Lecture 1

Physically Based Modeling
Differential Equation Basics

Initial Value Problems

The vector field

Differential equations describe the relationship between an unknown function and its derivatives.

$$x'(t) = f(x, t)$$
$$x'(t) = f(x(t))$$

where x is the state of the system, $x(t)$ and $x'(t)$ is in vector form. The initial value problem gives $x(t_0) = x_0$ and The differential equation above defines a vector field over x.

Initial Value Problems

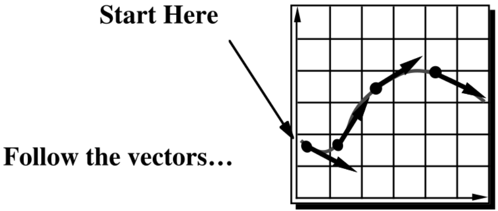


Figure: Starting from a point x_0 , move with the velocity specified by the vector field.

Numerical Solutions

- We take **discrete** time steps starting with the initial value $x(t_0)$.
- Take a step by using the derivative function $f(x, t)$ to **approximate** the change.
- **Deriavative evaluations** are performed at each time step.

Euler's Method

Simple derivative

Consider in time step t_0 and time step $t_0 + h$ Let us start writing the simple derivative function:

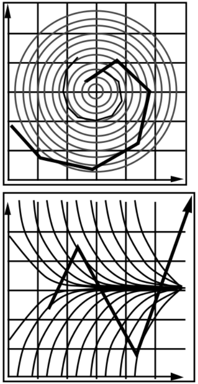
$$x'(t_0) = \frac{x(t_0 + h) - x(t_0)}{h}, h \to 0$$

Linear approxmiation

$$x(t_0 + h) \approx x(t_0) + hx'(t_0)$$

Euler's Method

- Linear approximation is bad for simulating curve path



Inaccuracy:
Error turns x(t) from a circle into the spiral of your choice.

Instability: off to Neptune!

Two Problems

Euler's Method

Questions

- Why is a linear approximation bad for spiral/curve path simulation?
- How can we increase the accuracy of simulation?

Euler's Method

The problem

- Increase the accuracy $\rightarrow$ decrease the size step h
- Decrease size step h $\rightarrow$ more step $\rightarrow$ more cost

The solution

- Increase the accuracy $\rightarrow$ improve the derivative evaluation h
- No need to change the size step h, or even increase h.

Taylor expansion series

- Euler is just the "shortened" form of the Tayler expansion series.
- Taylor expansion is used to approximate a function

Taylor general form

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$f(x) = \Sigma_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor expansion series

- Assuming $x(t)$ is smooth, we can express its value at the end of the step as an infinite sum involving the value and derivatives at the beginning:

Euler is just two terms in the series

$$x(t_0 + h) = x(t_0) + h \cdot x'(t_0) + \frac{h^2}{2!} x''(t_0) + \cdots + \frac{h^n}{n!} \frac{\partial^n x(t_0)}{\partial t^n}$$

- The error term, the difference between the Euler step and the full. In Euler method, it is $O(h^2)$ (read "Order h squared")

Evaluation

- We chop our stepsize in half $\rightarrow \frac{h}{2}$
- This produces only about one fourth the error because of $O(h^2)$.
- But we have to take twice as many steps over any given interval.

Intuition

So what we have to do is just to keep the same step size h, but improve from $O(h^2)$ to $O(h^3)$

The Midpoint Method

- We could acheive $O(h^3)$ accuracy instead of $O(h^2)$ simply by retaining one additional term in the truncated Taylor series.

Change error term

From $O(h^2)$

$$x(t_0 + h) = x(t_0) + h \cdot x'(t_0) + O(h^2)$$

To $O(h^3)$

$$x(t_0 + h) = x(t_0) + h \cdot x'(t_0) + \frac{h^2}{2!} x''(t_0) + O(h^3)$$

But we do not want to evaluate the second derivative of x since it is costly. Therefore, we do math.

The Midpoint Method

Proof

We have

$$x'(t) = f(x(t))$$

Then we take derivative of both sides to obtain $x''(t)$

$$x''(t) = f'(x(t))x'(t)$$
$$x''(t) = f'(x(t))f(x(t))$$
$$x'' = f'f$$

The Midpoint Method

Proof

We have

$$x(t_0 + h) = x(t_0) + h \cdot x'(t_0)$$

The the amount of which x changes from t_0 to $t_0 + h$ is

$$\Delta x = h \cdot x'(t_0) = h \cdot f(x(t_0))$$

The Midpoint Method

Proof

We have the equation need calculating:

$$x(t_0 + h) = x(t_0) + h \cdot x'(t_0) + O(h^2)$$

What if we try to take derivate of both sides:

$$x'(t_0 + h) = x'(t_0) + h \cdot x''(t_0) + O(h^2)$$
$$f(x(t_0 + h)) = f(x(t_0)) + h \cdot f(x(t_0))f'(x(t_0)) + O(h^2)$$
$$f(x(t_0) + h \cdot x'(t_0)) = f(x(t_0)) + h \cdot f(x(t_0))f'(x(t_0)) + O(h^2)$$
$$f(x(t_0) + \Delta x) = f(x(t_0)) + \Delta x f'(x(t_0)) + O(\Delta x^2)$$
$$f(x_0 + \Delta x) = f(x_0) + \Delta x f'(x_0) + O(\Delta x^2)$$

The Midpoint Method

Proof

We already know that we have:

$$\Delta x = h \cdot x'(t_0) = h \cdot f(x(t_0))$$

What if we try to take half the step size. So the new change of position is:

$$\Delta x = \frac{h}{2} \cdot x'(t_0) = \frac{h}{2} \cdot f(x(t_0)) = \frac{h}{2} \cdot f(x_0)$$

But why/how do we think of this? Did you remember the Taylor expansion series?

The Midpoint Method

Proof

Now, try to replace the old Δx with new Δx:

$$f(x_0 + \Delta x) = f(x_0) + \Delta x f'(x_0) + O(h^2)$$

$$f(x_0 + \frac{h}{2} \cdot f(x_0)) = f(x_0) + \frac{h}{2} \cdot f(x_0) f'(x_0) + O(h^2)$$

Now we can multiply both side with h, things would get more familiar:

$$hf(x_0 + \frac{h}{2} \cdot f(x_0)) = hf(x_0) + \frac{h^2}{2} \cdot f(x_0) f'(x_0) + O(h^3)$$

We now want to change back to x, not f anymore

The Midpoint Method

Proof

We rearrange the equation:

$$\frac{h^2}{2} \cdot f(x_0)f'(x_0) + O(h^3) = hf(x_0 + \frac{h}{2} \cdot f(x_0)) - h \cdot f(x_0)$$

Notice that $x''(t_0) = f(x_0)f'(x_0)$ and $h \cdot f(x_0) = h \cdot x'(t_0)$

$$\frac{h^2}{2} \cdot x''(t_0) + O(h^3) = hf(x_0 + \frac{h}{2} \cdot f(x_0)) - h \cdot x'(t_0)$$

The Midpoint method

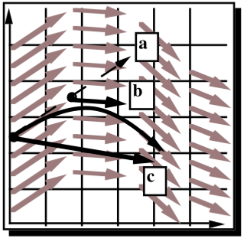
Proof

So that we finally approximate the second order derivative indirectly

$$
\begin{aligned}
x(t_0 + h) &= x(t_0) + h \cdot x'(t_0) + \frac{h^2}{2!} x''(t_0) + O(h^3) \\
&= x(t_0) + h \cdot x'(t_0) + hf(x_0 + \frac{h}{2} \cdot f(x_0)) - h \cdot x'(t_0) \\
&= x(t_0) + hf(x_0 + \frac{h}{2} \cdot f(x_0))
\end{aligned}
$$

Boom! This is the final equation of midpoint method.

The Midpoint Method



a. **Compute an Euler step**

$$\Delta \mathbf{x} = \Delta t\, \mathbf{f}(\mathbf{x}, t)$$

b. **Evaluate f at the midpoint**

$$\mathbf{f}_{mid} = \mathbf{f}\left(\frac{x + \Delta x}{2}, \frac{t + \Delta t}{2}\right)$$

c. **Take a step using the midpoint value**

$$\mathbf{x}(t + \Delta t) = \mathbf{x}(t) + \Delta t\, \mathbf{f}_{mid}$$

The Midpoint Method