CS 70 Discrete Mathematics and Probability Theory
Summer 2015 Chung-Wei Lin HW 1

Due Monday June 29 at Noon

1. Propositions and Quantifiers (**30 points, 5 points for each part**)

   Given $x, y \in \{1, 2, 3\}$ and the following propositions:

   - P_0: $x < y$.
   - P_1: both x and y are even.
   - P_2: y is divisible by x and x is odd.
   - P_3: $x = y$.
   - P_4: $x + y < 7$.

   (a) Take the following table as an example

P_0		x		
		1	2	3
y	1	F	F	F
	2	T	F	F
	3	T	T	F

   and complete the following tables.

   Answer:

P_1		x		
		1	2	3
y	1	F	F	F
	2	F	T	F
	3	F	F	F

P_2		x		
		1	2	3
y	1	T	F	F
	2	T	F	F
	3	T	F	T

P_3		x		
		1	2	3
y	1	T	F	F
	2	F	T	F
	3	F	F	T

P_4		x		
		1	2	3
y	1	T	T	T
	2	T	T	T
	3	T	T	T

   (b) Complete the following table.

   Answer:

	P_0	P_1	P_2	P_3	P_4
$\exists x \exists y\, P_i$	T	T	T	T	T
$\exists x \forall y\, P_i$	F	F	T	F	T
$\forall x \exists y\, P_i$	F	F	F	T	T
$\forall x \forall y\, P_i$	F	F	F	F	T

   (c) Prove it or provide a counterexample: for any proposition P, $\exists x \forall y\, P \implies \forall x \exists y\, P$.

   Answer: Counterexample: P_2. To find a counterexample, look in the tables for a proposition where a column has all T, but there is not a T in each column.

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(d) Prove it or provide a counterexample: for any proposition P, $\forall x \exists y\, P \implies \exists x \forall y\, P$.

Answer: Counterexample: P_3. To find a counterexample, look in the tables for a proposition where there is a T in each column, but no column has all T.

(e) Prove it or provide a counterexample: for any proposition P, $\exists x \forall y\, P \implies \forall y \exists x\, P$.

Answer: If $\exists x \forall y\, P$ is true, we can define an x' such that $\forall y\, P(x', y)$ is true, and thus $\forall y \exists x\, P$ is true. Using the tables, the statement says that if there is a column with all T, then each row has a T, which is logical because each T in the all-true column constitutes a T in each row.

(f) Prove it or provide a counterexample: for any proposition P, $\forall x \exists y\, P \implies \exists y \forall x\, P$.

Answer: Counterexample: P_3. To find a counterexample, look in the tables for a proposition where there is a T in each column, but there is not any row with all T.

2. Minesweeper (**20 points, 5 points for each part**)

In the following 3×3 squares, there is at most 1 mine in each square, and the number in a square is the number of mines in its neighboring squares (at most 8). Now, four squares have been revealed, and there is no mine in them. Assume X_i is the proposition that there is a mine in square S_i.

S_1	2	0
S_2	4	1
S_3	S_4	S_5

(a) What can the "2" tell you (without considering the "1" and "4")? Write the proposition using the notation covered in class.

Answer: $X_1 \wedge X_2 = T$.

The "2" tells us that there are 2 mines in its neighboring squares, and since S_1 and S_2 are the only 2 neighboring squares, both must have mines in them.

(b) What can the "1" tell you (without considering the "2" and "4")? Write the proposition using the notation covered in class.

Answer: $(X_4 \wedge \neg X_5) \vee (\neg X_4 \wedge X_5) = T$.

The "1" tells us that there is 1 mine in its neighboring squares, and since S_4 and S_5 are the only 2 neighboring squares, there is either a mine in S_4 or a mine in S_5. The first conjunction says that there is a mine in S_4 and there is no mine in S_5, and the second says there is a mine in S_5 and there is no mine in S_4.

(c) What can the "4" tell you (without considering the "1" and "2")? Write the proposition using the notation covered in class.

Answer: $(X_1 \wedge X_2 \wedge X_3 \wedge X_4 \wedge \neg X_5) \vee (X_1 \wedge X_2 \wedge X_3 \wedge \neg X_4 \wedge X_5) \vee (X_1 \wedge X_2 \wedge \neg X_3 \wedge X_4 \wedge X_5) \vee (X_1 \wedge \neg X_2 \wedge X_3 \wedge X_4 \wedge X_5) \vee (\neg X_1 \wedge X_2 \wedge X_3 \wedge X_4 \wedge X_5) = T$.

The "4" tells us that exactly 4 out of the 5 neighboring squares have a mine in them. Therefore, we create conjunctive clauses for each possible situation where we have exactly 4 out of the 5 squares having mines and connect them with OR's to show that one of the cases must be true.

(d) Can you decide the values of $X_1, \ldots, X_5$ such that all of the propositions above are satisfied? List all possible solutions.

Answer: $(X_1, X_2, X_3, X_4, X_5) = (T, T, T, T, F)$ or (T, T, T, F, T).

The "2" tells us that the squares S_1 and S_2 must have a mine. Then, the "1" tells us that only one of S_4 and S_5 can have a mine, so S_3 must have a mine since the "4" tells us that there are 4 mines total in the 5 squares. Therefore, we have two cases: both cases say that S_1, S_2, and S_3 have mines, and one says S_4 but not S_5 has a mine while the other says S_5 but not S_4 has a mine.

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3. Proofs (**20 points, 10 points for each part**)

   (a) If $n = 3k+2$ where $k \in \mathbb{N}$, then n is not a perfect square (which is of the form m^2 for some integer m).

   Answer: We will use proof by contraposition and proof by cases. The contrapositive of the statement is as follows: if n is a perfect square, then $n \neq 3k+2$ for any $k \in \mathbb{N}$. Assume n is a perfect square, which means $n = m^2$ for some integer m. We will now show that n cannot be written in the form $n = 3k+2$ for any $k \in \mathbb{N}$. There are three cases for m: m must be in the form $3j$, $3j+1$, or $3j+2$ for some integer j. Squaring each of these 3 cases, we get

   - $m = (3j)^2 = 9j^2 = 3(3j^2)$.
   - $m = (3j+1)^2 = 9j^2+6j+1 = 3(3j^2+2j)+1$.
   - $m = (3j+2)^2 = 9j^2+12j+4 = 3(3j^2+4j+1)+1$.

   In all three cases, the perfect square m^2 is either in the form $3k$ or $3k+1$, and never in the form $3k+2$. Therefore, since the contrapositive is proven true, the original statement is true.

   (b) There is no integer solution to the equation $x^2 - y^2 = 10$.

   Answer: Proof by contradiction. If there is an integer solution to $x^2 - y^2 = 10$, there must be integer solutions to $(x+y) = a$ and $(x-y) = b$ such that $ab = 10$. The possibilities for (a,b) are $(1,10)$, $(2,5)$, $(5,2)$, $(10,1)$, $(-1,-10)$, $(-2,-5)$, $(-5,-2)$, and $(-10,-1)$. In each of these cases, the summation (a) and the difference (b) of x and y have different parities (if a is even, b is odd, or vice-versa). This is impossible for integers (you do not need to prove it, but it is because $x+y = (x-y)+2y$). Therefore, there cannot be an integer solution to the original equation.

   Note 1: There is another approach first claiming $-5 \leq x \leq 5$ and $-5 \leq y \leq 5$ since, given an integer $z \geq 6$ or $z \leq -6$, the difference between z^2 and its closest perfect square must be larger than 10. Then, one can prove the original claim by proving there is no integer solution with all combinations in the ranges of $-5 \leq x \leq 5$ and $-5 \leq y \leq 5$.

   Note 2: One can also prove there is no non-negative integer solution first and then claim there is no integer solution because $x^2 - y^2 = (-x)^2 - y^2 = x^2 - (-y)^2 = (-x)^2 - (-y)^2$, *i.e.*, negative x and/or y will not result in any different value on the LHS.

4. Proof Checker (**15 points, 5 points for each part**)

   In this problem, you are asked to put yourself in the position of a reader! For each of the "proofs" below, say whether you think the proof is correct or incorrect. If you think the proof is incorrect, explain clearly and concisely where the logical error in the proof lies. (If you think the proof is correct, you do not need to give any explanation.) Recall from the notes and from class that simply saying that the claim (or the inductive hypothesis) is false is not a valid explanation.

   (a) **Claim**: for all $n \in \mathbb{N}$, ($2n+1$ is a multiple of 3) $\implies$ (n^2+1 is a multiple of 3).

   Proof: proof by contraposition. Assume $2n+1$ is not a multiple of 3.

   - If $n = 3k+1$ for $k \in \mathbb{N}$, then $n^2+1 = 9k^2+6k+2$ is not a multiple of 3.
   - If $n = 3k+2$ for $k \in \mathbb{N}$, then $n^2+1 = 9k^2+12k+5$ is not a multiple of 3.
   - If $n = 3k+3$ for $k \in \mathbb{N}$, then $n^2+1 = 9k^2+18k+10$ is not a multiple of 3.

   In all cases, we have concluded n^2+1 is not a multiple of 3, so we have proved the claim.

   Answer: The proof is incorrect. You want to prove an implication of the form $P(n) \implies Q(n)$ for every n, where $P(n)$ is "$2n+1$ is a multiple of 3" and $Q(n)$ is "n^2+1 is a multiple of 3". The

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contrapositive is $\neg Q(n) \implies \neg P(n)$. Your proof begins with $\neg P(n)$ and concludes with $\neg Q(n)$, so you have shown $\neg P(n) \implies \neg Q(n)$, which is the converse, not contrapositive.

There is another flaw: $n = 0$ is not covered in the proof by cases.

Furthermore, if $n = 3k+1$, then $2n+1 = 6k+3$ is a multiple of 3, so the case is redundant to prove $\neg P(n) \implies \neg Q(n)$. Bonus points will be given for pointing this out.

(b) **Claim**: for all $n \in \mathbb{N}$, $n < 2^n$.

Proof: the proof will be by induction on n.

- Base case: suppose that $n = 0$. $2^0 = 1$ which is greater than 0, so the statement is true for $n = 0$.
- Inductive hypothesis: assume $n < 2^n$.
- Inductive step: we need to show that $n+1 < 2^{n+1}$. By the inductive hypothesis, we know that $n < 2^n$. Plugging in $n+1$ in place of n, we get $n+1 < 2^{n+1}$, which is what we needed to show. This completes the inductive step.

Answer: Using induction requires showing that, given a true proposition $P(n)$, it follows that $P(n+1)$. This proof simply changes n to $n+1$, which is not valid justification for induction. The inductive hypothesis must assume that the theorem is true for some value of n, not for every value of n.

<u>Note</u>: Although you do not need to reprove the claim, one way to make this proof valid would be to show that, given $k < 2^k$ for some $k \in \mathbb{N}$, multiplying the right side by 2 will increase it by at least one. Then, it follows that $k+1 < 2^{k+1}$, which completes justification for induction.

(c) **Claim**: for all $x, y, n \in \mathbb{N}$, if $\max(x,y) = n$, then $x \leq y$.

Proof: the proof will be by induction on n.

- Base case: suppose that $n = 0$. If $\max(x,y) = 0$ and $x, y \in \mathbb{N}$, then $x = 0$ and $y = 0$, hence $x \leq y$.
- Inductive hypothesis: assume that, whenever we have $\max(x,y) = k$, then $x \leq y$ must follow.
- Inductive step: we must prove that if $\max(x,y) = k+1$, then $x \leq y$. Suppose x, y are such that $\max(x,y) = k+1$. Then, it follows that $\max(x-1, y-1) = k$, so by the inductive hypothesis, $x - 1 \leq y - 1$. In this case, we have $x \leq y$, completing the induction step.

Answer: The problem lies in the application of the inductive hypothesis. More specifically, the incorrect step is: "Then, it follows that $\max(x-1, y-1) = k-1$, so by the inductive hypothesis, $x-1 \leq y-1$." The problem is that $x-1$ or $y-1$ might be negative (this happens when $x = 0$ or $y = 0$). Then the inductive hypothesis no longer applies, since $x-1$ and $y-1$ are not both natural numbers, so we cannot conclude that $x - 1 \leq y - 1$.

<u>Note</u>: You may first find a counterexample $(x, y, n) = (1, 0, 1)$. Then, you can use (plug in) this counterexample to check where a flaw is.

5. Induction (**20 points, 10 points for each part**)

(a) Prove that, for any positive integer $n \geq 3$, $n^2 \geq 2n+1$.

Answer:

- Base Case ($n = 3$): $3^2 = 9$, $2(3)+1 = 7$, and $9 \geq 7$. It is true.
- Inductive hypothesis: Assume that $k^2 \geq 2k+1$ is true for some $n = k \geq 3$.

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- Inductive Step: We are trying to prove $(k+1)^2 \geq 2(k+1)+1$. We start with the inductive hypothesis that $k^2 \geq 2k+1$. Adding $2k+1$ to both sides gives $k^2+2k+1 \geq 4k+2$, where the LHS is exactly $(k+1)^2$. For the RHS, we use the fact that $k \geq 3$ to conclude that $2k \geq 1$ and thus $4k+2 = 2k+2k+2 \geq 2k+1+2 = 2k+3$. Now we have $(k+1)^2 \geq 4k+2 \geq 2k+3$, and we complete the proof.

(b) Prove that, for any positive integer $n \geq 4$, $2^n \geq n^2$.

Answer:

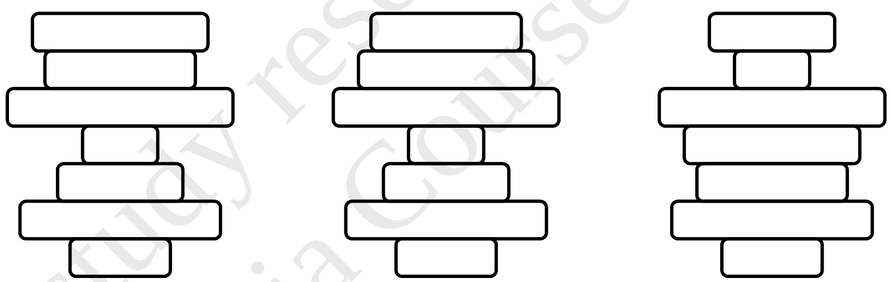
- Base Case ($n=4$): $2^4 = 16$, $4^2 = 16$, and $16 \geq 16$. It is true.
- Inductive hypothesis: Assume that $2^k \geq k^2$ is true for some $n = k \geq 4$
- Inductive Step: We are trying to prove $2^{k+1} \geq (k+1)^2$.

$$2^{k+1} = 2 \times 2^k \geq 2 \times k^2 = k^2 + k^2 \geq k^2 + 2k + 1 = (k+1)^2,$$

where the second step is by the inductive hypothesis, and the fourth step is by Part (a) (note that, in Part (a), the condition $n \geq 3$ covers all cases here). Therefore, since the inductive step is proven true, we complete the proof.

6. Pizza Flipping (**15 points**)

Philip J. Fry has a job at the local pizza parlor, where he tends to be a bit distractable. One day, he has a stack of unbaked pizza doughs and for some unknown reason, he decides to arrange them in order of size, with the largest pizza on the bottom, the next largest pizza just above that, and so on. He has learned how to place his spatula under one of the pizzas and flip over the whole stack above the spatula (reversing their order). The figure shows two sample flips.



initial stack — after flipping top two pizzas in initial stack — after flipping top five pizzas in second stack

This is the only move Fry can do to change the order of the stack; however, he is willing to keep repeating this move until he gets the stack in order. Is it always possible for him to get the pizzas in order via some sequence of moves, no matter how many pizzas he starts with and what order they are originally in? Prove your answer.

Answer: **Theorem**: Given n pizzas, Fry can always order them using some finite sequence of moves.

Proof: We induct on n.

- Base case: When $n = 1$ there is nothing to prove, since the pizza is already ordered.
- Inductive hypothesis: We may assume that $P(k)$ is true. That is, we assume Fry can put k pizzas into order.

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- Inductive step: Suppose Fry has $k+1$ pizzas, and suppose the largest pizza is the m-th pizza from the top. First, Fry flips the top m pizzas. This brings the largest pizza to the top of the stack. Then Fry flips the entire stack, bringing the largest pizza to the bottom. By the inductive hypothesis, he can then sort the other k pizzas on top of the largest one, without disturbing the largest one on the bottom. Then the stack is completely sorted.

By the principle of induction, the claim follows.

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