CU Denver Math Camp - Matrix Algebra

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Matrix Fundamentals

Transformations and their Inverses

Inverse of a Matrix

Span and Range of Matrices

Application 1: System of Equations

Application 2: Linear Regression

Matrix Fundamentals

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

- A matrix is a rectangular array of numbers.
- Size/Dimension: (rows) $\times$ (columns). E.g. A is a 2×3 matrix.
  $\rightarrow$ I remember by "Row your boat".
- The element in row i and column j is referruby to as a_{ij} or A_{ij}.

Matrix Addition and Subtraction

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

- Dimensions must match:

$$(r \times c) \pm (r \times c) \implies (r \times c)$$

- A and B are both 2×3 matrices, so we can add and subtract them:

$$A \pm B = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & a_{13} \pm b_{13} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & a_{23} \pm b_{23} \end{bmatrix}$$

Scalar Multiplication

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

- For any scalar c (a number like 2 or -4):

$$cA = \begin{bmatrix} c * a_{11} & c * a_{12} & c * a_{13} \\ c * a_{21} & c * a_{22} & c * a_{23} \end{bmatrix}$$

Matrix Multiplication

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \text{ and } D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \\ d_{31} & d_{32} \end{bmatrix}$$

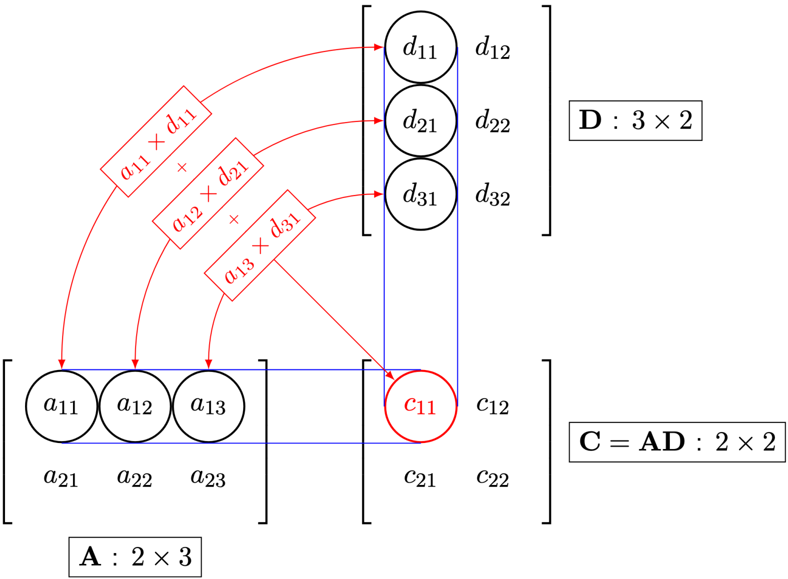
- Inner Dimensions must match:

$$(r \times \underline{c}) \times (\underline{c} \times p) \implies (r \times p)$$

- A is a 2×3 and D is a 3×2 matrix, so we can multiply (the 2s are equal):

$$A \times D = \begin{bmatrix} a_{11}d_{11} + a_{12}d_{21} + a_{13}d_{31} & a_{11}d_{12} + a_{12}d_{22} + a_{13}d_{32} \\ a_{11}d_{11} + a_{12}d_{21} + a_{13}d_{31} & a_{11}d_{12} + a_{12}d_{22} + a_{13}d_{32} \end{bmatrix}$$

Matrix Multiplication



Matrix Multiplication Practice

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 3 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 2 & 2 \end{bmatrix}$$

- What is $A \times B$? What is $B \times A$?

Transpose

$$A = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix}$$

- The tranpose of an $n \times m$ matrix A, labelled A^T or A', is a $m \times n$ matrix, where the columns in A are the rows in A^T.

$$A^T = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \end{bmatrix}$$

Transpose Practice

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 3 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 2 & 2 \end{bmatrix}$$

- What is the transpose of A and B?

Variance Covariance Matrix

- Consider a matrix of variable where each column is a (de-meaned) sample.

$$A = \begin{bmatrix} x_1 - \bar{x} & y_1 - \bar{y} & z_1 - \bar{z} \\ x_2 - \bar{x} & y_2 - \bar{y} & z_2 - \bar{z} \\ \vdots & \vdots & \vdots \\ x_n - \bar{x} & y_n - \bar{y} & z_n - \bar{z} \end{bmatrix},$$

where $\bar{x}$ is the mean of variable x.

Variance Covaraince Matrix

- The Variance-Covariance Matrix is $A^T A =$

$$\begin{bmatrix} \sum_{i=1}^{n}(x_i-\bar{x})^2 & \sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y}) & \sum_{i=1}^{n}(x_i-\bar{x})(z_i-\bar{z}) \\ \sum_{i=1}^{n}(y_i-\bar{y})(x_i-\bar{x}) & \sum_{i=1}^{n}(y_i-\bar{y})^2 & \sum_{i=1}^{n}(y_i-\bar{y})(z_i-\bar{z}) \\ \sum_{i=1}^{n}(z_i-\bar{z})(x_i-\bar{x}) & \sum_{i=1}^{n}(z_i-\bar{z})(y_i-\bar{y}) & \sum_{i=1}^{n}(z_i-\bar{z})^2 \end{bmatrix}$$

$$= \begin{bmatrix} Var(x) & Cov(x,y) & Cov(x,z) \\ Cov(y,x) & Var(y) & Cov(y,z) \\ Cov(z,x) & Cov(z,y) & Var(z) \end{bmatrix}$$

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Vectors

- Vectors are matrices with only one row or column. For example,the column vector:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- You can think of vectors as being a line in $\mathbb{R}^n$.
- For example, any point on x-y plane can be written as a 2×1 vector:

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Matrix Times a Vector (Transformations)

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- An $n \times n$ matrix, A, times a $n \times 1$ vector, x, is a transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. So A takes x, rotates it around and/or shrinks or extends the line.
- In general,

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} \in \mathbb{R}^2$$

Identity Matrix (A Special Transformation)

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

- This matrix has a special property. It is the matrix equivalent to 1.
- If you multiply a matrix of "conformable" size with the identity matrix, it returns the original matrix.
- Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 * 2 + 0 * 3 \\ 0 * 2 + 1 * 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Transformation Examples

- Reflection on the Y-axis:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

- Reflection 90 degrees clockwise:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix}$$

- Enlargement by scale factor a in the x direction and scale factor b in the y direction:

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ by \end{bmatrix}$$

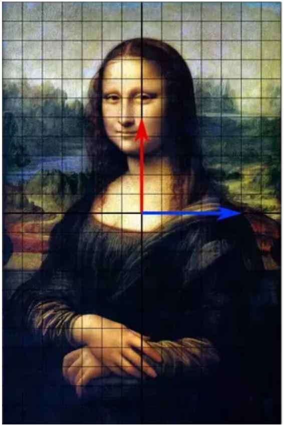
Combination of Transformations

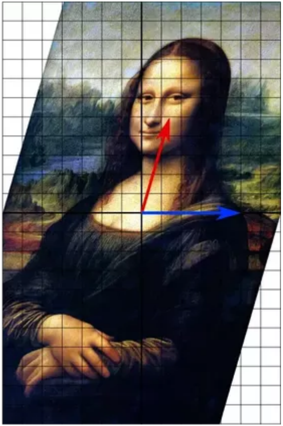
- Let's say I want to rotate a vector 90 degrees clockwise and then keep only the x direction (i.e. scale the y by 0.)
- I just multiply the matrices in the order I want to do them:

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{\text{Keep only } x} \overbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}^{\text{Rotate 90 degrees clockwise}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ -x \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix}$$

Determinant of a Matrix

- When we rotate and scale an image, we are just doing many many vectors times a transformation matrix. The determinant asks how much does the area change with out transformation:





Formula for 2×2 Determinant

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

- The Determinant of A is given by:

$$det(A) = a_{11} * a_{22} - a_{12} * a_{21}$$

- If $det(A) = 1$, then the transformation preserves area
- If $det(A)$ is greater than/smaller than 1, then the transformation grows/shrinks area.
- If $det(A) = 0$, then the transformation area shrinks to zero (you lose a dimension).

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Inverse of a Matrix

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Inverse of Matrix

- A square matrix, A, (i.e., dimension $n \times n$) has an inverse *if and only if* there exists an $n \times n$ matrix X such that $AX = I_n$ and $XA = I_n$. We label X as A^{-1}.
- The inverse of a matrix "undoes" the transformation done by A, i.e $AA^{-1}x = A^{-1}Ax = x$.
- If the determinant of a matrix is 0, then the transformation does not have an inverse. For example, the matrix that only keeps the x component can't be inverted (what is the correct y value?)

Inverse of 2x2 Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

- For $2x2$ matrices, there is a nice formula for the inverse:

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

- The fraction is 1 over the determinant of the matrix A.

Inverse Example

$$A = \begin{bmatrix} 2 & 4 \\ -4 & 10 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

- Find the inverse of A and verify it is indeed the inverse of A.
- What is the inverse of B? Does the inverse of B exist?

Verifying inverse matrices

- The formula for inverses get much more complicated and computers can do it much more easily, so we will instead just learn how to verify a matrix and its inverse. Show that the following matrices are inverses.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 2 & 4 \\ 6 & 8 & 0 \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} -8 & 6 & \frac{3}{2} \\ 6 & -\frac{9}{2} & -1 \\ -3 & \frac{5}{2} & \frac{1}{2} \end{bmatrix}$$

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Column Span

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- The other way to think of Ax is as a linear combination of the columns of A.

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} \in \mathbb{R}^2.$$

Column Span

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 3 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 3 & 6 \end{bmatrix}$$

- We can ask what elements of $\mathbb{R}^3$ can linear combinations of A make? We call this set the column span of A.

$$\text{span}(A) = \left\{ a \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} | a, b \in \mathbb{R} \right\} \subset \mathbb{R}^3$$

Linear Dependence

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 3 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 3 & 6 \end{bmatrix}$$

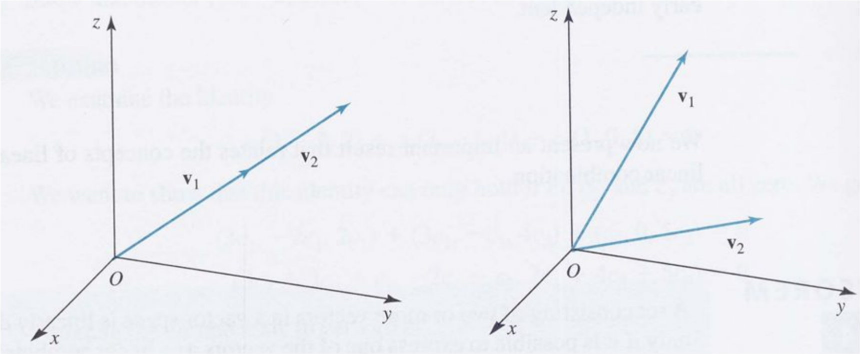
- Two columns are linearly dependent if one can be written as a scalar multiple of the other one.
- The columns of A are linearly independent because $a \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 3a \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ has no solution ($a = 1$ and $3a = 0$).
- The columns of B are linearly dependent because $2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

Range of a Matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 3 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 3 & 6 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- span$(C) = \left\{ a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} | a \in \mathbb{R} \right\}$ is a line along the x-axis.
- span$(A) = \left\{ a \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} | a, b \in \mathbb{R} \right\}$ is a plane in $\mathbb{R}^3$.
- span(B) is a line because the columns are linearly dependent.

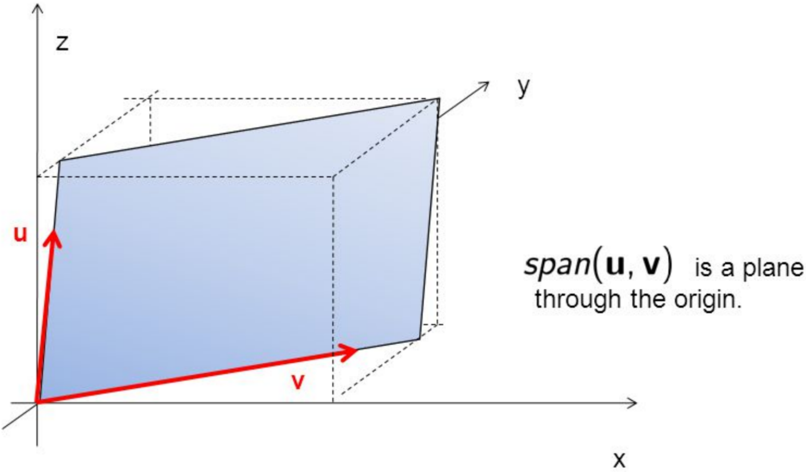
Linearly Dependent



$\{\mathbf{v}_1, \mathbf{v}_2\}$ linearly dependent; vectors lie on a line

$\{\mathbf{v}_1, \mathbf{v}_2\}$ linearly independent; vectors do not lie on a line

Column Span



Column Rank of a Matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 3 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 3 & 6 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- The column rank of a matrix is the number of linearly independent columns in a matrix (if two vectors are linearly dependent you only count one).
- rank$(A) = 2$, rank$(B) = 1$, and rank$(C) = 1$.

Finding Column Rank of a Matrix Practice

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 9 \\ 0 & 0 & 0 \end{bmatrix}$$

Span Example

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 0 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Do matrices A, B, and C span $\mathbb{R}^3$?

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System of Equations to Matrix

$$3x + y = 5$$

$$2x - y = 0$$

- You can write system of equations in a matrix form:

$$\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Solving Systems of Equations using Matrices

$$\underbrace{\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}}_{\equiv A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

- By multiplying the matrix by its inverse, we can solve for x and y:

$$A^{-1}A \begin{bmatrix} x \\ y \end{bmatrix} = I_2 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Solving System of Equations using Matrices

Solve the following:

$$\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

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Linear Regression

- In econometrics, we run a lot of regressions. We have a matrix of covariates X and an outcome variable y.
- Our basic model is:

$$y = X\beta + \epsilon$$

, where y is an $n \times 1$ vector, X is an $n \times k$ matrix of k variables, β is a $k \times 1$ vector of coefficients, and ϵ is an $n \times 1$ vector of unobserved effect on y (sum of the effect of other variables that affect y)

$$X\beta = \begin{bmatrix} \underbrace{\vec{x_1}}_{n\times 1} & \cdots & \underbrace{\vec{x_k}}_{n\times 1} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_k \end{bmatrix}$$

Projection Matrix

Our regression coefficient is $\hat{\beta} = (X^T X)^{-1} X^T y$ which is a matrix transformation applied to the vector y.

- When we want to predict y, we have $\hat{y} = X \underbrace{(X^T X)^{-1} X^T y}_{k \times 1}$ which is a linear combination of the columns of X.
- That is, the predicted values from a linear regression are just the $\vec{x_1}\hat{\beta}_1 + \cdots + \vec{x_k}\hat{\beta}_k$
- We call the matrix $X(X^T X)^{-1} X^T$ the projection matrix because it takes any vector and projects it to the span of X.