Topic #1 Assignment

ECON 5753 — University of Arkansas

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These assignments should be completed in groups of 1 or 2 but submitted individually.

Theoretical Questions

1. Let

$$\boldsymbol{A} = \begin{bmatrix} 6 & 7 & 9 \\ 1 & 2 & 3 \\ 8 & 4 & 6 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 10 & 9 & 8 \\ 7 & 5 & 4 \\ 1 & 7 & 6 \end{bmatrix}, \quad \text{and } \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 4 & 0 & 2 \end{bmatrix}.$$

Calculate the following:

i. $3\boldsymbol{B}^T + \boldsymbol{A}$

Answer:

$$\begin{bmatrix} 36 & 28 & 12 \\ 28 & 17 & 24 \\ 32 & 16 & 24 \end{bmatrix}$$

ii. $\boldsymbol{C}^T - 4\boldsymbol{A}$

Answer:

$$\begin{bmatrix} -23 & -28 & -32 \\ -4 & -7 & -12 \\ -30 & -16 & -22 \end{bmatrix}$$

iii. $(\boldsymbol{C}\boldsymbol{A})'$

Answer:

$$\begin{bmatrix} 22 & 1 & 40 \\ 15 & 2 & 36 \\ 21 & 3 & 48 \end{bmatrix}$$

2. Let

$$\boldsymbol{X} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}.$$

Verify the following two are equivalent:

$$X\beta \quad \text{and } X_{\cdot,1}\beta_1 + X_{\cdot,2}\beta_2 + X_{\cdot,3}\beta_3$$

Answer:

$$X\beta = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} x_{11}\beta_1 + x_{12}\beta_2 + x_{13}\beta_3 \\ x_{21}\beta_1 + x_{22}\beta_2 + x_{23}\beta_3 \\ x_{31}\beta_1 + x_{32}\beta_2 + x_{33}\beta_3 \end{bmatrix}$$

$$X_{\cdot,1}\beta_1 + X_{\cdot,2}\beta_2 + X_{\cdot,3}\beta_3 = \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} \beta_1 + \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} \beta_2 + \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} \beta_3 = \begin{bmatrix} x_{11}\beta_1 + x_{12}\beta_2 + x_{13}\beta_3 \\ x_{21}\beta_1 + x_{22}\beta_2 + x_{23}\beta_3 \\ x_{31}\beta_1 + x_{32}\beta_2 + x_{33}\beta_3 \end{bmatrix}$$

3. Match each matrix with its inverse:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 & 1 \\ -1 & 1 & -1 \\ 1 & 4 & 0 \end{bmatrix}, \quad \text{and } C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$a = \begin{bmatrix} -4 & -4 & 5 \\ 1 & 1 & -1 \\ 5 & 4 & -6 \end{bmatrix}, \quad b = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}, \quad \text{and } c = \text{ Not invertible.}$$

Answer: A and b; B and a; C and c. You can verify these by doing e.g. $Ab = bA = I$.

4. Let

$$\varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix} \right).$$

What is the distribution of $\begin{bmatrix} 1 & 1 \end{bmatrix} \varepsilon$?

Answer: Note $\begin{bmatrix} 1 & 1 \end{bmatrix} \varepsilon = \varepsilon_1 + \varepsilon_2$ is the sum of the two errors. The expectation is $\mathbb{E}[\varepsilon_1 + \varepsilon_2] = \mathbb{E}[\varepsilon_1] + \mathbb{E}[\varepsilon_2] = 2$.

The variance is

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_1^2 + \sigma_{12} \\ \sigma_{12} + \sigma_2^2 \end{bmatrix} = \sigma_1^2 + 2\sigma_{12} + \sigma_2^2$$

Therefore, we have $\begin{bmatrix} 1 & 1 \end{bmatrix} \varepsilon \sim \mathcal{N}(2, \sigma_1^2 + 2\sigma_{12} + \sigma_2^2)$.

5. Let $f(x, y) = x^2 + y^2 + 2xy$.

   i. What is $\frac{\partial}{\partial x} f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$?

   Answer: $\frac{\partial}{\partial x} f(x, y) = 2x + 2y$ and $\frac{\partial}{\partial y} f(x, y) = 2y + 2x$

   ii. Use this to create a taylor approximation of this function at (x_0, y_0). Write this in the form of:

$$f(x_0 + dx, y_0 + dy) \approx f(x_0, y_0) + \begin{bmatrix} \frac{\partial}{\partial x} f(x,y) & \frac{\partial}{\partial x} f(x,y) \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

Answer:

$$\begin{aligned} f(x_0 + dx, y_0 + dy) &\approx f(x_0, y_0) + \begin{bmatrix} \frac{\partial}{\partial x} f(x,y) & \frac{\partial}{\partial x} f(x,y) \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} \\ &= f(x_0, y_0) + \begin{bmatrix} 2x_0 + 2y_0 & 2y_0 + 2x_0 \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} \\ &= x_0^2 + y_0^2 + 2x_0 y_0 + (2x_0 + 2y_0)dx + (2y_0 + 2x_0)dy \end{aligned}$$

iii. Plug in $(x_0, y_0) = (0, 0)$ to create a linear approximation to this function. **Answer:** Evaluating the above expansion at $(x_0, y_0) = (0, 0)$ yields

$$f(x_0 + dx, y_0 + dy) = 0$$

Since $(0, 0)$ is the minimum of this function, the local linear approximation of it at this point is just the flat plane.