CS 70 Discrete Mathematics and Probability Theory
Summer 2016 Dinh, Psomas, and Ye HW 3

Due Monday July 11 at 1:59PM

1. (12 points: 3/3/6) **Planar Graph**

   A simple graph is *triangle-free* when it has no simple cycle of length three.

   (a) Prove for any connected triangle-free planar graph with $v > 2$ vertices and e edges, $e \leq 2v - 4$. *Hint*: Similar to the proof that $e \leq 3v - 6$.

   (b) Show that any connected triangle-free planar graph has at least one vertex of degree three or less.

   (c) Prove by induction on the number of vertices that any connected triangle-free planar graph is 4-colorable. *Hint*: use part b

   Answer:

   (a) Proved in Note 4. Use f to denote the number of faces in the graph, s_i to denote the number of sides on face i, and e to denote the number of edges in the graph. Now, if we sum the s_is we are going to get 2e, because each edge is counted twice, once for the face on its right and once for the face on its left:

   $$\sum_{i=1}^{f} s_i = 2e$$

   Since the planar graph is triangle-free, each face i has $s_i \geq 4$. Therefore :

   $$4f \leq \sum_{i=1}^{f} s_i = 2e$$

   Solve for f and plug into Euler's formula, we get $e \leq 2v - 4$.

   (b) If all the vertices have degree greater than or equal to 4, then e would be greater than or equal to $2v$. However, we have proved that $e \leq 2v - 4$, so there is at least one vertex of degree three or less.

   (c) *Proof* Induction on the number of vertices n.
   Base Case For $n \leq 4$ all vertices can be assigned different colors, so the claim holds true when $n \leq 4$.
   Inductive Hypothesis Connected triangle-free planar graphs of $5 \geq n \leq k$ vertices are 4-colorable.
   Indcutive Step Consider a connected triangle-free planar graph with $k+1$ vertices. We have proved in part (b) that there exists at least one vertex u with degree three or less. Remove u and its adjacent edges from the graph. The new graph is either a connected triangle-free planar graph or a disconnected triangle-free planar graph. If the graph G is connected, by our inductive

hypothesis, G is 4-colorable. Then we add u back to G and assign to u a color different from its neighbors. If the graph is disconnected, each of the connected componet in G has vertices fewer than k. Therefore, by the inductive hypothesis, all the connected componets are 4-colorable. Then we add u back to G, and assign u a color different from all its neighbors. Note that it is always possible to assign a valid color to u because it has at most 3 neighbors.
Therefore, any connected triangle-free planar graph is 4-colorable.

2. (16 points: 5/4/4/3) **Countability**

We say that a set S is **countable** if there is a bijection from $\mathbb{N}$, or a subset of $\mathbb{N}$, to S. Equivalently, S has the same cardinality as $\mathbb{N}$ or a subset of $\mathbb{N}$.

For the following, you may find these facts useful (you should be able to prove them from the definition above, **but you don't need to**).

If there exists a surjection from $\mathbb{N}$ to S, then S is countable. If there exists an injection from S to $\mathbb{N}$, then S is countable.

(a) Given two countable sets A and B, prove that the Cartesian Product $A \times B = \{(a,b) : a \in A, b \in B\}$ is countable.

(Hint: This should be reminiscent of the argument to show why rational numbers are countable.)

(b) Consider $\mathbb{Z}^m$, the set of all m-length vectors with integer elements for some positive integer m. Prove that $\mathbb{Z}^m$ is countable.

(Hint: Use induction.)

Quick note: Length m vectors here means a vector with m elements.

(c) Prove that the countable union of countable sets is countable. I.e, prove the following is countable:

$$A_1 \cup A_2 \cup A_3 \dots$$

where each A_i is countable and there are countably many of them.

(Hint: How does this relate to Cartesian Products?)

(d) If there are countably infinite (also known as denumerable) sets A_i, could the above union still be finite? If yes, explain briefly what kind of sets A_i for which this holds. If not, prove that union is always infinite.

Answer:

(a) First, let's show that $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$ (all pairs of natural numbers) have the same cardinality.

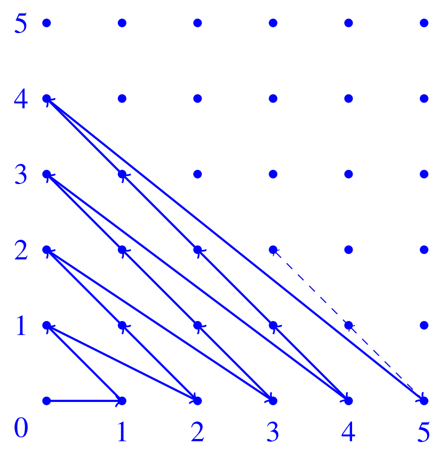
In the lecture notes, an injection was constructed from $\mathbb{Z} \times \mathbb{Z}$ (all pairs of integers) to $\mathbb{N}$ using a spiral map. We notice that $\mathbb{N} \times \mathbb{N}$ is a subset of $\mathbb{Z} \times \mathbb{Z}$ (all pairs of natural numbers are also pairs of integers).

Thus, the same spiral map can be considered, acting just on $\mathbb{N} \times \mathbb{N}$. As in, we consider the same map, but now, the inputs are just taken from $\mathbb{N} \times \mathbb{N}$. The outputs, as before, are still natural numbers. Since the entire spiral map was injective to begin with, even this map must be injective. That is, two distinct pairs of natural numbers must be mapped to distinct natural numbers. Thus, we have constructed an injection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$.

We can construct an injection from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$ very easily. Consider the map which takes every natural number n to $(0,n)$. Since each distinct natural number is mapped to a distinct pair in $\mathbb{N} \times \mathbb{N}$, this is injective.

Thus, from Cantor-Bernstein's theorem, we see that $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$ have the same cardinality.

Another way to show this is to explicitly define a bijection from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$. Consider the diagram below:



This diagram shows us a way to explicitly construct a bijection by walking along the counter diagonals of $\mathbb{N} \times \mathbb{N}$ written as a grid. This, again, shows that we have a bijection from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$. For an explicit formula of this, look at the Cantor pairing function: http://en.wikipedia.org/wiki/Cantor_pairin

Let Ψ be some bijection from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$ (for example, the Cantor pairing function).

Now, since A and B are countable, there must be surjections from $\mathbb{N}$ to A and from $\mathbb{N}$ to B. This is because every bijection is a surjection. If there is a bijection from a subset of $\mathbb{N}$ to A (or B), it can be made into a surjection from $\mathbb{N}$ to A by just making all the elements outside this subset map to the same element in A.

Let these surjections be Φ_A and Φ_B. With these, we can construct a surjection Φ from $\mathbb{N} \times \mathbb{N}$ to $A \times B$. This will just be the individuals surjections Φ_A and Φ_B mapping the two coordinates of an element in $\mathbb{N} \times \mathbb{N}$ independently.

I.e, $\Phi(i, j) = (\Phi_A(i), \Phi_B(j)) \in A \times B$. This is certainly a surjection as every element in $A \times B$ will be mapped to by some natural number under Φ, as Φ_A and Φ_B are surjections.

Thus, finally, we can create the surjective map $f : \mathbb{N} \to A \times B$ such that $f(n) = \Phi(\Psi(n))$. This takes it to the intermediate space of $\mathbb{N} \times \mathbb{N}$ and then maps it to $A \times B$. Since both Ψ and Φ are surjective, we know that the composite map f is also surjective.

Now that we have found a surjection from $\mathbb{N}$ to $A \times B$, this set is countable.

Note: You need not be this detailed in your answer. The important points to note are that we can find a surjection from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$ and then one from $\mathbb{N} \times \mathbb{N}$ to $A \times B$. You can briefly, but precisely construct these.

This allows us to find a composite map which is a surjection from $\mathbb{N}$ to $A \times B$ and thus, this set is countable.

(b) We did all the hard work of showing that the Cartesian product of two countable sets is countable. We can just use induction to show that the set of m-length vectors of elements taken from countable sets is countable.

Base case: When $m = 1$, $\mathbb{Z}^m$ is just $\mathbb{Z}$. This is definitely countable as discussed in the lecture notes.

Inductive hypothesis: Assume that when $m = k$, the set $\mathbb{Z}^k$ is countable.

Inductive step: We want to show that when $m = k+1$, the set $\mathbb{Z}^{k+1}$ is countable.

We know that $\mathbb{Z}$ is a countable set and $\mathbb{Z}^k$ is a countable set. From (a), $\mathbb{Z}^k \times \mathbb{Z}$ must be a countable set. I.e, all pairs where the first element is a $k-$vector of integers and the second element is a single integer is countable. But this is exactly the same as a length $(k+1)-$vector of integers.

If you are uncomfortable with that, we can construct a bijection from $\mathbb{Z}^k \times \mathbb{Z}$ to $\mathbb{Z}^{k+1}$ where the first k elements of the output $(k+1)$-vector are the same as those in the first item of the input pair and the last element in the output vector is the second item in the input pair.

Thus $\mathbb{Z}^{k+1}$ is countable.

So, by induction, we can say that for all positive integers m, the set $\mathbb{Z}^m$ is countable.

(c) The trick here is to re-label the elements in the sets and show that the set of all possible labels is countable. Our labelling should be injective in that, if two elements are distinct, then they must have different labels. Then, we can find a surjective mapping from labels to elements in the union. So, if we show that there is a surjection from $\mathbb{N}$ to the set of labels, we can show that there is a surjection from $\mathbb{N}$ to the elements in the union.

Let's adopt a labelling scheme with two natural-number indices as follows: $a_{i,j}$ is j^{th} element from A_i. The label indices look like a Cartesian product of natural numbers.

We have a small problem with our second index, j. If a countable set is not ordered, what is its j^{th} element? This problem is not present with the first index because our sets are already labelled by natural number indices. So we already know how to find the i^{th} set, from just the way they are defined.

To deal with unordered sets A_i, we can just induce an ordering from the natural numbers. Since each A_i is countable, there must be a bijection from some subset of $\mathbb{N}$ (possibly $\mathbb{N}$ itself) to A_i. We can now number each element in A_i by the natural number which maps to it under this bijection. Thus, we have induced an ordering over each set A_i and referring to the j^{th} element now makes sense.

Our labelling scheme just looks like a grid (like $\mathbb{N} \times \mathbb{N}$) where column i contains the ordered elements of set A_i. Each label is just a coordinate on the grid. Look at the illustration below.

5	$a_{1,5}$	$a_{2,5}$	$a_{3,5}$	$a_{4,5}$	$a_{5,5}$
4	$a_{1,4}$	$a_{2,4}$	$a_{3,4}$	$a_{4,4}$	$a_{5,4}$
3	$a_{1,3}$	$a_{2,3}$	$a_{3,3}$	$a_{4,3}$	$a_{5,3}$
2	$a_{1,2}$	$a_{2,2}$	$a_{3,2}$	$a_{4,2}$	$a_{5,2}$
1	$a_{1,1}$	$a_{2,1}$	$a_{3,1}$	$a_{4,1}$	$a_{5,1}$
	A_1	A_2	A_3	A_4	A_5

Now that we have our completed labelling scheme, let's see if it works. We see that we can represent every single element in every single set A_i uniquely. If any two elements are different, they would be on different coordinates and thus their labels will differ. Since every element is mapped to by some label (each element must appear somewhere on this grid), the map from labels to elements is a surjection. Can two labels map to the same element? Yes, that's possible. This happens if two or more sets share an element. This element would appear multiple times on the grid. But this is okay! We still have a valid surjection from labels to elements.

Now, we can find a surjection from $\mathbb{N}$ to the elements in the union, by using the labels as an intermediate space. the set of all labels is just a set of pairs of natural numbers (i, j). This is a subset of $\mathbb{N} \times \mathbb{N}$. Why is it a subset and not the entire thing? Some sets A_i could be finite. So elements will not exist at some coordinates (and thus, these coordinates will not be labels).

We can certainly find a surjection from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$ (as we did in part (a)). And since our set of labels is a subset of $\mathbb{N} \times \mathbb{N}$, we can easily construct another surjection from this, mapping elements from some subset of $\mathbb{N}$ to the set of labels. (Just take all the set of all elements in $\mathbb{N}$ which map to valid labels; the same surjective as above, with only these elements as input, will be a surjective map onto just the set of labels).

Finally, we have a surjection from $\mathbb{N}$ (or a subset of $\mathbb{N}$) to the set of labels and another surjection from the set of labels to the set of all elements in the union. Thus, the composite map of the two above is a surjection from $\mathbb{N}$ to the elements in the union. And so, the countable union of countable sets is countable.

(d) Yes, it can! It is finite only if there is some natural number N for which the following holds:

$$A_m \subset A_1 \cup A_2 \cup A_3 \dots A_N, \qquad \forall m > N$$

And, of course $A_1 \cup A_2 \cup A_3 \dots A_N$ itself must be finite. This is basically saying that only finite number of unique elements exist in all the sets together.

An example of this would be if each $A_i = \{1, 2, 3, 4, 5\}$. Then the union would be the same set.

3. (20 points:1/1/1/1/2/2/2/2/2/2/2/2) **Countability**

For each of the following sets, determine and briefly explain whether it is finite, countably infinite (i.e., countable and infinite), or uncountably infinite (i.e., uncountable)

(a) $\mathbb{R}$ (the set of all real numbers)

(b) $\mathbb{C}$ (the set of all complex numbers)

(c) $\{0, 1, 2\}^*$ (the set of all finite-length ternary strings)

(d) $\mathbb{Z}^3 = \{(a, b, c) : a, b, c \in \mathbb{Z}\}$ (the set of triples of integers)

(e) $S = \{p(x) : p(x) = ax^2 + bx + c, \text{ where } a, b, c \in \mathbb{Z}\}$ (the set of all polynomials of degree at most 2 with integer coefficients)

(f) $T = \{p(x) : p(x) = a_n x^n + \cdots + a_1 x + a_0, \text{ where } n \in \mathbb{N} \text{ and } a_0, a_1, \dots, a_n \in \mathbb{Z}\}$ (the set of all polynomials with integer coefficients, of any degree)

(g) Numbers that are the roots of nonzero polynomials with integer coefficients.

(h) $U = \{f : \mathbb{N} \to \{0, 1\}\}$ (the set of all functions that map each natural number to 0 or 1)

(i) $V = \{f : \mathbb{N} \to \mathbb{N}\}$ (the set of all functions that map each natural number to a natural number)

(j) Computer programs that halt.

(k) Computer programs that always correctly tell if a program halts or not.

(l) Computer programs that correctly return the product of their two integer arguments.

Answer:

(a) Uncountable. This can be proved using a diagonalization argument, as shown in class. See Note 6.

(b) Uncountable. $\mathbb{R} \subset \mathbb{C}$, and $\mathbb{R}$ is uncountable, so $\mathbb{C}$ must be uncountable too.

(If $\mathbb{C}$ was countable, there would be an enumeration of $\mathbb{C}$; but then we could cross off the numbers in that enumeration that aren't real numbers and obtain an enumeration of $\mathbb{R}$, which is impossible. In general, any subset of a countable set is countable, and any superset of an uncountable set is uncountable.)

(c) Countable and infinite. As explained in Note 6.

(d) Countable and infinite. If S and T are any countable sets, then $S \times T$ is countable. Since $\mathbb{Z}$ is countable, this means $\mathbb{Z} \times \mathbb{Z}$ is countable. That, in turn means that $\mathbb{Z}^3 = (\mathbb{Z} \times \mathbb{Z}) \times \mathbb{Z}$ is countable. Alternatively, we can enumerate the triples in $\mathbb{Z}^3$ by first ordering on the sum of absolute values of the three coordinates, and then breaking ties by lexicographic order on the three coordinates. For any $(x,y,z) \in \mathbb{Z}^3$, there are only a finite number of elements of $\mathbb{Z}^3$ preceding it in this enumeration, and hence its position in the enumeration is a natural number. This enumeration covers every element of $\mathbb{Z}^3$, so $\mathbb{Z}^3$ is countable.

(e) Countable and infinite. We can define a function f from S to $\mathbb{Z}^3$ which takes the degree 2 polynomial $p(x) = ax^2 + bx + c$ to $(a,b,c) \in \mathbb{Z}^3$. It is clear that f is a bijection, and so $|S| = |\mathbb{Z}^3|$. From the previous part, $|\mathbb{Z}^3| = |\mathbb{N}|$ and so $|S| = |\mathbb{N}|$.

(f) Countable and infinite. By a generalization of part (i), for each $d \in \mathbb{N}$ there is a bijective function f_d that maps from all polynomials of degree d to $\mathbb{N}$. Consider the function $g : T \to \mathbb{N}^2$ given by

$$g(a_n x^n + \cdots + a_1 x + a_0) = (n, f_n(a_n x^n + \cdots + a_1 x + a_0)).$$

The function g is a bijection, so $|T| = |\mathbb{N}^2|$. We saw earlier that $\mathbb{N}^2$ is countable, from which it follows that T must be countable as well.

Alternatively, we can enumerate all the polynomials in T by first ordering them by the sum of the degree and the absolute values of the coefficients, i.e., $n + |a_n| + \cdots + |a_1| + |a_0|$, and then by lexicographic order on the vector of coefficients. Any polynomial $p(x)$ in T will then appear at some finite index in the enumeration. Therefore, T is countable.

(g) Countably infinite. Polynomials with integer coefficients themselves are countably infinite. So let us list all polynomials with integer coefficients as $P_1, P_2, \ldots$. We can label each root by a pair (i, j) which means take the polynomial P_i and take its j-th root (we can have an arbitrary ordering on the roots of each polynomial). This means that the roots of these polynomials can be mapped in an injective manner to $\mathbb{N} \times \mathbb{N}$ which we know is countably infinite. So this set is either finite or countably infinite. But every natural number n is in this set (it is the root of $x - n$). So this set is countably infinite.

(h) Uncountable. We can apply diagonalization. Given any hypothesized enumeration $f_0, f_1, f_2, \ldots$ of U, we construct a new function g defined as $g(n) = 1 - f_n(n)$. Then $g \in U$ but g is not in the enumeration, so our hypothesized enumeration must not have been an enumeration after all. Contradiction. Therefore, U cannot be enumerated.

Alternatively, consider the mapping $g : U \to \mathscr{P}(\mathbb{N})$ defined by $g(f) = \{i : f(i) = 1\}$ for $f \in U$. It should be clear that g is a bijection from U to $\mathscr{P}(\mathbb{N})$. Hence, $|U| = |\mathscr{P}(\mathbb{N})|$. Note 6 shows that $\mathscr{P}(\mathbb{N})$ is uncountable.

(i) Uncountable. It's clear that $U \subset V$, where U is defined as in part (k), and we showed in part (k) that U is uncountable. Therefore, V is uncountable too.

(j) Countably infinite. The total number of programs is countably infinite, since each can be viewed as a string of characters (so for example if we assume each character is one of the 256 possible values, then each program can be viewed as number in base 257, and we know these numbers are countably infinite. So the number of halting programs, which is a subset of all programs, can be either finite or countably finite. But there are an infinite number of halting programs, for example for each number i the program that just prints i is different for each i. So the total number of halting programs is countably infinite.

(k) Finite. There is no such program because the halting problem cannot be solved.

(l) Countably infinite. As said previously the set of all programs is countably infinite. So this set is either finite or countably infinite. But it is not finite. Consider a program that multiplies its two integer arguments and add a simple for loop that runs for i steps to its beginning. It does not change the behavior of the program, but it will give us a different program for each i. Therefore this set is countably infinite.

4. (21 points: 1/1/1/1/1/1/1/1/1/1/1/2/2/2/2/2) **Counting, counting, and more counting**

The only way to learn counting is to practice, practice, practice, so here is your chance to do so. We encourage you to leave your answer as an expression (rather than trying to evaluate it to get a specific number).

(a) How many 10-bit strings are there that contain exactly 4 ones?

Answer: This is just the number of ways to choose 4 positions out of 10 positions to place the ones, and so is $\binom{10}{4}$.

(b) How many different 13-card bridge hands are there? (A bridge hand is obtained by selecting 13 cards from a standard 52-card deck. The order of the cards in a bridge hand is irrelevant.)

Answer: We have to choose 13 cards out of 52 cards, so this is just $\binom{52}{13}$.

(c) How many different 13-card bridge hands are there that contain no aces?

Answer: We now have to choose 13 cards out of 48 non-ace cards. So this is $\binom{48}{13}$.

(d) How many different 13-card bridge hands are there that contain all four aces?

Answer: We now require the four aces to be present. So we have to choose the remaining 9 cards in our hand from the 48 non-ace cards, and this is $\binom{48}{9}$.

(e) How many different 13-card bridge hands are there that contain exactly 6 spades?

Answer: We need our hand to contain 6 out of the 13 spade cards, and 7 out of the 39 non-spade cards, and these choices can be made separately. Hence, there are $\binom{13}{6}\binom{39}{7}$ ways to make up the hand.

(f) How many 99-bit strings are there that contain more ones than zeros?

Answer:

Answer 1: There are $\binom{99}{k}$ 99-bit strings with k ones and $99-k$ zeros. We need $k > 99-k$, i.e. $k \geq 50$. So the total number of such strings is $\sum_{k=50}^{99}\binom{99}{k}$.
This expression can however be simplified. Since $\binom{99}{k} = \binom{99}{99-k}$, we have $\sum_{k=50}^{99}\binom{99}{k} = \sum_{k=50}^{99}\binom{99}{99-k} = \sum_{l=0}^{49}\binom{99}{l}$ by substituting $l = 99-k$. Now $\sum_{k=50}^{99}\binom{99}{k} + \sum_{l=0}^{49}\binom{99}{l} = \sum_{m=0}^{99}\binom{99}{m} = 2^{99}$. Hence, $\sum_{k=50}^{99}\binom{99}{k} = \frac{1}{2}2^{99} = 2^{98}$.

Answer 2: Since the answer from above looked so simple, there must have been a more elegant way to arrive at it. Since 99 is odd, no 99-bit string can have the same number of zeros and ones.

Let A be the set of 99-bit strings with more ones than zeros, and B be the set of 99-bit strings with more zeros than ones. Now take any 99-bit string x with more ones than zeros i.e. $x \in A$. If all the bits of x are flipped, then you get a string y with more zeros than ones, and so $y \in B$. This operation of bit flips creates a one-to-one and onto function (called a bijection) between A and B. Hence, it must be that $|A| = |B|$. Every 99-bit string is either in A or in B, and since there are 2^{99} 99-bit strings, we get $|A| = |B| = \frac{1}{2}2^{99}$. The answer we sought was $|A| = 2^{98}$.

(g) How many different anagrams of FLORIDA are there? (An anagram of FLORIDA is any re-ordering of the letters of FLORIDA, i.e., any string made up of the letters F, L, O, R, I, D, and A, in any order. The anagram does not have to be an English word.)

Answer: This is the number of ways of rearranging 7 distinct letters and is $7!$.

(h) How many different anagrams of ALASKA are there?

Answer: In this 6 letter word, the letter A is repeated 3 times while the other letters appear once. Hence, the number $6!$ overcounts the number of different anagrams by a factor of $3!$ (which is the number of ways of permuting the 3 A's among themselves). Hence, there are $\frac{6!}{3!}$ different anagrams.

(i) How many different anagrams of ALABAMA are there?

Answer: In this 7 letter word, the letter A is repeated 4 times while the other letters appear once. Hence, the number $7!$ overcounts the number of different anagrams by a factor of $4!$ (which is the number of ways of permuting the 4 A's among themselves). Hence, there are $\frac{7!}{4!}$ anagrams.

(j) How many different anagrams of MONTANA are there?

Answer: In this 7 letter word, the letter A and N are each repeated 2 times while the other letters appear once. Hence, the number $7!$ overcounts the number of different anagrams by a factor of $2! \times 2!$ (one factor of $2!$ for the number of ways of permuting the 2 A's among themselves and another factor of $2!$ for the number of ways of permuting the 2 N's among themselves). Hence, there are $\frac{7!}{2! \times 2!}$ different anagrams.

(k) If we have a standard 52-card deck, how many ways are there to order these 52 cards?

Answer: The first position of the ordering has 52 choices for the card, the second position has 51 choices, and so on, until the last position where there is only one choice. Thus, there are $52!$ ways to order the deck.

(l) Two identical decks of 52 cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?

Answer: If we consider the $104!$ rearrangements of 2 identical decks, since each card appears twice, we would have overcounted each distinct rearrangement. Consider any distinct rearrangement of the 2 identical decks of 52 cards and see how many times this appears among the rearrangement of 104 cards where each card is treated as different. For each identical pair (such as the two Ace of spades), there are two ways they could be permuted among each other (since $2! = 2$). This holds for each of the 52 pairs of identical cards. So the number $104!$ overcounts the actual number of rearrangements of 2 identical decks by a factor of 2^{52}. Hence, the actual number of rearrangements of 2 identical decks is $\frac{104!}{2^{52}}$.

(m) We have 9 balls, numbered 1 through 9, and 27 bins. How many different ways are there to distribute these 9 balls among the 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).

Answer: Each ball has a choice of which bin it should go to. So each ball has 27 choices and the 9 balls can make their choices separately. Hence, there are 27^9 ways.

(n) We throw 9 identical balls into 7 bins. How many different ways are there to distribute these 9 balls among the 7 bins such that no bin is empty? Assume the bins are distinguishable (e.g., numbered 1 through 7).

Answer:

Answer 1: Since each bin is required to be non-empty, let's throw one ball into each bin at the outset. Now we have 2 identical balls left which we want to throw into 7 distinguishable bins. There are 2 cases to consider:
Case 1: The 2 balls land in the same bin. This gives 7 ways.
Case 2: The 2 balls land in different bins. This gives $\binom{7}{2}$ ways of choosing 2 out of the 7 bins for the balls to land in. Note that it is *not* 7×6 since the balls are identical and so there is no order on them.
Summing up the number of ways from both cases, we get $7 + \binom{7}{2}$ ways.

Answer 2: Since each bin is required to be non-empty, let's throw one ball into each bin at the outset. Now we have 2 identical balls left which we want to throw into 7 distinguishable bins. From class (see notes 10), we already saw that the number of ways to put k identical balls into n distinguishable bins is $\binom{n+k-1}{k}$. Taking $k = 2$ and $n = 7$, we get $\binom{8}{2}$ ways to do this.
EASY EXERCISE: Can you give an expression for the number of ways to put k identical balls into n distinguishable bins such that no bin is empty?

(o) How many different ways are there to throw 9 identical balls into 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).

Answer: Since there is no restriction on how many balls a bin needs to have, this is just the problem of throwing k identical balls into n distinguishable bins, which can be done in $\binom{n+k-1}{k}$ ways. Here $k = 9$ and $n = 27$, so there are $\binom{35}{9}$ ways.

(p) There are exactly 20 students currently enrolled in a class. How many different ways are there to pair up the 20 students, so that each student is paired with one other student?

Answer:

Answer 1: Let's number the students from 1 to 20. Student 1 has 19 choices for her partner. Let i be the smallest index among students who have not yet been assigned partners. Then no matter what the value of i is (in particular, i could be 2 or 3), student i has 17 choices for her partner. The next smallest indexed student who doesn't have a partner now has 15 choices for her partner. Continuing in this way, the number of pairings is $19 \times 17 \times 15 \times \cdots \times 1 = \prod_{i=1}^{10}(2i-1)$.

Answer 2: Arrange the students numbered 1 to 20 in a line. There are 20! such arrangements. We pair up the students at positions $2i-1$ and $2i$ for i ranging from 1 to 10. You should be able to see that the 20! permutations of the students doesn't miss any possible pairing. However, it counts every different pairing multiple times. Fix any particular pairing of students. In this pairing, the first pair had freedom of 10 positions in any permutation that generated it, the second pair had a freedom of 9 positions in any permutation that generated it, and so on. There is also the freedom for the elements within each pair i.e. in any student pair (x, y), student x could have appeared in position $2i-1$ and student y could have appeared in position $2i$ and also vice versa. This gives 2 ways for each of the 10 pairs. Thus, in total, these freedoms cause $10! \times 2^{10}$ of the 20! permutations to give rise to this particular pairing. This holds for each of the different pairings. Hence, 20! overcounts the number of different pairings by a factor of $10! \times 2^{10}$. Hence, there are $\frac{20!}{10! \cdot 2^{10}}$ pairings.

Answer 3: In the first step, pick a pair of students from the 20 students. There are $\binom{20}{2}$ ways to do this. In the second step, pick a pair of students from the remaining 18 students. There are $\binom{18}{2}$

ways to do this. Keep picking pairs like this, until in the tenth step, you pick a pair of students from the remaining 2 students. There are $\binom{2}{2}$ ways to do this. Multiplying all these, we get $\binom{20}{2}\binom{18}{2}\ldots\binom{2}{2}$. However, in any particular pairing of 20 students, this pairing could have been generated in 10! ways using the above procedure depending on which pairs in the pairing got picked in the first step, second step, ..., tenth step. Hence, we have to divide the above number by 10! to get the number of different pairings. Thus there are $\frac{\binom{20}{2}\binom{18}{2}\ldots\binom{2}{2}}{10!}$ different pairings of 20 students.

You may want to check for yourself that all three methods are producing the same integer, even though they are expressed very differently.

5. (11 points: 4/7) **Charm School Applications**

(a) n males and n females apply to the Elegant Etiquette Charm School (EECS) within UC Berkeley. The EECS department only has n seats available. In how many ways can it admit students? Use the above story for a combinatorial argument to prove the following identity:

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2$$

Answer: One way of counting is simply $\binom{2n}{n}$, since we must pick n students from $2n$.
The other way is to first pick i males, then $n-i$ females. Equivalently, choose i males to admit, and i females to NOT admit. For a fixed i, this yields $\binom{n}{i}\binom{n}{n-i} = \binom{n}{i}^2$ choices. Thus, over all choices of i:

$$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2$$

(b) Among the n admitted students, there is at least one male and at least one female. On the first day, the admitted students decide to carpool to school. The boy(s) get in one car, and the girl(s) get in another car. Use the above story for a combinatorial argument to prove the following identity:

$$\sum_{k=1}^{n-1} k \cdot (n-k) \cdot \binom{n}{k}^2 = n^2 \cdot \binom{2n-2}{n-2}$$

(Hint: Each car has a driver...)

Answer: Out of the n males and n females who applied, count the number of ways that accepted students can drive to school.
RHS: First pick one male driver and one female driver from the n male and n female applicants (n^2). Then pick the other $n-2$ accepted students from the pool of $2n-2$ remaining applicants.
LHS: Pick k males and $n-k$ females that were accepted: $\binom{n}{k}\binom{n}{n-k} = \binom{n}{k}^2$. Then pick a driver among the k males, and among the $n-k$ females. Because the problem statement says there is at least 1 girl and 1 boy, k can range from 1 to $n-1$.

6. (13 points: 2/3/4/4) **Getting to CS**

Harry, the chosen one, is chosen to drive the boys to the Charm School. The Flying Ford Anglia he's driving is behaving weirdly – it would only go south or east for at least a certain distance. Figure 1 shows the path the car could go from Harry's house (H) to the Charm School (CS).

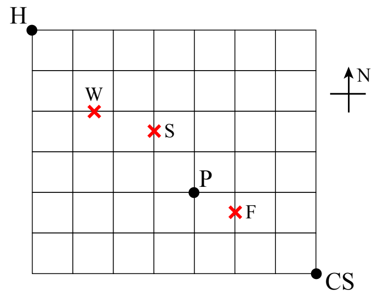


Figure 1: The map from Harry's house to the Charm School

(a) How many ways can he get there?

Answer: Harry needs to go south 6 times and east 7 times. We can think of a path from H to CS as a series of $6+7=13$ moves, 6 of which are going south and 7 of which are going east. Out of 13 total moves, there are $\binom{13}{6}=\binom{13}{7}=1,716$ ways to make such a path. (Either choose 6 out of 13 moves to be going south, or choose 7 out of 13 moves to be going east.). $\square$

(b) Harry has to pick up other students at point P. How many ways can he stop by point P and go to the Charm School?

Answer: Let $P_{x_1\to x_2\to\ldots\to x_n}$ denote the set of paths from x_1 to x_n that pass through $x_2,x_3,\ldots,x_{n-1}$. Then the number of ways Harry can get to CS by passing P is just the number of ways from H to P multiplied by the number of ways from P to CS.

$$\begin{aligned}|P_{H\to P\to CS}| &= |P_{H\to P}||P_{P\to CS}| \\ &= \binom{4+4}{4}\binom{2+3}{3}=\binom{8}{4}\binom{5}{3}=70\cdot 10=700.\end{aligned}$$

$\square$

(c) The Whomping Willow (W) will attack anything that comes near. Harry must not drive through it. How many ways can he pick up the students and then go to the Charm School now?

Answer: Figure 2 labels more points. Notice that point W doesn't affect the paths from P to

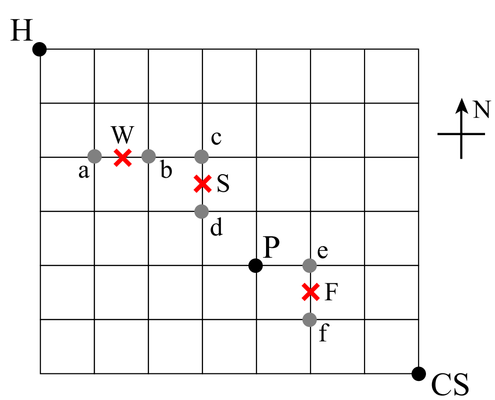


Figure 2: More elaborated map from Harry's house to the Charm School

CS at all, so we can just calculate how it affects the number of possible paths from H to P, then multiply with $|P_{P\to CS}|$. First, we find the total number of paths that pass through W. For any path to pass through W, it must pass point a then point b.

$$|P_{H\to a\to b\to P}| = |P_{H\to a}||P_{b\to P}| = \binom{3}{1}\binom{4}{2} = 3\cdot 6 = 18$$

The number of paths from H to P that don't pass W is just the total number of paths from H to P minus the number of paths that do pass W.

$$|P_{H\to P}| - |P_{H\to a\to b\to P}| = 70 - 18 = 52.$$

Therefore, the total number of paths from H to P to CS that don't pass $W = 52\cdot|P_{P\to CS}| = 52\cdot\binom{5}{3} = 52\cdot 10 = 520.$ □

(d) On top of the Whomping Willow, the Marauder's Map shows Professor Snape (S) and Filch (F) whom he doesn't want to drive past either. How many ways can Harry go to the Charm School without getting past the Whomping Willow (W), Professor Snape (S), or Filch (F), while still picking up other students?

Answer: Again, the path that passes S must pass through points c and d, and the path that passes F must pass through points e and f.

$$|P_{H\to c\to d\to P}| = \binom{5}{3}\binom{2}{1} = 20$$
$$|P_{P\to e\to f\to CS}| = \binom{3}{1} = 3$$

Observe that points W and S only affect the paths from H to P, and F only affects the paths from P to CS. We can calculate these separately.

<u>From H to P:</u> It is a little tricky to find the number of paths that don't pass through both W and S. First, we find the total number of paths that pass through both W and S,

$$|P_{H\to a\to b\to c\to d\to P}| = \binom{3}{1}\binom{2}{1} = 6$$

Next, we find the total number of paths that passes *either* W *or* S,

$$\begin{aligned}|P_{H\to a\to b\to P}\cup P_{H\to c\to d\to P}| &= |P_{H\to a\to b\to P}| + |P_{H\to c\to d\to P}| - |P_{H\to a\to b\to c\to d\to P}| \\ &= 18 + 20 - 6 = 32.\end{aligned}$$

Hence, the number of paths that don't pass W and S, i.e., any of the points a, b, c, and d, is,

$$\begin{aligned}|P_{H\to P}| - |(P_{H\to a\to b\to P}\cup P_{H\to c\to d\to P})| &= |P_{H\to P}| - |P_{H\to a\to b\to P}\cup P_{H\to c\to d\to P}| \\ &= 70 - 32 = 38\end{aligned}$$

<u>From P to CS:</u> The number of paths that don't pass F is $|P_{P\to CS}| - |P_{P\to e\to f\to CS}| = \binom{5}{3} - 3 = 7.$

Therefore, the answer is

$$(|P_{H\to P}| - |(P_{H\to a\to b\to P}\cup P_{H\to c\to d\to P})|)(|P_{P\to CS}| - |P_{P\to e\to f\to CS}|) = 38\cdot 7 = 266.$$

□

7. (7 points: 3/4) **Charming Star**

At the end of each day, students will vote for the most charming student. There are 5 candidates and 100 voters. Each voter can only vote once, and all of their votes weigh the same.

(a) How many possible voting combinations are there for the 5 candidates?

Answer: $\binom{100+5-1}{100}$.

Let x_i be the number of votes of the i-th candidates. We would like to find all possible combinations of $(x_1, x_2, x_3, x_4, x_5)$ such that

$$x_1 + x_2 + x_3 + x_4 + x_5 = 100.$$

It is equivalent to selecting $k = 100$ objects from $n = 5$ categories. The number of possible combinations is:

$$\binom{100+5-1}{100} = \binom{104}{100} = 4598126.$$

(b) How many possible voting combinations are there such that exactly one candidate gets more than 50 votes?

Answer: $\binom{5}{1}\binom{100-51+5-1}{100-51}$.

Now we have a constraint that one of the x_i should be at least 51. Say, let x_1 be at least 51. It is equivalent to giving the first candidate 51 votes at the beginning and then distributing the remaining 49 votes to them again. The number of possible combinations is $\binom{49+5-1}{49}$. Since one of the 5 candidates could have at least 51 votes, the total number of possible voting combinations such that exactly one candidate gets more than 50 votes is:

$$\binom{5}{1}\binom{49+5-1}{49} = 1464125.$$