

## Problem Set 2

Lecturer: Steven Heilman

Kyle Barron

**Exercise 1**

Let  $X, Y$  be random variables with  $\mathbb{E}[X^2] < \infty$  and  $\mathbb{E}[Y^2] < \infty$ . Prove the Cauchy-Schwarz inequality:

$$\mathbb{E}[XY] \leq \mathbb{E}[X^2]^{1/2} \mathbb{E}[Y^2]^{1/2}.$$

Then, deduce the following when  $X, Y$  both have finite variance:

$$|\text{cov}(X, Y)| \leq \text{var}(X)^{1/2} \text{var}(Y)^{1/2}.$$

(Hint: in the case that  $\mathbb{E}[Y^2] > 0$ , expand out the product  $\mathbb{E}[X - Y\mathbb{E}[XY]/\mathbb{E}[Y^2]]^2$ .)

**Proof:** Observe that we can rewrite the above inequality as

$$\mathbb{E}[XY]^2 \leq \mathbb{E}[X^2]\mathbb{E}[Y^2].$$

We can assume that  $\mathbb{E}[Y^2] \neq 0$ , because if  $\mathbb{E}[Y^2] = 0$ , then we have  $Y = 0$  with probability 1, and  $\mathbb{E}[XY] = 0$ , so the inequality holds. Then we have

$$\begin{aligned} 0 &\leq \mathbb{E} \left[ \left( X - \frac{\mathbb{E}[XY]}{\mathbb{E}[Y^2]} Y \right)^2 \right] \\ &= \mathbb{E} \left[ X^2 - 2 \frac{\mathbb{E}[XY]}{\mathbb{E}[Y^2]} XY + \frac{(\mathbb{E}[XY])^2}{(\mathbb{E}[Y^2])^2} Y^2 \right] \\ &= \mathbb{E}[X^2] - 2 \frac{\mathbb{E}[XY]}{\mathbb{E}[Y^2]} \mathbb{E}[XY] + \frac{(\mathbb{E}[XY])^2}{(\mathbb{E}[Y^2])^2} \mathbb{E}[Y^2] \\ &= \mathbb{E}[X^2] - \frac{(\mathbb{E}[XY])^2}{\mathbb{E}[Y^2]} \Rightarrow \\ &\mathbb{E}[XY]^2 \leq \mathbb{E}[X^2]\mathbb{E}[Y^2]. \end{aligned}$$

Then for the second part of the problem, let  $\tilde{X} = X - \mathbb{E}[X]$  and let  $\tilde{Y} = Y - \mathbb{E}[Y]$ . Using the Cauchy-Schwarz inequality, we have

$$\text{cov}(X, Y)^2 = \mathbb{E}[\tilde{X}\tilde{Y}]^2 \leq \mathbb{E}[\tilde{X}^2]\mathbb{E}[\tilde{Y}^2] = \text{var}(X)\text{var}(Y).$$

Therefore,

$$|\text{cov}(X, Y)| \leq \text{var}(X)^{1/2} \text{var}(Y)^{1/2}.$$

■

**Exercise 2**

Let  $X$  be a binomial random variable with parameters  $n = 2$  and  $p = 1/2$ . So  $\mathbf{P}(X = 0) = \frac{1}{4}$ ,  $\mathbf{P}(X = 1) = \frac{1}{2}$ , and  $\mathbf{P}(X = 2) = \frac{1}{4}$ . And  $X$  satisfies  $\mathbb{E}[X] = 1$  and  $\mathbb{E}[X^2] = \frac{3}{2}$ .

Let  $Y$  be a geometric random variable with parameter  $1/2$ . So, for any positive integer  $k$ ,  $\mathbf{P}(Y = k) = 2^{-k}$ . And  $Y$  satisfies  $\mathbb{E}[Y] = 2$  and  $\mathbb{E}[Y^2] = 6$ .

Let  $Z$  be a Poisson random variable with parameter 1. So, for any nonnegative integer  $k$ ,  $\mathbf{P}(Z = k) = \frac{1}{e} \frac{1}{k!}$ . And  $Z$  satisfies  $\mathbb{E}[Z] = 1$  and  $\mathbb{E}[Z^2] = 2$ .

Let  $W$  be a discrete random variable such that  $\mathbf{P}(W = 0) = 1/2$  and  $\mathbf{P}(W = 4) = 1/2$ , so that  $\mathbb{E}[W] = 2$  and  $\mathbb{E}[W^2] = 8$ .

Assume that  $X$ ,  $Y$ ,  $Z$ , and  $W$  are all independent. Compute  $\text{var}(X + Y + Z + W)$ .

**Solution:** Recall that we can compute variance as  $\text{var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ . Therefore,

$$\text{var}(X) = \frac{3}{2} - 1^2 = \frac{1}{2},$$

$$\text{var}(Y) = 6 - 2^2 = 2,$$

$$\text{var}(Z) = 2 - 1^2 = 1,$$

$$\text{var}(W) = 8 - 2^2 = 4.$$

Using Corollary 2.25 from the notes, when  $X_1, \dots, X_n$  are independent we have  $\text{var}(\sum_{i=1}^n X_i) = \sum_{i=1}^n \text{var}(X_i)$ . Therefore, since  $X$ ,  $Y$ ,  $Z$ , and  $W$  are independent we have

$$\text{var}(X + Y + Z + W) = \text{var}(X) + \text{var}(Y) + \text{var}(Z) + \text{var}(W) = \frac{1}{2} + 2 + 1 + 4 = \frac{15}{2}.$$

■

### Exercise 3

Let  $X_1, \dots, X_n$  be random variables with finite variance. Define an  $n \times n$  matrix  $A$  such that  $A_{ij} = \text{cov}(X_i, X_j)$  for any  $1 \leq i, j \leq n$ . Show that the matrix  $A$  is positive semidefinite. That is, show that for any  $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ , we have

$$y^T A y = \sum_{i,j=1}^n y_i y_j a_{ij} \geq 0.$$

**Solution:** Let  $A$  be an  $n \times n$  matrix where  $A_{ij} = \text{cov}(X_i, X_j)$  for any  $1 \leq i, j \leq n$ .

$$\begin{aligned} y^T A y &= \sum_{i,j=1}^n y_i y_j a_{ij} \\ &= \sum_{i,j=1}^n y_i y_j \text{cov}(X_i, X_j) \\ &= \sum_{j=1}^n y_j \text{cov}\left(\sum_{i=1}^n X_i, X_j\right) \\ &= \text{cov}\left(\sum_{i=1}^n X_i, \sum_{j=1}^n y_j X_j\right) \\ &= \text{var}\left(\sum_{i=1}^n X_i\right) \geq 0. \end{aligned}$$

■

### Exercise 4

(Another Total Expectation Theorem) Using the definition of  $\mathbb{E}(X|Y)$ , prove the following theorem, which can be considered as a version of a Total Expectation Theorem:

$$\mathbb{E}(\mathbb{E}(X|Y)) = \mathbb{E}(X).$$

**Proof:** Supposing that  $X$  and  $Y$  are continuous random variables,

$$\begin{aligned} \mathbb{E}[\mathbb{E}[X|Y]] &= \int_{-\infty}^{\infty} \mathbb{E}[Y|X = x] f_X(x) dx \\ &= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} y \frac{f_{X,Y}(x, y)}{f_X(x)} dy \right) f_X(x) dx \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f_{X,Y}(x, y) dx dy \\ &= \int_{-\infty}^{\infty} y f_Y(y) dy \\ &= \mathbb{E}[Y]. \end{aligned}$$

The proof is identical for discrete random variables by substituting  $\int dx$  for  $\sum_x$  and  $\int dy$  for  $\sum_y$ . ■

### Exercise 5

If  $X$  is a random variable, and if  $f(t) := \mathbb{E}[(X - t)^2]$ ,  $t \in \mathbb{R}$ , then the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is uniquely minimized when  $t = \mathbb{E}[X]$ . This follows e.g. by writing

$$\begin{aligned} \mathbb{E}[(X - t)^2] &= \mathbb{E}[(X - \mathbb{E}[X] + \mathbb{E}[X] - t)^2] \\ &= \mathbb{E}[X - \mathbb{E}[X]]^2 + (\mathbb{E}[X] - t)^2 + 2\mathbb{E}[(X - \mathbb{E}[X])(\mathbb{E}[X] - t)] \\ &= \mathbb{E}[X - \mathbb{E}[X]]^2 + (\mathbb{E}[X] - t)^2. \end{aligned}$$

So, the choice  $t = \mathbb{E}[X]$  is the smallest, and it recovers the definition of variance, since  $\text{var}(X) = \mathbb{E}[X - \mathbb{E}[X]]^2$ .

A similar minimizing property holds for conditional expectation. Let  $h : \mathbb{R} \rightarrow \mathbb{R}$ . Show that the quantity  $\mathbb{E}[X - h(Y)]^2$  is minimized among all functions  $h$  when  $h(Y) = \mathbb{E}[X|Y]$ . (Hint: Exercise 4 might be helpful.)

**Proof:**

$$\begin{aligned} \mathbb{E}[(X - h(Y))^2] &= \mathbb{E}[(X - \mathbb{E}[X|Y] + \mathbb{E}[X|Y] - h(Y))^2] \\ &= \mathbb{E}[(X - \mathbb{E}[X|Y])^2] + 2\mathbb{E}[(X - \mathbb{E}[X|Y])(\mathbb{E}[X|Y] - h(Y))] + \mathbb{E}[(\mathbb{E}[X|Y] - h(Y))^2]. \end{aligned}$$

It is sufficient to show that  $2\mathbb{E}[(X - \mathbb{E}[X|Y])(\mathbb{E}[X|Y] - h(Y))] = 0$ . Then it is clear that  $h(Y) = \mathbb{E}[X|Y]$  minimizes  $\mathbb{E}[(X - h(Y))^2]$ . From the textbook, we know that  $\mathbb{E}[Xh(Y)|Y] = h(Y)\mathbb{E}[X|Y]$ . Therefore,

$$\begin{aligned} 2\mathbb{E}[(X - \mathbb{E}[X|Y])(\mathbb{E}[X|Y] - h(Y))] &= 2\mathbb{E}\left[\mathbb{E}[(X - \mathbb{E}[X|Y])(\mathbb{E}[X|Y] - h(Y))]\right] \\ &= 2\mathbb{E}[(\mathbb{E}[X|Y] - h(Y))\mathbb{E}[X - \mathbb{E}[X|Y]|Y] \\ &= 2(\mathbb{E}[X|Y] - \mathbb{E}[X|Y])(\mathbb{E}[X|Y] - \mathbb{E}[X|Y]) \\ &= 0. \end{aligned}$$

Therefore  $\mathbb{E}[(X - h(Y))^2]$  is minimized when  $h(Y) = \mathbb{E}[X|Y]$ . ■

## Exercise 6

Toys are stored in small boxes, small boxes are stored in large crates, and large crates comprise a shipment. Let  $X_i$  be the number of toys in small box  $i \in \{1, 2, \dots\}$ . Assume that  $X_1, X_2, \dots$  all have the same CDF. Let  $Y_i$  be the number of small boxes in large crate  $i \in \{1, 2, \dots\}$ . Assume that  $Y_1, Y_2, \dots$  all have the same CDF. Let  $Z$  be the number of large crates in the shipment. Assume that  $X_1, X_2, \dots, Y_1, Y_2, \dots, Z$  are all independent, nonnegative integer-valued random variables, each with expected value 10 and variance 16.

Compute the expected value and variance of the number of toys in the shipment.

### Solution:

Let  $T_1$  be the number of toys in the large crate. Let  $y \in \mathbb{N}_+$ . Given that  $Y_1 = y$ , we know that  $T_1 = X_1 + \dots + X_y$ . So using independence, we have

$$\mathbb{E}[T_1|Y_1 = y] = \mathbb{E}[X_1 + X_2 + \dots + X_y|Y_1 = y] = y\mathbb{E}[X_1] = 10y.$$

So by the definition of conditional expectation,  $\mathbb{E}[T_1|Y_1] = 10Y_1$ . We know from Exercise 4 that

$$\mathbb{E}[T_1] = \mathbb{E}[\mathbb{E}[T_1|Y_1]] = \mathbb{E}[10Y_1] = 10\mathbb{E}[Y_1] = 100.$$

For the variance, from the definition of conditional variance and Corollary 2.25, we observe that

$$\begin{aligned} \text{var}(T_1|Y_1 = y) &= \mathbb{E}[(T_1 - 10y)^2|Y_1 = y] \\ &= \mathbb{E}[(X_1 + X_2 + \dots + X_y - 10y)^2|Y_1 = y] \\ &= \mathbb{E}[(X_1 + X_2 + \dots + X_y - \mathbb{E}[X_1 + \dots + X_y])^2] \\ &= \text{var}(X_1 + X_2 + \dots + X_y) \\ &= y\text{var}(X_1) \\ &= 16y. \end{aligned}$$

So,  $\text{var}(T_1|Y_1) = 16Y_1$ , and by Proposition 2.34 we have

$$\begin{aligned} \text{var}(T_1) &= \mathbb{E}[\text{var}(T_1|Y_1)] + \text{var}(\mathbb{E}[T_1|Y_1]) \\ &= \mathbb{E}[16Y_1] + \text{var}(10Y_1) \\ &= 16\mathbb{E}[Y_1] + 100\text{var}(Y_1) \\ &= 160 + 1600 = 1760. \end{aligned}$$

Now let  $W$  be the number of toys in the entire shipment. Let  $z \in \mathbb{N}_+$ . Given that  $Z = z$ , we know that  $W = T_1 + T_2 + \dots + T_z$ . So using independence we have

$$\mathbb{E}[W|Z = z] = \mathbb{E}[T_1 + T_2 + \dots + T_z|Z = z] = z\mathbb{E}[T_1] = 100z.$$

So by the definition of conditional expectation,  $\mathbb{E}[W|Z] = 100Z$ . By Exercise 4,

$$\mathbb{E}[W] = \mathbb{E}[\mathbb{E}[W|Z]] = \mathbb{E}[100Z] = 100\mathbb{E}[Z] = 1000.$$

From the definition of conditional variance and Corollary 2.25, we observe that

$$\begin{aligned} \text{var}(W|Z = z) &= \mathbb{E}[(W - 100z)^2|Z = z] \\ &= \mathbb{E}[(T_1 + T_2 + \dots + T_z - 100z)^2|Z = z] \\ &= \mathbb{E}[(T_1 + T_2 + \dots + T_z - \mathbb{E}[T_1 + \dots + T_z])^2] \\ &= \text{var}(T_1 + T_2 + \dots + T_z) \\ &= z\text{var}(T_1) \\ &= 1760z. \end{aligned}$$

So  $\text{var}(W|Z) = 1760Z$ , and by Proposition 2.34 we have

$$\begin{aligned}\text{var}(W) &= \mathbb{E}[\text{var}(W|Z)] + \text{var}(\mathbb{E}[W|Z]) \\ &= \mathbb{E}[1760Z] + \text{var}(100Z) \\ &= 1760\mathbb{E}[Z] + 10000\text{var}(Z) \\ &= 1760 \cdot 10 + 10000 \cdot 16 \\ &= 177600.\end{aligned}$$

So the expected number of toys in the shipment is 1000 with a variance of 177600. ■

## Exercise 7

Let  $0 < p < 1$ . Suppose you have a biased coin which has a probability  $p$  of landing heads, and probability  $1 - p$  of landing tails, each time it is flipped. Also, suppose you have a fair six-sided die (so each face of the cube has a distinct label from the set  $\{1, 2, 3, 4, 5, 6\}$ , and each time you roll the die, any face of the cube is rolled with equal probability.)

Let  $N$  be the number of coin flips you need to do until the first head appears. Now, roll the fair die  $N$  times. Let  $S$  be the sum of the results of the  $N$  rolls of the die. Compute  $\mathbb{E}[S]$  and  $\text{var}(S)$ .

### Solution:

Let  $N$  be the number of coin flips that it takes to make the first head appear. Let  $S$  be the sum of the results of the  $N$  rolls of the die. Then  $N$  is a geometric random variable with parameter  $p$ . Therefore we have

$$\mathbb{E}[N] = \frac{1}{p} \text{ and } \text{var}(N) = \frac{1-p}{p^2}.$$

Let  $D_i$  be the random variable corresponding to each roll of the dice. Note that  $\mathbb{E}[D_i] = 3.5$  and that  $\text{var}(D_i) = \frac{17.5}{6}$ . Let  $n \in \mathbb{N}_+$ , so that  $S = D_1 + D_2 + \dots + D_n$  when  $N = n$ . Then

$$\mathbb{E}[S|N = n] = \mathbb{E}[D_1 + \dots + D_n|N = n] = n\mathbb{E}[D_1] = 3.5n.$$

By the definition of conditional expectation, we have  $\mathbb{E}[S|N] = 3.5N$ . By Exercise 4 we have

$$\mathbb{E}[S] = \mathbb{E}[\mathbb{E}[S|N]] = \mathbb{E}[3.5N] = 3.5\mathbb{E}[N] = \frac{3.5}{p}.$$

From the definition of conditional variance and Corollary 2.25, we have

$$\begin{aligned}\text{var}(S|N = n) &= \mathbb{E}[(S - \mathbb{E}[S|N])^2|N = n] \\ &= \mathbb{E}[(D_1 + D_2 + \dots + D_n - 3.5N)^2|N = n] \\ &= \text{var}(D_1 + \dots + D_n) \\ &= n\text{var}(D_1) \\ &= n\frac{17.5}{6}.\end{aligned}$$

So then  $\text{var}(S|N) = N \frac{17.5}{6}$ . Then by proposition 2.34,

$$\begin{aligned} \text{var}(S) &= \mathbb{E}[\text{var}(S|N)] + \text{var}(\mathbb{E}[S|N]) \\ &= \mathbb{E}[N \frac{17.5}{6}] + \text{var}(3.5N) \\ &= \frac{17.5}{6} \mathbb{E}[N] + 3.5^2 \cdot \text{var}(N) \\ &= \frac{17.5}{6} \cdot \frac{1}{p} + \frac{49}{4} \cdot \frac{1-p}{p^2} \\ &= \frac{147 - 112p}{12p^2}. \end{aligned}$$

So we have  $\mathbb{E}[S] = 3.5 \frac{1-p}{p}$  and  $\text{var}(S) = \frac{147-112p}{12p^2}$ . ■

### Exercise 8

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a twice differentiable function. Assume that  $f$  is convex. That is,  $f''(x) \geq 0$ , or equivalently, the graph of  $f$  lies above any of its tangent lines. That is, for any  $x, y \in \mathbb{R}$ ,

$$f(x) \geq f(y) + f'(y)(x - y).$$

Let  $X$  be a discrete random variable. By setting  $y = \mathbb{E}[X]$ , prove **Jensen's inequality**:

$$\mathbb{E}[f(X)] \geq f(\mathbb{E}[X]).$$

In particular, choosing  $f(x) = x^2$ , we have  $\mathbb{E}[X^2] \geq (\mathbb{E}[X])^2$ .

**Proof:** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be twice differentiable and convex. Therefore, for all  $x, y \in \mathbb{R}$ ,

$$f(x) \geq f(y) + f'(y)(x - y).$$

Let  $x = X$  and let  $y = \mathbb{E}[X]$ . Then

$$f(X) \geq f(\mathbb{E}[X]) + f'(\mathbb{E}[X])(X - \mathbb{E}[X]).$$

Notice however, that  $\mathbb{E}[X]$  is a constant, so its derivative is zero. Now taking the expectation of both sides, we get

$$\begin{aligned} \mathbb{E}[f(X)] &\geq \mathbb{E}[f(\mathbb{E}[X]) + f'(\mathbb{E}[X])(X - \mathbb{E}[X])] \\ &= f(\mathbb{E}[X]) + \mathbb{E}[f'(\mathbb{E}[X])(X - \mathbb{E}[X])] \\ &= f(\mathbb{E}[X]). \end{aligned}$$

Therefore  $\mathbb{E}[f(X)] \geq f(\mathbb{E}[X])$  for a convex function  $f$ . ■