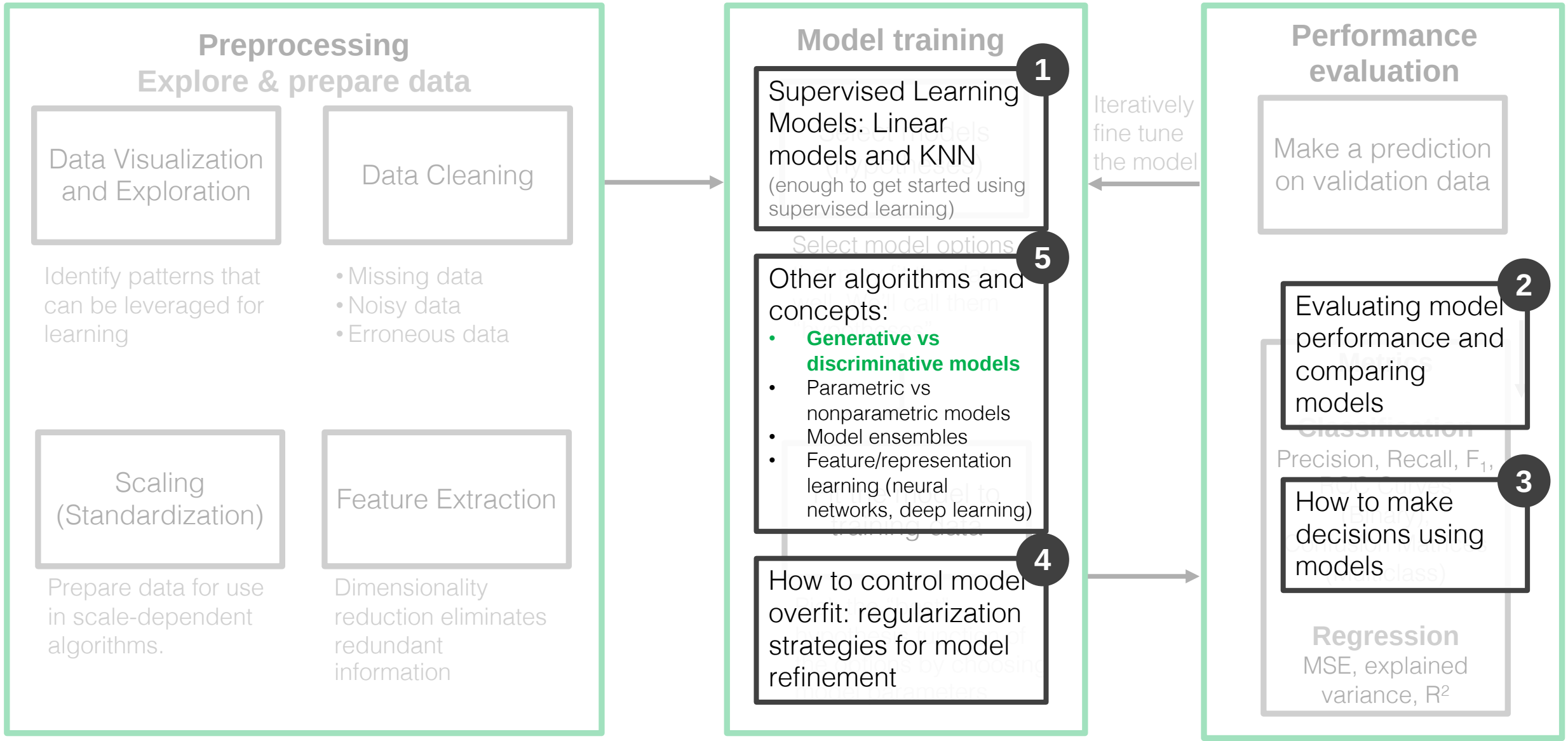
Generative Models for Classification

Supervised learning in practice



Classifiers

Covered so far

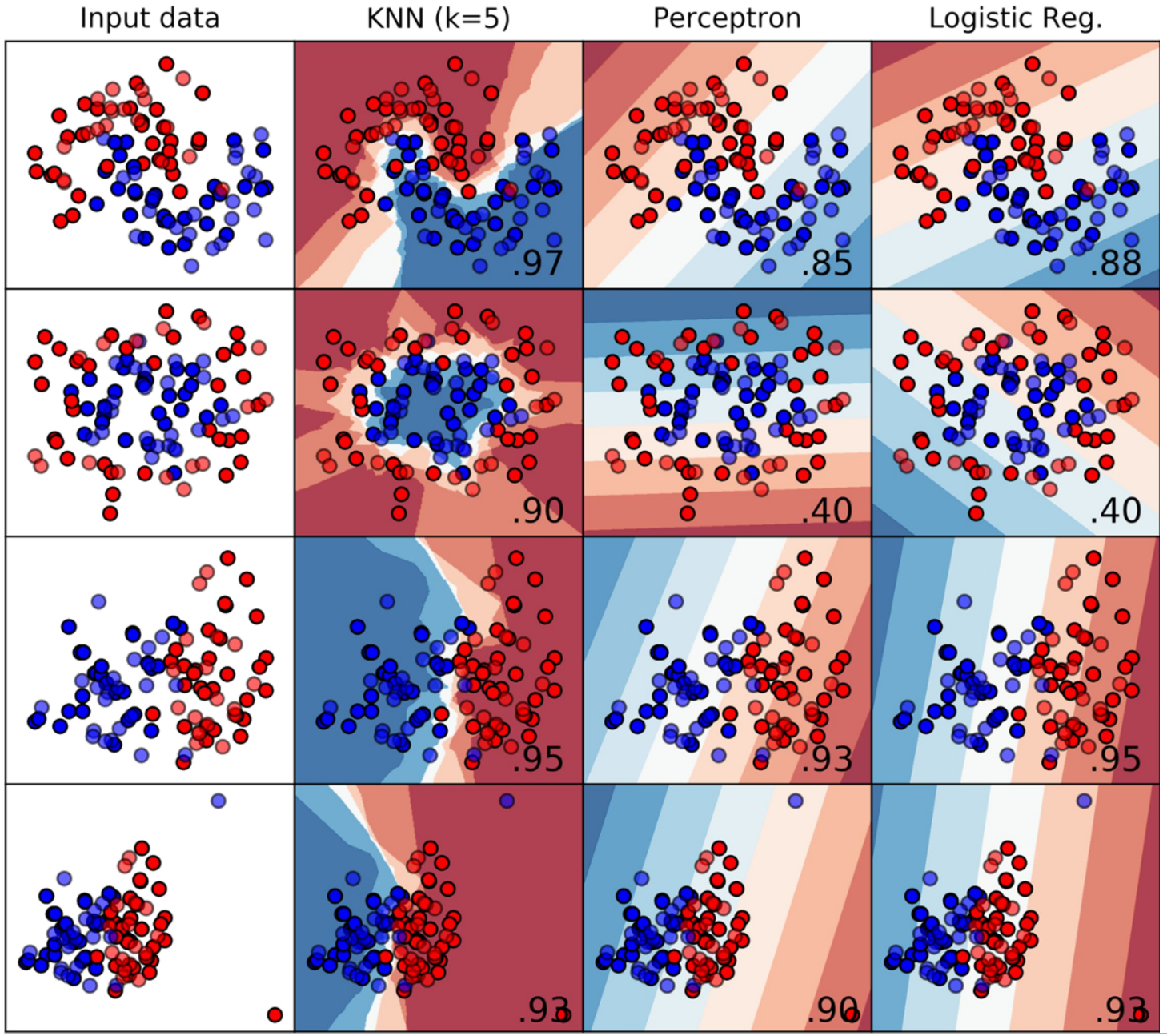
- K-Nearest Neighbors
- Perceptron
- Logistic Regression
- Linear Discriminant Analysis
- Quadratic Discriminant Analysis
- Naïve Bayes

Along the way…

Revisiting Bayes' Rule

Projections from higher dimensions

Multivariate normal distributions



Comparison of classifiers we have seen so far

Examples of deep generative models…

Face Synthesis

Image Synthesis (link)

Karras et al. 2018, NVIDIA: Progressive growing of GANS for improved quality, stability, and variation

These images are all synthetic

Synthetic Generation

Karras, T., Laine, S. and Aila, T., 2019. A style-based generator architecture for generative adversarial networks. In Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition (pp. 4401-4410).

Synthetic Images

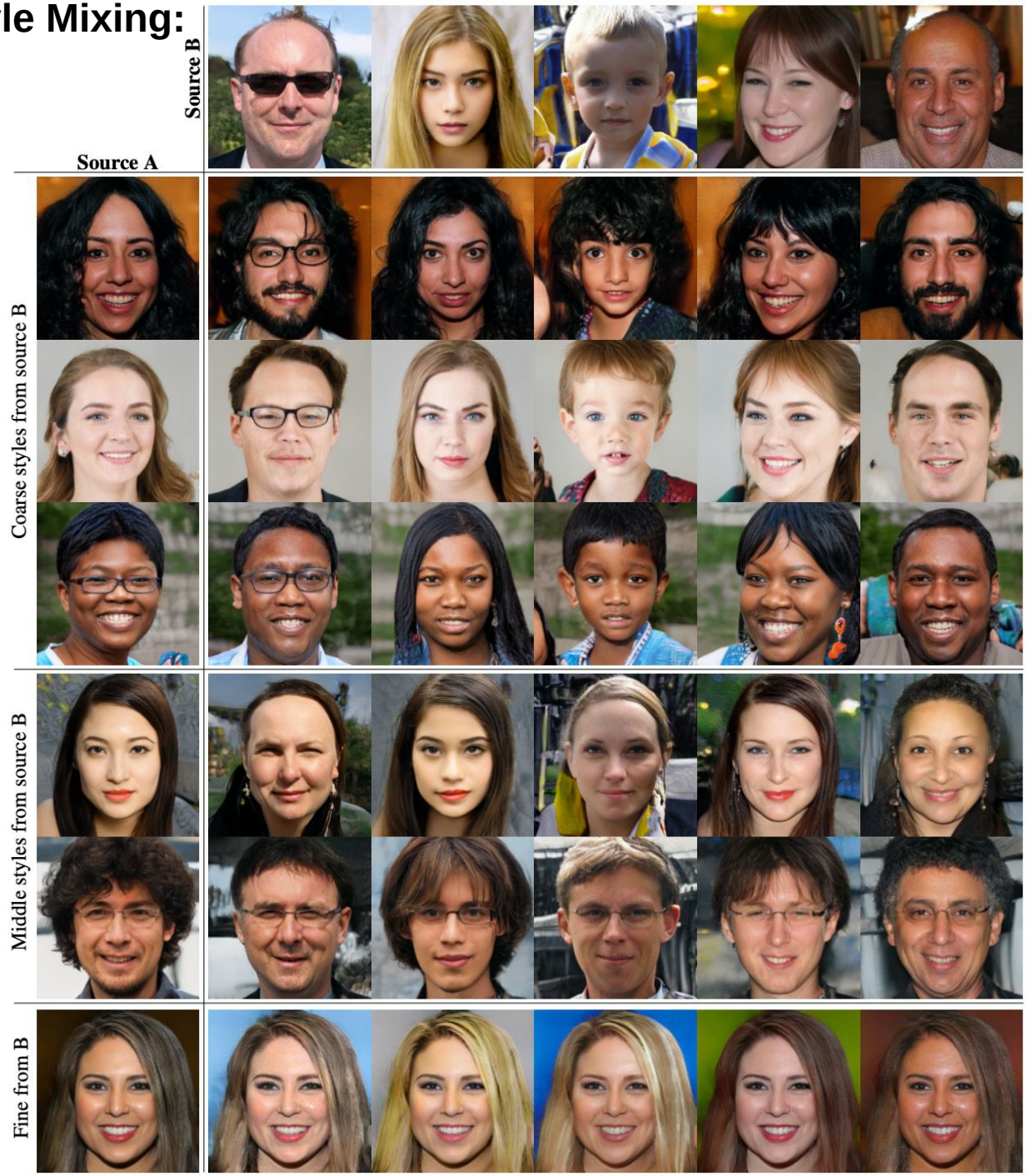
From source B:

Pose, general hair style, face shape, eyeglasses

Hair style, eyes open/closed

Color scheme and microstructure

Style Mixing:



Bayes' rule in the context of classification

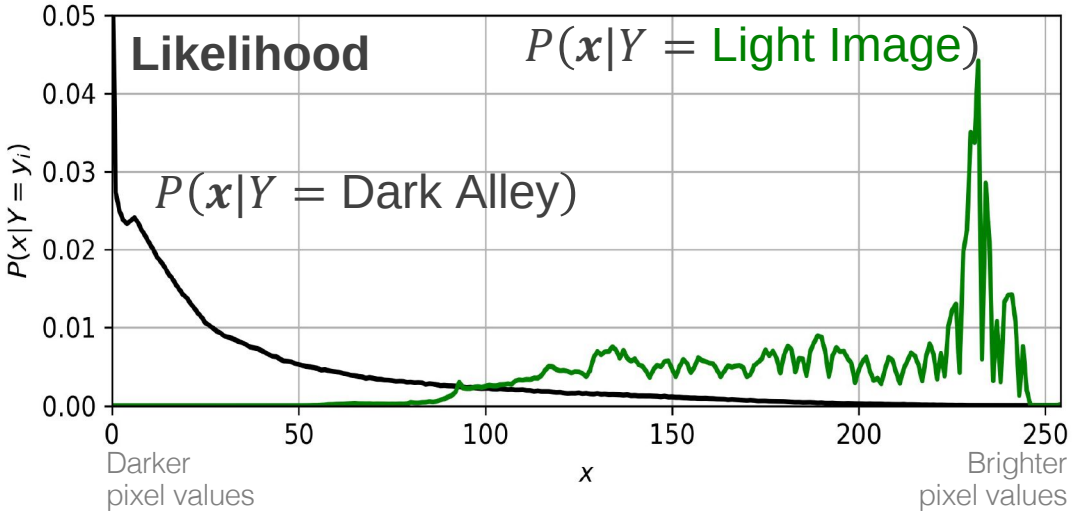
Class 1: Light Image

Class 0: Dark Image

Randomly draw a pixel from either of the images: $x_i = 149$

Darker pixel values are lower numbers (closer to 0), brighter pixels are higher numbers (closer to 255)

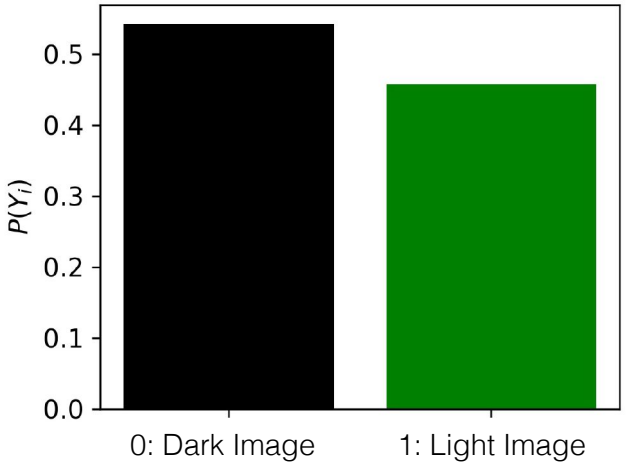
How do we determine which image the sample was most likely to have come from?



Class 1: Light Image y_1

Class 0: Dark Image y_0

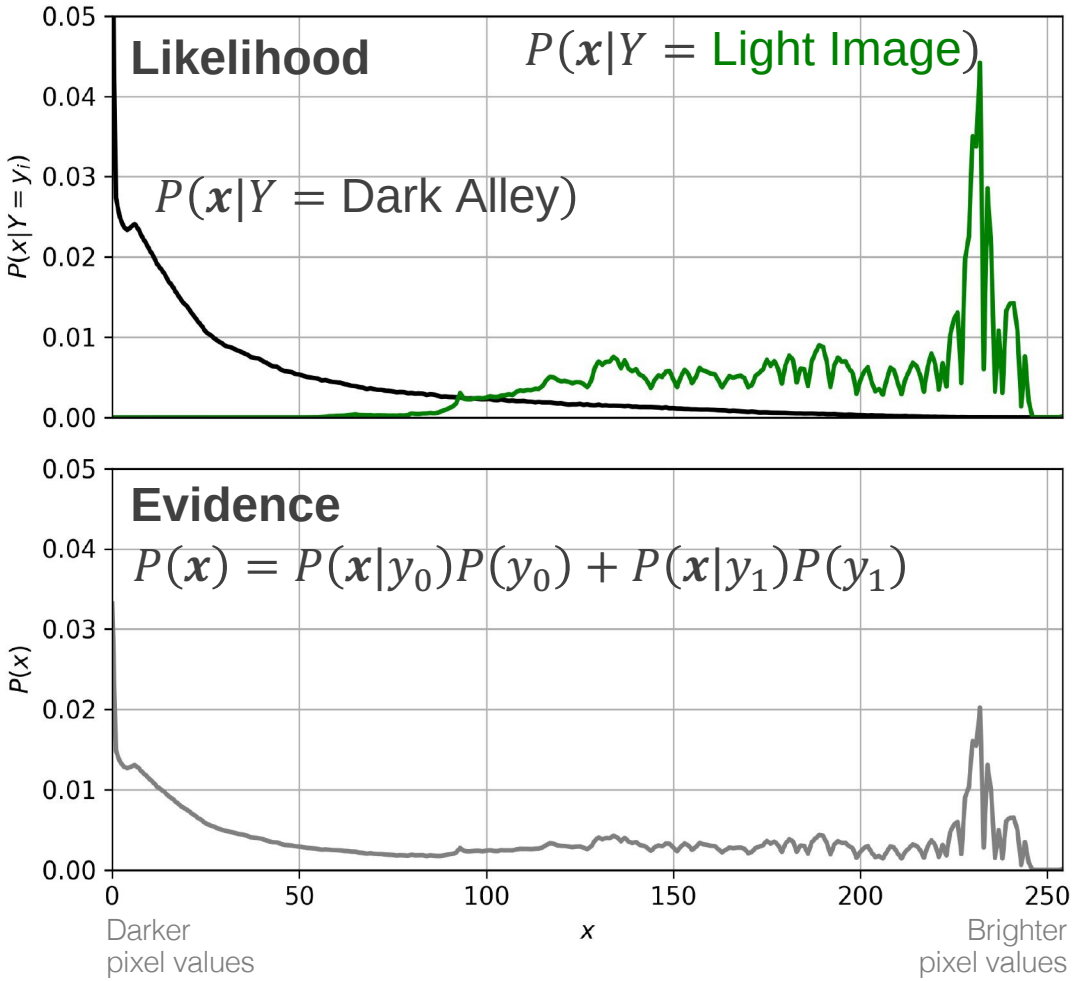
Prior: $P(Y = y_i)$



Bayes' Rule

$$P(Y = y_i|\boldsymbol{x}) = \frac{P(\boldsymbol{x}|Y = y_i)P(Y = y_i)}{P(\boldsymbol{x})}$$

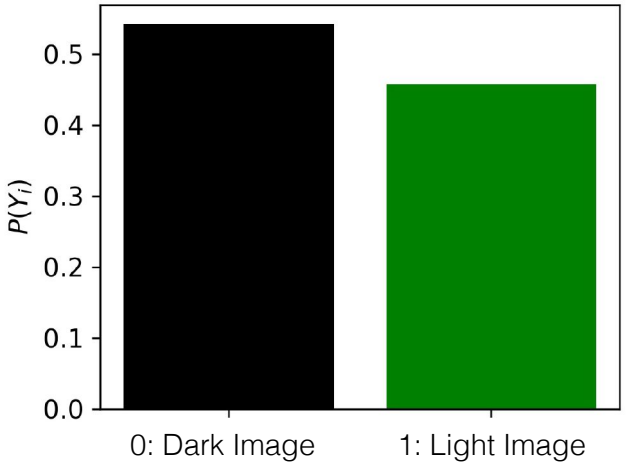
Posterior: $P(Y = y_i|\boldsymbol{x})$ — Likelihood: $P(\boldsymbol{x}|Y = y_i)$ — Prior: $P(Y = y_i)$ — Evidence: $P(\boldsymbol{x})$



Class 1: Light Image y_1

Class 0: Dark Image y_0

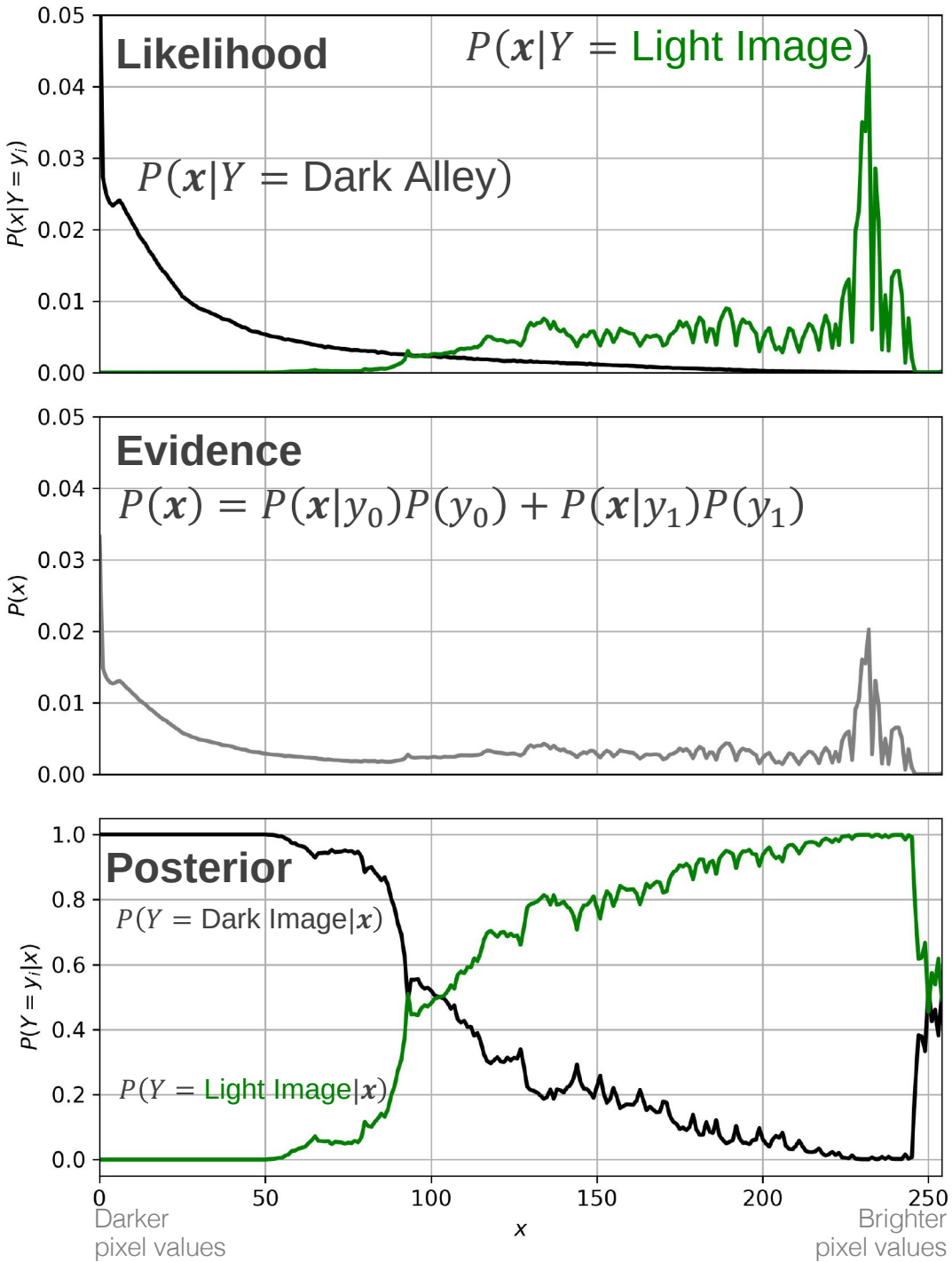
Prior: $P(Y = y_i)$



Bayes' Rule

$$P(Y = y_i|\boldsymbol{x}) = \frac{P(\boldsymbol{x}|Y = y_i)P(Y = y_i)}{P(\boldsymbol{x})}$$

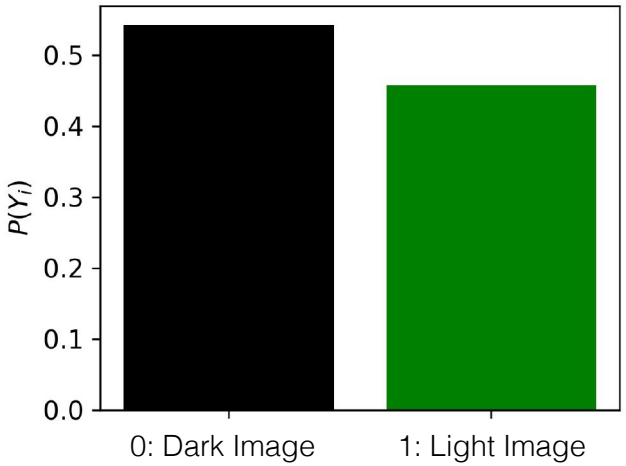
Posterior: $P(Y = y_i|\boldsymbol{x})$ — Likelihood: $P(\boldsymbol{x}|Y = y_i)$ — Prior: $P(Y = y_i)$ — Evidence: $P(\boldsymbol{x})$



Class 1: Light Image y_1

Class 0: Dark Image y_0

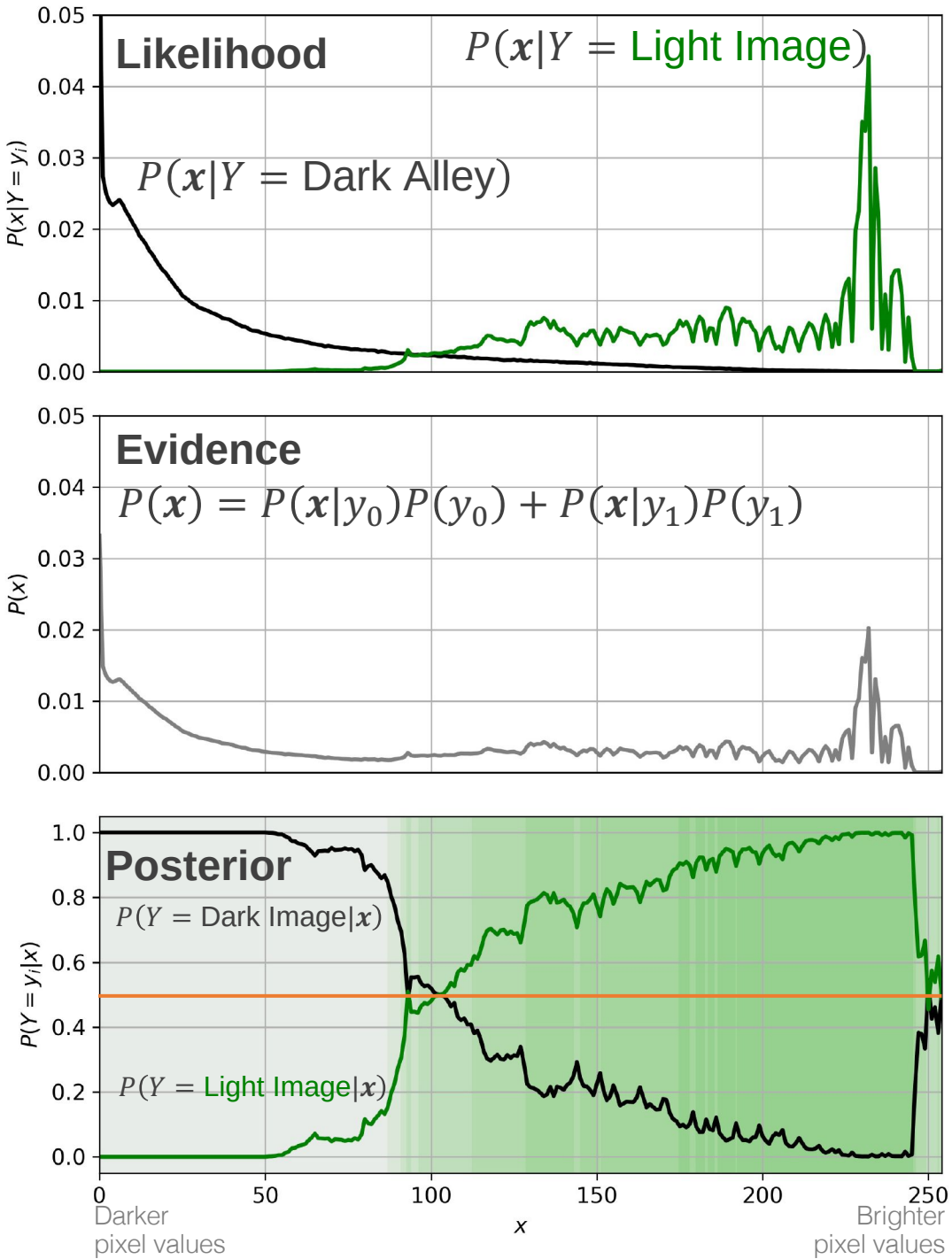
Prior: $P(Y = y_i)$



Bayes' Rule

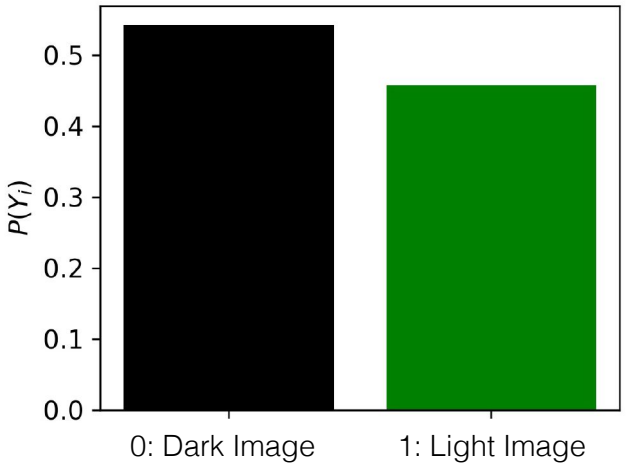
$$P(Y = y_i|\boldsymbol{x}) = \frac{P(\boldsymbol{x}|Y = y_i)P(Y = y_i)}{P(\boldsymbol{x})}$$

Likelihood · Prior · Posterior · Evidence



Class 1: Light Image y_1 **Class 0: Dark Image** y_0

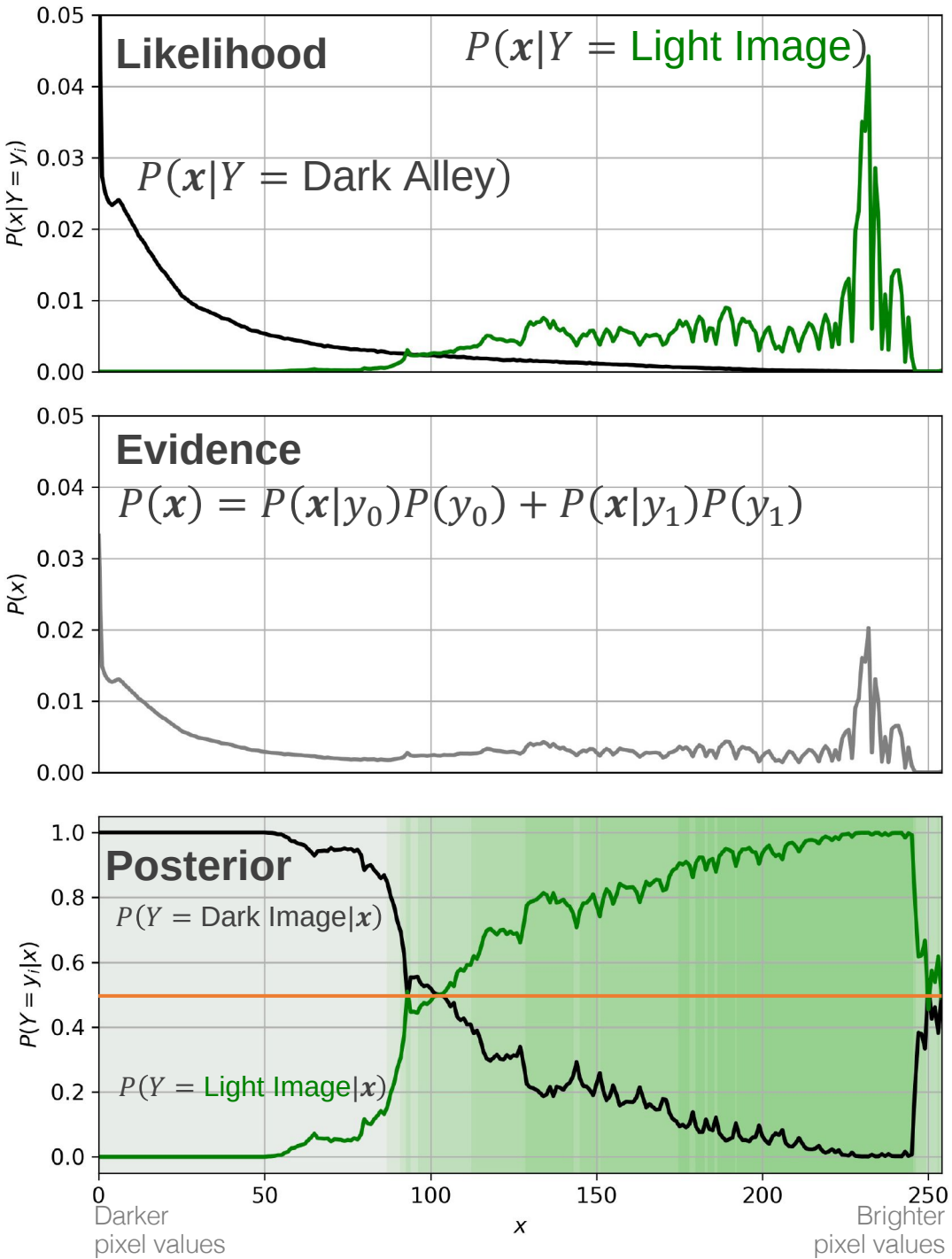
Prior: $P(Y = y_i)$



Decision rule:

If $P(Y = \text{Light Image}|\boldsymbol{x}) > P(Y = \text{Dark Image}|\boldsymbol{x})$ then Light Image

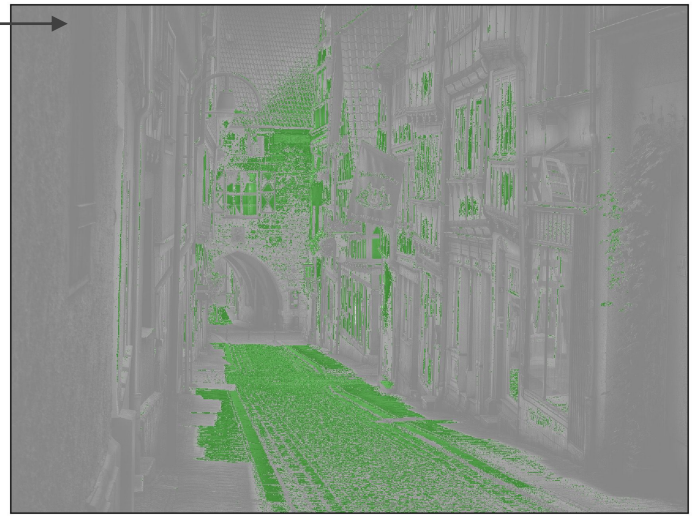
else Dark Image



Class 1: Light Image y_1

Class 0: Dark Image y_0

Green = classified as from Light Image
Grey = classified as from Dark Image

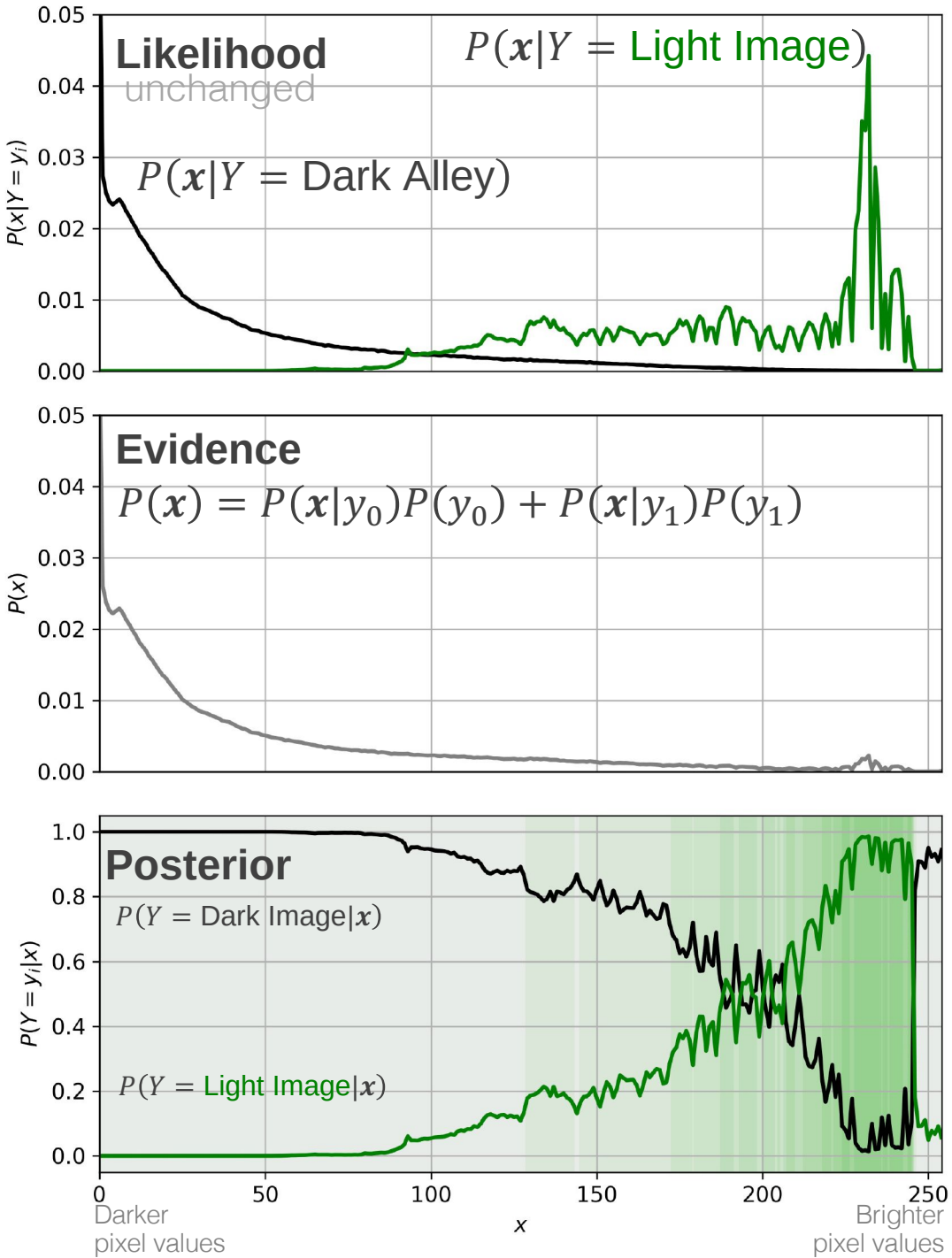


Classifying each of the individual pixels as being either from **Light Image** or **Dark Image** results in classification above

Decision rule:

If $P(Y = \text{Light Image}|\boldsymbol{x}) > P(Y = \text{Dark Image}|\boldsymbol{x})$ then Light Image

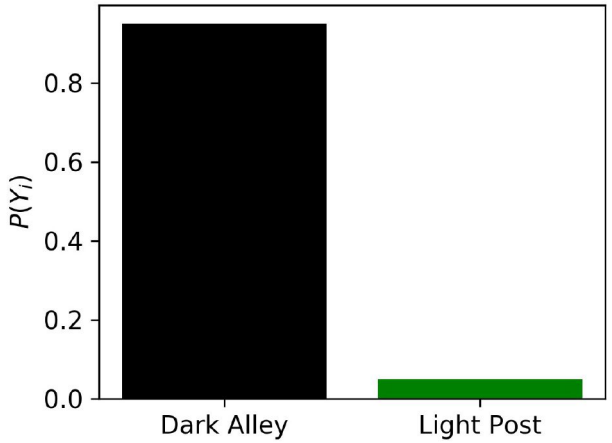
else Dark Image



Class 1: Light Image y_1

Class 0: Dark Image y_0

Prior: $P(Y = y_i)$

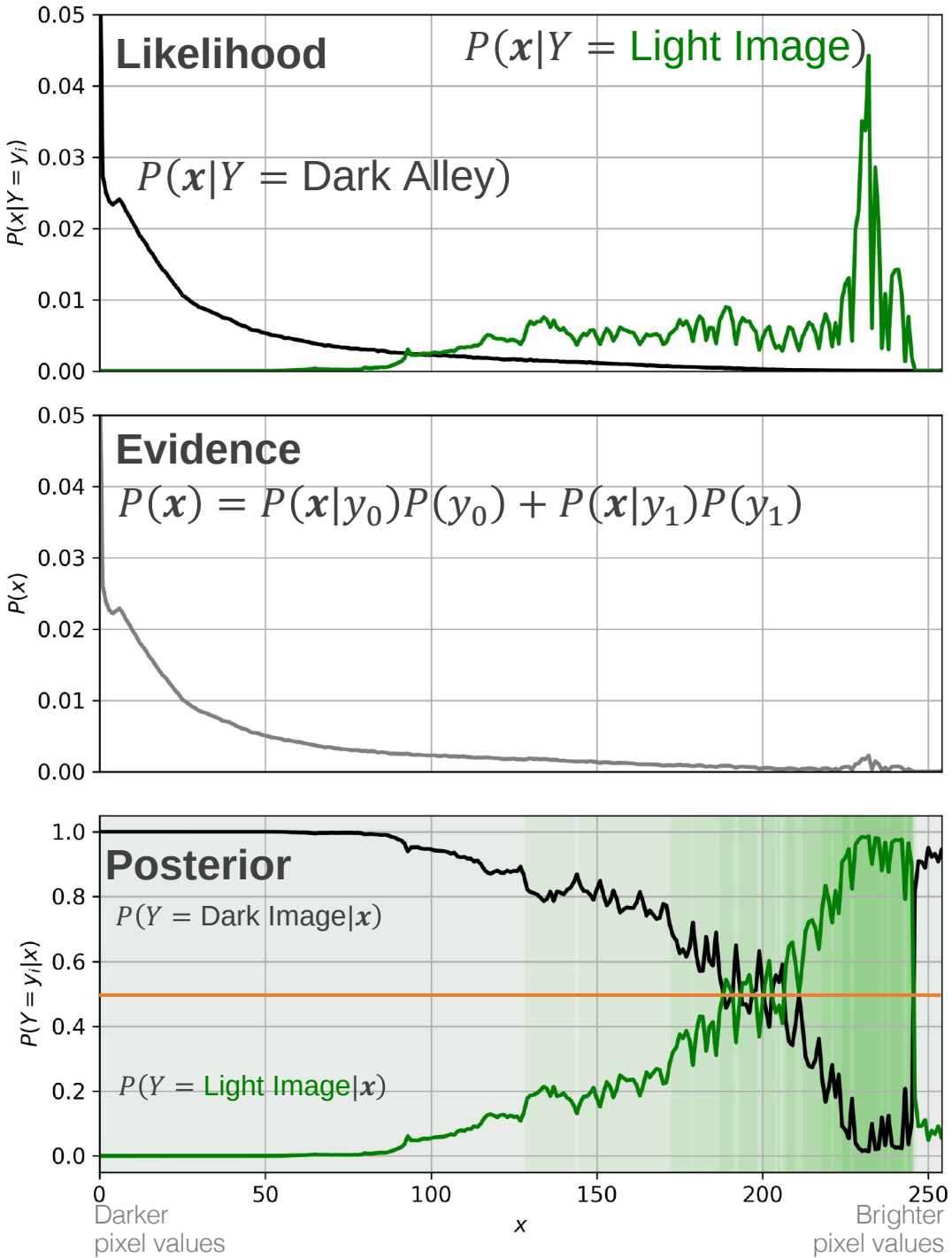


Let's assume the sampling of pixels occurred more from the **Dark Image**

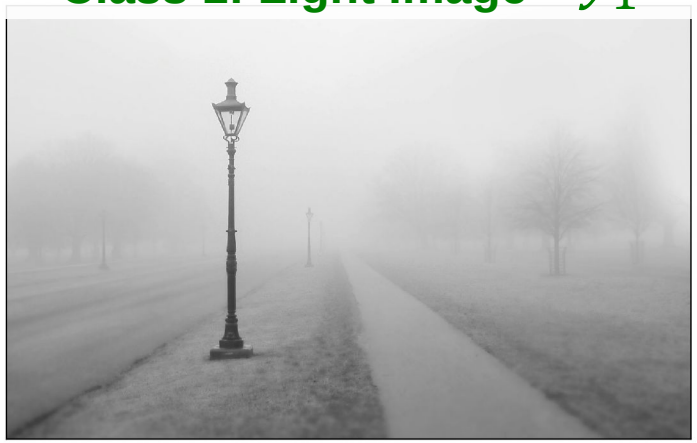
Bayes' Rule

$$P(Y = y_i|\boldsymbol{x}) = \frac{P(\boldsymbol{x}|Y = y_i)P(Y = y_i)}{P(\boldsymbol{x})}$$

Posterior = Likelihood × Prior / Evidence



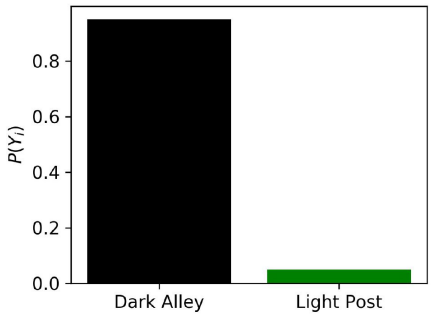
Class 1: Light Image y_1



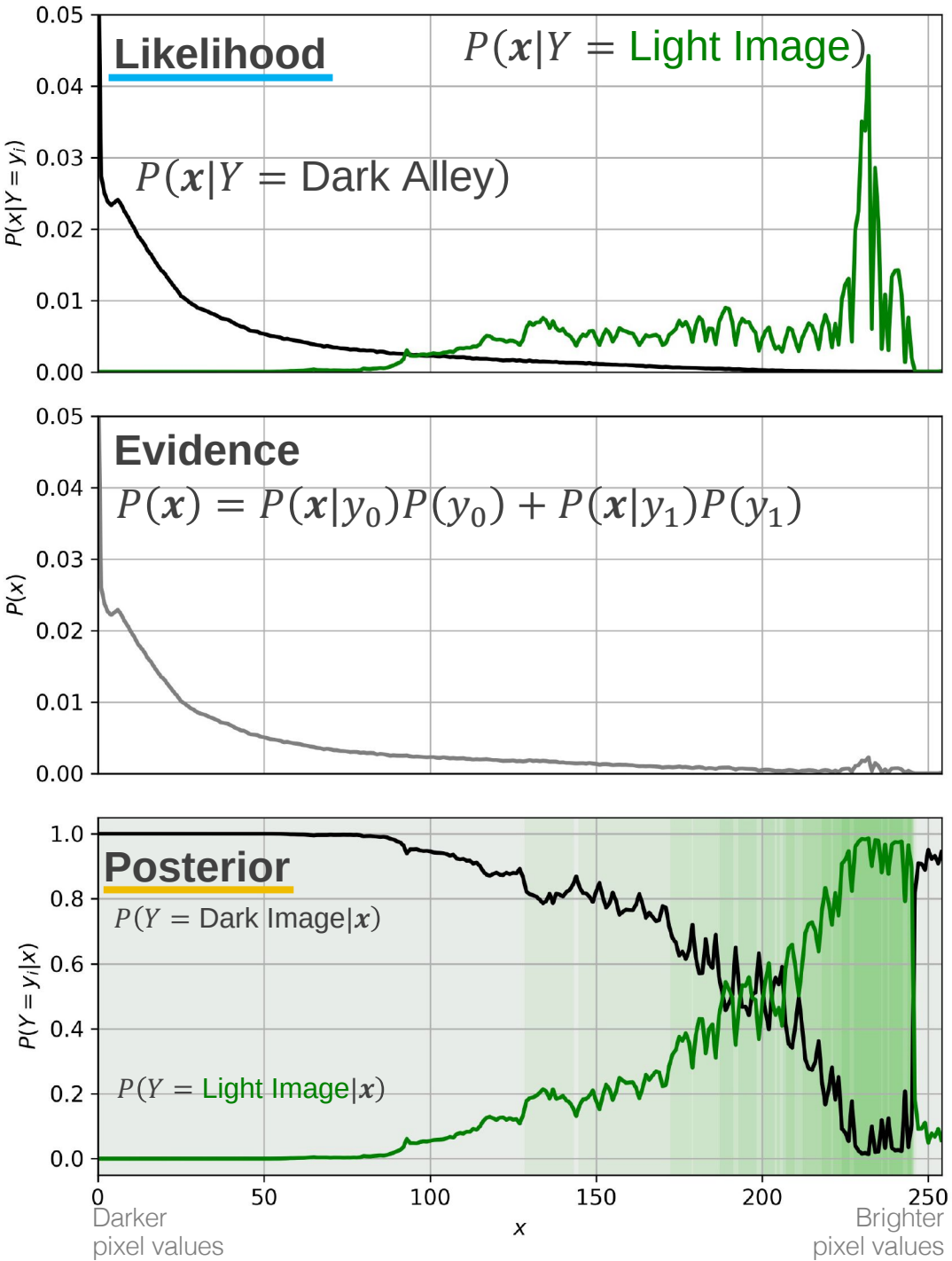
Class 0: Dark Image y_0

Green = classified as from Light Image
Grey = classified as from Dark Image

Prior: $P(Y = y_i)$



Assuming we the sampling of pixels occurred more from the **Dark Image**



Generative models model the **likelihood**

- These can also be used to generate synthetic data
- Often good performance when sample size is small

Examples: linear discriminant analysis, naïve Bayes, hidden Markov models, Guassian mixture models, Generative Adversarial Networks

Likelihood Prior

Posterior

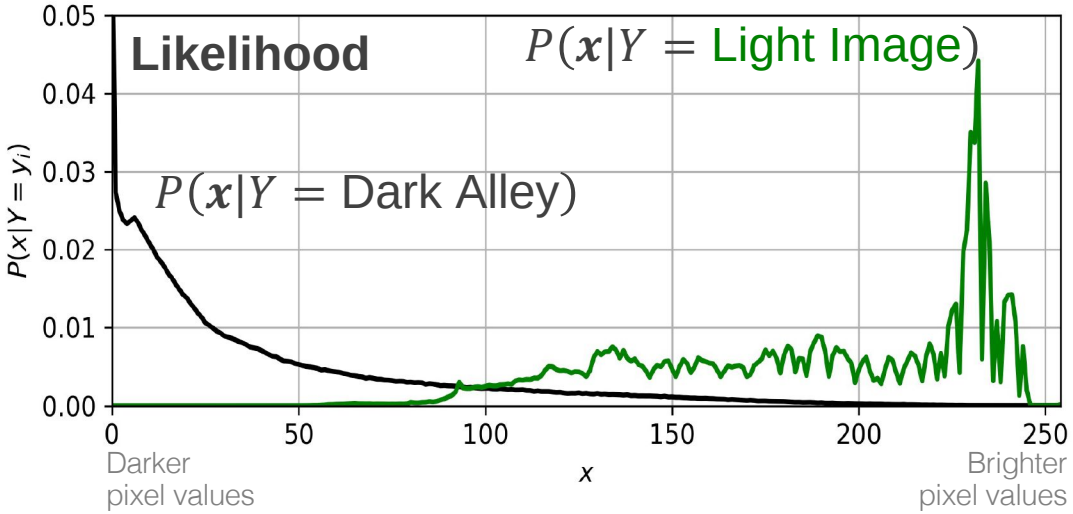
$$P(Y = y_i|\boldsymbol{x}) = \frac{P(\boldsymbol{x}|Y = y_i)P(Y = y_i)}{P(\boldsymbol{x})}$$

Evidence

Discriminative models model the **posterior**

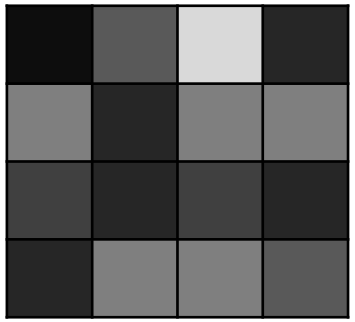
- Or they just directly estimate labels without a probabilistic interpretation, $f(\boldsymbol{x}) \rightarrow \boldsymbol{y}$
- Often better performance for large sample sizes

Examples: logistic regression, support vector machines, neural networks, k nearest neighbors

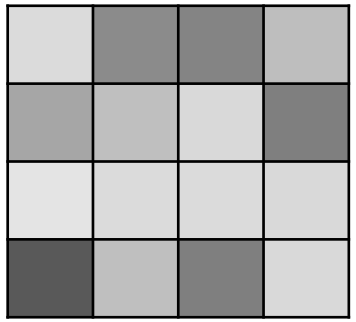


Class 1: Light Image y_1

Class 0: Dark Image y_0



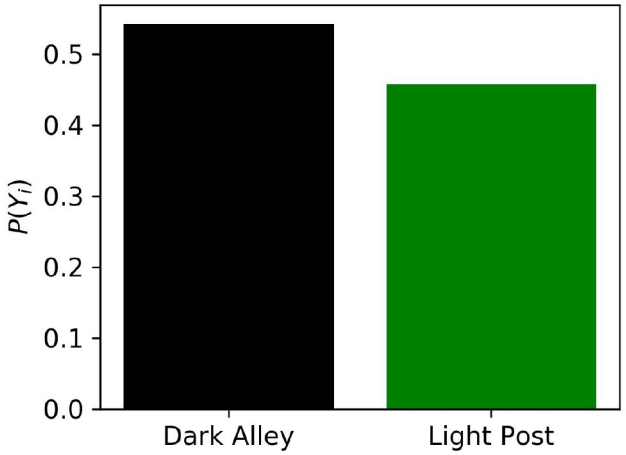
Synthetic "Image" sampled from **Class 0**



Synthetic "Image" sampled from **Class 1**

How to "Generate" Data?

Prior: $P(Y = y_i)$



Bayes' Rule

$$P(Y = y_i|\boldsymbol{x}) = \frac{\overbrace{P(\boldsymbol{x}|Y = y_i)}^{\text{Likelihood}}\overbrace{P(Y = y_i)}^{\text{Prior}}}{\underbrace{P(\boldsymbol{x})}_{\text{Evidence}}} \quad \text{Posterior}$$

Generative modeling for classification

Assume we have c different classes

For a new sample, classify it as the class with the **largest posterior** $P(y = i|\boldsymbol{x})$

$i = 1, \ldots, c$

Generative modeling for classification

If we have c different classes, we define a discriminant function, $d_i(\boldsymbol{x})$ for $i = 1, \ldots, c$

If $P(y = i|\boldsymbol{x}) > P(y = j|\boldsymbol{x})$ for $i \neq j$, then we classify feature $\boldsymbol{x}$ to class i

$$P(y = i|\boldsymbol{x}) = \frac{P(\boldsymbol{x}|y = i)P(y = i)}{P(\boldsymbol{x})}$$

$$= \frac{P(\boldsymbol{x}|y = i)P(y = i)}{\sum_{i=1}^{c} P(\boldsymbol{x}|y = i)P(y = i)}$$

Denominator is the same for all classes i, so it won't help us tell which class's posterior is higher relative to other classes, so we ignore it going forward

Bayes' Rule: $P(Y|\boldsymbol{X}) = \frac{P(\boldsymbol{X}|Y)P(Y)}{P(\boldsymbol{X})}$

Likelihood, Prior, Posterior, Evidence

We can define the discriminant function as:

$$d_i(x) = P(\boldsymbol{x}|y = i)P(y = i)$$

If we know the **true likelihood and prior** for our data, this process yields our **Bayes' classifier** (minimum misclassification error classifier)

Generative modeling for classification

$$d_i(x) = P(\boldsymbol{x}|y = i)P(y = i)$$

We **rarely** know our **true likelihood** for our data so we need to assume a form for the distributions

1. Assume a form for $P(\boldsymbol{x}|y = i)$

   Gaussian → Linear and Quadratic Discriminant Analysis
   Gaussian mixture models
   Nonparametric density estimates

   If we assume **independent features** → **Naïve Bayes**

(remember $d_i(x)$ proportional to the posterior)

2. Assign the class, i, for which $d_i(x)$ is largest

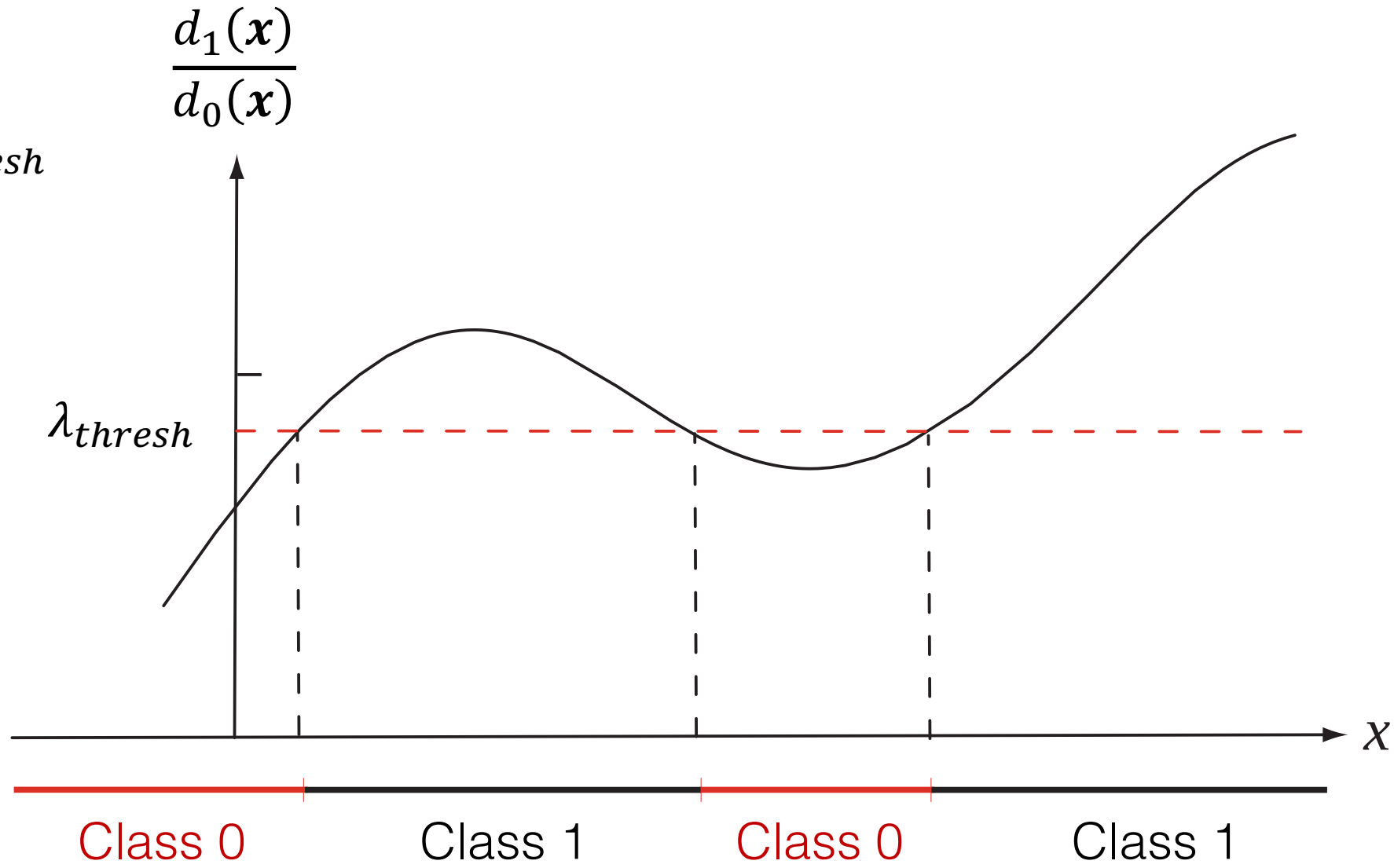
   Applies to both binary and multiclass problems

Discriminant Function: 2 classes

Decision rule:

Class 1 if: $\frac{d_1(\boldsymbol{x})}{d_0(\boldsymbol{x})} > \lambda_{thresh}$

Otherwise, class 0



Duda, Hart, and Stork, Pattern Classification, 2006

Discriminant Function: 2 classes

We build a classifier that assigns the class with the higher posterior probability:

If $\frac{d_1(\boldsymbol{x})}{d_0(\boldsymbol{x})} = \frac{P(\boldsymbol{x}|y=1)P(y=1)}{P(\boldsymbol{x}|y=0)P(y=0)} > 1$ Assign class 1, else class 0

Assumes these likelihoods are normal

$$\frac{P(\boldsymbol{x}|y=1)}{P(\boldsymbol{x}|y=0)} > \frac{P(y=0)}{P(y=1)}$$

$N(\mu_1, \Sigma_1)$

$N(\mu_0, \Sigma_0)$

Estimate the class-conditional mean and covariance matrix from the data

Discriminant Function: 2 classes

We build a classifier that assigns the class with the higher posterior probability:

Likelihood ratio:

$$\frac{P(\boldsymbol{x}|y=1)}{P(\boldsymbol{x}|y=0)} > \frac{P(y=0)}{P(y=1)}$$

$P(\boldsymbol{x}|y=1) \sim N(\mu_1, \Sigma_1)$

$P(\boldsymbol{x}|y=0) \sim N(\mu_0, \Sigma_0)$

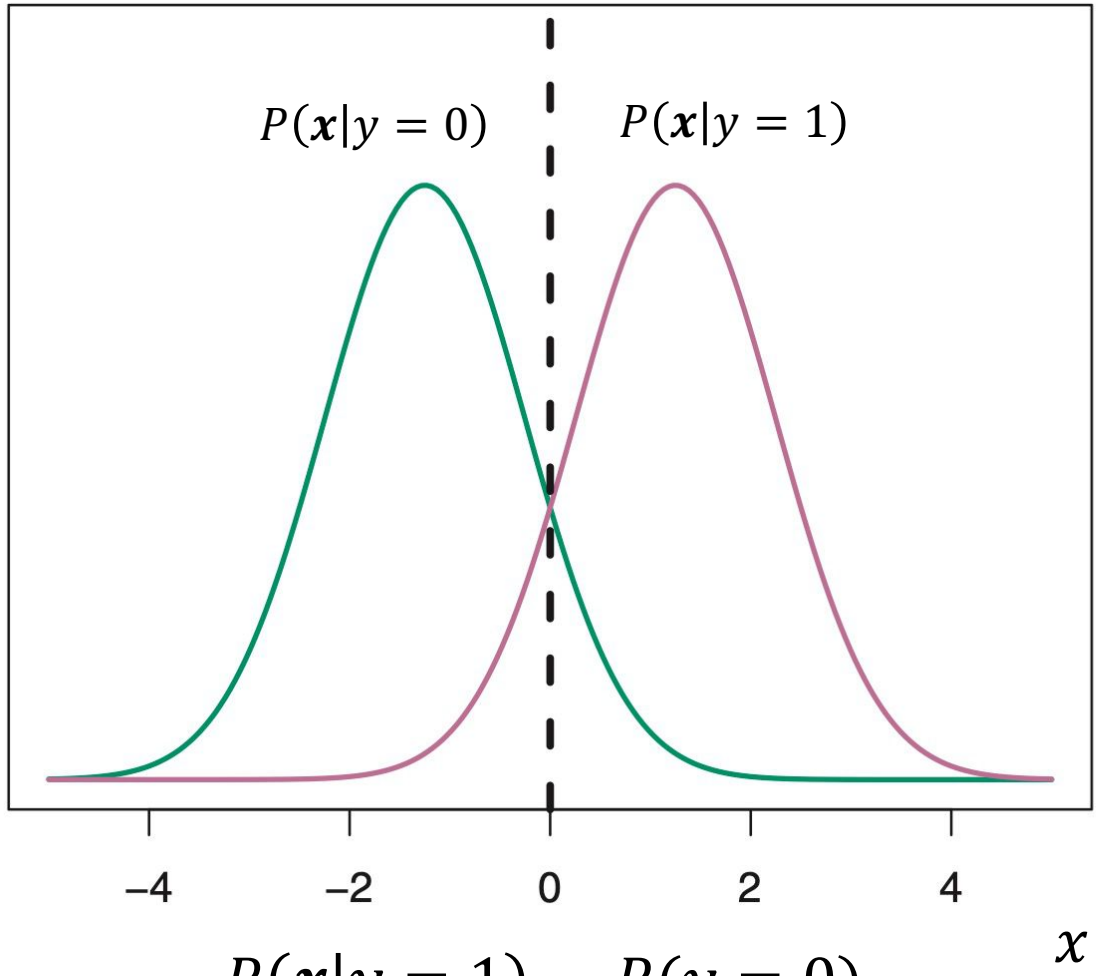
If we assume the class conditional distributions are **Gaussian**, this represents

Quadratic Discriminant Analysis

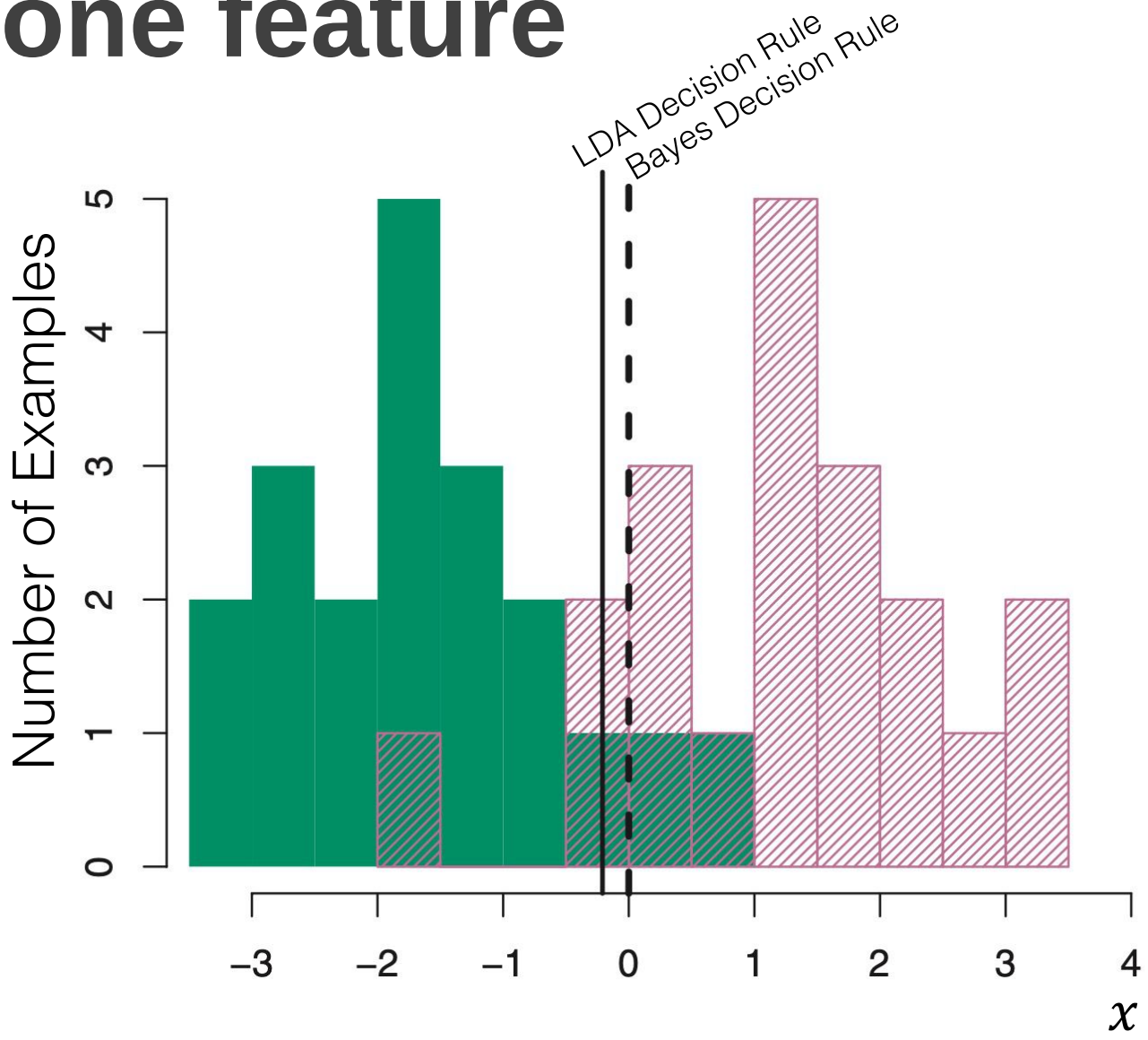
If we further assume the **covariance matrices for each class are the same**, $\Sigma_0 = \Sigma_1$, this represents

Linear Discriminant Analysis

Simple example with one feature



$$\frac{P(\boldsymbol{x}|y=1)}{P(\boldsymbol{x}|y=0)} > \frac{P(y=0)}{P(y=1)}$$



Figures from James et al. - Introduction to Statistical Learning

Multivariate Gaussian

Univariate Gaussian (1 predictor)

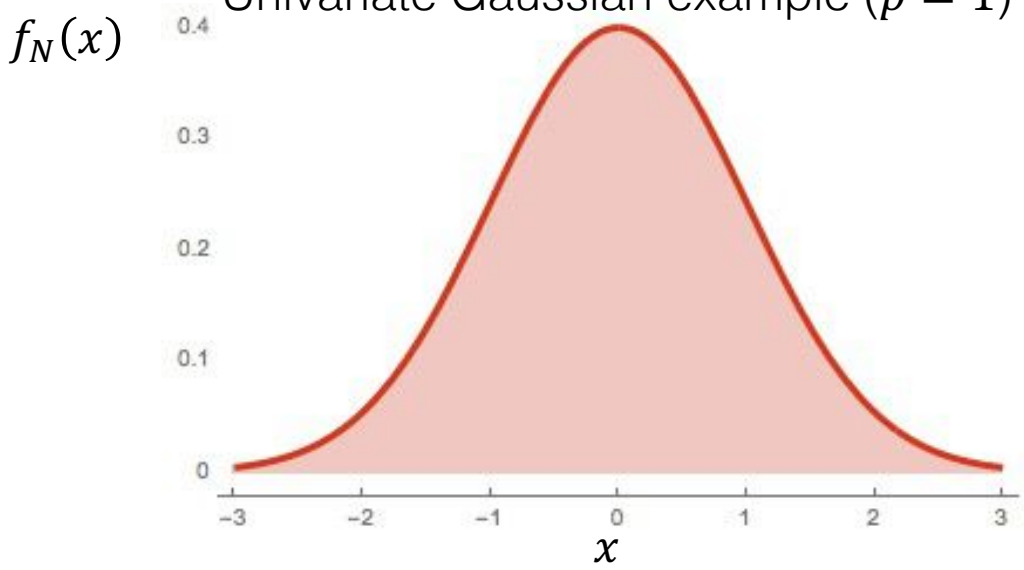
$$f_N(x) = \frac{1}{(2\pi)^{1/2}\sigma}\exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right]$$

Multivariate Gaussian (p predictors)

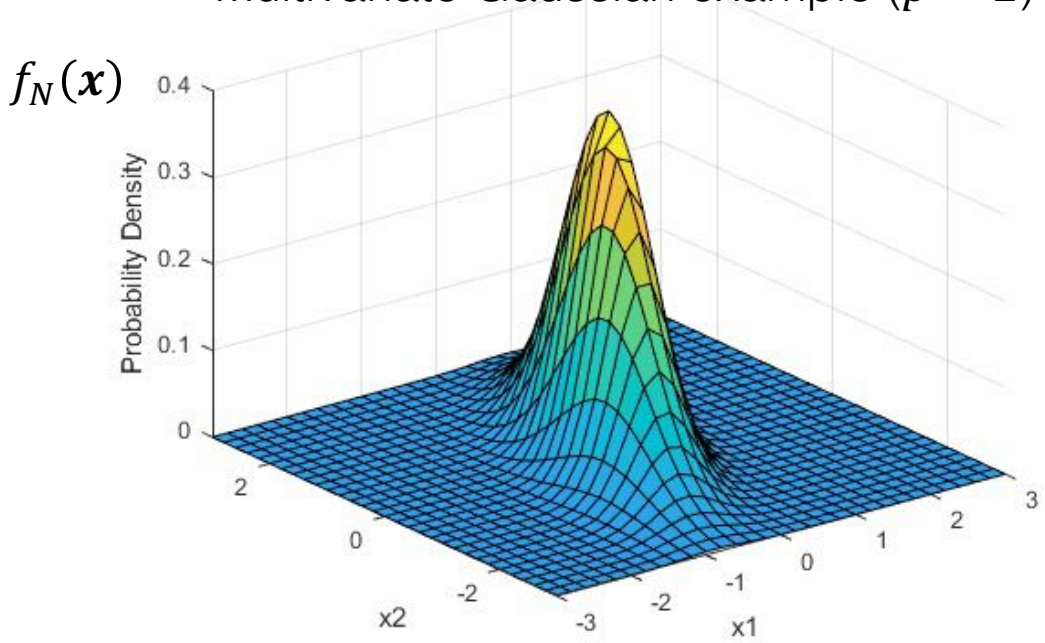
$$f_N(\boldsymbol{x}) = \frac{1}{(2\pi)^{p/2}|\boldsymbol{\Sigma}|^{1/2}}\exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right]$$

Estimate our means and variances and we can use this as our likelihood

Univariate Gaussian example ($p = 1$)



Multivariate Gaussian example ($p = 2$)



Images from Brilliant.org (top) and the Mathworks (bottom)

Linear Discriminant Analysis ($\Sigma_0 = \Sigma_1$)

We build a classifier that assigns the class with the higher posterior probability:

$$\frac{P(\boldsymbol{x}|y=1)}{P(\boldsymbol{x}|y=0)} = \frac{\frac{1}{(2\pi)^{d/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_1)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_1)\right]}{\frac{1}{(2\pi)^{d/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_0)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_0)\right]}$$

$$= \frac{\exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_1)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_1)\right]}{\exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_0)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_0)\right]}$$

$$\ln\left|\frac{P(\boldsymbol{x}|y=1)}{P(\boldsymbol{x}|y=0)}\right| = -\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_1)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_1) + \frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_0)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_0)$$

Derivation 1/4

Linear Discriminant Analysis ($\Sigma_0 = \Sigma_1$)

$$\ln\left|\frac{P(\boldsymbol{x}|y=1)}{P(\boldsymbol{x}|y=0)}\right| = -\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_1)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_1) + \frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_0)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_0)$$

Expanding this expression yields:

These combine since $\boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_i = \boldsymbol{\mu}_i^T\boldsymbol{\Sigma}^{-1}\boldsymbol{x}$ for symmetric matrices

$$= -\frac{1}{2}[\boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{x} - \boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_1 - \boldsymbol{\mu}_1^T\boldsymbol{\Sigma}^{-1}\boldsymbol{x} + \boldsymbol{\mu}_1^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_1$$
$$-\boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_0 + \boldsymbol{\mu}_0^T\boldsymbol{\Sigma}^{-1}\boldsymbol{x} - \boldsymbol{\mu}_0^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_0$$

$$= \frac{1}{2}[2\boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_0) - (\boldsymbol{\mu}_1+\boldsymbol{\mu}_0)\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_0)]$$

$$= \boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_0) - \frac{1}{2}(\boldsymbol{\mu}_1+\boldsymbol{\mu}_0)\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_0)$$

Derivation 2/4

Linear Discriminant Analysis: 2 Class

Σ is the same for both classes

$$\ln\left|\frac{P(\boldsymbol{x}|y=1)}{P(\boldsymbol{x}|y=0)}\right| = \boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_0) - \frac{1}{2}(\boldsymbol{\mu}_1+\boldsymbol{\mu}_0)\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_0)$$

Since our decision rule is to classify as class 1 if the following is true:

$$\ln\left|\frac{P(\boldsymbol{x}|y=1)}{P(\boldsymbol{x}|y=0)}\right| > \ln\left|\frac{P(y=0)}{P(y=1)}\right|$$

We can rewrite our decision rule as:

(see appendix slides for full derivation)

$$\underbrace{\boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_0)}_{\text{Project the data into a 1-dimensional space}} > \underbrace{\frac{1}{2}(\boldsymbol{\mu}_1+\boldsymbol{\mu}_0)\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_0) + \ln\left|\frac{P(y=0)}{P(y=1)}\right|}_{\text{Set the threshold for classifying as class 1 or 0}}$$

Derivation 3/4

Linear Discriminant Analysis: 2 Class

Σ is the same for both classes

$$\ln\left|\frac{P(\boldsymbol{x}|y=1)}{P(\boldsymbol{x}|y=0)}\right| = \boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_0) - \frac{1}{2}(\boldsymbol{\mu}_1+\boldsymbol{\mu}_0)\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_0)$$

Since our decision rule is to classify as class 1 if the following is true:

$$\ln\left|\frac{P(\boldsymbol{x}|y=1)}{P(\boldsymbol{x}|y=0)}\right| > \ln\left|\frac{P(y=0)}{P(y=1)}\right|$$

We can rewrite our decision rule as:

(see appendix slides for full derivation)

$$\boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_0) > \frac{1}{2}(\boldsymbol{\mu}_1+\boldsymbol{\mu}_0)\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_0) + \ln\left|\frac{P(y=0)}{P(y=1)}\right|$$

Or simply as: $\boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_0) > \lambda_{thresh}$

If we define $\boldsymbol{w} = \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_0)$, then this becomes $\boldsymbol{x}^T\boldsymbol{w} = \boldsymbol{w}^T\boldsymbol{x} > \lambda_{thresh}$

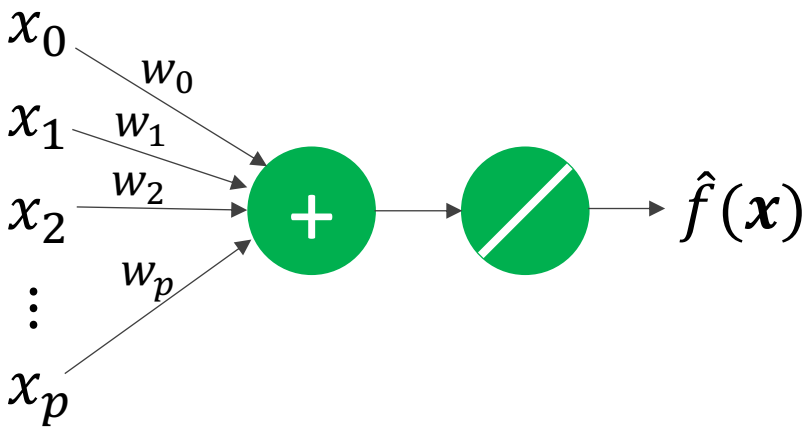
Derivation 4/4

Remember linear models?

Linear Regression

$$\hat{f}(\boldsymbol{x}) = \sum_{i=0}^{p} w_i x_i$$

$$\boldsymbol{w}^T \boldsymbol{x}$$

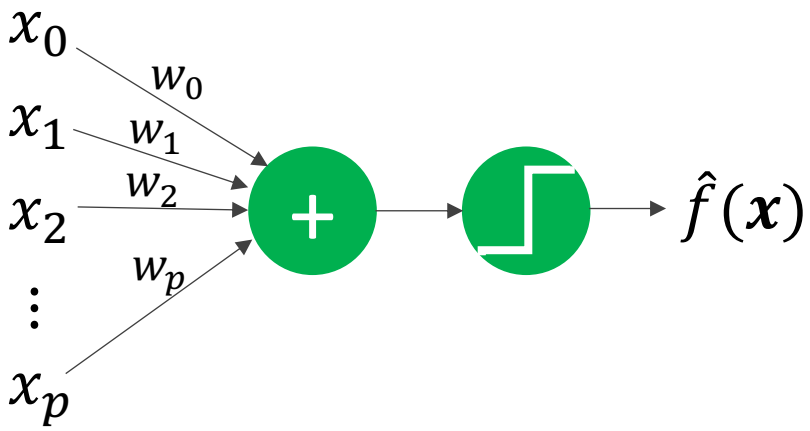


Linear Classification

Perceptron

$$\hat{f}(\boldsymbol{x}) = sign\left(\sum_{i=0}^{p} w_i x_i\right)$$

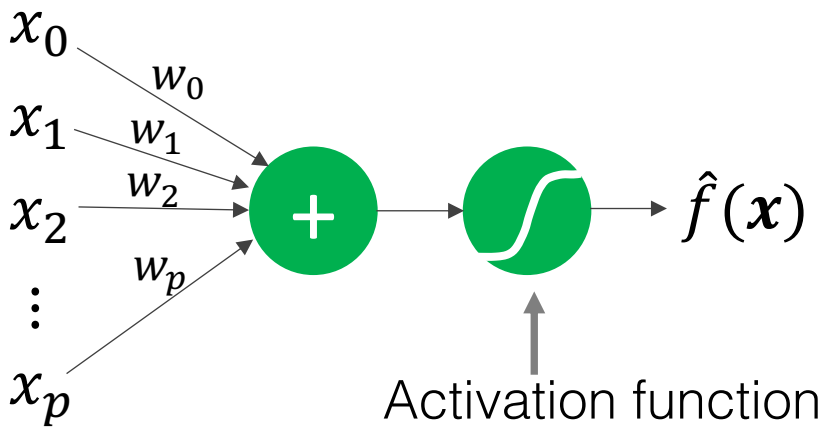
$$sign(x) = \begin{cases} 1 & x > 0 \\ -1 & \text{else} \end{cases}$$



Logistic Regression

$$\hat{f}(\boldsymbol{x}) = \sigma\left(\sum_{i=0}^{p} w_i x_i\right)$$

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$



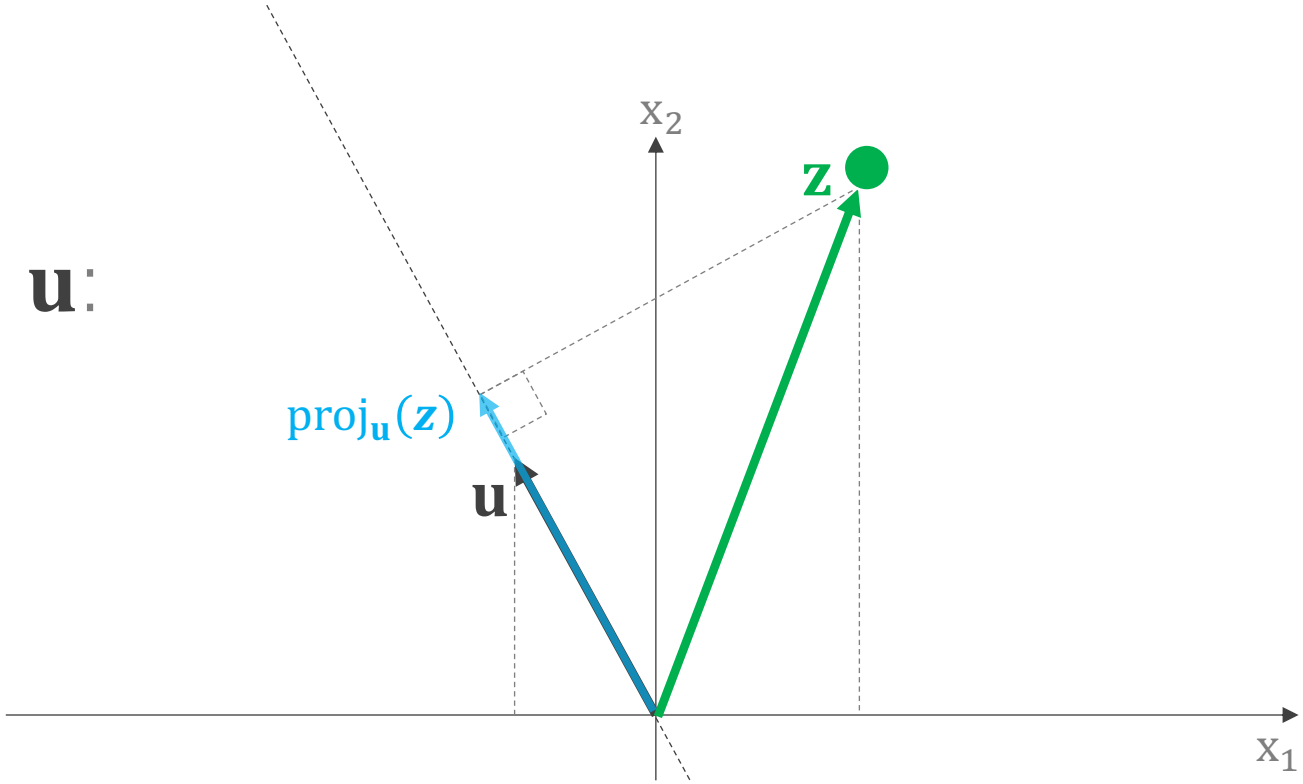
Source: Abu-Mostafa, Learning from Data, Caltech

Intuitively interpreting LDA in terms of projections

Projections

The vector projection of $\boldsymbol{z}$ onto $\mathbf{u}$:

$$\text{proj}_{\mathbf{u}}(\boldsymbol{z}) = \left(\frac{\mathbf{u}^T\mathbf{z}}{\|\mathbf{u}\|}\right)\frac{\mathbf{u}}{\|\mathbf{u}\|}$$



The scalar length (Euclidean or L_2 norm) of the vector $\mathbf{u}$ is $\|\mathbf{u}\|$

If we assume $\mathbf{u}$ is a unit vector then $\|\mathbf{u}\| = 1$

$$\text{proj}_{\mathbf{u}}(\boldsymbol{z}) = (\mathbf{u}^T\mathbf{z})\mathbf{u}$$

Magnitude of projection onto direction of $\mathbf{u}$

Projections

$$\mathbf{u}^T\mathbf{z} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

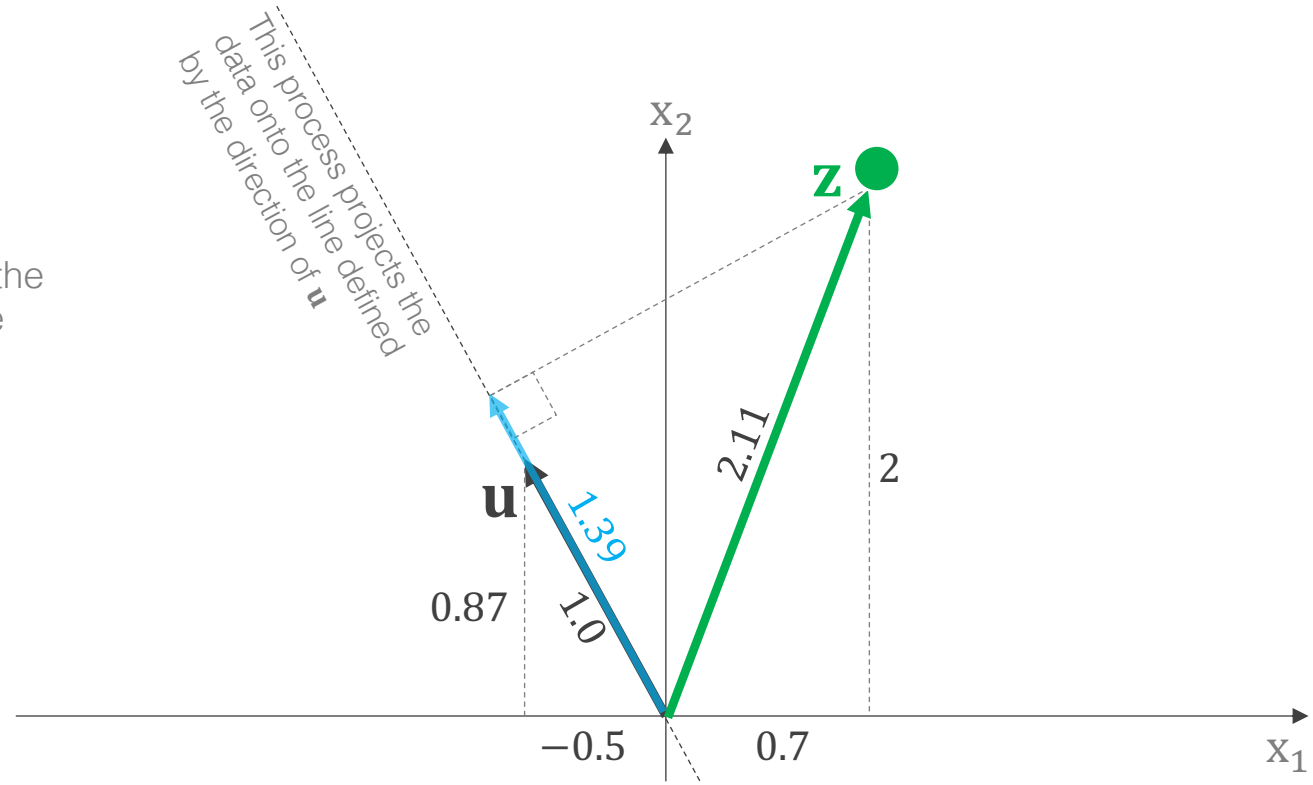
This is an inner product, but assuming $\mathbf{u}$ is a unit vector computes this as the magnitude (length) of the projection of $\mathbf{z}$ onto $\mathbf{u}$

$$= u_1 z_1 + u_2 z_2$$

$$= (-0.5)(0.7) + (0.87)(2)$$

$$= 1.39$$

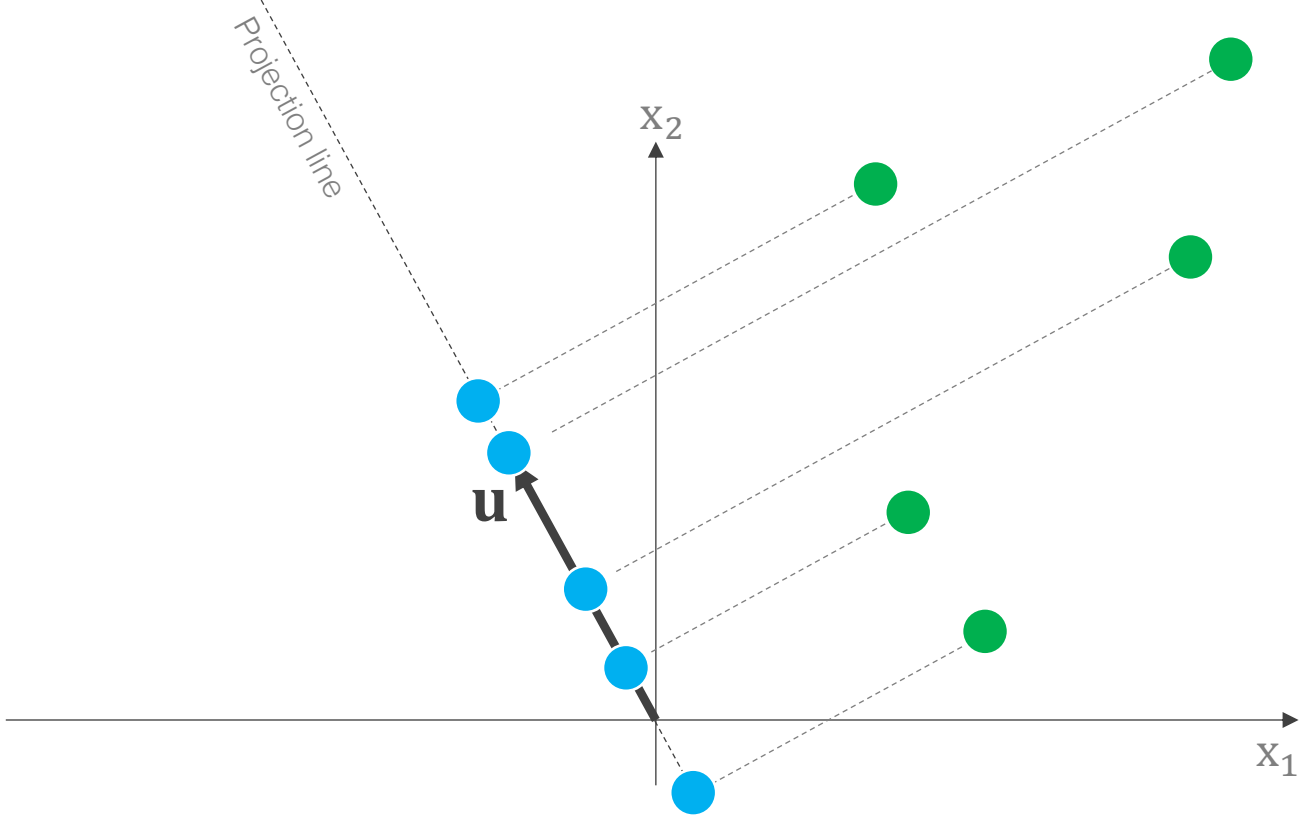
Length (magnitude) of the projection of $\mathbf{z}$ onto $\mathbf{u}$



This is valid because $\mathbf{u}$ is a unit vector (length is 1: $\|\mathbf{u}\|_2 = \sqrt{u_1^2 + u_2^2} = \sqrt{(-0.5)^2 + (0.87)^2} \cong 1$)

Projections

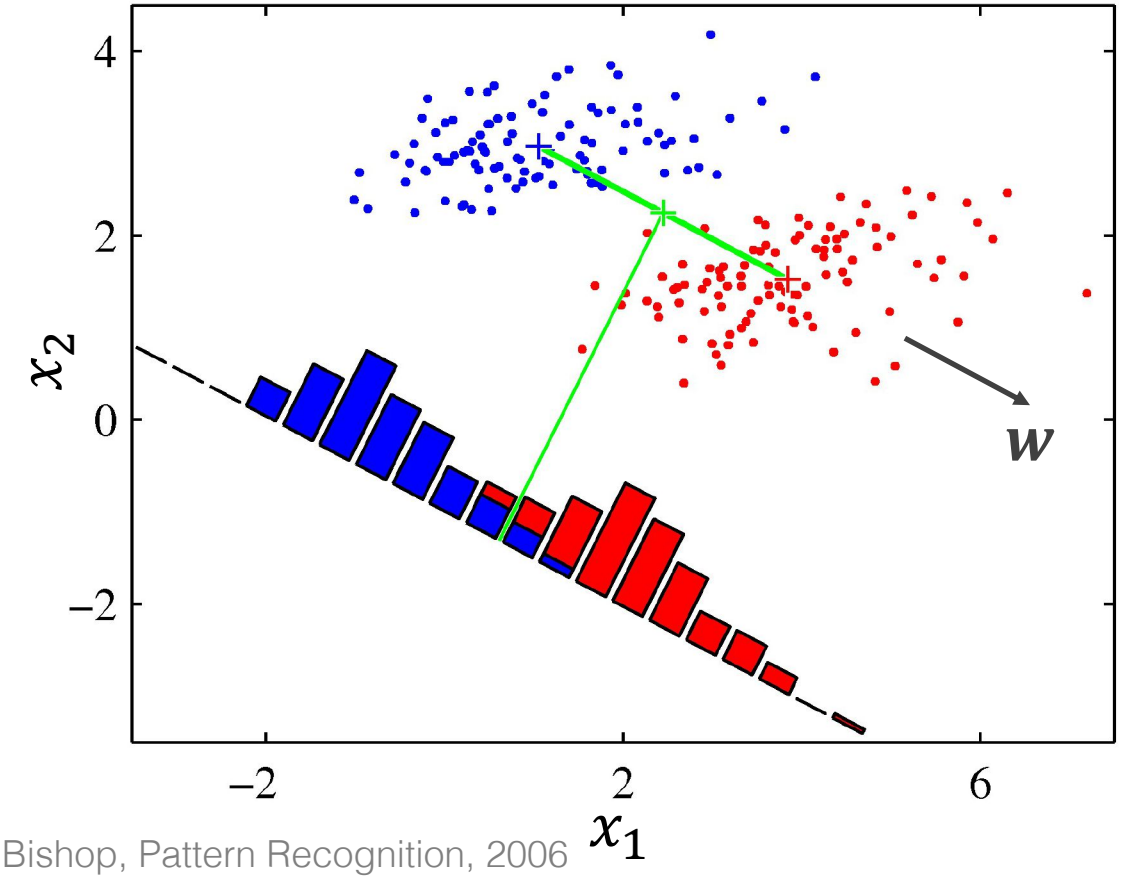
We could project any points in this space onto the line defined by the direction of unit vector $\mathbf{u}$



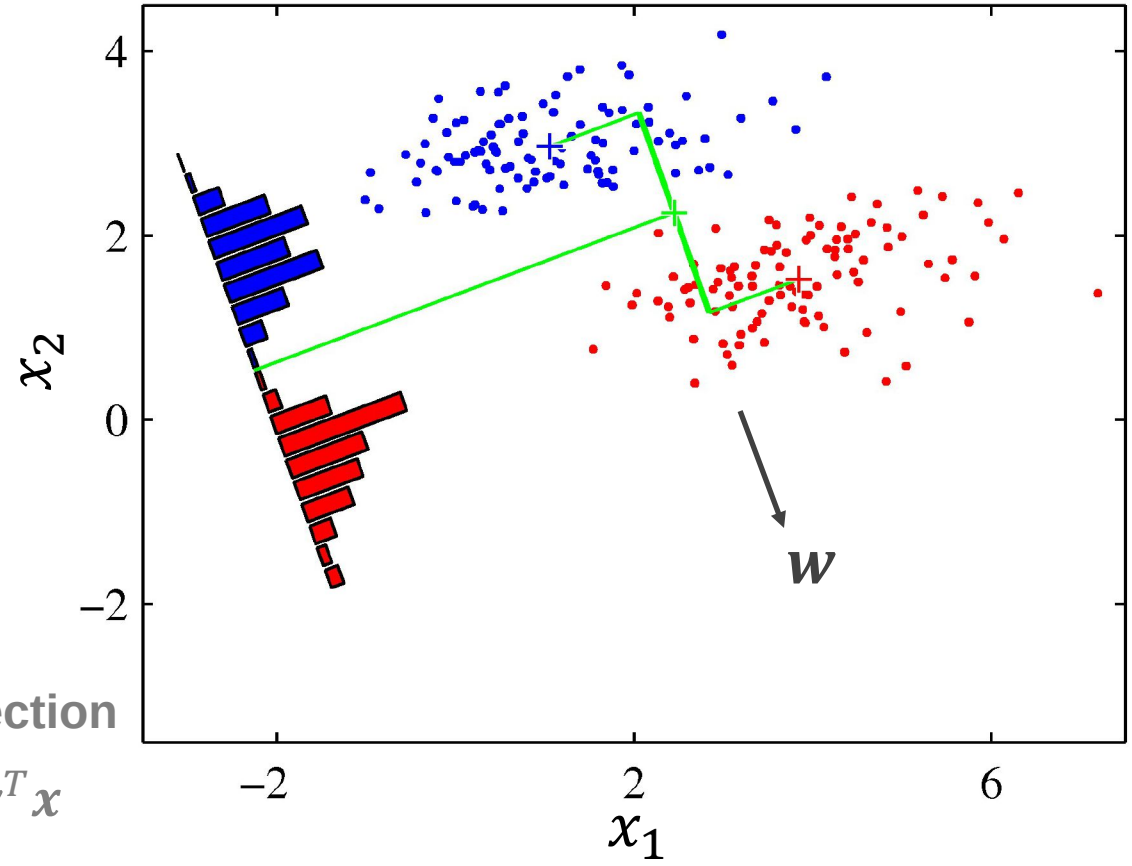
Linear Discriminant Analysis

Looks for the projection into the one dimension that "best" separates the classes

Projection onto line connecting the means



Projection onto a line providing improved class separation



Linear projection

$$\hat{f}(x) = w^T x$$

Bishop, Pattern Recognition, 2006

Linear Discriminant Analysis (LDA)

1. Finds a projection into a lower dimension that "best" separates the classes

$$\hat{f}(\boldsymbol{x}) = \boldsymbol{w}^T \boldsymbol{x}$$

Consider $\boldsymbol{w}$ is a unit vector of parameters

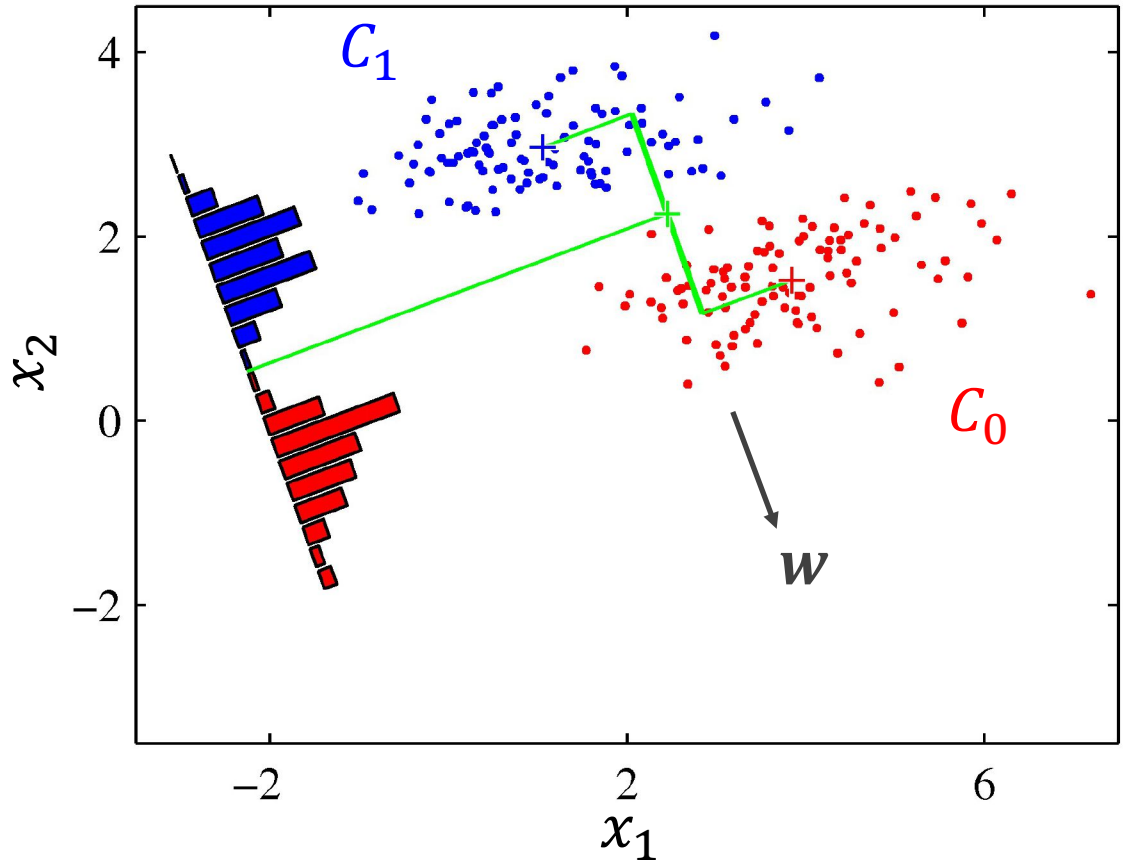
2. We then classify the data in this space linearly

Our decision rule becomes:

if	$\boldsymbol{w}^T \boldsymbol{x} > \lambda_{thresh}$	Class 1
	else	Class 0

2-Class LDA: what does vector w accomplish?

Bishop, Pattern Recognition, 2006



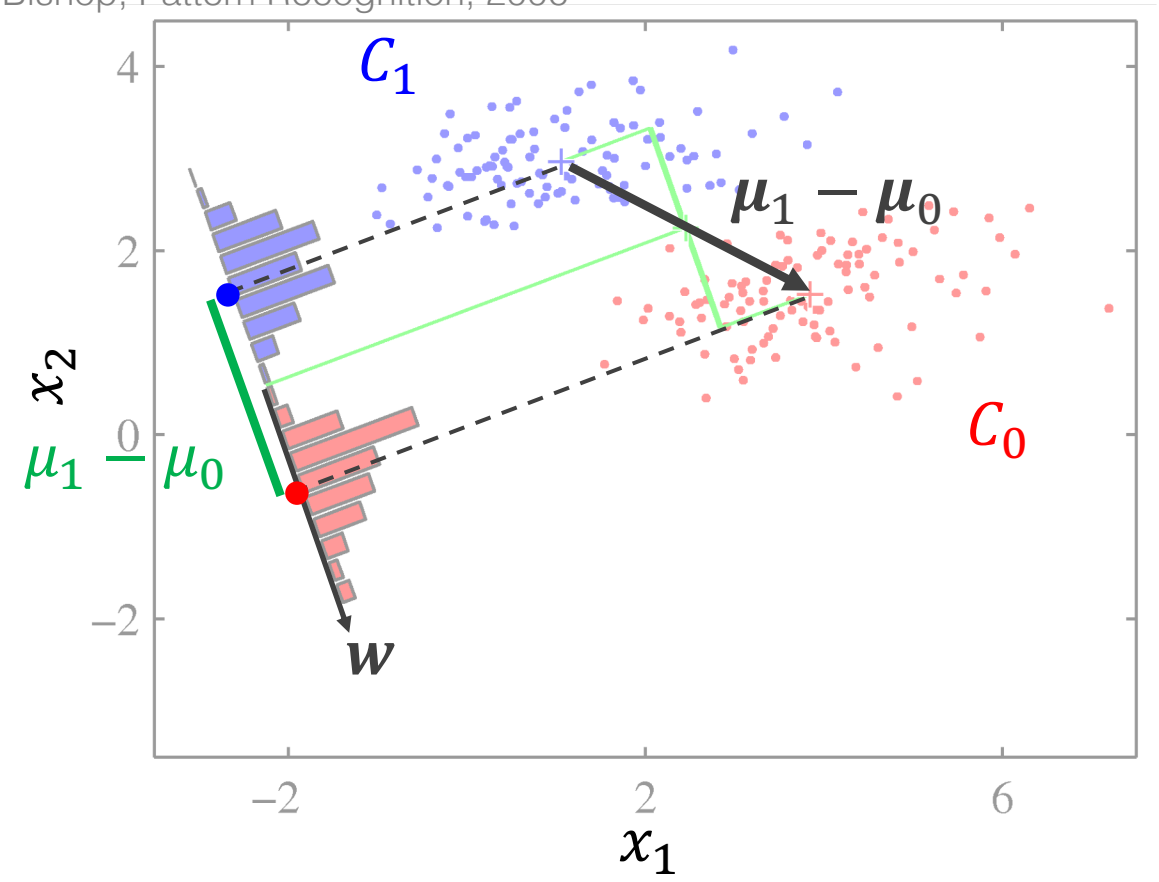
$$y = \hat{f}(\boldsymbol{x}) = \boldsymbol{w}^T \boldsymbol{x}$$

Increases the distance between class **means**

Decreases the **variance** within the classes

2-Class LDA: what does vector w accomplish?

Bishop, Pattern Recognition, 2006



$$y = \hat{f}(\boldsymbol{x}) = \boldsymbol{w}^T \boldsymbol{x}$$

Increase the distance between the **means**

$$\boldsymbol{\mu}_1 = \frac{1}{N_1} \sum_{i \in C_1} \boldsymbol{x}_i$$

mean of class 1

$$\boldsymbol{\mu}_0 = \frac{1}{N_0} \sum_{i \in C_0} \boldsymbol{x}_i$$

mean of class 0

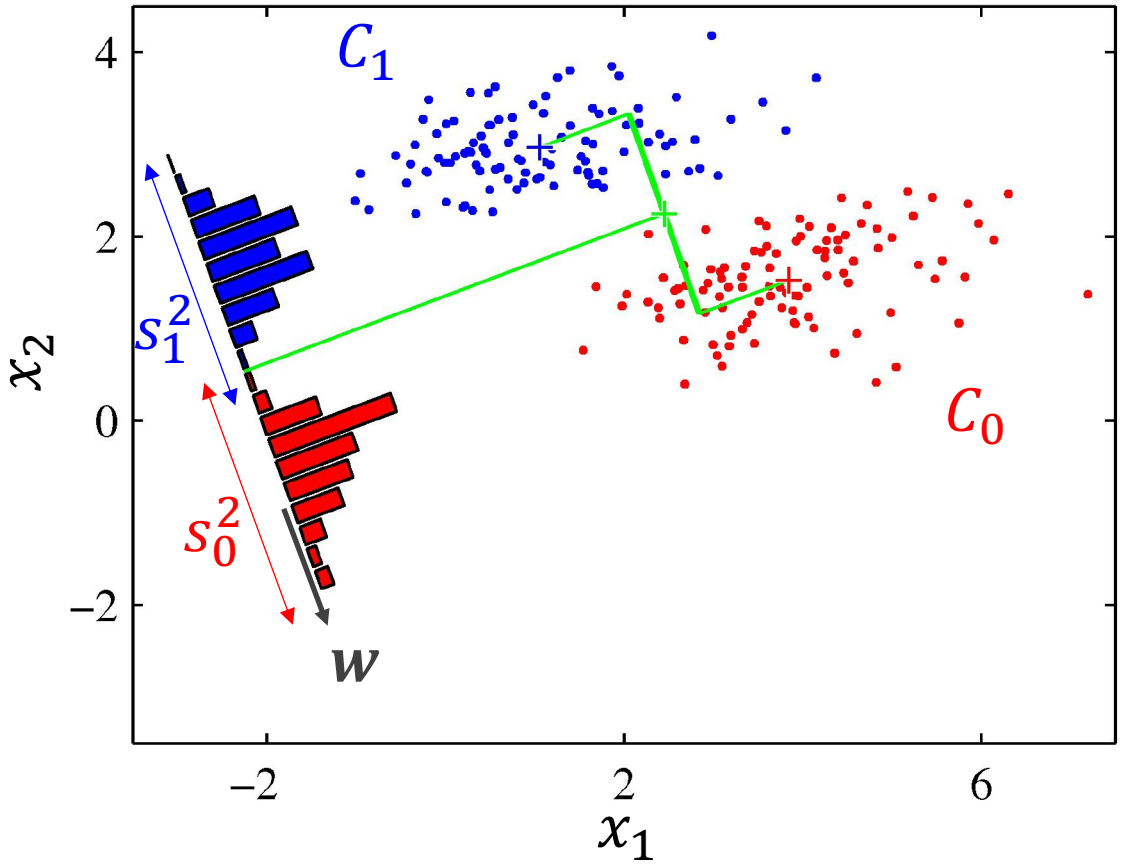
The means projected onto $\boldsymbol{w}$: $\mu_k = \boldsymbol{w}^T \boldsymbol{\mu}_k$

The distance between the means:

$$\mu_1 - \mu_0 = \boldsymbol{w}^T (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)$$

2-Class LDA: what does vector w accomplish?

Bishop, Pattern Recognition, 2006



$$y = \hat{f}(\boldsymbol{x}) = \boldsymbol{w}^T \boldsymbol{x}$$

Decreases the **variance** within the classes

The "scatter" of the **projected** data:

$$s_k^2 = \sum_{i \in C_k} (y_i - \mu_k)^2$$

where $\mu_k = \boldsymbol{w}^T \boldsymbol{\mu}_k$

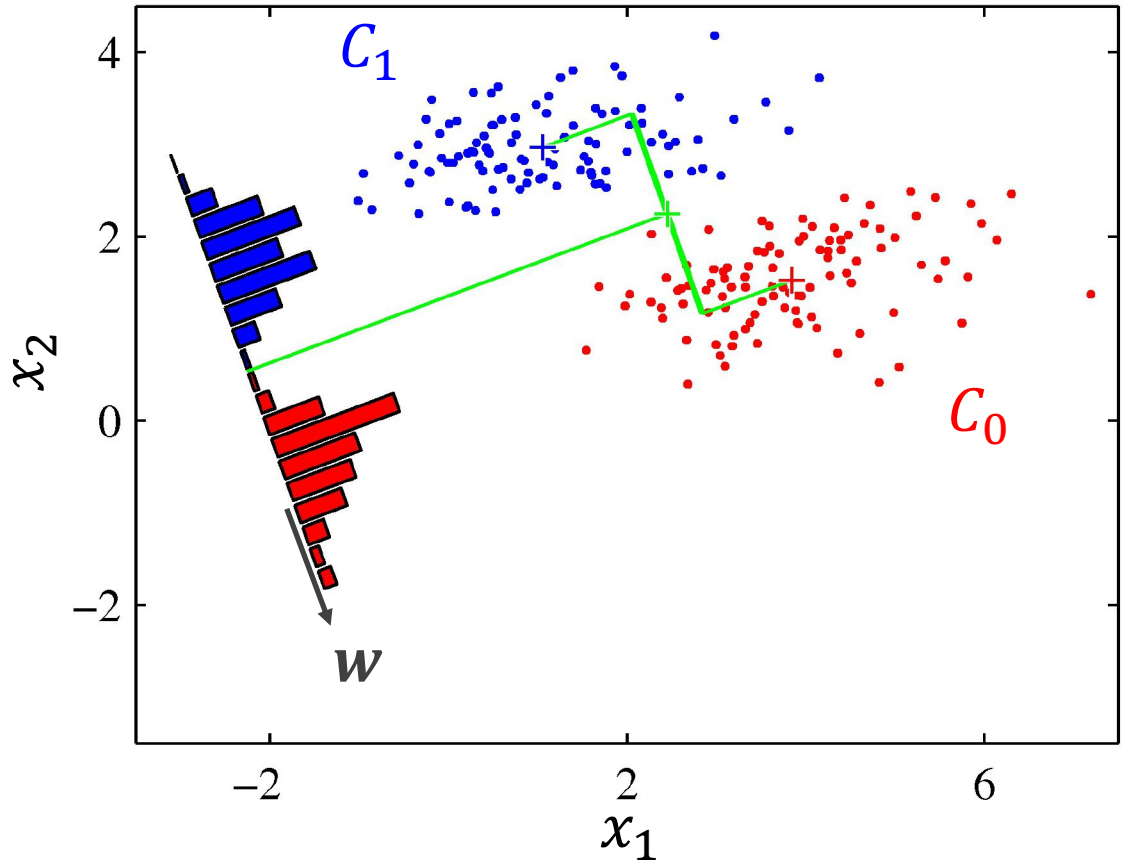
$$y_i = \boldsymbol{w}^T \boldsymbol{x}_i$$

We assume these are the same:

$$S = s_1^2 = s_0^2$$

2-Class LDA: what does vector w accomplish?

Bishop, Pattern Recognition, 2006



$$y = \hat{f}(\boldsymbol{x}) = \boldsymbol{w}^T \boldsymbol{x}$$

Increase the distance between the **means**

$$\mu_1 - \mu_0 = \boldsymbol{w}^T(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)$$

Decrease the **variance** within each class

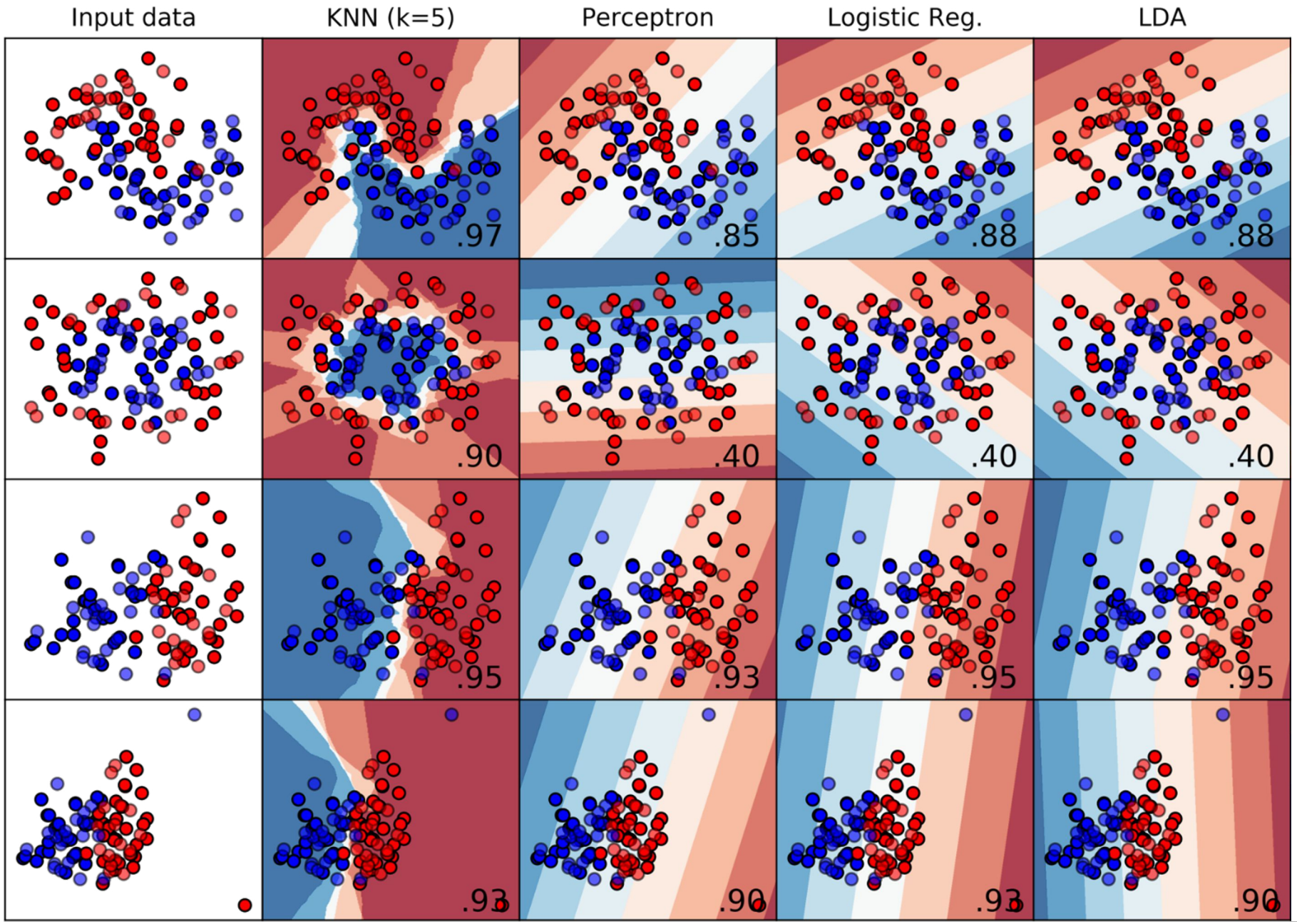
$$S = s_1^2 = s_0^2$$

$$\boldsymbol{w} = \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)$$

We use this to project the features into one dimension for classification, $\boldsymbol{w}^T\boldsymbol{x}$

This approach is a supervised dimensionality reduction technique that we use for classification

(To extend to >2 classes pick the one with the largest $d_i(x)$)


Input data
KNN (k=5)
Perceptron
Logistic Reg.
LDA
.97
.85
.88
.88
.90
.40
.40
.40
.95
.93
.95
.95
.93
.90
.93
.90

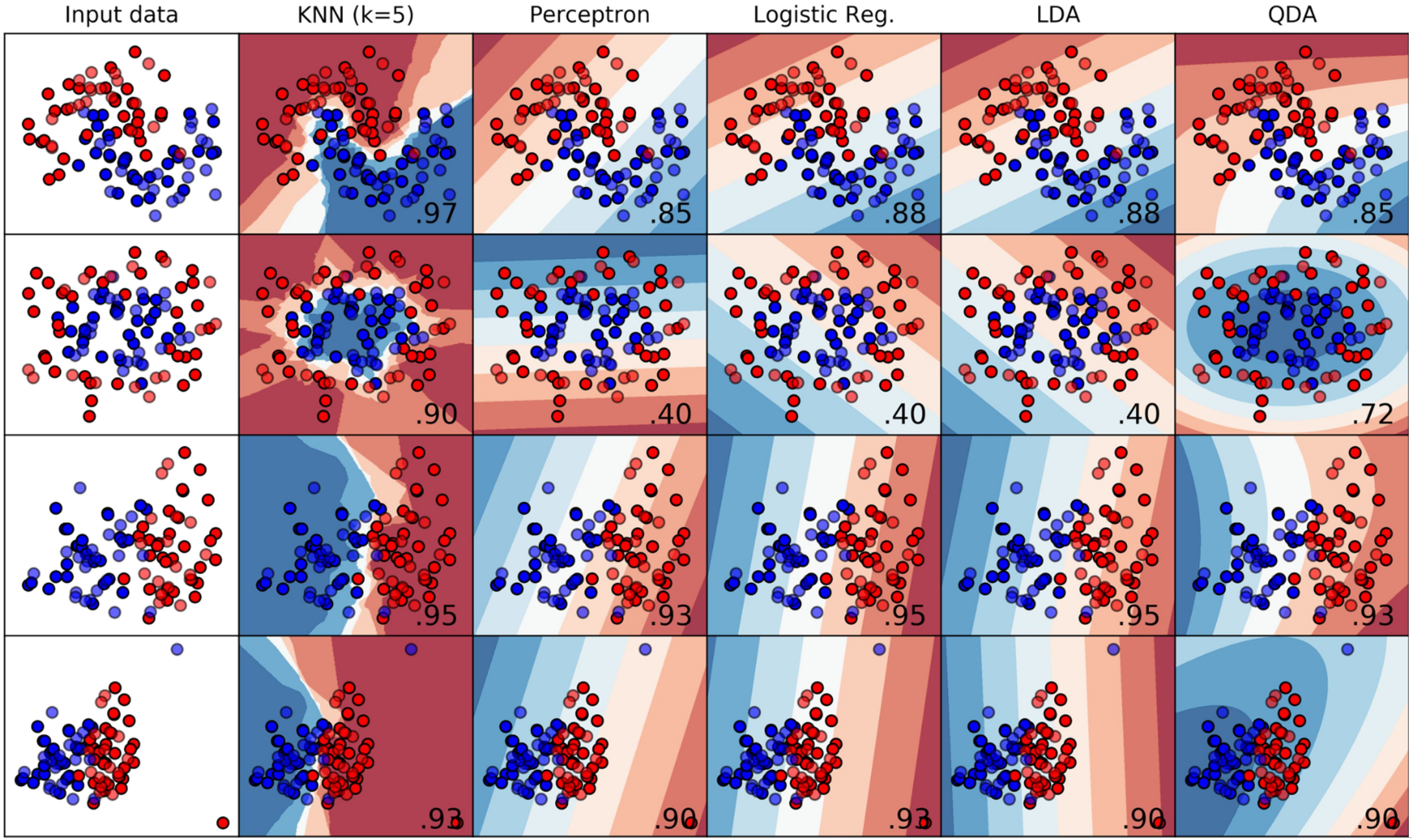
Quadratic Discriminant Analysis

We build a classifier that assigns the class with the higher posterior probability:

$$d_k(\boldsymbol{x}) = \frac{1}{(2\pi)^{d/2}|\boldsymbol{\Sigma}_k|^{1/2}} \exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_k)^T \boldsymbol{\Sigma}_k^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_k)\right]\pi_k$$

We assume a normal distribution, but **different covariance matrices**

Produces a quadric decision boundary


Input data
KNN (k=5)
Perceptron
Logistic Reg.
LDA
QDA
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Joint vs Marginal Densities

The marginal densities don't factor in relationships between features

What if the joint density is too hard to estimate?

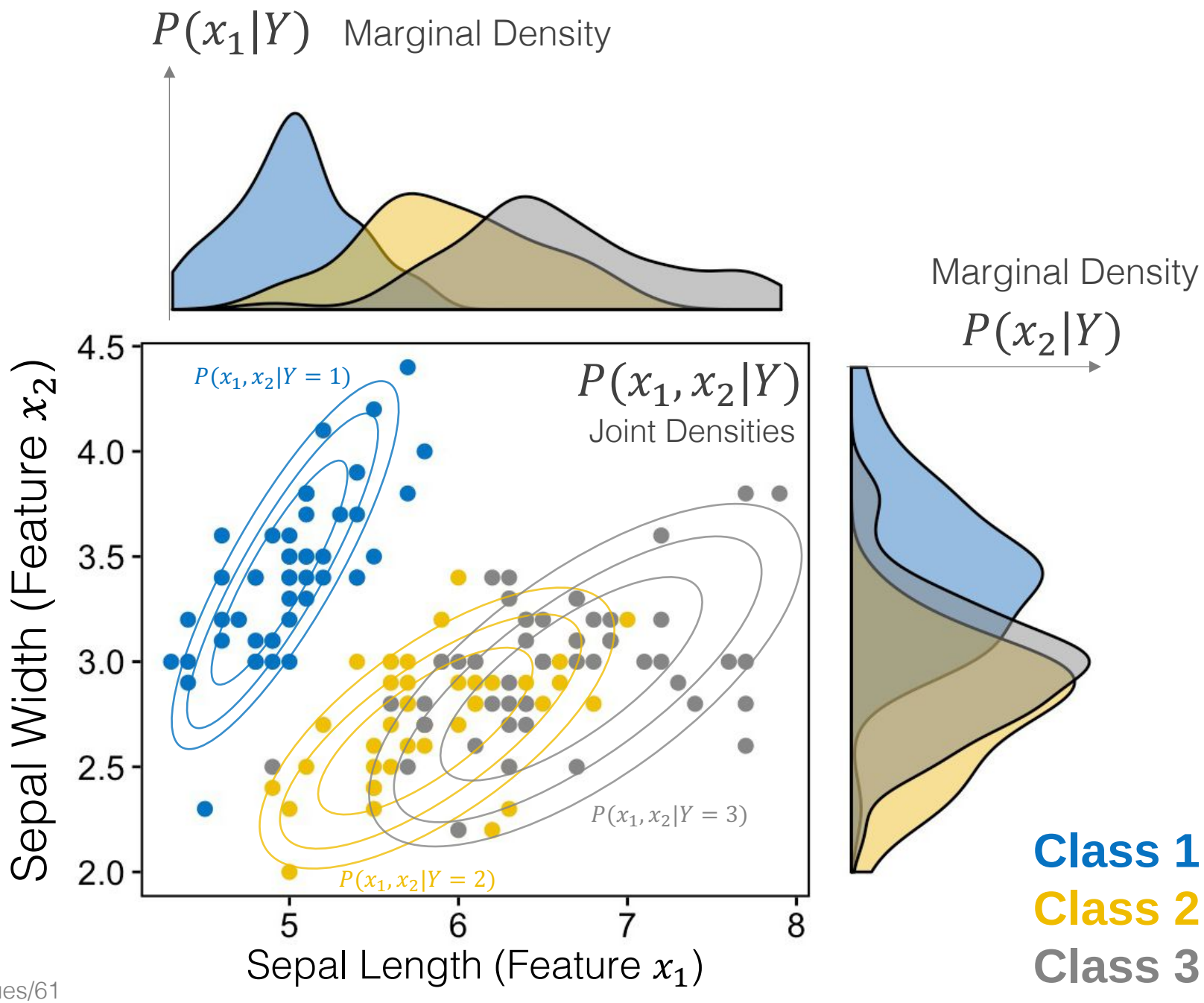


Image adapted from: https://github.com/daattali/ggExtra/issues/61

Naïve Bayes

For independent events: A, B, and C

$$P(A \text{ and } B \text{ and } C) = P(A)P(B)P(C)$$

Start with our original expression for our discriminant function
(proportional to posterior distribution)

$$d_i(\boldsymbol{x}) = P(\boldsymbol{x}|y = i)P(y = i)$$

Write out the full expression with all the terms in $\boldsymbol{x}$
(assume p predictors/features)

$$d_i(x_1, x_2, \dots, x_p) = P\big(x_1, x_2, \dots, x_p\big|y = i\big)P(y = i)$$

Assumption: Given the class, the features are independent

$$d_i(x_1, x_2, \dots, x_p) = P(y = i)\prod_{j=1}^{p} P\big(x_j\big|y = i\big)$$

Predict the class with the highest discriminant function (i.e. posterior probability)

Naïve Bayes

We assign the class that has the largest discriminant (i.e. posterior probability)

$$d_i(x_1, x_2, \ldots, x_p) = P(y = i) \prod_{j=1}^{p} P(x_j | y = i)$$

This implies we estimate the density of each feature **separately**

This independence assumption is a strong assumption that is rarely valid

Considerably simplifies computation and data needs

Is flexible to allow for different distributional forms (i.e. Gaussian) or nonparametric techniques for the likelihood $P(x_j | y = i)$

Naïve Bayes: Gaussian example

We assign the class that has the largest discriminant (i.e. posterior probability)

$$d_i(x_1, x_2, \ldots, x_p) = P(y = i) \prod_{j=1}^{p} P\left(x_j \middle| y = i\right)$$

This implies we estimate the density of each feature **separately**

If $P\left(x_j \middle| y = i\right)$ is $N(\mu_{ji}, \sigma_{ji}^2)$, so for each class we estimate one mean and variance for each of the p features and for each class. We multiply **univariate** distributions together

$$d_i(x_1, x_2, \ldots, x_p) = P(y = i) \prod_{j=1}^{p} N(\mu_{ji}, \sigma_{ji}^2)$$

Naïve Bayes: Parameters

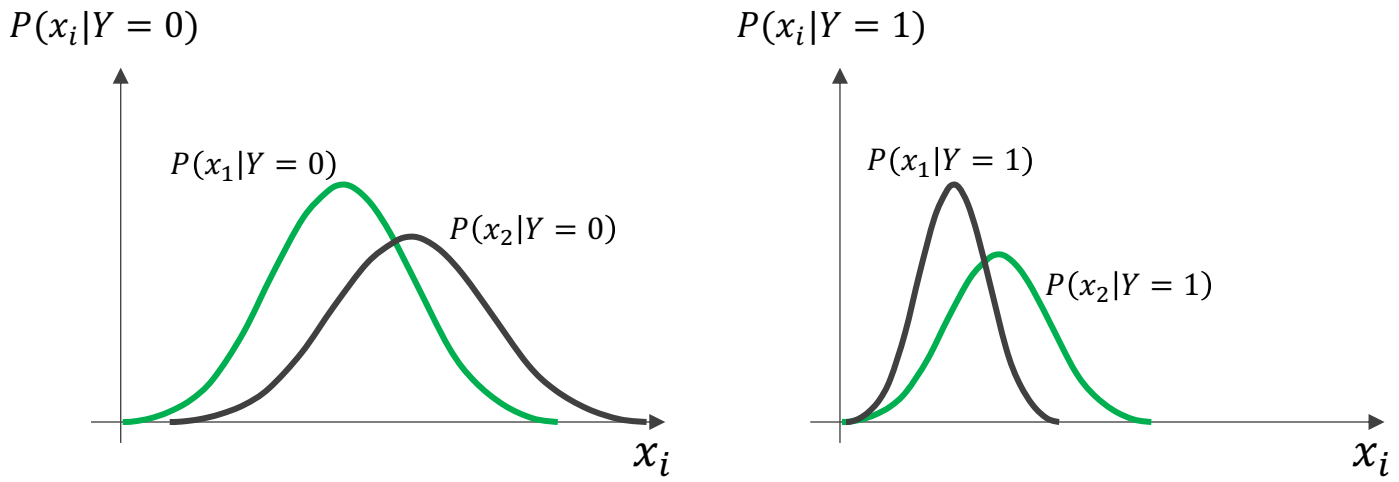
p predictors, c classes

For each predictor, x_i, and class, y_j:

$$(\mu_{ij}, \sigma_{ij}^2)$$

Total parameters = $2cp$

Naïve Bayes ($p = 2$)



Without the Naïve Bayes independence assumption, each class would be a multivariate Gaussian with $(\boldsymbol{\mu}_j, \boldsymbol{\Sigma}_j)$

$$\boldsymbol{\mu}_j = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix}, \boldsymbol{\Sigma}_j = \begin{bmatrix} \sigma_{11}^2 & \sigma_{12}^2 & \cdots & \sigma_{1p}^2 \\ \sigma_{21}^2 & \sigma_{22}^2 & \cdots & \sigma_{2p}^2 \\ \vdots & \vdots & & \vdots \\ \sigma_{p1}^2 & \sigma_{p2}^2 & \cdots & \sigma_{pp}^2 \end{bmatrix}$$

Total parameters = $c\left(p + \frac{p^2 - p}{2} + p\right)$

Multivariate Gaussian ($p = 2$)

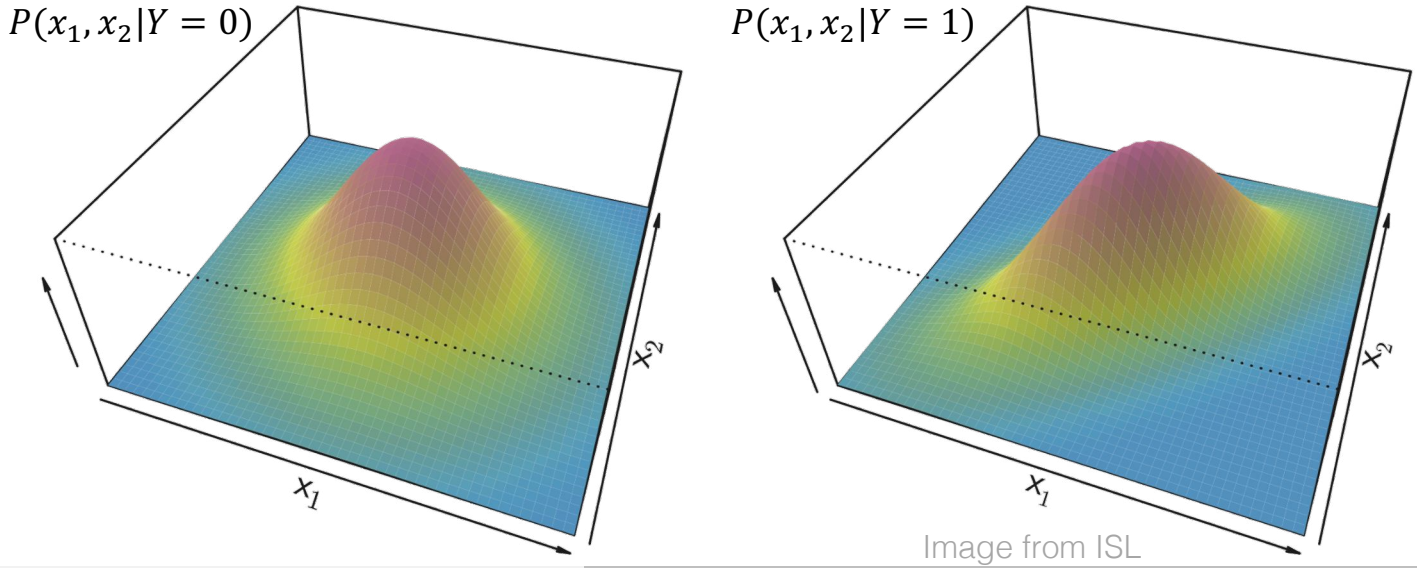
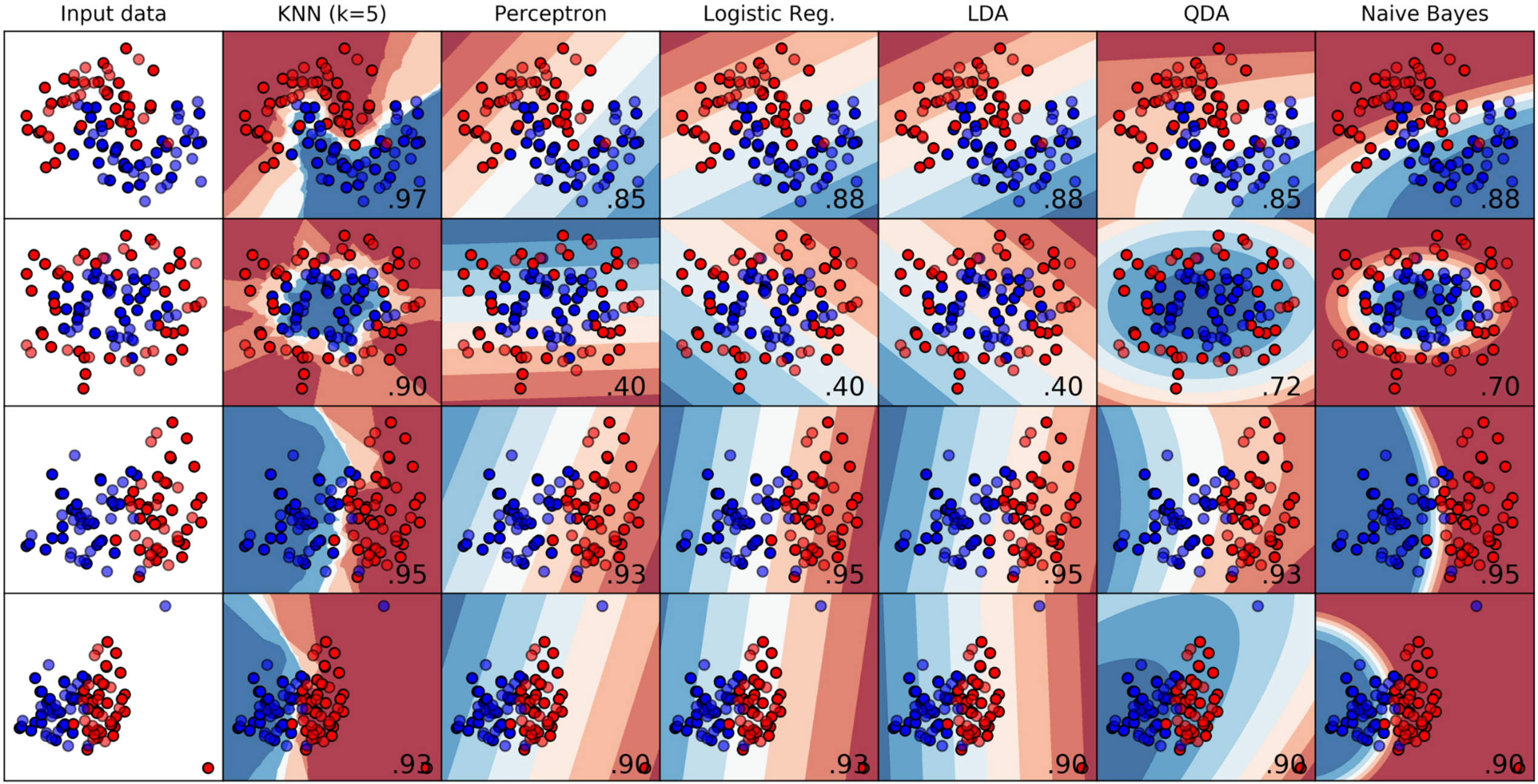


Image from ISL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Classifiers

Covered so far

- K-Nearest Neighbors
- Perceptron
- Logistic Regression
- Linear Discriminant Analysis
- Quadratic Discriminant Analysis
- Naïve Bayes

Have closed-form solutions

- Apply to multiclass problems
- No hyperparameters
- Fast to train

- Requires small amounts of training data
- Only choice is the form of P(X|Y) (otherwise no parameter choices)
- Fast to train