Stokes vectors, Mueller matrices, and polarized scattered light

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poles, electric and magnetic, in both media. The Brewster angle is given by Eqs. (22) and (23), using relative values for the permittivity and permeability.

V. CONCLUSION

Fresnel's equations can be written in a factored form that facilitates comparison with the scattering interpretation of reflection and refraction. Although they are very different in appearance, the factored forms are mathematically identical to the original forms. Both forms of Fresnel's equations hold for arbitrary media.

In transparent media, the two factors have a simple physical interpretation. They separately express the dynamical and wave kinematical aspects of reflection and transmission considered as a scattering phenomenon. One factor expresses the scattering pattern of the individual electric and magnetic dipoles, the other the coherent radiation pattern of the dipole array.

The physical interpretation of the separate factors suggests improvements to the well-known scattering model of Brewster's law. Because it is based directly on Fresnel's equations, the new scattering model is applicable to either magnetic or nonmagnetic media and to both external and internal reflections.

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The complete characterization of scattered light is described in the context of Stokes vectors and Mueller matrices which highly motivates the measuring procedures. The most general form of the scattering matrix coupled with polarizers and quarter wave plates elegantly demonstrates the physical relationship among the matrix elements and polarization measurements.

I. INTRODUCTION

Research with polarized scattered light deals with the entire scattering process in the context of Stokes vectors, Mueller matrices, reference angles, intensities, and fractional and percent polarizations. An introduction to optical polarization often starts with a description of the optical elements which physically act as polarizers and retarders. The Mueller matrix–Stokes vector calculus is then used to show mathematically how these optical elements affect a light beam. Such an approach often creates an impression that these optical elements are novel curiosities and that the matrix math is nothing more than an exercise in short-cut manipulations. Understanding the effects that occur when polarizers and retarders are manipulated requires a precise understanding of the vector nature of light and the coordinate systems involved.

One of the first goals in studying polarization would be to understand the procedures necessary to determine the kinds of polarization states in a complex light beam, as is succinctly done by Longhurst.[1] However we can go one important step farther and learn how to measure exactly the intensities of each polarization component in a light beam and determine the fractional and percent polarizations.

During our light scattering research[2] with perfect spheres, fibers, and irregular scatterers we have developed a pedagogical approach which demonstrates the relationship among these quantities, their physical significance, and the simplicity of interpreting experimental data.

A practical introduction to polarized light might start with the question: "Is light from the moon polarized?" To answer this question one needs an operational definition of polarization and a highly motivated set of measurements. The experiments that could be done are illustrated in Fig. 1 and described below.

(1) Place a linear polarizer between the moon and eye (or

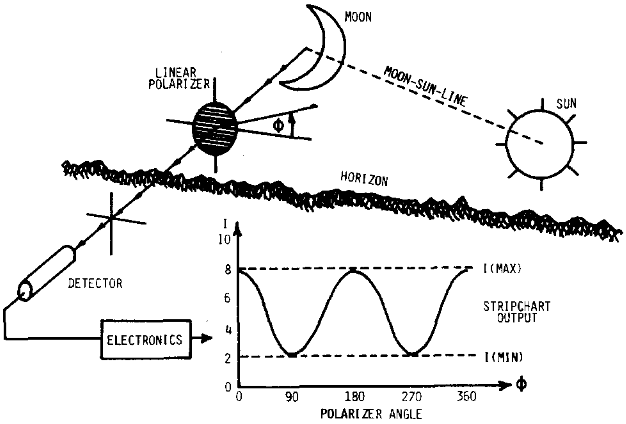

Fig. 1. The relationship between polarizer axis and a reference line for a polarization measurement.

another detector). Rotate the polarizer through angle ϕ. An intensity change with ϕ indicates that the moon light is polarized.

(2) Measure the angle between the polarizer axis and *some* reference axis such as the horizon or moon–sun line when the intensity is a maximum and a minimum. This angle determines the axis and direction of the linear polarization. The example in Fig. 1 shows maximum intensity occurring when the polarizer axis is parallel to the horizon.

(3) Measure the maximum intensity $I(\max)$ and minimum intensity $I(\min)$ of polarized light with a detector. The detector output will give a signal (in volts, amps, counts/s, deflection, etc.) proportional to the intensity. From these measurements determine the fractional polarizations,

$$f(\max) = \frac{I(\max)}{I(\max) + I(\min)} = \frac{I(\max)}{I(\text{total})} = 0.8\,,$$

$$f(\min) - \frac{I(\min)}{I(\max) + I(\min)} = \frac{I(\min)}{I(\text{total})} = 0.2\,.$$

(4) The percent polarization is defined as

$$P = \frac{I(\max) - I(\min)}{I(\max) + I(\min)} \times 100\%$$

$$= [f(\max) - f(\min)] \times 100\% = 60\%\,.$$

The above procedure can establish the linear polarization of reflected, scattered, and transmitting light from natural objects such as the moon, stars, sky, clouds, comet tails, interstellar dust, etc.

However, there is much more polarization information available in the detected light than obtained in the above simple experiment. First, the polarization could be wavelength dependent. Second, other polarizations such as circular and elliptical could be present and would not be detected by a single linear polarizer. Third, the polarization could be angle dependent—dependent on both the angle of illumination of and scatter at the object. Fourth, the incident illuminating radiation might itself be polarized. Consequently the results of the above measurement would be different if the moon were illuminated with a single polarization state and if polarization of the light at a particular wavelength and scattering angle were measured. Narrowband interference filters could be used to study the wavelength dependence—assuming that the interaction is linear (no fluorescence or nonlinear effects), but we cannot choose the polarization state to illuminate such natural objects. In addition there is usually very little control of the angle of illumination and scatter. Nevertheless limited measurements, often the best that can be done, necessarily produce limited information. Yet it can be valuable for characterizing the nature of distant objects.

Polarization measurements of light scattered in laboratory experiments are much different. We can control the angles of illumination and scatter to an arbitrarily high degree of accuracy, choose the illuminating wavelength and polarization condition exactly, and measure the desired polarization in the scattered light. Although a large number of different measurements are possible if one considers all input and exit beam polarizer combinations and their orientations, only a few measurements *of the right kind* are needed to completely characterize the polarization of any light beam. The necessary procedure is elegantly demonstrated by the Stokes vector–Mueller matrix approach to polarization and light scattering. By exhausting all alternatives we will define a set of 16 independent measurements.

II. THE MUELLER MATRICES AND STOKES VECTORS

When a suitable coordinate system is established each optical component in a particular orientation can be mathematically represented by its appropriate 4×4 sixteen-element Mueller matrix. Each light beam can be prepared into a pure polarization state and represented by its appropriate four-component Stokes vector. The matrices for certain orientations of various optical components are well known.[3] Those for a clear plate of glass (open hole), opaque sheet, linear polarizer, quarter wave plate, and circular polarizer are given in Fig. 2. In each case the unpolarized beam passes through the optical element from left to right, emerging in the pure polarization state shown. The Mueller matrix and Stokes vector for each geometrical orientation is written below the optical component and symbol describing the polarized light. Note that right-hand circular polarization occurs when the E vector rotates counterclockwise as seen looking *down the beam*. (At fixed point in space the E vector would be seen as rotating counterclockwise looking down the beam.) Also note the importance of the relative angles between the linear polarizer and quarter wave plate for making left- or right-handed circularly polarized light. In a later section, we will distinguish between an analyzer and a polarizer. The perspective views in Fig. 2 identify the coordinate system, axis, and direction of the light beam for each case.

We now define the symbols to be used throughout this discussion. First we define the following Stokes vectors $|V|$: $|V_t|$, $|V_h|$, $|V_v|$, $|V_+|$, $|V_-|$, $|V_r|$, $|V_l|$, where t, h, v, $+$, $-$, r, and l stand, respectively, for unpolarized, horizontal, vertical, $+45°$, $-45°$, right-hand circular, and left-hand circular. For example $|V_-|$ is the Stokes vector for $-45°$ linear polarized light. Stokes vectors, some of which are listed in Fig. 2, are usually defined as a column with components I, Q, U, V.

We also define the following Mueller matrices: $[t]$, $[0]$, $[h]$, $[v]$, $[+]$, $[-]$, $[\lambda/4]$, $[r]$, $[l]$, where $[t]$, $[0]$, and $[\lambda/4]$ are the matrices for a clear plate of glass (open hole), opaque plate, and quarter wave plate, respectively. For example, $[-]$ is the Mueller matrix for a $-45°$ linear polarizer. We assume all optical components are perfect and that their

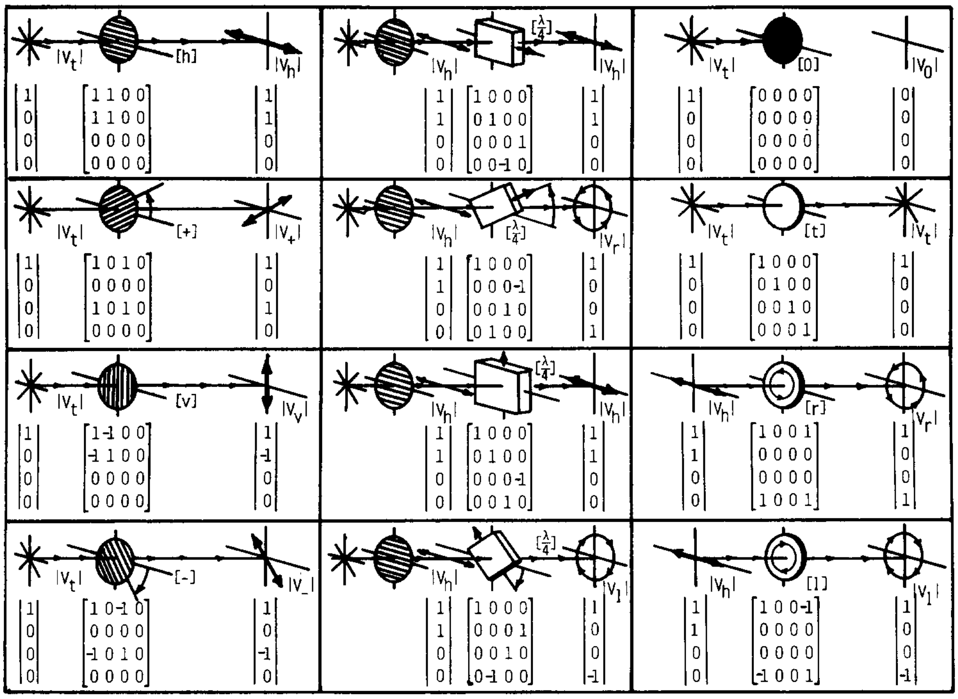

Fig. 2. Geometrical arrangement of some optical elements and their Mueller matrices. The Stokes vector is listed below the vector that represents a particular polarization state.

absorption effects are zero. We are also going to ignore the normalization constants that go with each matrix. Later we will show how each polarization signal measured with a photo detector can be calibrated and normalized to yield the proper value for the polarization.

The following calculation demonstrates the Mueller matrix–Stokes vector mathematics.

A quarter wave plate $[\lambda/4]$ with its fast axis $+45°$ to a horizontally polarized beam $|V_h|$ produces right-hand circularly polarized light $|V_r|$.

$$[\lambda/4]|V_h| = |V_r|,$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{vmatrix} 1 \\ 1 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 1 \end{vmatrix} = \begin{vmatrix} I \\ Q \\ U \\ V \end{vmatrix} = \begin{vmatrix} I \\ 0 \\ 0 \\ V \end{vmatrix}.$$

This says that the beam is totally right-hand circularly polarized and that $I = V$ $(Q = U = 0)$. The relationship $I^2 \geqslant Q^2 + U^2 + V^2$ exists among the components of all Stokes vectors. The equality occurs for totally polarized light, the inequality occurs when an unpolarized component exists. (The Jones matrices cannot treat unpolarized light.)

The Stokes vectors and their components are related to the electric field vectors:

$$\begin{aligned} I &= \langle E_h E_h^* + E_v E_v^* \rangle \\ Q &= \langle E_h E_h^* - E_v E_v^* \rangle \\ U &= \langle E_h E_v^* + E_v E_h^* \rangle \\ V &= \langle i(E_h E_v^* - E_v E_h^*) \rangle \end{aligned} \equiv \begin{vmatrix} I \\ Q \\ U \\ V \end{vmatrix},$$

where the angle brackets represent vector time averages over time periods longer than the period $T = 1/f$, where f is the optical frequency $(\sim 10^{13}/\mathrm{s})$.

Some of the Stokes components have an obvious interpretation. For example, $I = \langle E_h E_h^* + E_v E_v^* \rangle$ is equal to the total intensity since it is the time average of the sum of products of E-field amplitudes. Since any light beam can be represented as an instantaneous sum of its E_h and E_v vectors, we have

$$I_h = \langle E_h E_h^* \rangle, \quad \text{when } E_v E_v^* = 0,$$
$$I_v = \langle E_v E_v^* \rangle, \quad \text{when } E_h E_h^* = 0,$$

then $I_t = I_h + I_v$.

Also we see that $Q = \langle E_h E_h^* - E_v E_v^* \rangle$ is related to the linear polarization since

$$Q = \langle E_h E_h^* - E_v E_v^* \rangle = I_h - I_v.$$

The linear polarization P is then

$$\frac{I_h - I_v}{I_h + I_v} = \frac{I(\max) - I(\min)}{I(\max) + I(\min)} = \frac{P}{100\%} = \frac{Q}{I}.$$

The matrices listed in Fig. 2 correspond to certain standard orientations of the optical elements. Note that certain matrix elements seem to "oscillate" between their maximum and minimum values as the optical element is rotated through $\phi = 360°$. For example, element 21 (2nd row, 1st column) of $[h]$ has values $+1, 0, -1, 0, +1$, for $\phi = 0$, 45°, 90°, 135°, and 180°, respectively. Its value "oscillates" one full cycle for a 180° rotation of the polarizer. Element 31 (3rd row, 1st column) oscillates one full cycle also for a 180° rotation, but out of phase by 90° with respect to element 21. Some matrix elements remain zero for all ϕ, while some others oscillate only between 0 and $+1$.

The details of these oscillations are displayed by the general mathematical form of the Mueller matrices for polarizers and quarter wave plates.[3] We have

for a linear polarizer

$$\begin{bmatrix} 1 & C_2 & S_2 & 0 \\ C & C_2^2 & C_2S_2 & 0 \\ S_2 & C_2S_2 & S_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for a quarter wave plate

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C_2^2 & C_2S_2 & -S_2 \\ 0 & C_2S_2 & S_2^2 & C_2 \\ 0 & S_2 & -C_2 & 0 \end{bmatrix}$$

where $C_2 = \cos 2\phi$, $S_2 = \sin 2\phi$, $C_2^2 = \cos^2 2\phi$, and $S_2^2 = \sin^2 2\phi$.

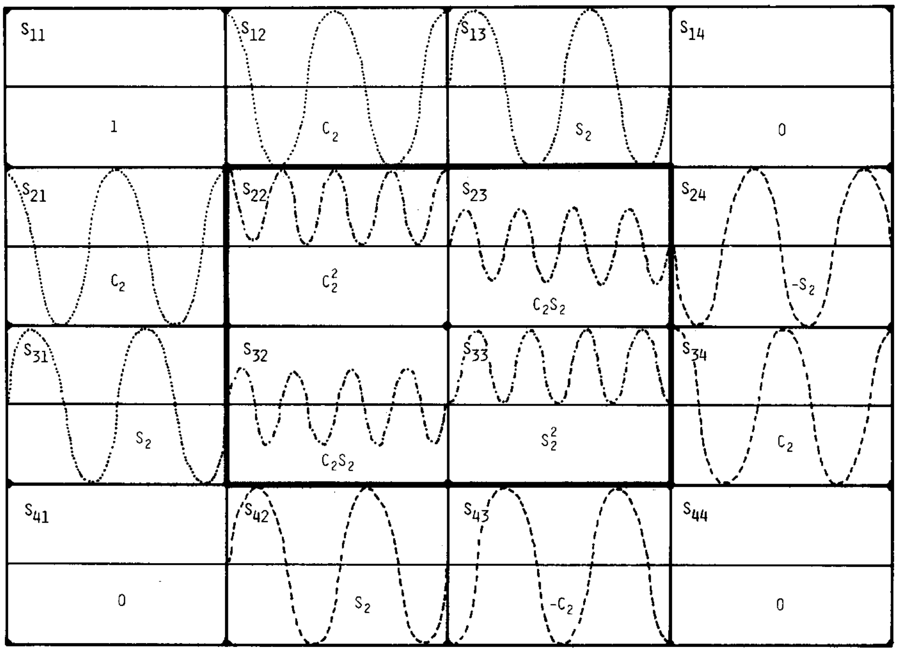

Fig. 3. The Mueller matrix for a linear polarization (·····) and a quarter wave plate (- - - - -). The center box contains the matrix elements that are common to both (- · · · · ·).

Figure 3 displays in a matrix array the actual curves of these mathematical expressions. The ϕ dependence of each function is shown as ϕ varies from 0° to 360° along the abcissa of each matrix element position, corresponding to a full 360° rotation of each optical element. The ordinate varies between ± 1. Note that the middle matrix elements S_{ij} (i and $j \neq 1$) are identical for both the linear polarizer and quarter wave plate. The characteristics specific to a linear polarizer lie in the first row, first column elements; those specific to a quarter wave plate lie in the fourth row, fourth column elements. This observation will be important for interpretation later on.

III. THE SCATTERING MATRIX

For completeness and maximum generality we introduce an "optical element" called a scatterer.[4] It can depolarize the incident beam, mix its polarization states, and change its direction (scatter it into all θ and ϕ). The scatterer is also represented by a matrix called the scattering matrix $[S] = [S(\theta\phi)]$. The position of each matrix element is indicated by S_{ij}, none of which are zero for the most general case. All $S_{ij} = S_{ij}(\theta\phi)$,

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix}.$$

Depending on the symmetry and certain optical properties of the scattering element $[S]$, some S_{ij} could be equal to others and some might be zero. For example, if the scatterer were to act exactly like a pure linear horizontal polarizer $[h]$, the matrix would have $S_{11} = S_{12} = S_{21} = S_{22} = 1$ with all other $S_{ij} = 0$. If the scatterer were to act *somewhat* as a circular polarizer, one or more of the S_{i4} or S_{4j} would be nonzero as shown in Fig. 2.

IV. THE EXPERIMENTAL SYSTEM

The experimental system that can measure polarized light scattered into θ and ϕ from a scatterer is shown in Fig. 4. Only three definite Stokes vector components are needed to uniquely establish the polarization state of a light beam.

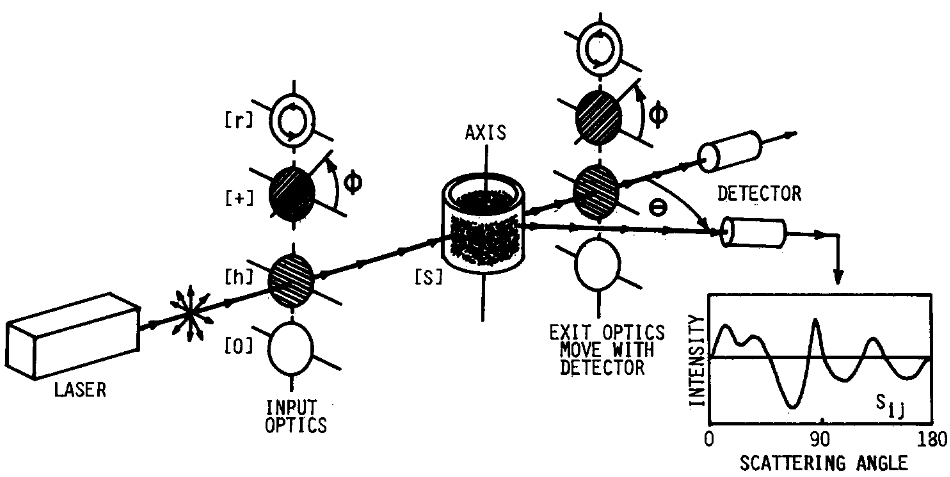

Fig. 4. The experimental optical system used to measure all elements of the scattering matrix and polarization states. The detector can be swung through angle $\theta = 180°$ while its signal is displayed on a strip chart.

In general they are: (1) a linear polarization (relative to any axis), (2) a linear polarization at 45° to the first axis, and (3) a left- or right-handed circular polarization. It is very convenient to choose the following three: horizontal$\equiv|V_h|$, $+45°\equiv|V_+|$, and right-hand circular$\equiv|V_r|$ since these polarizations are represented by Stokes vectors with *positive* components as shown in Fig. 2. Note that these vectors are the ones created from an unpolarized beam with Mueller matrices also having *positive* elements.

Since the scattering interaction mixes the initially pure polarization state into combinations of linearly and circularly polarized components, we have a choice as to which ones to measure. Again only three definite components are needed and it is convenient to also measure the same Stokes vectors in the scattered beam. Experimentally one of the three input vectors $|V_h|$, $|V_+|$, or $|V_r|$ are chosen to illuminate the scatterer $[S]$. For each particular input vector, three output vectors, $|V_h|$, $|V_+|$, or $|V_r|$, can be measured after scattering. These 3×3 input–output combinations make a set of nine polarization measurements. In addition, the scatterer can also be illuminated with completely unpolarized light (total intensity) and the total intensity output can be measured. These total intensity measurements, together with the nine polarization combinations give a total of $4\times4=16$ possible input–output Stokes vector combinations to measure to get the total information about the scatterer. The total intensity measurement (open hole on input and exit optics shown in Fig. 4) will be redundant if $I^2=Q^2+U^2+V^2$. However, an intensity calibrated system is needed to verify this relationship.

V. THE SCATTERER

The scatterer is usually an unknown. However it is important to test the scattering instrument (optics, electronics, and mechanical stability), using perfect exactly known scattering systems such as small spheres or fibers of micron diameter.[5] Such perfect systems can be used to test theory and evaluate the entire scattering measurement because their interaction with the incident light can be exactly predicted from fundamental electromagnetic theory. Unknown scatterers could consist of smoke, dust, bioparticles, or any other particulate suspended in air or in solution.[6] Regardless of the nature of the scatterer, in most scattering experiments it is illuminated with a beam of light with wavelength λ_0 prepared into a particular incident Stokes vector while a particular scattered Stokes vector is measured as a function of θ. The scatterer $[S]$ mixes the initially pure polarization state and produces a mixed final Stokes vector. The final Stokes vector will differ from the incident Stokes vector in intensity I, polarization π, and direction θ. For the case of elastic scattering discussed here, $\lambda_0=\lambda$ scattered. The main scattering problem is to define the scatterer in terms of its matrix elements via polarization measurements. The following example shows how this is done.

Illuminate the scatterer $[S]$ with $+45°$ linear polarized light $|V_+|$ and analyze the scattered radiation with a RH circular polarizer $[r]$. We have

$$[S]|V_+|=|V_s|,$$

where $|V_s|$ is the scattered Stokes vector

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14}\\ S_{21} & S_{22} & S_{23} & S_{24}\\ S_{31} & S_{32} & S_{33} & S_{34}\\ S_{41} & S_{42} & S_{43} & S_{44}\end{bmatrix}\begin{vmatrix}1\\0\\1\\0\end{vmatrix} = \begin{vmatrix} S_{11}+S_{13}\\ S_{21}+S_{23}\\ S_{31}+S_{33}\\ S_{41}+S_{43}\end{vmatrix} = \begin{vmatrix} I\\ Q\\ U\\ V\end{vmatrix} = |V_s|\,.$$

The scattered Stokes vector is a mixture of polarization

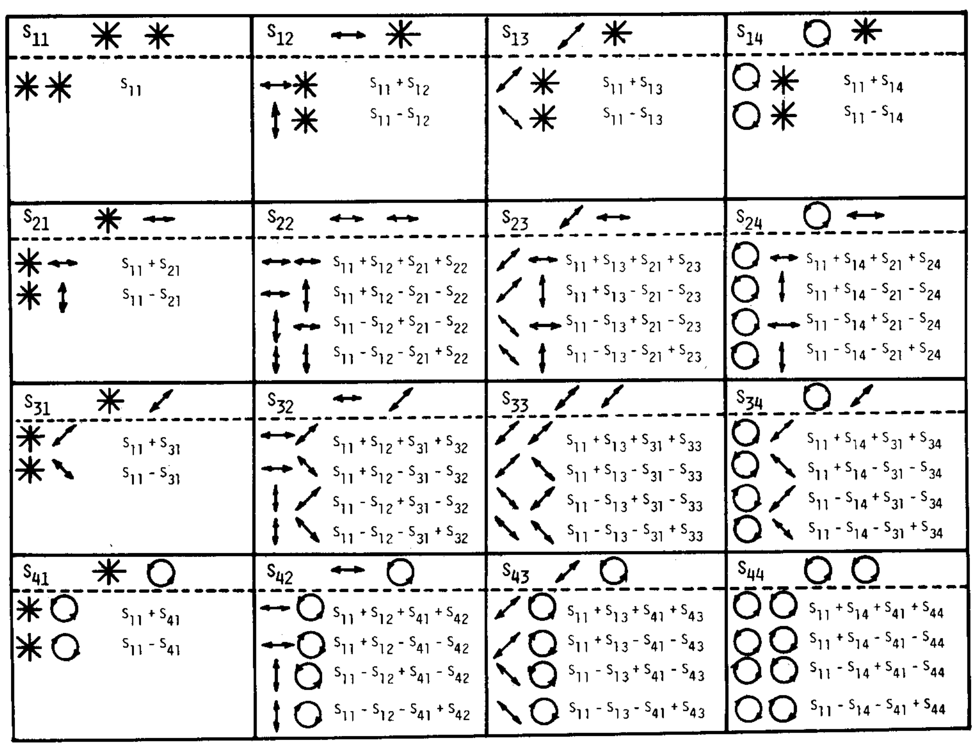

Fig. 5. A matrix array showing the matrix element combination measured for various arrangements of input and output optical elements (open hole, linear, and circular polarizers).

states since its components Q, U, and V, created from the initially pure $+45°$ polarized Stokes vector, are nonzero. It is also a mixture of certain matrix elements. Now analyze the scattered light with a RH circular analyzer $[r]$ and measure the intensity I as a function of scattering angle θ.

$$[r]|V_s| = |V_d|,$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{vmatrix} S_{11}+S_{13} \\ S_{21}+S_{23} \\ S_{31}+S_{33} \\ S_{41}+S_{43} \end{vmatrix}$$

$$= \begin{vmatrix} (S_{11}+S_{13})+(S_{41}+S_{43}) \\ 0 \\ 0 \\ (S_{11}+S_{13})+(S_{41}+S_{43}) \end{vmatrix} = \begin{vmatrix} I \\ Q \\ U \\ V \end{vmatrix}_d = |V_d|.$$

In this case the Stokes vector $|V_d|$ entering the detector consists of only RH circular polarized light ($Q = U = 0$). But most important is that information about it contained in the matrix element combination $(S_{41}+S_{43})$, *has been brought up into the first component* by the RH circular analyzer. The total intensity seen by the detector is now $I = (S_{11}+S_{13}) + (S_{41}+S_{43})$, which can be plotted onto a strip chart as a function of scattering angle θ.

We put the results of such calculations for all 16 Stokes vector combinations into a matrix array shown in Fig. 5. Each matrix element label S_{ij} is in the uppermost left-hand corner of each matrix element block. The symbols to the immediate right of S_{ij} represent the kind of light involved in the measurement. For example, the elements in matrix element position S_{23} are measured by illuminating the scatterer with $\pm 45°$ linear polarized light while analyzing the scattered light with a linear horizontal or vertical polarizer. Symbols below the dotted line in each box show the complementary orientations of the input–output polarizations. The actual matrix element combinations involved in that measurement are given on the right of each symbol pair.

There are four interesting observations regarding the matrix array:

(1) The kind of polarized light (incident and detected) used in a particular measurement establishes uniquely the matrix elements that will be mixed by that measurement. For example, illuminating the scatterer with $+45°$ polarization and analyzing for horizontal polarization will mix only elements S_{11}, S_{13}, S_{21}, and S_{23}. This combination occurs *only* in matrix element location S_{23}.

(2) Various complimentary orientations of any polarizer configuration give matrix element combinations that differ *only in the sign of the* elements mixed. For example, for S_{23}, changing the final polarizer to pass vertical polarized light instead of horizontal, changes only the sign of elements S_{21} and S_{23}.

(3) Each matrix element combination has a unique location in the matrix. S_{11} occurs in every location, S_{ij} $(i=j)$ occur only along the diagonal. A combination of four matrix elements belongs in the location indicated by the S_{ij} in that combination that contains i and $j \neq 1$.

(4) The matrix elements S_{ij} where i and/or $j = 4$ contain information about circularly polarized light. This is already evident from the Mueller matrix for a quarter wave plate (circular polarizer) which contains nonzero elements for row 4 and column 4.

VI. THE POLARIZATION MATRIX

Each matrix element combination listed in Fig. 5 is the *first component* (total intensity term) of the output Stokes vector. Physically it contains a mixture of polarization states (and matrix elements) and is proportional to the intensity measured by the detector. Each one can be plotted on a strip chart as the detector is scanned from $\theta = 0°$ to $\theta = 180°$. At this point we might ask: "Which are more important or significant, the matrix elements or the polarization intensities?" The answer is—it depends on which you are most comfortable with. A definite relationship exists between the matrix elements and intensities. This is demonstrated by examining how each individual matrix element is measured. For example, a detector set up to measure the *total intensity*, scattered from a sample illuminated with horizontal linear polarized light measures the polarization intensity I_{h0} which connects the matrix elements S_{11} and S_{12}, giving $I_{h0} = S_{11} + S_{12}$. Similarly when the input polarizer is rotated 90° to illuminate the scatterer with *vertical linear* polarized light the detector measures the polarization intensity I_{v0} which connects the same matrix elements S_{11} and S_{12}, but with $I_{v0} = S_{11} - S_{12}$. Subtracting these two signals gives $I_{h0} - I_{v0} = (S_{11}+S_{12}) - (S_{11}-S_{12}) = 2S_{12}$. This matrix element will be zero for scatterers that treat vertical and horizontal input polarization equally, that is, for scatterers equally efficient in completely depolarizing both vertical or horizontally polarized light.

A scatterer can be completely described by measuring all of its matrix elements as a function of θ. This is often the goal of a scattering study. Only two measurements as shown above are needed to determine the matrix elements of the first row S_{1j} and first column S_{i1}. S_{11} can be determined from a single measurement using unpolarized light for illumination and measuring the total scattered intensity. Four measurements are needed to determine each of the other nine matrix elements S_{ij} (i and $j \neq 1$). For example, in order to compute the polarization measurements needed to determine S_{32} proceed as follows:

(1) Measure $I_{h+} = S_{11} + S_{12} + S_{31} + S_{32}$ (horizontal goes to $+45°$),

(2) Measure $I_{h-} = S_{11} + S_{12} - S_{31} - S_{32}$ (horizontal goes to $-45°$),

(3) Subtract $I_{h+} - I_{h-} = 2(S_{31} + S_{32})$,

(4) Measure $I_{v+} = S_{11} - S_{12} + S_{31} - S_{32}$ (vertical goes to $+45°$),

(5) Measure $I_{v-} = S_{11} - S_{12} - S_{31} + S_{32}$ (vertical goes to $-45°$),

(6) Subtract $I_{v+} - I_{v-} = 2(S_{31} - S_{32})$,

(7) Compute $(I_{h+} - I_{h-}) - (I_{v+} - I_{v-}) = (I_{h+} + I_{v-}) - (I_{v+} + I_{h-}) = 4S_{32}$.

Similar measurements and calculations will yield all 16 matrix elements of the scattering matrix $[S]$. The matrix which shows the relationship between the S_{ij} and the polarization measurements is given in Fig. 6.

Not all matrix elements are independent. Generally the matrix is diagonal with some $S_{ij} = \pm S_{ji}$ or zero. The degree of matrix symmetry and the number of nonzero elements depends on the degree of particle (or particle distribution) symmetry of the scatterer. As mentioned above, a particle or particle distribution symmetry that treats various kinds of polarized light equally will have a zero for

S_{11}	S_{12}	S_{13}	S_{14}
I_{00}	$I_{H0} - I_{V0}$	$I_{+0} - I_{-0}$	$I_{L0} - I_{R0}$
S_{21}	S_{22}	S_{23}	S_{24}
$I_{0H} - I_{0V}$	$(I_{HH} + I_{VV}) - (I_{VH} + I_{HV})$	$(I_{+H} + I_{-V}) - (I_{-H} + I_{+V})$	$(I_{LH} + I_{RV}) - (I_{RH} + I_{LV})$
S_{31}	S_{32}	S_{33}	S_{34}
$I_{0+} - I_{0-}$	$(I_{H+} + I_{V-}) - (I_{V+} + I_{H-})$	$(I_{++} + I_{--}) - (I_{-+} + I_{+-})$	$(I_{L+} + I_{R-}) - (I_{R+} + I_{L-})$
S_{41}	S_{42}	S_{43}	S_{44}
$I_{0L} - I_{0R}$	$(I_{HL} + I_{VR}) - (I_{VL} + I_{HR})$	$(I_{+L} + I_{-R}) - (I_{-L} + I_{+R})$	$(I_{LL} + I_{RR}) - (I_{RL} + I_{LR})$

Fig. 6. A matrix array showing the polarization measurements necessary to measure each particular matrix element.

those matrix elements that are involved with that kind of polarized light—whether the polarized light be on the input or exit beam. This means that if $S_{ij} = 0$, then $S_{ji} = 0$.

Making the above 16 measurements of polarized light from a scatterer will produce 16 matrix element curves. Each one is a θ-dependent intensity measurement for a particular arrangement of input–output optics. These 16 curves contain all the information that can be learned from an elastic scattering experiment. Choosing input–output optical combinations, different than the ones described above, will produce a set of curves drastically different in appearance but not fundamentally different in information content. They will simply be related by various coordinate transformations.

When the 16 matrix elements are measured the data are ready for the theorist. For certain perfect particles like spheres, fibers, and mixtures of perfect particles, the matrix elements can be exactly predicted.[7] The constants which appear in the theory are the radius r, refractive index n, absorption coefficient μ, and number ratio R, which exactly describe the particle and particle system. For irregular particles where r is not a constant and μ and n are not homogeneous, approximations must be made.[6] However, the experimentalist can often measure what the theorist

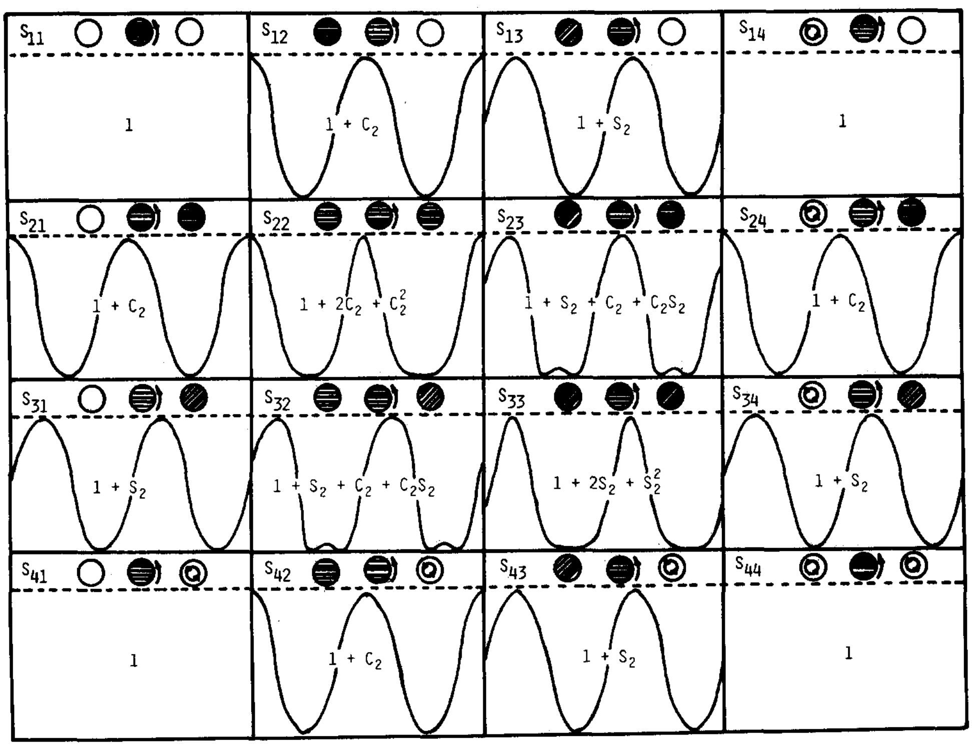

Fig. 7. The intensity matrix that occurs when a horizontal linear polarizer is rotated $\theta = 360°$ between the input and exit optics of Fig. 2. The signals are normalized to make the maximum signal transmitted equal to full vertical scale deflection.

cannot calculate, so the set of 16 measurements will stand as the signature of the scatterer as described by polarized scattered light. An important area of light scattering research is to characterize scattering systems by their matrix elements and to use the matrix element signals to probe for changes in the system.[8] This is a very powerful application of light scattering for biological and other complex systems even though the system is not amenable to theoretical treatment. Generally experimentally measured light scattering curves cannot be inverted to determine the optical constants of the scattering system. This is because light scattering curves are often not unique and because certain large features of the scattering system could completely mask smaller features. An important current research area deals with establishing the information content of the various matrix element signals.

VII. CALIBRATION OF THE OPTICAL SYSTEM

The light scattering signals measured with the apparatus shown in Fig. 2 are intensity curves for a particular input–output optical combination as a function of scattering angle. The optical system shown in Fig. 5 can be calibrated to measure the intensity of polarized light from a "perfect system" by setting the detector and exit optics arm at $\theta = 0°$ (straight through) while rotating through $0 \leqslant \phi \leqslant 360°$ at the scattering position, a "perfect" linear polarizer, quarter wave plate, circular polarizer, or other known optical element. The total intensity seen by the detector, which is the first element in the output Stokes vector $|V_d|$, can be plotted on a strip chart. This signal can then be compared with the mathematical expression corresponding to the interaction of a perfect optical element, that is, one for which no scattering, absorption, fluorescence or mixing of polarization states occur.

VIII. CALIBRATION FOR A HORIZONTAL AND + 45° LINEAR POLARIZER

Consider for example the intensity variation that occurs in matrix element position S_{23} when a linear polarizer $[h(\phi)]$ is rotated $0 < \phi < 360°$ between a fixed $+45°$ input polarizer and a fixed horizontal exit polarizer. The matrix mathematics is

$$[h][h(\phi)][+]|V_t| = |V_d|,$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & C_2 & S_2 & 0 \\ C_2 & C_2^2 & C_2S_2 & 0 \\ S_2 & C_2S_2 & S_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix} = \begin{bmatrix} 1+S_2+C_2+C_2S_2 \\ 1+S_2+C_2+C_2S_2 \\ 0 \\ 0 \end{bmatrix} = \begin{vmatrix} I \\ Q \\ U \\ V \end{vmatrix},$$

where $I = 1 + \sin 2\phi + \cos 2\phi + \sin 2\phi \times \cos 2\phi$ is the total intensity seen by the detector. To calibrate the optical electronic system adjust the zero and gain control on the strip chart to make the signal vary between 0 and $I_{\max}$ (say 100) as the optical element is rotated through angle ϕ. When properly adjusted the strip chart will produce the curve in the S_{23} matrix element position shown in Fig. 7. Similar calculations for all other matrix element combinations shown in Fig. 4 yield curves which are displayed in the matrix array shown in Fig. 7. The vertical scale in each matrix element position represents the intensity ratio $I(\phi)/I_{\max}$ which has been normalized by setting $I_{\max} = 100$. The accuracy of the experimental measure-

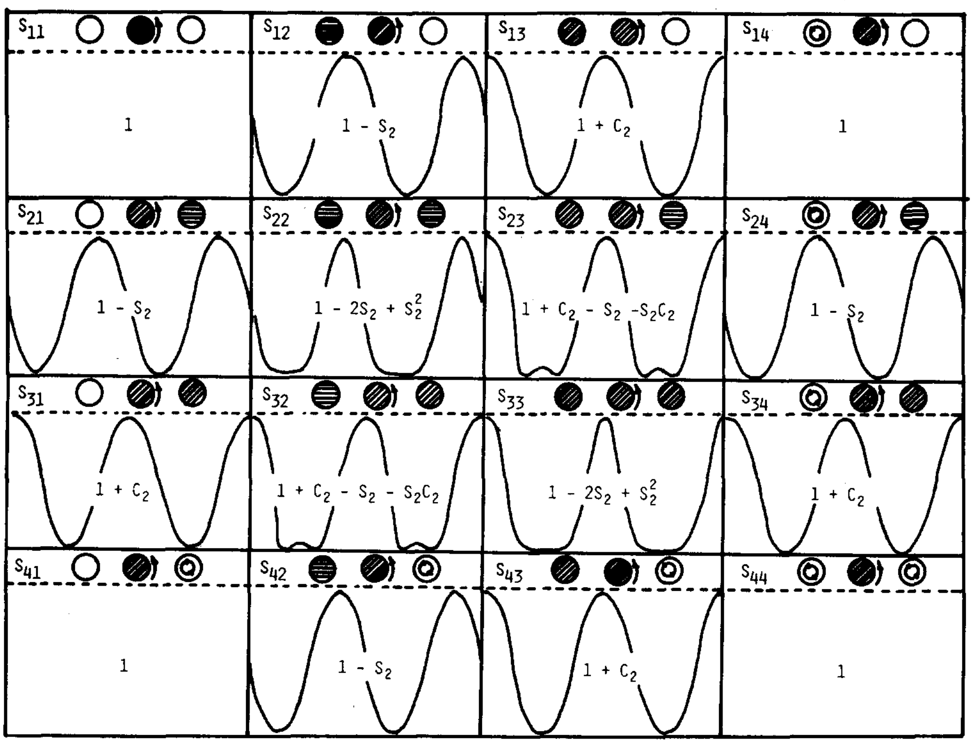

Fig. 8. The intensity matrix that occurs when a + 45° linear polarizer is rotated $\phi = 360°$ between the input and exit optics of Fig. 2. The signals are normalized to make the maximum signal transmitted equal to full-scale vertical deflection.

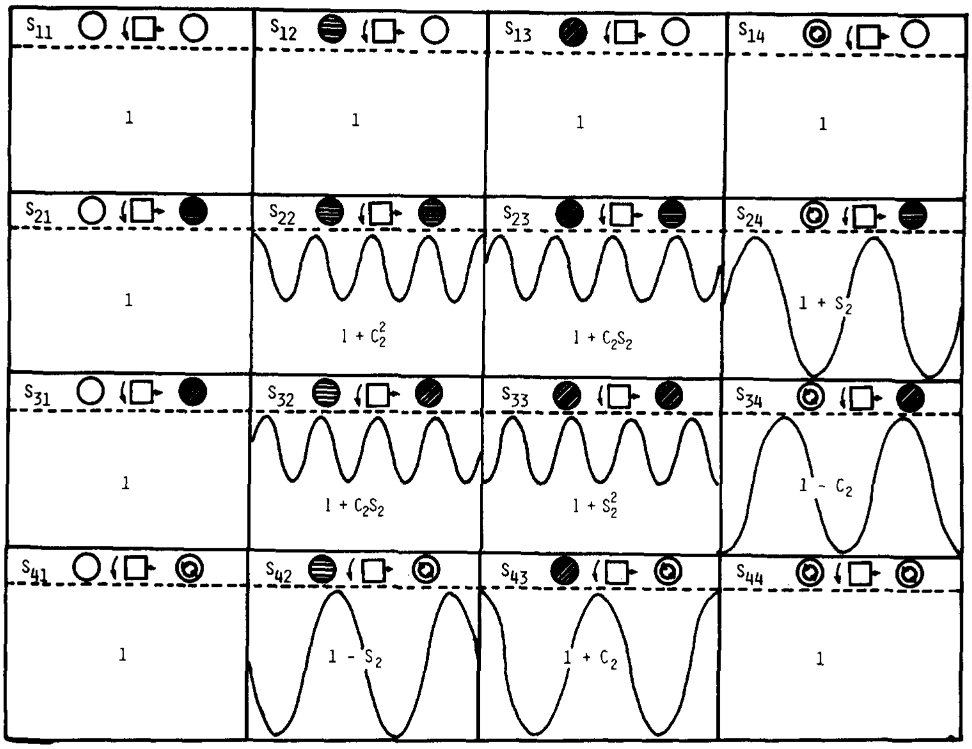

Fig. 9. The intensity matrix that occurs when a quarter wave plate (fast axis indicated by arrow) is rotated $\phi = 360°$ between the input and exit optics of Fig. 2. The signals are normalized to make the maximum signal transmitted equal to full vertical scale deflection. The signal minimum is $\frac{1}{2}I$ max for the center part of the array.

ment can be assessed by comparing the experimental curves with the theoretical values of each $S_{ij}(\phi)$. The signals that occur when a linear polarizer $[h(\phi + 45°)]$ is rotated between the same optical element combination is shown in Fig. 8. As expected all signals are phase shifted by 45°.

IX. CALIBRATION FOR A QUARTER WAVE PLATE

To calibrate the system for a quarter wave plate, consider the intensity variation that occurs in matrix element position S_{23} when a quarter wave plate $[(\lambda/4)(\phi)]$ is rotated $0 < \phi < 360°$ between the $+45°$ and horizontal polarizer. The matrix mathematics is

$$[h][\lambda/4(\phi)][+]|V_t| = |V_d|,$$

$$\begin{bmatrix}1&1&0&0\\1&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&C_2^2&C_2S_2&-S_2\\0&C_2S_2&S_2^2&C_2\\0&S_2&-C_2&0\end{bmatrix}\begin{bmatrix}1&0&1&0\\0&0&0&0\\1&0&1&0\\0&0&0&0\end{bmatrix}\begin{vmatrix}1\\0\\0\\0\end{vmatrix}=\begin{vmatrix}1+C_2^2\\1+C_2^2\\0\\0\end{vmatrix}=\begin{vmatrix}I\\Q\\U\\V\end{vmatrix},$$

where $I = 1 + \cos^2 2\phi$ is the total intensity seen by the detector. Similar calculations for all other matrix element combinations are shown in Fig. 9.

X. CALIBRATION FOR A CIRCULAR POLARIZER

To calibrate the system for a circular polarizer consider the intensity variation that occurs in the matrix element S_{23} when a RH circular polarizer $[r]$ is rotated $0 \leqslant \phi \leqslant 360°$ between the $+45°$ and horizonal polarizer. To do this we rotate *as a unit*, the linear polarizer–quarter wave plate combination shown in the second row second column of Fig. 2. It is represented by the matrix combination $[[\lambda/4(+45)][h]] = [r(\phi)]$. The matrix mathematics is

$$[h][r(\phi)][+]|V_t| = |V_d|,$$

$$\begin{bmatrix}1&1&0&0\\1&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}1&C_2&S_2&0\\0&0&0&0\\0&0&0&0\\1&C_2&S_2&0\end{bmatrix}\begin{vmatrix}1&0&1&0\\0&0&0&0\\1&0&1&0\\0&0&0&0\end{vmatrix}\begin{vmatrix}1\\0\\0\\0\end{vmatrix}=\begin{vmatrix}1+S_2\\1+S_2\\0\\0\end{vmatrix}=\begin{vmatrix}I\\Q\\U\\V\end{vmatrix},$$

where $I = 1 + \sin 2\phi$ is the total intensity seen by the detector. Similar calculations for all other matrix element combinations are shown in Fig. 10.

Figures 7, 8, 9, and 10 show the intensity variations that occur at $\theta = 0$ when perfect optical elements are rotated 360° at the scatter position. A scatterer placed at the scattering position will both transmit some light straight through ($\theta = 0$) and scatter it with all θ and ϕ. The scattered intensity is measured by scanning the detector from $0 \leqslant \theta \leqslant 180°$ with the input and exit optics set in the desired

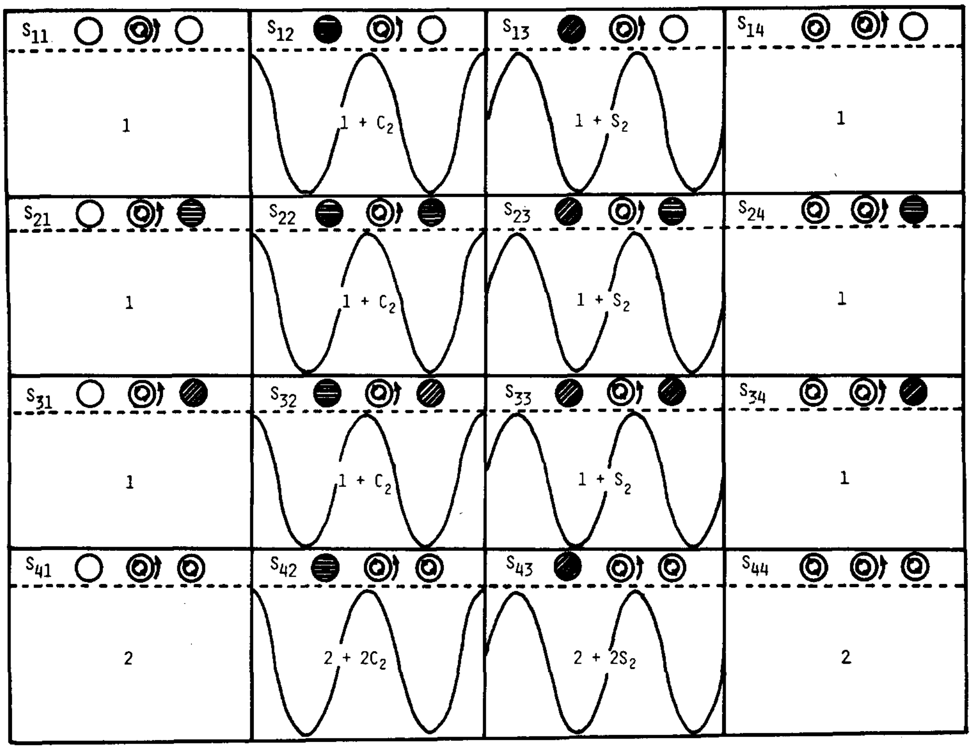

Fig. 10. The intensity matrix that occurs when a circular polarizer is rotated $\phi = 360°$ between the input and exit optics of Fig. 2. The signals are normalized to make the maximum signal transmitted equal to full vertical scale deflection.

orientation. These measurements will generate the 16 light scattering curves which characterize the scatterer.

XI. THE CIRCULAR POLARIZER

It is worthwhile to discuss in some detail the role of the rather esoteric circular polarizer and describe how it is used in the Stokes vector–Mueller matrix context.

A circular polarizer is an optical element that converts unpolarized (or any kind of) light into circularly polarized light. For example,

$$[r]|V_t| = |V_r|,$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 1 \end{vmatrix}.$$

Here $[r]$ is the matrix for a circular polarizer shown in Fig. 2.

We note in Fig. 2 that a circular polarizer is made from a quarter wave plate and linear polarizer in proper combination. For example, $[\lambda/4(+45)][h] = [r]$,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{vmatrix}.$$

This optical combination produces circularly polarized light as is demonstrated in the second row, second column diagram of Fig. 2.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 1 \end{vmatrix}.$$

However, this matrix is not the one usually listed for a circular polarizer[3] and it does not fulfill all that is required of a circular polarizer. The significant difference between the two matrices

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

becomes apparent when they act on a beam that contains a mixture of pure polarization states represented by the Stokes vector $|IQUV|$. We have

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{vmatrix} I \\ Q \\ U \\ V \end{vmatrix} = \begin{vmatrix} I+Q \\ 0 \\ 0 \\ I+Q \end{vmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{vmatrix} I \\ Q \\ U \\ V \end{vmatrix} = \begin{vmatrix} I+V \\ 0 \\ 0 \\ I+V \end{vmatrix}.$$

This is an interesting observation and important fact. Both matrices create a circularly polarized Stokes vector but only the "correct one" brings the information about circular polarization, the fourth component V, into the first component creating a total intensity signal $I + V$ for the detector. The other one creates the mixture $I + Q$ which contains information about *horizontal* polarization even though it is carried in a circularly polarized beam. In order to characterize a system by its matrix elements, it is more important to mix the proper matrix elements than it is to create a particular Stokes vector. In fact, suitable manipulations of input and exit optics could create most any matrix element mixture in any Stokes vector component.

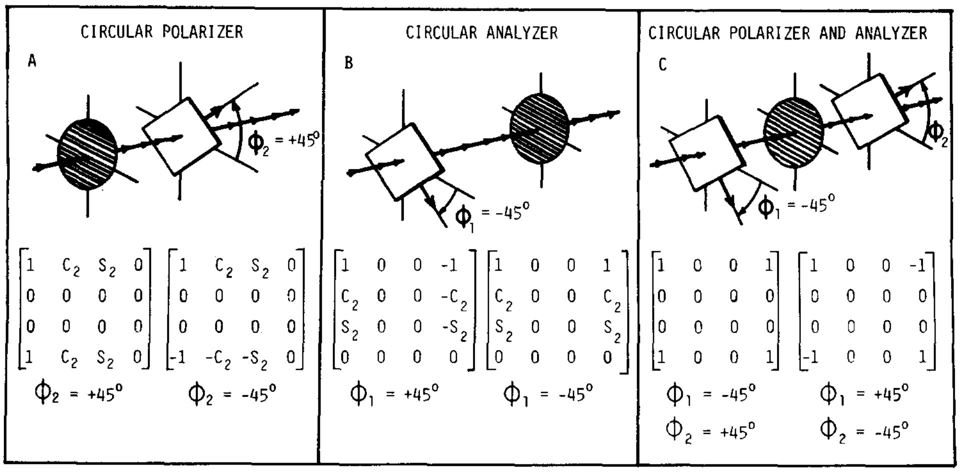

Fig. 11. The geometrical arrangement of a linear polarizer and quarter wave plate for a circular polarizer A, circular analyzer B, and circular polarizer–analyzer combination C. The Mueller matrix describes the above optical combination for a particular orientation of ϕ_1 and ϕ_2.

A circular polarizer also has the property that it is not its own analyzer. This means that the element used to create polarized light is not the same one that can analyze it—in contrast with a linear polarizer. Figure 11 shows the relationship between a circular polarizer and analyzer. Figure 11A shows a circular polarizer (quarter wave plate–linear polarizer combination) that will create circularly polarized light from unpolarized (or any other kind of) light. Right-handed and left-handed circular polarizations are created for $\phi_1 = +45°$ and $-45°$, respectively. Figure 11B shows a circular *analyzer* that will create linearly polarized light from circularly polarized light. When $\phi_1 = -45°$ the combination will "unwind" LH circular polarization, to create a linear polarization. When $\phi = +45°$ it will "unwind" RH circular polarization. This is easily demonstrated by multiplying the Stokes vector $|1 \;\; 0 \;\; 0 \;\; +1|$ by each of the matrices. For the circular polarizer, *all* the linearly polarized light between the polarizer and quarter wave plate gets converted to circular polarization. If the polarizer were removed, the intensity after the quarter wave plate would *increase* by a factor of 2. For the circular analyzer, *all* the circularly polarized light entering the quarter wave plate gets converted to linear polarization. If the quarter wave plate were removed the intensity would *decrease* by a factor of 2.

A circular polarizer–analyzer combination can be created from two quarter wave plates and a linear polarizer in proper combination as is shown in Fig. 11C. We have

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 1 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 1\end{bmatrix}.$$

This optical combination and matrix satisfies three requirements:

(1) It is a true polarizer and analyzer.
(2) It creates circular polarized light.
(3) It brings the circularly polarized component V into the first component of the Stokes vector which can be measured by the PM tube.

XII. CONCLUSION

We described a complete polarization–scattering experiment with highly motivated measurements to show the intimate relationship between the Stokes vectors and Mueller matrices. It is seen that which is more fundamental or useful in describing a system—intensities of polarization states, percent polarization matrix elements, or Stokes vectors—is essentially a matter of choice. We have taken care to establish unambiguously the coordinate systems involved, the redundancy of certain measurements and the importance of particular orientations of optical element combinations. We believe that these concepts are important for understanding and fully appreciating optical polarization and that this approach is attractive because it discusses the many alternative procedures that are equally valid.

ACKNOWLEDGMENTS

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