EECS 70 Discrete Mathematics and Probability Theory
Fall 2015 Satish Rao

Discussion 2B Solution

1. (Stable Marriage) Consider the following list of preferences:

Men	Preferences
A	$4 > 2 > 1 > 3$
B	$2 > 4 > 3 > 1$
C	$4 > 3 > 1 > 2$
D	$3 > 1 > 4 > 2$

Women	Preferences
1	$A > D > B > C$
2	$D > C > A > B$
3	$C > D > B > A$
4	$B > C > A > D$

1. Is $\{(A,4),(B,2),(C,1),(D,3)\}$ a stable pairing?
   No. Rogue pair: $(C,3)$.
2. Find a stable matching by running the Traditional Propose & Reject algorithm.
   A men-optimal pairing: $\{(A,2),(B,4),(C,3),(D,1)\}$.
3. Show that there exist a stable matching where women 1 is matched to men A.
   A pairing can be: $\{(A,1),(B,4),(C,3),(D,2)\}$. This is stable because each woman gets their first preference. In other words, man A is the optimal man for woman 1, the best man can do in a pairing.

2. (Objective Preferences) Imagine that in the context of stable marriage all men have the same preference list. That is to say there is a global ranking of women, and men's preferences are directly determined by that ranking.

1. Prove that the first woman in the ranking has to be paired with her first choice in any stable pairing.
   If the first woman is not paired with her first choice, then she and her first choice would form a rogue couple, because her first choice prefers her over any other woman, and vice versa.
2. Prove that the second woman has to be paired with her first choice if that choice is not the same as the first woman's first choice. Otherwise she has to be paired with her second choice.
   If the first and second women have different first choices, then the second woman must be matched to her first choice. Otherwise she and her first choice would form a rogue couple (since her first choice is not matched to the first woman, he would prefer the second woman over his current match).
   If the first choices are the same, then the second woman must be paired with her second choice, otherwise she and her second choice would form a rogue couple (neither of them are matched to their first choices, and they are each other's second choice).
3. Continuing this way, assume that we have determined the pairs for the first $k-1$ women in the ranking. Who should the k-th woman be paired with?
   The k-th woman should be paired with the first man on her list who has not been matched yet (with the first $k-1$ women). If she's not matched to him, they would form a rogue couple. This is because the man would have to be matched to a woman ranked worse than k, so she would prefer the k-th woman over his current partner, and the k-th woman obviously prefers him to whoever she's matched with.
4. Prove that there is a unique stable pairing.
   In the previous parts, we saw that for each woman, given the pairs for the lower-ranked women, her pair would be determined uniquely. So there is only one stable pairing.

This can be stated and proved more rigorously using induction. Namely that there is a unique pairing for the first k women, assuming stability. An induction on k would prove this.

3. (Odd degree vertices)

Claim: Let $G = (V,E)$ be an undirected graph. The number of vertices of G that have odd degree is even.

Prove the claim above using:

(i) Direct proof (e.g., counting the number of edges in G)

(ii) Induction on $m = |E|$ (number of edges)

(iii) Induction on $n = |V|$ (number of vertices)

(iv) Well-ordering principle

Let $V_{\text{odd}}(G)$ denote the set of vertices in G that have odd degree. We prove that $|V_{\text{odd}}(G)|$ is even.

(i) Let d_v denote the degree of vertex v (so $d_v = |N_v|$, where N_v is the set of neighbors of v). Observe that

$$\sum_{v \in V} d_v = 2m$$

because every edge is counted exactly twice when we sum the degrees of all the vertices. Now partition V into the odd degree vertices $V_{\text{odd}}(G)$ and the even degree vertices $V_{\text{odd}}(G)^{\complement}$, so we can write

$$\sum_{v \in V_{\text{odd}}(G)} d_v = 2m - \sum_{v \notin V_{\text{odd}}(G)} d_v.$$

Both terms in the righthand side above are even ($2m$ is even, and each term d_v is even because we are summing over even degree vertices $v \notin V_{\text{odd}}(G)$). So for the lefthand side $\sum_{v \in V_{\text{odd}}(G)} d_v$ to be even, we must have an even number of terms, since each term in the summation is odd. Therefore, there must be an even number of odd-degree vertices, namely, $|V_{\text{odd}}(G)|$ is even.

(ii) We use induction on $m \geq 0$.

Base case $m = 0$: If there are no edges in G, then all vertices have degree 0, so $V_{\text{odd}}(G) = \emptyset$.

Inductive hypothesis: Assume $|V_{\text{odd}}(G)|$ is even for all graphs G with m edges.

Inductive step: Let G be a graph with $m+1$ edges. Remove an arbitrary edge $\{u,v\}$ from G, so the resulting graph G' has m edges. By the inductive hypothesis, we know $|V_{\text{odd}}(G')|$ is even. Now add the edge $\{u,v\}$ to get back the original graph G. Note that u has one more edge in G than it does in G', so $u \in V_{\text{odd}}(G)$ if and only if $u \notin V_{\text{odd}}(G')$. Similarly, $v \in V_{\text{odd}}(G)$ if and only if $v \notin V_{\text{odd}}(G')$. The degrees of all other vertices are unchanged in going from G' to G. Therefore,

$$V_{\text{odd}}(G) = \begin{cases} V_{\text{odd}}(G') \cup \{u,v\} & \text{if } u,v \notin V_{\text{odd}}(G') \\ V_{\text{odd}}(G') \setminus \{u,v\} & \text{if } u,v \in V_{\text{odd}}(G') \\ (V_{\text{odd}}(G') \setminus \{u\}) \cup \{v\} & \text{if } u \in V_{\text{odd}}(G'), v \notin V_{\text{odd}}(G') \\ (V_{\text{odd}}(G') \setminus \{v\}) \cup \{u\} & \text{if } u \notin V_{\text{odd}}(G'), v \in V_{\text{odd}}(G') \end{cases}$$

so we see that $|V_{\text{odd}}(G)| - |V_{\text{odd}}(G')| \in \{-2, 0, 2\}$. Since $|V_{\text{odd}}(G')|$ is even, we conclude $|V_{\text{odd}}(G)|$ is also even.

(iii) We use induction on $n \geq 1$.

Base case $n = 1$: If G only has 1 vertex, then that vertex has degree 0, so $V_{\text{odd}}(G) = \emptyset$.

Inductive hypothesis: Assume $|V_{\text{odd}}(G)|$ is even for all graphs G with n vertices.

Inductive step: Let G be a graph with $n+1$ vertices. Remove a vertex v and all edges adjacent to it from G. The resulting graph G' has n vertices, so by the inductive hypothesis, $|V_{\text{odd}}(G')|$ is even. Now add the vertex v and all edges adjacent to it to get back the original graph G. Let $N_v \subseteq V$ denote the neighbors of v (i.e., all vertices adjacent to v). Among the neighbors N_v, the vertices in the intersection $A = N_v \cap V_{\text{odd}}(G')$ had odd degree in G', so they now have even degree in G. On the other hand, the vertices in $B = N_v \cap V_{\text{odd}}(G')^{\complement}$ had even degree in G', and they now have odd degree in G. The vertex v itself has degree $|N_v|$, so $v \in V_{\text{odd}}(G)$ if and only if $|N_v|$ is odd. We now consider two cases:

(a) Suppose $|N_v|$ is even, so $v \notin V_{\text{odd}}(G)$. Then

$$V_{\text{odd}}(G) = (V_{\text{odd}}(G') \setminus A) \cup B$$

so $|V_{\text{odd}}(G)| = |V_{\text{odd}}(G')| - |A| + |B|$. Note that A and B are disjoint and their union equals N_v, so $|A| + |B| = |N_v|$. Therefore, we can write $|V_{\text{odd}}(G)|$ as

$$|V_{\text{odd}}(G)| = |V_{\text{odd}}(G')| + |N_v| - 2|A|$$

which is even, since $|V_{\text{odd}}(G')|$ is even by the inductive hypothesis, and $|N_v|$ is even by assumption.

(b) Suppose $|N_v|$ is odd, so $v \in V_{\text{odd}}(G)$. Then

$$V_{\text{odd}}(G) = (V_{\text{odd}}(G') \setminus A) \cup B \cup \{v\}$$

so, again using the relation $|A| + |B| = |N_v|$, we can write

$$|V_{\text{odd}}(G)| = |V_{\text{odd}}(G')| - |A| + |B| + 1 = |V_{\text{odd}}(G')| + (|N_v| + 1) - 2|A|$$

which is even, since $|V_{\text{odd}}(G')|$ is even by the inductive hypothesis, and $|N_v|$ is odd by assumption.

This completes the inductive step and the proof.

Note how this proof is more complicated than the proof in part (i), even though they are both using induction. This tells you that choosing the right variable to induct on can simplify the proof.

(iv) Here we give a well-ordering proof using the number of edges m as the notion of "size" of G, so this is equivalent to the proof in part (i) using induction on m. (You can also try to give a well-ordering proof using n as the size of G.)

Suppose the contrary that the claim is false for some graphs. This means the set M is not empty, where M is the set of $m \in \mathbb{N}$ for which there exists a graph G with m edges that is a counterexample to the claim. Thus, we have a nonempty subset M of $\mathbb{N}$, so by the well-ordering principle, M has a smallest element m'. Note that $m' > 0$, since the claim is true for all graphs with 0 edges.

Let G be a graph with m' edges for which the claim is false, i.e., $|V_{\text{odd}}(G)|$ is odd (here we know such a G must exist from the definition of $m' \in M$). Remove one edge from G to obtain a smaller graph G' with $m' - 1$ edges (here we need $m' \geq 1$, which we have seen above). By our choice of m' as the smallest element of M, we know that $m' - 1 \notin M$, so the claim holds for G', namely, $|V_{\text{odd}}(G')|$ is even. Now add the removed edge to get back G. By the same argument as in the inductive step in part (i), this implies that $|V_{\text{odd}}(G)|$ is also even, a contradiction.