Modeling linear and non-linear drag in horizontal oscillatory motion

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Background

Exponentially damped oscillatory motion is often encountered early in a physics student's education. The closed form solution in the case of linear-in-velocity drag makes this an attractive first model, but realistic systems typically include constant- and/or quadratic-in-velocity drag. The equations of motion for non-linear drag can be solved numerically so that data can be fit using more accurate models. A data set is presented and analyzed which students can use to explore topics such as differing models of friction, numerical modeling techniques, and evaluating and quantifying the agreement between a model and data. As an example, we aim to determine whether linear or quadratic drag better models the motion of a cart.

Code and data are provided for physics instructors' use.

Methods: fitting

We model the motion of our system using Newton's second law with a spring force described by Hooke's law and various candidate models of drag

$$m\ddot{x} = -k\left(x - x_{\text{eq}}\right) - c_1\dot{x} - \left(c_0 + c_n\left|\frac{\dot{x}}{1\ \text{m/s}}\right|^n + c_2\dot{x}^2\right)\operatorname{sign}\left(\dot{x}\right). \tag{1}$$

There is no easy to work with, closed-form solution to (1) in the case of any non-linear drag(s), so we use Runga-Kutta to obtain a numerical solution. Our goal is to determine system parameters, which we collect into a vector

$$\vec{A} = \langle x_0, v_0, m, k, x_{\text{eq}}, \ldots\rangle. \tag{2}$$

We can quantify the goodness of a fit using

$$\chi^2 = \sum_{i=1}^{N}\Delta_i^2 = \sum_{i=1}^{N}\left(x_i - f(t_i;\vec{A})\right)^2, \tag{3}$$

where $f(t;\vec{A})$ solves (1). Gradient descent finds $\vec{A}$ to minimize χ^2 by iterating

$$\vec{A}_{\text{new guess}} = \vec{A}_{\text{old guess}} - \alpha\cdot\vec{\nabla}_A\chi^2(\vec{A}_{\text{old guess}}). \tag{4}$$

Methods: data collection

Position vs time data were collected using the Vernier motion encoder system and LoggerPro.

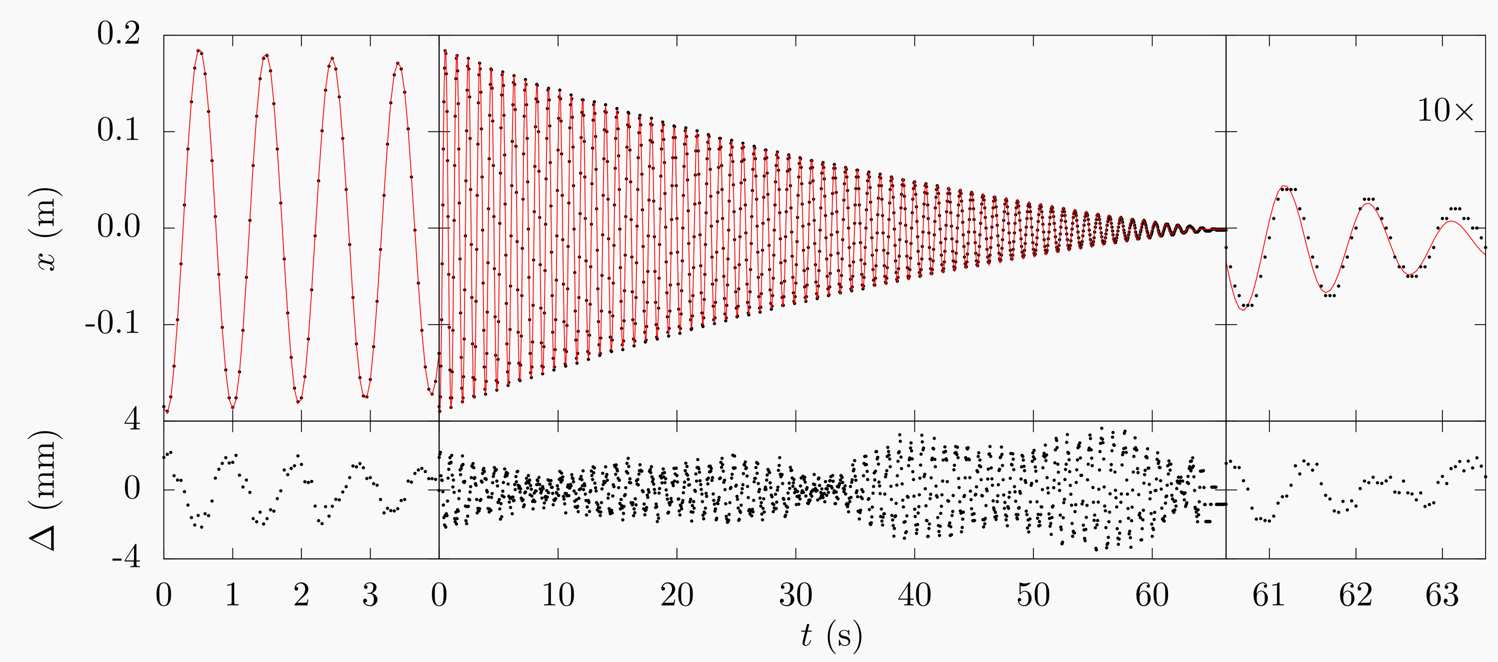

Figure 1: Fit for model CON. Top: data (black points) and fit (red line); bottom: residuals $\Delta_i = x_i - f(t_i;\vec{A})$; center: full duration; left and right: focus on early and late times. The vertical axis of the top right plot has been zoomed in by an order of magnitude to better show the data.

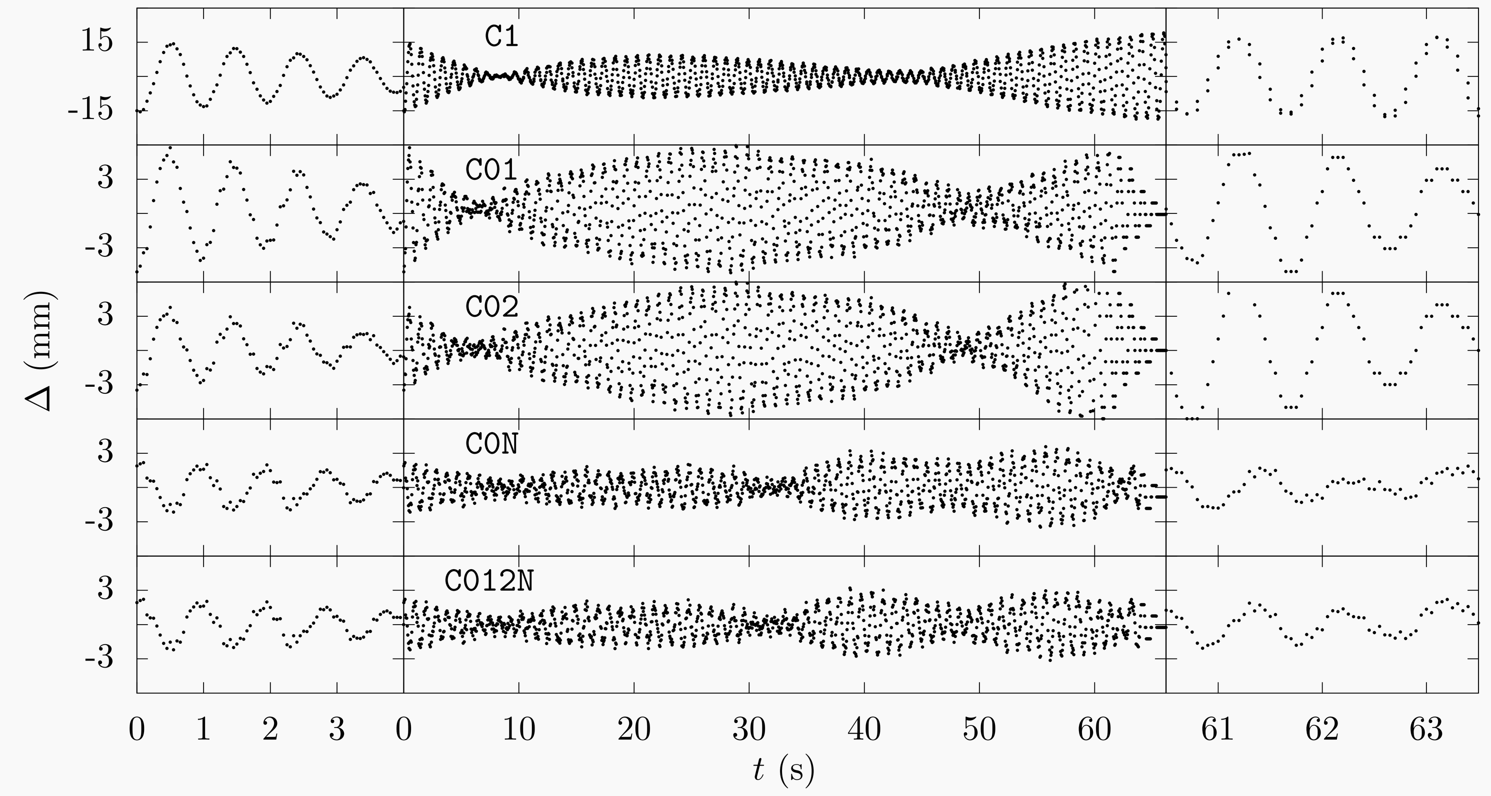

Figure 2: Residuals for five different models of drag. Left, right, and center are as in Fig. 1; each row of graphs corresponds to a column in Table 1 and is labeled with the forms of friction which are present. Note that the residuals for C1 have a different vertical scale.

Analysis

We used five models to fit the same data set, differentiated by which type(s) of friction are present: C1, C01, C02, CON, and C012N. C1 was analysed using LoggerPro's fit to exponentially damped harmonic oscillation

$$Ae^{-Bt}\sin\left(Ct + D\right) + E; \tag{5}$$

all other models were analysed using our code. C1 and C012N were used to determine reasonable bounds on χ^2.

Table 1: Fit parameters for the five models whose residuals are shown in Fig. 2; data columns here correspond to rows there. Spring constant values were fixed in the fits.

	C1	C01	C02	CON	C012N		
k	48.84	48.84	48.84	48.84	48.84		N/m
x_0	-2.000	-1.800	-1.815	-1.869	-1.869	$\times10^{-1}$	m
v_0	-3.253	-3.037	-3.051	-3.051	-3.052	$\times10^{-1}$	m/s
m	1.147	1.147	1.146	1.147	1.147		kg
x_{eq}	-1.604	-2.059	-1.806	-1.588	-1.574	$\times10^{-3}$	m
c_0	–	2.601	3.057	2.243	2.209	$\times10^{-2}$	N
c_1	83.12	21.33	–	–	7.436	$\times10^{-3}$	kg/s
c_2	–	–	20.31	–	5.139	$\times10^{-3}$	kg/m
c_n	–	–	–	4.033	2.822	$\times10^{-2}$	N
n	–	–	–	1.500	1.500	–	
χ^2	56.74	8.700	9.722	2.229	1.945	$\times10^{-3}$	m^2

Conclusions

- Reynolds number (~ 2000), χ^2 of C01 and C02, and n from CON suggest the cart's motion is near the laminar/turbulent transition and drag is neither linear nor quadratic.
- Fit values for mass were consistently statistically significantly larger than the measured value of (1.12 ± 0.01) kg.
- Raises a number of interesting pedagogical points.

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Code and data available at https://github.com/kyle-slinker/teaching-resources.