Math 170B **Winter 2017**

Problem Set 1

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Exercise 1

Using the De Moivre-Laplace Theorem, estimate the probability that 1,000,000 coin flips of fair coins will result in more than 501,000 heads. (Some of the following integrals may be relevant: $\int_{-\infty}^{0} e^{-t^2/2}dt/\sqrt{2\pi} = 1/2$, $\int_{-\infty}^{1} e^{-t^2/2}dt/\sqrt{2\pi} \approx .8413$, $\int_{-\infty}^{2} e^{-t^2/2}dt/\sqrt{2\pi} \approx .9772$, $\int_{-\infty}^{3} e^{-t^2/2}dt/\sqrt{2\pi} \approx .9987$.)

Solution: Since coin flips of fair coins are independent Bernoulli random variables, we can use the De Moivre-Laplace Theorem. Let $n = 1{,}000{,}000$.

$$\begin{aligned}
\mathbb{P}(X_1 + ... + X_n > 501000) &= 1 - \mathbb{P}(X_1 + ... + X_n \leq 501000) \\
&= 1 - \mathbb{P}(X_1 + ... + X_n - 500000 \leq 1000) \\
&= 1 - \mathbb{P}\left(\frac{X_1 + ... + X_n - 500000}{1000} \leq 1\right) \\
&= 1 - \mathbb{P}\left(\frac{X_1 + ... + X_n - 500000}{1000 \cdot \sqrt{1/4}} \leq 2\right) \\
&= 1 - \mathbb{P}\left(\frac{X_1 + ... + X_n - (1/2)n}{\sqrt{n} \cdot \sqrt{1/4}} \leq 2\right) \\
&\approx 1 - \int_{-\infty}^{2} e^{-t^2/2} \frac{dt}{\sqrt{2\pi}} \\
&= 1 - .9772 = .0228.
\end{aligned}$$

(Where the approximation is given by the De Moivre-Laplace Theorem, and $a = 2$.) ■

Exercise 2

Let X and Y be nonnegative random variables. Recall that we can define

$$\mathbb{E}[X] := \int_0^\infty \mathbb{P}(X > t)dt.$$

Assume that $X \leq Y$. Conclude that $\mathbb{E}[X] \leq \mathbb{E}[Y]$.

More generally, if X satisfies $\mathbb{E}[|X|] < \infty$, we define $\mathbb{E}[X] := \mathbb{E}[\max(X, 0)] - \mathbb{E}[\max(-X, 0)]$. If X, Y, are any random variables with $X \leq Y$, $\mathbb{E}[|X|] < \infty$ and $\mathbb{E}[|Y|] < \infty$, show that $\mathbb{E}[X] \leq \mathbb{E}[Y]$.

Solution: Suppose that $X \leq Y$. We want to show that $\mathbb{E}[X] \leq \mathbb{E}[Y]$. Well by the definition of $\mathbb{E}[X]$ above, we can write

$$\begin{aligned}
\mathbb{E}[X] \leq \mathbb{E}[Y] &\Leftarrow \int_0^\infty \mathbb{P}(X > t)dt \leq \int_0^\infty \mathbb{P}(Y > t)dt \\
&\Leftarrow \mathbb{P}(X > t) \leq \mathbb{P}(Y > t), \forall t \\
&\Leftarrow \{X > t\} \subseteq \{Y > t\}, \forall t \\
&\Leftarrow \text{ if } X > t, Y \geq X > t, \forall t \\
&\Leftarrow X \leq Y.
\end{aligned}$$

That is, since $X \leq Y$, then if $X > t$, then $Y \geq X > t$. That means that the set of all X that is greater than t is contained within the set of all Y that is greater than t. Hence $\mathbb{P}(X > t) \leq \mathbb{P}(Y > t), \forall t$. Thus $\mathbb{E}[X] \leq \mathbb{E}[Y]$.

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Exercise 3

Using the definition of convergence, show that the sequence of numbers

$$1, 1/2, 1/3, 1/4, ...$$

converges to 0.

Proof: Let $x_n = \frac{1}{n}$, let $\epsilon > 0$, and let $M = \frac{1}{\epsilon}$. Then for $n > M$ we have

$$|x_n - x| = \left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{M} = \epsilon.$$

Therefore the sequence (x_n) with $x_n = \frac{1}{n}$ converges to 0. ■

Exercise 4

Let $x_1, x_2, ...$ be a sequence of real numbers. Let $x, y \in \mathbb{R}$. Assume that $x_1, x_2, ...$ converges to x. Assume also that $x_1, x_2, ...$ converges to y. Prove that $x = y$. That is, a sequence of real numbers cannot converge to two different real numbers.

Proof: Call (x_n) the sequence of $x_1, x_2, ...$ Let $x, y \in \mathbb{R}$. Suppose that $x_n \to x$ and $x_n \to y$. Suppose for contradiction that $x \neq y$.

Let $\epsilon = \frac{|x-y|}{3}$. Since $x_n \to x$, there exists M_x such that $n > M_x \Rightarrow |x_n - x| < \epsilon$. Since $x_n \to y$, there exists M_y such that $n > M_y \Rightarrow |x_n - y| < \epsilon$. Let $Z = \max\{M_x, M_y\}$. Then when $n > Z$,

$$|x - y| = |x - x_n + x_n - y| \leq |x_n - x| + |x_n - y| < \frac{|x-y|}{3} + \frac{|x-y|}{3} < |x - y|,$$

where the second relation is from the triangle inequality and the third is using the convergence of x_n to x and y. This is a contradiction so $x = y$. ■

Exercise 5

Let X be a uniformly distributed random variable on $[-1, 1]$. Let $Y := X^2$. Find f_Y.

Solution: Let X be uniformly distributed on $[-1, 1]$, let $g(x) = x^2$, and let $Y = g(X)$. Since X is continuous, x^2 is continuous and F_Y is differentiable we can use Proposition 2.6 from the notes.

First note that

$$f_X(x) = \begin{cases} \frac{1}{2}, & x \in [-1, 1] \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} \int_{\{x \in \mathbb{R}: g(x) \leq y\}} f_X(x) dx \\ &= \frac{d}{dy} \int_{\{x \in [-1,1]: x^2 \leq y\}} \frac{1}{2} dx. \end{aligned}$$

If $y < 0$, the integral is zero. If $y > 1$ the integral is 1. If $y \in [0, 1]$ we have

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} \frac{1}{2} \int_{-y^{1/2}}^{y^{1/2}} dx \\ &= \frac{1}{2} \frac{d}{dy} [y^{1/2} + y^{1/2}] \\ &= \frac{1}{2\sqrt{y}}. \end{aligned}$$

So the PDF of Y is

$$f_Y = \begin{cases} 0, & y < 0 \\ \frac{1}{2\sqrt{y}}, & y \in [0, 1] \\ 0, & y > 1. \end{cases}$$

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Exercise 6

Let X be a uniformly distributed random variable on $[0, 1]$. Let $Y := 4X(1 - X)$. Find f_Y.

Solution: We wish to find f_Y. We'll first find F_Y and then take the derivative. First note that when $x \in [0, 1]$, the image of $4x(1 - x)$ is $[0, 1]$. So we only have to deal with y values between 0 and 1. Note that $f_X(x) = 1$ on $x \in [0, 1]$.

$$\begin{aligned} \mathbb{P}(Y \leq y) &= \mathbb{P}(4X(1 - X) \leq y) \\ &= \mathbb{P}\left(0 \leq x \leq \frac{1 - \sqrt{1 - y}}{2}\right) + \mathbb{P}\left(\frac{1 + \sqrt{1 - y}}{2} \leq x \leq 1\right) \\ &= \frac{1 - \sqrt{1 - y}}{2} + 1 - \frac{1 + \sqrt{1 - y}}{2} \\ &= 1 - \sqrt{1 - y} \text{ for } y \in [0, 1]. \end{aligned}$$

Therefore, the PDF of Y is:

$$f_Y(y) = F_Y'(y) = \begin{cases} \frac{1}{\sqrt{1-y}}, & y \in [0, 1] \\ 0, & \text{otherwise.} \end{cases}$$

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Exercise 7

Let X be a uniformly distributed random variable on $[0,1]$. Find the PDF of $-\log(X)$.

Solution:

Recall that the CDF of X is $F_X(x) = \begin{cases} 0, & x \leq 0 \\ x, & x \in [0,1] \\ 1, & x \geq 1. \end{cases}$.

Then to find the CDF of Y,

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(-\log(X) \leq y) = \mathbb{P}(X \geq e^{-y}) = 1 - \mathbb{P}(X \leq e^{-y}) = 1 - e^{-y}.$$

Therefore $F_Y(y) = \begin{cases} 0, & y < 0 \\ 1 - e^{-y}, & y \geq 0 \end{cases}$, so $f_Y(y) = \begin{cases} 0, & y < 0 \\ e^{-y}, & y \geq 0 \end{cases}$.

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Exercise 8

Let X be a standard normal random variable. Find the PDF of e^X.

Solution: Let X be standard normal. Let $Y = e^X$.

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} \int_{\{x \in \mathbb{R}: g(x) \leq y\}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx \\ &= \frac{d}{dy} \int_{\{x \in \mathbb{R}: e^x \leq y\}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx \\ &= \frac{d}{dy} \int_{\{x \in \mathbb{R}: x \leq \ln(y)\}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx \\ &= \frac{d}{dy} \int_{\infty}^{\ln(y)} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{\ln(y)^2}{2}} \\ &= \frac{1}{\sqrt{2\pi}} e^{\ln(y^{-1/2})^2} \\ &= \frac{1}{\sqrt{2\pi}} \left(e^{\ln(y^{-1/2})}\right)^{\ln(y^{-1/2})} \\ &= \frac{1}{\sqrt{2\pi}} \left(y^{-1/2}\right)^{\ln(y^{-1/2})}. \end{aligned}$$

Note that $f_Y(y)$ is defined this way on $y \in \mathbb{R}$. ■

Exercise 9

Let X, Y, Z be independent standard Gaussian random variables. Find the PDF of $\max(X, Y, Z)$.

Solution: Let X, Y, Z be independent standard Gaussian random variables. Denote $W = \max\{X, Y, Z\}$. First I'll find the CDF of W and then I'll take the derivative to get the PDF of W.

$$\begin{aligned}
F_W(w) &= \mathbb{P}(W \leq w) \\
&= \mathbb{P}(X \leq w, Y \leq w, Z \leq w) \\
&= \mathbb{P}(X \leq w)\mathbb{P}(Y \leq w)\mathbb{P}(Z \leq w) \\
&= F_X(w)F_Y(w)F_Z(w) \\
&= F_X(w)^3. \\
f_W(w) &= 3F_X(w)^2 \cdot f_X(w).
\end{aligned}$$

Where $f_X(w) = \frac{1}{\sqrt{2\pi}} e^{-w^2/2}$ and $F_X(w) = \int_{-\infty}^{w} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$.

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Exercise 10

Let X be a random variable uniformly distributed in $[0, 1]$ and let Y be a random variable uniformly distributed in $[0, 2]$. Suppose X and Y are independent. Find the PDF of X/Y^2.

Solution: Let X be uniformly distributed on $[0, 1]$ and let Y be distributed uniformly on $[0, 2]$. Let $Z = \frac{X}{Y^2}$. First I'll find the CDF of Z and then I'll find the PDF of Z. Since X and Y are both positive, the range for Z is $[0, \infty)$.

To find the CDF of Z we need to do

$$\mathbb{P}(Z \leq z) = \mathbb{P}\left(\frac{X}{Y^2} \leq z\right) = \int\int_{\{0 \leq x \leq 1, 0 \leq y \leq 2, \frac{x}{y^2} \leq z\}} f_{X,Y}(x, y) dxdy = \int\int_{\{0 \leq x \leq 1, 0 \leq y \leq 2, \frac{x}{y^2} \leq z\}} f_X(x)f_Y(y) dxdy$$

where the last equality is because X and Y are independent. The substituting in the two PDF's, the integral becomes $\frac{1}{2}\int\int dxdy$.

We now need to break this up into cases.

- Case 1: $z \cdot y^2 \geq 1$. Then $y \geq \sqrt{\frac{1}{z}} \Rightarrow 0 \leq X \leq 1$.
- Case 2: $z \cdot y^2 < 1$. Then $y < \sqrt{\frac{1}{z}} \Rightarrow 0 \leq x \leq z \cdot y^2$.

Those are the cases for z and y jointly but we also have the cases just depending on z:

- Case A: if $\sqrt{\frac{1}{z}} \geq 2$, only case 2 exists above.

$$\begin{aligned}
F_Z(z) &= \frac{1}{2}\left(\int_0^2 \int_0^{zy^2} dxdy\right) \\
&= \frac{1}{2}\left(\int_0^2 zy^2 dy\right) \\
&= \frac{1}{2}\left[\frac{zy^3}{3}\right]_{y=0}^{y=2} \\
&= \frac{4z}{3}, \text{ for } 0 \leq z \leq \frac{1}{4}.
\end{aligned}$$

- Case B: if $\sqrt{\frac{1}{z}} < 2$, both case 1 and 2 above exist.

$$\begin{aligned}
F_Z(z) &= \frac{1}{2}\left(\int_{\sqrt{\frac{1}{z}}}^{2}\int_0^1 dxdy + \int_0^{\sqrt{\frac{1}{z}}}\int_0^{zy^2} dxdy\right)\\
&= \frac{1}{2}\left(2 - \sqrt{\frac{1}{z}} + \left[\frac{zy^3}{3}\right]_{y=0}^{y=\sqrt{\frac{1}{z}}}\right)\\
&= \frac{1}{2}\left(2 - \sqrt{\frac{1}{z}} + \frac{z \cdot z^{-3/2}}{3}\right)\\
&= \frac{1}{2}\left(2 - \sqrt{\frac{1}{z}} + \frac{z^{-1/2}}{3}\right)\\
&= \frac{3\sqrt{z}-1}{3\sqrt{z}}, \text{ for } z > \frac{1}{4}.
\end{aligned}$$

So therefore, taking the derivative of each we get

$$f_Z(z) = \begin{cases} \frac{1}{6z^{3/2}}, & z > \frac{1}{4} \\ \frac{4}{3}, & 0 \leq z \leq \frac{1}{4}. \end{cases}$$

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