Homework 5

Due: Monday, July 28, 11:59pm

CS 70: Discrete Mathematics and Probability Theory, Summer 2014

1. **More counting!** Leave your answers as tidy expressions involving factorials, binomial coefficients etc., rather than evaluating them as decimal numbers. Also, you should explain clearly how you arrived at your answers; bald solutions with no explanation will receive no credit.

1a. Two identical decks of 52 cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?

Answer: There are 104! permutations of the whole deck, and each of the 52 cards has two permutations (since there are two idential cards in this double-deck). Therefore there are $104!/2^{52}$ ways to order the deck.

1b. How many 66-bit strings are there that contain more ones than zeros?

Answer: One approach is to consider all strings with 34 or more ones, so there are $\sum_{i=34}^{i=66} \binom{66}{i}$ bitstrings. Another approach is to consider that all bitstrings have either more ones, more zeros, or an equal amount of each, and the number with more ones is equal to the number with more zeros. The number of bitstrings is 2^{66}, excluding bitstrings with 33 ones is $2^{66} - \binom{66}{33}$, and dividing by 2 yields $\dfrac{2^{66} - \binom{66}{33}}{2}$.

1c. We have 8 balls, numbered 1 through 8, and 24 bins. How many different ways are there to distribute these 8 balls among the 24 bins?

Answer: The balls are distinct. Each ball can go into any of the 24 bins, so there are 24^8 different distributions.

1d. How many different ways are there to throw 8 identical balls into 24 bins?

Answer: The balls are indistinguishable. There are $\binom{24+8-1}{8} = \binom{31}{8}$ different ways.

1e. We throw 8 identical balls into 5 bins. How many different ways are there to distribute these 8 balls among the 5 bins such that no bin is empty?

Answer: Since no bin is empty, there is at least one ball in each bin, and three balls remain after that. Therefore there are $\binom{7}{3}$ ways to distribute the balls as described.

1f. There are 30 students currently enrolled in a class. How many different ways are there to pair up the 30 students, so that each student is paired with one other student?

Answer: There are many ways to solve this problem. Here are three. **Solution 1:** This problem can be thought of as throwing 30 distinct balls (students) into 15 indistinguishable bins (pairs) such that each bin has 2 balls. There are 15! permutations of bins and $\frac{30!}{2^{15}}$ ways to throw the balls into bins as described, so there are $\frac{30!}{2^{15}15!}$ different pairings total.

Solution 2: Another way to think of the problem is by randomly selecting 15 students to send to Mars, then matching each student on Earth with a student on Mars. There are $\binom{30}{15}$ ways to select students to send to Mars. There are 15! ways to pair the students on Earth each with a student on Mars. However, this counts matching Alice on Mars with Bob on Earth as different from matching Bob on Mars with Alice on Earth. For each of the 15 students, we counted twice, so we need to divide by 2^{15}. Our final answer is $\binom{30}{15}\frac{15!}{2^{15}}$, which is indeed equal to $\frac{30!}{2^{15}15!}$.

Solution 3: One more way to think of this problem is to order the students arbitrarily (say by height). Randomly match the tallest student with one of the remaining 29 students. Then randomly match the tallest remaining student with one of the remaining 27 students. Continue this until all students are matched. The final answer is $29 \cdot 27 \cdot 25 \cdot \ldots \cdot 3 \cdot 1$. You can check that this is equal to the other answers.

2. **Oops! I Did It Again** Let p be a prime number and let k be a positive integer.

2a. We have an endless supply of beads. The beads come in k different colors. All beads of the same color are indistinguishable. We have a piece of string. We want to make a pretty decoration by threading p many beads onto the string (from left to right). We can choose any sequence of colors, subject only to one rule: the p beads must not all have the same color. How many different ways are there to construct such a sequence of beads?

Answer: Without the color restriction, each of the p beads represents a choice of k colors, so there are k^p total combinations. There are k ways to make a string with all beads of one color, so there are $k^p - k$ ways to make a string without all beads of one color.

2b. Now we tie the two ends of the string together, forming a circular necklace. This lets us freely rotate the beads around the necklace. We'll consider two necklaces equivalent if the sequence of colors on one can be obtained by rotating the beads on the other. (For instance, if we have $k = 3$ colors — red (R), green (G), and blue (B) — then the length $p = 5$ necklaces RGGBG, GGBGR, GBGRG, BGRGG, and GRGGB are all equivalent, because these are cyclic shifts of each other.) Count how many non-equivalent necklaces there are, if the p beads must not all have the same color. (Hint: What can you conclude if rotating all the beads on a necklace to another position produces an identical looking necklace? Recall that p is prime.)

Answer: Since p is prime, the only way for any two non-monochromatic necklaces in a group to be identical is by rotating the beads p times. Let's prove this. Suppose that you could rotate the beads some minimal number m times with $0 < m < p$ such that the resulting necklace is exactly identical to the original. Since $m < p$, we know that $p = qm + r$ for some positive q and nonnegative $r < m$ (this is division with a remainder). Since p rotations generates the same necklace and qm rotations generates the same necklace, r rotations must also generate the same necklace. Since $r < m$ and m is the minimal number of steps to generate an identical necklace, $r = 0$. This means that $p = qm$ for some positive integer q, or in other words $m|p$. Since p is prime, either $m = p$ or $m = 1$. If $m = 1$ the necklace would be monochromatic, and thus we have shown that the smallest number of rotations to generate the original necklace is p. This means that every group of identical non-monochromatic necklaces will have exactly p necklaces in that group. This means that there are $\dfrac{k^p - k}{p}$ different necklaces that don't use all beads of one color.

2c. How can you use the above reasoning to give a new proof of Fermat's Little Theorem? (Recall that the theorem says that if $a \not\equiv 0 \pmod{p}$ then $a^{p-1} \equiv 1 \pmod{p}$.)

Answer: We just determined that $\dfrac{k^p - k}{p}$ is an integer number for a prime p and arbitrary integer k (since there can't be a fraction amount of necklaces). Let's let c be that integer, so $\dfrac{k^p - k}{p} = c$. Multiplying by p, we get $k^p - k = cp$, or $k^p - k \equiv 0 \pmod{p}$. If $k \neq 0$, we can divide by it, and get $k^{p-1} - 1 \equiv 0 \pmod{p}$. Adding one to both sides results in exactly Fermat's Little Theorem, thus proving it.

3. **Combinatorial Proofs, Part 1** Prove the the following statements using a combinatorial argument. That is, interpret the quantities on either side of the $=$ or $\leq$ as answers to two counting problems, and explain why the equation or inequality in each part holds.

3a. Prove that $\binom{n}{a}\binom{m}{b} \leq \binom{n+m}{a+b}$

Answer: Consider a set S with n elements, and a set T with m elements.

Next consider the set X of ways of choosing a elements from S and b elements from T. The cardinality of this set is $\binom{n}{a}\binom{m}{b}$.

Also consider the set Y of ways of choosing $a+b$ elements from $S+T$, with cardinality $\binom{n+m}{a+b}$.

Consider the subset Z of Y where we first partition $S+T$ into its components S and T, and choose a elements from S and b elements from T. In fact, $Z = X$, and so X is a subset of Y. Therefore $|X| \leq |Y|$ and so $\binom{n}{a}\binom{m}{b} \leq \binom{n+m}{a+b}$.

3b. Prove that $\binom{n}{a}\binom{n-a}{b-a} = \binom{n}{b}\binom{b}{a}$

Answer: We consider a set S with n elements. The left side of the equation describes the number of ways you can partition S into three subsets, S_1, S_2, and S_3, where each element of S appears in exactly one of the S_i's. $\binom{n}{a}$ describes choosing a elements from S and putting them into S_1. Then, $\binom{n-a}{b-a}$ describes choosing $b-a$ elements from the remaining elements $S - S_1$ and putting them in S_2. $a + (b-a) = b$ elements were chosen to go in either S_1 or S_2. The remaining $n-b$ elements go in S_3.

The right side does the same thing in a different way. First, with $\binom{n}{b}$, we choose the b elements that will go in either S_1 or S_2. Then, with $\binom{b}{a}$, we choose which a of those b elements will go in S_1.

3c. Prove that $a(n-a)\binom{n}{a} = n(n-1)\binom{n-2}{a-1}$

Answer: We consider a set S with n elements. We'll partition it into two sets, S_1 and S_2, (where each element of S appears in exactly one of the S_i's—$|S_1| = a$ and $|S_2| = n-a$) and select one special element from each. Both sides of the equation describe doing this in two different ways.

The left side describes partitioning S and then choosing the two special elements. $\binom{n}{a}$ describes choosing the a elements to go in S_1, a is choosing the special element in S_1, and $n-a$ is choosing the special element in S_2.

The right side describes choosing the special element in S_1, (n) choosing the special element in S_2, ($n-1$) then choosing the remaining $a-1$ elements to go in S_1 from the remaining $n-2$ elements. ($\binom{n-2}{a-1}$)

4. **Combinatorial Proofs, Part 2** Consider the following identity:

$$\binom{2n}{2} = 2\binom{n}{2} + n^2$$

4a. Prove the identity by algebraic manipulation (using the formula for binomial coefficients).

Answer:

$$\begin{aligned}
\binom{2n}{2} &= \frac{(2n)!}{2!(2n-2)!} \\
&= \frac{2n \cdot (2n-1) \cdot (2n-2)!}{2 \cdot (2n-2)!} \\
&= 2 \cdot \frac{n \cdot (2n-1)}{2} \\
&= 2n^2 - n \\
&= n^2 - n + n^2 \\
&= n(n-1) + n^2 \\
&= 2 \cdot \frac{n \cdot (n-1)}{2} + n^2 \\
&= 2\binom{n}{2} + n^2
\end{aligned}$$

4b. Prove the identity using a combinatorial argument.

Answer: The left side represents choosing any 2 objects out of a set of $2n$ objects. The right side represents an arbitrary partition into two groups of n objects. It then chooses any 2 objects out of the first n objects, or chooses any 2 objects out of the second n objects, or chooses one object from the first n and one from the second n objects. The right side is therefore the enumeration of three different ways to choose 2 objects out of $2n$ total objects.

5. **Baby Needs A New Pair Of Shoes** Your friend proposes the following game. She will roll six fair dice. If the number of different numbers that show up is exactly four, then you win \$1 from her. Otherwise, she wins \$1 from you. Would you play this game? Justify your answer with a calculation. (Be very careful in counting the number of ways that exactly four distinct numbers can show up! You should find that the game is rather finely balanced.)

Answer: We will consider order mattering for this problem. In an order-matters rolling of six fair dice, there are 6^6 equally likely outcomes. We want exactly four numbers to show up. We will see how many ways that can happen without order mattering, then apply order-mattering. First, there are $\binom{6}{4}$ different choices of exactly four numbers showing up. Since all of the dice fall on one of those numbers, and there are two other dice to roll, both the remaining dice must fall on one of the four numbers. Either the two can fall on the same number, or the two can fall on different numbers. There are 4 different ways those two dice can fall on the same number. There are $\binom{4}{2}$ different ways those dice can fall on different numbers. Now let's consider how order affects this. If the two remaining dice had fallen on the same number, there would be $6!/3!$ different orderings for the six dice, since one number appears three times and the others appear once. If the two remaining dice had fallen on different numbers, there would be $6!/(2! \cdot 2!)$ different orderings for the six dice, since two numbers appear twice each and the others appear once. This means that there are $\binom{6}{4} \cdot (4 \cdot 6!/3! + \binom{4}{2} \cdot 6!/(2! \cdot 2!))$ different order-matters ways to roll exactly four different numbers with six dice. Dividing the two numbers gives us the probability of rolling exactly four numbers with six dice, which is about 0.501543. Since 0.501543 is greater than 50%, you'll end up on winning some money if you play this game long enough (though with the odds so close to 50%, you might as well get a job instead to guarantee some income).

6. **Balls in Boxes** You have two boxes. The first box contains two identical red balls and one blue ball, and the second box contains one red ball and two identical blue balls. You pick the first box with probability $1/4$ and the second box with probability $3/4$, and then you draw two balls uniformly at random without replacement from your chosen box. That is, you pull out one ball at random, and then pull out another ball at random from the same box, without putting the first ball back.

   6a. Compute the probability of each outcome for this experiment. An outcome should specify which box was chosen, the color of the first ball, and the color of the second ball. (There are eight different outcomes, but you should find that two of those eight have zero probability.)

   Answer: Let's make a table.

Box	Ball 1	Ball 2	Calculations	Probability
1	r	r	$\frac{1}{4} \cdot \frac{2}{3} \cdot \frac{1}{2}$	$\frac{1}{12}$
1	r	b	$\frac{1}{4} \cdot \frac{2}{3} \cdot \frac{1}{2}$	$\frac{1}{12}$
1	b	r	$\frac{1}{4} \cdot \frac{1}{3} \cdot \frac{2}{2}$	$\frac{1}{12}$
1	b	b	$\frac{1}{4} \cdot \frac{1}{3} \cdot 0$	0
2	r	r	$\frac{3}{4} \cdot \frac{1}{3} \cdot 0$	0
2	r	b	$\frac{3}{4} \cdot \frac{1}{3} \cdot \frac{2}{2}$	$\frac{1}{4}$
2	b	r	$\frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{2}$	$\frac{1}{4}$
2	b	b	$\frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{2}$	$\frac{1}{4}$

6b. Compute Pr[the two chosen balls have different colors].

Answer: Adding values from the table:

$$\Pr[\text{the two chosen balls have different colors}] = \frac{1}{12} + \frac{1}{12} + \frac{1}{4} + \frac{1}{4} = \frac{2}{3}$$

7. **Grace Hopper Book Club** Suppose you arrange 12 different books on a bookshelf, uniformly at random. Three of the books are about discrete math, four of the books are about data structures, and the other five are about underwater basket weaving. What is the probability that the three discrete math books are all together?

   Answer: There are 12! possible ways to arrange the 12 books. Now, there are $12-3+1 = 10$ different places the discrete math books could go (positions 1,2,3, positions 2,3,4, all the way until positions 10,11,12). For each of those 10 places, there are 3! ways to arrange the discrete math books and 9! ways to arrange the other 9 books in the 9 remaining spots. So, in total, there are $10 \cdot 3! \cdot 9!$ ways of arranging the books so that the 3 discrete math books are together. So the probability we get what we want is $\frac{10 \cdot 3! \cdot 9!}{12!} \approx .0455$

8. **Dice Probabilities** We roll two fair 6-sided dice. Each one of the 36 possible outcomes is assumed to be equally likely.

   8a. Find the probability that doubles were rolled.

8b. Given that the roll resulted in a sum of 4 or less, find the conditional probability that doubles were rolled.

8c. Given that the two dice land on different numbers, find the conditional probability that at least one die is a 6.

Answer:

(a) $\Pr[\text{doubles}] = 6 \cdot \Pr[\text{two 1's}] = \frac{6}{36} = \boxed{\mathbf{\frac{1}{6}}} \approx .167$

(b) $\Pr[\text{doubles} \,|\, \text{sum} \leq 4] = \frac{\Pr[\text{sum}\leq 4 \,|\, \text{doubles}] \cdot \Pr[\text{doubles}]}{\Pr[\text{sum}\leq 4]}$. Each of the 6 doubles is equally likely, and only (1,1) and (2,2) sum to less than 4, so $\Pr[\text{sum} \leq 4 \,|\, \text{doubles}] = \frac{2}{6} = \frac{1}{3}$. Enumerating them, there are 6 ways to roll and get sum$\leq$ 4. So $\Pr[\text{sum} \leq 4] = \frac{6}{36} = \frac{1}{6}$. Now finally, plugging in, we get that $\Pr[\text{doubles} \,|\, \text{sum} \leq 4] = \frac{\left(\frac{1}{3}\right) \cdot \left(\frac{1}{6}\right)}{\left(\frac{1}{6}\right)} = \boxed{\mathbf{\frac{1}{3}}}$

(c) $\Pr[\geq \text{one } 6 \,|\, \text{diff numbers}] = \frac{\Pr[\text{diff numbers} \cap \geq \text{one } 6]}{\Pr[\text{diff numbers}]}$. Simply enumerating out the possibilities, we see that $\Pr[\text{diff numbers} \cap \geq \text{one } 6] = \frac{10}{36} = \frac{5}{18}$ and $\Pr[\text{diff numbers}] = \frac{30}{36} = \frac{5}{6}$. So $\Pr[\geq \text{one } 6 \,|\, \text{diff numbers}] = \frac{\left(\frac{5}{18}\right)}{\left(\frac{5}{6}\right)} = \boxed{\mathbf{\frac{1}{3}}}$.

9. **Fun With Conditional Probability** This problem will give you practice with conditional probability.

9a. I have a bag containing either a \$5 bill (with probability 1/3) or a \$10 bill (with probability 2/3). I then add a \$5 bill to the bag, so it now contains two bills. The bag is shaken, and you randomly draw a bill from the bag (without looking in the bag). Suppose it turns out to be a \$5 bill. If a second student draws the remaining bill from the bag, what is the probability that it too is a \$5 bill? Show your calculations.

9b. Your gambling buddy found a website online where he could buy trick coins that are heads or tails on both sides. He puts three coins into a bag: one coin that is heads on both sides, one coin that is tails on both sides, and one that is heads on one side and tails on the other side. You shake the bag, draw out a coin at random, put it on the table without looking at it, then look at the side that is showing. Suppose you notice that the side that is showing is heads. What is the probability that the other side is heads? Show your calculations.

Answer:

(a) Let's use the law of total probability!
$\Pr[\text{2nd bill is \$5} \,|\, \text{1st bill is \$5}] = \frac{\Pr[\text{1st bill is \$5 and 2nd bill is \$5}]}{\Pr[\text{1st bill is \$5}]}$. Let's look at each of these. $\Pr[\text{1st bill is \$5 and 2nd bill is \$5}] = \Pr[\text{chose \$5 bag}] = \frac{1}{3}$. $\Pr[\text{1st bill is \$5}] = \Pr[\text{1st bill is \$5} \,|\, \text{in the \$5 bag}] \cdot \Pr[\text{in the \$5 bag}] + \Pr[\text{1st bill is \$5} \,|\, \text{in the \$10 bag}] \cdot \Pr[\text{in the \$10 bag}]$ by the law of total probability. This gives $\Pr[\text{1st bill is \$5}] = 1 \cdot \left(\frac{1}{3}\right) + \left(\frac{1}{2}\right) \cdot \left(\frac{2}{3}\right) = \frac{2}{3}$. So finally, we have $\Pr[\text{2nd bill is \$5} \,|\, \text{1st bill is \$5}] = \frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{3}\right)} = \boxed{\mathbf{\frac{1}{2}}}$

(b) The probability is **not** $\frac{1}{2}$ as one might intuitively think! We can realize that there are 3 ways for the side we see to be heads, and 2 of those ways will yield a heads on the other side. Or, we can blindly compute the conditional probability. $\Pr[\text{other side heads} \mid \text{this side heads}] = \frac{\Pr[\text{both sides are heads}]}{\Pr[\text{this side is heads}]} = \frac{(\frac{1}{3})}{(\frac{1}{2})} = \boxed{\mathbf{\frac{2}{3}}}$. Using the law of total probability on $\Pr[\text{this side is heads}]$, conditioning on which coin you picked, will show why you get $\frac{1}{2}$ for it.