EECS 70 Discrete Mathematics and Probability Theory
Fall 2015 Jean Walrand

Discussion 15B Solution

1. (Conditional expectation) Suppose that X and Y are independent geometric random variables with parameter p. Suppose that $n > 2$ is a positive integer.

What is $E(X \mid X+Y=n)$?

First note that for $1 \leq i \leq n-1$, $P(X+Y=n, X=i) = P(X=i, Y=n-i) = P(X=i)P(Y=n-i) = p(1-p)^{i-1}p(1-p)^{n-i-1} = p^2(1-p)^{n-2}$.

This gives $P(X+Y=n) = (n-1)p^2(1-p)^{n-2}$.

Therefore, $P(X=i \mid X+Y=n) = \frac{p^2(1-p)^{n-2}}{(n-1)p^2(1-p)^{n-2}} = \frac{1}{n-1}$

Finally, $E(X \mid X+Y=n) = \frac{1}{n-1}[1+2+3+...+(n-1)] = \frac{n}{2}$.

Remark: Indeed, the result is the same whenever X and Y are i.i.d. by symmetry. Since, $E[X|X+Y=n] = E[Y|X+Y=n]$ and these two numbers add up to $E[X+Y|X+Y=n] = n$.

2. (Continuous random variables) Let the probability density function for a random variable X is

$$f(x) = \begin{cases} 2/3, & \text{if } 0 \leq x < 1; \\ 1/3, & \text{if } 1 \leq x < 2; \\ 0, & \text{otherwise.} \end{cases}$$

$\Pr[X=0] =$? $\Pr[\frac{1}{2} \leq X \leq \frac{3}{2}] =$? What is the expectation of X? What is the variance of X?

$$\Pr[X=0] = 0$$

$$\Pr[\frac{1}{2} \leq X \leq \frac{3}{2}] = \frac{2}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2} = \frac{1}{2}$$

$$E(X) = \int_0^1 \frac{2}{3} x dx + \int_1^2 \frac{1}{3} x dx = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

$$E(X^2) = \int_0^1 \frac{2}{3} x^2 dx + \int_1^2 \frac{1}{3} x^2 dx = \frac{2}{9} + \frac{7}{9} = 1$$

$$Var(X) = E(X^2) - E(X)^2 = 1 - (\frac{5}{6})^2 = \frac{11}{36}$$

3. (Triangle)

You are given a one meter-long stick. You choose two points $X < 1$ and $Y < 1$ randomly along the stick and cut the stick at those two points.

What is the probability that you can make a triangle with the three pieces?

Let us first look at when $X < Y$. Then, the stick is broken up into three pieces of lengths X, $Y - X$ and $1 - Y$.

The triangle inequality states that the sum of any two of the sides of a triangle is greater than than the third side. So, let's apply this to each of the three sides we found above.

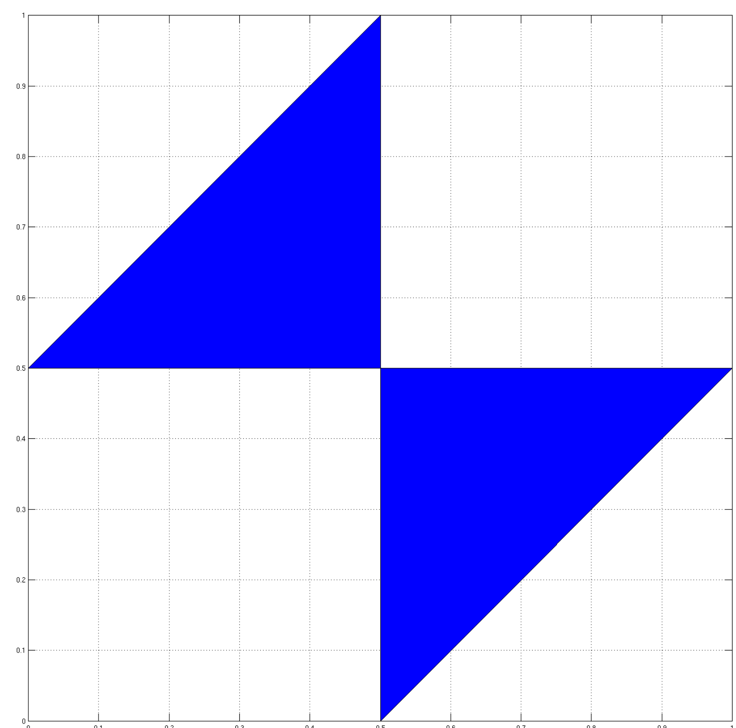
1. $X + (Y - X) > (1 - Y) \implies 2Y > 1 \implies Y > 0.5$
2. $(Y - X) + (1 - Y) > X \implies 1 > 2X \implies X < 0.5$
3. $X + (1 - Y) > (Y - X) \implies 1 > 2(Y - X) \implies Y < X + 0.5$

By symmetry, the case where $Y < X$ has the following conditions after applying triangle inequality.

1. $X > 0.5$
2. $Y < 0.5$
3. $X < Y + 0.5$

Now, let's look at the unit square as representing the different values that X and Y can take. Since X and Y take values uniformly between 0 and 1, every point on the square is equally likely.

In the plot below, shaded in blue, are the feasible regions we found above. That is, if we cut the stick at $X = x, Y = y$ from some point in those regions, we will have a valid triangle.



Now, we see that the total area that this feasible region covers is $\frac{1}{4}$. Since all points in the square are equally likely, the area tells us exactly what the probability of the feasible region is. Thus, the total probability of choosing a valid triangle is $\frac{1}{4}$.

Also, one can evaluate the probability as P(valid triangle) $= 2\int_0^{0.5}\int_{0.5}^{x+0.5} 1 dy dx = 2\int_0^{0.5} x dx = 2 \times (\frac{0.5^2}{2}) = \frac{1}{4}$