Math 170B **Winter 2017**

Problem Set 4

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Exercise 1

Compute the characteristic function of a uniformly distributed random variable on $[-1, 1]$. (Some of the following formulas might help to simplify your answer: $e^{it} = \cos(t) + i\sin(t)$, $\cos(t) = [e^{it} + e^{-it}]/2$, $\sin(t) = [e^{it} - e^{-it}]/[2i]$, $t \in \mathbb{R}$.) (Here $i := \sqrt{-1}$.)

Solution:

$$\begin{aligned}
\phi_X(t) &= \mathrm{E}[e^{itx}] \\
&= \frac{1}{2}\int_{-1}^{1} e^{itx}dx \\
&= \frac{1}{2it}e^{itx}\Big|_{-1}^{1} \\
&= \frac{1}{2it}\left(e^{it} - e^{-it}\right) \\
&= \frac{\sin(t)}{t}.
\end{aligned}$$

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Exercise 2

Let X be a random variable. Assume we can differentiate under the expected value of $\mathrm{E}e^{itX}$ any number of times. For any positive integer n, show that

$$\frac{d^n}{dt^n}\phi_X(t)\Big|_{t=0} = i^n \mathrm{E}(X^n).$$

So, in principle, all moments of X can be computed just by taking derivatives of the characteristic function.

Proof:

$$\begin{aligned}
\frac{d^n}{dt^n}\phi_X(t) &= \frac{d^n}{dt^n}\int_{-\infty}^{\infty} e^{itx}f_X(x)dx \\
&= \int_{-\infty}^{\infty} \frac{d^n}{dt^n}e^{itx}f_X(x)dx \\
&= \int_{-\infty}^{\infty} (ix)^n e^{itx}f_X(x)dx.
\end{aligned}$$

Now considering the special case where $t = 0$, we get that

$$\begin{aligned}\frac{d^n}{dt^n}\phi_X(t)\bigg|_{t=0} &= \int_{-\infty}^{\infty}(ix)^n f_X(x)dx \\ &= i^n\int_{-\infty}^{\infty}x^n f_X(x)dx \\ &= i^n \mathrm{E}[X^n]\end{aligned}$$

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Exercise 3

Let X be a random variable such that $\mathrm{E}[|X|^3] < \infty$. Prove that for any $t \in \mathbb{R}$,

$$\mathrm{E}e^{itX} = 1 + it\mathrm{E}X - t^2\mathrm{E}X^2/2 + o(t^2).$$

That is,

$$\lim_{t\to 0} t^{-2}|\mathrm{E}e^{itX} - [1 + it\mathrm{E}X - t^2\mathrm{E}X^2/2]| = 0.$$

(Hint: it may be helpful to use Jensen's inequality to first justify that $\mathrm{E}|X| < \infty$ and $\mathrm{E}X^2 < \infty$. Then, use the Taylor expansion with error bound: $e^{iy} = 1 + iy - y^2/2 - (i/2)\int_0^y (y-s)^2 e^{is}ds$, which is valid for any $y \in \mathbb{R}$.)

Actually, this same bound holds only assuming $\mathrm{E}X^2 < \infty$, but the proof of that bound requires things we have not discussed.

Solution: Let X be a random variable with $\mathrm{E}[|X|^3] < \infty$. Recall that Jensen's inequality says that for a twice differentiable, convex function f, $\mathrm{E}[f(X)] \geq f(\mathrm{E}[X])$.

I'll first show that $\mathrm{E}[|X|] < \infty$. Let $f(X) = |X|^3$; note that this is a convex function. Then $\mathrm{E}[f(|X|)] = \mathrm{E}[|X|^3]$ and $f(\mathrm{E}[|X|]) = |\mathrm{E}[|X|]|^3 = \mathrm{E}[|X|]^3$. We can now use Jensen's inequality.

$$\infty > \mathrm{E}[|X|^3] \geq |\mathrm{E}[|X|]|^3 = \mathrm{E}[|X|]^3.$$

Therefore, $\mathrm{E}[|X|] < \infty$.

I'll do a similar procedure to show that $\mathrm{E}[X^2] < \infty$. Let $g(X) = |X|^{\frac{3}{2}}$. This is also a convex function. Then $\mathrm{E}[g(X^2)] = \mathrm{E}[|X|^3]$ and $g(\mathrm{E}[X^2]) = |\mathrm{E}[X^2]|^{\frac{3}{2}} = \mathrm{E}[X^2]^{\frac{3}{2}}$. We can now use Jensen's inequality.

$$\infty > \mathrm{E}[|X|^3] \geq |\mathrm{E}[X^2]|^{\frac{3}{2}} = \mathrm{E}[X^2]^{\frac{3}{2}}.$$

Therefore, $\mathrm{E}[X^2] < \infty$.

Using the Taylor expansion of $\mathrm{E}[e^{itX}]$, we have

$$\mathrm{E}[e^{itX}] = 1 + it\mathrm{E}[X] - \frac{t^2\mathrm{E}[X^2]}{2} - \frac{it}{2}\int_0^{tX}(tX - s)^2 e^{is}ds.$$

Now plugging this into the limit we want to prove,

$$\lim_{t\to 0}\frac{1}{t^2}\left|\mathrm{E}e^{itX}-\left[1+it\mathrm{E}X-\frac{t^2\mathrm{E}X^2}{2}\right]\right| =$$
$$\lim_{t\to 0}\frac{1}{t^2}\left|1+it\mathrm{E}[X]-\frac{t^2\mathrm{E}[X^2]}{2}-\frac{it}{2}\int_0^{tX}(tX-s)^2e^{is}ds-\left[1+it\mathrm{E}X-\frac{t^2\mathrm{E}X^2}{2}\right]\right| =$$
$$\lim_{t\to 0}\frac{1}{t^2}\left|\frac{it}{2}\int_0^{tX}(tX-s)^2e^{is}ds\right| =$$
$$\lim_{t\to 0}\frac{1}{t}\left|\frac{i}{2}\int_0^{tX}(tX-s)^2e^{is}ds\right|.$$

Clearly this limit is equal to zero. ■

Exercise 4

(Convolution is Associative.) Let $g, h, d\colon \mathbb{R}\to\mathbb{R}$. Then for any $t\in\mathbb{R}$,

$$((g*h)*d)(t) = (g*(h*d))(t)$$

Proof:

$$\begin{aligned}((g*h)*d)(t) &= \int_{-\infty}^{\infty}(g*h)(x)d(t-x)dx\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(g(s)h(x-s)ds)d(t-x)dx\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(s)h(x-s)d(t-x)dsdx.\end{aligned}$$

Now switching the order of integration gives:

$$\begin{aligned}((g*h)*d)(t) &= \int_{-\infty}^{\infty}g(s)\left[\int_{-\infty}^{\infty}h(x-s)d(t-x)dx\right]ds\\ &= \int_{-\infty}^{\infty}g(s)\left[\int_{-\infty}^{\infty}h((x+s)-s)d(t-(x+s))dx\right]ds\\ &= \int_{-\infty}^{\infty}g(s)\left[\int_{-\infty}^{\infty}h(x)d((t-s)-x)dx\right]ds\\ &= \int_{-\infty}^{\infty}g(s)\left[(h*d)(t-s)\right]ds\\ &= (g*(h*d))(t)\end{aligned}$$

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Exercise 5

Let X, Y, Z be independent and uniformly distributed on $[0,1]$. Note that f_X is not a continuous function.

Using convolution, compute f_{X+Y}. Draw f_{X+Y}. Note that f_{X+Y} is a continuous function, but it is not differentiable at some points.

Using convolution, compute f_{X+Y+Z}. Draw f_{X+Y+Z}. Note that f_{X+Y+Z} is a differentiable function, but it does not have a second derivative at some points.

Make a conjecture about how many derivatives $f_{X_1+\cdots+X_n}$ has, where $X_1,\ldots,X_n$ are independent and uniformly distributed on $[0,1]$. You do not have to prove this conjecture. The idea of this exercise is that convolution is a kind of average of functions. And the more averaging you do, the more derivatives $f_{X_1+\cdots+X_n}$ has.

Solution: Let X,Y,Z be independent and uniformly distributed over $[0,1]$. First I'll compute f_{X+Y}. Recall that Proposition 2.60 says that $f_{X+Y}(t) = (f_X * f_Y)(t)$, for all $t \in \mathbb{R}$. Recall that by Definition 2.59, $(g*h)(t) \coloneqq \int_{-\infty}^{\infty} g(x)h(t-x)dx$, for all $t \in \mathbb{R}$. Therefore,

$$\begin{aligned} f_{X+Y}(t) &= (f_X * f_Y)(t) \\ &= \int_{-\infty}^{\infty} f_X(x) f_Y(t-x)dx \\ &= \int_0^1 1 \cdot f_Y(t-x)dx \\ &= \int_{x\in[0,1]\cap[t-1,t]} 1dx. \end{aligned}$$

Therefore, $f_{X+Y}(t-x) = \begin{cases} t-x & 0 \le t-x \le 1; t-1 \le x \le t \\ 2-(t-x) & 1 \le t-x \le 2; t-2 \le x \le t-1. \end{cases}$.

We then convolute again to find $f_{X+Y+Z}(t)$.

$$f_{X+Y+Z}(t) = \int_{s\in[0,1]\cap[t-1,t]} (t-1)ds + \int_{s\in[0,1]\cap[t-2,t]} 2-(t-s)ds$$

We now have three cases:

1. $t \in [0,1]$:

$$\begin{aligned} f_{X+Y+Z} &= \int_{s\in[0,1]\cap[-1,1]} (t-s)ds + \int_{s\in[0,1]\cap[-2,1]} 2-(t-s)ds \\ &= \int_0^1 (t-s)ds + \int_0^2 2-(t-s)ds \\ &= t - \frac{1}{2} + 4 - 2t + 2 \\ &= \frac{11}{2} - t. \end{aligned}$$

2. $t \in [1,2]$:

$$\begin{aligned} f_{X+Y+Z} &= \int_{s\in[0,1]\cap[0,2]} (t-s)ds + \int_{s\in[0,1]\cap[-1,2]} 2-(t-s)ds \\ &= 2. \end{aligned}$$

3. $t \in [2,3]$:

$$\begin{aligned}
f_{X+Y+Z} &= \int_{s\in[0,1]\cap[1,2]} (t-s)ds + \int_{s\in[0,1]\cap[0,3]} 2-(t-s)ds \\
&= 0 + \int_0^1 2-(t-s)ds \\
&= 2-t+\frac{1}{2} \\
&= \frac{5}{2} - t.
\end{aligned}$$

Therefore, we have

$$f_{X+Y+Z}(t) = \begin{cases} \frac{11}{2} - t, & \text{if } t \in [0,1] \\ 2, & \text{if } t \in [1,2] \\ \frac{5}{2} - t, & \text{if } t \in [2,3]. \end{cases}$$

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Exercise 6

Construct two random variables X, Y such that X and Y are each uniformly distributed on $[0,1]$, and such that $\mathbf{P}(X+Y=1)=1$.

Then construct two random variables W, Z such that W and Z are each uniformly distributed on $[0,1]$, and such that $W+Z$ is uniformly distributed on $[0,2]$.

(Hint: there is a way to do each of the above problems with about one line of work. That is, there is a way to solve each problem without working very hard.)

Solution: For the first part, let X be uniformly distributed on $[0,1]$ with $f_X(x)=1$ for $x\in[0,1]$. Let $Y=1-X$. Then Y is also distributed uniformly on $[0,1]$ and $X+Y=X+1-X=1$.

For the second part, let W be distributed as $f_W(x)=1$ for $x\in[0,1]$. Let $Z=W$. Then

$$\mathbf{P}(W+Z=t) = \mathbf{P}(2W=t) = \mathbf{P}\left(W=\frac{t}{2}\right).$$

So $f_{W+Z}(t)=\frac{1}{2}$ for $t\in(0,2)$. ■