CS 70 Discrete Mathematics and Probability Theory
Fall 2012 Vazirani **HW 3**

1 Generalized DeMorgan's Laws: An Exercise in Induction

Prove using induction

$$\bigcup_{i=1}^{n} {A_i}^c = (\bigcap_{i=1}^{n} A_i)^c$$

Where A^c means "A complement", the set of elements which lie in the space but which do not lie in A. For instance, in the space of real numbers ($\mathbb{R}$), the complement of rational numbers ($\mathbb{Q}$) is the irrational numbers ($\mathbb{Q}^c$). Note that a set and it's complement always union to equal the entire space.

Hint: To show that two sets are equal, it suffices to show that their elements are the same. Recall what it means to be in the union of sets, and the intersection of sets. (This has an obvious correspondence with the "and" and "or" logical operators.) Tease out these definitions to establish the base case. Then use induction.

Food for thought: can you see how DeMorgan's Law corresponds to the logic operations of negating existential quantifiers to equal universal quantifiers? This question is to give you a flavour of set theoretic notation, in anticipation of Lecture 24. We will post this lecture up for your reference.

–Solution–

Proof by induction.
Base case: Prove for $n = 1$ and $n = 2$. $n = 1$ is trivial. $n = 2$: Observe

$$[\neg(x \in A_1) \vee \neg(x \in A_2)] \Leftrightarrow \neg\,(x \in A_1 \wedge x \in A_2)$$

Therefore, $A_1^c \bigcup A_2^c = (A_1 \bigcap A_2)^c$.
Inductive hypothesis: $\bigcup_{i=1}^{n} A_i^c = \left(\bigcap_{i=1}^{n} A_i\right)^c$
Inductive step:

$$\begin{aligned} \bigcup_{i=1}^{n+1} A_i^c &= \left(\bigcup_{i=1}^{n} A_i^c\right) \bigcup A_{n+1}^c \\ &= \left(\bigcap_{i=1}^{n} A_i\right)^c \bigcup A_{n+1}^c \\ &= \left(\bigcap_{i=1}^{n+1} A_i\right)^c \end{aligned}$$

□

Logical version of De Morgan's Law:

A	B	$\neg A \vee \neg B$	$\neg(A \wedge B)$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

Proof: Observe the following:

$$\begin{aligned} x \in A \bigcup B &\Leftrightarrow x \in A \vee x \in B \\ x \in A^c &\Leftrightarrow \neg(x \in A) \end{aligned}$$

This converts statements about unions/complements to logic, which we can now apply to De Morgan's law:

$$\begin{aligned} A^c \bigcup B^c &\Leftrightarrow (x \in A^c) \vee (x \in B^c) \\ &\Leftrightarrow \neg(x \in A) \vee \neg(x \in B) \\ &\Leftrightarrow \neg\left((x \in A) \wedge (x \in B)\right) \end{aligned}$$

Base Case: $n = 1$. $A_i^c = (A_i)^c$ trivially. $n = 2$ proven by De Morgan's law above.
Inductive Hypothesis: Assume $\bigcup_{i=1}^n A_i^c = (\bigcap_{i=1}^n A_i)^c$ for $n \geq 2$.
Inductive Step: Consider $\bigcup_{i=1}^{n+1} A_i^c$.

$$\bigcup_{i=1}^{n+1} A_i^c = \left(\bigcup_{i=1}^{n} A_i^c\right) \bigcup A_{n+1}^c \tag{1}$$

By the inductive hypothesis,

$$\bigcup_{i=1}^{n+1} A_i^c = \left(\bigcap_{i=1}^{n} A_i\right)^c \bigcup A_{n+1}^c \tag{2}$$

Recognize $(\bigcap_{i=1}^k A_i)^c$ as just another set and apply the base case, or $n = 2$:

$$\bigcup_{i=1}^{n+1} A_i^c = \left[\left(\bigcap_{i=1}^{n} A_i\right) \bigcap A_{n+1}\right]^c = \left(\bigcap_{i=1}^{n+1} A_i\right)^c \tag{3}$$

$\square$

2 More Induction with Fibonnaci Sequence

Prove that $F(n) \geq 2^{(n-1)/2}$ for $n \geq 3$.

–Solution–

Proof by strong induction, base case for $n = 3, 4$.
Inductive hypothesis $F(k) \geq 2^{(k-1)/2}$ $\forall k \in [3, n]$.
Inductive step: Utilize definition of Fibonacci sequence: $F(k+1) = F(k) + F(k-1)$ and the inductive hypothesis:
$F(k+1) \geq 2^{(n-1)/2} + 2^{(n-2)/2}$. Note the RHS is $\geq \frac{2^{n/2}}{2} + \frac{2^{n/2}}{2} = 2^{n/2}$, completing the proof.

–Exemplar–

Proof by strong induction.

Proof: Fibonacci sequence is defined as: $F(0) = 0$, $F(1) = 1$, and
$\forall k \geq 2$, $F(k) = F(k-1) + F(k-2)$.
Base case: $n = 3$, $F(3) = 2 \geq 2^{2/2} = 2$, and $n = 4$, $F(4) = 3 \geq 2^{3/2}$.
Inductive hypothesis: Assume $F(k) \geq 2^{k-1}/2$ for all $3 \leq k \leq n$.
Inductive step: Consider $F(n+1) = F(n) + F(n-1)$.

$$\begin{aligned}
F(n+1) &= F(n) + F(n-1) \\
F(n+1) &\geq 2^{(n-1)/2} + 2^{(n-2)/2} \quad \text{by inductive hypothesis.} \\
&\geq \frac{2^{n/2}}{\sqrt{2}} + \frac{2^{n/2}}{2} \\
&\geq \frac{2^{n/2}}{2} + \frac{2^{n/2}}{2} \quad \text{after multiplying first term by } 1/\sqrt{2}\text{, making term yet smaller.} \\
&\geq 2^{n/2}
\end{aligned}$$

By induction, $\forall k \geq 3$, $F(k) \geq 2^{(n-1)/2}$. □

3 Programming and Induction

Consider the program $F_2(n)$ in the notes (and copied below). Let the running time of the program be $O(T(n))$. What is the best such function $T(n)$? Prove the bound.

```
function F2(n)
  if n=0 then return 0
  if n=1 then return 1
  a = 1
  b = 1
  for k = 2 to n do
    tmp = a
    a = a + b
    b = tmp
  od
  return a
```

–Solution–

Define a constant time unit τ for each iteration of the loop.

For $n = 2$, the loop runs 1 time, taking time τ. Assert that $R(n)$, the running time of the function, is approximately $(n-1)\tau$.
Select $K > \tau$. Then, $R(n) = \tau(n-1) \leq K(n-1) \leq Kn$ for all $n \geq 2$. Therefore, $R(n) = O(n)$.

–Solution–

Define time unit τ', which is the time to calculate the two if-statements and the two assignments, and to enter the for-loop. Define time unit τ to be the time it takes to calculate one pass through the for-loop. Let $R(n)$ represent the run-time of the function with argument n. $R(0)$ is roughly τ'. $R(1)$ is roughly τ'. $R(2) = \tau' + \tau$. Assert that $R(n) = \tau' + (n-1)\tau$ for all $n \geq 2$.
In general, $R(n+1) = \tau' + \tau + (n-1)\tau = \tau' + n\tau$, as each time through the loop adds a constant time τ.

Let $K = (\tau' + \tau)$, and $g(n) = Kn$. We now prove that $R(n) \leq g(n)\ \forall n \geq 1$.
For $n = 1$: $R(1) = \tau' \leq \tau' + \tau = g(1)$.
Consider $n \geq 1$:

$$R(n) = \tau' + (n-1)\tau \leq n(\tau' + \tau) = g(n)$$

$R(n) \leq Kn$ for all $n \geq 1$. Therefore, $T(n)$ is $O(n)$. Running time grows linearly with the problem size.

4 Running time on the Stable Marriage Algorithm

Run the traditional propose and reject algorithm on the following example.

Men's preference list:

1	A	B	C	D
2	B	C	A	D
3	C	A	B	D
4	A	B	C	D

Women's preference list:

A	2	3	4	1
B	3	4	1	2
C	4	1	2	3
D	1	2	3	4

Please put answer in the form of tuples, that is $\{(i, J)$ where $i \in \{1, 2, 3, 4\}$ and $J \in \{A, B, C, D\}\}$

–Solution & Exemplar–

1. Pairs: (1,∅), (2,B), (3,C), (4,A)
2. Pairs: (1,B), (2,∅), (3,C), (4,A)
3. Pairs: (1,B), (2,C), (3,∅), (4,A)
4. Pairs: (1,B), (2,C), (3,A), (4,∅)
5. Pairs: (1,∅), (2,C), (3,A), (4,B)
6. Pairs: (1,C), (2,∅), (3,A), (4,B)
7. Pairs: (1,C), (2,A), (3,∅), (4,B)
8. Pairs: (1,C), (2,A), (3,B), (4,∅)
9. Pairs: (1,∅), (2,A), (3,B), (4,C)
10. Pairs: (1,D), (2,A), (3,B), (4,C)

Observe this to be a stable matching.

5 Proving Sharper Bounds for Stable Marriage

In lecture, we showed that a propose and reject algorithm must terminate after at most n^2 days. Prove a sharper bound showing that the algorithm must terminate after at most $n(n-1)+1$ days.

–Solution–

Observe and prove that each man must end with at least one woman still on his list. Thus, there may be at most $n(n-1)$ rejections. The final proposal takes another day, so there are at most $n(n-1)+1$ days to complete the algorithm.

–Exemplar–

At the end, every man must have at least one woman left on his list, by the following Lemma.

Lemma 1 *No man may be rejected by every woman.*

Proof: Proved in lecture. □

After each day, at least one woman is crossed off of a man's list; if that's not the case, nobody is rejected, the algorithm terminates with a stable marriage. By the above Lemma, there may be at most $n(n-1)$ rejections, because each man may be rejected at most $n-1$ times. Each rejection takes 1 day, and the final proposal takes 1 day, so there are at most $n(n-1)+1$ days to complete the algorithm.

6 Extra credit: Common Knowledge

In distributed computing, which processor "knows" what information is a central issue. In this question we explore how interesting and paradoxical such issues of common knowledge can be through a hat puzzle:

There are 20 people in a room, each with either a blue or red hat on their head. Each person can clearly see the hat on everyone else's head, but cannot see their own. In fact 8 people have blue hats and 12 have red. The game is played in rounds, and in each round the people who know the color of their hats raise their hands. The game proceeds this way an exhausting number of rounds and no one raises their hands.

Then a child wanders into the room and exclaims "a blue hat!" Eight rounds later all eight people with blue hats raise their hands.

1. Explain why this happened. i.e. show that if there were k blue hats all k people would raise their hands in the k-th round.
2. What do you expect to happen in the 9th round?
3. Everyone in the room already knew that there was at least one person had a blue hat, since everyone could see at least seven blue hats. Then why did the child's exclamation have any effect?

–Solution–

1. $P(k)$ indicates that there are k blue hats.
   Suppose $P(1)$. The one person with the blue hat knows immediately they are the one with the blue hat. For $k = n + 1$, an arbitrary person with a blue hat knows $P(k) \vee P(k + 1)$. They watch what could be a game with n blue hats. After the nth round, and nobody raises their hand, they know they have a blue hat. All $n + 1$ realize at the same time, and $n + 1$ raise their hands on the $n + 1$th round.

2. On the 9th round, everyone knows there were only 8 blue hats, so all the red hats raise their hands.

3. With $k = 1$, it is not common knowledge that there was at least 1 blue hat. The red hats know it, but the blue hat does not. When the child states there is at least one blue hat, then the blue hat can deduce they are wearing a blue hat.
   With $k = 2$, it is common knowledge that there is at least 1 blue hat, as everyone sees that there is a blue hat. However, it is NOT the case that everyone knows that everyone knows there is at least 1 blue hat. After the child states that there is at least one blue hat, then everyone knows that everyone knows there is at least 1 blue hat. After the first day has passed and nobody has left, now each blue hat knows that the other blue hat knows there are at least 2 blue hats, and they both raise their hands.
   With $k = 3$, on the first day, each blue hat knows that another blue hat knows there is a blue hat. On the second day, each blue hat knows that another blue hat knows that another blue hat knows there is another blue hat. On the third day, each blue hat knows that another blue hat knows that another blue hat knows that another blue hat knows there is a blue hat, therefore each blue hat knows there to be at least 3 blue hats. Seeing only 2, each knows that they have a blue hat.

1. Proof by induction. Base case: Suppose $k = 1$. The person with the blue hat sees only red hats, and is not sure if there is a blue hat ($k = 1$) or not ($k = 0$). With knowledge that a blue hat exists, the sole blue hat knows that $k = 1$ and raises their hand.

   Suppose that for $k = n$, n people raise their hand on the nth round.

   Consider the case for $k = n + 1$. Each person with a blue hat sees n blue hats, and must figure out if there are n blue hats or $n+1$ blue hats. Assuming themselves to have red hats, they watch a game which should have $k = n$ blue hats. After n rounds, when n players don't raise their hands, by the inductive hypothesis, this player knows this is a game with $n + 1$ blue hats, and raises their hand the next round. This holds for every player with a blue hat, and so all raise their hands at the same time.

   Therefore, after k rounds, k players raise their hand, for $k \geq 1$.

2. In the 9th round, the remaining players raise their hand, knowing they have red hats. As they saw 8 blue hats, they had been thinking "either there are 8 blue hats, or I am wearing a blue hat!". Knowing there to be 8 blue hats induces knowledge that this player is wearing a red hat.

3. With $k = 1$, it is not common knowledge that there was at least 1 blue hat. The red hats know it, but the blue hat does not. When the child states there is at least one blue hat, then the blue hat can deduce they are wearing a blue hat.
   With $k = 2$, it is common knowledge that there is at least 1 blue hat, as everyone sees that there is a blue hat. However, it is NOT the case that everyone knows that everyone knows there is at least 1 blue hat. After the child states that there is at least one blue hat, then everyone knows that everyone knows there is at least 1 blue hat. After the first day has passed and nobody has left, now each blue hat knows that the other blue hat knows there are at least 2 blue hats, and they both raise their hands.
   With $k = 3$, on the first day, each blue hat knows that another blue hat knows there is a blue hat. On the second day, each blue hat knows that another blue hat knows that another blue hat knows there is another blue hat. On the third day, each blue hat knows that another blue hat knows that another blue hat knows that another blue hat knows there is a blue hat, therefore each blue hat knows there to be at least 3 blue hats. Seeing only 2, each knows that they have a blue hat.
   One doesn't know they have a blue hat until they know that everyone knows they have a blue hat.