Please provide complete and well-written solutions to the following exercises.

Due January 12, in the discussion section.

(This Review Assignment will be collected, but this Review Assignment will not be graded.)

Preliminary Review Assignment

Exercise 1. As needed, refresh your knowledge of proofs and logic by reading the following document by Michael Hutchings: http://math.berkeley.edu/~hutching/teach/proofs.pdf

Exercise 2. Take the following quizzes on logic, set theory, and functions. (This material should be review from 115A.):

http://scherk.pbworks.com/w/page/14864234/Quiz%3A%20Logic
http://scherk.pbworks.com/w/page/14864241/Quiz%3A%20Sets
http://scherk.pbworks.com/w/page/14864227/Quiz%3A%20Functions

(These quizzes are just for your own benefit; you don't need to record your answers anywhere.)

Exercise 3. Prove the following assertion by induction:

For any natural number n, $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$.

Proof. Step 1. The above formula clearly works for $n = 1$.
Step 2. Suppose it works for some $n \geq 1$. Then

$$\begin{aligned} 1^2 + \cdots + (n+1)^2 &= (1^2 + \cdots + n^2) + (n+1)^2 \\ &= \frac{1}{6}n(n+1)(2n+1) + (n+1)^2 = \frac{1}{6}(n+1)(n+2)(2n+3). \end{aligned}$$

Thus the formula works for $n + 1$.

□

Exercise 4. Prove that the set of real numbers $\mathbf{R}$ can be written as the countable union

$$\mathbf{R} = \bigcup_{j=1}^{\infty} [-j, j].$$

(Hint: you should show that the left side contains the right side, and also show that the right side contains the left side.)

Prove that the singleton set $\{0\}$ can be written as

$$\{0\} = \bigcap_{j=1}^{\infty} [-1/j, 1/j].$$

Proof. (a) First, for any $j \in \mathbb{N}$, $\mathbf{R} \supseteq [-j, j]$. Thus $\mathbf{R} \supseteq \bigcup_{j=1}^{\infty} [-j, j]$.
Second, by the archimedean property of real numbers, for any $x \in \mathbf{R}$, there exists $i \in \mathbb{N}$ such that $|x| \leq i$. Thus $x \in [-i, i] \subseteq \bigcup_{j=1}^{\infty} [-j, j]$. Thus $\mathbf{R} \subseteq \bigcup_{j=1}^{\infty} [-j, j]$.
Thus $\mathbf{R} = \bigcup_{j=1}^{\infty} [-j, j]$.
(b) First, for any $j \in \mathbb{N}$, $\{0\} \subseteq [-1/j, 1/j]$. Thus $\{0\} \subseteq \bigcap_{j=1}^{\infty} [-1/j, 1/j]$.
Second, by the archimedean property of real numbers again, for any $x \in (\mathbf{R} \setminus \{0\})$, there exists $i \in \mathbb{N}$ such that $i > 1/|x|$. This implies that $x \notin [-1/i, 1/i]$, which further implies that $x \notin \bigcap_{j=1}^{\infty} [-1/j, 1/j]$. Thus $(\mathbf{R} \setminus \{0\}) \subseteq (\mathbf{R} \setminus \bigcap_{j=1}^{\infty} [-1/j, 1/j])$, which implies $\{0\} \supseteq \bigcap_{j=1}^{\infty} [-1/j, 1/j]$.
Thus $\{0\} = \bigcap_{j=1}^{\infty} [-1/j, 1/j]$.

$\square$

Exercise 5 (Continuity of a Probability Law)**.** Let $\mathbf{P}$ be a probability law on a sample space Ω. Let $A_1, A_2, \ldots$ be sets in Ω which are increasing, so that $A_1 \subseteq A_2 \subseteq \cdots$. Then

$$\lim_{n \to \infty} \mathbf{P}(A_n) = \mathbf{P}(\cup_{n=1}^{\infty} A_n).$$

In particular, the limit on the left exists. Similarly, let $A_1, A_2, \ldots$ be sets in Ω which are decreasing, so that $A_1 \supseteq A_2 \supseteq \cdots$. Then

$$\lim_{n \to \infty} \mathbf{P}(A_n) = \mathbf{P}(\cap_{n=1}^{\infty} A_n).$$

Proof. (a) For $A_1 \subseteq A_2 \subseteq \cdots$, define $B_1 = A_1$, $B_n = A_n \setminus A_{n-1}$ for $n \geq 2$. Then $A_n = B_1 \uplus \cdots \uplus B_n$. (Here $\uplus$ stands for disjoint union.) Thus by additivity of the probability law

$$\mathbf{P}(A_n) = \mathbf{P}(B_1 \uplus \cdots \uplus B_n) = \mathbf{P}(B_1) + \cdots + \mathbf{P}(B_n). \tag{1}$$

And $\cup_{n=1}^{\infty} A_n = \uplus_{n=1}^{\infty} B_n$. By additivity of the probability law again

$$\mathbf{P}(\cup_{n=1}^{\infty} A_n) = \mathbf{P}(\uplus_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mathbf{P}(B_n). \tag{2}$$

(2) implies that the limit of (1) exists to be (2). That is $\lim_{n \to \infty} \mathbf{P}(A_n) = \mathbf{P}(\cup_{n=1}^{\infty} A_n)$.
(b) For $A_1 \supseteq A_2 \supseteq \cdots$, define $B_n = A_n^c$. Then $B_1 \subseteq B_2 \subseteq \cdots$, thus by Part (a), we have

$$\lim_{n \to \infty} \mathbf{P}(B_n) = \mathbf{P}(\cup_{n=1}^{\infty} B_n).$$

But $\mathbf{P}(B_n) = 1 - \mathbf{P}(B_n^c) = 1 - \mathbf{P}(A_n)$, and $\mathbf{P}(\cup_{n=1}^{\infty} B_n) = 1 - \mathbf{P}((\cup_{n=1}^{\infty} B_n)^c) = 1 - \mathbf{P}(\cap_{n=1}^{\infty} A_n)$. Thus $\lim_{n \to \infty}(1 - \mathbf{P}(A_n)) = 1 - \mathbf{P}(\cap_{n=1}^{\infty} A_n)$. This implies $\lim_{n \to \infty} \mathbf{P}(A_n) = \mathbf{P}(\cap_{n=1}^{\infty} A_n)$.

$\square$

Exercise 6. Retake at least one of the finals I gave when I taught math 170A:

http://www.math.ucla.edu/ heilman/teach/170afinal.pdf
http://www.math.ucla.edu/ heilman/teach/170afinalsoln.pdf
http://www.math.ucla.edu/ heilman/teach/170afinalv2.pdf
http://www.math.ucla.edu/ heilman/teach/170afinalv2soln.pdf