Midterm 2 Prep

Production Functions

1. Consider the production function $Q(K, L) = 5K + L$.

   (a) What is the marginal product of capital and labor? Interpret these in words.

   (b) What is marginal rate of technical substitution $MRTS_{K,L}$?

   (c) What does this production function tell you about the substitutability of labor and capital?

   (d) Does this production function exhibit constant, increasing or decreasing returns to scale?

   (e) Draw 3 isoquant curves for this production function.

   (f) Holding fixed capital at $K = 10$, what happens to the marginal product of labor as the firm increases L?

   Solution:

   (a) $MP_K = 5$ and $MP_L = 1$. For each additional unit of labor, one additional unit of output is produced. For each additional unit of capital used, one additional unit of output is produced.

   (b) $MRTS_{K,L} = \frac{MP_L}{MP_K} = 1/5$. For every unit of labor the producer gives up, they need $1/5$ unit of capital to produce the same output as before.

   (c) Increasing returns to scale. $Q(\phi K, \phi L) = 5\phi K \phi L = \phi^2 Q(K, L)$.

   (d) Lines with a slop of $-1/5$.

   (e) Stays constant at 1.

2. Consider the production function $Q(K, L) = 5KL$.

   (a) What is the marginal product of capital and labor? Interpret these in words.

   (b) What is marginal rate of technical substitution $MRTS_{K,L}$?

   (c) Does this production function exhibit constant, increasing or decreasing returns to scale?

   (d) Draw 3 isoquant curves for this production function.

   (e) Holding fixed capital at $K = 10$, what happens to the marginal product of labor as the firm increases L?

   (f) If the production function becomes $Q(K, L) = 5K^{3/2}L$, is this capital-biased, labor-biased, or neutral technological change?

   Solution:

   (a) $MP_K = 5L$ and $MP_L = 5K$. For each additional unit of labor, $5K$ additional unit of output is produced. For each additional unit of capital used, $5L$ additional unit of output is produced.

   (b) $MRTS_{K,L} = \frac{MP_L}{MP_K} = K/L$. For every unit of labor the producer gives up, they need K/L unit of capital to produce the same output as before.

(c) Curvy macaroni shapes

(d) Holding $K = 10$, the marignal product of labor increases with L

(e) $MRTS_{K,L} = \frac{MP_L}{MP_K} = \frac{5K^{3/2}}{5*3/2K^{1/2}L} = \frac{2}{3}\frac{K}{L}$. This means, for every unit of labor the producer gives up, they need $\frac{2}{3}K/L$ unit of capital to produce the same output as before. This is less than before, so capital is more productive than before. Therefore this is capital-biased technological change.

3. Let T represent car tires and F represent car frames. The production of a car requires 4 tires and 1 frame.

(a) Draw 3 isoquant curves for this production function.

(b) Write this as a mathematical production function.

Solution:

(a) A bunch of "brackets" along the line $4F = T$.

(b) $Q(F,T) = \min\{4F, T\}$

Cost Minimization

1. For the following scenarios, find the cost-minimizing input bundles and the total costs of producing the given quantity

(a) $Q(K,L) = 2L + 4K$, $r = 8$, $w = 2$, and $\bar{Q} = 40$

(b) $Q(K,L) = 2L + 4K$, $r = 8$, $w = 6$, and $\bar{Q} = 40$

(c) $Q(K,L) = K^{1/2}L^{1/2}$, $r = 8$, $w = 8$, and $\bar{Q} = 40$

(d) $Q(K,L) = \min(K, 2L)$, $r = 8$, $w = 2$, and $\bar{Q} = 40$

Solution:

(a) $\frac{MP_K}{r} = 1/2$ and $\frac{MP_L}{w} = 1$, so use only labor and $K^* = 0$. $2L + 4*0 = 40 \implies L^* = 20$.

(b) $\frac{MP_K}{r} = 1/2$ and $\frac{MP_L}{w} = 1/3$, so use only capital and $L^* = 0$. $2*0 + 4K = 40 \implies K^* = 10$.

(c) Our optimality condition is given by

$$\frac{MP_K}{r} = \frac{MP_L}{w} \implies \frac{1/2K^{-1/2}L^{1/2}}{8} = \frac{1/2L^{-1/2}K^{1/2}}{8} \implies K = L$$

Plugging this into our isoquant constraint, we have:

$$40 = K^{1/2}K^{1/2} \implies K^* = 40 \text{ and } L^* = 40$$

(d) $K^* = 40$ and $2L = 40 \implies L^* = 20$

2. Describe in words, why a firm producing with cobb-douglas technology needs to have $MP_k/r = MP_L/w$.

Solution: The marginal product per dollar of each input has to be equal. If one was larger, then you could save money by using more of the input with a larger marginal product of labor and less of the other input.

3. A firm is producing the required amount of output, $\bar{Q}$ units with $MP_k/r = 2$ and $MP_L/w = 4$. Is this firm producing at the lowest-possible cost? If not, explain how the firm could shift between inputs and lower costs.

4. Consider the production function $Q(K, L) = KL^{1/2}$.

   (a) Solve for the conditional input demand functions.

   (b) Are labor and capital normal or inferior inputs?

   (c) In the short-run, the firm's capital is fixed at $\bar{K} = 10$. Find the short-run conditional labor demand function.

   (d) What is the labor demanded to produce $\bar{Q} = 20$ units in the short-run when $\bar{K} = 10$.

   Solution:

   (a) The optimality condtion is given by

   $$\frac{L^{1/2}}{r} = \frac{1/2L^{-1/2}K}{w} \implies K = 2\frac{w}{r}L$$

   Plugging that into the production constraint gives

   $$\bar{Q} = 2\frac{w}{r}L^{3/2} \implies L^*(\bar{Q}, w, r) = \big(\frac{\bar{Q}r}{2w}\big)^{2/3}$$

   Likewise, $K^*(\bar{Q}, w, r) = 2\frac{w}{r}\big(\frac{\bar{Q}r}{2w}\big)^{2/3}$

   (b) They are normal inputs since $\partial L^*/\partial\bar{Q} > 0$ and $\partial K^*/\partial\bar{Q} > 0$.

   (c) $\bar{Q} = 10L^{1/2} \implies L^*(\bar{Q}) = \frac{\bar{Q}^2}{100}$

   (d) $L^*(10) = \frac{10^2}{100} = 1$

Cost Curves

1. Consider the production function $Q(K, L) = K^{1/2}L^{1/2}$.

   (a) Let $w = 2$ and $r = 4$. Solve for the total cost function of producing $\bar{Q}$ units.

   (b) Now, solve for the long-run toral cost curve as a function of $\bar{Q}$, w, and r.

   (c) In the short-run, the firm's capital is fixed at $\bar{K} = 16$. Find the short-run conditional labor demand function.

   (d) Find the short-run total cost curve when $\bar{K} = 16$. Is it true that $TC(\bar{Q}, w, r) \leq SRTC(\bar{Q}, w, r)$?

   (e) In words, explain why the short-run total cost curve has to be larger than the long-run total cost curve.

   Solution:

   (a) The optimality condition is $L = 2K$. Plugging this into the production constraint yields

   $$\bar{Q} = (2K)^{1/2}K^{1/2} = \sqrt{2}K \implies K^* = \bar{Q}/\sqrt{2}$$

   Pluggin K^* back into the optimality condition yields $L^* = \sqrt{2}\bar{Q}$.
   Thus, the total cost is given by

   $$TC(\bar{Q}) = wL^* + rK^* = 2\sqrt{2}\bar{Q} + 4/\sqrt{2}\bar{Q}$$

(b) The optimality condition is $L = \frac{r}{w}K$. Plugging this into the production constraint yields

$$\bar{Q} = (\frac{r}{w}K)^{1/2}K^{1/2} = \sqrt{\frac{r}{w}}K \implies K^* = \sqrt{\frac{w}{r}}\bar{Q}$$

Pluggin K^* back into the optimality condition yields $L^* = \sqrt{\frac{w}{r}}\bar{Q}$.
Thus, the total cost is given by

$$TC(\bar{Q}) = wL^* + rK^* = w\sqrt{\frac{w}{r}}\bar{Q} + r\sqrt{\frac{w}{r}}\bar{Q}$$

(c) The production constraint is given by $\bar{Q} = 16^{1/2}L^{1/2} \implies L^* = \bar{Q}^2/16$

(d) The short-run cost curve is given by

$$SRTC(\bar{Q}, w, r) = r * 16 + w * \bar{Q}^2/16.$$

(e) The short-run total cost curve must be weakly larger than the total cost curve since you can not chose the optimal amount of capital in the short-run. Therefore, the optimality condition for cost minimization is not likely to hold.

2. Consider the total cost curve $TC(Q) = \frac{Q}{25}\sqrt{wr}$.

   (a) What is the marginal cost of producing the 11th unit of output when $w = 4, r = 4$?

   (b) What is the firm's average total cost of producing 40 units when $w = 2$ and $r = 8$? Interpret this in words

   (c) Does this firm experience economies of scale?

Solution:

(a) $MC(Q) = \frac{\sqrt{wr}}{25}$ which when $w = 4$ and $r = 4$ yields $MC(11) = \frac{4}{25} = 0.16$. This means the 11th unit costs 16 cents to make.

(b) $ATC(Q) = TC(Q)/Q = \frac{\sqrt{wr}}{25}$. When $w = 2$ and $r = 4$, we have $ATC(40) = \frac{4}{25} = 0.16$. This means the first 40 units cost on average 16 censs to make.

(c) Economies of scale is when $\partial ATC(Q)/\partial Q < 0$. In this case, the ATC is constant, so $\partial ATC(Q)/\partial Q = 0$. Therefore, the firm does not experience economies of scale.

3. Consider the total cost curve $TC(Q) = Q^2 - 2Q + 10$.

   (a) What is the average total cost curve, $ATC(Q)$ and the marginal cost curve, $MC(Q)$?

   (b) What is the quantity where the firm is producing at the minimum efficient scale?

   (c) When is the firm experiencing economies of scale and diseconomie of scale? (*Hint:* use the previous question)

   (d) In the short run, the firm can not change the amount of labor they use because of worker's contracts. What is the relationship between the short-run marginal cost and the long-run marginal cost?

Solution:

(a) $MC(Q) = 2Q - 2$ and $ATC(Q) = TC(Q)/Q = Q - 2 + 10/Q$.

(b) Minimum efficient scale is when $MC(Q^{MES}) = ATC(Q^{MES})$. This implies

$$2Q - 2 = Q - 2 + 10/Q \implies Q = 10/Q \implies Q^2 = 10 \implies Q^{MES} = \sqrt{10}$$

(c) The firm is experiencing economies of scale when $Q < \sqrt{10}$ and diseconomies of scale when $Q > \sqrt{10}$.

(d) In the short run, the firm's short-run marginal cost must be weakly greated than the long-run marginal cost.