Topic 4: Multiple Regression Analysis

ECON 4753 — University of Arkansas

Prof. Kyle Butts

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Multiple Variable Regression

Polynomials

Interactions

Multiple Regression

The multiple variable linear regression model is written as

$$y_i = \beta_0 + X_{1,i}\beta_1 + \cdots + X_{K,i}\beta_K = u_i$$

There are K covariates that we want to *jointly* estimate their associations with y

- Note these don't have to be 'distinct' variables, but could be polynomials, binned data, interactions between variables, etc.

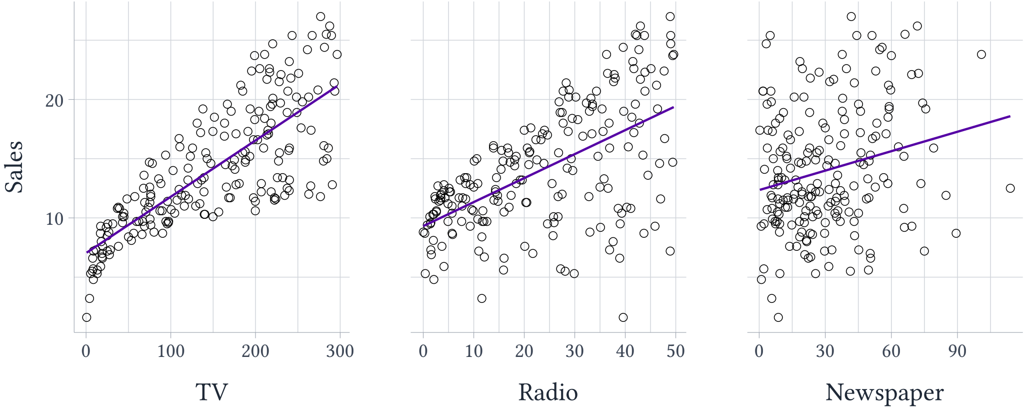
Why Multiple Variable Regression?

Advertising Example

Let's given an example. Say you're a business and you want to use advertising to boost sales. You have a bunch of different markets (e.g. cities) and you have data on how you've spent your advertising budget in those markets and the sales in that market

Advertising Example

Single-variable predictors



Advertising Example

We see that sales are higher in markets that have spending on TV, Radio, and Newspaper ads separately.

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- Are there synergies between different advertising strategies (are they substitutes or complements to one another)?
- Do places with more TV ads also have more radio ads? Then how can we tell if it is TV ads that are helping or if it is really radio ads?

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Key takeaway: Forecasting models get better the more carefully you think about the context you are in

Intuition of multiple linear regression

> *Do places with more TV ads also have more radio ads? Then how can we tell if it is TV ads that are helping or if it is really radio ads ?*

Multiple regression will help with this. To estimate the relationship between sales and radio ads, what we really want to do is compare places with *the same TV ads* but different amounts of radio ads and see how their sales differ

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- The language typically used by economists is ‘controlling for TV ads, what is the relationship between radio ads and sales?’

Correlation between X

More generally, we want to use multiple regression when multiple X variables are correlated with one another. Say $X_{k,i}$ and $X_{\ell,i}$ are correlated with each other.

- In the data, when we view someone with higher $X_{k,i}$, they also have higher $X_{\ell,i}$ (or lower in the case of negative correlation)

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- In the data, when we view someone with higher $X_{k,i}$, they also have higher $X_{\ell,i}$ (or lower in the case of negative correlation)

Multiple regression will let us understand the relationship between $X_{k,i}$ and Y while "keeping $X_{\ell,i}$ constant"

Marginal Effects with Multiple Variables

Say we have two variables in our linear model $\hat{y} = \beta_0 + \beta_1 X_1 + \beta_2 + X_2$.

Our predictions differ by

$$\hat{y}_j - \hat{y}_i = \beta_1(X_{1,j} - X_{1,i}) + \beta_2(X_{2,j} - X_{2,i})$$

Marginal Effects with Multiple Variables

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Let's think about a simple version. Take two individuals with the same X_2 and X_1 differing by 1 unit (say $X_{1,j} - X_{1,i} = 1$)

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Then, our estimated change is

$$\Delta\hat{y} = \beta_1$$

- We refer to the β_k as "marginal effects", i.e. the predicted change in y from a 1 unit increase in X, holding *fixed* all the other variables

Least Squares problem

How do we estimate the least-squares coefficients? Our goal, just like before is to minimize the mean-squared prediction error:

$$\text{MSPE}(\boldsymbol{b}) = \frac{1}{n}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 X_{1,i} - \cdots - \beta_K X_{K,i})^2$$

- Searching over $b = (b_0, b_1, \ldots, b_K)'$

First-order conditions

Denote $\hat{u}_i$ to be the difference between y_i and $b_0 + b_1 X_{1,i} + \cdots + b_K X_{K,i}$

Our first-order conditions become:

1. $b_0 = \bar{y} - \bar{X}_1 b_1 + \ldots \bar{X}_K b_K$
2. $0 = \sum_{i=1}^{n} X_{k,i} \hat{u}_i$ for $k = 1, \ldots, K$

In words, 1. the intercept makes the mean values fall on the regression line and 2. each X variable is uncorrelated with the residual

Solving multiple regression estimates

These $K+1$ equations can be solved, but less simply than before. Instead we end up with the following solution:

$$(\text{var}\left(\boldsymbol{X}_i\right))^{-1}\,\text{cov}(\boldsymbol{X}_i, Y_i)$$

where:

1. $\text{cov}(\boldsymbol{X}_i, Y_i)$ is the $K+1$ vector of covariances between each $X_{k,i}$ including the intercept, and
2. $\text{var}\left(\boldsymbol{X}_i\right)$ is the $K+1 \times K+1$ matrix with typical element $\text{cov}(X_{k,i}, X_{\ell,i})$

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What is happening

$$(\mathrm{var}\,(\boldsymbol{X}_i))^{-1}\,\mathrm{cov}(\boldsymbol{X}_i, Y_i)$$

Admittedly, this equation is a bit intimidating. But the intuition is that:

1. $\mathrm{var}\,(\boldsymbol{X}_i)$ notices that some variables are correlated with each-other (the two variables co-move)
2. So by inverting this matrix, we are essentially removing this co-movement, aka holding “all else equal”

Dependent Variable:	Sales			
Constant	7.033***	9.312***	12.35***	2.939***
	(0.4578)	(0.3882)	(0.6338)	(0.3365)
TV	0.0475***			0.0458***
	(0.0027)			(0.0019)
Radio		0.2025***		0.1885***
		(0.0217)		(0.0108)
Newspaper			0.0547***	-0.0010
			(0.0186)	(0.0064)
R^2	0.61188	0.33203	0.05212	0.89721

Signif. Codes: ***: 0.01, **: 0.05, *: 0.1

Interpreting regression results

Notice on the last table, in the bivariate regression, higher newspaper ads spending is associated with higher sales

But, after controlling for TV and Radio ads, there is no relationship between Newspaper ads and sales

- It seems like the bivaraite relationship is being driven by newspaper ads being correlated with TV and Radio sales. Holding them fixed removed the relationship between Newspaper ads and sales

Inference on regression coefficients

Conducting inference on regression coefficients is the same as before

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- We get point estimates and standard errors just as before

However, the standard errors are going to depend on other covariates.

- Whenever the covariate of interest is highly correlated with other variables, the standard errors on our estimate will grow
  - $\rightarrow$ there is not much independent variation left after controlling for the other variables (so our estimates are noisier)

Multicollinearity and Standard Errors

Consider a regression of SAT scores among middle and high-school students. Imagine regressing SAT score on the age of the student and the student's grade

Age and a student's grade are highly correlated, so it's very hard to distinguish between the two

$\implies$ the standard errors on each will be very large

“All Else Equal”

The latin word you’ll sometimes see is *ceteris parabus* meaning “other things being equal”.

For example, think about when we learned about the demand curve

- The realtionship between price and quantity demanded *at a point in time*
- You are changing the price of the good but not changing the state of the economy

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- The realtionship between price and quantity demanded *at a point in time*
- You are changing the price of the good but not changing the state of the economy

In our data, when we see prices change, this is usually driven by changes in the economy (e.g. inflation or supply-chain shocks)

- Not ceterus parabus!

Causal Effects and Multiple Regression

Once again, we face the issue of interpreting regressions causally. Say you want to estimate the causal effect of college attendance on wages. You control for a bunch of other determinants of earnings that might correlate with college attendance (e.g. family income and SAT scores)

Causal Effects and Multiple Regression

Once again, we face the issue of interpreting regressions causally. Say you want to estimate the causal effect of college attendance on wages. You control for a bunch of other determinants of earnings that might correlate with college attendance (e.g. family income and SAT scores)

You may be tempted to say that you have isolated the “causal effect” of gender on wages, i.e. the effect of sexism on wages

- However, it could be the case that other *unobservable* variables are correlated with college attendance that you did not control for in your regression (e.g. work-ethic and parental investment)

Causal Effects and Multiple Regression

The upshot: causal effects require *all else equal*, not just the things you control for

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Quadratic terms

Say you have the following model with wages as a quadratic function of age

$$w_i = \beta_0 + \beta_1 \text{age}_i + \beta_2 \text{age}_i^2 + u_i$$

Before we were discussing the idea of changing one variable, while holding the rest "equal"

- How does one change age without changing age^2?

Marginal Effects

Really what we were doing was writing out the predicted value as

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_{1,i} + \cdots + \hat{\beta}_K X_{K,i} + u_i$$

Then the marginal effect of the ℓ-th variable was the change in $\hat{y}_i$ when changing $X_{\ell,i}$ but holding the rest of the variables *fixed*

- This is exactly what the partial-derivative tells us

Marginal Effects

That is, our marginal effect was $\frac{\partial \hat{y}_i}{\partial X_{\ell,i}}$ evaluated at the original covariate values: $X_{1,i}, \ldots, X_{K,i}$.

Marginal effects with quadratic terms

$$\hat{w}_i = \hat{\beta}_0 + \hat{\beta}_1 \mathsf{age}_i + \hat{\beta}_2 \mathsf{age}_i^2$$

From calculus, we know that the partial derivative of $\hat{w}_i$ with respect to age_i is given by

$$\frac{\partial \hat{w}_i}{\partial \mathsf{age}_i} = \hat{\beta}_1 + 2\hat{\beta}_2 \mathsf{age}_i$$

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The change in predicted wage of a worker as they age is given by $\hat{\beta}_1 + 2\hat{\beta}_2 \text{age}_i$ which depends on their age

- In words, how much predicted wages change as a worker gets a year older changes over a worker's lifetime

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Why interactions

Wages Example

Consider a model where we want to understand how wages are influenced by both being a female and being a college-educated worker. We can write the model as:

$$w_i = \beta_0 + \beta_1 \text{female}_i + \beta_2 \text{college}_i + \beta_3 (\text{female}_i \times \text{college}_i) + u_i$$

Here, β_3 captures the interaction effect between female and college-education status on wages

- This means that the effect of females on wages may differ depending on whether the worker has a college-degree, and vice versa.

Interactions

Wages Example

Consider the difference in predicted wages for non-college educated male vs. non-college educated workers:

$$w_{i,NC,F} - w_{i,NC,M} = (\beta_0 + \beta_1) - \beta_0 = \beta_1$$

Compare this to the difference in predicted wages for college educated male vs. college educated workers:

$$w_{i,C,F} - w_{i,C,M} = (\beta_0 + \beta_1 + \beta_2 + \beta_3) - (\beta_0 + \beta_2) = \beta_1 + \beta_3$$

Interactions

Wages Example

Wage-gap for college educated workers is $\beta_1 + \beta_3$ and the wage-wagp for non-college educated workers is β_1

- β_3 represents the difference in wage gaps of college-educated vs. non-college-educated workers.

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Wages Example

Wage-gap for college educated workers is $\beta_1 + \beta_3$ and the wage-wagp for non-college educated workers is β_1

- β_3 represents the difference in wage gaps of college-educated vs. non-college-educated workers.

More generally, can interpret interactions how one variable changes the marginal effect of another variable

- This is similar to when you have a quadratic function of X, the marginal effect depends where you are along the distribution of X.

Interactions

Partial Derivative

$$\hat{w}_i = \hat{\beta}_0 + \hat{\beta}_1 \text{female}_i + \hat{\beta}_2 \text{college}_i + \hat{\beta}_3 (\text{female}_i \times \text{college}_i)$$

We can derive this result using partial derivatives:

$$\frac{\partial \hat{w}_i}{\partial \text{female}_i} = \hat{\beta}_1 + \hat{\beta}_3 \text{college}_i$$

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$$\frac{\partial \hat{w}_i}{\partial \text{female}_i} = \hat{\beta}_1 + \hat{\beta}_3 \text{college}_i$$

In this case, it's a little weird to think of "small" changes in the female variable. Instead, we will think of it as a 1 unit change (from 0 to 1)

Continuous interacted with a discrete variable

Let X_i be a continuous variable and D_i be a dummy variable and consider the regression

$$y_i = \beta_0 + D_i\beta_1 + X_i\beta_2 + X_iD_i\beta_3 + u_i$$

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In this case, the marginal effect of X_i is given by $\frac{\partial \hat{y}_i}{\partial X_i} = \hat{\beta}_2 + D_i\hat{\beta}_3$

- The marginal effect for group $D_i = 0$ is given by $\hat{\beta}_2$
- The marginal effect for group $D_i = 1$ is given by $\hat{\beta}_2 + \hat{\beta}_3$

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- The marginal effect for group $D_i = 0$ is given by $\hat{\beta}_2$
- The marginal effect for group $D_i = 1$ is given by $\hat{\beta}_2 + \hat{\beta}_3$
- Therefore, $\hat{\beta}_3$ is the difference in marginal effects between $D_i = 1$ relative to $D_i = 0$

Continuous interacted with a discrete variable

$$y_i = \beta_0 + D_i\beta_1 + X_i\beta_2 + X_iD_i\beta_3 + u_i$$

Exercise:

- In words, how would you interpret a t-test for the null that $\hat{\beta}_2 = 0$?
- In words, how would you interpret a t-test for the null that $\hat{\beta}_3 = 0$?

`mtcars` example

```
OLS estimation, Dep. Var.: mpg
              Estimate Std. Error   t value   Pr(>|t|)
(Intercept) 26.624848   1.346754 19.769644  < 2.2e-16 ***
am::1        5.217653   2.324898  2.244250 3.2904e-02 *
hp          -0.059137   0.008957 -6.602265 3.6781e-07 ***
am::1:hp     0.000403   0.013362  0.030152 9.7616e-01
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Exercise: Does the estimate relationship between being a car's horsepower and miles per gallon depend on whether it is an automatic? How do you know?

Interaction terms always should have ‘main effects’

When including an interaction term, it is important to (almost) always include the main effects

The main effects are the variables by themselves. E.g. if you interact gender with race, you want to include race and gender as separate terms as well

- The main effects are what let us interpret the interaction term as the ‘difference’ in marginal effects

Continuous-Continuous interactions

Now consider two continuous variables being interacted:

$$y_i = \beta_0 + X_{1,i}\beta_1 + X_{2,i}\beta_2 + X_{1,i}X_{2,i}\beta_3 + u_i$$

This is common when you think there are complementarities between variables

- E.g. y is crop-yield, X_1 is the amount of fertilizer applied, and X_2 is the amount of water given. Does it help to do more of both (complements)
- y is a measure of job performance, X_1 is a measure of experience, and X_2 is a measure of training

Continuous-Continuous interactions

$$\frac{\partial \hat{y}_i}{\partial X_{1,i}} = \hat{\beta}_1 + X_{2,i}\hat{\beta}_3 \text{ and } \frac{\partial \hat{y}_i}{\partial X_{2,i}} = \hat{\beta}_2 + X_{1,i}\hat{\beta}_3$$

Can interpret it in two ways:

1. The marginal effect of X_1 grows/shrinks with the value of X_2 (depending on the sign of $\hat{\beta}_3$)