

Problem Set 5

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Exercise 1

Let X be a standard Gaussian random variable. Let $t > 0$ and let n be a positive even integer. Show that

$$\mathbf{P}(X > t) \leq \frac{(n-1)(n-3) \cdots (3)(1)}{t^n}.$$

That is, the function $t \mapsto \mathbf{P}(X > t)$ decays faster than any monomial.

Proof: Let X be a standard Gaussian random variable and let $t > 0$. Let n be a positive even integer. Recall that from problem set 3, question 2, we found that for n even, $\mathbf{E}[X^n] = (n-1)(n-3) \cdots 1$. Now using the Markov Inequality we have

$$\mathbf{P}(X > t) \leq \mathbf{P}(|X| > t) = \mathbf{P}(X^n > t^n) \leq \frac{\mathbf{E}[X^n]}{t^n} = \frac{(n-1)(n-3) \cdots 1}{t^n},$$

where the second equality is since n is even, and the inequality is using the Markov Inequality, since X^n is nonnegative. ■

Exercise 2

Let X be a random variable. Let $t > 0$. Show that

$$\mathbf{P}(|X| > t) \leq \frac{\mathbf{E}X^4}{t^4}.$$

Proof: Let X be a random variable and let $t > 0$.

$$\begin{aligned} \mathbf{P}(|X| > t) &= \mathbf{P}(|X|^4 > t^4) \\ &= \mathbf{P}(X^4 > t^4) \\ &\leq \frac{\mathbf{E}[X^4]}{t^4}. \end{aligned}$$

where the first equality is true because x^4 is an increasing function when $x \geq 0$ (and here both $|X|$ and t are nonnegative), the second equality is true because $x^4 \geq 0$ always, and the third line uses the Markov Inequality. ■

Exercise 3

(The Chernoff Bound.) Let X be a random variable and let $r > 0$. Show that, for any $t > 0$,

$$\mathbf{P}(X > r) \leq e^{-tr} M_X(t).$$

Consequently, if X_1, \dots, X_n are independent random variables with the same CDF, and if $r, t > 0$,

$$\mathbf{P} \left(\frac{1}{n} \sum_{i=1}^n X_i > r \right) \leq e^{-trn} (M_{X_1}(t))^n.$$

For example, if X_1, \dots, X_n are independent Bernoulli random variables with parameter $0 < p < 1$, and if $r, t > 0$,

$$\mathbf{P} \left(\frac{X_1 + \dots + X_n}{n} - p > r \right) \leq e^{-trn} (e^{-tp} [pe^t + (1-p)])^n.$$

And if we choose t appropriately, then the quantity $\mathbf{P} \left(\frac{1}{n} \sum_{i=1}^n (X_i - p) > r \right)$ becomes exponentially small as either n or r become large. That is, $\frac{1}{n} \sum_{i=1}^n X_i$ becomes very close to its mean. Importantly, the Chernoff bound is much stronger than either Markov's or Cheyshev's inequality, since they only respectively imply that

$$\mathbf{P} \left(\left| \frac{X_1 + \dots + X_n}{n} - p \right| > r \right) \leq \frac{2p(1-p)}{nr}, \quad \mathbf{P} \left(\left| \frac{X_1 + \dots + X_n}{n} - p \right| > r \right) \leq \frac{p(1-p)}{nr^2}.$$

Exercise 4

Let X_1, X_2, \dots be independent random variables, each with exponential distribution with parameter $\lambda = 1$. For any $n \geq 1$, let $Y_n := \max(X_1, \dots, X_n)$. Let $0 < a < 1 < b$. Show that $\mathbf{P}(Y_n \leq a \log n) \rightarrow 0$ as $n \rightarrow \infty$, and $\mathbf{P}(Y_n \leq b \log n) \rightarrow 1$ as $n \rightarrow \infty$. Conclude that $Y_n / \log n$ converges to 1 in probability as $n \rightarrow \infty$.

Exercise 5

We say that random variables X_1, X_2, \dots converge to a random variable X in L_2 if

$$\lim_{n \rightarrow \infty} \mathbf{E}|X_n - X|^2 = 0.$$

Show that, if X_1, X_2, \dots converge to X in L_2 , then X_1, X_2, \dots converges to X in probability.

Is the converse true? Prove your assertion.

Proof: Let X, X_1, X_2, \dots be random variables such that X_1, X_2, \dots converge to X in L_2 . That is,

$$\lim_{n \rightarrow \infty} \mathbf{E}[|X_n - X|^2] = 0.$$

We want to show that X_1, X_2, \dots converges to X in probability, that is, if for all $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbf{P}(|X_n - X| > \epsilon) = 0.$$

First recall that the Markov Inequality says that

$$\mathbf{P}(|Z| \geq t) \leq \frac{\mathbf{E}[|Z|^2]}{t^2}, \forall t > 0.$$

Let $\epsilon > 0$, let $t = \epsilon$, and let $Z = X_n - X$. So then we have

$$\mathbf{P}(|X_n - X| > \epsilon) \leq \frac{\mathbf{E}[(X_n - X)^2]}{\epsilon^2}.$$

Since we know that X_1, X_2, \dots converge to X in L^2 ,

$$\lim_{n \rightarrow \infty} \mathbf{E}[(X_n - X)^2] = 0 \Rightarrow \lim_{n \rightarrow \infty} \mathbf{P}(|X_n - X| > \epsilon) = 0.$$

So X_1, X_2, \dots converge in probability.

Recall that

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Exercise 6

Let X_1, X_2, \dots be independent, identically distributed random variables such that $E|X| < \infty$ and $\text{var}(X) < \infty$. For any $n \geq 1$, define

$$Y_n := \frac{1}{n} \sum_{i=1}^n X_i^2.$$

Show that Y_1, Y_2, \dots converges in probability. Express the limit in terms of EX and $\text{var}(X)$.

Proof: Let X_1, X_2, \dots be independent and identically distributed with $E[|X|] < \infty$ and $\text{Var}(X) < \infty$. Define

$$Y_n := \frac{1}{n} \sum_{i=1}^n X_i^2.$$

Let $\epsilon > 0$. To show that Y_1, Y_2, \dots converges in probability we need to show that $\lim_{n \rightarrow \infty} \mathbf{P}(|Y_n - Y| > \epsilon) = 0$. I'll show that Y , the limit of Y_n as $n \rightarrow \infty$, is equal to the second moment of X_1 (which is the same for all X since they're independent and identically distributed).

$$\begin{aligned} \mathbf{P}(|Y_n - Y| > \epsilon) &= P\left(\left|\frac{x_1^2 + \dots + x_n^2}{n} - E[X_1^2]\right| > \epsilon\right) \\ &= \frac{1}{\epsilon^2} E\left[\left|\frac{x_1^2 + \dots + x_n^2}{n} - E[X_1^2]\right|^2\right] \\ &= \frac{1}{\epsilon^2} E\left[\left(\frac{x_1^2 + \dots + x_n^2}{n} - \frac{nE[X_1^2]}{n}\right)^2\right] \\ &= \frac{1}{\epsilon^2} E\left[\left(\frac{x_1^2 + \dots + x_n^2}{n} - \frac{E[X_1^2] + \dots + E[X_n^2]}{n}\right)^2\right] \\ &= \frac{1}{\epsilon^2} E\left[\left(\frac{x_1^2 + \dots + x_n^2}{n} - \frac{E[X_1^2 + \dots + X_n^2]}{n}\right)^2\right] \\ &= \frac{1}{\epsilon^2 n^2} E\left[(x_1^2 + \dots + x_n^2 - E[X_1^2 + \dots + X_n^2])^2\right] \\ &= \frac{1}{\epsilon^2 n^2} \text{Var}(X_1^2 + \dots + X_n^2) \\ &= \frac{1}{\epsilon^2 n} \text{Var}(X_1^2). \end{aligned}$$

Now letting $n \rightarrow \infty$, we have that $\frac{\text{Var}(X_1^2)}{\epsilon^2 n} \rightarrow 0$. Therefore, Y_1, Y_2, \dots converges in probability to $E[X_1^2] = \text{Var}(X_1) + E[X_1]^2$. ■