses, $2 \cdot \binom{6}{3}$ ways to include exactly one man with glasses, and $1 \cdot \binom{6}{2}$ ways to include both men with glasses. We subtract these bad scenarios from the total ways to get a final probability of $\frac{\left(\binom{8}{4}-\binom{7}{3}\right)\left(\binom{8}{4}-2\cdot\binom{6}{3}-\binom{6}{2}\right)}{\binom{8}{4}\binom{8}{4}}$, which is equal to a more intuitive approach: $\frac{\binom{7}{4}\binom{6}{4}}{\binom{8}{4}\binom{8}{4}}$.

6. **Random Team** (9 points, 3 points for each part)

There is a set of 100 people, all *strictly ordered* in some way (the first person is the best). You randomly pick a team of 50.

(a) What is the probability that at least one member of your team will be better than the median?

(b) What is the probability that at least one member of your team will be better than the k-th person for $1 \leq k \leq 51$?

(c) What is the probability that at least one member of your team will be better than the k-th person for $52 \le k \le 100$?.

Answer:

(a) $1 - \frac{\binom{50}{50}}{\binom{100}{50}} = 1 - \frac{1}{\binom{100}{50}}$. There are exactly 50 people strictly better than the median and 50 not. A team that has no people better than the median must be made from within these bottom 50 people. So the probability is the number of teams that can be made from within only the bottom 50, divided by all possible teams made from all 100.

(b) $1 - \frac{\binom{100-k+1}{50}}{\binom{100}{50}}$. This is the complement of the event that you pick all your people from within the set $\{k, k+1, \ldots, 100\}$.

(c) 1. You must pick 50 people for your team so at least one of them must be in the set $\{1, 2, \ldots, 51\}$.

7. **Roll the Caltrop** (18 points, 3 points for each part)

You are playing Dungeons and Dragons with your friends. As usual, you spend far too long designing your character, but this time you have a good reason: you are thinking about the probabilities involved in rolling a 4 sided die (sometimes called a caltrop) three times.

(a) How large is the sample space of rolling the caltrop three times?

(b) How likely is it that you will roll the same number all three times?

(c) What is the chance that you will roll exactly two 3's?

(d) It turns out your friend has a caltrop which he tinkered with to make it more likely to land showing a 1. It lands on 1 half of the time, and 2, 3, and 4 each have an equal probability of $\frac{1}{6}$. How large is the sample space of rolling the caltrop three times?

(e) Following Part (d), how likely is it that you will roll the same number all three times? Is it larger than, equal to, or smaller than the answer in Part (b)?

(f) Following Part (d), what is the chance that you will roll exactly two 3's? Is it larger than, equal to, or smaller than the answer in Part (c)?

Answer:

(a) We are sampling with replacement, so there are $4^3 = 64$ total sample points in the sample space.

(b) Each sample point has probability $\frac{1}{64}$. There are four sample points that satisfy the requirement $((1,1,1)$, $(2,2,2)$, $(3,3,3)$, and $(4,4,4))$. Since there it is a uniform probability space, we just divide the number of points that satisfy by the total number and get $\frac{4}{64} = \frac{1}{16}$.

Note: You can also think of this as rolling the first die arbitrarily, and then having $\frac{1}{4^2}$ probability of having the next two match that number.

(c) Each roll is independent, so the probability of rolling two 3's and something not a 3 is $\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{64}$. There are $\binom{3}{1} = 3$ different ways to place the roll that is not a 3 $((X,3,3)$, $(3,X,3)$, and $(3,3,X))$, so the probability is $3 \cdot \frac{3}{64} = \frac{9}{64}$.

(d) 64, which is the same.

(e) Take the summation of the probability of the corresponding sample points ($(1,1,1)$, $(2,2,2)$, $(3,3,3)$, and $(4,4,4)$). $\frac{1}{2^3} + \frac{1}{6^3} + \frac{1}{6^3} + \frac{1}{6^3} = \frac{30}{216}$, which is larger than $\frac{1}{16}$.

(f) Each roll is independent, so the probability of rolling two 3's and something not a 3 is $\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} = \frac{5}{216}$. There are $\binom{3}{1} = 3$ different ways to place the roll that is not a 3, so the probability is $3 \cdot \frac{5}{216} = \frac{15}{216}$, which is smaller than $\frac{9}{64}$.