Math 170B	**Winter 2017**

Problem Set 5

Lecturer: Steven Heilman — *Kyle Barron*

Exercise 1

Let X be a standard Gaussian random variable. Let $t > 0$ and let n be a positive even integer. Show that

$$\mathbf{P}(X > t) \leq \frac{(n-1)(n-3)\cdots(3)(1)}{t^n}.$$

That is, the function $t \mapsto \mathbf{P}(X > t)$ decays faster than any monomial.

Exercise 2

Let X be a random variable. Let $t > 0$. Show that

$$\mathbf{P}(|X| > t) \leq \frac{\mathrm{E}X^4}{t^4}.$$

Exercise 3

(The Chernoff Bound.) Let X be a random variable and let $r > 0$. Show that, for any $t > 0$,

$$\mathbf{P}(X > r) \leq e^{-tr} M_X(t).$$

Consequently, if $X_1, \ldots, X_n$ are independent random variables with the same CDF, and if $r, t > 0$,

$$\mathbf{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i > r\right) \leq e^{-trn}(M_{X_1}(t))^n.$$

For example, if $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter $0 < p < 1$, and if $r, t > 0$,

$$\mathbf{P}\left(|\frac{X_1 + \cdots + X_n}{n} - p| > r\right) \leq e^{-trn}(p(1-p)(1+e^t))^n.$$

And if we choose $t = 1/10$, then the quantity $\mathbf{P}\left(\frac{1}{n}|\sum_{i=1}^{n}(X_i - p)| > r\right)$ becomes exponentially small as either n or r become large. That is, $\frac{1}{n}\sum_{i=1}^{n} X_i$ becomes very close to its mean. Importantly, the Chernoff bound is much stronger than either Markov's or Cheyshev's inequality, since they only respectively imply that

$$\mathbf{P}\left(|\frac{X_1 + \cdots + X_n}{n} - p| > r\right) \leq \frac{2p(1-p)}{nr}, \quad \mathbf{P}\left(|\frac{X_1 + \cdots + X_n}{n} - p| > r\right) \leq \frac{p(1-p)}{nr^2}.$$

Exercise 4

Let $X_1, X_2, \ldots$ be independent random variables, each with exponential distribution with parameter $\lambda = 1$. For any $n \geq 1$, let $Y_n := \max(X_1, \ldots, X_n)$. Let $0 < a < 1 < b$. Show that $\mathbf{P}(Y_n \leq a \log n) \to 0$ as $n \to \infty$, and $\mathbf{P}(Y_n \leq b \log n) \to 1$ as $n \to \infty$. Conclude that $Y_n / \log n$ converges to 1 in probability as $n \to \infty$.

Exercise 5

We say that random variables $X_1, X_2, \ldots$ converge to a random variable X in L_2 if

$$\lim_{n\to\infty} \mathrm{E}|X_n - X|^2 = 0.$$

Show that, if $X_1, X_2, \ldots$ converge to X in L_2, then $X_1, X_2, \ldots$ converges to X in probability.

Is the converse true? Prove your assertion.

Exercise 6

Let $X_1, X_2, \ldots$ be independent, identically distributed random variables such that $\mathrm{E}|X| < \infty$ and $\mathrm{var}(X) < \infty$. For any $n \geq 1$, define

$$Y_n := \frac{1}{n} \sum_{i=1}^{n} X_i^2.$$

Show that $Y_1, Y_2, \ldots$ converges in probability. Express the limit in terms of $\mathrm{E}X$ and $\mathrm{var}(X)$.