Math 170B **Winter 2017**

Problem Set 5

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Exercise 1

Let X be a standard Gaussian random variable. Let $t > 0$ and let n be a positive even integer. Show that

$$\mathbf{P}(X > t) \leq \frac{(n-1)(n-3)\cdots(3)(1)}{t^n}.$$

That is, the function $t \mapsto \mathbf{P}(X > t)$ decays faster than any monomial.

Proof: Let X be a standard Gaussian random variable and let $t > 0$. Let n be a positive even integer. Recall that from problem set 3, question 2, we found that for n even, $\mathrm{E}[X^n] = (n-1)(n-3)\cdots 1$. Now using the Markov Inequality we have

$$\mathbf{P}(X > t) \leq \mathbf{P}(|X| > t) = \mathbf{P}(X^n > t^n) \leq \frac{\mathrm{E}[X^n]}{t^n} = \frac{(n-1)(n-3)\cdots 1}{t^n},$$

where the second equality is since n is even, and the inequality is using the Markov Inequality, since X^n is nonnegative. ■

Exercise 2

Let X be a random variable. Let $t > 0$. Show that

$$\mathbf{P}(|X| > t) \leq \frac{\mathrm{E}X^4}{t^4}.$$

Proof: Let X be a random variable and let $t > 0$.

$$\begin{aligned}\mathbf{P}(|X| > t) &= \mathbf{P}(|X|^4 > t^4) \\ &= \mathbf{P}(X^4 > t^4) \\ &\leq \frac{\mathrm{E}[X^4]}{t^4}.\end{aligned}$$

where the first equality is true because x^4 is an increasing function when $x \geq 0$ (and here both $|X|$ and t are nonnegative), the second equality is true because $x^4 \geq 0$ always, and the third line uses the Markov Inequality. ■

Exercise 3

(The Chernoff Bound.) Let X be a random variable and let $r > 0$. Show that, for any $t > 0$,

$$\mathbf{P}(X > r) \leq e^{-tr} M_X(t).$$

Consequently, if $X_1, \dots, X_n$ are independent random variables with the same CDF, and if $r, t > 0$,

$$\mathbf{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i > r\right) \leq e^{-trn}(M_{X_1}(t))^n.$$

For example, if $X_1, \dots, X_n$ are independent Bernoulli random variables with parameter $0 < p < 1$, and if $r, t > 0$,

$$\mathbf{P}\left(\frac{X_1 + \cdots + X_n}{n} - p > r\right) \leq e^{-trn}(e^{-tp}[pe^t + (1-p)])^n.$$

And if we choose t appropriately, then the quantity $\mathbf{P}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i - p) > r\right)$ becomes exponentially small as either n or r become large. That is, $\frac{1}{n}\sum_{i=1}^{n} X_i$ becomes very close to its mean. Importantly, the Chernoff bound is much stronger than either Markov's or Cheyshev's inequality, since they only respectively imply that

$$\mathbf{P}\left(\left|\frac{X_1 + \cdots + X_n}{n} - p\right| > r\right) \leq \frac{2p(1-p)}{nr}, \quad \mathbf{P}\left(\left|\frac{X_1 + \cdots + X_n}{n} - p\right| > r\right) \leq \frac{p(1-p)}{nr^2}.$$

Proof: Let X be a random variable and let $r > 0$. Then we have that

$$\mathbf{P}(X \geq r) = \mathbf{P}(e^{tX} \geq e^{tr}) \leq \frac{\mathrm{E}[e^{tX}]}{e^{tr}} = e^{-tr}M_X(t),$$

where the first equality uses that $t > 0$ so the exponential function is increasing, and the inequality uses the Markov inequality (which is valid since $e^x \geq 0$ for all x).

Now suppose that $X_1, ...X_n$ are independent and identically distributed. Then

$$\begin{aligned}
\mathbf{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i > r\right) &= \mathbf{P}\left(\sum_{i=1}^{n} X_i > rn\right) \\
&\leq e^{-trn} M_{\sum_{i=1}^{n} X_i}(t) \\
&= e^{-trn}\left(M_{X_1}(t)\right)^n
\end{aligned}$$

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Exercise 4

Let $X_1, X_2, \dots$ be independent random variables, each with exponential distribution with parameter $\lambda = 1$. For any $n \geq 1$, let $Y_n \coloneqq \max(X_1, \dots, X_n)$. Let $0 < a < 1 < b$. Show that $\mathbf{P}(Y_n \leq a \log n) \to 0$ as $n \to \infty$, and $\mathbf{P}(Y_n \leq b \log n) \to 1$ as $n \to \infty$. Conclude that $Y_n / \log n$ converges to 1 in probability as $n \to \infty$.

Proof: Let $X_1, X_2, ...$ be independent random variables, each exponentially distributed with $\lambda = 1$. For any $n \geq 1$, define $Y_n \coloneqq \max(X_1, ..., X_n)$. First let $c > 0$. Then

$$\begin{aligned}
\mathbf{P}(Y_n \leq c \log n) &= \mathbf{P}(\max(X_1, ..., X_n) \leq c \log n) \\
&= \mathbf{P}(X_1 \leq c \log n \cap X_2 \leq c \log n \cap \cdots \cap X_n \leq c \log n) \\
&= \mathbf{P}(X_1 \leq c \log n) \cdots \mathbf{P}(X_n \leq c \log n) \\
&= \mathbf{P}(X_1 \leq c \log n)^n \\
&= (1 - e^{-c \log n})^n \\
&= (1 - ne^{-c})^n
\end{aligned}$$

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Exercise 5

We say that random variables $X_1, X_2, \ldots$ converge to a random variable X in L_2 if

$$\lim_{n\to\infty} \mathrm{E}|X_n - X|^2 = 0.$$

Show that, if $X_1, X_2, \ldots$ converge to X in L_2, then $X_1, X_2, \ldots$ converges to X in probability.

Is the converse true? Prove your assertion.

Proof: Let $X, X_1, X_2, ...$ be random variables such that $X_1, X_2, ...$ converge to X in L_2. That is,

$$\lim_{n\to\infty} \mathrm{E}[|X_n - X|^2] = 0.$$

We want to show that $X_1, X_2, ...$ converges to X in probability, that is, if for all $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbf{P}(|X_n - X| > \epsilon) = 0.$$

First recall that the Markov Inequality says that

$$\mathbf{P}(|Z| \geq t) \leq \frac{\mathrm{E}[|Z|^2]}{t^2}, \forall t > 0.$$

Let $\epsilon > 0$, let $t = \epsilon$, and let $Z = X_n - X$. So then we have

$$\mathbf{P}(|X_n - X| > \epsilon) \leq \frac{\mathrm{E}[(X_n - X)^2]}{\epsilon^2}.$$

Since we know that $X_1, X_2, ...$ converge to X in L_2,

$$\lim_{n\to\infty} \mathrm{E}[(X_n - X)^2] = 0 \Rightarrow \lim_{n\to\infty} \mathbf{P}(|X_n - X| > \epsilon) = 0.$$

So $X_1, X_2, ...$ converge in probability.

The converse is not true: convergence in probability does not imply convergence in L_2Recall that

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Exercise 6

Let $X_1, X_2, \ldots$ be independent, identically distributed random variables such that $\mathrm{E}|X| < \infty$ and $\mathrm{var}(X) < \infty$. For any $n \geq 1$, define

$$Y_n := \frac{1}{n}\sum_{i=1}^{n} X_i^2.$$

Show that $Y_1, Y_2, \ldots$ converges in probability. Express the limit in terms of $\mathrm{E}X$ and $\mathrm{var}(X)$.

Proof: Let $X_1, X_2, ...$ be independent and identically distributed with $\mathrm{E}[|X|] < \infty$ and $\mathrm{Var}(X) < \infty$. Define

$$Y_n := \frac{1}{n}\sum_{i=1}^{n} X_i^2.$$

Let $\epsilon > 0$. To show that $Y_1, Y_2, ...$ converges in probability we need to show that $\lim_{n\to\infty} \mathbf{P}(|Y_n - Y| > \epsilon) = 0$. I'll show that Y, the limit of Y_n as $n \to \infty$, is equal to the second moment of X_1 (which is the same for all X since they're independent and identically distributed). I use Markov's inequality with $n = 2$ and then simplify:

$$\begin{aligned}
\mathbf{P}(|Y_n - Y| > \epsilon) &= P\left(\left|\frac{x_1^2 + ... + x_n^2}{n} - \mathrm{E}[X_1^2]\right| > \epsilon\right) \\
&\leq \frac{1}{\epsilon^2}\mathrm{E}\left[\left|\frac{x_1^2 + ... + x_n^2}{n} - \mathrm{E}[X_1^2]\right|^2\right] \\
&= \frac{1}{\epsilon^2}\mathrm{E}\left[\left(\frac{x_1^2 + ... + x_n^2}{n} - \frac{n\mathrm{E}[X_1^2]}{n}\right)^2\right] \\
&= \frac{1}{\epsilon^2}\mathrm{E}\left[\left(\frac{x_1^2 + ... + x_n^2}{n} - \frac{\mathrm{E}[X_1^2] + ... + \mathrm{E}[X_n^2]}{n}\right)^2\right] \\
&= \frac{1}{\epsilon^2}\mathrm{E}\left[\left(\frac{x_1^2 + ... + x_n^2}{n} - \frac{\mathrm{E}[X_1^2 + ... + X_n^2]}{n}\right)^2\right] \\
&= \frac{1}{\epsilon^2 n^2}\mathrm{E}\left[\left(x_1^2 + ... + x_n^2 - \mathrm{E}[X_1^2 + ... + X_n^2]\right)^2\right] \\
&= \frac{1}{\epsilon^2 n^2}\mathrm{Var}(X_1^2 + ...x_n^2) \\
&= \frac{1}{\epsilon^2 n}\mathrm{Var}(X_1^2).
\end{aligned}$$

Now letting $n \to \infty$, we have that $\frac{\mathrm{Var}(X_1^2)}{\epsilon^2 n} \to 0$. Therefore, $Y_1, Y_2, ...$ converges in probability to $\mathrm{E}[X_1^2] = \mathrm{Var}(X_1) + \mathrm{E}[X_1]^2$. (Note that this is well defined since both the variance and mean of X_1 are finite.) ■