Homework 1 Solutions

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CS 70: Discrete Mathematics and Probability Theory, Summer 2014

1. 1a. **Answer**: Not equivalent.

P	Q	$P \wedge (Q \vee P)$	$P \wedge Q$
T	T	T	T
T	F	T	F
F	T	F	F
F	F	F	F

1b. **Answer**: Not equivalent.

P	Q	R	$(P \Rightarrow Q) \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	F	T

1c. **Answer**: Equivalent.

P	Q	R	$(P \Rightarrow Q) \Rightarrow (P \Rightarrow R)$	$P \Rightarrow (Q \Rightarrow R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

1d. **Answer**: Equivalent.

P	Q	$(P \wedge \neg Q) \Leftrightarrow (\neg P \vee Q)$	$(Q \wedge \neg P) \Leftrightarrow (\neg Q \vee P)$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	F	F

2. 2a. (I) "No one, who is going to a party, ever fails to brush his or her hair"
Answer: $\forall x \; \big(P(x) \Rightarrow B(x)\big)$

(II) "No one looks fascinating, if he or she is untidy."
Answer: $\forall x \ \big(U(x) \Rightarrow \neg F(x)\big)$

(III) "Opium-eaters have no self-command."
Answer: $\forall x \ \big(O(x) \Rightarrow N(x)\big)$

(IV) "Everyone who has brushed his or her hair looks fascinating"
Answer: $\forall x \ \big(B(x) \Rightarrow F(x)\big)$

(V) "No one wears kid gloves, unless he or she is going to a party"
Answer: $\forall x \ \big(K(x) \Rightarrow P(x)\big)$

(VI) "A person is always untidy if he or she has no self-command."
Answer: $\forall x \ \big(N(x) \Rightarrow U(x)\big)$

2b. (I) $\forall x \ \big(\neg B(x) \Rightarrow \neg P(x)\big)$
(II) $\forall x \ \big(F(x) \Rightarrow \neg U(x)\big)$
(III) $\forall x \ \big(\neg N(x) \Rightarrow \neg O(x)\big)$
(IV) $\forall x \ \big(\neg F(x) \Rightarrow \neg B(x)\big)$
(V) $\forall x \ \big(\neg P(x) \Rightarrow \neg K(x)\big)$
(VI) $\forall x \ \big(\neg U(x) \Rightarrow \neg N(x)\big)$

2c. **Answer**: A person who wears kid gloves is not an opium-eater.
Derivation: $K(x) \Rightarrow P(x) \Rightarrow B(x) \Rightarrow F(x) \Rightarrow \neg U(x) \Rightarrow \neg N(x) \Rightarrow \neg O(x)$

3. 3a. **Claim**: $\forall x \exists y \ (xy \geq x^2)$
Answer: True.
Proof: Let $y = x$. It is trivially true that $\forall x \ (x^2 \geq x^2)$. ■

3b. **Claim**: $\exists y \forall x \ (xy \geq x^2)$
Answer: False.
Proof: The proposition cannot be true for some $y < 0$, since $x^2 \geq 0$ and $xy < 0$ for $x > 0$ and $y < 0$. The proposition similarly cannot be true for some $y > 0$, since $x^2 \geq 0$ and $xy < 0$ for $x < 0$ and $y < 0$. The proposition is obviously not true for $y = 0$, since $x^2 > 0$ for $x \neq 0$. Since the proposition cannot be true for any real number y, the proposition is false. ■

3c. **Claim**: $\neg \forall x \exists y \ (xy > 0 \Rightarrow y > 0)$
Answer: False.
Proof: This is easiest to approach by looking at the proposition before negation, then applying negation. The proposition before negation is $\forall x \exists y \ (xy > 0 \Rightarrow y > 0)$. The implication in this proposition is vacuously true for $y = 0$. Because of this, the proposition before negation is true, so the negation of that proposition is false. ■

4. 4a. **Claim**: $\neg \forall x \ \exists y \ \big(P(x) \Rightarrow \neg Q(x, y)\big) \ \equiv \ \exists x \ \forall y \ \big(P(x) \wedge Q(x, y)\big)$
Answer: The equivalence holds.
Justification: Truth tables show that $P(x) \Rightarrow \neg Q(x, y) \ \equiv \ \neg P(x) \vee \neg Q(x, y)$. Using

De Morgan's Law to distribute the negation on the left side yields $\exists x\ \forall y\ \big(\neg\neg P(x) \wedge \neg\neg Q(x,y)\big)$, which is equivalent to the right side.

4b. **Claim**: $\forall x\ \exists y\ \big(P(x) \Rightarrow Q(x,y)\big) \ \equiv\ \forall x\ \big(P(x) \Rightarrow (\exists y\ Q(x,y))\big)$
Answer: The equivalence holds.
Justification: We can rewrite the claim as $\forall x\ \exists y\ \big(\neg P(x) \vee Q(x,y)\big) \ \equiv\ \forall x\ \big(\neg P(x) \vee (\exists y\ Q(x,y))\big)$. Clearly, the two sides are the same if $\neg P(x)$ is true. If $\neg P(x)$ is false, then the two sides are still the same, because $\forall x\ \exists y\ \big(\text{False} \vee Q(x,y)\big) \ \equiv\ \forall x\ \big(\text{False} \vee (\exists y\ Q(x,y))\big)$.

4c. **Claim**: $\forall x\ \exists y\ \big(Q(x,y) \Rightarrow P(x)\big) \ \equiv\ \forall x\ \big((\exists y\ Q(x,y)) \Rightarrow P(x)\big)$
Answer: The equivalence does not hold.
Justification: We can rewrite the claim as $\forall x\ \exists y\ \big(\neg Q(x,y) \vee P(x)\big) \ \equiv\ \forall x\ \big((\neg(\exists y\ Q(x,y))) \vee P(x)\big)$. By De Morgan's Law, distributing the negation on the right side of the equivalence changes the $\exists y$ to $\forall y$, and the two sides are clearly not the same. Another approach to the problem is to consider by linguistic example. Let x and y span the universe of all people, and let $Q(x,y)$ mean "Person x is Person y's offspring", and let $P(x)$ mean "Person x likes tofu". The left side claims that, for all Persons x, there exists some Person y such that either Person x is not Person y's offspring or that Person x likes tofu. The right side claims that, for all Persons x, if there exists a parent of Person x, then Person x likes tofu. Obviously, these are not the same.

5. 5a. **Claim**: For all natural numbers n, if n is odd then $n^2 + 2n$ is odd.
Answer: True.
Proof: We will use a direct proof. Assume n is odd. By the definition of odd numbers, $n = 2k + 1$ for some natural number k. Substituting into the expression $n^2 + 2n$, we get $(2k+1)^2 + 2 \times (2k+1)$. Simplifying the expression yields $4k^2 + 8k + 3$. This can be rewritten as $2 \times (2k^2 + 4k + 1) + 1$. Since $2k^2 + 4k + 1$ is a natural number, by the definition of odd numbers, $n^2 + 2n$ is odd. ■

5b. **Claim**: For all natural numbers n, $n^2 + 7n + 1$ is odd.
Answer: True.
Proof: We will use a proof by cases. Let n be an even number. By the definition of even numbers, $n = 2k$ for some natural number k. Substituting into the expression $n^2 + 7n + 1$, we get $(2k)^2 + 7 \times (2k) + 1$. Simplifying the expression yields $4k^2 + 14k + 1$. This can be rewritten as $2 \times (2k^2 + 7k) + 1$, which is an odd number. Therefore, if n is even, then $n^2 + 7n + 1$ is odd. Now let n be an odd number. By the definition of odd numbers, $n = 2k + 1$ for some natural number k. Substituting into the expression $n^2 + 7n + 1$, we get $(2k+1)^2 + 7 \times (2k+1) + 1$. Simplifying the expression yields $4k^2 + 18k + 9$. This can be rewritten as $2 \times (2k^2 + 9k + 4) + 1$, which is an odd number. Therefore, if n is odd, then $n^2 + 7n + 1$ is odd. Since $n^2 + 7n + 1$ is odd when n is even or when n is odd, $n^2 + 7n + 1$ is odd for all natural numbers n. ■

5c. **Claim**: For all real numbers a, b, if $a + b \leq 10$ then $a \leq 7$ or $b \leq 3$.
Answer: True.
Proof: We will use a proof by contraposition. Suppose that $a > 7$ and $b > 3$ (note that

this is equivalent to $\neg\big(a \leq 7 \vee b \leq 3\big)$). Since $a > 7$ and $b > 3$, $a + b > 10$ (note that $a + b > 10$ is equivalent to $\neg\big(a + b \leq 10\big)$). Thus, if $a + b \leq 10$, then $a \leq 7$ or $b \leq 3$ (or both, as "or" is not "exclusive or" in this case). ■

5d. **Claim**: For all real numbers r, if r is irrational then $r + 1$ is irrational.
Answer: True.
Proof: We will use a proof by contraposition. Assume that $r + 1$ is rational. Since $r + 1$ is rational, it can be written in the form a/b where a and b are integers. Then r can be written as $(a - b)/b$. By the definition of rational numbers, r is a rational number, since both $a - b$ and b are integers. By contraposition, if r is irrational, then $r + 1$ is irrational. ■

5e. **Claim**: For all natural numbers n, $10n^2 > n!$.
Answer: False.
Proof: We will use proof by counterexample. Let $n = 6$. $10 \times 6^2 = 360$. $6! = 720$. Since $10n^2 < n!$, the claim is false. ■

6. 6a. **Claim**: $\forall n \in \mathbf{Z}\ \big(n \geq 1 \Rightarrow \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}\big)$
Proof: Define $P(x)$ to be the predicate

$$\sum_{i=1}^{x} \frac{1}{i(i+1)} = \frac{x}{x+1}$$

For the base case, let $n = 1$.

$$\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{2} = \frac{n}{n+1}$$

so the base case is true. For our inductive hypothesis, assume that $P(n)$ is true for some $n \geq 1$. Adding $\frac{1}{(n+1)(n+2)}$ to both sides yields

$$\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n^2 + 2n + 1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2}$$

This means that $\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \frac{n+1}{n+2}$, which means that $P(n) \Rightarrow P(n+1)$. By the principle of induction, $\forall n \in \mathbf{Z}\ \big(n \geq 1 \Rightarrow \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}\big)$. ■

6b. **Claim**: $\forall n \in \mathbf{N}\ \big(5|(8^n - 3^n)\big)$
Proof: For our base case, let $n = 0$. $8^0 - 3^0 = 0$, which is obviously divisible by 5, so the base case is true for $n = 0$. For our inductive hypothesis, assume that $5|(8^n - 3^n)$ for some $n \geq 0$. Multiplying $8^n - 3^n$ by 8, we get $8^{n+1} - 8 \times 3^n$. Note that since we multiplied by an integer, this is still divisible by 5. Subtracting 5×3^n yields $8^{n+1} - 3 \times 3^n = 8^{n+1} - 3^{n+1}$.

Note that since we subtracted a multiple of 5, $8^{n+1} - 3^{n+1}$ is still divisible by 5. By the principle of induction, $\forall n \in \mathbf{N}\ \big(5|(8^n - 3^n)\big)$. ■

6c. Problem moved to HW2.

7. **Claim**: $\forall k \in \mathbf{Z}\ \forall r \in \mathbf{R}\ \big((k > 0 \wedge r > 0) \Rightarrow (\text{There are finitely many solutions to } \frac{1}{n_1} + \cdots + \frac{1}{n_k} = r, n_i \in \mathbf{Z}, n_i > 0)\big)$
Proof: We will prove this by induction on k. For our base case, $k = 1$. In the base case, iff r can be written as $\frac{1}{n_1}$ when n_1 is a positive integer, then there is exactly one solution, $n_1 = \frac{1}{r}$. If r cannot be written in that form, then there are exactly zero solutions. In all cases, there is a finite number of solutions. For the inductive hypothesis, assume that there are finitely many solutions for some $k \geq 1$ for all r. Each real number r_1 either can or cannot be written as the sum of $k+1$ integers' inverses. If r_1 cannot be written in that form, then there are exactly zero solutions. If r_1 can be written in that form, then the integers' inverses can be ordered. Since r_1 is the sum of $k+1$ integers' inverses, the largest $\frac{1}{n_i}$ must be at least $\frac{r_1}{k+1}$. This means that the smallest n_i must be at most $\frac{k+1}{r_1}$, which means that the smallest n_i has finitely many possible values. For each of the possible smallest n_i values, there is a real number $r_1 - \frac{1}{n_i}$ that can be written as the sum of k integers' inverses in finitely many ways (using the induction hypothesis). This means that there are only finitely many possible solutions for $k+1$ (combining all solutions (finitely many) for each possible smallest n_i values (finitely many)). By the principle of induction, there are finitely many solutions for all k for all r.

8. 8a. Problem moved to HW2.

   8b. Problem moved to HW2.

   8c. Problem moved to HW2.

9. 9a. The proof is incorrect. The use of $max(x-1, y-1)$ is not correct, since $x-1$ and $y-1$ will fall outside the range of natural numbers when $x = 0$ or $y = 0$. Since this is not a situation that was shown in the base case, the proof does not hold.

   9b. The proof is incorrect. Using induction requires showing that, given a true proposition $P(n)$, it follows that $P(n+1)$. This "proof" simply changed n to $n+1$, which is not valid justification for induction. The inductive hypothesis must assume that the theorem is true for some value of n, not for every value of n. One way to make this proof valid would be to show that, given $n < 2^n$ for some $n \geq 0$, multiplying the right side by 2 will increase it by at least one. Then it follows that $n + 1 < 2^{n+1}$, which completes justification for induction.

   9c. The proof is incorrect. You want to prove an implication of the form $P(n) \implies Q(n)$ for every n, where $P(n)$ is "$2n+1$ is a multiple of 3" and $Q(n)$ is "n^2+1 is a multiple of 3".

The contrapositive is $\neg Q(n) \implies \neg P(n)$. Your proof begins with $\neg P(n)$ and concludes with $\neg Q(n)$, so you have shown $\neg P(n) \implies \neg Q(n)$, which is the contrapositive (it's actually the contrapositive to the converse).

Furthermore the theorem is not true. Your argument in the proof shows that $\neg Q(n)$ always holds (for $n > 0$ - be careful about the trivial case $n = 0$!). Yet, $\neg P(n)$ does not hold for any n of the form $3k + 1$, so the contrapositive implications fails for those n.