EECS 70 Discrete Mathematics and Probability Theory
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Discussion 9b

1. Flipping coins

1. You have a fair coin, and you flip it 4 times. What is the probability that the number of heads is always ahead of the number of tails in the 4 flips? For example, the sequence HHHT has fits the description, but HTTH does not

   Solution: Let X_i denote the outcome of the i^{th} toss. X_1 must be H, since otherwise you have 1 head and 0 tails. T X_2 must also be H, since otherwise you have 1 head and 1 tail, so #heads is not strictly ahead. If X_3 is H, then X_4 can be either H or T. If X_3 is T, then X_4 must be H (otherwise #heads is not strictly ahead)

   So total probability is

   $$P(X_1 = H)P(X_2 = H)(P(X_3 = H)(P(X_4 = H) + P(X_4 = T)) + P(X_3 = T)(P(X_4) = H))$$

   which evaluates to

   $$\frac{1}{2} \cdot \frac{1}{2}(\frac{1}{2}(\frac{1}{2} + \frac{1}{2}) + \frac{1}{2} \cdot \frac{1}{2}) = \frac{3}{16}$$

2. What is the probability of getting 4 heads out of 4 flips, given that there are at least 2 heads?

   Solution:

   $$P(4 \text{ heads}, \text{at least 2 heads}) = P(4 \text{ heads}) = (\frac{1}{2})^4 = \frac{1}{16}$$

   $$\begin{aligned} &P(\text{at least 2 heads}) \\ =&1 - P(\text{exactly 1 head}) - P(\text{exactly 0 heads}) \\ =&1 - 4 \cdot (\frac{1}{2})^4 - (\frac{1}{2})^4 \\ =&1 - \frac{1}{4} - \frac{1}{16} \\ =&\frac{11}{16} \end{aligned}$$

   Using Bayes rule

   $$P(4 \text{ heads}|atleast2heads) = \frac{P(4 \text{ heads}, \text{at least 2 heads}}{P(\text{at least 2 heads})} = \frac{1/16}{11/16} = \frac{1}{11}$$

3. Now assume that you are given two identical looking coins, but one is fair and the other is loaded, with $P(H) = 0.6$. You pick one uniformly at random, and toss it 3 times, getting 3 heads. What is the probability that you picked the loaded coin?

   Solution:

   $$P(\textbf{loaded}, \textbf{3 heads}) = P(\textbf{3 heads}|\textbf{loaded})P(\textbf{loaded}) = 0.6^3 \cdot 0.5 = 0.108$$

   $$P(\textbf{fair}, \textbf{3 heads}) = P(\textbf{3 heads}|\textbf{fair})P(\textbf{fair}) = 0.5^3 \cdot 0.5 = 0.0625$$

Using bayes rule

$$\begin{aligned}
&P(\textbf{loaded}|\textbf{3 heads})\\
=&\frac{P(\textbf{loaded},\textbf{3 heads})}{P(\textbf{3 heads})}\\
=&\frac{P(\textbf{loaded},\textbf{3 heads})}{P(\textbf{loaded},\textbf{3 heads})+P(\textbf{fair},\textbf{3 heads})}\\
=&\frac{0.6^3\cdot 0.5}{0.6^3\cdot 0.5+0.5^4}\\
=&0.633 \quad \text{(roughly)}
\end{aligned}$$

2. Pairwise Independence

The events A_1, A_2, A_3 are *pairwise independent* if, for all $i \neq j$, A_i is independent of A_j. However, pairwise independence is a weaker statement than *mutual independence*, which requires the additional condition that $P(A_1, A_2, A_3) = P(A_1)P(A_2)P(A_3)$.

Try to construct an example where three events are pairwise independent but not mutually independent.

Here is one potential starting point: Let A_1, A_2 be the respective results of flipping two fair coins. Can you come up with an event A_3 that works?

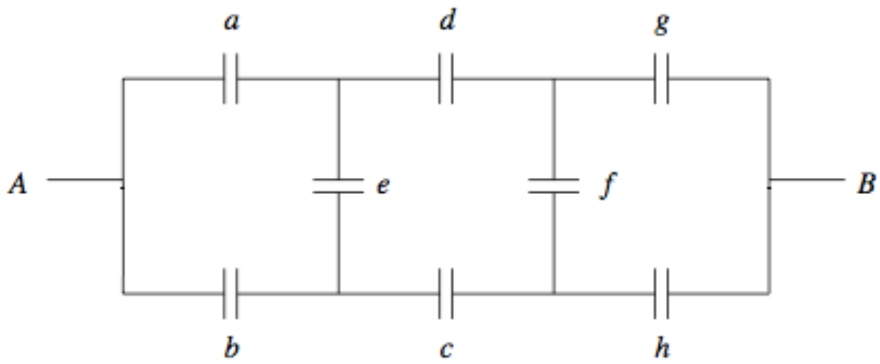
Solution: A_1: the first result is Head; A_2: the second result is Head; A_3: both results are the same.

3. Balls in Bins: Independent? You have k balls and n bins labelled $1, 2, \ldots, n$, where $n \geq 2$. You drop each ball uniformly at random into the bins.

1. What is the probability that bin n is empty?
   Solution: $(\frac{n-1}{n})^k$
2. What is the probability that bin 1 is non-empty? Argue this both by counting, and by independence.
   Solution: $1 - (\frac{n-1}{n})^k$
3. What is the probability that both bin 1 and bin n are empty?
   Solution: $(\frac{n-2}{n})^k$
4. What is the probability that bin 1 is non-empty and bin n is empty?
   Solution: $(\frac{n-1}{n})^k - (\frac{n-2}{n})^k$
5. What is the probability that bin 1 is non-empty given that bin n is empty?
   Solution: $\dfrac{\left(\frac{n-1}{n}\right)^k - \left(\frac{n-2}{n}\right)^k}{\left(\frac{n-1}{n}\right)^k} = 1 - \left(\dfrac{n-2}{n-1}\right)^k$

4. Communication network

In the communication network shown below, link failures are independent, and each link has a probability of failure of p. Consider the physical situation before you write anything. A can communicate with B as long as they are connected by at least one path which contains only in-service links.



1. Given that exactly five links have failed, determine the probability that A can still communicate with B.

   Solution: There are only two paths of 3 links from A to B. And there are $\binom{8}{5}$ ways of the links messing up.

   So the probability is $\frac{2}{56} = \frac{1}{28}$.

   This is because every single case of exactly 5 links being down have the same probability. So it's a uniform distribution over all possibilities.

2. Given that exactly five links have failed, determine the probability that either g or h (*but not both*) is still operating properly.

   Solution: Fix g as down and h as working. There are $\binom{6}{4}$ ways to have 4 out of the remaining go down. Symmetric argument for h down and g up.

   So probability is $\frac{30}{56} = \frac{15}{28}$.

3. Given that *a, d* and h have failed (but no information about the information of the other links), determine the probability that A can communicate with B.

   Solution: We would just want the 4 on the only remaining path from A to B not to be down.

   The probability of this happening is $(1-p)^4$.