Math 170B **Winter 2017**

Problem Set 3

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Exercise 1

Let X be a random variable. Assume that $M_X(t)$ exists for all $t \in \mathbb{R}$, and assume we can differentiate under the expected value any number of times. For any positive integer n, show that

$$\frac{d^n}{dt^n}\bigg|_{t=0} M_X(t) = \mathrm{E}[X^n].$$

So, in principle, all moments of X can be computed just by taking derivatives of the moment generating function.

Solution: Recall from the definition of a power series that we can write a function about x_0 using

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + ... + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \end{aligned}$$

Therefore we may write the moment generating function as

$$M_X(t) = \sum_{n=0}^{\infty} \frac{(M_X(t))^{(n)}\big|_{t=0}}{n!} t^n = \sum_{n=0}^{\infty} \frac{\mathrm{E}[X^n]}{n!} t^n.$$

Therefore

$$\frac{d^n}{dt^n}\bigg|_{t=0} M_X(t) = \mathrm{E}[X^n].$$

■

Exercise 2

Let X be a standard Gaussian random variable. Compute an explicit formula for the moment generating function of X. (Hint: completing the square might be helpful.) From this explicit formula, compute an explicit formula for all moments of the Gaussian random variable. (The $2n^{th}$ moment of X should be something resembling a factorial.)

Solution: Let X be a standard Gaussian random variable. So $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Then

$$\begin{aligned} M(t) &= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx - x^2/2} dx. \end{aligned}$$

I now want to complete the square on $tx - x^2/2$ to make the integral easier to compute.

$$\begin{aligned} tx - x^2/2 &= -\frac{1}{2}x^2 + tx \\ &= -\frac{1}{2}\left(x + \frac{t}{2(-\frac{1}{2})}\right)^2 + \frac{t^2}{2} \\ &= -\frac{1}{2}(x-t)^2 + \frac{t^2}{2}. \end{aligned}$$

Now substituting this back into the integral, we get

$$\begin{aligned} M(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-t)^2 + \frac{t^2}{2}}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-t)^2} e^{\frac{t^2}{2}}\, dx \\ &= e^{\frac{t^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-t)^2}\, dx \\ &= e^{\frac{t^2}{2}}. \end{aligned}$$

Note that the third inequality is true because $e^{\frac{t^2}{2}}$ does not depend on x and the fourth is true because the integral represents the PDF of a normally distributed random variable, which integrates to 1.

Now we can represent $M(t)$ using a Taylor series:

$$\begin{aligned} M(t) &= M(0) + M'(0)t + M''(0)\frac{t^2}{2!} + M'''(0)\frac{t^3}{3!} + ... \\ &= M(0) + \mathrm{E}[X]t + \mathrm{E}[X^2]\frac{t^2}{2!} + \mathrm{E}[X^3]\frac{t^3}{3!} + ... \end{aligned}$$

But recall that we can also represent $e^{t^2/2}$ in terms of the definition of the exponential:

$$\begin{aligned} M(t) = e^{t^2/2} &= \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{t^2}{2}\right)^j \\ &= \sum_{j=0}^{\infty} \frac{(2j)!}{j!2!} \cdot \frac{t^{2j}}{(2j)!}. \end{aligned}$$

Putting the above two equations together, we see that

$$\sum_{j=0}^{\infty} \frac{(2j)!}{j!2!} \cdot \frac{t^{2j}}{(2j)!} = M(0) + \mathrm{E}[X]t + \mathrm{E}[X^2]\frac{t^2}{2!} + \mathrm{E}[X^3]\frac{t^3}{3!} + ...$$

Note that the left hand side only has even powers, so the terms on the right hand side with odd powers of t are zero. Therefore $\mathrm{E}[X^n]$ for n odd is zero. So

$$\begin{aligned} \mathrm{E}[X^{2j}] &= \frac{(2j)!}{j!2^j} \\ \mathrm{E}[X^j] &= \frac{j!}{2^{j/2}(j/2)!} \text{ for } j \text{ even.} \end{aligned}$$

$\blacksquare$

Exercise 3

Construct two random variables $X, Y : \Omega \to \mathbb{R}$ such that $X \neq Y$ but $M_X(t)$, $M_Y(t)$ exist for all $t \in \mathbb{R}$, and such that $M_X(t) = M_Y(t)$ for all $t \in \mathbb{R}$.

Solution: Let X be uniformly distributed on $[-1, 1]$, and let $Y = -X$. So Y is also uniformly distributed on $[-1, 1]$. Note that X and Y are different functions on the sample space but have the same distribution. Since they have the same distribution, $M_X(t) = M_Y(t)$ for all t. ■

Exercise 4

Unfortunately, there exist random variables X, Y such that $\mathrm{E}[X^n] = \mathrm{E}[Y^n]$ for all $n = 1, 2, 3, ...$ but such that X, Y do not have the same CDF. First, explain why this does not contradict the Lévy Continuity Theorem, Weak Form. Now, let $-1 < a < 1$, and define a density

$$f_a(x) := \begin{cases} \frac{1}{x\sqrt{2\pi}} e^{-\frac{(\log x)^2}{2}} (1 + a\sin(2\pi \log x)), & \text{if } x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Suppose X_a has density f_a. If $-1 < a, b < 1$, show that $\mathrm{E}[X_a^n] = \mathrm{E}[X_b^n]$ for all $n = 1, 2, 3, ...$ (Hint: write out the integrals, and make a change of variables $s = \log(x) - n$.)

Solution: In exercise 1, we proved that the n^{th} moment is equal to the n^{th} derivative of the moment generating function evaluated at zero. This is only true, however, when the moment generating function equals its power series, and its power series converges. If $\mathrm{E}[X^n] = \mathrm{E}[Y^n]$ for all $n \in \mathbb{Z}_+$, but the moment generating function does not exist (i.e. the integral to create it goes to infinity), then X and Y may not have the same CDF.

Now let $-1 < a < 1$ and define

$$f_a(x) := \begin{cases} \frac{1}{x\sqrt{2\pi}} e^{-\frac{(\log x)^2}{2}} (1 + a\sin(2\pi \log x)), & \text{if } x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Define a change of variables $s = \log(x) - n$. Then $ds = \frac{1}{x}$ and $x = e^{s+n}$.

$$\begin{aligned}
\mathrm{E}[X_a^n] &= \int_0^\infty x^n \frac{1}{x\sqrt{2\pi}} e^{-\frac{(\log x)^2}{2}} (1 + a\sin(2\pi \log x)) dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(s+n)n} e^{-(s+n)^2/2} (1 + a\sin(2\pi(s+n))) ds \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{sn+n^2-\frac{s^2}{2}-sn-\frac{n^2}{2}} (1 + a\sin(2\pi(s+n))) ds \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{n^2-s^2}{2}} (1 + a\sin(2\pi(s+n))) ds \\
&= \frac{1}{\sqrt{2\pi}} e^{\frac{n^2}{2}} \int_{-\infty}^{\infty} e^{\frac{-s^2}{2}} (1 + a\sin(2\pi(s+n))) ds \\
&= \frac{1}{\sqrt{2\pi}} e^{\frac{n^2}{2}} \int_{-\infty}^{\infty} \left(e^{\frac{-s^2}{2}} + e^{\frac{-s^2}{2}} a\sin(2\pi(s+n)) \right) ds.
\end{aligned}$$

Note however, that sin integrates to cos, and cos is an even function. Therefore $\int_{-\infty}^{\infty} \sin(x)dx = 0$. This means that the right side of the above integral evaluates to zero. Therefore,

$$\begin{aligned} \mathrm{E}[X_a^n] &= e^{\frac{n^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-s^2}{2}}\, ds \\ &= e^{\frac{n^2}{2}}. \end{aligned}$$

Therefore, $\mathrm{E}[X_a^n]$ does not depend on a, so $\mathrm{E}[X_a^n] = \mathrm{E}[X_b^n]$ for all $n = 1, 2, 3, ...$, for any $-1 < a, b < 1$. ■