A Geometric Introduction to Relativity

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October 28, 2022

Contents

A Note About Units 4

Activity 1: Spacetime Diagrams – Instructions 5

Activity 1: Spacetime Diagrams – Answer Sheet 15

Activity 2: Relativistic Energy and Momentum – Instructions 24

Activity 2: Relativistic Energy and Momentum – Answer Sheet 28

Activity 3: Black Hole Spacetime Diagrams – Instructions 30

Activity 3: Black Hole Spacetime Diagrams – Answer Sheet 34

Activity 4: Cosmological Spacetime Diagrams – Instructions 40

Activity 4: Cosmological Spacetime Diagrams – Answer Sheet 43

Problems 49

The following is a set of activities which are designed to walk you through relativity. Along the way, you should encounter a number of bizarre conclusions – all of which seem to be confirmed by experiment.

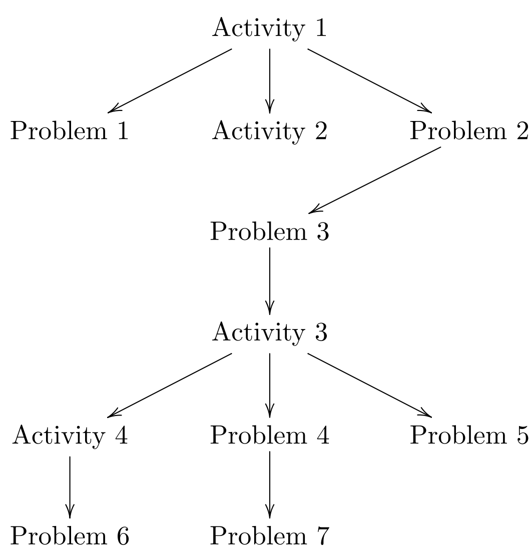
The first two activities concentrate on special relativity. You will investigate how the geometry of spacetime is described and some of the consequences of tying space and time together. You will also investigate what relativity has to say about familiar physical quantities such as energy and momentum.

The last two activities are focused on general relativity. This is a theory which describes gravity. In particular, you look at spacetime near a black hole and – by tracing the path of light near the black hole – uncover a number of their bizarre properties. You also look at how gravity models the expansion of the Universe.

There are also a few homework-type problems included after the last activity. Some of these extend ideas you have seen in activities and some introduce ideas you will need for subsequent activities. The suggested order in which to do all the activities and problems is given below (reading top to bottom, left to right), with arrows showing which problems and activities depend on ideas introduced in others.

This introduction takes a very geometric approach, more so than a typical introduction to relativity. This is intentional: many of the things explored in the first activity are done in such a way that they lay a foundation for the types of thinking you will need to do in future activities. You do not need calculus to work through this, but you will need to be comfortable with algebra (including graphing, transcendental functions, vectors, and polynomials with multiple solutions) and physics (including describing motion with constant or zero acceleration and mass, energy, and momentum for Activity 2).

Throughout, you will be confronted with a number of very subtle points. If none of your answers seem counter-intuitive, you may not be thinking about the questions deeply enough.



A Note About Units

Theoretical physicists often use what are called *natural units*, which are unit systems based around fundamental physical constants. The most common way in which this starts is by considering the speed of light, c.

Because we have a fundamental speed scale, this gives us a natural relationship between lengths and times. If, instead of talking about a time t, we always talk about the combination ct, we can "measure time in meters." Of course, we don't literally do this; we measure time in seconds or years or the like. But if I tell you a length (e.g., ct) and we have agreed that we will use c as the natural speed scale, you can always get back the actual SI value for the corresponding time (e.g., t).

It may seem somewhat strange to "measure time in meters," but we actually do things similar to this on a regular basis in everyday life. One example is converting between kilograms and pounds. Kilograms are a unit of mass and pounds are a unit of force; kilograms actually have the same dimensions as slugs and pounds actually have the same dimensions as Newtons. But because most of us never leave Earth, we have an assumed gravitational field strength $g = 9.8\ \text{m/s}^2$ which lets us translate between masses and forces. It is also very commonplace to give the distance between two cities in terms of the time it takes to drive between them, effectively "measuring distance in seconds" (the opposite direction physicists usually do this conversion). A light-year is a unit of distance which formalizes this idea: it is the distance which naturally (using the speed of light) corresponds to the time 1 year.

Where this starts to become helpful in a practical way is when we take the idea so seriously we decide to treat c as a conversion factor. If $1\ \text{s} = 3 \times 10^8\ \text{m}$, then $c = 1$. At that point, instead of having to try and keep all our t's as ct's we can just drop the c entirely (since that would be multiplying by one). In many calculations, there are so many c's they become cumbersome when doing the algebra. As long as it is understood that "$c = 1$," we can do all our calculations ignoring c. If we want to make a comparison with a real-world result we would need to convert the answer in natural units back to SI units by "putting the c's back in," in the sense mentioned in the last sentence of the second paragraph on this page.

Many of the calculations in these activities can be simplified using natural units. But you should probably not use natural units in your other science classes unless your instructor knows what you are doing. Otherwise, it may just look like you are unable to accurately handle units. Some people may (very reasonably) see natural units as simply being sloppy with units. Before you can abuse units in this way, you need to be very sure that you are able to use units correctly. Besides, units are more than just a chore; they can actually be a useful tool to double check that you are doing your algebra correctly. Natural units are a trade off in simplicity in one place for more complexity in another. So think carefully about whether you want to use them here.

As you work through this document, and especially in Activity 3, we start to pick up many factors of G. We can do a similar thing and "set $G = 1$" in addition to $c = 1$. This has the effect of having time, length, and mass all measured in units of length. When you have both $c = 1$ and $G = 1$, G/c^2 essentially becomes a conversion factor saying that $1\ \text{kg} = 7.43 \times 10^{-28}\ \text{m}$. You have to multiply and/or divide by some power of both c and G to get from natural units back to SI units.[1]

[1] As an aside, though everything is measured in meters in this document, gravitational physicists typically choose to measure everything in kilograms or solar masses when they set $c = 1$ and $G = 1$.

Activity 1: Spacetime Diagrams – Instructions

In this activity, you will be drawing a number of *spacetime diagrams.* At their core, these diagrams are simply plots which show the motion of objects through space as a function of time (you will focus on only one spatial dimension here). They show the motion both through space and time, hence 'spacetime.' This idea should be fairly familiar to you from your experiences with position vs time graphs, but when drawing spacetime diagrams there are a few somewhat odd conventions which physicists follow:

- The time axis is vertical and the space axis is horizontal. This is the opposite of how you have probably previously thought of position as a function of time.
- Physicists generally use ct on the time-axis instead of just t. This gives both axes dimensions of length. It is also common practice to make both axes scaled the same way (i.e., 1 m on the space-axis is the same length on your paper as 1 m on the time-axis).
- The line representing an object's path through spacetime (i.e., its time vs position graph) is called its *worldline*.
- A point in spacetime is called an *event*, whether or not anything particularly interesting happens at that point. This represents a specific place at a specific time.
- For whatever reason, people who are moving tend to ride trains (or, sometimes spaceships).

Make all spacetime diagrams as *quantitatively* accurate as possible and always label all important worldlines and events. Important features of a plot may difficult to see if the plot is too small. As you work through this activity, make sure you are answering the questions; they are italicized in the Instructions and collected on an Answer Sheet for reference. Keep in mind that the Answer Sheet may not give full context, so work from the Instructions. The ideas in this activity build on each other fairly rapidly, so **make sure you are fully comfortable with something before moving on**.

Diagram 1.1

Draw a spacetime diagram representing the following situation:

- You are standing still at $x = 0$ m.
- Your friend is riding a train which is passing you with velocity $v = +0.5c$.
- Your friend passes you at $ct = 0$ m.
- A photon passes you at the same time your friend passes you; it is moving in the positive x-direction.

Question 1.1: What is the slope of the photon's worldline in Diagram 1.1? Some of the conventions listed above are chosen to make sure this answer is what it is.

Diagram 1.2

Re-draw Diagram 1.1, but using your friend's perspective (i.e., they are at rest at the origin).

Question 1.2: How are Diagrams 1.1 and 1.2 similar? How are they different?

Diagram 1.3

Draw a spacetime diagram representing the following situation:

- You are standing still at $x = 6$ m.
- Your friend is riding a train which is passing you with velocity $v = +0.5c$.
- Your friend passes $x = 0$ m at $ct = 0$ m.
- The instant your friend passes the origin, they throw a ball at you with $v = +0.6c$.
- The instant the ball hits you, you throw it back at your friend with $v = -0.6c$.

Question 1.3a: How is the speed of an object related to the slope of its worldline?

Question 1.3b: At what time does the ball get back to your friend? Give your answer twice: once with units of meters and once with units of seconds (i.e., what is ct and what is t?). You can either measure the coordinates from your diagram or use algebra to find them, it is up to you which you would like to do. This not meant to be a hugely time consuming a diagram, so if you find yourself spending a lot of time here you may be worrying about too much detail.

Diagram 1.4

Draw a spacetime diagram representing the following situation:

- You are standing still at $x = 0$ m.
- Your friend is riding a train which is passing you with velocity $v = +0.5c$.
- Your friend passes you at $ct = 0$ m.
- There are two flashlights aimed at you, one at $x = 5$ m and one at $x = -5$ m. At $ct = 0$ m, both flashlights are turned on for a single instant.
- Show on the diagram all events where the flashes of light were emitted or observed (assume the flashes of light travel through people when they are observed) and the worldlines the light followed.

Because both flashes of light were emitted at $ct = 0$ m, you know that the events corresponding to their emission were simultaneous. **Question 1.4a: How can you be sure that the two events are simultaneous based on measurements you could actually take (such as measuring the distances between things and noting the time at which you observe flashes of light). Hint: think about $\Delta x = v\,(t_f - t_i)$?**

Question 1.4b: What does your friend conclude about whether or not the flashes of light were emitted simultaneously?

Diagram 1.5

Draw a spacetime diagram representing the following situation:

- You are standing still at $x = 0$ m.
- Your friend is riding a train which is passing you with velocity $v = +0.5c$.
- Your friend passes you at $ct = 0$ m.
- When your friend passes you, the front of their train car is at $x = 5$ m and the back of their train car is at $x = -5$ m. Show on the diagram worldlines for the front and back of the train car.
- There are two flashlights sitting at either end of the train car pointing at your friend. They emit flashes of light, which your friend sees at the same time.
- Show on the diagram all events which you consider to be simultaneous with $ct = 0$ m (this is one of your *lines of simultaneity*).
- Show on the diagram your friend's line of simultaneity for all events which are simultaneous with the emission of the flashes of light.

By construction, your friend should think that the events where the flashes of light are emitted are simultaneous (does the argument you developed after drawing Diagram 1.4 confirm this?). **Question 1.5: What do you conclude about whether or not these flashes of light were emitted at the same time?**

Stop!

Diagrams 1.4 and 1.5 are trying to walk you to one of the most important and subtle points in all of special relativity. This point is very counter-intuitive; so if you were not surprised by any of your answers to the last few questions, you may have missed something. Make sure you understand what 'simultaneous' means and how the lines of simultaneity are set up! Constructing your friend's line of simultaneity is not easy, so be careful when you draw Diagram 1.5 and make sure to think about the logic you used when drawing Diagram 1.4 and the events where the flashes of light are emitted on Diagram 1.5.

Diagram 1.6

Two events are said to be *timelike separated* if an object with $|v| < c$ can go through both events.

Two events are said to be *lightlike separated* if light can go through both events.

Two events are said to be *spacelike separated* if nothing with $|v| \leq c$ can go through both events.

The *(squared) spacetime interval* between two events is defined as[2]

$$\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2. \tag{1}$$

Draw a spacetime diagram and show the following:

- Show on the diagram all events which are timelike separated from the origin. Pick a few of these events and compute the spacetime interval between them and the origin.
- Show on the diagram all events which are lightlike separated from the origin. Pick a few of these events and compute the spacetime interval between them and the origin.
- Show on the diagram all events which are spacelike separated from the origin. Pick a few of these events and compute the spacetime interval between them and the origin.

Question 1.6: Give a relationship between the spacetime interval and timelike, lightlike, and spacelike separation.

Diagram 1.7

Draw a spacetime diagram and show the following:

- Show all events which are separated from the origin by $\Delta s^2 = 1\ \mathrm{m}^2$.
- Show all events which are separated from the origin by $\Delta s^2 = -1\ \mathrm{m}^2$.

Recall that a circle centered at the origin with radius 1 is the collection of points in space which are separated from the origin by $\Delta x^2 + \Delta y^2 = 1\ \mathrm{m}^2$. This means that what you have drawn on your diagram is like the spacetime version of a circle. The geometry of spacetime is encoded in (1); the minus sign in this equation leads to a number of interesting consequences, some of which you have already seen in the diagrams you have drawn so far.

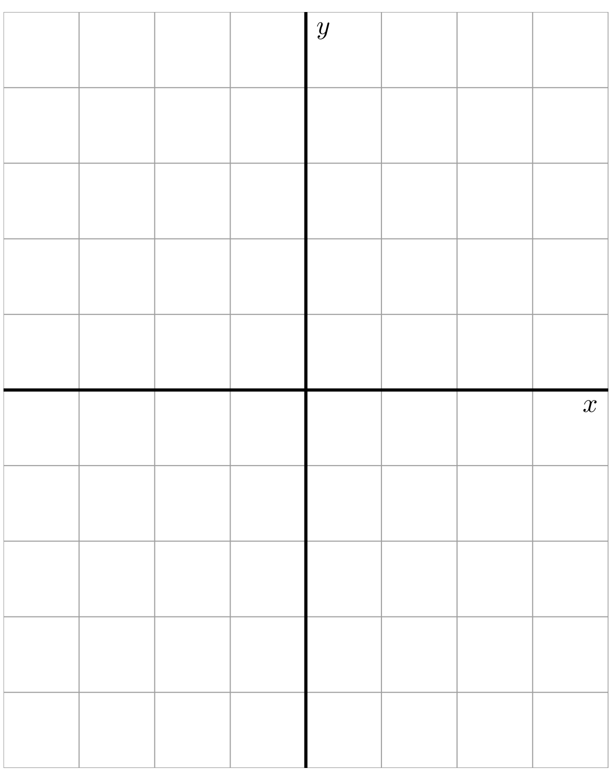
[2]Because of the minus sign in (1), it is possible for Δs to be imaginary. By convention, physicists generally only ever talk about values for Δs^2 – and not Δs – so that they do not have to worry about this.

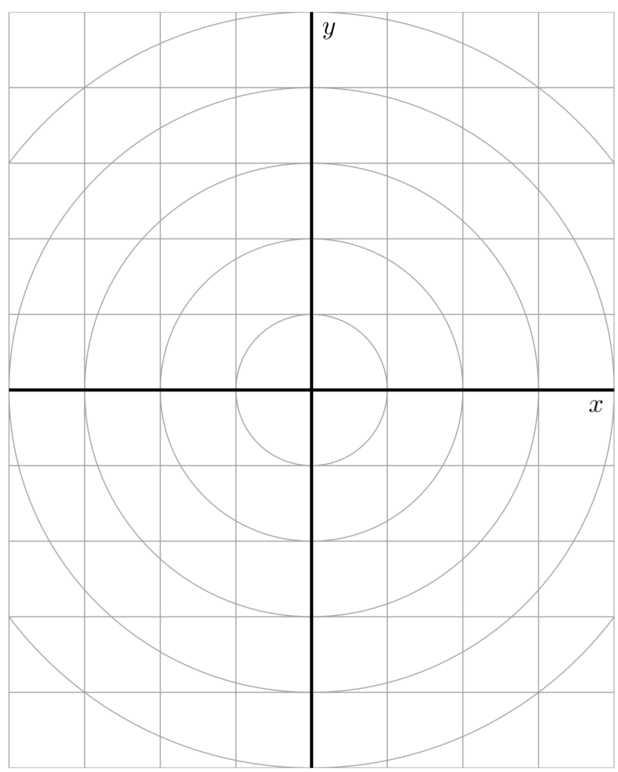
Diagram 1.8

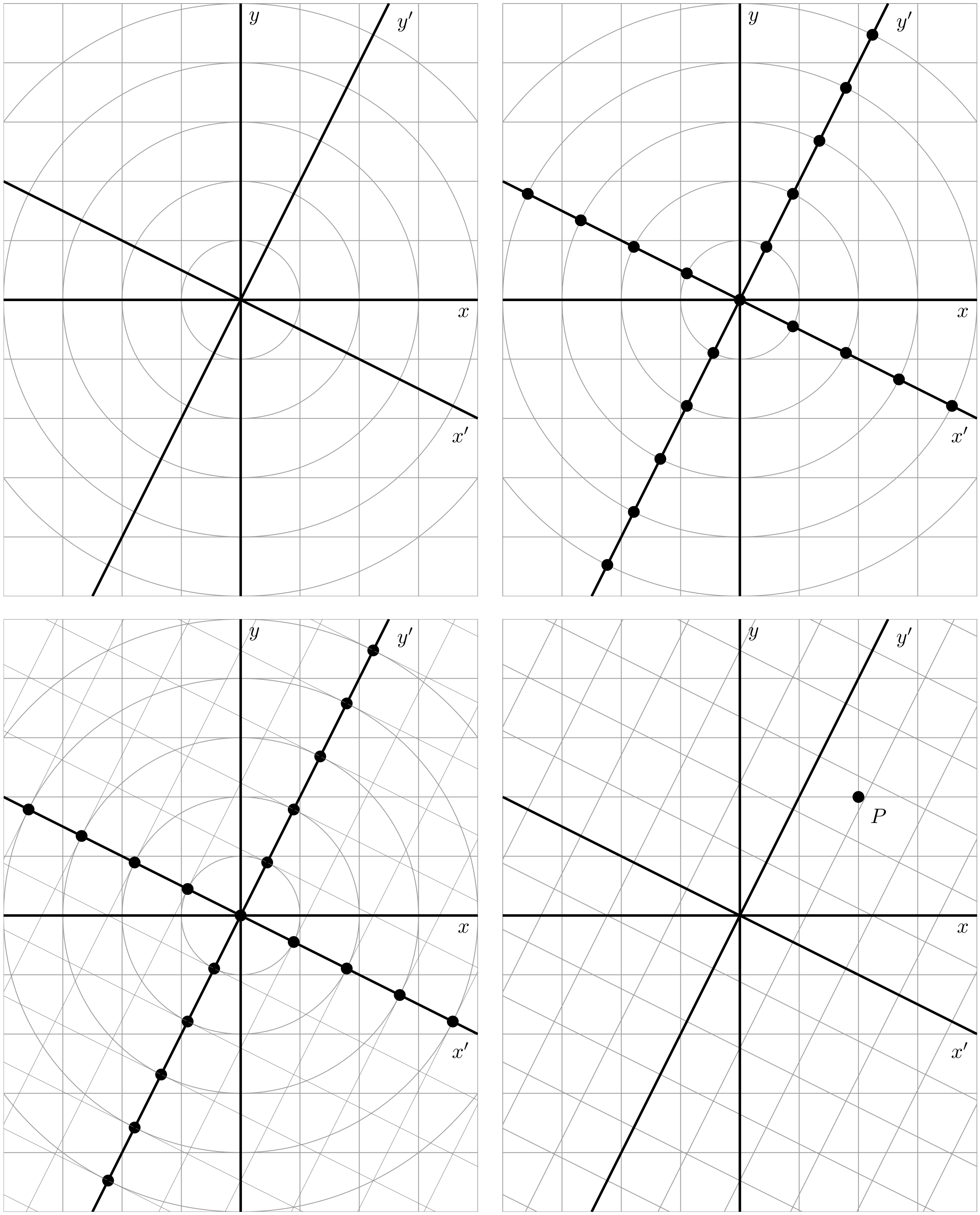
To see why you might care so much about these 'circles,' take a minute to think about how you could take a coordinate system in space and build a second coordinate system which is rotated relative to the first one. The six diagrams below show each of these steps.

1. You start out with the x- and y-axes.
2. Each time a gridline passes through the axis, you draw a circle that goes through this point and is centered around the origin.
3. You draw a set of rotated axes, and call them the x'- and y'-axes.
4. Mark where these new axes intersect the circles you drew in step 2.
5. At each of these intersections, you draw new gridlines which are parallel to the axes you drew in step 3.
6. Now that you have both grids, you can identify a point using either coordinate system. For instance, the point P shown on the last diagram can be located at $(x = 2, y = 2)$ or, using the other coordinate system, at $(x' = 0.894427, y' = 2.68328)$.

Recreate this process to create a second coordinate system for a spacetime diagram. The answer sheet has a spacetime diagram with a number of the spacetime circle analogs drawn on it to help you out with this. Another major difference between space and spacetime – in addition to the spacetime not-actually-circles – is that the spacetime axes do not rotate the same way the space axes do. Instead of looking like the x'- and y'-axes in the third figure, the ct'-axis will be your friend's worldline and the x'-axis will be (parallel to) your friend's line of simultaneity from Diagram 1.5.







y
y'
x
x'
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y'
x
x'
y
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x
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y
y'
P
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x'

When you finish Diagram 1.8, you will have a spacetime with the coordinates associated with your friend's reference frame. For the next few diagrams, you will be provided with a spacetime showing both your and your friends reference frames (coordinate systems).

Diagram 1.9

Draw a spacetime diagram representing the following situation:

- You are standing still at $x = 0$ m.
- Your friend is riding a train which is passing you with velocity $v = +0.5c$.
- Your friend passes you at $ct = 0$ m.
- There is a 5 m bar lying next to you with one of its edges at $x = 1$ m.
- There is another 5 m bar lying on the train next to your friend with one of its edges at $x' = 1$ m.

Question 1.9a: How long does your friend measure your bar to be? How long do you measure your friend's bar to be? Determine your answers by making measurements on Diagram 1.9; carefully show how you are making these measurements on the diagram, including the events between which you are measuring. Comment on these measurements. Hint: when you draw the bars, make sure to draw worldlines for *both* ends of the bars.

What you have (hopefully) uncovered here is the famous *length contraction*

$$\gamma \Delta x = \Delta x_0, \tag{2}$$

where you will need the *Lorentz factor*

$$\gamma \equiv \frac{1}{\sqrt{1 - (v/c)^2}} \tag{3}$$

and the *proper length* Δx_0. The proper length of an object is its length *as measured in the reference frame where it is stationary.* There can only be one reference frame in which an object is at rest, so the proper length is unique. The distance between the events Δx can be measured in any frame, but the proper length Δx_0 can only be measured in one. In this context "proper" does not mean "correct." The "proper length" is a technical term, as is "proper time."

The length of your bar in the diagram corresponds to the distance between a pair of events, where each event is at one end of the bar. But you can only measure the length of an object by looking at two events which are simultaneous. Since you and your friend cannot agree on the simultaneity of events you measure different lengths. Stop and think about this, it is a very important and subtle point!

It is important to note that there are no forces acting on the bar to squish it and it does not matter what the bar is made of; this is a fundamental property of spacetime. **Question 1.9b: Is it you or your friend who measures the proper length of your bar? What about the proper length of your friend's bar? Do the lengths you and your friend measure (as read off your diagram) agree with the length contraction formula?**

Diagram 1.10

Draw a spacetime diagram representing the following situation:

- You are standing still at $x = 0$ m.
- Your friend is riding a train which is passing you with velocity $v = +0.5c$.
- Your friend passes you at $ct = 0$ m.
- You and your friend are both carrying clocks and they both read zero when you pass each other.
- Mark the event where the clock you are carrying says that $ct = 6$ m of time has passed.
- Mark the event where your friend's clock says that $ct' = 6$ m of time has passed.

Question 1.10a: When your clock reads $ct = 6$ m, what time does your friend think it is? What time do you think it is when your friend's clock reads $ct' = 6$ m? Determine your answers by making measurements on Diagram 1.10; carefully show how you are making these measurements on the diagram, including the events between which you are measuring. Comment on these measurements.

What you have (hopefully) uncovered here is the famous *time dilation*

$$\Delta t = \gamma \Delta t_0, \tag{4}$$

where Δt_0 is the *proper time*. The proper time between two (timelike separated) events is the amount of time between those events *as measured by a clock which experiences both events*. This is similar to how proper length is defined. As with proper length, there is only one reference frame in which the proper time can be measured, meaning it is also a unique value.

Again, this effect is a fundamental property of spacetime; it has nothing to do with how the clock works and applies equally to mechanical clocks, digital clocks, and biological clocks. **Question 1.10b: Is it you or your friend who measures the proper time for your clock? What about the proper time of your friend's clock? Do the time intervals you and your friend measure (as read off your diagram) agree with the time dilation formula?**

Diagram 1.11

On this spacetime diagram mark the following events (put the events on the x and ct gridlines so that their (ct, x) coordinates are round, easy to measure values):

- Event 1 on the ct-axis but not on the x-axis.
- Event 2 on the x-axis but not on the ct-axis.
- Event 3 on the ct'-axis but not on the x'-axis.
- Event 4 on the x'-axis but not on the ct'-axis.
- Event 5 which is not on any axis, $x' < 0$ m, $x > 0$ m.
- Event 6 which is not on any axis, $ct' < 0$ m, $ct > 0$ m.
- Event 7 which is not on any axis and $ct' > 0$ m and $x' > 0$ m.

The Lorentz transformations

$$ct' = \gamma \left(ct - x \cdot \frac{v}{c} \right) \quad (5)$$

$$x' = \gamma \left(x - vt \right) \quad (6)$$

relate the coordinates of events in the two different coordinate systems. **Question 1.11: For each event, record the (ct, x) coordinates in the table on the answer sheet. Also measure the (ct', x') coordinates off of Diagram 1.11 and record them. Finally, compute the (ct', x') coordinates from the (ct, x) coordinates using the Lorentz transformation and compare this to the values you measured (explicitly show your work for at least two events). Enter all values in the table with units of meters.**

To review, in this activity you explored the geometry of spacetime and saw that:

- Spacetime diagrams show the motion of objects.
- The notion of simultaneity is subtle and observers may not always agree what is simultaneous.
- If an object experiences an event in spacetime it may or may not be able to experience other events; the relationship between events tells you if they are timelike, lightlike, or spacelike separated.
- The spacetime version of a circle is a hyperbola. The spacetime version of a rotation is using a different worldline for the time axis and a different line of simultaneity for the space axis.
- An observer who sees an object moving will measure that object to be physically shorter than an observer who does not see the object moving.
- An observer who sees a clock moving will see it running slower than an observer who sees the clock as stationary.
- The coordinates one observer measures for an event can me transformed to the coordinates another observer would measure using the Lorentz transformations.

All of the geometry of spacetime comes from the spacetime interval $\Delta s^2 = -c^2\Delta t^2 + \Delta x^2$ which measures 'distance' in spacetime the same way the Pythagorean theorem $r^2 = \Delta x^2 + \Delta y^2$ measures distance in space. The minus sign in the spacetime interval is what leads to all the odd behavior you have seen. In future activities, you will investigate how this spacetime interval affects physics quantities you are already familiar with and also what happens in the vicinity of a black hole, which changes the spacetime interval itself.

Activity 1: Spacetime Diagrams – Answer Sheet

Make all spacetime diagrams as *quantitatively* accurate as possible and always label all important worldlines and events. Important features of a plot may difficult to see if the plot is too small. As you work through this activity, make sure you are answering the questions; they are italicized in the Instructions and collected on an Answer Sheet for reference. Keep in mind that the Answer Sheet may not give full context, so work from the Instructions. The ideas in this activity build on each other fairly rapidly, so **make sure you are fully comfortable with something before moving on**.

Diagram 1.1

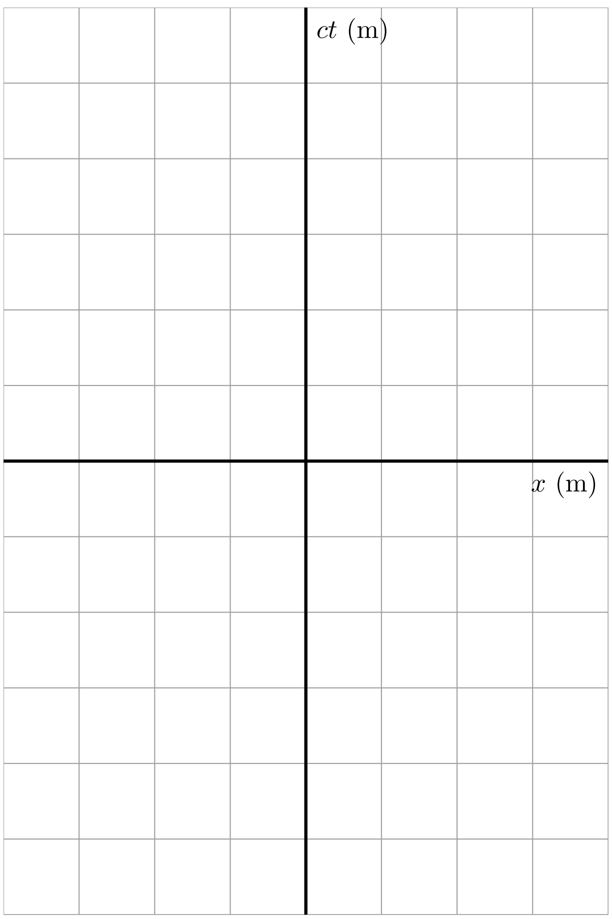
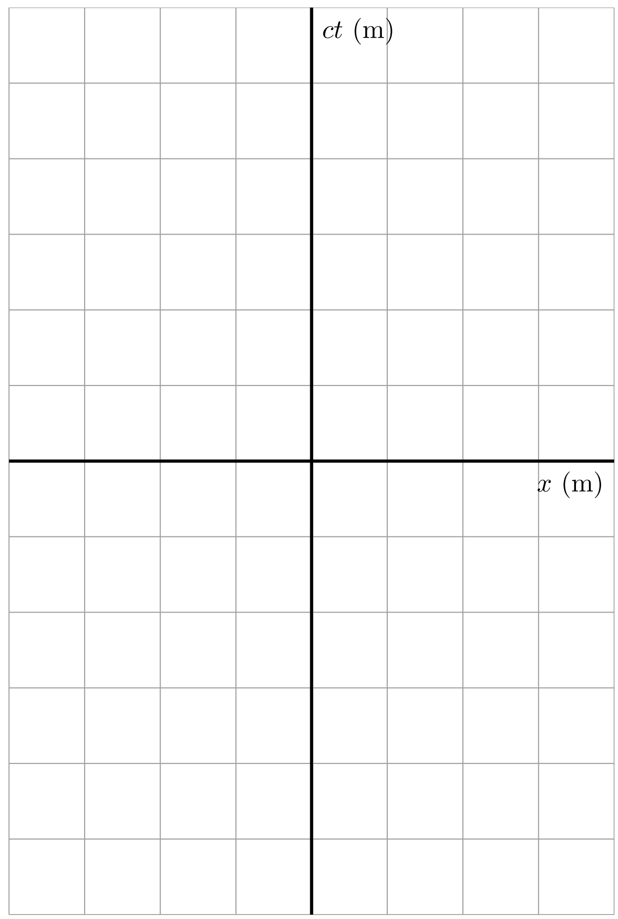


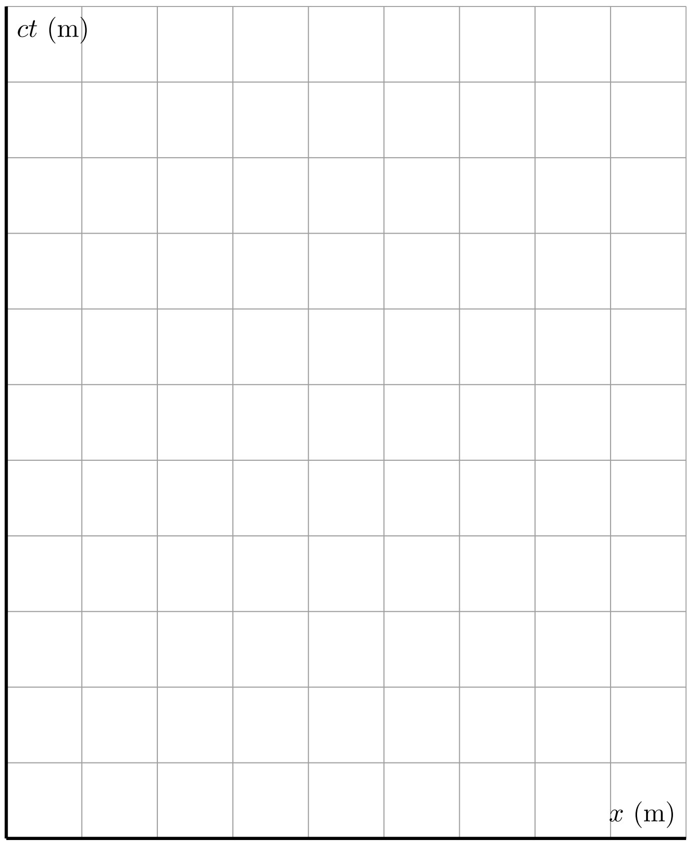
Diagram 1.2



Question 1.1: What is the slope of the photon's worldline in Diagram 1.1?

Question 1.2: How are Diagrams 1.1 and 1.2 similar? How are they different?

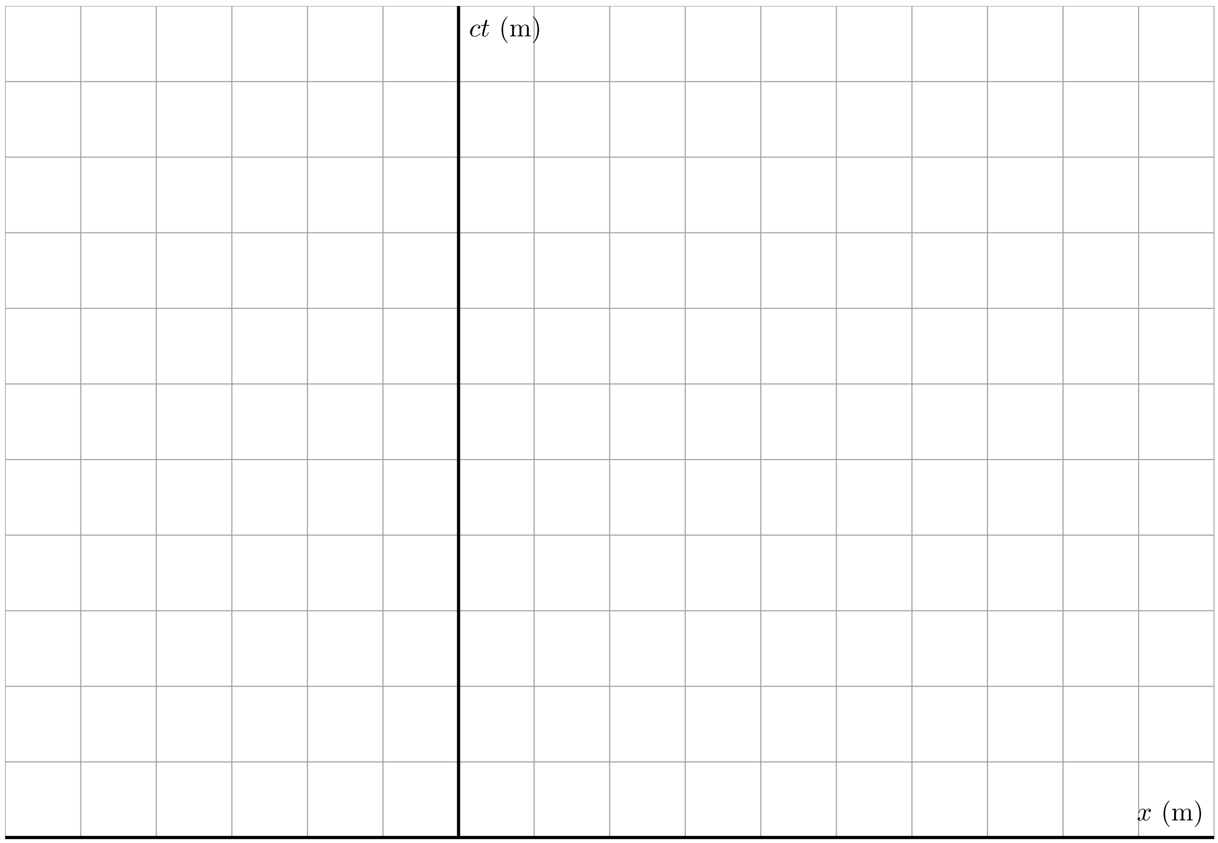
Diagram 1.3



Question 1.3a: How is the speed of an object related to the slope of its worldline?

Question 1.3b: At what time does the ball get back to your friend? Give your answer twice: once with units of meters and once with units of seconds (i.e., what is ct and what is t?).

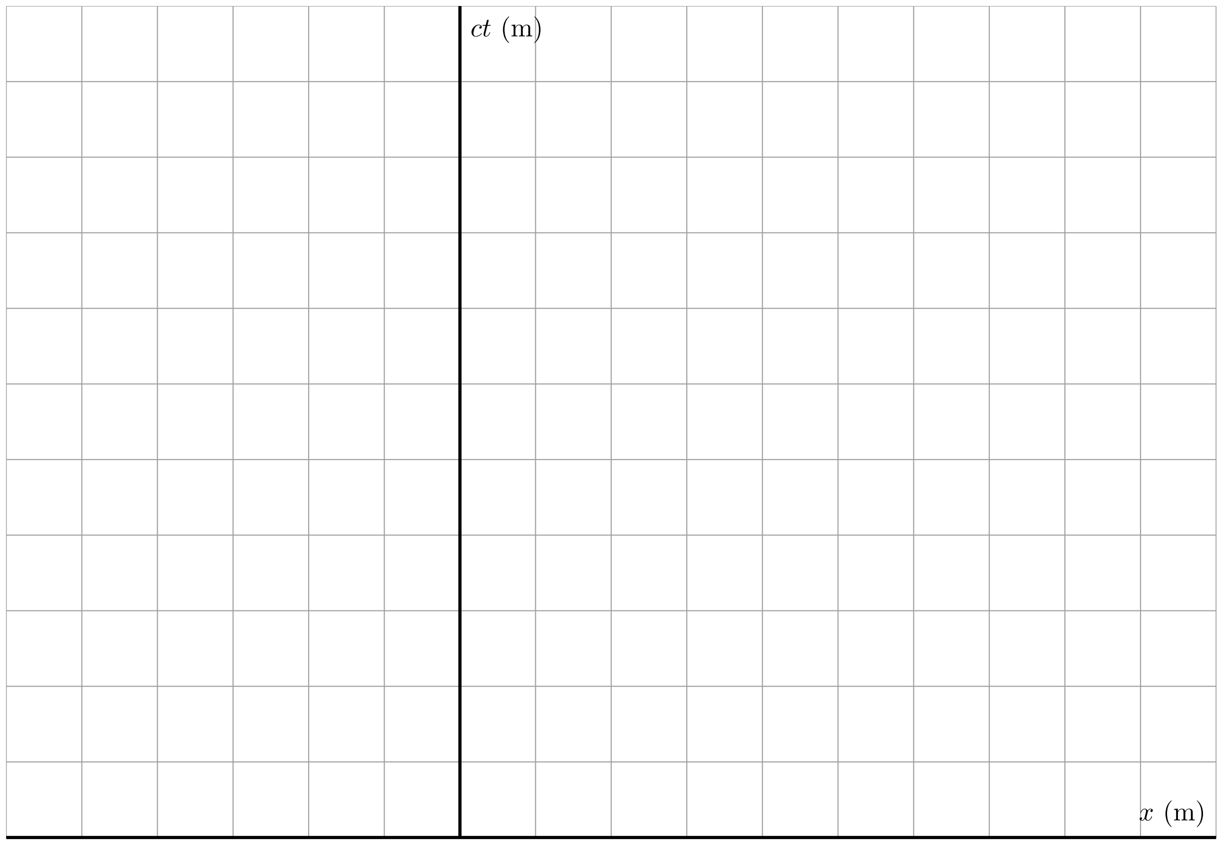
Diagram 1.4



Question 1.4a: How can you be sure that the two events are simultaneous based on measurements you could actually take (such as measuring the distances between things and noting the time at which you observe flashes of light). Hint: think about $\Delta x = v\,(t_f - t_i)$?

Question 1.4b: What does your friend conclude about whether or not the flashes of light were emitted simultaneously?

Diagram 1.5



Question 1.5: What do you conclude about whether or not these flashes of light were emitted at the same time?

Stop!

Before continuing, make sure you read the information just after Diagram 5 on the instructions page.

Diagram 1.6

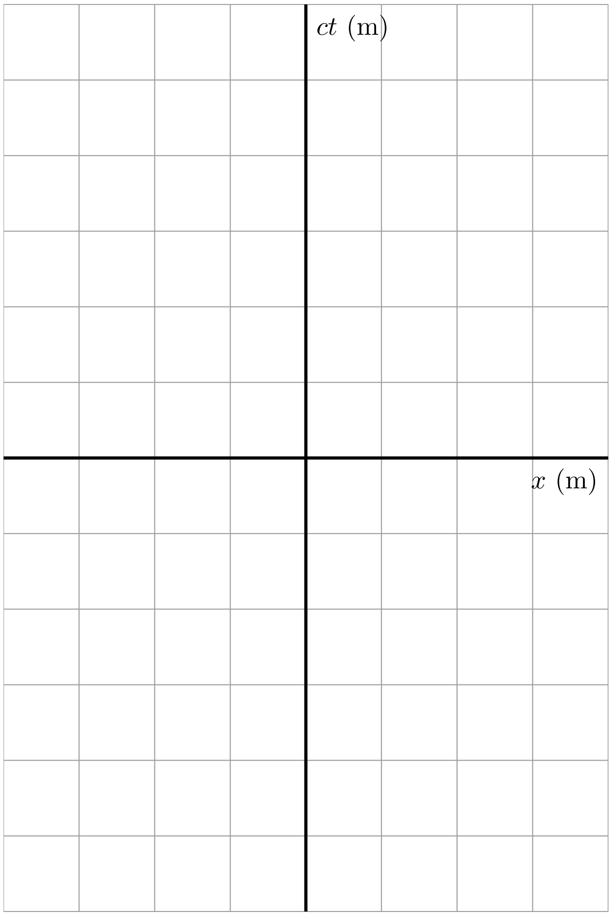
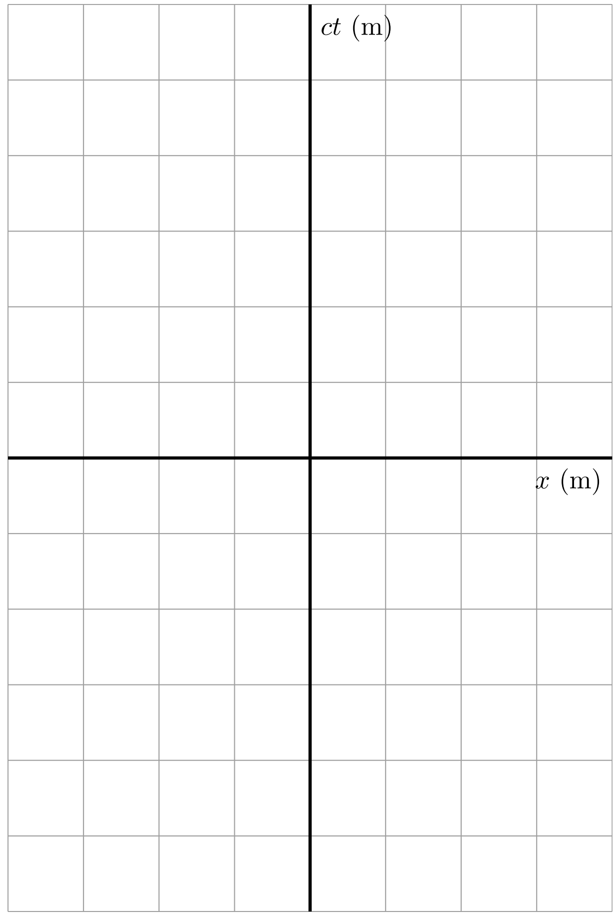


Diagram 1.7



Question 1.6: Give a relationship between the spacetime interval and timelike, lightlike, and spacelike separation.

Diagram 1.8

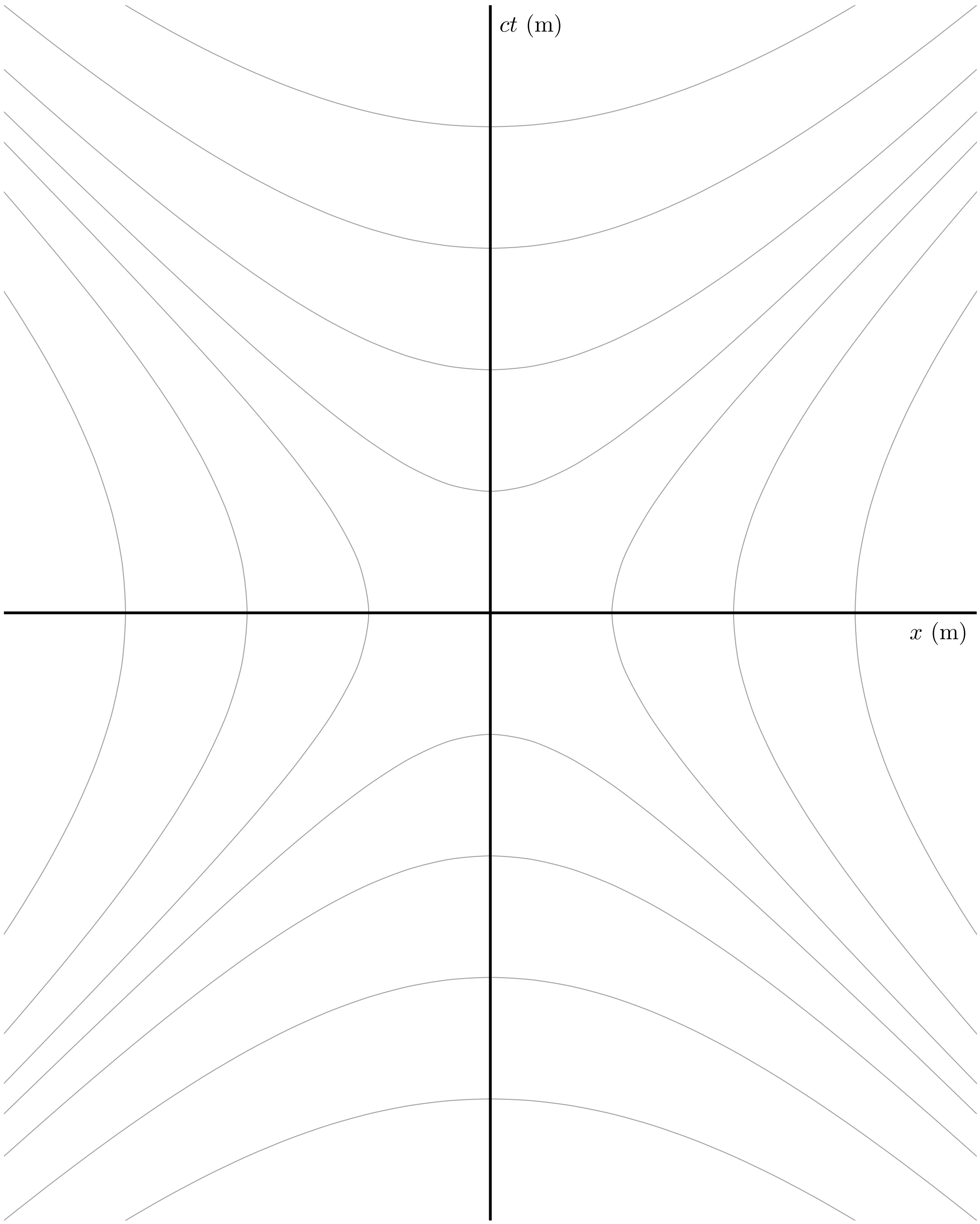
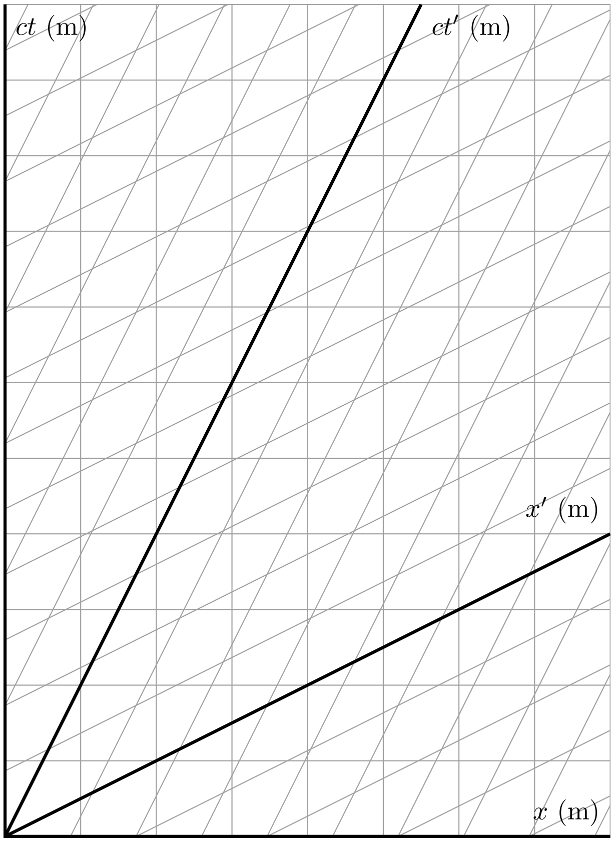


Diagram 1.9



Question 1.9a: How long does your friend measure your bar to be? How long do you measure your friend's bar to be? Determine your answers by making measurements on Diagram 1.9; carefully show how you are making these measurements on the diagram, including the events between which you are measuring. Comment on these measurements.

Question 1.9b: Is it you or your friend who measures the proper length of your bar? What about the proper length of your friend's bar? Do the lengths you and your friend measure (as read off your diagram) agree with the length contraction formula?

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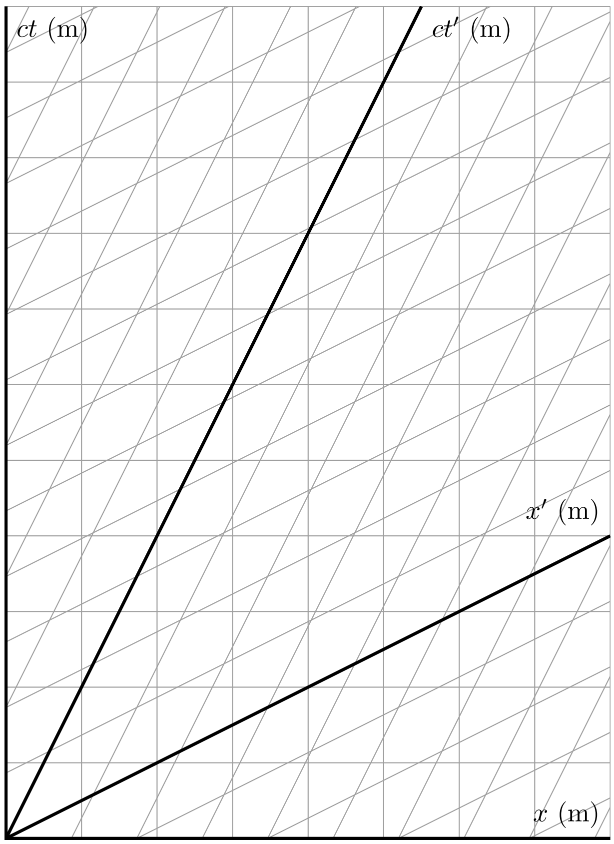

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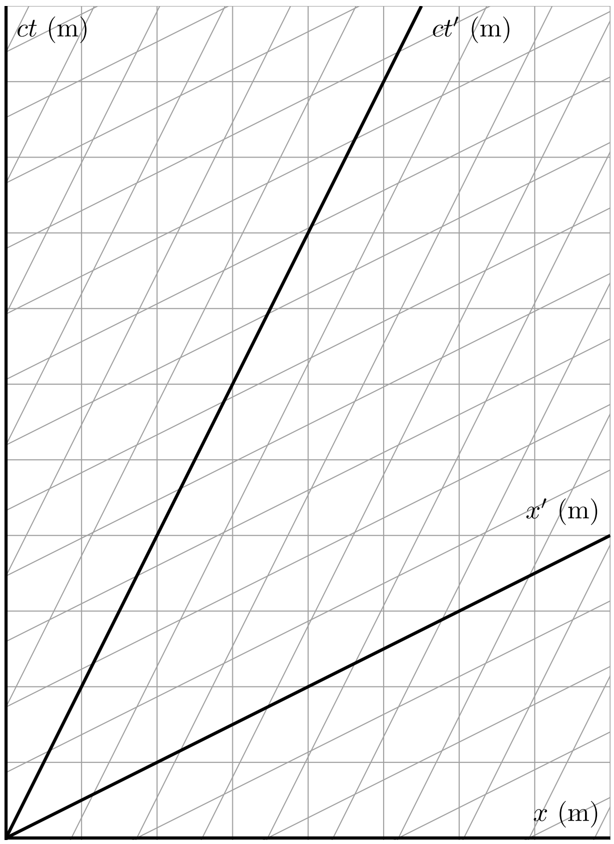
Diagram 1.11



Question 1.11: For each event, record the (ct, x) coordinates in the table on the answer sheet. Also measure the (ct', x') coordinates off of Diagram 1.11 and record them. Finally, compute the (ct', x') coordinates from the (ct, x) coordinates using the Lorentz transformation and compare this to the values you measured (explicitly show your work for at least two events). Enter all values in the table with units of meters.

Event	ct	x	ct' (measured)	x' (measured)	ct' (computed)	x' (computed)
1						
2						
3						
4						
5						
6						
7						

Activity 2: Relativistic Energy and Momentum – Instructions

In this activity, you will continue your investigation of the geometry of spacetime. This time you will focus on what spacetime can tell you about some of the physical quantities you have encountered in previous classes (such as mass, momentum, and energy). In order to determine this, you will analyse the motion of your friend, who is moving past you with a speed v.

As you work through this activity, make sure you are answering the questions; they are italicized in the Instructions and collected on an Answer Sheet for reference. Keep in mind that the Answer Sheet may not give full context, so work from the Instructions. The ideas in this activity build on each other fairly rapidly, so **make sure you are fully comfortable with something before moving on**.

You will need a number of results from the previous spacetime activity; in particular, remember one of the last lines in the instructions for that activity: "All of the geometry of spacetime comes from the spacetime interval $(\Delta s)^2 = -c^2 (\Delta t)^2 + (\Delta x)^2$ which measures 'distance' in spacetime the same way the Pythagorean theorem $d^2 = (\Delta x)^2 + (\Delta y)^2$ measures distance in space." Also, remember the shorthand

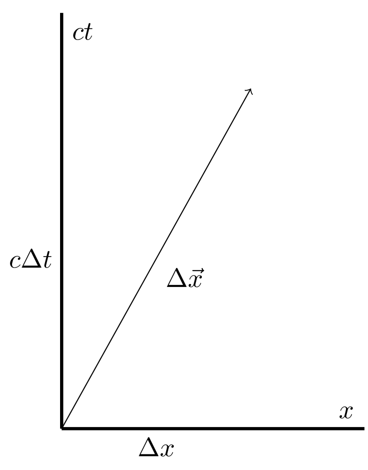
$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \tag{7}$$

which will show up heavily in the calculations you are about to do. Do not forget that if an expression contains the *Lorentz factor* γ, it also depends on speed v.

When talking about space represented by an xy-plane, you can define vectors which live in the space and have x- and y-components. Now that you are considering spacetime with a tx-plane, you will move to talking about vectors with t- and x-components. Below is a table which shows some space-to-spacetime correspondences you built in the previous activity along with some which you will build in this activity.

Space	Spacetime
point	event
circle	hyperbola
rotation	boost
Pythagorean theorem	$\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2$
three-vector	four-vector
dot product	$\vec{a} \cdot \vec{b} = -a_t b_t + a_x b_x$
(three-)velocity v	four-velocity $\vec{u} = \langle u_t, u_x \rangle$
(three-)momentum p	four-momentum $\vec{q} = \langle q_t, q_x \rangle$

Three-vector and *four-vector* may seem like strange names, since the four-vector only has two components. But remember, you are ignoring the y- and z-directions. This means you only need to worry about one of the three components of your three-vectors and only two of the four components of your four-vectors.



To the left is a spacetime diagram with your friend's worldline. It can be interpreted as a four-vector representing their displacement between two events (one of which is the origin). You can write this vector in terms of coordinates as[3]

$$\Delta\vec{x} = \langle c\Delta t, \Delta x \rangle \, . \tag{8}$$

As with three-vectors, you can find the 'length' of a four-vector by computing its dot product with itself, where you use the special relativistic definition of the dot product. For any two four-vectors $\vec{a}$ and $\vec{b}$,

$$\vec{a} \cdot \vec{b} = -a_t b_t + a_x b_x. \tag{9}$$

[3]Some ways to write $\Delta\vec{x}$ using other notations you may have seen for vectors: $\Delta\vec{x} = (c\Delta t, \Delta x)$, $\Delta\vec{x} = [c\Delta t, \Delta x]$, or

Question 2.1: Compute $\Delta\vec{x}\cdot\Delta\vec{x}$. Your answer should look familiar. **Question 2.2: Where have you seen the 'length' of a displacement four-vector before?**

When working in space you look at the displacement over a given time interval to construct an object's velocity. You will do the same in order to define a spacetime version of velocity. It would be nice to avoid having this vector depend on who measured it, so build it using something which is invariant. The time measured by different observers may differ, but the proper time gives you something everyone can agree on. Therefore, you can define your friend's *four-velocity* using their spacetime displacement and proper time

$$\vec{u} \equiv \frac{\Delta\vec{x}}{\Delta\tau}. \tag{10}$$

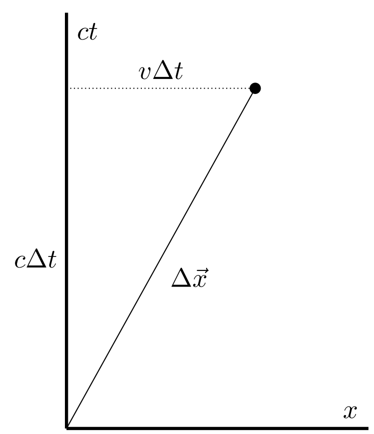
See if you can work out what the components of this vector are. You will need to start by dividing each component of the displacement by the proper time, $\Delta\tau$, and multiplying each component by a factor of $\Delta t/\Delta t$ (which is 1)

$$\vec{u} = \left\langle \frac{c\Delta t}{\Delta\tau}, \frac{\Delta x}{\Delta\tau} \right\rangle = \left\langle \frac{c\Delta t}{\Delta\tau}\frac{\Delta t}{\Delta t}, \frac{\Delta x}{\Delta\tau}\frac{\Delta t}{\Delta t} \right\rangle = \left\langle \frac{c\Delta t}{\Delta t}\frac{\Delta t}{\Delta\tau}, \frac{\Delta x}{\Delta t}\frac{\Delta t}{\Delta\tau} \right\rangle = \frac{\Delta t}{\Delta\tau}\left\langle c\frac{\Delta t}{\Delta t}, \frac{\Delta x}{\Delta t} \right\rangle. \tag{11}$$

But $\Delta x/\Delta t = v$ is just the normal definition of velocity, so

$$\vec{u} = \frac{\Delta t}{\Delta\tau}\left\langle c\cdot 1, v\right\rangle = \frac{\Delta t}{\Delta\tau}\left\langle c, v\right\rangle. \tag{12}$$

Thus, you should be able to get a pretty good handle on the components of $\vec{u}$ if you can work out what $\Delta t/\Delta\tau$ is.



In order to do this, reconsider the diagram you looked at just above. Over a time interval Δt your friend will move a distance $v\Delta t$. For events which are timelike separated (which you saw in the previous activity must be the case for any two events on your friend's worldline), $\Delta s^2 < 0$. In these cases, it can be helpful to relate the invariant spacetime 'distance' to an invariant notion of time. This is essentially what proper time is! The proper time and spacetime interval are related by

$$-c^2\Delta\tau^2 = \Delta s^2. \tag{13}$$

Question 2.3: Use the 'length' of $\Delta\vec{x}\cdot\Delta\vec{x}$ and the relationship between proper time and the spacetime interval to compute $\Delta t/\Delta\tau$. Where have you seen this result before?

The previous answer, along with (12), should allow you to find expressions for the components of your friend's four-velocity which look a bit simpler. **Question 2.4: Give expressions for the components of $\vec{u}$ in terms of only c, v, and γ.** What is the 'length' of $\vec{u}$? **Question 2.5: Compute $\vec{u}\cdot\vec{u}$. Is the value what you would expect? How does it depend on your friend's speed?**

Briefly consider $\vec{u}$ in your friend's reference frame. **Question 2.6: What values does your friend compute for the components of their velocity vector $\vec{u}$? What do they compute for $\vec{u}\cdot\vec{u}$?**

Having constructed a spacetime notion of your friend's velocity, see if you can extend it to a spacetime notion of momentum. Just like in your previous physics classes, you will get momentum by multiplying mass and velocity. In the case of *four-momentum*,

$$\vec{q} \equiv m\vec{u}. \tag{14}$$

$\Delta\vec{x} = (c\Delta t)\,\hat{t} + (\Delta x)\,\hat{x}$.

Based on previous calculations, you should be able to determine fairly easily that the components of your friend's four-momentum are

$$\vec{q} = \langle \gamma mc, \gamma mv \rangle \,. \tag{15}$$

Now see if you can make sense of these components.

To do so, you can check to see what happens if your friend is moving slowly, so that your previous experience with physics is applicable. You can accomplish this by looking to see what happens when v/c is very close to 0.[4] You will need two approximations which are valid when x is small[5]

$$(1+x)^n = 1 + nx + \mathcal{O}\left(x\right)^2 \tag{16}$$

$$x\left(1-x^2\right)^n = x + \mathcal{O}\left(x\right)^3 . \tag{17}$$

The $\mathcal{O}\left(x\right)$ notation is a formal way of quantifying how small the error is when using these approximations. If $x = 10^{-3}$, then $\mathcal{O}\left(x\right)^3$ means the error will be about $(10^{-3})^3 = 10^{-9}$. You may be familiar with this from Taylor series in calculus. If not, all you really need to know here is that

$$(1+x)^n \approx 1 + nx \tag{18}$$

$$x\left(1-x^2\right)^n \approx x. \tag{19}$$

Use the first approximation, along with $x = -v^2/c^2$, to determine approximately what γmc is equal to. Compare your answer to standard physics formulas. **Question 2.7: What physics equation(s) do you recognize in your approximation for γmc? What do you think γmc corresponds to?**

Use the second approximation, along with $x = v/c$, to determine approximately what γmv is equal to. Compare your answer to standard physics formulas. **Question 2.8: What physics equation(s) do you recognize in your approximation for γmv? What do you think γmv corresponds to?**

Based on your last two answers, you should be finding that it is plausible that $E = \gamma mc^2$ and $p = \gamma mv$. It turns out that the familiar formulas for energy and momentum you have seen in other physics classes are not quite accurate and only apply when an object is moving much slower than the speed of light. To see how different the relativistic and non-relativistic formulas are, take a look at how kinetic energy depends on the speed of an object both in terms of the relativistic and non-relativistic kinetic energies. The rest mass is often not considered to be a part of an object's relativistic kinetic energy; in this case, an object's relativistic kinetic energy is $KE_R = \gamma mc^2 - mc^2$. **Question 2.9: Draw a graph of $KE/(mc^2)$ vs v/c, showing both the relativistic kinetic energy $KE_R = \gamma mc^2 - mc^2$ and the non-relativistic kinetic energy $KE_N = \frac{1}{2}mv^2$. Use the axes and scales provided. How are these two curves different? Similar? How much energy does it take to accelerate an object to the speed of light?**

Given your previous answers, you can write

$$\vec{q} = \left\langle \frac{E}{c}, p \right\rangle . \tag{20}$$

This shows that energy and momentum are related in much the same way that time and space are related (with energy playing a time-like role and momentum playing a space-like role). It also shows that the usual Laws of Conservation of Energy and Conservation of (Three-)Momentum can be bundled together into one Law of Conservation of Four-Momentum.

Check to see what the 'length' of $\vec{q}$ tells you. Use (14), (20), and $\vec{q} \cdot \vec{q}$ to show that

$$-m^2c^2 = -\frac{E^2}{c^2} + p^2. \tag{21}$$

[4] It is good practice to find a unitless quantity if you want to say that something is small or large. Is a meter large? It is hard to say. Are you a cosmologist or a particle physicist? A meter is pretty small to a cosmologist but it is pretty big to a particle physicist. You need some representative scale to compare to. In this case, v/c lets you compare your friend's speed to the speed of light, which is a natural scale for speeds.

[5] Here x is your unitless quantity. Do not confuse it for the x-direction.

Solve this equation for E^2. **Question 2.10: Where have you seen this energy formula before?** This is one consequence of the Law of Conservation of Four-Momentum.

To review, in this activity you further explored the geometry of spacetime and saw that:

- Four-dimensional vectors can be defined which have components in the timelike direction in addition to the usual spacelike directions.
- A relativistic notion of the dot product has a minus sign in it which is related to the minus sign in the spacetime interval.
- The four-velocity of an object describes its path through spacetime, similar to the way a three-velocity describes the motion of an object through space.
- The four-momentum of an object combines the notions of energy and momentum, similar to the way space and time are combined.
- The usual formulas you have seen for energy and momentum of an object are only approximately correct and are valid only when the object is moving very slowly.
- Combining energy and momentum leads to the famous $E = mc^2$ formula, which describes how nuclear reactions convert mass to energy as a consequence of the Law of Conservation of Four-Momentum.

Again, all of the geometry of spacetime comes from the spacetime interval $\Delta s^2 = -c^2\Delta t^2 + \Delta x^2$ which measures 'distance' in spacetime. The minus sign in the spacetime interval makes its way into the relativistic version of a dot product, leading to some of the results you found in this activity. In a future activity, you will start to investigate black holes – which change the spacetime interval – and the bizarre effects of the geometry of spacetime in their vicinity.

Activity 2: Relativistic Energy and Momentum – Answer Sheet

As you work through this activity, make sure you are answering the questions; they are italicized in the Instructions and collected on an Answer Sheet for reference. Keep in mind that the Answer Sheet may not give full context, so work from the Instructions. The ideas in this activity build on each other fairly rapidly, so **make sure you are fully comfortable with something before moving on**.

Question 2.1: Compute $\Delta\vec{x} \cdot \Delta\vec{x}$.

Question 2.2: Where have you seen the 'length' of a displacement four-vector before?

Question 2.3: Use the 'length' of $\Delta\vec{x}\cdot\Delta\vec{x}$ and the relationship between proper time and the spacetime interval to compute $\Delta t/\Delta\tau$. Where have you seen this result before?

Question 2.4: Give expressions for the components of $\vec{u}$ in terms of only c, v, and γ.

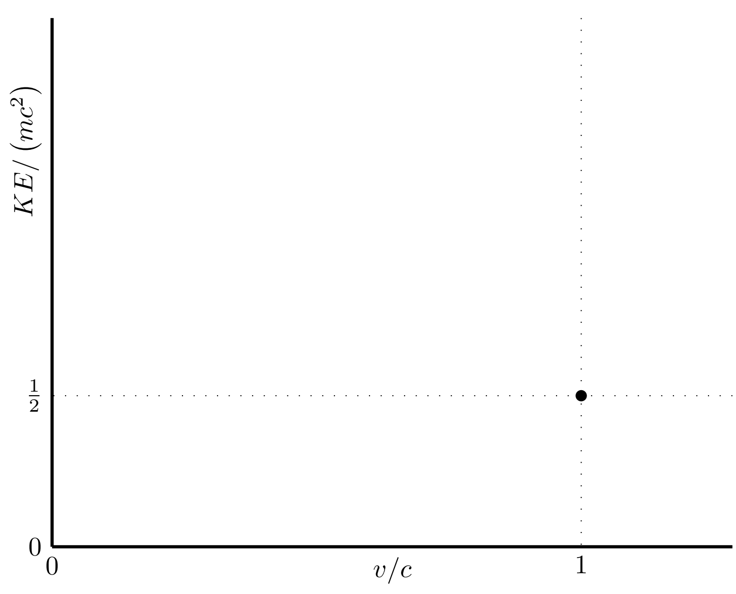
Question 2.5: Compute $\vec{u} \cdot \vec{u}$. Is the value what you would expect? How does it depend on your friend's speed?

Question 2.6: What values does your friend compute for the components of their velocity vector $\vec{u}$? What do they compute for $\vec{u} \cdot \vec{u}$?

Question 2.7: What physics equation(s) do you recognize in your approximation for γmc? What do you think γmc corresponds to?

Question 2.8: What physics equation(s) do you recognize in your approximation for γmv? What do you think γmv corresponds to?

Question 2.9: Draw a graph of $KE/(mc^2)$ vs v/c, showing both the relativistic kinetic energy $KE_R = \gamma mc^2 - mc^2$ and the non-relativistic kinetic energy $KE_N = \frac{1}{2}mv^2$. Use the axes and scales provided. How are these two curves different? Similar? How much energy does it take to accelerate an object to the speed of light?



Question 2.10: Where have you seen this energy formula before?

Activity 3: Black Hole Spacetime Diagrams – Instructions

In this activity, you will be drawing more spacetime diagrams. The difference between these diagrams and the ones you drew previously is that your Diagram 3.2 and Diagram 3.3 in this activity will represent spacetime near a black hole. This activity also uses slightly different conventions than the previous set of spacetime diagrams:

- The time axis is still vertical and the space axis is still horizontal.
- Gravitational physicists generally measure everything in meters and multiply things by G and c to eliminate kilograms and seconds. You will still have both axes scaled the same way (i.e., 1 m on the space-axis is the same length on your paper as 1 m on the ct-axis).
- The line representing an object's path through spacetime (i.e., its time vs position graph) is still its worldline.
- A point in spacetime is still an event.

Make all spacetime diagrams as *quantitatively* accurate as possible and always label all important worldlines and events. Important features of a plot may difficult to see if the plot is too small. As you work through this activity, make sure you are answering the questions; they are italicized in the Instructions and collected on an Answer Sheet for reference. Keep in mind that the Answer Sheet may not give full context, so work from the Instructions. The ideas in this activity build on each other fairly rapidly, so **make sure you are fully comfortable with something before moving on**.

Diagram 3.1

Recall the spacetime interval you looked at in your previous diagrams

$$\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2 \tag{22}$$

and that you found light moves along paths for which $\Delta s^2 = 0$.

Pick five evenly spaced events on the t-axis. For each event, draw the worldlines for light moving through the event in the positive and negative x directions. At each event where these worldlines intersect, note that they create a triangular wedge with the event in question at one vertex. This region is called a *light cone* and represents the timelike future of the event. Note that a light cone could be drawn for any event in spacetime, what you've essentially done here is draw a representative sample. The light cones are a new way to look at some of the ideas from Activity 1 which will be helpful when investigating black holes.Make sure you get a handle on the units here; it is only going to get more complicated.

Question 3.1a: Give an explicit formula for how Δt and Δx are related for the light rays in Diagram 3.1.

Question 3.1b: Draw worldlines for you and your friend: you have a non-zero velocity with no acceleration but your friend is accelerating. How do you and your friend's worldlines relate to the light cones you drew?

Diagram 3.2

In the vicinity of a black hole, the geometry of spacetime changes and (22) is no longer accurate. Instead, the spacetime interval is given by

$$\Delta s^2 = -c^2\left(1 - \frac{2GM}{c^2 r}\right)\Delta t^2 + \frac{4GM}{cr}\Delta t\Delta r + \left(1 + \frac{2GM}{c^2 r}\right)\Delta r^2. \tag{23}$$

Two comments about (23): First, you are switching from x to r and are only worried how far something is away from the origin. Second, the black hole's mass is M, which you can assume is $\sim 1.3 \times 10^{27}$ kg (so that $GM/c^2 = 1$ m).

For the photon worldlines in Diagram 3.2, the relationship between Δt and Δr is given by

$$\frac{c\Delta t}{\Delta r} = -1 \qquad \text{or} \qquad \frac{c\Delta t}{\Delta r} = -\frac{2GM/c^2 + r}{2GM/c^2 - r}. \tag{24}$$

Question 3.2a: Derive the relationship between Δt and Δr given by (24). (Hint: Divide (23) by Δr^2 then treat the result as a quadratic equation in the variable $c\Delta t/\Delta r$).

You may recognize that (24) are differential equations. You do not need to worry about solving differential equations here; the important thing is the slopes give you a relationship for how t and r are related:

$$ct = -r + k_1 \qquad \text{or} \qquad ct = r + \frac{4GM}{c^2}\ln\left|\frac{c^2 r}{2GM} - 1\right| + k_2, \tag{25}$$

where k_1 and k_2 are just some numbers which you are free to choose. These functions tell you what the photon worldlines look like in the black hole spacetime. Note that you need to take the absolute value of the argument of the logarithm.

Pick a few evenly spaced values for k_1 and k_2 between -15 m and 15 m. Plot both equations of (25) for all of these k_1 and k_2 values. A blank graph is provided on the answer sheet.

Compare the photon worldlines in your blackhole spacetime to the one you made for Diagram 3.1. **Question 3.2b: How are the photon worldlines in Diagrams 3.1 and 3.2 different? How are they similar?**

The line $r = 2GM/c^2$ is called the black hole's *event horizon*. Draw the event horizon on Diagram 3.2. Something funny happens here. **Question 3.2c: Describe what is going on near the event horizon in Diagram 3.2.**

Pick an event where two of your photon worldlines intersect inside the event horizon; call this Event A. Pick a second event where two of your photon worldlines intersect, this time outside the event horizon; call this Event B.

Look at the area immediately around Events A and B. For both Events, the worldlines should divide the spacetime into four regions. **Question 3.2d: For Event A, record which regions are spacelike and timelike separated from Event A; and, for regions which are timelike separated, record if they are in the future or past of Event A. Justify your answers by picking an event close to Event A in each of the four regions and computing Δs^2 between that event and Event A (it's important that the events you choose are close to Event A). This technically is only correct in the infinitesimal limit (i.e., doing calculus), so take a look at your answers and make sure they are reasonable conceptually; you may need to discount some of your math if the answers you are getting are nonsense. Do the same for Event B.** Use the information you figured out here to draw the light cones on Diagram 3.2. **Question 3.2e: What happens to the light cones near the event horizon on Diagram 3.2 as you approach the event horizon from outside the black hole?**

Congratulations; you have drawn a black hole! The differences in the way the light cones behave in Diagrams 3.1 and 3.2 reflect the change in the geometry of the spacetime due to the presence of the black hole. You can now use Diagram 3.2 to investigate some of the properties of black holes.

You may want to get a separate sheet of paper to put on top of your black hole so you can experiment with worldlines which you draw. Once you have a good feel for the worldlines you want to draw, add them to Diagram 3.2.

You and your friend are floating outside your black hole to admire it. Excited, you want to investigate it; draw your worldline, starting outside of the black hole and going into the black hole. Pay attention to the light cones you just drew. **Question 3.2f: What happens when you cross over the event horizon in Diagram 3.2? Once you are inside the event horizon can you get out? Where can you go?** Your friend thought it might not be safe[6] so they did not want to follow you into the black hole. **Question 3.2g: Can your friend get away to safety? Or are they doomed to fall in with you? Where can they go?**

Diagram 3.3

In a previous activity, one thing you looked at was what happens with spacetime diagrams when you change coordinate systems. You can do the same thing with black hole spacetimes, though the relationship between coordinate systems is much more complicated.[7] You will change from t and r coordinates to a different kind of time and radial coordinates, which you should call T and R; the different t's and r's mean different things, but there is a relationship between them, which is given implicitly by

$$e^{c^2r/2GM}\left(\frac{c^2r}{2GM}-1\right)=-\frac{c^4}{G^2M^2}\left(cT-R\right)\left(cT+R\right). \tag{26}$$

This transcendental equation does not allow you to solve for R as a function of r directly, but it will let you figure out many properties of this new coordinate system. The spacetime interval in these coordinates is

$$\Delta s^2=-\frac{32GM}{c^2r}e^{-c^2r/2GM}\left(c\Delta T-\Delta R\right)\left(c\Delta T+\Delta R\right). \tag{27}$$

You should have found before that the black hole's event horizon exists at $r = 2GM/c^2$. **Question 3.3a: Find where the event horizon is in terms of the new T and R coordinates.**

The center of the black hole – its singularity – is at $r = 0$. **Question 3.3b: Find where the singularity is in terms of the new T and R coordinates.**

Question 3.3c: Find where $r = GM/c^2$ and $r = 3GM/c^2$ are in terms of the new T and R coordinates.

Question 3.3d: Find the relationship between ΔT and ΔR for the photon worldlines.

You should now be in a position where you can draw Diagram 3.3 – the black hole spacetime in these new coordinates. For technical reasons, you only need to worry about the part of this diagram where $cT \geq -R$. When drawing the light cones, make sure you think about where Δs^2 is positive and where it is negative and which direction the future is in. **Question 3.3e: Do the light cones change when you are inside or outside the event horizon in Diagram 3.3?**

[6]Maybe they have heard of *spaghettification*, one of the best terms in science.

[7]The 'special' and 'general' in special and general relativity refer to the fact that you are only allowed to make a restricted, special kind of coordinate change in special relativity, but you are allowed to use any general kind of coordinate change in general relativity. For the record, the first black hole coordinate system you looked at is called 'Ingoing Eddington-Finkelstein coordinates.' The second system you are looking at is called 'Kruskal coordinates.' The homework assignment uses systems called 'Schwarzschild coordinates' and 'Outgoing Eddington-Finkelstein coordinates.'

Think about what worldlines for the following things would look like and add them to your new diagram:

- A flash of light which is sent into the black hole from its exterior.
- A flash of light which is emitted from inside the black hole moving outward.
- A flash of light which is emitted from outside the black hole moving outward.
- You and your friend's worldlines from Diagram 3.2.

Question 3.3f: How are Diagrams 3.2 and 3.3 different? How are they similar? Does the location of the black hole's singularity in each one make sense? Make sure to indicate the regions on Diagram 3.3 corresponding to the interior and exterior of the black hole.

Diagram 3.3 highlights an important relationship between the event horizon and photon worldlines: they have the same slope. **Question 3.3g: Comment on this relationship between the event horizon and photon worldlines and describe what it tells you about the event horizon.**

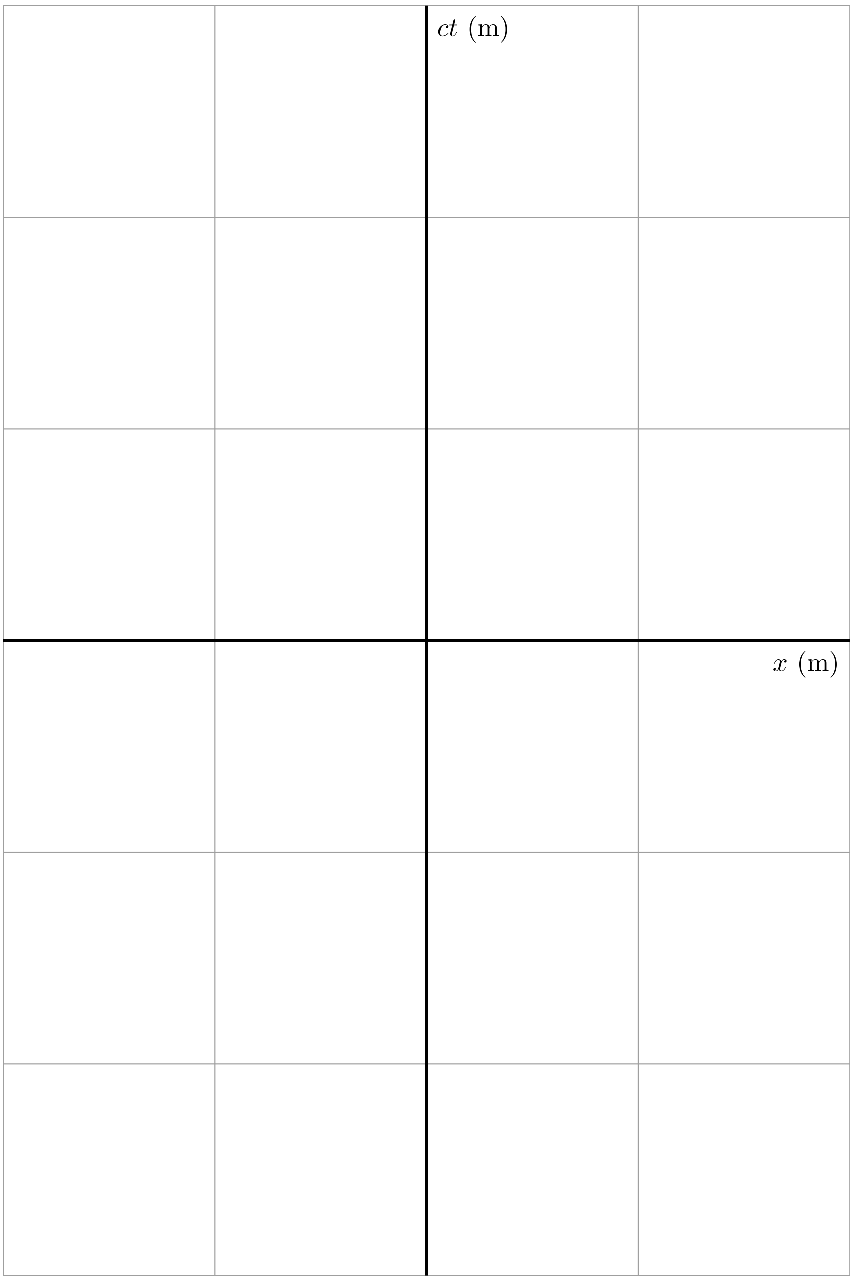
To review, in this activity you explored the geometry of spacetime near black holes and saw that:

- Light cones give another way to describe how observers move through a spacetime.
- When black holes are present, the spacetime interval is no longer $\Delta s^2 = -c^2\Delta t^2 + \Delta r^2$. This causes the trajectories of light (and observers) to bend. This is how general relativity describes a gravitational interaction.
- Spacetime near a black hole can be described using multiple coordinate systems, similar to the way you and your friend's reference frames gave two different grids in the previous spacetime diagrams activity.
- The event horizon describes the boundary of the region from which nothing can escape – not even light. This is the defining property of a black hole.

Activity 3: Black Hole Spacetime Diagrams – Answer Sheet

Make all spacetime diagrams as *quantitatively* accurate as possible and always label all important worldlines and events. Important features of a plot may difficult to see if the plot is too small. As you work through this activity, make sure you are answering the questions; they are italicized in the Instructions and collected on an Answer Sheet for reference. Keep in mind that the Answer Sheet may not give full context, so work from the Instructions. The ideas in this activity build on each other fairly rapidly, so **make sure you are fully comfortable with something before moving on**.

Diagram 3.1

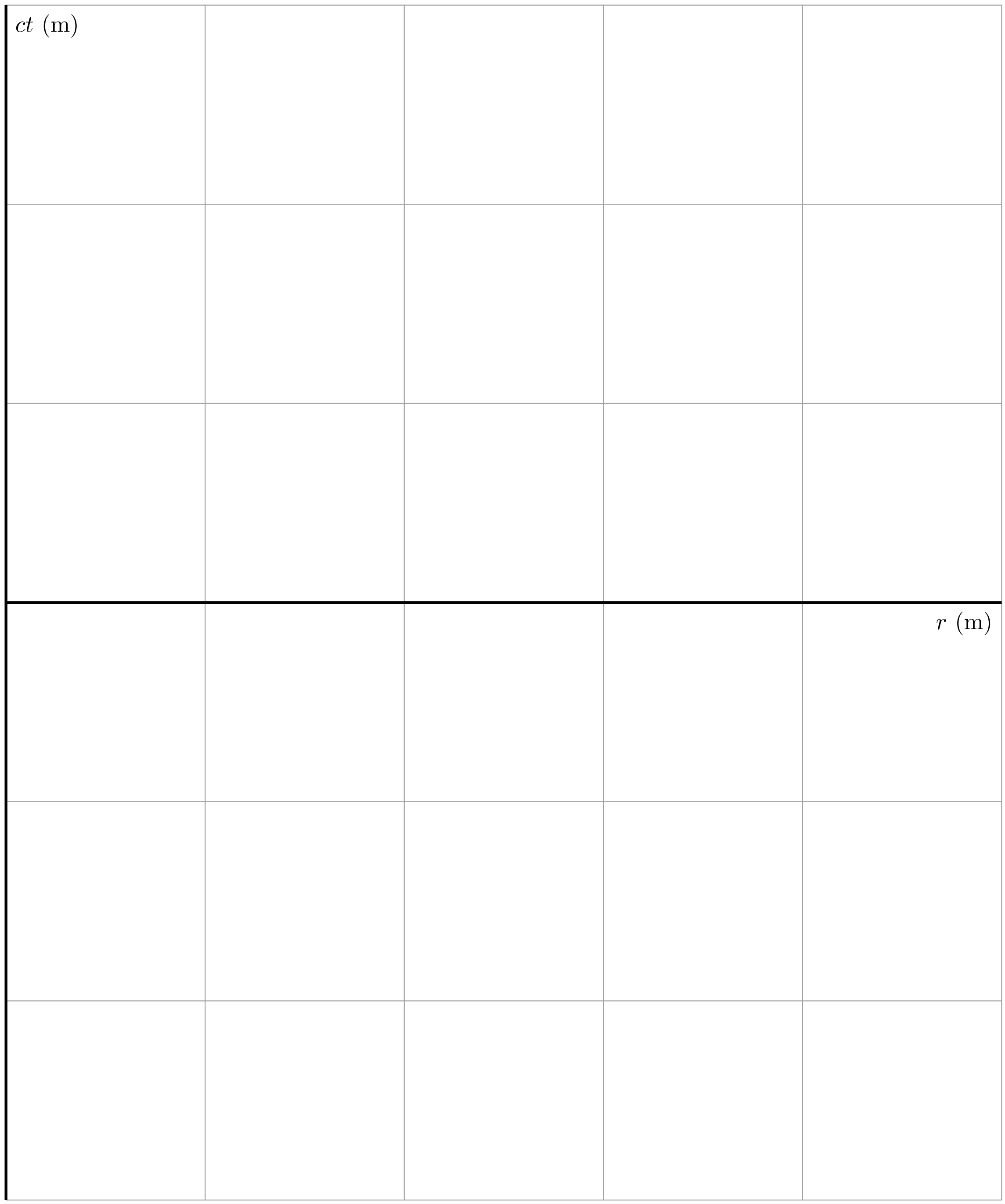


Question 3.1a: Give an explicit formula for how Δt and Δx are related for the light rays in Diagram 3.1.

Question 3.1b: Draw worldlines for you and your friend: you have a non-zero velocity with no acceleration but your friend is accelerating. How do you and your friend's worldlines relate to the light cones you drew?

Question 3.2a: Derive the relationship between Δt and Δr given by (24). (Hint: Divide (23) by Δr^2 then treat the result as a quadratic equation in the variable $c\Delta t/\Delta r$).

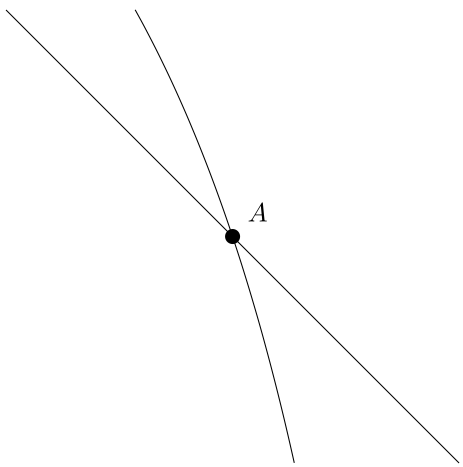
Diagram 3.2

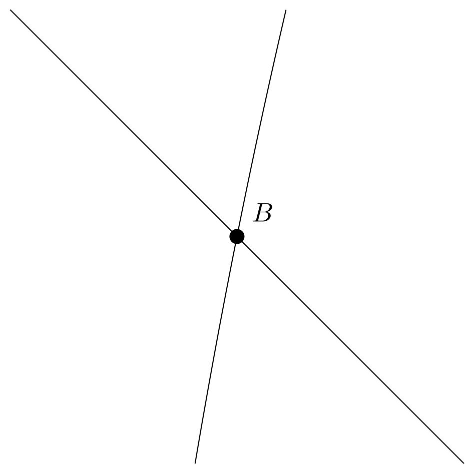


Question 3.2b: How are the photon worldlines in Diagrams 3.1 and 3.2 different? How are they similar?

Question 3.2c: Describe what is going on near the event horizon in Diagram 3.2.

Question 3.2d: For Event A, record which regions are spacelike and timelike separated from Event A; and, for regions which are timelike separated, record if they are in the future or past of Event A. Justify your answers by picking an event close to Event A in each of the four regions and computing Δs^2 between that event and Event A (it's important that the events you choose are close to Event A). This technically is only correct in the infinitesimal limit (i.e., doing calculus), so take a look at your answers and make sure they are reasonable conceptually; you may need to discount some of your math if the answers you are getting are nonsense. Do the same for Event B.



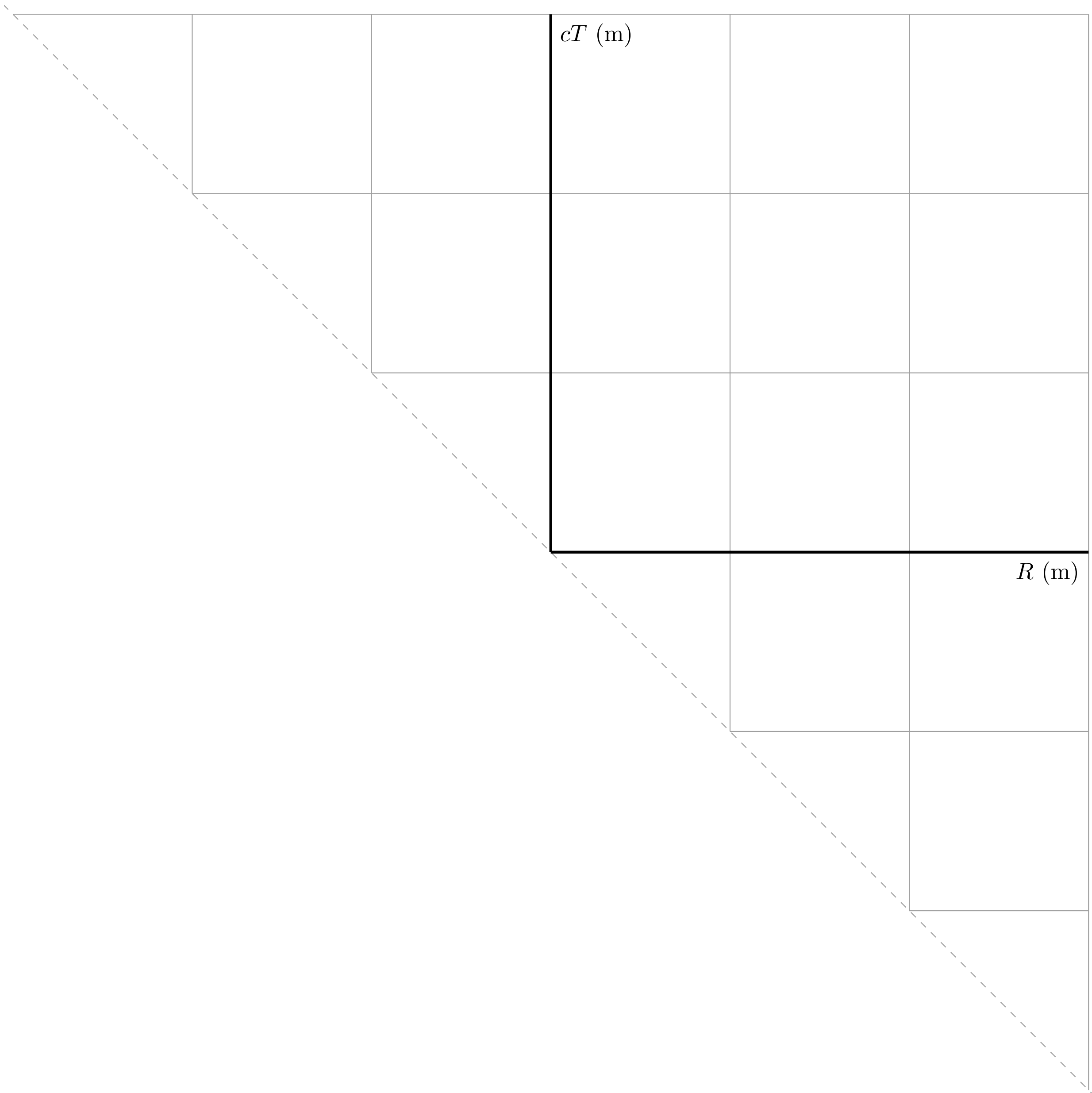


Question 3.2e: What happens to the light cones near the event horizon on Diagram 3.2 as you approach the event horizon from outside the black hole?

Question 3.2f: What happens when you cross over the event horizon in Diagram 3.2? Once you are inside the event horizon can you get out? Where can you go?

Question 3.2g: Can your friend get away to safety? Or are they doomed to fall in with you? Where can they go?

Diagram 3.3



Question 3.3a: Find where the event horizon is in terms of the new T and R coordinates.

Question 3.3b: Find where the singularity is in terms of the new T and R coordinates.

Question 3.3c: Find where $r = GM/c^2$ and $r = 3GM/c^2$ are in terms of the new T and R coordinates.

Question 3.3d: Find the relationship between ΔT and ΔR for the photon worldlines.

Question 3.3e: Do the light cones change when you are inside or outside the event horizon in Diagram 3.3?

Question 3.3f: How are Diagrams 3.2 and 3.3 different? How are they similar? Does the location of the black hole's singularity in each one make sense? Make sure to indicate the regions on Diagram 3.3 corresponding to the interior and exterior of the black hole.

Question 3.3g: Comment on this relationship between the event horizon and photon worldlines and describe what it tells you about the event horizon.

Activity 4: Cosmological Spacetime Diagrams – Instructions

In this activity, you will be drawing more spacetime diagrams. The difference between these diagrams and the ones you drew previously is that these will model an expanding spacetime. This activity uses the same set conventions used in the Black Hole Spacetime Diagram activity:

- The time axis is still vertical and the space axis is still horizontal.
- Gravitational physicists generally measure everything in meters and multiply things by G and c to eliminate kilograms and seconds. You will still have both axes scaled the same way (i.e., 1 m on the space-axis is the same length on your paper as 1 m on the ct-axis).
- The line representing an object's path through spacetime (i.e., its time vs position graph) is still its worldline.
- A point in spacetime is still an event.

Make all spacetime diagrams as *quantitatively* accurate as possible and always label all important worldlines and events. Important features of a plot may difficult to see if the plot is too small. As you work through this activity, make sure you are answering the questions; they are italicized in the Instructions and collected on an Answer Sheet for reference. Keep in mind that the Answer Sheet may not give full context, so work from the Instructions. The ideas in this activity build on each other fairly rapidly, so **make sure you are fully comfortable with something before moving on**.

Diagram 4.1

In an empty, expanding Universe, the spacetime interval is given by

$$\Delta s^2 = -c^2\left(1-\frac{\Lambda r^2}{3}\right)\Delta t^2 - \frac{2c\Lambda r^2}{3}\Delta t\Delta r + \left(1+\frac{\Lambda r^2}{3}\right)\Delta r^2. \tag{28}$$

Obviously, our Universe is not empty, so this is only a rough model. It does capture the expansion of the Universe, which matches cosmological observations. It also captures that this expansion is (in part) driven by *dark energy*, which is a property of spacetime (even empty spacetime) which is controlled by Λ, the *cosmological constant*.

For the photon worldlines, the relationship between Δt and Δr is given by

$$\frac{c\Delta t}{\Delta r} = 1 \qquad \text{or} \qquad \frac{c\Delta t}{\Delta r} = -\frac{3+\Lambda r^2}{3-\Lambda r^2}. \tag{29}$$

Question 4.1a: Derive the relationship between Δt and Δr given by (29).

You may recognize that (29) are differential equations. You do not need to worry about solving differential equations here; the important thing is the slopes give you a relationship for how t and r are related:

$$ct = r + k_3 \qquad \text{or} \qquad ct = r - \sqrt{\frac{3}{\Lambda}}\ln\left|\frac{\sqrt{3}+r\sqrt{\Lambda}}{\sqrt{3}-r\sqrt{\Lambda}}\right| + k_4, \tag{30}$$

where k_3 and k_4 are just some numbers which you are free to choose. These functions tell you what the photon worldlines look like in the cosmological spacetime. Note that you need to take the absolute value of the argument of the logarithm.

Pick a few evenly spaced values for k_3 and k_4 between -15 m and 15 m. Use a value of $\Lambda = \frac{1}{3}$ m^{-2}. Plot both equations of (30) for all of these k_3 and k_4 values. A blank graph is provided on the answer sheet.

This spacetime contains a horizon which is similar to the event horizon around the black hole. Since this is meant to model the Universe at large, this horizon is called a *cosmological horizon.* **Question 4.1b: What happens to the light cones near the cosmological horizon on Diagram 4.1?**

Question 4.1c: Does the cosmological horizon prevent you from returning the same way the event horizon does? Which way(s) can objects move across the horizon?

Question 4.1d: Where is the cosmological horizon? Give an r value. One way to find where a horizon will be in a spacetime is to look at the slope of the photons' worldlines. **Question 4.1e: What happens to (29) near this r value?** This is a characteristic of horizons. Use this to figure out what the horizon would be for other values of Λ. **Question 4.1f: What is the relationship between the r value for the horizon and Λ?** Picking $\Lambda = \frac{1}{3}$ m^{-2} was mostly to make plotting easier. Measurements for Λ are more like $\sim 10^{-52}$ m^{-2}. **Question 4.1g: How would your diagram change if you had plotted with the more realistic Λ?**

One way in which the expansion of the Universe can be detected is that distant galaxies appear to be moving away from us; furthermore, the further away they are the faster they are moving away from us. It turns out that the speed at which galaxies move away from us depends on distance as

$$\frac{v}{c} = r\sqrt{\frac{\Lambda}{3}}. \tag{31}$$

Question 4.1h: What happens to the speed of galaxies given by (31) at the cosmological horizon?

Thinking about the speed should give you some indication of why the cosmological horizon exists.[8] **Question 4.1i: Based on the observed velocities of galaxies, why does the cosmological horizon exist where it does?**

Diagram 4.2

Of course, (28) says nothing about whether or not the paper which the spacetime diagram is drawn on is folded or curved, nor does any other Δs^2 you have looked at so far. For simplicity here, go back to the simpler case where $\Delta x^2 = -c^2\Delta t^2 + \Delta x^2$. Assume that the space axis is wrapped up in a circle. On Diagram 4.2, x completes a full circle between $x = -\pi$ and $x = \pi$ (but leave the ct-axis straight for now). This will make the spacetime the surface of a cylinder. You may want to actually wrap up your paper to better visualize this.

Question 4.2a: Determine the timelike future for the event $(x, ct) = (\pi/2 \text{ m}, -3\pi/4 \text{ m})$ on Diagram 4.2. Draw this lightcone on the Diagram.

Question 4.2b: For an observer who is stationary at $x = -\pi/4$ m, show that they can see their past self by showing how light could reach them from an event on their own worldline. Draw this on the Diagram and explain what is happening.

[8] Similar to modeling a black hole as something with a escape velocity equal to the speed of light.

Diagram 4.3

For Diagram 4.3 wrap up the ct-axis similar to the way you wrapped up the x-axis in Diagram 4.2. You should have a cylinder again but this time time is wrapped around and space is not.

Question 4.3a: Determine the timelike future for the event $(x, ct) = (0\text{ m}, \pi/4\text{ m})$ **on Diagram 4.3. Draw this lightcone on the Diagram.**

Question 4.3b: What happens when an observer always remains stationary in Diagram 4.3? This sort of situation is sometimes described as a form of time travel. Explain. Did the observer ever move backward through time?

Question 4.3c: Do you think it is reasonable that our Universe would be wrapped up as in Diagram 4.2? What about Diagram 4.3? Could it be both? Neither?

To review, in this activity you explored the geometry of spacetime in an expanding Universe and saw that:

- Light cones give another way to describe how observers move through a spacetime.
- In an expanding Universe, the spacetime interval is no longer $\Delta s^2 = -c^2\Delta t^2 + \Delta r^2$. This causes the trajectories of light (and observers) to bend. This is how general relativity describes a gravitational interaction.
- An expanding Universe contains a horizon which shares similarities with black holes' event horizons. This cosmological horizon comes from the rate of expansion of the Universe increasing the further away an observer looks, eventually becoming faster than the speed of light.
- A Universe may be wrapped up so that time and/or space 'repeat' themselves.

Activity 4: Cosmological Spacetime Diagrams – Answer Sheet

Make all spacetime diagrams as *quantitatively* accurate as possible and always label all important worldlines and events. Important features of a plot may difficult to see if the plot is too small. As you work through this activity, make sure you are answering the questions; they are italicized in the Instructions and collected on an Answer Sheet for reference. Keep in mind that the Answer Sheet may not give full context, so work from the Instructions. The ideas in this activity build on each other fairly rapidly, so **make sure you are fully comfortable with something before moving on**.

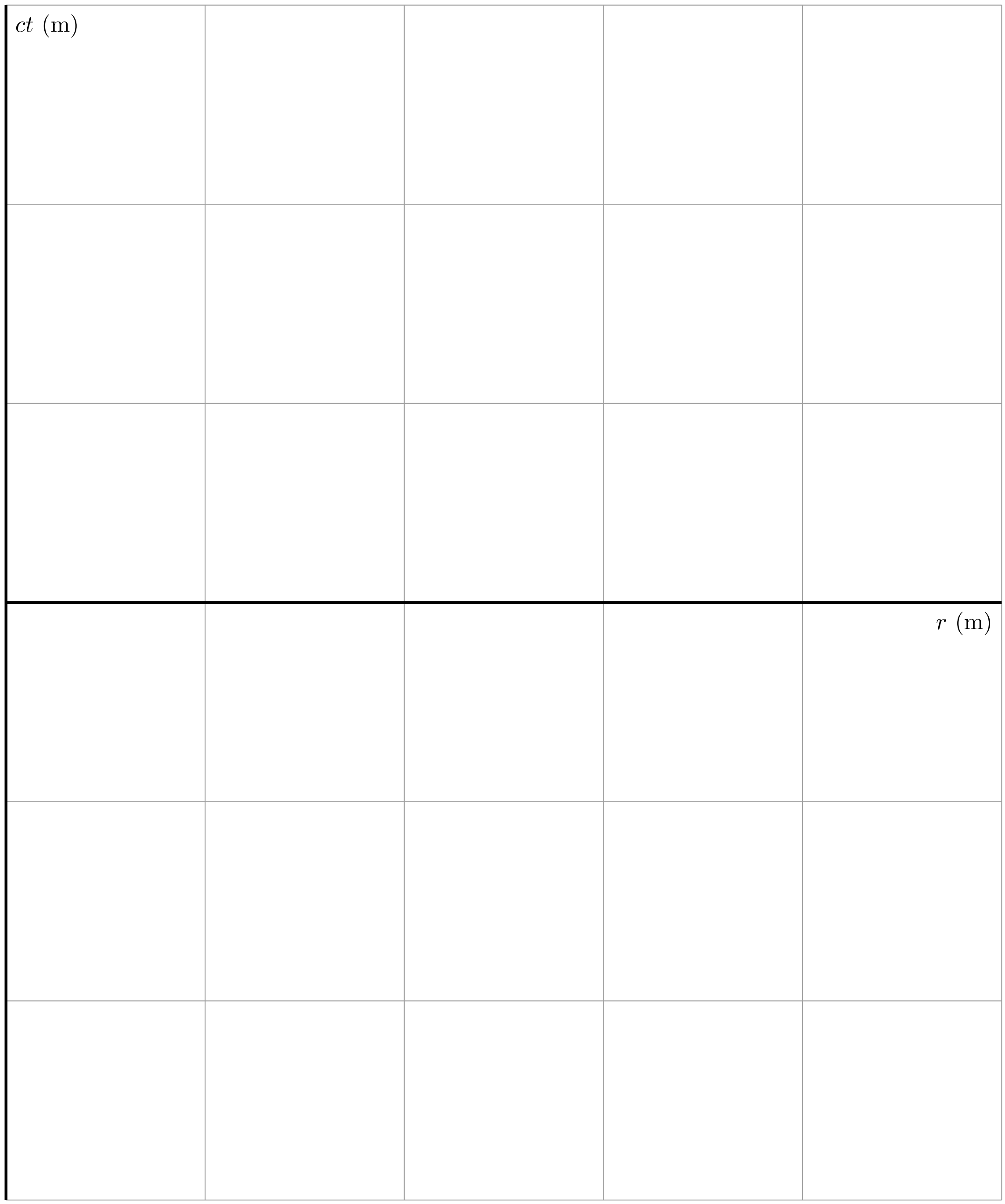
Question 4.1a: Derive the relationship between Δt and Δr given by (29).

Question 4.1b: What happens to the light cones near the cosmological horizon on Diagram 4.1?

Question 4.1c: Does the cosmological horizon prevent you from returning the same way the event horizon does? Which way(s) can objects move across the horizon?

Question 4.1d: Where is the cosmological horizon? Give an r value.

Diagram 4.1



Question 4.1e: What happens to (29) near this r value?

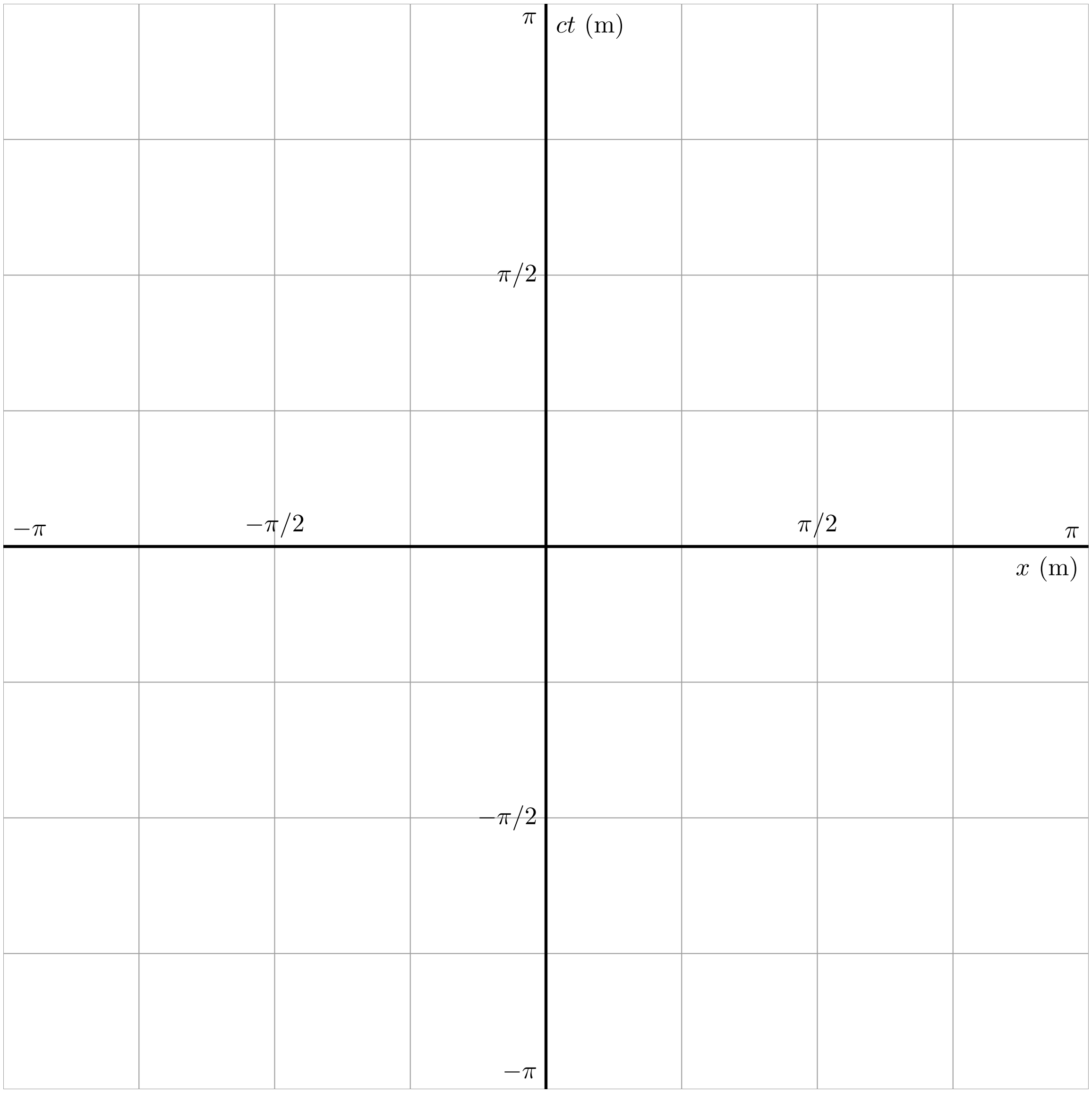
Question 4.1f: What is the relationship between the r value for the horizon and Λ?

Question 4.1g: How would your diagram change if you had plotted with the more realistic Λ?

Question 4.1h: What happens to the speed of galaxies given by (31) at the cosmological horizon?

Question 4.1i: Based on the observed velocities of galaxies, why does the cosmological horizon exist where it does?

Diagram 4.2



Question 4.2a: Determine the timelike future for the event $(x, ct) = (\pi/2 \text{ m}, -3\pi/4 \text{ m})$ on Diagram 4.2. Draw this lightcone on the Diagram.

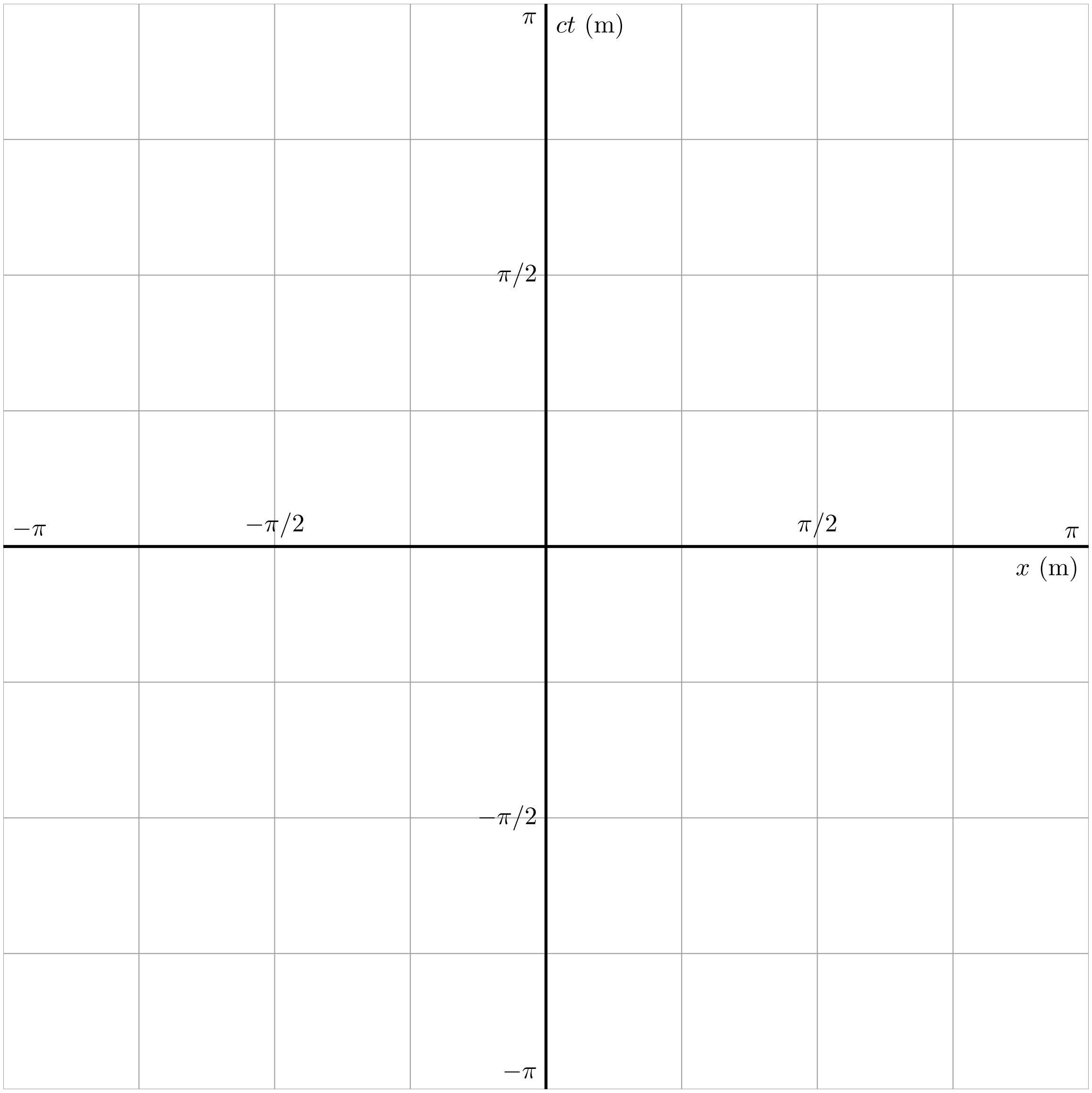
Question 4.2b: For an observer who is stationary at $x = -\pi/4$ m, show that they can see their past self by showing how light could reach them from an event on their own worldline. Draw this on the Diagram and explain what is happening.

Question 4.3a: Determine the timelike future for the event $(x, ct) = (0 \text{ m}, \pi/4 \text{ m})$ on Diagram 4.3. Draw this lightcone on the Diagram.

Question 4.3b: What happens when an observer always remains stationary in Diagram 4.3? This sort of situation is sometimes described as a form of time travel. Explain. Did the observer ever move backward through time?

Question 4.3c: Do you think it is reasonable that our Universe would be wrapped up as in Diagram 4.2? What about Diagram 4.3? Could it be both? Neither?

Diagram 4.3



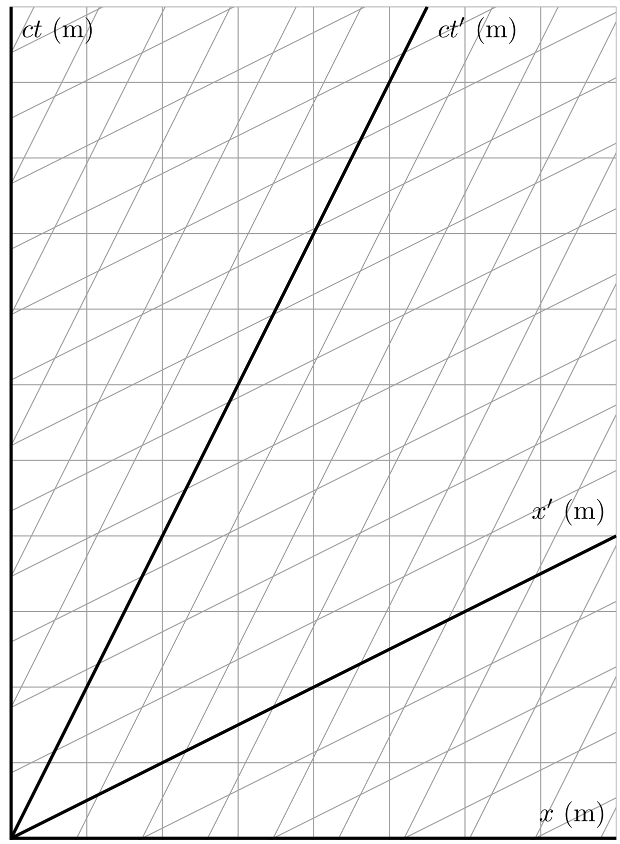
Problems

1. Assume you see two of your friends, Friend 1 and Friend 2, speeding by you in rocket ships. Friend 1 is moving with velocity $+c/2$ and Friend 2 is moving with velocity $+4c/5$. The three of you pass each other when your watches all read zero.

   (a) On the diagram to the right, draw your friends' two worldlines. The graph paper here shows your coordinate system as well as the coordinate system of Friend 1.

   (b) How fast does Friend 1 measure Friend 2 to be going? Give the explicit events and calculations Friend 1 uses to determine Friend 2's speed.

   (c) Are you surprised by Friend 1's answer? What does this tell you?



2. From the Spacetime Diagrams activity, you are pretty familiar with measuring everything in meters. But astrophysicists do not stop there in making weird systems of units. When working with black holes we typically also measure mass in meters. Please read A Note About Units on page 4 and then show, by multiplying and/or dividing by G and c, (a) 1 $M_\odot$ corresponds to about 1.5 km and (b) 1 $M_\odot$ corresponds to about 5 μs. (c) Comment on the scale of these three values (are they everyday values?); consider the numerical values for c and G (in SI units). (d) The average American is about 175 cm tall, about 38 years old, and masses about 81 kg; what are these values in their distance counterparts in meters?

3. Model a black hole as a spherical object with radius R and mass M. Assume that this object has an escape velocity which is the speed of light c. Find a relationship between the radius and mass of the black hole. This radius is called the Schwarzschild radius. (As a note, this gives the right answer for not really the right reason; similar to the Bohr model for Hydrogen.)

4. The spacetime interval for a black hole of mass M gives the relationship between an observer's proper time τ, coordinate time t, and the radius r

$$-c^2\Delta\tau^2 = -c^2\left(1-\frac{2GM}{c^2 r}\right)\Delta t^2 + \left(1-\frac{2GM}{c^2 r}\right)^{-1}\Delta r^2. \tag{32}$$

   For an observer who is fixed at $r = 4GM/c^2$, does time move faster or slower than for an observer at $r=\infty$? By what factor? (Hint: What are Δr and Δt for two events on the worldline of the observer at $r = 4GM/c^2$? What about for the observer at infinity, assuming the two events happen at the same coordinate time t as the events experienced by the $r = 4GM/c^2$ observer?) For an actual application of this effect, see Problem 7.

5. (**Challenge!**) The spacetime interval

$$\Delta s^2 = -c^2\left(1-\frac{2GM}{c^2r}\right)\Delta t^2 - \frac{4GM}{cr}\Delta t\Delta r + \left(1+\frac{2GM}{c^2r}\right)\Delta r^2. \tag{33}$$

can be used as an equally valid model for a black hole (this version of the spacetime interval corresponds to yet another coordinate system). Repeat the procedure for Diagram 3.2 with (33) replacing (23) as your starting point. (a) How is Diagram 3.2 different in this case? (b) There is an event horizon in this spacetime as well. Which way can people cross over this event horizon? (c) This solution is often referred to as a *white hole*. Explain why. (d) This solution is needed in order to interpret the lower half ($cT < -R$) of Diagram 3.3. Extend the coordinate system for Diagram 3.3 and include all solutions to the equations which were ignored in the activity. What does the white hole solution tell you about the extended Diagram 3.3?

6. (**Challenge!**) A black hole in an expanding Universe can be modeled by the spacetime interval

$$\Delta s^2 = -c^2\left(1-\frac{2GM}{c^2r}-\frac{\Lambda r^2}{3}\right)\Delta t^2 + c\left(\frac{4GM}{c^2r}-\frac{2\Lambda r^2}{3}\right)\Delta t\Delta r + \left(1+\frac{2GM}{c^2r}+\frac{\Lambda r^2}{3}\right)\Delta r^2. \tag{34}$$

To simplify the algebra for this Diagram a bit, assume $GM/c^2 = 30/31$ m (expressed more normally, that would be $M = 1.3\times10^{27}$ kg) and $\Lambda = 3/124\ \mathrm{m}^{-2}$ and you can be sloppy with and/or ignore units.[9] Repeat the procedure for Diagram 3.2 with (34) replacing (23) as your starting point. Note,

$$\frac{c\Delta t}{\Delta r} = \frac{r^3 \pm 4r\sqrt{961-60r}-240}{r^3-124r+240} \tag{35}$$

(with r and Δr in meters and Δt in seconds) leads to

$$ct = r + \frac{29}{7}\ln\left|29\pm\sqrt{961-60r}\right| + \frac{95}{11}\ln\left|19\mp\sqrt{961-60r}\right| - \frac{984}{77}\ln\left|41\mp\sqrt{961-60r}\right| + k, \tag{36}$$

(with r and in meters and t in seconds). Compare (34) to (23) and (28); what happens if you set $M = 0$ or $\Lambda = 0$? How is this Diagram different than Diagrams 4.1 and 3.2? What features do they share?

7. (**Challenge!**) GPS satellites orbit the Earth ($M_\oplus = 5.97\times10^{24}$ kg, $R_\oplus = 6.37\times10^6$ m) twice a day. [Note that the answers you get when working through this problem are likely order of magnitude estimates at best.]

   (a) Determine the altitude and speed of a GPS satellite's orbit. What is the relative speed of the satellite compared to an observer on the ground? If you've worked Problem 1, note that these speeds are small enough that simple velocity addition and Kepler's laws are valid.

   (b) Based on the speed of a satellite and the speed of an observer on the ground, does special relativity predict that the satellite's clock will run slower or faster than a clock near the Earth's surface? By what fraction? Most calculators cannot store enough digits to compute the needed Lorentz factor to sufficient precision, so make use of $(1+x)^n \approx 1+nx$ for small x. Hint: think about time dilation.

   (c) Based on the altitude of a satellite and the altitude of an observer on the ground, does general relativity predict that the satellite's clock will run slower or faster than a clock near the Earth's surface? By what fraction? You'll need to use $(1+x)^n \approx 1+nx$ again. Hint: think about Problem 4.

   (d) Over the course of one day, what is the total accumulated time the satellite's clock gains or loses relative to a clock on the ground?

   (e) If these effects were not accounted for, how much error in location finding would GPS accumulate each day?

[9] Note that this value for Λ is unrealistically large in order to make graphing easier. This is a somewhat artificial model anyway, but does provide the simplest model which includes both a black hole and an expanding Universe.