Introduction to Forecasting

ECON 5753 — University of Arkansas

Prof. Kyle Butts

April 2025

Forecasting

Goals of Forecasting

Fitting Models

- Model Selection: Adveristing Example

Sample Distribution

Types of Data

Problem of Prediction

The goal of forecasting is to learn the relationship between input variables and outcome variable(s) so that we can predict the outcome variables when we do not observe it.

Problem of Prediction

The goal of forecasting is to learn the relationship between input variables and outcome variable(s) so that we can predict the outcome variables when we do not observe it.

E.g., Learn about who are potential customers to advertise to based on their observable characteristics

→ Input: observable characteristics

→ Outcome: whether they purchase a product

Problem of Prediction

The goal of forecasting is to learn the relationship between input variables and outcome variable(s) so that we can predict the outcome variables when we do not observe it.

E.g., Learn about who are potential customers to advertise to based on their observable characteristics

$\rightarrow$ Input: observable characteristics

$\rightarrow$ Outcome: whether they purchase a product

Predict values of a variable in the future, e.g. time-series of stock prices

$\rightarrow$ Input: the time-period

$\rightarrow$ Output: stock price

Model

We have an outcome variable y and a set of p different predictor variables $X = (X_1, X_2, \dots, X_p)$.

$\rightarrow$ For some observations we observe both X and Y

- Essential to *fit* the model, i.e. learn the relationship between the two

We can write the model in a general form as

$$y = f_0(X) + \varepsilon,$$

where f_0 is some unknown function of X and ε is the error term.

Model

$$y = f_0(X) + \varepsilon,$$

Here we think of $f_0(X)$ as the 'true' model: this is the best prediction of y given information on X. In this case we assume $\mathbb{E}[y - f_0(X) \mid X] = \mathbb{E}[\varepsilon \mid X] = 0$

$\rightarrow$ Once we know $f_0(X)$, we can not predict any more of the variation in y using X

Prediction of f

We will use a training sample of data to predict f_0. We will denote our estimate of f_0 as $\hat{f}$.

For a given unit, we predict

$$\hat{y} = \hat{f}(X)$$

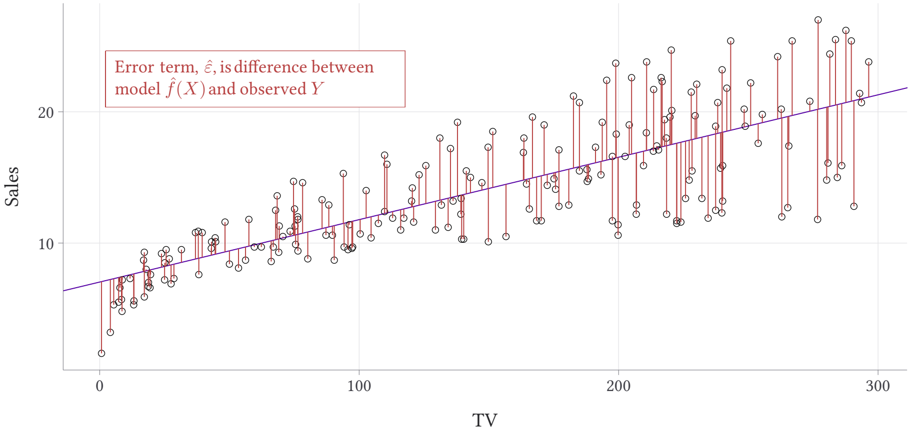
Error term

The term 'error term' is a bit overloaded. It's worth trying to clarify:

1. If we knew the 'true' model, $f_0(X)$, then $\varepsilon \equiv y - f_0(X)$ represents the things that are unpredictable given X
   - This is the "true" error term

2. If $\hat{f}(X)$ is an estimated model, $\hat{\varepsilon} = y - \hat{f}(X)$ is the difference between that unit's y_i and their predicted y, $f(X_i)$
   - This is better called the prediction error

Error term


Error term, $\hat{\varepsilon}$, is difference between model $\hat{f}(X)$ and observed Y
Sales
20
10
0
100
200
300
TV

Goals of estimation of f

There are two related goals when predicting f:

1. Predict y as well as possible (prediction)
   - Think of prediction as a 'black box' where the goal is to do as good of a job at predicting y as possible
   - Want to learn the 'true' $f_0(X)$, leaving no information on the table

2. Understand the relationship between X and y (inference)
   - If our goal is being able to describe the relationship between x and y; i.e. we care about understanding f_0 and not just $\hat{y}$
   - Worth approximating f_0 with a more 'simple' functional form so that we can better convey the results to stakeholders.
     - Creates "heuristics" that helps decision-makers

Examples of Prediction vs. Inference

Say we are thinking about housing:

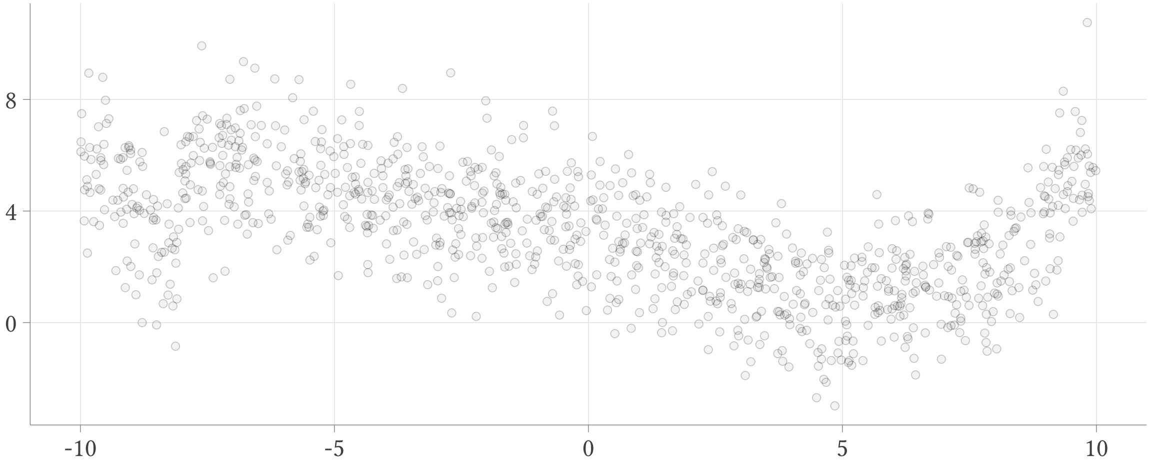
$\rightarrow$ Example of prediction: "Is the house over/under-priced"

- Only care about $\hat{\text{price}} = f(X)$

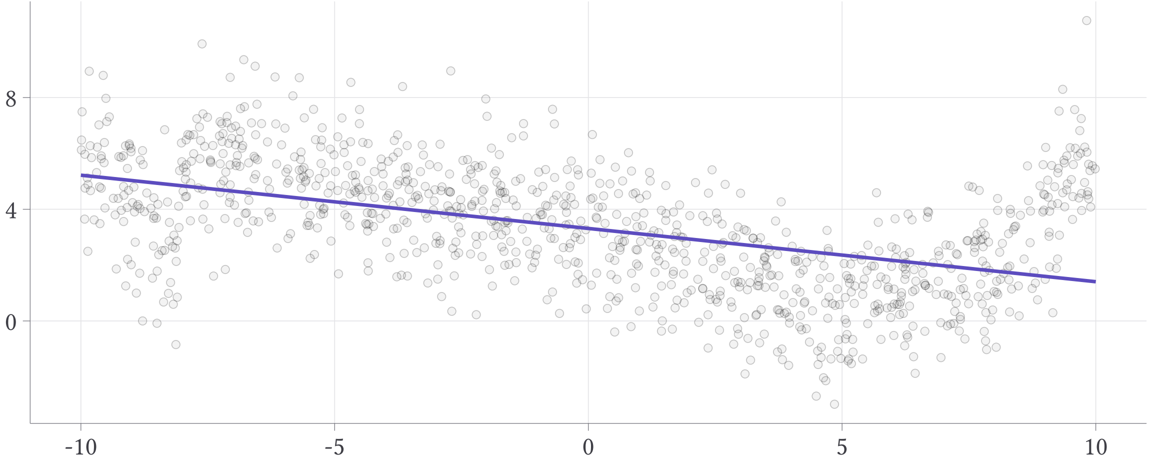
$\rightarrow$ Example of inference: "How much more do homes with a river view sell for?"

- Need to know information about f to answer

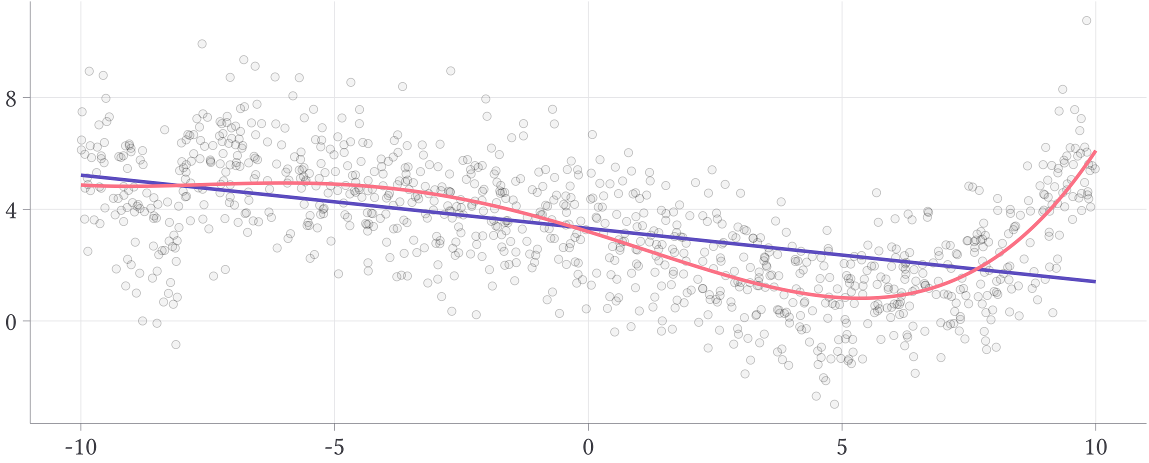
Examples of $\hat{f}$:



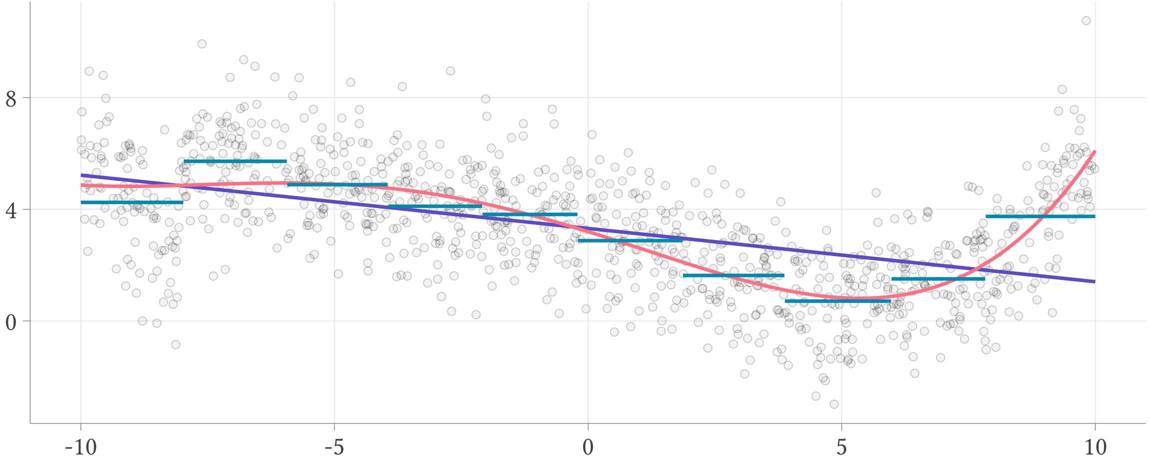
Examples of $\hat{f}$: Line



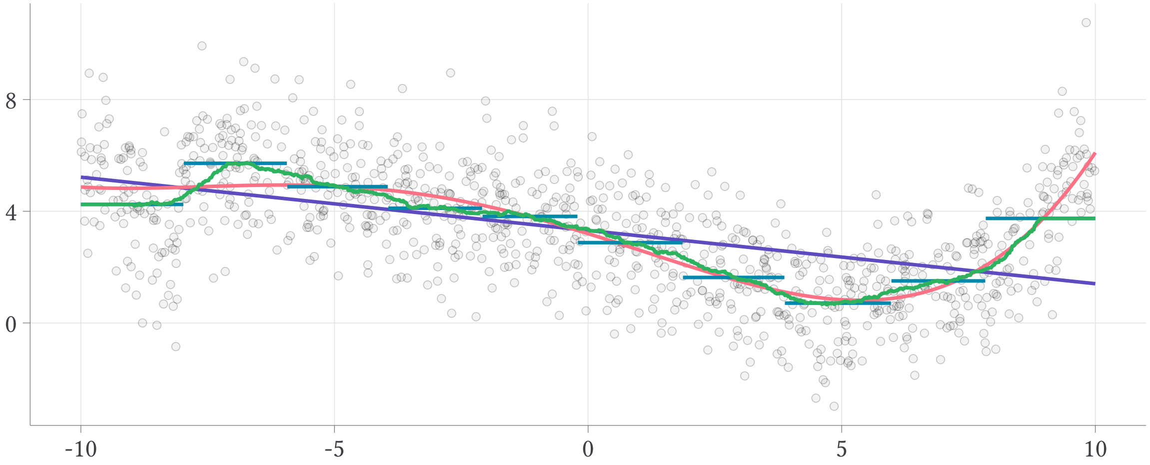
Examples of $\hat{f}$: Line, Polynomial (x^4)



Examples of $\hat{f}$: Line, Polynomial (x^4), Bins of x



Examples of $\hat{f}$: Line, Polynomial (x^4), Bins of x, KNN of x



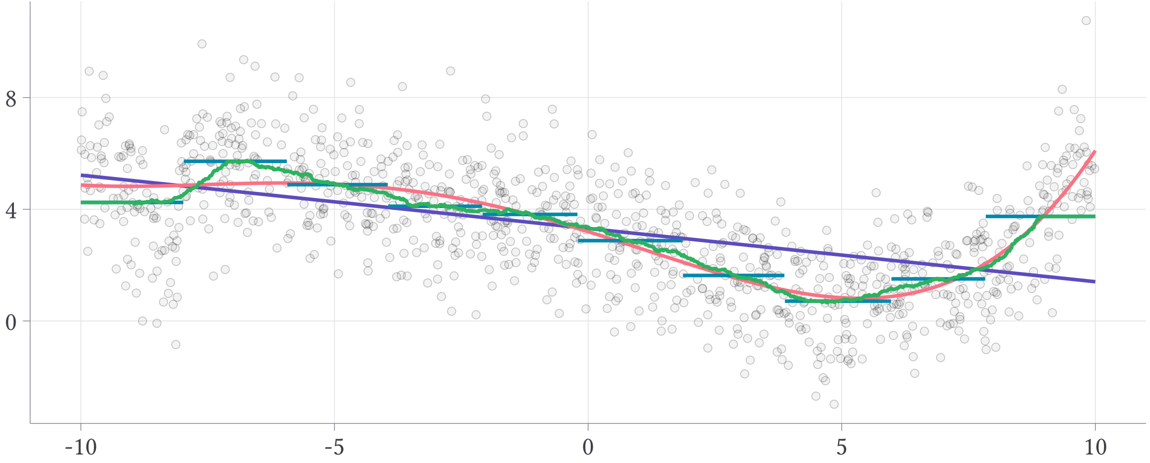
Model Flexibility

There is a limit to how flexible we can make our model

1. If our goal is prediciton, we only have a finite amount of data to use to fit the model, so there's a limit on how much we can learn
   - Face the risk of overfitting the data (chasing after the random noise ε)

2. If our goal is inference, then added flexibility is harder to summarize to stakeholders.

Examples of $\hat{f}$: Line, Polynomial (x^4), Bins of x, KNN of x



High-level of fitting models

The general framework for forecasting is as follows:

1. Collect a set of data $\{(y_i, X_i)\}_{i=1}^n$ for a set of observations.

2. Using knowledge of the topic, select a class of models $\mathcal{F}$ that you want to select from
   - That is, $\hat{f}$ must be one of the functions in $\mathcal{F}$, e.g. linear functions of X_i

3. Using your data, select $\hat{f} \in \mathcal{F}$ that *predicts* y "best"
   - "best" is defined by a loss function, e.g. mean-squared prediction error

Selecting class of models

Let's break this down. First, we have to select a class of models $\mathcal{F}$

$\rightarrow$ This involves selecting variables we want to include in the model

$\rightarrow$ Specifying a functional form for the model

Selecting class of models

Let's break this down. First, we have to select a class of models $\mathcal{F}$

$\rightarrow$ This involves selecting variables we want to include in the model

$\rightarrow$ Specifying a functional form for the model

For example, we might think about a simple *linear model* of our X variables:

$$f(X_i) = \alpha + X_{i1}\beta_1 + X_{i2}\beta_2$$

$\mathcal{F}$ would consist of all the models of this form, i.e. we are selecting over values of $(\alpha, \beta_1, \beta_2)$

title

For example, we might think about a simple *linear model* of our X variables:

$$f(X_i) = \alpha + X_{i1}\beta_1 + X_{i2}\beta_2$$

This is a restrictive model:

$\rightarrow$ no polynomials of X_1 (e.g. wages are non-linear in age)

$\rightarrow$ no interaction between X_1 and X_2 (e.g. college degree changes return to experience)

Selecting class of models

Perhaps, we want to look within a wider class of $\mathcal{F}$:

$$f(X_i) = \alpha + X_{i1}\beta_1 + X_{i1}^2\beta_2 + X_{i2}\beta_3 + X_{i2}^2\beta_4 + X_{i1}X_{i2}\beta_5 + u_i$$

$\mathcal{F}$ consists of all functions of this form

$\rightarrow$ some of the coefficients can be 0, so this class is *strictly* more general than the last.

Selecting class of models

Perhaps, we want to look within a wider class of $\mathcal{F}$:

$$f(X_i) = \alpha + X_{i1}\beta_1 + X_{i1}^2\beta_2 + X_{i2}\beta_3 + X_{i2}^2\beta_4 + X_{i1}X_{i2}\beta_5 + u_i$$

$\mathcal{F}$ consists of all functions of this form

$\rightarrow$ some of the coefficients can be 0, so this class is *strictly* more general than the last.

Can imagine creating a bunch more terms or having things other than quadratics

$\rightarrow$ e.g. $\mathbb{1}[20 < \text{Age}_i \leq 30]$

Prediction Error

We want to be able to evaluate which model in our selected class, $\mathcal{F}$, does the “best” job at predicting y.

Given a model $f \in \mathcal{F}$, we want to evaluate how good our model does at predicting observations y. For this, define the prediction error as

$$\hat{\varepsilon} = \underbrace{y}_{\text{true value}} - \underbrace{f(X)}_{\text{predicted value}}$$

$\rightarrow$ The prediction error depends on the choice of model $f \in \mathcal{F}$

Prediction Error

$$\hat{\varepsilon} = \underbrace{y}_{\text{true value}} - \underbrace{f(X)}_{\text{predicted value}}$$

Large $\hat{\varepsilon}$ means you did a poor job of predicting that observation. That could be because of

1. Reducible errors: The model is bad at predicting y, i.e. $f(X) \neq f_0(X)$
2. Irreducible errors: Or, the true noise ε is making y far away from the systematic component $f(X)$ for this observation

Prediction Error

We can rewrite our prediction error as

$$
\begin{aligned}
\hat{\varepsilon} &= y - f(X) \\
&= f_0(X) + \varepsilon - f(X) \\
&= \underbrace{f_0(X) - f(X)}_{\text{reducible}} + \underbrace{\varepsilon}_{\text{irreducible}}
\end{aligned}
$$

Remember: we do not know f_0, so we can not separate the two.

Loss functions

To provide a summary measure of fit, we want to *average* prediction error over many observations. This will find a ‘average’ prediction error

If we took the simple mean of prediction error, positive and negative prediction errors would cancel out

$\rightarrow$ An error of -1 and 1 would be just as bad as -4 and 4.

Loss functions: Mean-squared prediction error

The most common loss-function is the mean-square (prediction) error (MSE):

$$\text{MSE} \equiv \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2 = \frac{1}{n}\sum_{i=1}^{n}\hat{\varepsilon}_i^2 \tag{1}$$

Loss functions: Mean-squared prediction error

The most common loss-function is the mean-square (prediction) error (MSE):

$$\text{MSE} \equiv \frac{1}{n} \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = \frac{1}{n} \sum_{i=1}^{n} \hat{\varepsilon}_i^2 \tag{1}$$

If we collect $\hat{\varepsilon}_i$ as a vector, we can use linear algebra to more simply write it as

$$\text{MSE} = \frac{1}{n} \hat{\varepsilon}' \hat{\varepsilon}$$

Mean-square prediction error

y_i	$\hat{y}_i$	$\hat{\varepsilon}_i$
3.7	4.20	
4.1	4.18	
5.6	5.48	
2.9	3.29	
8.8	8.81	

Calculate mean-square prediction error:

y_i	$\hat{y}_i$	$\hat{\varepsilon}_i$
3.7	4.20	0.5
4.1	4.18	0.08
5.6	5.48	-0.12
2.9	3.29	0.39
8.8	8.81	0.01

Calculate mean-square prediction error:

$$\begin{aligned} \text{MSPE} &= \frac{1}{5}\left(0.5^2 + 0.08^2 + -0.12^2 + 0.39^2 + 0.01^2\right) \\ &= 0.0846 \end{aligned}$$

Loss functions

The mean-squared prediction error is not the only loss-function:

→ The mean-absolute prediction error $\frac{1}{n}\sum_{i=1}^{n}|y_i - \hat{y}_i|$

- Will estimate the *median* of y given X

Loss functions

The mean-squared prediction error is not the only loss-function:

- $\rightarrow$ The mean-absolute prediction error $\frac{1}{n}\sum_{i=1}^{n} |y_i - \hat{y}_i|$
  - Will estimate the *median* of y given X
- $\rightarrow$ Imagine a setting where you're predicting whether someone has a disease; you would want to penalize false-negatives more than false-positives

In-sample vs. Out-of-sample prediction error

As a forecaster, you will fit a model using a set of observations $\{(x_1, y_1), \ldots, (x_n, y_n)\}$. This is called the training data.

We can calculate the in-sample MSE by formula (1) averaging over all observations in the training data.

$\rightarrow$ This tells us how good we do at predicting the data *we trained the model on.*

In-sample vs. Out-of-sample prediction error

If our goal is prediction, we really want to know how the model would predict on *new* observations that we *have not seen before*

$\rightarrow$ It is common to hold out a set of test data that is NOT used for training the model, but just for evaluating it's performance

Why use ‘test data’?

It is common to try and ‘pick’ from a set of models based on how they do at in-sample prediction:

$\rightarrow$ That is, select the model with the smallest *in-sample MSE*.

Why use 'test data'?

It is common to try and 'pick' from a set of models based on how they do at in-sample prediction:

→ That is, select the model with the smallest *in-sample MSE*.

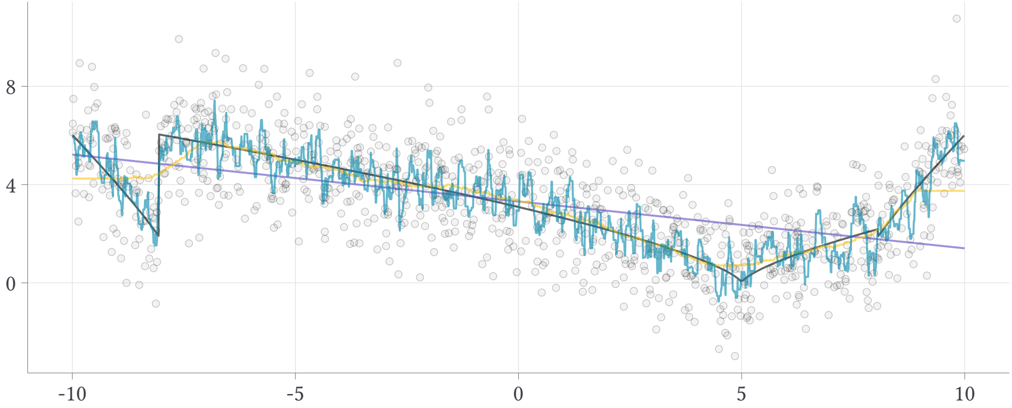
This is *a bad thing to do*; by focusing on fitting the current sample very well, you are risking overfitting the data

Flexibility vs. Overfitting

True $f(x)$

Flexibility vs. Overfitting

True $f(x)$, Line, Somewhat flexible, Highly flexible



Flexibility vs. Overfitting

By making the model more and more *flexible*, you risk overfitting more and more

$\rightarrow$ Your model tries to improve the *in-sample* mean-squared error, but it worstens your *out-of-sample* MSE.

A solution is to evaluate your model fit using outside 'testing data'

Sample-splitting

We will not spend much time in this course discussing sample-splitting/cross-validation and model selection, but I want to give just one example so you're aware of it

$\rightarrow$ A lot of you might recognize this from Hyunseok's class

Sample-splitting

We will not spend much time in this course discussing sample-splitting/cross-validation and model selection, but I want to give just one example so you're aware of it

→ A lot of you might recognize this from Hyunseok's class

Say you want to fit a polynomial, but are concerned with over-fitting. We can tackle this with sample-splitting:

→ Take a random half your data and fit a polynomial of order k

→ Evaluate MSE on the other half your data (test set)

Do this for $k = 1, \ldots, K$ and pick the polynomial degree that minimizes test-data MSE

→ See ILSR section 5.1 for cross-validation

Sample-splitting

This technique is not as common when your model is more simple (e.g. regression model with a few terms)

→ In some sense, you are preventing yourself from overfitting by making the model simple

High-level of fitting models

The general framework for forecasting is as follows:

1. Collect a set of data (y, X) for a set of observations.
2. Using knowledge of the topic, select a class of models $\mathcal{F}$ that you want to select from
   - That is, $\hat{f}$ must be one of the functions in $\mathcal{F}$.
3. Using your data (*perhaps on a training sample*), select $\hat{f} \in \mathcal{F}$ that minimizes the loss function
   - E.g. mean-squared prediction error (*perhaps on testing sample*)

Bias-variance trade-off

This discussion of increasing flexiblity leading to increasing the noise of the model fit is a well-known problem. It is called the Bias-Variance Tradeoff:

1. Bias: When the model we fit, $\hat{f}(x)$, does a poor job fitting the true model $f_0(x)$
2. Variance: The variability of the model we fit, $\hat{f}(x)$, across samples
   - Repeated sampling: the model we estimate varies from estimate to estimate

Bias-variance trade-off

This discussion of increasing flexiblity leading to increasing the noise of the model fit is a well-known problem. It is called the Bias-Variance Tradeoff:

1. Bias: When the model we fit, $\hat{f}(x)$, does a poor job fitting the true model $f_0(x)$
2. Variance: The variability of the model we fit, $\hat{f}(x)$, across samples
    - Repeated sampling: the model we estimate varies from estimate to estimate

This is a 'trade-off'. To lower bias by adding flexibility, you're adding variance (noisiness) to the estimate

Forecasting

Goals of Forecasting

Fitting Models

Model Selection: Adveristing Example

Sample Distribution

Types of Data

Model Selection

Our general approach seems to follow:

→ Select class of models to choose from, $\mathcal{F}$.

→ Find $\hat{f} \in \mathcal{F}$ that minimizes the in- or out-of- sample MSPE

The secret sauce of forecasting is in selecting $\mathcal{F}$

→ E.g. sports teams all have a ton of data to help them draft the best players; all will minimize MSPE to fit their model; all will do sample-splitting; etc.

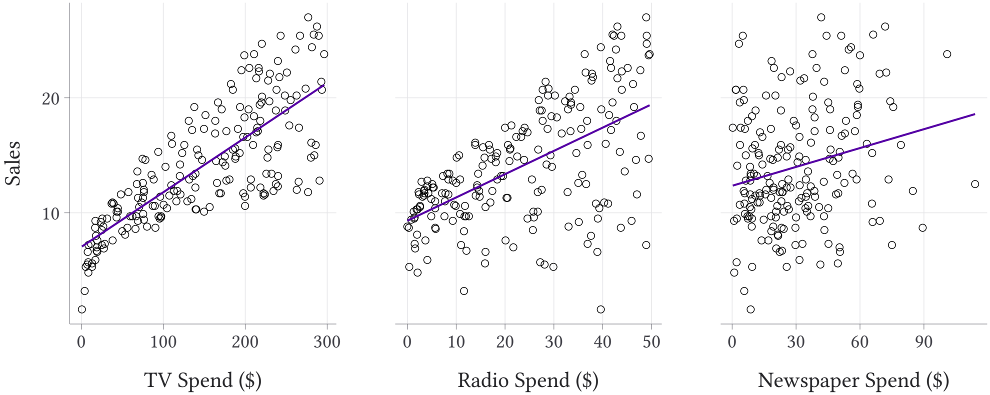
- The advantage is who can pick the best variables to include in their model
- Often called "feature selection"

Model Selection: Advertising Example

Say you’re a business and you want to use advertising to boost sales. You have a bunch of different markets (e.g. cities) and you have data on how you’ve spent your advertising budget in those markets and the sales in that market

Advertising Example

Single-variable predictors



Advertising Example

We see that sales are higher in markets that have spending on TV, Radio, and Newspaper ads separately.

Advertising Example

We see that sales are higher in markets that have spending on TV, Radio, and Newspaper ads separately.

These single scatter plots with line of best-fits are a somewhat poor model:

- $\rightarrow$ Are there synergies between different advertising strategies (are they substitutes or complements to one another)?
- $\rightarrow$ Do places with more TV ads also have more radio ads? Then how can we tell if it is TV ads that are helping or if it is really radio ads

Advertising Example

We see that sales are higher in markets that have spending on TV, Radio, and Newspaper ads separately.

These single scatter plots with line of best-fits are a somewhat poor model:

→ Are there synergies between different advertising strategies (are they substitutes or complements to one another)?

→ Do places with more TV ads also have more radio ads? Then how can we tell if it is TV ads that are helping or if it is really radio ads

Key takeaway: Forecasting models get better the more carefully you think about the context you are in

Interactions Matter

Over the next few weeks, we will learn a lot about regression methodology. We will do so for a set of covariates, X_i.

$\rightarrow$ These could be a set of variables like age, height, batting average, etc.

$\rightarrow$ But, these could also be functions of variables, e.g. $\mathbb{1}[\text{Age}_i = 21]$ or height $\times$ weight

It is important to remember that the world can feature a lot of non-linearities and interactive effects

$\rightarrow$ Your model should reflect those too!

Estimands vs. Estimators

Estimands are functions of the population data distribution. What you would estimate if you observed *everyone* in your population of interest

$\rightarrow$ E.g. a population mean or population line of best fit

Estimands vs. Estimators

Estimands are functions of the population data distribution. What you would estimate if you observed *everyone* in your population of interest

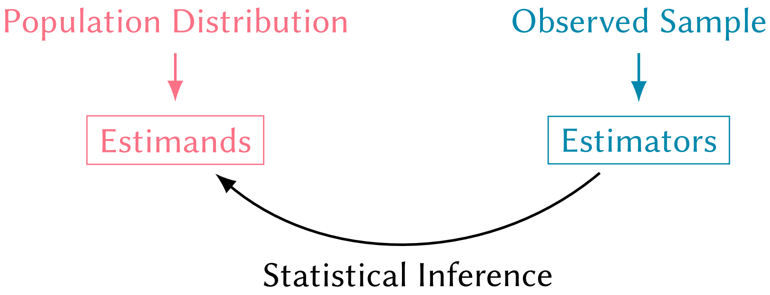
$\rightarrow$ E.g. a population mean or population line of best fit

Estimators are functions of the observed data itself (the "sample")

$\rightarrow$ E.g. a sample mean or OLS coefficients

Since your sample is random, so is your estimator. Each estimator has a distribution that we will call the *sample distribution*

The Lay of the Land



Population Regression

The OLS *estimator* $\hat{\beta}_{\text{OLS}}$ consistently estimates the regression *estimand* β_{OLS} under relatively weak conditions

Our statistical software uses a sample to estimate $\hat{\beta}_{\text{OLS}}$ and with our estimate we *infer* about β_{OLS}. With inference, we can say:

1. Our best guess at β_{OLS} is $\hat{\beta}_{\text{OLS}}$
2. With 95% confidence, β_{OLS} falls within the range

$$[\hat{\beta}_{\text{OLS}} - 1.96 * SE(\hat{\beta}_{\text{OLS}}), \hat{\beta}_{\text{OLS}} + 1.96 * SE(\hat{\beta}_{\text{OLS}})]$$

Repeated Sampling

When doing forecasting, we will observe *a single random sample* from the population. But, for conducting inference about the population parameter, it is useful to use the repeated sampling perspective:

$\rightarrow$ Imagine drawing a bunch of random samples of the sample size from the population. Let b denote each random sample.

$\rightarrow$ For each sample, form that sample's estimate $\hat{\theta}_b$

Repeated Sampling

When doing forecasting, we will observe *a single random sample* from the population. But, for conducting inference about the population parameter, it is useful to use the repeated sampling perspective:

- → Imagine drawing a bunch of random samples of the sample size from the population. Let b denote each random sample.
- → For each sample, form that sample's estimate $\hat{\theta}_b$

Since each sample is different, you have a distribution of $\hat{\theta}_b$. This is called the sampling distribution of the estimator.

Sample Distribution

Population mean

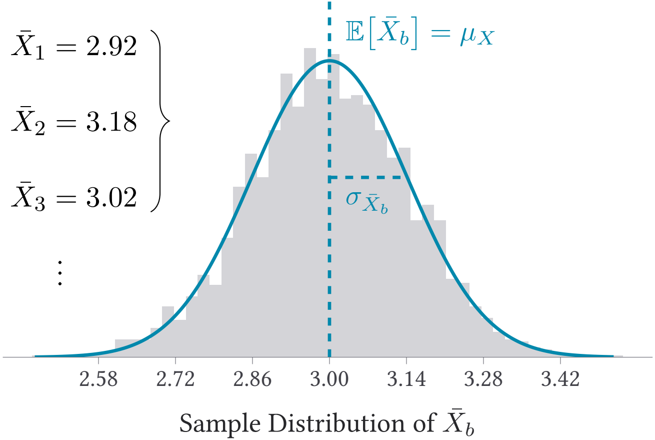
$\mu = 3$

sample of size n — $\bar{X}_1 = 2.92$

sample of size n — $\bar{X}_2 = 3.18$

sample of size n — $\bar{X}_3 = 3.02$

$\vdots$



Sample Distribution

The remarkable thing about sample distributions is that *most of the time* our estimators have a sample distribution that is a *normal distribution.* This is due to the central limit theorem

Unbiased Estimators

An estimator is unbiased if $\mathbb{E}\left[\hat{\theta}_b\right] = \theta$, the estimator is on average (across repeated samples) equal to the estimand

Consistent Estimators

An estimator is consistent if as $n \to \infty$

1. $\mathbb{E}\left[\hat{\theta}_b\right] \to \theta$ and
2. the standard deviation of the sample distribution of $\hat{\theta}_b$ collapse to 0

If you have a large enough sample, your sample estimator approaches the estimand

Cross-sectional Data

Cross-sectional data consists of many different *units* viewed at a point in time:

school_id	avg_sat_math	pct_white	pct_black
01M539	657	28.6%	13.3%
02M294	395	11.7%	38.5%
02M308	418	3.1%	28.2%
⋮	⋮	⋮	⋮
01M292	410	3.9%	24.4%
01M696	634	45.3%	17.2%
02M305	389	2.7%	41.9%

Time-series Data

Time-series data consists of a single observational unit viewed over multiple points in time:

month	day	hour	bikers	temp
1	1	0	16	0.24
1	1	1	40	0.22
1	1	2	32	0.22
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
12	31	21	52	0.40
12	31	22	38	0.38
12	31	23	31	0.36

Panel Data

Panel data is like time-series data, but for many different observational units:

fund_manager	month	return
1	1	-3.34%
1	2	3.76%
1	3	12.97%
⋮	⋮	⋮
2000	48	-3.76%
2000	49	2.25%
2000	50	6.68%