CS 70 Discrete Mathematics and Probability Theory
Spring 2015 Vazirani HW 10

Reminder: Deadline for HW9 self-grade is Thursday, April 2 at noon.

Reading: Note 14 and Note 15.

Due Monday April 6

1. **Card Game**

   A game is played with six double-sided cards. One card has "1" on one side and "2" on the other. Two cards have "2" on one side and "3" on the other. And the last three cards have "3" on one side and "4" on the other. A random card is then drawn and held in a random orientation between two players, each of whom sees only one side of the card. The winner is the one seeing the smaller number. If the card that was drawn was a "2/3" card, compute the probabilities each player thinks he/she has for winning.

   Answer: For the player that sees the "2" side:

$$\Pr[\text{Win} \mid \text{Player sees } 2] = \frac{\Pr[\text{Win} \cap \text{Player sees } 2]}{\Pr[\text{Player sees } 2]} = \frac{\Pr[\text{Card is "2/3"} \cap \text{Player sees } 2]}{\Pr[\text{Player sees } 2]}$$
$$= \frac{\Pr[\text{Card is "2/3"}] \cdot \Pr[\text{Player sees 2} \mid \text{Card is "2/3"}]}{\Pr[\text{Card has a "2"}] \cdot \Pr[\text{The side with "2" is chosen} \mid \text{Card has a "2"}]}$$
$$= \frac{\frac{2}{6} \times \frac{1}{2}}{\frac{3}{6} \times \frac{1}{2}} = \frac{2}{3}$$

   For the player that sees the "3" side:

$$\Pr[\text{Win} \mid \text{Player sees } 3] = \frac{\Pr[\text{Win} \cap \text{Player sees } 3]}{\Pr[\text{Player sees } 3]} = \frac{\Pr[\text{Card is "3/4"} \cap \text{Player sees } 3]}{\Pr[\text{Player sees } 3]}$$
$$= \frac{\Pr[\text{Card is "3/4"}] \cdot \Pr[\text{Player sees 3} \mid \text{Card is "3/4"}]}{\Pr[\text{Card has a "3"}] \cdot \Pr[\text{The side with "3" is chosen} \mid \text{Card has a "3"}]}$$
$$= \frac{\frac{3}{6} \times \frac{1}{2}}{\frac{5}{6} \times \frac{1}{2}} = \frac{3}{5}$$

2. **Poker Game**

   During a poker game a kibitzer manages to get a brief glimpse of one of the hands (and no other hands). In this glimpse he sees only that one card in the hand is an ace. He did not notice which ace it was. What is the probability that the hand has at least two aces? If the kibitzer noticed that one card was a black ace, what is the probability that the hand has at least two aces? Finally suppose the kibitzer saw that the hand had the ace of Spades. Now what is the probability that the hand has at least two aces?

Reflect on the fact that the (conditional) probability of events change considerably when one learns kinds of information that have no obvious relevance.

Answer: If the kibitzer saw one ace in the hand,

$$\begin{aligned}
&\Pr[\text{at least 2 aces} \mid \text{at least 1 ace}] = 1 - \Pr[\text{less than 2 aces} \mid \text{at least 1 ace}] \\
&= 1 - \frac{\Pr[\text{less than 2 aces} \cap \text{at least 1 ace}]}{\Pr[\text{at least 1 ace}]} = 1 - \frac{\Pr[\text{exactly 1 ace}]}{\Pr[\text{at least 1 ace}]} = 1 - \frac{\Pr[\text{exactly 1 ace}]}{1 - \Pr[\text{no ace}]} \\
&= 1 - \frac{\frac{\binom{4}{1}\cdot\binom{48}{4}}{\binom{52}{5}}}{1 - \frac{\binom{48}{5}}{\binom{52}{5}}} \approx 0.122
\end{aligned}$$

If the kibitzer saw one black ace,

$$\begin{aligned}
&\Pr[\text{at least 2 aces} \mid \text{at least 1 black ace}] = 1 - \Pr[\text{less than 2 aces} \mid \text{at least 1 black ace}] \\
&= 1 - \frac{\Pr[\text{less than 2 aces} \cap \text{at least 1 black ace}]}{\Pr[\text{at least 1 black ace}]} = 1 - \frac{\Pr[\text{1 black ace and no other aces}]}{\Pr[\text{at least 1 black ace}]} \\
&= 1 - \frac{\Pr[\text{1 black ace and no other aces}]}{1 - \Pr[\text{no black ace}]} = 1 - \frac{\frac{\binom{2}{1}\cdot\binom{48}{4}}{\binom{52}{5}}}{1 - \frac{\binom{50}{5}}{\binom{52}{5}}} \approx 0.190
\end{aligned}$$

If the kibitzer saw the ace of Spades,

$$\begin{aligned}
&\Pr[\text{at least 2 aces} \mid \text{ace of Spades in the hand}] = 1 - \Pr[\text{less than 2 aces} \mid \text{ace of Spades in the hand}] \\
&= 1 - \frac{\Pr[\text{less than 2 aces} \cap \text{ace of Spades in the hand}]}{\Pr[\text{ace of Spades in the hand}]} = 1 - \frac{\Pr[\text{ace of Spades and no other aces}]}{\Pr[\text{ace of Spades in the hand}]} \\
&= 1 - \frac{\Pr[\text{ace of Spades and no other aces}]}{1 - \Pr[\text{no ace of Spades}]} = 1 - \frac{\frac{1\cdot\binom{48}{4}}{\binom{52}{5}}}{1 - \frac{\binom{51}{5}}{\binom{52}{5}}} \approx 0.221
\end{aligned}$$

(Note: To gain some intuition, consider a simpler case where the deck contains only hearts and spades. If we know that at least one of the two aces is in the hand, the probability that both are in the hand is

$$\Pr[\textit{Both aces} \mid \textit{At least one}] = \frac{\Pr[\textit{Both aces}]}{\Pr[\textit{At least one}]} = \frac{\Pr[\textit{Both aces}]}{\Pr[\textit{Only } A\spadesuit] + \Pr[\textit{Only } A\heartsuit] + \Pr[\textit{Both aces}]}$$

However, if we know specifically that $A\spadesuit$ is in the hand,

$$\Pr[\textit{Both aces} \mid A\spadesuit \textit{ in the hand}] = \frac{\Pr[\textit{Both aces}]}{\Pr[A\spadesuit \textit{ in the hand}]} = \frac{\Pr[\textit{Both aces}]}{\Pr[\textit{Only } A\spadesuit] + \Pr[\textit{Both aces}]}$$

This clearly demonstrates that $\Pr[\textit{Both aces} \mid A\spadesuit \textit{ present}]$ is greater than $\Pr[\textit{Both aces} \mid \textit{At least one}]$. Intuitively, this happens because if we know that at least one ace is in the hand, we know that we are in one of the three cases: only $A\spadesuit$, only $A\heartsuit$, or both aces. In contrast, if we know specifically that $A\spadesuit$ is in the hand, this time we are in one of the two cases: only $A\spadesuit$ or both aces. Therefore, relatively there are more chances in the latter case of having both aces.)

3. **Boys and Girls**

   There are three children in a family. A friend is told that at least two of them are boys. What is the probability that all three are boys? The friend is then told that the two are the oldest two children. Now what is the probability that all three are boys? Use Bayes' Law to explain this. Assume throughout that each child is independently either a boy or a girl with equal probability.

   Answer: Let A be the information that you are told and B the event that all three are boys. By Bayes' rule, we have $\Pr[B \mid A] = \frac{\Pr[B]\cdot\Pr[A|B]}{\Pr[A]}$. However, note that whether $A =$ "at least two boys" or $A =$ "oldest two are boys", $\Pr[A \mid B]$ is simply 1. So in either case we have $\Pr[B \mid A] = \frac{\Pr[B]}{\Pr[A]}$.

   Hence,

$$\Pr[\text{All 3 boys} \mid \text{At least 2 boys}] = \frac{\Pr[\text{All 3 boys}]}{\Pr[\text{At least 2 boys}]} = \frac{\frac{1}{8}}{\frac{4}{8}} = \frac{1}{4}$$

$$\Pr[\text{All 3 boys} \mid \text{Oldest 2 are boys}] = \frac{\Pr[\text{All 3 boys}]}{\Pr[\text{Oldest 2 are boys}]} = \frac{\frac{1}{8}}{\frac{2}{8}} = \frac{1}{2}.$$

   In this case we saw Bayes' rule simplify to $\Pr[B \mid A] = \frac{\Pr[B]}{\Pr[A]}$. Since B is a subset of A, the formula directly shows that this conditional probability depends only on the number of possibilities contained in A. When we are told that at least two children are boys, any of the three children could be a girl. In contrast, if we are told that the oldest two children are boys, then only the youngest child has the possibility of being a girl. Therefore the latter case has fewer possibilities and therefore larger conditional probability.

4. **Lie Detector**

   A lie detector is known to be 80% reliable when the person is guilty and 95% reliable when the person is innocent. If a suspect is chosen from a group of suspects of which only 1% have ever committed a crime, and the test indicates that the person is guilty, what is the probability that he is innocent?

   Answer: Let A denote the event that the test indicates that the person is guilty, and B the event that the person is innocent. Note that

$$\Pr[B] = 0.99, \quad \Pr[\overline{B}] = 0.01, \quad \Pr[A \mid B] = 0.05, \quad \Pr[A \mid \overline{B}] = 0.8$$

   Using the Bayesian Inference Rule, we can compute the desired probability as follows:

$$\Pr[B \mid A] = \frac{\Pr[B]\Pr[A \mid B]}{\Pr[B]\Pr[A \mid B] + \Pr[\overline{B}]\Pr[A \mid \overline{B}]} = \frac{0.99 \cdot 0.05}{0.99 \cdot 0.05 + 0.01 \cdot 0.8} \approx 0.86$$

5. **Cliques in random graphs**

   Consider a graph $G(V,E)$ on n vertices which is generated by the following random process: for each pair of vertices u and v, we flip a fair coin and place an (undirected) edge between u and v if and only if the coin comes up heads. So for example if $n = 2$, then with probability $1/2$, $G(V,E)$ is the graph consisting of two vertices connected by an edge, and with probability $1/2$ it is the graph consisting of two isolated vertices.

   (a) What is the size of the sample space?

(b) A k-clique in graph is a set of k vertices which are pairwise adjacent (every pair of vertices is connected by an edge). For example a 3-clique is a triangle. What is the probability that a particular set of k vertices forms a k-clique?

(c) Prove that the probability that the graph contains a k-clique for $k = 4\lceil \log n \rceil + 1$ is at most $1/n$.

Answer:

(a) There are two choices for each of the $\binom{n}{2}$ pairs of vertices, so the size of the sample space is $2^{\binom{n}{2}}$.

(b) For a fixed set of k vertices to be a k-clique, all of the $\binom{k}{2}$ pairs of those vertices have to be connected by an edge. The probability of this event is $1/2^{\binom{k}{2}}$.

(c) Let A_S denote the event that S is a k-clique, where $S \subseteq V$ is of size k. Then, the event that the graph contains a k-clique can be described as the union of A_S's over all $S \subseteq V$ of size k. Using the union bound,

$$\Pr\left[\bigcup_{S\subseteq V, |S|=k} A_S\right] \le \sum_{S\subseteq V, |S|=k} \Pr[A_S] = \sum_{S\subseteq V, |S|=k} \frac{1}{2^{\binom{k}{2}}}.$$

Now, since there are $\binom{n}{k}$ ways of choosing a subset $S \subseteq V$ of size k, the right-hand side of the above equality is

$$\frac{\binom{n}{k}}{2^{\binom{k}{2}}} = \frac{\binom{n}{k}}{2^{\frac{k(k-1)}{2}}} \le \frac{n^k}{\left(2^{\frac{(k-1)}{2}}\right)^k} \le \frac{n^k}{\left(2^{\frac{(4\log n+1-1)}{2}}\right)^k} = \frac{n^k}{(2^{2\log n})^k} = \frac{n^k}{n^{2k}} = \frac{1}{n^k} \le \frac{1}{n}.$$

6. **Midterm 2**

Solve question 4 or 5 on midterm 2; whichever you did worse on.

Answer: Please refer to Midterm 2 Solutions.

7. **Extra Credit**

Consider the following game: The dealer shuffles a regular deck of 52 cards and successively turns over one card at a time. After any card, you are allowed to say "stop". If the next card is red, you win, and if it is black, you lose. You must say stop at some point during the game. What is your optimal strategy (i.e. when should you say "stop") and what is your probability of winning under this strategy?

Note: A strategy can be any stopping rule, for example: say "stop" the first time the majority of cards turned over are black.

Answer: Every strategy is an optimal strategy because no matter what you do, the probability of winning is always $1/2$.

Indeed, after we have seen k cards, although we know how many red cards and black cards remain in the deck, we have no information about the ordering of those cards. Thus, the conditional probability that the next card is red would be the same as the conditional probability that any other remaining card is red. This means that at any point in the game, the probability that you will win by saying "stop" is the same as, for example, the probability that the last card is red.

This shows us that this game is actually equivalent to a modified version of the game where when we say "stop" we check the last card in the deck rather than the immediate next one. And it is obvious that the probability of winning that game is $1/2$ independent of your strategy.