Power law ρ and $\Delta\Sigma$

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For a power law density profile:

$$\rho(r) = \rho_0 r^\alpha \tag{1}$$

The surface density profile is:

$$\Sigma(R) = \sqrt{\pi}\rho_0 \frac{\Gamma\left(-\frac{\alpha+1}{2}\right)}{\Gamma\left(-\frac{\alpha}{2}\right)} R^{\alpha+1} \tag{2}$$

And the mean enclosed surface density is:

$$\frac{2\sqrt{\pi}\rho_0}{\alpha+3} \frac{\Gamma\left(-\frac{\alpha+1}{2}\right)}{\Gamma\left(-\frac{\alpha}{2}\right)} R^{\alpha+1} \tag{3}$$

So the slopes are related very simply:

$$\beta = \frac{\mathrm{d}\log\Delta\Sigma}{\mathrm{d}\log R} = \frac{\mathrm{d}\log\rho}{\mathrm{d}\log r} + 1 \tag{4}$$

And given a measurement of $\Delta\Sigma$ at a location R and an estimate of its slope β at the same point, then the normalization constant of the density is:

$$\rho_0 = \frac{\Delta\Sigma}{\sqrt{\pi}\left(\frac{2}{\beta+2}\right)\frac{\Gamma\left(-\frac{\beta}{2}\right)}{\Gamma\left(-\frac{\beta-1}{2}\right)} R^\beta} \tag{5}$$

The enclosed mass is:

$$M(<r) = 4\sqrt{\pi} r^2 \Delta\Sigma \left(\frac{\Gamma\left(-\frac{\beta-1}{2}\right)}{\Gamma\left(-\frac{\beta}{2}\right)}\right)\left(\frac{2+\beta}{-\beta}\right) \tag{6}$$

In the singular isothermal sphere case ($\beta = -1$), this reduces to $M(<r) = 4r^2\Delta\Sigma$, as expected. Finally, the acceleration is:

$$g(r) = 4\sqrt{\pi} G \Delta\Sigma \left(\frac{\Gamma\left(-\frac{\beta-1}{2}\right)}{\Gamma\left(-\frac{\beta}{2}\right)}\right)\left(\frac{2+\beta}{-\beta}\right) \tag{7}$$

Note that actually using this as an enclosed mass estimator or acceleration estimator on any $\Delta\Sigma$ profile that is not a pure power law is a Really Bad Idea and is likely to imply the presence of negative mass (just like the SIS estimator does if $\mathrm{d}\log\Delta\Sigma/\mathrm{d}\log R < -2$).