

Problem Set 4

*Lecturer: Steven Heilman**Kyle Barron***Exercise 1**

Compute the characteristic function of a uniformly distributed random variable on $[-1, 1]$. (Some of the following formulas might help to simplify your answer: $e^{it} = \cos(t) + i \sin(t)$, $\cos(t) = [e^{it} + e^{-it}]/2$, $\sin(t) = [e^{it} - e^{-it}]/[2i]$, $t \in \mathbb{R}$.) (Here $i := \sqrt{-1}$.)

Solution:

$$\begin{aligned}
 \phi_X(t) &= \mathbb{E}[e^{itx}] \\
 &= \frac{1}{2} \int_{-1}^1 e^{itx} dx \\
 &= \frac{1}{2it} e^{itx} \Big|_{-1}^1 \\
 &= \frac{1}{2it} (e^{it} - e^{-it}) \\
 &= \frac{\sin(t)}{t}.
 \end{aligned}$$

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Exercise 2

Let X be a random variable. Assume we can differentiate under the expected value of $\mathbb{E}e^{itX}$ any number of times. For any positive integer n , show that

$$\left. \frac{d^n}{dt^n} \phi_X(t) \right|_{t=0} = i^n \mathbb{E}(X^n).$$

So, in principle, all moments of X can be computed just by taking derivatives of the characteristic function.

Proof:

$$\begin{aligned}
 \frac{d^n}{dt^n} \phi_X(t) &= \frac{d^n}{dt^n} \int_{-\infty}^{\infty} e^{itx} f_X(x) dx \\
 &= \int_{-\infty}^{\infty} \frac{d^n}{dt^n} e^{itx} f_X(x) dx \\
 &= \int_{-\infty}^{\infty} (ix)^n e^{itx} f_X(x) dx.
 \end{aligned}$$

Now considering the special case where $t = 0$, we get that

$$\begin{aligned}\left.\frac{d^n}{dt^n}\phi_X(t)\right|_{t=0} &= \int_{-\infty}^{\infty} (ix)^n f_X(x) dx \\ &= i^n \int_{-\infty}^{\infty} x^n f_X(x) dx \\ &= i^n E[X^n]\end{aligned}$$

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Exercise 3

Let X be a random variable such that $E[|X|^3] < \infty$. Prove that for any $t \in \mathbb{R}$,

$$Ee^{itX} = 1 + itEX - t^2 EX^2/2 + o(t^2).$$

That is,

$$\lim_{t \rightarrow 0} t^{-2} |Ee^{itX} - [1 + itEX - t^2 EX^2/2]| = 0.$$

(Hint: it may be helpful to use Jensen's inequality to first justify that $E|X| < \infty$ and $EX^2 < \infty$. Then, use the Taylor expansion with error bound: $e^{iy} = 1 + iy - y^2/2 - (i/2) \int_0^y (y-s)^2 e^{is} ds$, which is valid for any $y \in \mathbb{R}$.)

Actually, this same bound holds only assuming $EX^2 < \infty$, but the proof of that bound requires things we have not discussed.

Solution: Let X be a random variable with $E[|X|^3] < \infty$. Recall that Jensen's inequality says that for a twice differentiable, convex function f , $E[f(X)] \geq f(E[X])$.

I'll first show that $E[|X|] < \infty$. Let $f(X) = |X|^3$; note that this is a convex function. Then $E[f(|X|)] = E[|X|^3]$ and $f(E[|X|]) = |E[|X|]|^3 = E[|X|]^3$. We can now use Jensen's inequality.

$$\infty > E[|X|^3] \geq |E[|X|]|^3 = E[|X|]^3.$$

Therefore, $E[|X|] < \infty$.

I'll do a similar procedure to show that $E[X^2] < \infty$. Let $g(X) = |X|^{\frac{3}{2}}$. This is also a convex function. Then $E[g(X^2)] = E[|X|^3]$ and $g(E[X^2]) = |E[X^2]|^{\frac{3}{2}} = E[X^2]^{\frac{3}{2}}$. We can now use Jensen's inequality.

$$\infty > E[|X|^3] \geq |E[X^2]|^{\frac{3}{2}} = E[X^2]^{\frac{3}{2}}.$$

Therefore, $E[X^2] < \infty$.

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Exercise 4

(Convolution is Associative.) Let $g, h, d: \mathbb{R} \rightarrow \mathbb{R}$. Then for any $t \in \mathbb{R}$,

$$((g * h) * d)(t) = (g * (h * d))(t)$$

Proof:

$$\begin{aligned}
 ((g * h) * d)(t) &= \int_{-\infty}^{\infty} (g * h)(x) d(t - x) dx \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (g(s) h(x - s) ds) d(t - x) dx \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(s) h(x - s) d(t - x) ds dx
 \end{aligned}$$

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Exercise 5

Let X, Y, Z be independent and uniformly distributed on $[0, 1]$. Note that f_X is not a continuous function.

Using convolution, compute f_{X+Y} . Draw f_{X+Y} . Note that f_{X+Y} is a continuous function, but it is not differentiable at some points.

Using convolution, compute f_{X+Y+Z} . Draw f_{X+Y+Z} . Note that f_{X+Y+Z} is a differentiable function, but it does not have a second derivative at some points.

Make a conjecture about how many derivatives $f_{X_1+\dots+X_n}$ has, where X_1, \dots, X_n are independent and uniformly distributed on $[0, 1]$. You do not have to prove this conjecture. The idea of this exercise is that convolution is a kind of average of functions. And the more averaging you do, the more derivatives $f_{X_1+\dots+X_n}$ has.

Solution: Let X, Y, Z be independent and uniformly distributed over $[0, 1]$. First I'll compute f_{X+Y} . Recall that Proposition 2.60 says that $f_{X+Y}(t) = (f_X * f_Y)(t)$, for all $t \in \mathbb{R}$. Recall that by Definition 2.59, $(g * h)(t) := \int_{-\infty}^{\infty} g(x) h(t - x) dx$, for all $t \in \mathbb{R}$. Therefore,

$$\begin{aligned}
 f_{X+Y}(t) &= (f_X * f_Y)(t) \\
 &= \int_{-\infty}^{\infty} f_X(x) f_Y(t - x) dx \\
 &= \int_0^1 1 \cdot f_Y(t - x) dx \\
 &= \int_{x \in [0, 1] \cap [t-1, t]} 1 dx
 \end{aligned}$$

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Exercise 6

Construct two random variables X, Y such that X and Y are each uniformly distributed on $[0, 1]$, and such that $\mathbf{P}(X + Y = 1) = 1$.

Then construct two random variables W, Z such that W and Z are each uniformly distributed on $[0, 1]$, and such that $W + Z$ is uniformly distributed on $[0, 2]$.

(Hint: there is a way to do each of the above problems with about one line of work. That is, there is a way to solve each problem without working very hard.)

Solution: For the first part, let X be uniformly distributed on $[0, 1]$ with $f_X(x) = 1$ for $x \in [0, 1]$. Let $Y = 1 - X$. Then Y is also distributed uniformly on $[0, 1]$ and $X + Y = X + 1 - X = 1$.

For the second part, let W be distributed as $f_W(x) = 1$ for $x \in [0, 1]$. Let $Z = W$. Then

$$\mathbf{P}(W + Z = t) = \mathbf{P}(2W = t) = \mathbf{P}\left(W = \frac{t}{2}\right).$$

So $f_{W+Z}(t) = \frac{1}{2}$ for $t \in (0, 2)$. ■