CS 70 Discrete Mathematics and Probability Theory
Spring 2015 Vazirani HW 1

Due Monday January 26 at noon

1. Short Answer

a) For each question, circle whether the statement is true or false.

T F $\forall x \in \mathbb{Z}, x+1 \in \mathbb{Z}$.

Answer: True. The statement, when translated into English, reads "for all integers x, $x+1$ is an integer."

T F $\forall x \in \mathbb{Z}$, if $\frac{x}{2} \in \mathbb{Z}$ then $\frac{x+1}{2} \in \mathbb{Z}$.

Answer: False. The statement reads "for all integers x, if $\frac{x}{2}$ is an integer, then $\frac{x+1}{2}$ is an integer." To disprove this statement, we simply need to show a counterexample. For example if $x = 2$, then $\frac{x}{2} = 1$ is an integer, but $\frac{x+1}{2} = \frac{3}{2}$ is not an integer.

T F $\forall x \in \mathbb{Z}$, if $\frac{x}{2} \notin \mathbb{Z}$ then $\frac{x+1}{2} \in \mathbb{Z}$.

Answer: True. The statement reads "for all integers x, if $\frac{x}{2}$ is not an integer, then $\frac{x+1}{2}$ is an integer." If $\frac{x}{2}$ is not an integer, it means that x is not even. But every number is either even or odd, so x must be odd. Therefore $x+1$ is even, and so $x+1$ is divisible by 2. Hence $\frac{x+1}{2}$ is an integer.

T F $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}\ s.t.\ xy = 1$.

Answer: False. The statement reads "for all integers x, there exists an integer y such that $xy = 1$." To disprove this, we simply need to provide a counterexample x. If we take $x = 2$, then no matter what y is, xy is going to be an even number, so xy can never be 1 for any integer y.

T F $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Q}\ s.t.\ xy = 1$.

Answer: False. The statement reads "for all integers x, there exists a rational number y such that $xy = 1$." This statement is valid for almost all x's; but it has a single counterexample. If we take $x = 0$, then no matter what y is, $xy = 0$. So for that particular x, $\nexists y \in \mathbb{Q}\ s.t.\ xy = 1$.

b) The expression $\sum_{i=0}^{n} f(i)$ is equal to which of the following expressions (circle all that apply)?

- $f(0) \cdot f(1) \cdot f(2) \cdots f(n-1) \cdot f(n)$

  Answer: Not equal. This expression is the product of $f(0), f(1), \ldots, f(n)$, whereas we were looking for the sum. As a counter example take $n = 1$, and $f(0) = 0$ and $f(1) = 1$. Then $\sum_{i=0}^{n} f(i) = 1$, whereas $f(0) \cdot f(1) = 0$.

- $\sum_{i=0}^{n-1} (f(i) + f(n))$

  Answer: Not equal. The term $f(n)$ gets repeated $n-1$ times (once for each value of i) in this expression whereas it should appear just once. As a counter example take $n = 2$, and f to be the constant function 1. Then $\sum_{i=0}^{n} f(i) = 3$. But $\sum_{i=0}^{n-1} (f(i) + f(n)) = \sum_{i=0}^{1} 2 = 4$.

This study resource was shared via CourseHero.com
https://www.coursehero.com/file/12305819/cs70sp15sol1/

- $\left(\sum_{i=0}^{n-1} f(i)\right) + f(n)$

  Answer: Equal. The sum of n terms is always the sum of the first $n-1$ terms added with the last term. This expression is just that.

- $f(0) + f(1) + \cdots + f(n)$

  Answer: Equal. This is the sum of the n terms expanded in an explicit way.

- $\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} f(i) + \sum_{j=\lfloor \frac{n}{2} \rfloor + 1}^{n} f(j)$

  Answer: Equal. For any integer $0 \leq c < n$, we can break the sum $\sum_{i=0}^{n} f(i)$ into two parts: the sum of the terms where $i \leq c$ and the sum of the terms where $i \geq c+1$. Thus $\sum_{i=0}^{n} f(i) = \sum_{i=0}^{c} f(i) + \sum_{j=c+1}^{n} f(j)$. The expression in this statement is the special case where $c = \lfloor \frac{n}{2} \rfloor$

c) The expression $\prod_{i=0}^{n} f(i)$ is equal to which of the following expressions (circle all that apply)?

- $f(0) \cdot f(1) \cdot f(2) \cdots f(n-1) \cdot f(n)$

  Answer: Equal. This is simply the product of $f(i)$ for $0 \leq i \leq n$ written in an explicit way.

- $f(n) \prod_{i=0}^{n-1} f(i)$

  Answer: Equal. The product of n terms is simply the product of the first $n-1$ terms multiplied by the last term. This expression is just that.

- $\frac{f(n)}{f(0)}$

  Answer: Not equal. As a counter example take $n = 1$ and let f be the constant 2 function. Then $\prod_{i=0}^{n} f(i) = 2 \times 2 = 4$, but $\frac{f(n)}{f(0)} = 1$.

- $f(n) \prod_{i=0}^{\frac{n}{2}-1} \prod_{j=2i}^{2i+1} f(j)$ (for n even).

  Answer: Equal. Expanding the innermost product we get

  $$f(n) \prod_{i=0}^{\frac{n}{2}-1} f(2i) f(2i+1).$$

  If we now expand the product further we get

  $$f(n)\left((f(0)f(1))(f(2)f(3)) \ldots (f(n-2)f(n-1))\right)$$

  So we can see the expanded product contains every term from $f(0)$ through $f(n-1)$, and the extra $f(n)$ that is multiplied makes the whole product $\prod_{i=0}^{n} f(i)$.

- $\prod_{i=0}^{\lfloor \frac{n}{2} \rfloor} f(i) \prod_{j=\lfloor \frac{n}{2} \rfloor + 1}^{n} f(j)$

  Answer: Equal. The first product is $f(0)f(1)\ldots f(\lfloor \frac{n}{2} \rfloor)$ and the second product is $f(\lfloor \frac{n}{2} \rfloor + 1) f(\lfloor \frac{n}{2} \rfloor + 2) \ldots f(n)$. So multiplied together they give us $f(0) \ldots f(\lfloor \frac{n}{2} \rfloor) f(\lfloor \frac{n}{2} \rfloor + 1) \ldots f(n)$ which is just $\prod_{i=0}^{n} f(i)$.

This study resource was shared via CourseHero.com

https://www.coursehero.com/file/12305819/cs70sp15sol1/

2. Propositions

For each of the following propositions: (a) Explain in English what the statement says. (b) Negate the proposition and simplify (c) State whether the resulting (negated) proposition is true or false.

Answer: In order to negate these statements, we use the facts that $\neg\exists xP \equiv \forall x\neg P$ and $\neg\forall xP \equiv \exists x\neg P$ to move the negation inside (each time changing the quantifier), and once the negation is past all of the quantifiers, we use De Morgan's laws to appropriately change the conjunction or disjunction and simplify.

(a) $(\forall x \in \mathbf{Z})((x < \sqrt{48}) \vee (x > 7))$

Answer: The statement reads "for all integers x, x is less than $\sqrt{48}$ or x is greater than 7." The negation is derived below.

$$\begin{aligned}\neg(\forall x \in \mathbf{Z})((x < \sqrt{48}) \vee (x > 7)) &\equiv (\exists x \in \mathbf{Z})\neg((x < \sqrt{48}) \vee (x > 7))\\ &\equiv (\exists x \in \mathbf{Z})(\neg(x < \sqrt{48}) \wedge \neg(x > 7))\\ &\equiv (\exists x \in \mathbf{Z})((x \geq \sqrt{48}) \wedge (x \leq 7))\end{aligned}$$

Note that often in math, $(x \geq \sqrt{48}) \wedge (x \leq 7)$ is simply written as $\sqrt{48} \leq x \leq 7$. So the statement becomes $(\exists x \in \mathbf{Z})(\sqrt{48} \leq x \leq 7)$. The resulting proposition is true. To prove it we just need to provide an example x. Take $x = 7$. Then $x \leq 7$ and $x = \sqrt{49} \geq \sqrt{48}$.

(b) $(\forall x \in \mathbf{R})(\forall y \in \mathbf{R})(\exists z \in \mathbf{R})(x < z < y)$

Answer: The statement reads "for all real numbers x and y there exists a real number z such that z is greater than x and less than y." The negation is derived below.

$$\begin{aligned}\neg(\forall x \in \mathbf{R})(\forall y \in \mathbf{R})(\exists z \in \mathbf{R})(x < z < y) &\equiv (\exists x \in \mathbf{R})\neg(\forall y \in \mathbf{R})(\exists z \in \mathbf{R})(x < z < y)\\ &\equiv (\exists x \in \mathbf{R})(\exists y \in \mathbf{R})\neg(\exists z \in \mathbf{R})(x < z < y)\\ &\equiv (\exists x \in \mathbf{R})(\exists y \in \mathbf{R})(\forall z \in \mathbf{R})\neg(x < z < y)\\ &\equiv (\exists x \in \mathbf{R})(\exists y \in \mathbf{R})(\forall z \in \mathbf{R})((z \leq x) \vee (z \geq y))\end{aligned}$$

Note that in the last part we implicitly rewrote $x < z < y$ as $x < z \wedge z < y$, because $x < y < z$ is just a convenient notation for the latter. The resulting proposition is true. We just need to provide an example x and y such that $(\forall z \in \mathbf{R})((z \leq x) \vee (z \geq y))$. We can simply take $x = y = 0$. Then $(\forall z \in \mathbf{R})((z \leq 0) \vee (z \geq 0))$, because every real number z is either nonnegative or nonpositive.

(c) $(\forall x \in \mathbf{Z})(\exists y \in \mathbf{Z})(x - y > 16)$

Answer: The statement reads "for every integer x there exists an integer y which is at more than 16 units less than x." The negation is derived below.

$$\begin{aligned}\neg(\forall x \in \mathbf{Z})(\exists y \in \mathbf{Z})(x - y > 16) &\equiv (\exists x \in \mathbf{Z})\neg(\exists y \in \mathbf{Z})(x - y > 16)\\ &\equiv (\exists x \in \mathbf{Z})(\forall y \in \mathbf{Z})\neg(x - y > 16)\\ &\equiv (\exists x \in \mathbf{Z})(\forall y \in \mathbf{Z})(x - y \leq 16)\\ &\equiv (\exists x \in \mathbf{Z})(\forall y \in \mathbf{Z})(y \geq x - 16)\end{aligned}$$

This statement is false. To disprove it, we can prove the original statement (without the negation). If we take any integer x, then we can let y be $x - 17$ (which is an integer). But then $x - y = 17 > 16$. So the original statement is true.

(d) $(\exists x \in \mathbf{R})(\forall y \in \mathbf{R})(xy > 1)$

This study resource was shared via CourseHero.com
https://www.coursehero.com/file/12305819/cs70sp15sol1/

Answer: The statement reads "there is a real number x whose product with all real numbers y is greater than 1." The negation is derived below.

$$\begin{aligned}\neg(\exists x \in \mathbf{R})(\forall y \in \mathbf{R})(xy > 1) &\equiv (\forall x \in \mathbf{R})\neg(\forall y \in \mathbf{R})(xy > 1)\\ &\equiv (\forall x \in \mathbf{R})(\exists y \in \mathbf{R})\neg(xy > 1)\\ &\equiv (\forall x \in \mathbf{R})(\exists y \in \mathbf{R})(xy \leq 1)\end{aligned}$$

The resulting proposition is true. Because for every x we can simply pick y to be 0, and then $xy = 0 \leq 1$.

(e) $(\forall y \in \mathbf{R})(\exists x \in \mathbf{R})(xy > 1)$

Answer: The statement read "for every real number y, there is a real number x whose product with y is greater than 1." The negation is derived below.

$$\begin{aligned}\neg(\forall y \in \mathbf{R})(\exists x \in \mathbf{R})(xy > 1) &\equiv (\exists y \in \mathbf{R})\neg(\exists x \in \mathbf{R})(xy > 1)\\ &\equiv (\exists y \in \mathbf{R})(\forall x \in \mathbf{R})\neg(xy > 1)\\ &\equiv (\exists y \in \mathbf{R})(\forall x \in \mathbf{R})(xy \leq 1)\end{aligned}$$

The statement is true. We can take y to be 0. Then for any real number x, we have $xy = 0 \leq 1$.

3. Propositional logic and Boolean circuits
The exclusive OR (written as XOR or $\oplus$) is just what it sounds like: $P \oplus Q$ is true when exactly one of P, Q is true.

(a) Show, using a truth table, that $P \oplus Q$ is equivalent to $(P \vee Q) \wedge \neg(P \wedge Q)$.

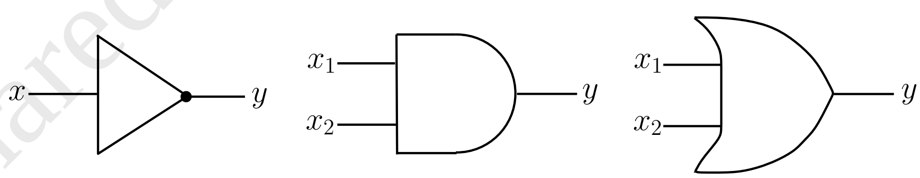
Answer: The truth tables can be found below.

P	Q	$P \vee Q$	$P \wedge Q$	$\neg(P \wedge Q)$	$(P \vee Q) \wedge \neg(P \wedge Q)$
false	false	false	false	true	false
false	true	true	false	true	true
true	false	true	false	true	true
true	true	true	true	false	false

P	Q	$P \oplus Q$
false	false	false
false	true	true
true	false	true
true	true	false

The truth tables result in the same truth values. Therefore the two expressions are equivalent.

(b) Logic is key to many fundamental areas of computer science such as digital circuits. Below you can see the symbols for some of the common gates used in digital circuits. The wires all carry boolean values (true or false), and the ones coming in from the left are *input*s and the ones exiting from the right are *output*s.



These gates, from left to right, are NOT, AND, and OR. Below you can find their truth tables

x	y
false	true
true	false

x_1	x_2	y
false	false	false
false	true	false
true	false	false
true	true	true

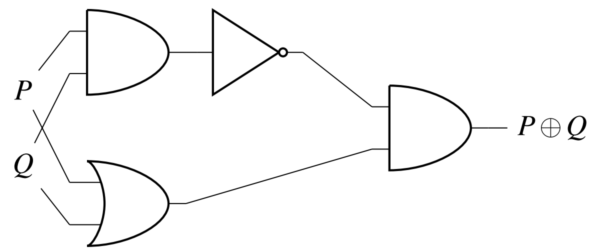
x_1	x_2	y
false	false	false
false	true	true
true	false	true
true	true	true

This study resource was shared via CourseHero.com

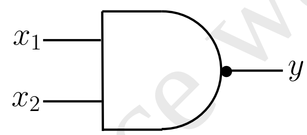
https://www.coursehero.com/file/12305819/cs70sp15sol1/

Using these logical gates, implement XOR. Assume that two input wires P and Q are given to you. Connect them using AND, OR, and NOT gates and produce an output wire whose value is always $P \oplus Q$. Draw the circuit you designed using the standard symbols for the gates.

Answer: We simply need to implement the expression from the previous part, which uses only $\neg, \vee, \wedge$ which correspond directly to NOT, OR, and AND gates. Below you can find the diagram corresponding to such implementation.



(c) (Extra credit) Another common gate, most readily available in hardware chips, is the NAND gate, which can be thought of as an AND whose output is inverted. Below you can find the symbol and the truth table for the NAND gate.



x_1	x_2	y
false	false	true
false	true	true
true	false	true
true	true	false

Implement XOR using the minimal number of NAND gates. You may only use NAND gates, and no other gate.

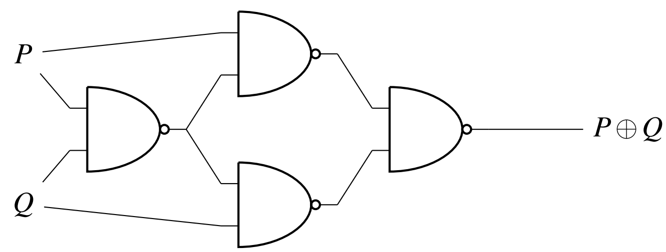
Answer: We can implement each of the AND, OR, and NOT gates using NAND gates. But that does not necessarily use the minimum number of NAND gates. One can simply try to construct the truth tables for expressions built from just the NAND gate. Below you can find some of them.

P	Q	$\text{NAND}(P,Q)$	$\text{NAND}(P,P)$	$\text{NAND}(Q,Q)$	$\text{NAND}(P,\text{NAND}(P,Q))$
false	false	true	true	true	true
false	true	true	true	false	true
true	false	true	false	true	false
true	true	false	false	false	true

We can continue to write down the expressions that use at most two NANDs. Then we can see which ones combine to give us $P \oplus Q$. If we write down the expression $\text{NAND}(Q,\text{NAND}(P,Q))$ and take its NAND with $\text{NAND}(P,\text{NAND}(P,Q))$ we will get $P \oplus Q$. Note that this can be actually implemented with 4 NANDs because these two expressions share the $\text{NAND}(P,Q)$ part. A little bit of search shows that using 3 NANDs we cannot implement the truth table for $P \oplus Q$. The diagram below depicts the answer with 4 NANDs.

This study resource was shared via CourseHero.com

https://www.coursehero.com/file/12305819/cs70sp15sol1/



4. Portia's Caskets

In Shakespeare's Merchant of Venice, Portia had three caskets, gold, silver and lead, inside one of which was her portrait. Her suitor was asked to choose one of the caskets, and if he chose the portrait, he could claim Portia as his bride.

Here is Portia's casket test again with a twist. Suppose there are two casket makers, Bellini and Cellini. Bellini always writes true statements on his caskets, while Cellini always writes false ones.

(a) Suppose one of the caskets contains, not a portrait, but a dagger, and the suitor's job is to avoid choosing that casket. The caskets are inscribed as follows:

Gold: The dagger is in this casket

Silver: This casket is empty.

Lead: At most one of these three caskets was made by Bellini.

Which casket should the suitor choose? Give a proof by cases. A clever choice of cases will greatly reduce the number of cases you have to consider.

Answer: The suitor should choose the lead casket. We will give a proof by cases that this casket does not contain the dagger.

There are two cases.

i. The lead casket was made by Bellini.
ii. The lead casket was made by Cellini.

In the first case, the statement on the lead casket must be true. So at most one casket must be made by Bellini. But we know the lead casket is made by Bellini, so this means the other two are made by Cellini. So their statements are false. Therefore the statement that "the silver casket is empty" is false, which means that the silver casket contains the dagger. So lead and gold are are both safe choices in this case.

In the second case, the statement on the lead casket must be false. So more than one (i.e. at least two) caskets must be made by Bellini. We know lead is not one of them. So the other two caskets must both be made by Bellini. But the statement on the gold casket says "the dagger is in this casket", which must be true. So the gold casket contains the dagger which means that lead and silver are both safe choices.

Since lead was a safe choice in both cases, the suitor can safely choose lead.

(b) Suppose Portia placed her portrait in one of the caskets and the suitor's job is to select the casket with the portrait and to determine the maker of each of the three caskets. The caskets are inscribed as follows:

Gold: The portrait is in here.

Silver: The portrait is in here.

Lead: At least two of these caskets were made by Cellini.

This study resource was shared via CourseHero.com

https://www.coursehero.com/file/12305819/cs70sp15sol1/

How should the suitor answer to pass the test? Prove your answer. What type of proof technique did you use?

Answer: We will first prove that the lead casket is made by Bellini. We use proof by contradiction for this part. Assuming the negative, that the lead casket was made by Cellini, it means that the statement on it must be false. So less than two (i.e. at most one) casket is made by Cellini. But lead is one of them, so the other two must be made by Bellini. But the statements on the gold and silver casket cannot both be true, because the portrait cannot be in both of them. So we arrived at a contradiction.

Now that we have proved lead is made by Bellini, we see that its statement must be true, which means that at least two caskets are made by Cellini and lead is not one of them. So both gold and silver must be made by Cellini which means that they are both false. So gold and silver do not contain the portrait, and hence the portrait must be in the lead casket.

We used proof by contradiction for the first part of the proof, and a direct proof for the second part.

5. Proof by?
Prove that if $x, y \in \mathbb{Z}$, if 6 does not divide xy, then 6 does not divide x and 6 does not divide y. In notation: $(\forall x, y \in \mathbb{Z})\ 6 \nmid xy \implies (6 \nmid x \ \wedge\ 6 \nmid y)$. What proof technique did you use?

Answer: We will use proof by contraposition. For any arbitrary given x and y, the statement $6 \nmid xy \implies (6 \nmid x \wedge 6 \nmid y)$ is equivalent using contraposition to $\neg(6 \nmid x \wedge 6 \nmid y) \implies \neg(6 \nmid xy)$. Moving the negations inside, this becomes equivalent to $(6 \mid x \vee 6 \mid y) \implies 6 \mid xy$.

Now for this part, we give a proof by cases. Assuming that $6 \mid x \vee 6 \mid y$, one of the two cases must be true.

(a) $6 \mid x$: in this case $x = 6k$ for some $k \in \mathbb{Z}$. Therefore $xy = 6ky$ which is a multiple of 6. So $6 \mid xy$.

(b) $6 \mid y$: in this case $y = 6k$ for some $k \in \mathbb{Z}$. Therefore $xy = 6kx$ which is a multiple of 6. So $6 \mid xy$.

Therefore assuming $6 \mid x \vee 6 \mid y$ we proved $6 \mid xy$.

We used proof by cases and proof by contraposition.

6. Implication
Let $C(x)$, $S(x)$ and $E(x)$ be the statements "x is a clear explanation," "x is satisfactory," and "x is an excuse," respectively. Suppose that the universe of discourse for x is the set of all English text. Express each of the following statements using quantifiers, logical connectives, and $C(x)$, $S(x)$ and $E(x)$.

(a) All clear explanations are satisfactory

Answer: $\forall x(C(x) \implies S(x))$. The statement reads as "for any English text, if it is clear then it is satisfactory." Note that this statement does not say anything about English texts that are not clear; it only says those that are clear are satisfactory.

(b) Some excuses are unsatisfactory

Answer: $\exists x(E(x) \wedge \neg S(x))$. The statement reads as "there is some English text which is an excuse and is unsatisfactory." Note the difference between this part and the previous part. When talking about all x that satisfy a property, we put that property behind an implication. But when talking about the existence of some x that satisfy a property, we put that property behind an and.

This study resource was shared via CourseHero.com

https://www.coursehero.com/file/12305819/cs70sp15sol1/

(c) Some excuses are not clear explanations

Answer: $\exists x(E(x) \wedge \neg C(x))$. The statement reads as "there is some English text which is an excuse and is not clear." Similar to the previous part we were faced with $\exists$, so we put the property describing the English text (being an excuse) behind an and.

Does (c) follow from (a) and (b)? If not, is there a correct conclusion?

Answer: Yes. Part (c) follows from (a) and (b). Part (b) says that $\exists x(E(x) \wedge \neg S(x))$. Let x_0 be an example, i.e. let x_0 be such that $E(x_0) \wedge \neg S(x_0)$. So we know $\neg S(x_0)$. But from (a), and using contraposition we get that $\neg S(x_0) \implies \neg C(x_0)$. So we know that $\neg C(x_0)$ is true. But we also know $E(x_0)$. So $E(x_0) \wedge \neg C(x_0)$ is true. Therefore $\exists x(E(x) \wedge \neg C(x))$ is true.

7. Winning
A two person game is a game like checkers or go, in which two players Amy and Bob alternately take turns making moves; first Amy takes a turn and then Bob takes a turn and then Amy and Bob and so on until the game ends and one of them is declared the winner, or the game ends in a tie. We say that Amy has a winning strategy if no matter what moves Bob makes (no matter how clever he is), Amy can always win. Similarly Bob has a winning strategy if he can always force a win, no matter how Amy plays. Another possibility is that Amy can guarantee that she does not lose (no matter how clever is, she either ties or wins). Remarkably, it can be proved that for any two person game, either Amy has a winning strategy or Bob has a winning strategy or both can guarantee a tie. (What does this say about chess?) In this question we will prove this result in a simplified setting: we consider only games which never result in ties, and we assume that the number of rounds of the game is a fixed number n.

(a) Assume that the game consists of two turns. First, Amy takes a turn, and then Bob, and then one of them is declared the winner. Use quantifiers to write the statement that Amy has a winning strategy. Write the statement that Bob has a winning strategy. Prove that either Amy has a winning strategy or Bob does.

Answer: Amy has a winning strategy if she has a move x that wins against any move that Bob makes in response. This statement can be written as a proposition.

$$(\exists x \in \text{Amy's moves})(\forall y \in \text{Bob's moves})(\text{Amy wins the game whose moves are } x, y)$$

On the other hand Bob has a winning strategy if he can find a response move that wins him the game for every move that Amy can make. It can be written as a proposition.

$$(\forall x \in \text{Amy's moves})(\exists y \in \text{Bob's moves})(\text{Bob wins the game whose moves are } x, y)$$

We will show that the second proposition is the negation of the first one. Therefore exactly one of them must be true. If we negate the first statement, and move the negation inside, using the rules of negation and quantifiers, all quantifiers get flipped. So the negation of the first proposition becomes:

$$(\forall x \in \text{Amy's moves})(\exists y \in \text{Bob's moves})\neg(\text{Amy wins the game whose moves are } x, y)$$

But because every game is won by either Amy or Bob, Amy not winning is equivalent to Bob winning. So we can further simplify this to

$$(\forall x \in \text{Amy's moves})(\exists y \in \text{Bob's moves})(\text{Bob wins the game whose moves are } x, y)$$

which is exactly the statement that Bob has a winning strategy.

This study resource was shared via CourseHero.com

https://www.coursehero.com/file/12305819/cs70sp15sol1/

(b) (Extra credit)
Prove that if the game consists of n turns for an arbitrarily large value for n, still either Amy has a winning strategy or Bob does.

Answer: In general, if we show Amy's moves by $x_1,\ldots,x_n$ and Bob's moves by $y_1,\ldots,y_n$, then Amy having a winning strategy can be written as

$$\exists x_1 \forall y_1 \exists x_2 \forall y_2 \ldots \forall y_n (\text{Amy wins with moves } x_1, y_1, \ldots, x_n, y_n).$$

And Bob having a winning strategy can be written as

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 \ldots \exists y_n (\text{Bob wins with moves } x_1, y_1, \ldots, x_n, y_n).$$

Again it can be seen using the rules of negation and quantifiers that the second statement is the negation of the first one. So exactly one of the two is true.

To explain further why we could write the propositions we wrote, take Amy's proposition for example. Once Amy and Bob each make a move, the remaining game has $n-1$ rounds. Now assuming we know how to write the statement that Amy has a winning strategy in an $n-1$ round game, we can write that statement for the n round game as

$$\exists x_1 \forall y_1 (\text{Amy has a winning strategy in the remaining part of the game}).$$

Because a strategy for Amy would be a first move, that guarantees no matter what Bob does, leads to a winning strategy for Amy in the remaining part of the game. So if we further expand the inside expression, on and on, we get what we wrote at first.

8. Extra Credit

There are two doors, one leads to heaven and one leads to hell. There are two guards, one for each door. One always lies and the other always tells the truth. You are allowed to ask one of the guards a single question and must determine which door is which. What question would you ask?

Answer: We can pick one of the guards, point to a door, and ask "if I asked the other guard whether this door leads to heaven would he say yes?" The door leads to heaven if and only if the answer is no.

To prove this, we use proof by cases.

(a) The guard we asked the question from is the liar. In this case, the other guard would have told us the truth, so the other guard would have said yes to "whether this door leads to heaven" if and only if that door led to heaven. But since we are asking the liar, the answer will get flipped. So the door leads to heaven if and only if the answer is no.

(b) The guard we asked the question from is the truth-teller. In this case, the other guard would have answered "whether this door leads to heaven" with no if and only if it led to heaven. Since we are asking the truth-teller, we get this directly from him, so the final answer is no if and only if the door leads to heaven.

This study resource was shared via CourseHero.com
https://www.coursehero.com/file/12305819/cs70sp15sol1/