Portfolio Optimization with Conditional Value-at-Risk Budgets[1]

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> The art of successful portfolio management is not only to be able to identify opportunities, but also to balance them against the risks that they create in the context of the overall portfolio.
>
> - Robert Litterman [1996, p. 73]

Risk budgets are a central tool to estimation and management of the portfolio risk allocation. They decompose total portfolio risk into the risk contribution of each component position. The focus of this paper is on the construction and use of ex ante risk budgets for portfolio allocation using portfolio (component) Conditional Value at Risk. The minimum CVaR concentration portfolio is proposed as the portfolio that minimizes the largest component CVaR value. This portfolio is shown to balance the investor's minimum risk, risk diversification, and overall return objectives. For a portfolio invested in bonds, equity and commodities, the minimum CVaR concentration portfolio offers an attractive compromise between the good risk-adjusted return properties of the minimum CVaR portfolio in down markets and the upward return potential and low portfolio turnover of the equal-weight portfolio.

The outline of the paper is as follows. Section 1 studies the construction and interpretation of CVaR portfolio budgets. Section 2 develops several portfolio allocation strategies that use the portfolio component CVaR risk budget as an objective or constraint in the portfolio optimization problem. Section 3 evaluates the

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risk budget optimized portfolios on the January 1976 – June 2010 monthly total USD returns of Merrill Lynch Domestic Master index, the S&P 500 index, the MSCI EAFE index and the S&P Goldman Sachs Commodity index.

PORTFOLIO CVaR BUDGETS

DEFINITIONS

The first step in the construction of a risk budget is the definition of how portfolio risk and its risk contributions should be measured. Multiple risk decomposition approaches have been suggested in the literature. A naïve approach is to set the risk contribution equal to the stand-alone risk of each portfolio component. This approach is overly simplistic and neglects important diversification or multiplication effects of the component units being exposed differently to the underlying risk factors. An alternative approach is to measure the risk contribution as the weight of the position in the portfolio times the partial derivative of the portfolio risk R_w with respect to that weight:

$$C_{(i)}R_w = w_{(i)} \frac{\partial R_w}{\partial w_{(i)}}. \tag{1}$$

The standard deviation, VaR and CVaR of a portfolio are all linear in position size. By Euler's theorem we have that for such risk measures the total portfolio risk equals the sum of the risk contributions. Standard deviation and CVaR are subadditive risk measures, meaning that the portfolio risk is always less than the sum of the risks of the underlying assets. The allocation problem is to apportion this diversification advantage to the assets in a fair manner, yielding, for each asset, a risk contribution that accounts for diversification. Using game theory, Denault [2001] has shown that the risk contributions given in (1) are the unique satisfactory risk allocation principle.

Previous work by Chow and Kritzman [2001], Litterman [1996], Maillard, Roncalli and Teiletche [2010], Peterson and Boudt [2009] and Scherer [2002] study the use of portfolio standard deviation and value-at-risk (VaR) budgets. In his book "Risk budgeting", Pearson [2002, p.7] notes that "value-at-risk has some well known limitations, and it may be that some other risk measure eventually supplants value-at-risk in the risk budgeting process". Unlike value-at-risk, conditional value-at-risk (CVaR) has all the properties a risk measure should have to be coherent and is a convex function of the portfolio weights (see Artzner, Delbaen, Eber and Heath [1999] and Pflug [2000]). Moreover, CVaR provides less incentive to load on to tail risk above the VaR level.

We develop a risk budgeting framework for portfolio Conditional Value at Risk (CVaR). Portfolio CVaR can be expressed in monetary value or percentage returns. Our goal is to apply the CVaR budget in an investment strategy based on quantitative analysis of the assets returns. We therefore choose to define CVaR in percentage returns.

Denote by r_{wt} the return at time t on the portfolio with weight vector w. To simplify notation, we omit the time index whenever no confusion is possible and assume that the density function of r_w is continuous. At a preset probability level denoted α, which typically is between 1 and 5 percent, the portfolio VaR is the negative value of the α-quantile of the portfolio returns. The portfolio CVaR is the negative value of the expected portfolio return when that return is less than its α-quantile:

$$CVaR_w(\alpha) = -E[r_w \mid r_w \leq -VaR_w(\alpha)], \tag{2}$$

with E the expectation operator. The CVaR contribution is the weight of the position in the portfolio times the partial derivative of the portfolio CVaR with respect to that weight:

$$C_{(i)}CVaR_w = w_{(i)} \frac{\partial CVaR_w}{\partial w_{(i)}}, \tag{3}$$

where $w_{(i)}$ is the portfolio weight of position i and there are N assets in the investment universe ($i = 1,..., N$). For ease of interpretation, the CVaR contributions are standardized by the total CVaR. This yields the percentage CVaR contributions:

$$\%C_{(i)}CVaR_w = \frac{C_{(i)}CVaR_w}{CVaR_w}. \tag{4}$$

An interesting summary statistic of the portfolio's CVaR allocation is what we call the portfolio CVaR Concentration, defined as the largest Component CVaR of all positions:

$$C_w(\alpha) = \max_i C_{(i)}CVaR_w(\alpha). \tag{5}$$

As we will show later, minimizing the portfolio CVaR concentration leads to portfolios with a realatively low CVaR and a balanced CVaR allocation.

INTERPRETATION

The risk contributions in (1) can be interpreted as the marginal risk impact of the corresponding position. Because of transaction costs, traders and portfolio managers often update their portfolios incrementally, which makes the marginal interpretation of risk contribution useful in practice (Litterman [1996], Stoyanov, Rachev and Fabozzi [2009]). Our proposal to use the CVaR contributions as an objective or constraint in portfolio optimization is based on the result of Scaillet [2002] that the contributions to CVaR can also be interpreted as the conditional expectation of the return of the portfolio component when the portfolio loss is larger than its VaR loss. More formally, let $r_{(i)}$ be the return on position i. We have that:

$$C_{(i)}CVaR_w(\alpha) \;=\; -E[w_{(i)}r_{(i)} \mid r_w \leq -VaR_w(\alpha)]. \tag{6}$$

From (1) and (6) it follows that the percentage CVaR contribution can be rewritten as the ratio between the expected return on the position at the time the portfolio experiences a beyond VaR loss and the expected value of these beyond VaR portfolio losses:

$$\%C_{(i)}CVaR_w(\alpha) \;=\; \frac{E[w_{(i)}r_{(i)} \mid r_w \leq -VaR_w(\alpha)]}{E[r_w \mid r_w \leq -VaR_w(\alpha)]}. \tag{7}$$

In almost all practical cases, the denominator in (7) is negative such that a high positive percentage CVaR contribution indicates that the position has a large loss when the portfolio also has a large loss. The higher the percentage CVaR, the more the portfolio downside risk is concentrated on that asset and vice versa.

ESTIMATION

The actual risk contributions can be estimated in two ways. A first approach is to estimate the risk contributions in (6) by replacing the expectations with the sample counterparts evaluated at historical or simulated data. In a portfolio optimization setting the risk contributions needs to be evaluated for a large number of possible weights and therefore fast and explicit estimators are needed. A more elegant approach for optimization problems is therefore to derive the analytical formulae of the risk contributions. If the returns at time t are conditionally normally distributed with mean μ_t and covariance matrix Σ_t, then CVaR at time t is given by:

$$CVaR_w(\alpha) \;=\; -w'\mu_t + \sqrt{w'\Sigma_t w}\,\frac{\phi(z_\alpha)}{\alpha}, \tag{8}$$

with z_α the α-quantile of the standard normal distribution and ϕ the standard normal density function. The contribution to CVaR is then:

$$C_{(i)}CVaR_w(\alpha) \;=\; w_{(i)}\left[-\mu_{(i)t} + \frac{(\Sigma_t w)_{(i)}}{\sqrt{w'\Sigma_t w}}\frac{\phi(z_\alpha)}{\alpha}\right]. \tag{9}$$

Financial returns are usually non-normally distributed. In Boudt, Peterson and Croux [2008] we used Cornish-Fisher expansions to construct an explicit estimator of portfolio CVaR and its contributions that accounts for the observed skewness and excess kurtosis in the data. To save space, we refer the reader to Boudt, Peterson and Croux [2008] for the exact definition of this "Cornish-Fisher CVaR" estimator and to the Appendix for details on the implementation of this estimation method as used in this paper. Throughout the paper, we set the loss probability α to 5%.

CVaR RISK BUDGETS IN PORTFOLIO OPTIMIZATION

Previously, risk budgets based on portfolio standard deviation and value-at-risk have been used either as an *ex post* or *ex ante* tool for tuning the portfolio allocation.

In the ex post approach, the portfolio is first optimized without taking the risk allocation into account. Next the risk budget of the optimal portfolio is estimated and risk budget violations are adjusted on a marginal basis. This process is described by Litterman [1996]. He notes that by definition, component risk measures are proportional to the marginal impact of that position on portfolio risk. When the risk contribution of a position is zero, Litterman [1996] calls this the "best hedge" position for that portfolio component. The positions with the largest risk contributions are called "hot spots™". If a risk contribution is negative, a small increase in the corresponding portfolio weight leads to a decrease in the portfolio risk. Keel and Ardia [2009] show however that reallocation of the portfolio based on these risk contributions is limited in two ways. First of all, as a sensitivity measure, they are only precise for infinitesimal changes, but for realistic reallocations, these approximations can be poor. Second, they assume changing a single position keeping fixed all other positions. In the presence of a full investment constraint, this is unrealistic.

The ex ante use of risk budgets in portfolio allocation is more recent and has focused on constructing a portfolio that allocates portfolio variance equally across asset classes, including stocks, bonds and commodities. Qian [2005] calls this the "Risk Parity Portfolio" and Maillard, Roncalli and Teileche [2010] describe it as the "Equally-Weighted Risk Contribution Portfolio" or, simply, the Equal-Risk Contribution (ERC) portfolio. We follow the latter approach and focus on the use of ex ante CVaR budgets to control the downside risk allocation.

For realistic portfolio optimization problems, the ERC constraint that

$$\%C_{(1)}CVaR_w(\alpha) = ... = \%C_{(N)}CVaR_w(\alpha) = 1/N \tag{10}$$

may be too restrictive and even incompatible with other constraints such as a minimum return target constraint or a weight or position weight limits. We study next two alternative strategies.

As a first alternative, we propose the Minimum CVaR Concentration portfolio (MCC), which uses the ERC criterion as an objective rather than a constraint. More formally, the MCC portfolio allocation is given by:

$$w^{MCC} = \arg\min_w C_w(\alpha), \tag{11}$$

with the portfolio's CVaR concentration $C_w(\alpha)$ as defined in (5). The second strategy consists of imposing bound constraints on the percentage CVaR contributions. This may be viewed as a direct substitute for a risk diversification approach based on position limits. The next paragraphs develop the properties of these two approaches in more detail. In the empirical section, we will compare these CVaR budget based portfolio allocation rules with the more standard Minimum CVaR (MC) and Equal-Weight (EW) portfolios:

$$w^{MC} = \arg\min_w CVaR_w(\alpha) \text{ and } w^{EW} = (1/N,...,1/N)'. \tag{12}$$

PROPERTIES OF THE MINIMUM CVaR CONCENTRATION PORTFOLIO

Minimizing the portfolio CVaR concentration strikes a balance between the objectives of portfolio risk diversification and total risk minimization. To see this, we rewrite the portfolio CVaR concentration as the portfolio CVaR times the largest percentage CVaR contribution:

$$C_w(\alpha) = CVaR_w(\alpha)\max\{\%C_{(1)}CVaR_w(\alpha),...,\%C_{(N)}CVaR_w(\alpha)\}. \tag{13}$$

The first factor in (12) is minimum at the MC portfolio. The second factor attains its lowest value when the portfolio has the ERC property, since $\max\{\%C_{(1)}CVaR_w(\alpha),...,\%C_{(N)}CVaR_w(\alpha)\} \geq 1/N$.

The first order conditions of the MCC portfolio give some interesting insights:

$$\frac{\partial C_w(\alpha)}{\partial w} = CVaR_w(\alpha)\frac{\partial \max\{\%C_{(1)}CVaR_w(\alpha),...,\%C_{(N)}CVaR_w(\alpha)\}}{\partial w}$$

$$+\max\{\%C_{(1)}CVaR_w(\alpha),...,\%C_{(N)}CVaR_w(\alpha)\}\frac{\partial CVaR_w(\alpha)}{\partial w} = 0 \tag{14}$$

We see that a necessary condition for the MCC portfolio to have the exact ERC property is that also the derivative of the portfolio CVaR is zero. Since CVaR is a convex function, this only happens if the (unconstrained) minimum CVaR portfolio has the ERC characteristic, in which case it coincides with the MCC portfolio. Compared with the unconstrained minimum CVaR portfolio, we have that the CVaR of the MCC portfolio is higher, but the risk is less concentrated. In fact, we show in Appendix that the percentage CVaR of the minimum CVaR portfolio coincides with the component's portfolio weight:

$$\%C_{(i)}CVaR_{w^{MC}} = w_{(i)}^{MC}. \tag{15}$$

It is well known that the minimum CVaR portfolio generally suffers from the drawback of portfolio concentration. By (14) this carries directly over to the CVaR allocation.

Note also in (13) that there can be multiple portfolio weights for which the first order conditions are satisfied. For this reason, a global optimizer is needed to find the MCC portfolio. We used the differential evolution algorithm developed by Price, Storn and Lampinen [2008].

The minimum CVaR portfolio under the ERC constraint is an alternative way to the MCC portfolio for attaining a balance between the objectives of portfolio risk diversification and total risk minimization. On our data examples, the two portfolios were always very similar.

The advantage of the MCC portfolio over the ERC constrained minimum CVaR portfolio is that it is computationally simpler and also will yield a solution if the ERC constraint is not feasible or conflicts with other constraints. E.g. the minimum CVaR concentration objective can easily be combined with a return target. This serves the general purpose of maximizing return subject to some level of risk, while also minimizing risk concentration at that risk level. We define the mean/CVaR concentration efficient frontier as the collection of all portfolios that achieve the lowest degree of CVaR concentration for a return objective. For a given return target $\bar{r}$, the mean/CVaR concentration efficient portfolio solves:

$$\min_w C_w(\alpha) \quad \text{s.t.} \quad w'\mu \geq \bar{r}. \tag{16}$$

PORTFOLIO ALLOCATION USING PERCENTAGE CVaR CONSTRAINTS

The risk allocation can also be controlled by imposing explicit constraints on the percentage CVaR allocations. This process operates in much the same way that portfolio managers impose weight constraints on portfolios.

As an example of a percentage CVaR constraint, consider first the ERC constraint as an alternative to the equal-weight portfolio:

$$\min_w CVaR_w(\alpha) \quad \text{s.t.} \quad \%C_{(1)}CVaR_w(\alpha) = ... = \%C_{(N)}CVaR_w(\alpha) = 1/N. \tag{17}$$

This portfolio strategy was recently studied in detail by Maillard, Roncalli and Teileche [2010], but for standard deviation rather than CVaR as a risk measure. Since CVaR is convex, it follows from their paper that the ERC weights are equivalently determined by first solving

$$\min_w CVaR_w(\alpha) \quad \text{s.t.} \quad \sum_{i=1}^{n} \ln w_i \geq c, \tag{18}$$

(with c an arbitrary constant) and without imposing any other constraint on the portfolio weights. The ERC weights are the solution of (17) standardized by dividing by the sum of all weights. Note that that for a portfolio that has the ERC property, the relative weights are inversely proportional to the marginal impact of the position on the portfolio CVaR. When all portfolio percentage CVaR contributions are the same, the component CVaRs coincide. Since $C_{(i)}CVaR_w(\alpha) = w_{(i)}\partial CVaR_w(\alpha)/\partial w_{(i)}$, we have that:

$$\frac{w_{(i)}}{w_{(j)}} = \frac{\partial CVaR_w(\alpha)/\partial w_{(j)}}{\partial CVaR_w(\alpha)/\partial w_{(i)}}. \tag{19}$$

It follows that the equal risk allocation strategy yields portfolios that give higher weights to assets with a small marginal risk impact and down-weights the investments with a high marginal risk (the so-called "hot spots" in Litterman [1996]).

Consider now the more general case of percentage CVaR contributions that are constrained to be between l and u, with $0 \leq l \leq 0.5 \leq u \leq 1.$ These percentage CVaR contribution constraints reduce the feasible space in a way that depends on the return characteristics. Stoyanov, Rachev and Fabozzi [2009] study in detail the effect of the component return characteristics on the total portfolio CVaR. To build further intuition, we plot in Exhibit 1 the percentage CVaR contributions for a two-asset portfolio with asset returns that have a bivariate normal distribution with means μ_1 and μ_2, standard deviations σ_1 and σ_2 and a correlation of 0.5. Of course, the percentage CVaR contribution is zero and one if the weight is zero and one, respectively. In between these values, the percentage CVaR displays an S-shape. The dotted lines in this figure illustrate the effect on the feasible space for portfolio weight 1 of imposing an upper 60% bound on the percentage CVaR contributions of the two assets. This implies that the percentage CVaR contribution of asset 1 has to be between 40% and 60%. In the top figure, the two assets are identical. We see that in this case, the feasible space is centered around the equal-weight portfolio. In the middle and bottom figure, asset 1 is more attractive than asset 2 since it offers either a lower volatility or a higher expected return. We see that this leads to a shift of the feasible space to the right, with allowed portfolio weights around 60%. The set of possible weights allowed by the percentage CVaR constraints changes thus in an intuitively appealing way when differences in return and volatility are allowed.

In the special case of a portfolio of normal assets with mean zero, the percentage CVaR constraints simplify to bound constraints on quadratic functions of the portfolio weights. Combining (8) and (9) and setting $\mu_t = 0$, yields that the feasible space is given by the portfolio weights for which:

$$lw'\Sigma_t w \leq w_{(i)} (\Sigma_t w)_{(i)} \leq uw'\Sigma_t w, \tag{20}$$

for all $i = 1,..., N.$ We show in Appendix that the ERC portfolio ($l = u = 1/N$) is given by:

$$w_{(1)}^{ERC} = \frac{\sigma_2}{\sigma_1 + \sigma_2} \text{ and } w_{(2)}^{ERC} = \frac{\sigma_1}{\sigma_1 + \sigma_2}. \tag{21}$$

For general portfolios with non-normal returns, there is no such explicit representation of the percentage CVaR constraint as weight constraint available for investment. A general purpose portfolio solver that can handle such percentage CVaR contribution constraints is available in the R package PortfolioAnalytics of Boudt, Carl and Peterson [2009].

EMPIRICAL RESULTS

In this section we apply the CVaR decomposition methodology to optimise portfolios that allocate across asset classes. The analysis is based on the January 1976 – June 2010 monthly total USD returns of Merrill Lynch Domestic Master index (bonds), the S&P 500 index (US stocks), the MSCI EAFE index (Europe, Asia and Far East stocks) and the S&P Goldman Sachs Commodity index. The data are obtained from Datastream. We will start with a static two asset bond-equity portfolio, and expand to a larger portfolio for studying the effects of rebalancing under various constraints and objectives. Throughout the application, we impose in all portfolio allocations the full investment constraint and exclude short sales.

STATIC BOND-EQUITY PORTFOLIO

We first consider in Exhibit 2 the following static US bond-equity portfolios: equal-weight, 60/40 weight allocation, minimum CVaR, minimum CVaR concentration and a 60/40 risk allocation portfolio. All moments are estimated by their historical sample counterpart. Over the January 1976 – June 2010 period, the annualized average monthly return of the bond and US equity is 7.55% and 10.25%, respectively. This difference in average return is traded off with the higher CVaR of the equity (9.97%) compared to the one of the bond (2.46%). We see in Exhibit 2 that for the classical portfolios, the risk allocation are heavily concentrated either in the equity component (resp. 97% and 100% of portfolio CVaR caused by the equity investment in the equal-weight and 60/40 portfolios) or the bond (the bond allocation is responsible for 97% of portfolio CVaR in the minimum CVaR portfolio). These extreme risk allocations are avoided by the risk budget optimised portfolios. The minimum CVaR concentration portfolio is an equal risk contribution portfolio with a 23% part in equity. The 60/40 risk allocation portfolio has a slightly higher allocation to equity.

MEAN-CVaR CONCENTRATION EFFICIENT FRONTIER

In Exhibit 3 we add a return target to the minimum CVaR concentration portfolio and consider all four assets. The GSCI has a relatively low annualised monthly return (5.41%) and high risk (monthly CVaR of 12.78%). With an annualized return of 8.68% and monthly CVaR of 10.87%, the EAFE offers a higher return than the bond, but a lower return and higher risk than the S&P 500.

Interestingly, we find that on this sample, the mean-CVaR concentration efficient frontier has three distinguishable segments. Unconstrained, the mean-CVaR concentration efficient frontier is an equal risk contribution portfolio that is invested for 64% in the bond, 13% in the S&P 500, 10% in the EAFE and 12% in the commodities index. It has an annualised return of 7.77% and its monthly 95% CVaR is 3.47%. For a target return between 7.77% and 8.13%, the portfolio CVaR concentration increases from 0.87% to 1.12%, but the portfolio CVaR decreases from 3.47% to 3.35%. This is due to a reallocation from the more risky commodity investment into bonds. At the end of this segment, the portfolio is no longer invested in commodities. Bonds dominate the portfolio budget allocation with a 71% share. On the second segment, the bond allocation shrinks to zero, while the shares of the S&P 500 and the EAFE raise from 16% to 51% and from 13% to 49%, respectively. On this angle portfolio, the S&P 500 and EAFE contribute each for 50% to the portfolio CVaR, which is now 9.8% compensated by a target return of 9.48%. The portfolios on the final segment of the frontier replace gradually the EAFE investment with the S&P 500. Since this asset offers the highest return, it is also the endpoint of the long-only constrained efficient frontier.

DYNAMIC INVESTMENT STRATEGIES

Let us now consider a dynamic portfolio invested in bonds, US equity, Europe, Asia and Far East equity and commodities. The portfolio is rebalanced quarterly to satisfy either an equal-weight, minimum CVaR or minimum CVaR concentration (MCC) objective. The portfolios are all fully invested and short sales are excluded. The risk budgets that are optimised are all conditional on the information available at the time of rebalancing. We give more details on the estimation in the Appendix. Since part of the estimation is based on rolling samples of eight years and the data span is January 1976 – June 2010, the optimized weights are only available for the quarters 1984Q1 – 2010Q3. In all aspects, the MCC portfolio and ERC constrained minimum

CVaR portfolios are very similar. We'll therefore discuss in the text only the results for the MCC portfolio, but for completeness the exhibits show the results for both portfolios.

The left and right panels of Exhibit 4 plot the weight and CVaR allocations of the equal-weight, minimum CVaR and MCC portfolios. We find that for almost all periods the minimum CVaR portfolio is highly invested in the bond index, while the MCC portfolio is more balanced across all asset classes. As predicted by theory, the risk allocation of the minimum CVaR portfolio coincides with its weight allocation and the risk allocation of the MCC portfolio is close to the ERC state. The CVaR of the equal-weight portfolio is dominated by the US and EAFE stocks. The diversification potential of the bond is not fully exploited by the equal-weight portfolio, since for many quarters it has a negative risk contribution. This means that increasing the weight of the bond would marginally decrease portfolio risk. The reason for the bad performance of weight constraints in ensuring ex ante risk diversification is the non-linear dependence of portfolio CVaR contributions on the weights. Therefore direct constraints on the risk budgets rather than on the weights are more efficient in reaching a portfolio manager's goal of ensuring risk diversification.

Exhibit 5 plots the ex ante portfolio risk estimates. As expected, the CVaR of the MCC portfolio is for all quarters in between the CVaR of the minimum CVaR portfolio and the CVaR of the equal-weight portfolio.

Consider now in Exhibit 6 the out-of-sample performance of these portfolios. The solid black lines in the lower and upper panels of Exhibit 6 plot the ratio of the monthly cumulative out-of-sample returns of the minimum CVaR and MCC portfolios versus the cumulative returns of the equal-weight portfolio over the period January 1984-June 2010. The value of the chart is less important than the slope of the line. If the slope is positive, the strategy in the numerator is outperforming the equal-weight strategy, and vice versa. The vertical grey bars denote bear markets defined by Ellis [2005] as periods with a decline in the S&P 500 of 12 per cent or more. The left side of the bar corresponds to the market peaks and the right side to the stock market trough.[2] We see in Exhibit 5 that the minimum CVaR portfolio, having a large allocation to the bond, outperforms the equal-weight and MCC portfolios at times of serious stock market downturn. The performance of the MCC portfolio seems to be a middle ground between the performance of the equal-weight and minimum CVaR portfolios. It offers an attractive compromise between the good performance of the minimum CVaR portfolio in downturn markets and the upward potential of the equal-weight portfolio. A final observation is that periods where one

[2] For our sample, the bear market periods are September-November 1987, June-October 1990, July-August 1998 and November 2007-February 2009.

strategy is outperforming the other are relatively long and indicate the possibility of applying market timing strategies on top of these allocations.

Recall from Exhibit 4 that the risk of the minimum CVaR portfolio is concentrated on the bond risk. An upper bound on portfolio weights or CVaR contributions reduces this risk concentration. The upper two plots in Exhibit 7 present the weight allocations of the minimum CVaR portfolios, constrained by either an upper 40% position limit or an upper 40% CVaR allocation limit. The choice of 40% is arbitrary, but is consistent with a stylistic maximum 60/40 allocation between equity and commodities on the one hand and bonds on the other hand. We see in Exhibit 7 that the 40% upper bound on the portfolio weights and risk allocations is stringent for almost all periods. Under these constraints, the component CVaR contribution of the minimum CVaR portfolio no longer coincides with the weight allocation. The investment in the bond typically contributes less to CVaR risk than its portfolio weight. Its contribution is for some months even negative under the position limit.

The MCC portfolio has an ideal ex ante risk diversification, namely the portfolio allocation such that all assets contribute almost equally to portfolio CVaR. In some cases, the investor might be interested to have the most diversified portfolio subject to a number of weight constraints. The last figure in Exhibit 7 gives the weight and risk allocation of the MCC portfolio under a 40% upper bound on the portfolio weights. We see that also under the constraint, the risk of the MCC portfolio is more equally spread out than for the minimum CVaR portfolio where for some periods the S&P 500 and EAFE investments cause more than half of portfolio risk.

The lower panel of Exhibit 5 shows the ex ante CVaR of these constrained portfolios. Under the 40% position limit, the CVaR of the minimum CVaR and MCC portfolios is close to the CVaR of the equal-weight portfolio. The upper 40% risk contribution constraint has less effect on the CVaR of the minimum CVaR portfolio. It is still relatively low for all quarters.

The grey lines in Exhibit 6 plot the relative performance of these constrained minimum CVaR and MCC portfolios with respect to the equal-weight portfolio. The constrained portfolios having an allocation that is closer to the equal-weight portfolio, it is no surprise that their relative performance deviates less from the unit line than the unconstrained portfolios. Note that especially the position limit constraint yields a very different return stream with higher returns in upward market (by limiting the bond exposure) and larger down movements in periods of market crisis.

A synthesis of the portfolio performance of the eight investment styles is given in Exhibit 8. This table reports the summary statistics of the monthly out-of-sample returns over the period January 1984-June 2010. Consider first the average return/risk statistics. The historical CVaR is the average out-of-sample portfolio return when the return is below its 5% empirical quantile. When computed over the whole period, we see that the annualized average return on all portfolios is similar. It is the lowest for the equal-weight portfolio (7.32% annualized mean) and the highest for the MCC portfolio (8.23%). When we split up the sample in the bear market subsample and all other months, we find a much bigger difference in the return of the portfolios. All position weight constrained portfolios, have large negative returns: -17% for the position limit constrained minimum CVaR and MCC portfolios and -24% for the equal-weight portfolio. In bear markets, the minimum CVaR portfolio has a positive return of 6.31%. The return on the MCC portfolio is -3.79% in bear markets and 10.52% at normal/bull market times. This confirms our previous observation that the MCC portfolio can be seen as a compromise between the good performance of the minimum CVaR portfolio in downturn markets and the upward potential of the equal-weight portfolio.

The standard deviation and CVaR are the lowest for the minimum CVaR portfolio (resp. 1.31% and 2.34%) and the second lowest for the MCC portfolio (1.67% and 3.35%). Imposing a weight constraint increases significantly the risk of the portfolio. The standard deviation and CVaR of the equal-weight portfolio is 3.01% and 7.42%, respectively. The equal-weight portfolio has extremely high drawdowns. Over the sample it has 4 drawdowns higher than 10%, while the minimum CVaR and MCC only have one. In the credit crisis the equal-weight portfolio suffered a drawdown of 48%, which is triple the drawdown of the MCC portfolio. If we impose an upper 40% position limit on the minimum CVaR portfolios, the CVaR increases from 2.34% to 6.15% and its maximum drawdown from 9% to 48%, thus imposing weight constraints significantly deteriorates overall risk/return profile of the portfolio.

In Exhibit 8 we also report a Herfindahl statistic measuring the out-of-sample concentration. It is defined as follows. For each month in which the portfolio return is below its 5% empirical quantile, we compute the return contributions of each component. The average squared value of these return contributions yields the Herfindahl Index (HI),

$$\text{HI of Hist. } C_{(i)}CVaR_w = \frac{\sum_{t=1}^{T}\sum_{i=1}^{N} w_{(i)t}^2 r_{(i)t}^2 I[w_t' r_t \le q_{w_t}]}{N\sum_{t=1}^{T} I[w_t' r_t \le q_{w_t}]}, \tag{22}$$

where q_{w_t} is the 5% empirical quantile of the returns on the portfolio with weights w_t. The higher the HI, the higher the out-of-sample concentration of downside risk. It is of interest to compare them with the ex ante CVaR allocation budgets in the right panels of Exhibits 4 and 7. Conform the ex ante predictions, we see that the out-of-sample concentration is the highest for the minimum CVaR portfolio (0.21%). Interestingly however, position limits seem to be more effective in ensuring ex post diversification than would be expected from the ex ante estimates. The lowest HI is attained for the equal-weight portfolio (0.06%) and the position constrained MCC portfolio (0.07%). The HI of the MCC portfolio is 0.12%. As such it effectively reaches a good trade-off between a low total portfolio risk and a high diversification.

Finally, we consider the portfolio turnover of the strategies, defined by DeMiguel, Garlappi and Uppal [2009] as the average sum of the absolute value of the trades across the *N* available assets:

$$\text{Turnover} = \frac{1}{NT_*}\sum_{t=1}^{T_*-1}\sum_{i=1}^{N}\left|w_{(i)t+1} - w_{(i)t+}\right|, \tag{23}$$

where $w_{(i)t+1}$ is the weight of asset i at the start of rebalancing period $t+1$, $w_{(i)t+}$ is the weight of that asset before rebalancing at $t+1$ and T_* is the total number of rebalancing periods. This turnover quantity can be interpreted as the average percentage of wealth traded in each period. The portfolio turnover is the lowest for the equal-weight portfolio (1.26%) and the highest for the position constrained minimum CVaR portfolio (3.55%). The MCC portfolio has a significantly lower turnover (1.74%) than the minimum CVaR portfolio (2.14%).

In conclusion, the minimum CVaR portfolio has the lowest out-of-sample risk but a high risk concentration and turnover. The equal-weight strategy has the lowest turnover and risk concentration, but highest total risk. The proposed MCC portfolio is on all these dimensions the second best and therefore an ideal compromise between a low risk, high diversification and low turnover objective. Moreover, it combines the good return/risk properties of the minimum CVaR in downturn markets and the upward potential of the equal-weight portfolio.

CONCLUSION

Portfolio risk budgeting is nowadays mainly based on an ex post analysis of the Euler decomposition of the portfolio value-at-risk. This paper shows that in a portfolio allocation setting, optimization of ex ante conditional value-at-risk (CVaR) budgets can be very useful. A first strategy is to impose bound constraints on the percentage CVaR contributions. This is a direct substitute to the risk diversification approach based on position limits. A second strategy consists in minimizing the largest CVaR contribution. This leads to portfolios that are close to portfolios for which the contributions to CVaR of all assets are identical. For a portfolio invested in bonds, equity and commodities, we find that diversification strategies based on risk budgets yield portfolios that offer an attractive compromise between the good return/risk properties of the minimum CVaR portfolio in downturn markets and the upward potential and low portfolio turnover of the equal-weight portfolio.

ENDNOTES

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REFERENCES

Artzner, Philippe, Delbaen, Freddy, Eber, Jean-Marc and David Heath. "Coherent measures of risk", *Mathematical Finance*, Vol. 6 (1998), pp. 203-228.

Boudt, Kris, Peterson, Brian G. and Christophe Croux. "Estimation and decomposition of downside risk for portfolios with non-normal returns." *The Journal of Risk*, Winter 2008, pp. 79-103.

Boudt, Kris, Carl, Peter and Peterson, Brian G. PortfolioAnalytics. R package version 0.0.6, 2010.

Carl, Peter and Peterson, Brian G. PerformanceAnalytics: Econometric tools for performance and risk analysis. R package version 1.0.2.1, 2010.

Chow, George, and Mark Kritzman. "Risk budgets." *The Journal of Portfolio Management*, Winter 2001, pp. 56-60.

DeMiguel, Victor, Garlappi, Lorenzo and Raman Uppal. "Optimal versus naïve diversification: How inefficient in the 1/N portfolio strategy?" *Review of Financial Studies*, Vol. 22 (2009), pp. 1915-1953.

Denault, Michel. "Coherent allocation of risk capital." *The Journal of Risk,* Fall 2001, pp. 1-33.

Ellis, J.H. Ahead of the curve: a commonsense guide to forecasting business and market cycles. Harvard Business Press, 2005.

Keel, Simon and David Ardia. "Generalized marginal risk." Aeris Capital working paper, 2009.

Litterman, Robert B. "Hot Spots^{TM} and hedges." *The Journal of Portfolio Management*, 1996 Special Issue, pp. 52-75.

Maillard, Sébastien, Roncalli, Thierry and Jérôme Teiletche. "On the properties of equally-weighted risk contributions portfolios." *The Journal of Portfolio Management*, Summer 2010, pp. 60-70.

Pearson, N.D. Risk Budgeting. New York: John Wiley and Sons, 2002.

Peterson, Brian, and Kris Boudt. "Component VaR for a non-normal world." *RISK*, November 2008, pp. 78-81.

Pflug, G. Ch. "Some remarks on the value-at-risk and the conditional value-at-risk." In S. Uryasev, ed., Probabilistic Constrained Optimization: Methodology and Applications, Dordrecht: Kluwer, 2000, pp. 272-281.

Price, K.V., Storn, R.M. and Lampinen, J.A. Differential Evolution - A Practical Approach to Global Optimization. Springer-Verlag, 2006.

Qian, Edward E. "Risk parity portfolios: Efficient portfolios through true diversification of risk." Panagora Asset Management, September.

Scaillet, Olivier. "Nonparametric estimation and sensitivity analysis of expected shortfall." *Mathematical Finance*, Vol. 14 (2002), pp. 74-86.

Scherer, B. Portfolio Construction & Risk Budgeting. London: Risk Books, 2002.

Stoyanov, Stoyan, Rachev, Svetlozar T. and Frank J. Fabozzi. "Sensitivity of portfolio VaR and CVaR to portfolio return characteristics." Universität Karlsruhe working paper, 2009.

APPENDIX

CVaR BUDGET OF MINIMUM CVaR PORTFOLIO

The first order conditions of the minimum CVaR portfolio are that

$$\frac{\partial CVaR_w}{\partial w_{(1)}} = ... = \frac{\partial CVaR_w}{\partial w_{(N)}} = 0. \tag{24}$$

By l'Hôpital's rule we have that

$$\%C_{(i)}CVaR_w = \frac{w_{(i)}\dfrac{\partial CVaR_w}{\partial w_{(i)}}}{\sum_{i=1}^{N} w_{(i)}\dfrac{\partial CVaR_w}{\partial w_{(i)}}} \to \frac{w_{(i)}}{\sum_{i=1}^{N} w_{(i)}} \quad when \quad \max\{\frac{\partial CVaR_w}{\partial w_{(1)}},...,\frac{\partial CVaR_w}{\partial w_{(N)}}\} \to 0. \tag{25}$$

Hence the weight and CVaR allocation coincide for the minimum CVaR portfolio.

EQUAL CVaR CONTRIBUTION PORTFOLIO UNDER A ZERO MEAN NORMAL DISTRIBUTION

Consider a two-asset portfolio with asset returns that have a normal distribution with means μ_1 and μ_2, correlation ρ and standard deviations σ_1 and σ_2. We will show that there is a unique portfolio that satisfies the ERC constraint. Let Σ be the corresponding covariance matrix. In this setting, the condition $C_{(1)}CVaR_w(\alpha) = C_{(2)}CVaR_w(\alpha)$ is equivalent to:

$$-w_{(1)}\mu_{(1)} + \frac{w_{(1)}^2\sigma_1^2}{\sqrt{w'\Sigma_t w}}\frac{\phi(z_\alpha)}{\alpha} = -w_{(2)}\mu_{(2)} + \frac{w_{(2)}^2\sigma_2^2}{\sqrt{w'\Sigma_t w}}\frac{\phi(z_\alpha)}{\alpha}. \qquad (26)$$

If we assume the means to be zero, the ERC condition in (26) simplifies to $w_{(1)}^2\sigma_1^2 = w_{(2)}^2\sigma_2^2$. For a fully invested portfolio, this implies solving the quadratic equation $(\sigma_1^2 - \sigma_2^2)w_{(1)}^2 + 2\sigma_2^2 w_{(1)} - \sigma_2^2 = 0$, with roots $w_{(1)}^{ERC} = \sigma_2/(\sigma_2 - \sigma_1)$ and $w_{(1)}^{ERC} = \sigma_2/(\sigma_2 + \sigma_1)$. The first solution does not satisfy the constraint that the weights have to be between 0 and 1. Multiplying numerator and denominator of this weight with $\sigma_1^{-1}\sigma_2^{-1}$ yields an equivalent expression, namely: $w_{(1)}^{ERC} = \sigma_1^{-1}/(\sigma_1^{-1} + \sigma_2^{-1})$, which corresponds to the ERC portfolio using standard deviation as a risk measure (Maillard, Roncalli and Teiletche [2010]).

DETAILS ON ESTIMATION METHOD

Because of the non-normality in the data, we use the Cornish-Fisher CVaR estimator of Boudt, Peterson and Croux [2008]. Its implementation requires an estimate of the first four moments of the portfolio returns. For the static portfolio, the moment estimates are their samples counterparts. For the dynamic portfolio allocation, time-varying conditional moment estimates are obtained as follows. We first center the returns around an exponentially weighted average of the returns over the past eight years. The centered returns are modelled as a GARCH(1,1) process whose parameters are estimated by Gaussian quasi-maximum likelihood using all data available from inception up to the time for which the CVaR estimate is needed. We then compute the innovations as the centered returns divided by their volatility estimate. The correlation, coskewness and cokurtosis matrices of these innovations are then estimated as the rolling eight year sample correlation, coskewness and cokurtosis matrix of a winsorized version of these innovations. The winsorization ensures the outlier-robustness of the estimates and is described in Boudt, Peterson and Croux [2008].