EECS 70 Discrete Mathematics and Probability Theory
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Discussion 8A

Hi, all! Please do not feel all the problems need be done in section. We added a couple to allow for extra practice as some students requested this in the midterm check-in survey. Enjoy!

0. Warmup: Clothes and stuff

1. Say we've decided to do the whole capsule wardrobe thing and we now have only 5 different items of clothing that we wear (jeans, tees, shoes, jackets, and floppy hats, obvi). We have 3 variations on each of the items, and we wear one of each item every day. How many different outfits can we make?

   Solution: 3^5

2. It turns out 3 floppy hats really isn't enough of a selection, so we've bought 11 more, and we now have 14 floppy hats. Now how many outfits can we make?

   Solution: $14 \cdot 3^4$

3. If we own k different items of clothing, with n_1 varitions of the first item, n_2 variations of the second, n_3 of the third, and so on, how many outfits can we make?

   Solution: $n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_k$

4. We love our floppy hats so much that we've decided to also use them as wall art, so we're picking 4 of our 14 hats to hang in a row on the wall. How many such arrangements could we make? (Order matters, because no one really wants to see that burgundy one next to our favorite forest green fedora.)

   Solution: $14!/10!$

5. Ok, now we're packing for vacation to Iceland, and we only have space for 4 of our 14 floppy hats. How many sets of 4 could we bring? (Yeah, yeah, we knew you were going to use that notation. Now tell us the number as a function of a, your answer from the previous part.)

   Solution: $\binom{14}{4}$ or written as a function of the previous part, $a/4!$

6. Ok, turns out the check-in person for our flight to Iceland is being *very* unreasonable about the luggage weight restrictions, and we're going to have to leave some hats behind. Despite our best intentions, and having packed only 4 hats, we actually bought 18 additional floppy hats at the airport (6 in burgundy, 6 in forest green, and 6 in classic black). We'll keep our 4 hats that we brought from home, but we'll have to return all but 6 of the airport hats. How many color configurations can there be for the 6 airport hats that we keep?

   Solution: $\binom{8}{6}$. But let's be serious, you should just keep the black ones – so much more versatile.

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1. Counting

1. How many ways are there to arrange n 1s and k 0s into a sequence?
   Solution: $\binom{n+k}{k}$

2. How many solutions does

   $x_0 + x_1 + \ldots + x_k = n$

   have, if all xs must be non-negative integers?

   Solution: $\binom{n+k}{k}$. There is a bijection between a sequence of n ones and k plusses and a solution to the equation: x_0 is the number of ones before the first plus, x_1 is the number of ones between the first and second plus, etc. A key idea is that if a bijection exists between two sets they must be the same size, so counting the elements of one tells us how many the other has.

3. How many solutions does

   $x_0 + x_1 = n$

   have, if all xs must be *strictly positive* integers?

   Solution: $n-1$. It's easy just to enumerate the solutions here. x_0 can take values $1, 2, \ldots, n-1$ and this uniquely fixes the value of x_1. So, we have $n-1$ ways to do this. But, this is just an example of the more general question below.

4. How many solutions does

   $x_0 + x_1 + \ldots + x_k = n$

   have, if all xs must be *strictly positive* integers?

   Solution: $\binom{(n-(k+1))+k}{k} = \binom{n-1}{k}$. By subtracting 1 from all $k+1$ variables, and $k+1$ from the total required, we reduce it to problem with the same form as the previous problem. Once we have a solution to that we reverse the process, and adding 1 to all the non-negative variables gives us positive variables.

2. Fermat's necklace

Let p be a prime number and let k be a positive integer. We have an endless supply of beads. The beads come in k different colors. All beads of the same color are indistinguishable.

1. We have a piece of string. As a relaxing study break, we want to make a pretty garland by threading p beads onto the string. How many different ways are there construct such a sequence of p beads of k different colors?

   Solution: k^p.

2. Now let's add a restriction. We want our garland to be exciting and multicolored. Now how many different sequences exist? (Your answer should be a simple function of k and p.)

   Solution: $k^p - k$. You can have k sequences of a beads with only one color.

3. Now we tie the two ends of the string together, forming a circular necklace which lets us freely rotate the beads around the necklace. We'll consider two necklaces equivalent if the sequence of colors on one can be obtained by rotating the beads on the other. (For instance, if we have $k = 3$ colors—red (R), green (G), and blue (B)—then the length $p = 5$ necklaces RGGBG, GGBGR, GBGRG, BGRGG, and GRGGB are all equivalent, because these are cyclic shifts of each other.)

   How many non-equivalent sequences are there now? Again, the p beads must not all have the same color. (Your answer should be a simple function of k and p.)

   [Hint: What follows if rotating all the beads on a necklace to another position produces an identical looking necklace?]

   Solution: Since p is prime, rotating any sequence by less than p spots will produce a new sequence. As in, there is no number x smaller than p such that rotating the beads by x would cause the pattern to look the same. So, every pattern which has more than one color of beads can be rotated to form $p-1$ other patterns. So the total number of patterns equivalent with some bead sequence is p. Thus, the total number of non-equivalent patterns are $\frac{k^p - k}{p}$.

4. Use your answer to part (c) to prove Fermat's little theorem. (Recall that Fermat's little theorem says that if p is prime and $a \not\equiv 0 \pmod{p}$, then $a^{p-1} \equiv 1 \pmod{p}$.)
   Solution: $\frac{k^p - k}{p}$ from above has to be an integer. Hence, $k(k^{p-1} - 1)$ has to be divisible by p. If k is coprime with p, then it follows that $k^{p-1} - 1$ is divisible by p which is equivalent to FLT.

3. Story Problems

Prove the following identities by combinatorial argument:

1. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$
   Solution: The left hand side is the number of ways to choose k elements out of n. Looking at this another way, we look at the first element and decide whether we are going to choose it or not. If we choose it, then we need to choose $k-1$ more elements from the remaining $n-1$. If we don't choose it, then we need to choose all our k elements from the remaining $n-1$. We are not double counting, since in one of our cases we chose the first element and in the other, we did not.
2. $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$
   Solution: RHS: From n people, pick one team-leader and some (possibly empty) subset of other people on his team.
   LHS: First pick k people on the team, then pick the leader among them.
3. $\sum_{k=j}^{n} \binom{n}{k}\binom{k}{j} = 2^{n-j}\binom{n}{j}$
   Solution: RHS: Form a team as follows: Pick j leaders from n people. Then pick some (possibly empty) subset of the remaining people.
   LHS: First pick $k \geq j$ people on the team, then pick the j leaders among them.

4. Countability Basics

1. Is $f : \mathbb{N} \to \mathbb{N}$, defined by $f(n) = n^2$ an injection (one-to-one)? Briefly justify.
   Solution: Yes. One way to illustrate is by drawing the one-to-one mapping from n to n^2. More formally we can show that the preimage is unique by showing that $m \neq n \implies f(m) \neq f(n)$.
   We'll do proof by contraposition. $f(m) = f(n) \implies m = n$.
   $$f(m) = f(n) \implies m^2 = n^2 \implies m^2 - n^2 = 0 \implies (m-n)(m+n) = 0 \implies m = \pm n$$
   Since n can't be negative, we have an injection.
2. Is $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^3 + 1$ a surjection (onto)? Briefly justify.
   Solution: Yes. For any value of y, there always exists a corresponding input x. If $y = x^3 + 1$, we know that $x = \sqrt[3]{y-1}$. Thus for any value of y, there exists this value of x which maps to it.