CHEG304 Homework 3

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1

Looking at the equation for the poisson distribution again

$$\frac{\lambda^x \exp(-\lambda)}{x!} \tag{poisson distribution}$$

Now, to find the expected value of this distribution we need to do an infinite sum of the poisson evaluated at each discrete x multiplied by x

$$E[x] = \sum_{i=1}^{\infty} x_i \frac{\lambda^{x_i} \exp(-\lambda)}{x_i!}$$

and using the properties of the factorial this becomes

$$E[x] = \exp(-\lambda) \sum_{i=1}^{\infty} \frac{\lambda^{x_i}}{(x_i - 1)!} = \exp(-\lambda)\lambda \sum_{i=1}^{\infty} \frac{\lambda^{x_i - 1}}{(x_i - 1)!}$$

it's not painfully obvious, but the sum is the taylor series of $\exp(\lambda)$,

$$\exp(\lambda) = \sum_{i=1}^{\infty} \frac{\lambda^i}{i!} \tag{taylor series of e^λ}$$

which makes our sum

$$E[x] = \lambda \exp(-\lambda) \exp(\lambda) = \lambda$$

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2

a

the valid variable space is all $t > 0$, so

$$\begin{aligned} \int_0^y \eta \exp(-\eta t)\ dt &= [-\exp(-\eta t)]_0^y \\ &= \exp(0) - \exp(-\eta y) \end{aligned}$$

now we want to know our integral to infinity, so we take the limit as y approaches infinity

$$\lim_{y \to \infty} 1 - \exp(-\eta y) = 1 - 0 = 1$$

b

i'll start by converting η into units of days

$$\eta = 1\frac{\text{failure}}{3\text{ years}} \times \frac{1\text{ year}}{365\text{ days}} = \frac{1}{1095}\frac{\text{failure}}{\text{day}}$$

now we (probably) want to take the integral from day 0 to day 90 of our pdf

$$\begin{aligned}\int_0^{90} \eta \exp(-\eta t)\ dt &= [-\exp(-\eta t)]_{t=0}^{t=90} \\ &= \exp(0) - \exp(-90\eta) \\ &= \boxed{0.0789}\end{aligned}$$

so about 8%

3

the joint pdf can be constructed by realizing if each day for X_1 is equally likely, the day for X_2 is just another perfectly random choice from the remaning days. We just need to keep the constraint in mind when putting together the table. now to show they are not independent consider $f(X_1 = 1, X_2 = 1)$.

	$X_2 = 1$	$X_2 = 2$	$X_2 = 3$	$X_2 = 4$	$X_2 = 5$	$f_1(X_1)$
$X_1 = 1$	1/25	1/25	1/25	1/25	1/25	1/5
$X_1 = 2$	0	1/20	1/20	1/20	1/20	1/5
$X_1 = 3$	0	0	1/15	1/15	1/15	1/5
$X_1 = 4$	0	0	0	1/10	1/10	1/5
$X_1 = 5$	0	0	0	0	1/5	1/5
$f_2(X_2)$	0.040	0.090	0.157	0.257	0.457	

$$\begin{aligned}f(X_1 = 1, X_2 = 1) &\stackrel{?}{=} f_1(X_1 = 1) \cdot f_1(X_2 = 1) \\ 0.04 &\stackrel{?}{=} (0.20) \cdot (0.04) \\ 0.04 &\neq 0.008\end{aligned}$$

therefore they are not independent. there is another interpretation I originally had where every possible combination of days had the same probability, 1/15. here is that PDF: (hopefully I don't get docked points for sharing both even though the wording is inprecise.)

	$X_2 = 1$	$X_2 = 2$	$X_2 = 3$	$X_2 = 4$	$X_2 = 5$	$f_1(X_1)$
$X_1 = 1$	1/15	1/15	1/15	1/15	1/15	5/15
$X_1 = 2$	0	1/15	1/15	1/15	1/15	4/15
$X_1 = 3$	0	0	1/15	1/15	1/15	3/15
$X_1 = 4$	0	0	0	1/15	1/15	2/15
$X_1 = 5$	0	0	0	0	1/15	1/15
$f_2(X_2)$	1/15	2/15	3/15	4/15	5/15	

now to show they are not independent consider $f(X_1 = 1, X_2 = 1)$.

$$f(X_1 = 1, X_2 = 1) \stackrel{?}{=} f_1(X_1 = 1) \cdot f_2(X_2 = 1)$$
$$1/15 \neq (5/15) \cdot (1/15)$$

therefore they are not independent.

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4

looking at our pdf

$$f(t_1, t_2) = c \exp(-0.1t_1 - 0.1t_2)$$

we know that for this to be valid the probabilities must sum to 1, which in math looks like

$$\int_{t_2=0}^{t_2=\infty} \int_{t_1=0}^{t_1=\infty} f(t_1, t_2)\ dt_1 dt_2 = 1$$

now, after mentally preparing for MATH243 war flashbacks, and using k and l to represent our upper bounds of integration (limit incoming later)

$$\begin{aligned}
c \int_{t_2=0}^{t_2=k} \int_{t_1=0}^{t_1=l} \exp(-0.1t_1 - 0.1t_2)\ dt_1 dt_2 &= c \int_{t_2=0}^{t_2=\infty} \Big[-10 \exp(-0.1t_1 - 0.1t_2) \Big]_{t_1=0}^{t_1=l} \\
&= 10c \int_{t_2=0}^{t_2=k} \exp(-0.1t_2) + \exp(-0.1l - 0.1t_2)\ dt_2 \\
&= 10c \int_{t_2=0}^{t_2=k} \exp(-0.1t_2)\ dt_2 + 10c \int_{t_2=0}^{t_2=k} \exp(-0.1l - 0.1t_2)\ dt_2 \\
&= 10c \Big[10 \exp(-0.1t_2) \Big]_{t_2=k}^{t_2=0} + 10c \Big[10 \exp(-0.1l - 0.1t_2) \Big]_{t_2=k}^{t_2=0} \\
&= 10c \big[10 - 10 \exp(-0.1k) \big] + 10c \big[10 \exp(-0.1l) - 10 \exp(-0.1l - 0.1k) \big] \\
&= 100c - 100 \exp(-0.1k) + 100c \exp(-0.1l) - 100c \exp(-0.1l - 0.1k)
\end{aligned}$$

now we want to take the limit as k, l approach infinity so we take an infinite integral

$$\lim_{k,l \to \infty} 100c - 100 \exp(-0.1k) + 100c \exp(-0.1l) - 100c \exp(-0.1l - 0.1k) = 100c$$

and now going back to our original statement,

$$\int_{t_2=0}^{t_2=\infty} \int_{t_1=0}^{t_1=\infty} f(t_1, t_2)\ dt_1 dt_2 = 1$$

$$\boxed{100c = 1 \longrightarrow c = 0.01}$$

now to find the marginal pdfs we need to integrate over all values of the other t_i

$$
\begin{aligned}
f_1(t_1) &= \lim_{k\to\infty} \int_0^k f(t_1, t_2)\ dt_2 \\
&= \lim_{k\to\infty} 0.01 \Big[10 \exp(-0.1t_1 - 0.1t_2) \Big]_{t_2=k}^{t_2=0} \\
&= \lim_{k\to\infty} 0.1 \Big[\exp(-0.1t_1) - \exp(-0.1t_1 - 0.1k) \Big] \\
&= 0.1 \exp(-0.1t_1)
\end{aligned}
$$

$$
\begin{aligned}
f_2(t_2) &= \lim_{l\to\infty} \int_0^l f(t_1, t_2)\ dt_1 \\
&= \lim_{l\to\infty} 0.01 \Big[10 \exp(-0.1t_1 - 0.1t_2) \Big]_{t_1=l}^{t_1=0} \\
&= \lim_{l\to\infty} 0.1 \Big[\exp(-0.1t_2) - \exp(-0.1t_2 - 0.1l) \Big] \\
&= 0.1 \exp(-0.1t_2)
\end{aligned}
$$

for independence it is sufficient to show that

$$f(t_1, t_2) = f_1(t_1) \cdot f_2(t_2)$$

so, using the derived marginal pdfs

$$
\begin{aligned}
0.01 \exp(-0.1t_1 - 0.1t_2) &\stackrel{?}{=} f_1(t_1) \cdot f_2(t_2) \\
&= (0.1 \exp(-0.1t_1)) \cdot (0.1 \exp(-0.1t_2)) \\
&= 0.01 \exp(-0.1t_1) \exp(-0.1t_2) \\
&= 0.01 \exp(-0.1t_1 - 0.1t_2) && \text{(properties of exp)}
\end{aligned}
$$

$\blacksquare$