1. A roulette of apples You bought 20 apples at your local farmer's market. Due to the organic nature of the apples, they are infested with worms. In particular, each apple contains a single worm with probability 0.3 (mutually independent). (**leave answers unevaluated**)

a) You pick up two apples a_1 and a_2.

i) What is the probability that there are exactly 2 worms? What is the probability that there is exactly 1 worm? 0 worms?
Solution: $P(2 \text{ worms}) = 0.09$, $P(1 \text{ worm}) = 0.42$, $P(0 \text{ worms}) = 0.49$

ii) What is $P(a_1 \text{ has worm} | a_2 \text{ has worm})$?
Solution: 0.3 by independence

iii) What is $P(a_1 \text{ has worm} | \text{there is exactly 1 worm among } a_1 \text{ and } a_2)$?
Solution: $\frac{P(a_1 \text{ has worm, } a_2 \text{ has no worm})}{P(\text{exactly 1 worm})} = \frac{0.3*0.7}{0.42} = 0.5$

b) You eat all 20 apples.

i) What is the probability that you end up eating no worms?
Solution: $0.7^{20} \approx 0.0008$

ii) what is the probability that you end up eating exactly 1 worm?
Solution: $\binom{20}{1} * 0.7^{19} * 0.3 \approx 0.007$

iii) what is the probability that you end up eating exactly 2 worms?
Solution: $\binom{20}{2} * 0.7^{18} * 0.3^2 \approx 0.028$

iv) How many apples can you eat if you want the probability of eating no worms to be at least 0.2?
Solution: Let k be the number of apples you eat. We want $0.7^k > 0.2$ so $k < \frac{\log(0.2)}{\log(0.7)} \approx 4.5$, so you can eat at most 4 apples.

c) You pick a single apple at random and slice it into 3 slices. If the apple has a worm, it will be hidden in one of the slices. You bravely eat the slices one by one. Let s_1, s_2, s_3 denote the three slices.

i) What is $P(s_1 \text{ has no worm} | \text{apple has worm})$
Solution: $2/3$

ii) What is $P(s_1 \text{ has no worm})$?
Solution:

$$\begin{aligned}
& P(s_1 \text{ has no worm}) \\
\textbf{(by total probability)} = \; & P(s_1 \text{ has no worm} \mid \text{apple has no worm}) \times P(\text{apple has no worm}) \\
& + P(s_1 \text{ has no worm} \mid \text{apple has worm}) \times P(\text{apple has worm}) \\
= \; & 1 \times 0.7 + 2/3 \times 0.3 \\
= \; & 0.9
\end{aligned}$$

iii) What is $P(\text{apple has no worm} | s_1 \text{ has no worm})$. Compare your answer to $P(\text{apple has no worm})$.
Solution: $\frac{P(\text{apple has no worm, } s_1 \text{ has no worm})}{P(s_1 \text{ has no worm})} = \frac{P(\text{apple has no worm})}{P(s_1 \text{ has no worm})} = \frac{0.7}{0.9} \approx 0.778$

Why is the first equality true? Because if the apple has no worm means that the first slice has no worm. We can see this using a similar argument to ii)

$$\begin{aligned}&P(\text{apple has no worm , } s_1 \text{ has no worm})\\&= P(\text{apple has no worm}) * P(s_1 \text{ has no worm} \mid \text{apple has no worm})\\&= P(\text{apple has no worm}) * 1\\&= P(\text{apple has no worm})\end{aligned}$$

Note how it is higher than $P(\text{apple has no worm}) = 0.7$

iv) What is $P(s_2 \text{ has no worm}|s_1 \text{ has no worm})$ Compare your answer to $P(s_2 \text{ has no worm})$.
Solution: $P(s_2 \text{ has no worm, } s_1 \text{ has no worm}) = 0.7 + 0.3 * 1/3 = 0.8$
note that $0.3*1/3$ came from $P(\text{apple has worm}) * P(s_1 \text{ has no worm, } s_2 \text{ has no worm} \mid \text{apple has worm})$
Using bayes rule, we get:
$P(s_2 \text{ has no worm, } s_1 \text{ has no worm}|s_1 \text{ has no worm}) = \frac{P(s_2 \text{ has no worm, } s_1 \text{ has no worm})}{P(s_1 \text{ has no worm})} = \frac{0.8}{0.9} \approx 0.8888$
Interestingly, the probability got lower!

v) Using your previous answer, what is the safest way to eat 2 slices of apples?
Solution: Take 2 apples, slice each into 3 slices, eat one slice from each apple.

2. Probability Practice

1. If we put 5 math, 6 biology, 8 engineering, and 3 physics books on a bookshelf at random, what is the probability that all the math books are together?
2. A message source M of a digital communication system outputs a word of length 8 characters, with the characters drawn from the ternary alphabet $\{0,1,2\}$, and all such words are equally probable. What is the probability that M produces a word that looks like a byte (*i.e.*, no appearance of '2')?
3. If five numbers are selected at random from the set $\{1,2,3,\ldots,20\}$ with replacement, what is the probability that their minimum is larger than 5?

Solution: Solution Sketch:

1. $\frac{18!5!}{22!} = \frac{1}{1463}$. The 18! comes from 18 "units": 3 physics books, 8 engineering books, 6 biology books and 1 block of math books. The 5! comes from number of ways to arrange the 5 math books within the same block. 22! is just the total number of ways to arrange the books.
2. $\left(\frac{2}{3}\right)^8 = \frac{256}{6561}$. This is just by independence.
3. $\left(\frac{3}{4}\right)^5 = \frac{243}{1024}$. For a single number, we can choose 6,7...20, so 15 valid outcomes out of 20 total outcomes => 3/4 probability. Then apply independence as in 2.2.

3. Card Counting

Consider a deck with just the four aces (red: hearts, diamonds; black: spades, clubs). Melissa shuffles the deck and draws the top two cards.

1. Given that Melissa has the ace of hearts, what is the probability that she has both red cards?
2. Given that Melissa has the ace of diamonds, what is the probability that she has both red cards?
3. Given that Melissa has at least one red card, what is the probability that she has both red cards?

4. Melissa speaks a very odd language; during the A.M. hours, "flurg" means "hearts" and "grue" means "diamonds", and during the P.M. hours, "flurg" means "diamonds" and "grue" means "hearts". You have no idea what time it is, and consider A.M. and P.M. equally likely. Given that Melissa has the ace of "flurg", what is the probability she also has the ace of "grue"?

Solution: Solution Sketch:

1. 1/3: 12 equally possible scenarios overall, of which 6 include the ace of hearts, and 2 includes both red cards.
2. 1/3; same as above.
3. 1/5; 10 scenarios include at least one red card, of which 2 includes both red cards.
4. 1/3; there are 24 equally possible scenarios (for the time of day and the 2 cards); in 12, Melissa has the ace of "flurg", and in 4 of those, Melissa also has the ace of "grue".