1. James Bond

   James Bond is imprisoned in a cell from which there are three possible ways to escape: an air-conditioning duct, a sewer pipe and the door (which is unlocked). The air-conditioning duct leads him on a two-hour trip whereupon he falls through a trap door onto his head, much to the amusement of his captors. The sewer pipe is similar but takes five hours to traverse. Each fall produces amnesia and he is returned to the cell immediately after each fall. Assume that he always immediately chooses one of the three exits from the cell with probability $\frac{1}{3}$. On average, how long does it take before he opens the unlocked door and escapes?

   Answer:

   Due to the memorylessness of the scenario, i.e., what James Bond chose in the previous attempt has nothing to do with his current attempt and all the following attempts, it is like starting all over again, as every attempt is just like his first attempt. Let X be the random variable of the amount of time he needs to escape. We can start by considering the outcome of Bond's first attempt, and see how the expected time to escape, $E[X]$, changes as a result of this. We have

$$\begin{aligned} E[X] &= E[X|A]\Pr[A] + E[X|S]\Pr[S] + E[X|D]\Pr[D] \\ &= \frac{1}{3}(E[X|A] + E[X|S] + E[X|D]) \end{aligned}$$

   where $E[X|A]$ means the expected time to escape given that Bond went through the AC-duct in his first attempt, and similarly for $E[X|S]$ and $E[X|D]$. Now we take a closer look at these conditional expectations. Clearly, $E[X|D] = 0$ because Mr. Bond escapes immediately. Meanwhile, we have $E[X|A] = 2 + E[X]$ because given that 007 chose AC-duct in his first attempt, he wastes 2 hours and have to try again (which triggers another completely fresh attempt with no memory and thus takes $E[X]$ hours to escape). Similarly, we have $E[X|S] = 5 + E[X]$. So now, we have derived a recursive relation for the expected time to escape

$$E[X] = \frac{1}{3}(2 + E[X] + 5 + E[X] + 0)$$

   Solving this for $E[X]$, we get that $E[X] = \boxed{\mathbf{007}}$.

2. Family Planning

   Mr and Mrs Brown decide to continue having children until they either have their first girl or until they have five children. Assume that each child is equally likely to be a boy or a girl, independent of all other children, and that there are no multiple births. Let B and G denote the numbers of boys and girls respectively that the Browns have. Let C be the total number of children they have.

   (a) Write down the sample space together with the probability of each sample point.

Answer: The sample space in this problem is the set of all possible sequences of children that the Browns can have; we enumerate them to get:

$$\Omega = \{g, bg, bbg, bbbg, bbbbg, bbbbb\}$$

The probabilities of all of these sample points are as follows:

$$\begin{aligned}
\Pr[g] &= \frac{1}{2} \\
\Pr[bg] &= \left(\frac{1}{2}\right)^2 = \frac{1}{4} \\
\Pr[bbg] &= \left(\frac{1}{2}\right)^3 = \frac{1}{8} \\
\Pr[bbbg] &= \left(\frac{1}{2}\right)^4 = \frac{1}{16} \\
\Pr[bbbbg] &= \left(\frac{1}{2}\right)^5 = \frac{1}{32} \\
\Pr[bbbbb] &= \left(\frac{1}{2}\right)^5 = \frac{1}{32}
\end{aligned}$$

(b) Write down the distributions of the random variables B, G and C.

Answer: First, let us think about what values B, G, and C can take on. Looking at our sample space, we see that the possible values for G are 0 and 1, the possible values for B range between 0 and 5, and finally, the possible values for C range between 1 and 5. To list the distributions, we need to find the probabilities of all of these happenings. We can just read off this information from what we have above. First, we calculate the distribution for G:

$$\begin{aligned}
\Pr[G = 0] &= \Pr[bbbbb] = \frac{1}{32} \\
\Pr[G = 1] &= \Pr[g] + \Pr[bg] + \Pr[bbg] + \Pr[bbbg] + \Pr[bbbbg] = \frac{31}{32}
\end{aligned}$$

Next we calculate the distribution for B:

$$\begin{aligned}
\Pr[B = 0] &= \Pr[g] = \frac{1}{2} \\
\Pr[B = 1] &= \Pr[bg] = \frac{1}{4} \\
\Pr[B = 2] &= \Pr[bbg] = \frac{1}{8} \\
\Pr[B = 3] &= \Pr[bbbg] = \frac{1}{16} \\
\Pr[B = 4] &= \Pr[bbbbg] = \frac{1}{32} \\
\Pr[B = 5] &= \Pr[bbbbb] = \frac{1}{32}
\end{aligned}$$

Finally the distribution of C, the total number of children is:

$$\begin{aligned}
\Pr[C=1] &= \Pr[g] = \frac{1}{2} \\
\Pr[C=2] &= \Pr[bg] = \frac{1}{4} \\
\Pr[C=3] &= \Pr[bbg] = \frac{1}{8} \\
\Pr[C=4] &= \Pr[bbbg] = \frac{1}{16} \\
\Pr[C=5] &= \Pr[bbbbg] + \Pr[bbbbb] = 2 * \frac{1}{32} = \frac{1}{16}
\end{aligned}$$

(c) Write down the joint distribution of G and C.

Answer: Here is a table giving the joint distribution of G and C.

Joint distribution of G and C					
	$C=1$	$C=2$	$C=3$	$C=4$	$C=5$
$G=1$	$\Pr[g]=\frac{1}{2}$	$\Pr[bg]=\frac{1}{4}$	$\Pr[bbg]=\frac{1}{8}$	$\Pr[bbbg]=\frac{1}{16}$	$\Pr[bbbbg]=\frac{1}{32}$
$G=0$	0	0	0	0	$\Pr[bbbbb]=\frac{1}{32}$

(d) Write down the conditional distributions of C given $G=i$ for all possible values i that G can take on.

Answer: We need to give both the conditional distribution of C given $G=0$, and the conditional distribution of C given $G=1$. In the first case, we have

$$\Pr[C|G=1] = \frac{\Pr[C \cap G=1]}{\Pr[G=1]} = \frac{\Pr[C \cap G=1]}{\frac{31}{32}}$$

based on Bayes rule, and plugging in $\Pr[G=1]$ from (b). But this means the conditional distribution of C given $G=1$ is just the distribution given in the top row of (d), with all the values divided by $\frac{31}{32}$. In other words, to condition on the event that we get an event from the top row, as opposed to from either row, we just normalize all the probabilities in the top row so they add up to 1. Thus, we have

Conditional distribution of C given $G=1$					
	$C=1$	$C=2$	$C=3$	$C=4$	$C=5$
$G=1$	$\frac{16}{31}$	$\frac{8}{31}$	$\frac{4}{31}$	$\frac{2}{31}$	$\frac{1}{31}$

Similarly, $\Pr[C|G=0] = \frac{\Pr[C \cap G=0]}{\Pr[G=0]} = \frac{\Pr[C \cap G=0]}{\frac{1}{32}}$, so we can think of taking the bottom row of the table in (d), multiplied by a normalization factor of $\frac{1}{32}$. So we have

Conditional distribution of C given $G=0$					
	$C=1$	$C=2$	$C=3$	$C=4$	$C=5$
$G=0$	0	0	0	0	1

In the second case, knowing that no girls are born means the Browns stop trying after having $C = 5$ children.

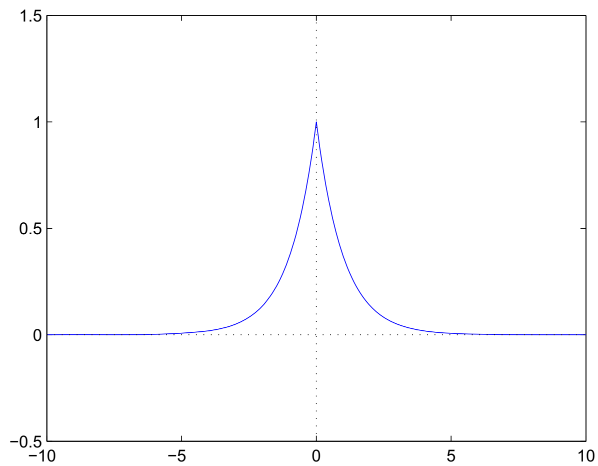
3. Continuous Probability
   Let X be a continuous r.v. with probability density function of the form :

$$f(x) = ce^{\lambda|x|}, \qquad -\infty < x < +\infty$$

   (a) Can λ take on any arbitrary value? If so, explain. If not, specify the possible values λ can take on.
   No. We need $\lambda < 0$.

   Otherwise, we would have an infinite area under the curve $f(x)$ over the range $-\infty < x < +\infty$. This is very easy to check: just over the range $0 < x < +\infty$, $f(x) = ce^{\lambda x}$, with infinite area under the curve unless $\lambda < 0$.

   (b) Sketch the probability density function.



   (c) Given a particular λ, can c take on any arbitrary value? If so, explain. If not, specify the possible values that c can take on (in terms of λ).
   No. For $f(x)$ to be a valid pdf, it must be normalized so that $\int_{-\infty}^{+\infty} f(x)dx = 1$. Thus we have

$$\int_{-\infty}^{+\infty} ce^{\lambda|x|}dx = \int_{-\infty}^{0} ce^{\lambda(-x)}dx + \int_{0}^{+\infty} ce^{\lambda x}dx$$

$$2\int_{0}^{+\infty} ce^{\lambda x}dx = \frac{2c}{\lambda}e^{\lambda x}\Big]_{0}^{+\infty} = -\frac{2c}{\lambda} = 1$$

$$\Longrightarrow c = -\frac{\lambda}{2}.$$

   (d) Evaluate $\mathbf{P}[X = 2]$.
   It is zero. There is zero probability of a continuous random variable taking a specific real number value.

(e) Evaluate the mean of X.

By symmetry, we see that the mean of X is zero, as the pdf is a symmetric function about $X = 0$. In other words, there is equal density at any $X = k$ as at $X = -k$.

We could also double check the math:

$$E[X] = \int_{-\infty}^{+\infty} ce^{\lambda|x|}xdx = \int_{-\infty}^{0} ce^{\lambda(-x)}xdx + \int_{0}^{+\infty} ce^{\lambda x}xdx$$

$$= -\int_{0}^{\infty} ce^{\lambda x}xdx + \int_{0}^{+\infty} ce^{\lambda x}xdx = 0.$$